Barotropic instability of shear flows

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Abstract

We consider barotropic instability of shear flows for incompressible fluids with Coriolis effects. For a class of shear flows, we develop a new method to find the sharp stability conditions. We study the flow with Sinus profile in details and find sharp stability boundary in the whole parameter space, which correct previous results in the fluid literature. The addition of the Coriolis force is found to bring some fundamental changes to the stability of shear flows. Moreover, we study the bifurcation of nontrivial traveling wave solutions and the linear inviscid damping near the shear flows. The first ingredient of our proof is a careful classification of the neutral modes. The second one is to write the linearized fluid equation in a Hamiltonian form and then use an instability index theory for general Hamiltonian PDEs. The last one is to study the singular and non-resonant neutral modes by using hypergeometric functions and singular Sturm-Liouville theory.

1 Introduction

When studying the large-scale motion of ocean and atmosphere, the rotation of the earth may affect the dynamics of the fluids significantly and therefore, Coriolis effects must be taken into account ([35]). In this paper, we study stability and instability of shear flows under Coriolis forces. We consider the fluids in a strip or channel denoted by

$$D = \{(x, y) \mid y \in [y_1, y_2]\},$$

where x is periodic. The fluid motion is modeled by the two-dimensional inviscid incompressible Euler equation with rotation

$$\partial_t \vec{u} + (\vec{u} \cdot \nabla)\vec{u} = -\nabla P - \beta y J \vec{u}, \quad t \times (x, y) \in [0, +\infty) \times D, \tag{1.1}$$

where $\vec{u} = (u_1, u_2)$ is the fluid velocity, P is the pressure,

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is the rotation matrix, and β is the Rossby number. Here, the term $-\beta y J \vec{u}$ denotes the Coriolis force under the beta-plane approximation. We assume the incompressible condition $\nabla \cdot \vec{u} = 0$ and the nonslip boundary condition

$$u_2 = 0, \text{ on } \partial D = \{y = y_1, y_2\}. \tag{1.2}$$

The vorticity ω is defined as $\omega := \operatorname{curl} \vec{u} = \partial_x u_2 - \partial_y u_1$, and the stream function ψ is introduced such that $\vec{u} = \nabla^\perp \psi = (\psi_y, -\psi_x)$. The vorticity form of (1.1) is

$$\partial_t \omega + (\vec{u} \cdot \nabla)\omega + \beta u_2 = 0, \tag{1.3}$$

which is also called the quasi-geostrophic equation in geophysical fluids ([35]). Consider a shear flow $\vec{u}_0 = (U(y), 0)$, $U \in C^2[y_1, y_2]$, which is a steady solution of (1.3). The linearized equation of (1.3) around the shear flow $\vec{u}_0$ is

$$\partial_t \omega + U \partial_x \omega - U'' u_2 + \beta u_2 = 0, \tag{1.4}$$

or

$$\partial_t \Delta\psi + U \partial_x \Delta\psi + (\beta - U'')\psi_x = 0, \tag{1.5}$$

in terms of the stream function ψ. To study the linear instability, it suffices to consider the normal mode solution $\psi(x, y, t) = \phi(y) e^{i\alpha(x - ct)}$, where $\alpha > 0$

is the wave number in the x-direction and $c = c_r + ic_i$ is the complex wave speed. Then (1.5) is reduced to the Rayleigh-Kuo equation

$$-\phi'' + \alpha^2\phi - \frac{\beta - U''}{U - c}\phi = 0, \tag{1.6}$$

with the boundary conditions

$$\phi(y_1) = \phi(y_2) = 0. \tag{1.7}$$

When $\beta = 0$, (1.6) becomes the classical Rayleigh Equation ([40]), which has been studied extensively (cf. [10, 15, 17, 18, 19, 38]).

The shear flow U is linear unstable if there exists a nontrivial solution to (1.6)–(1.7) with $\operatorname{Im} c > 0$. This so called barotropic instability is important for the dynamics of atmosphere and oceans. It has been a classical problem in geophysical fluid dynamics ([29, 30, 35]) since 1940s. Rossby first recognized the nature of barotropic instability and derived the linearized vorticity equation in [41]. Later, Kuo formulated the equation (1.6)–(1.7), and did some early studies in [29]. In particular, he gave a necessary condition for instability that $\beta - U''$ must change sign in the domain $[y_1, y_2]$, which generalized the classical Rayleigh criterion ([40]) for $\beta = 0$. In [33], Pedlosky showed that any unstable wave speed $c = c_r + ic_i$ ($c_i > 0$) must lie in the following semicircle

$$\left(c_r - \frac{U_{\min} + U_{\max}}{2}\right)^2 + c_i^2 \leq \left(\frac{U_{\max} - U_{\min}}{2} + \frac{|\beta|}{2\alpha^2}\right)^2, \tag{1.8}$$

which is a generalization of Howard's semicircle theorem [14] for $\beta = 0$. Here, $U_{\min} = \min U$ and $U_{\max} = \max U$. Additionally, the following characterization for the unstable wave speeds is given in [29, 32, 34].

Lemma 1.1 *If $\beta > 0$, then there is no nontrivial solution of (1.6)–(1.7) for $c_r > U_{\max}$; if $\beta < 0$, then there is no nontrivial solution of (1.6)–(1.7) for $c_r < U_{\min}$.*

Although there are several necessary conditions as indicated above, there has been very few sufficient conditions for the barotropic instability of shear flows. In the fluid literature, the linear instability was studied for some special shear flows. The barotropic instability of Bickley jet ($U(y) = \operatorname{sech}^2 y$) was studied by numerical computations and asymptotic analysis (cf. [2, 4, 11, 16, 25, 30, 31]). The stability boundary of hyperbolic-tangent shear flow was studied in [7, 16, 30]. Other references on the barotropic instability include [8, 9, 23, 24, 37]. In this paper, we consider the barotropic stability of the following class of shear flows.

Definition 1.1 *The flow U is in class $\mathcal{K}$ if $U \in C^3[y_1, y_2]$ and for each $\beta \in [\min U'', \max U'']$, there exists some $U_\beta \in [U_{\min}, U_{\max}]$ such that*

$$K_\beta(y) := \frac{\beta - U''(y)}{U(y) - U_\beta}$$

is nonnegative and bounded on $[y_1, y_2]$. Furthermore, U is said to be in class $\mathcal{K}^+$ if U is in class $\mathcal{K}$ and K_β is positive on $[y_1, y_2]$ for each $\beta \in [\min U'', \max U'']$.

Flows in class $\mathcal{K}^+$ include $U(y) = \sin y,\ \tanh y$, and more generally any $U(y)$ satisfying the ODE $U'' = g(U)$ with $g \in C^1([U_{\min}, U_{\max}])$ and $g' < 0$ on $[U_{\min}, U_{\max}]$. One important conclusion for class $\mathcal{K}^+$ flows is that there is a uniform H^2 bound for the unstable solutions of (1.6)–(1.7), see Lemma 2.4. Neutral modes are the solutions of (1.6)–(1.7) with $c \in \mathbf{R}$. In the study of stability of a shear flow $U(y)$, it is often useful to locate the neutral modes which are limits of a sequence of unstable modes. These so called neutral limiting modes determine the stability boundary. In Theorem 2.1, the neutral limiting modes, actually all H^2 neutral modes for a general shear flow, are classified into four types by their phase speed c: 1) $c = U_\beta$; 2) $c = U(y_1)$ or $U(y_2)$; 3) c is a critical value of U; 4) c is outside the range of U. Here, the neutral modes of types 2) and 3) might be singular and type 4) is called non-resonant since the phase speed c causes no interaction with the basic flow $U(y)$. This contrasts greatly with the non-rotating case $\beta = 0$, where it was shown in [20] that for neutral modes in H^2, c must be an inflection value of U.

In the literature, it is common to look for unstable modes near neutral modes. A useful approach to determine the stability boundary is to study the bifurcation of unstable modes near each neutral limiting wave number and then combine these information to detect the stability/instability at any wave number. In [18], this approach was used to show that: when $\beta = 0$, any flow $U(y)$ in class $\mathcal{K}^+$ is linearly stable if and only if $\alpha \geq \alpha_{\max}$, where $-\alpha_{\max}^2$ is the first eigenvalue of the operator $-\frac{d^2}{dy^2} - K_0(y)$. However, when $\beta \neq 0$, there are several difficulties in using this approach. First, we might need to deal with the subtle perturbation problem near singular neutral modes. Second, for non-resonant neutral modes, the phase speed c is to be determined. Moreover, near these non-resonant neutral modes, the bifurcation of unstable modes is usually non-smooth (see Remark 2.3). In some literature (e.g. [39]), it was believed that these non-resonant neutral modes are not adjacent to unstable modes. This turns out to be not true by our study of the Sinus flow in Section 4, see also Remark 2.3.

In this paper, we use a new approach to study the barotropic stability of shear flows. First, we write the linearized equation in a Hamiltonian form $\partial_t \omega = JL\omega$, where J is anti-self-adjoint and L is self-adjoint as defined in (3.4). For a fixed wave number α, by taking the ansatz $\omega = \omega_\alpha (y,t) e^{i\alpha x}$, the linearized equation is reduced to the Hamiltonian form $\partial_t \omega_\alpha = J_\alpha L_\alpha \omega_\alpha$, where J_α, L_α are defined in (3.8). Then by the instability index theorem recently developed in [22] for general Hamiltonian PDEs, we get the index formula (3.14). This formula implies that to determine the instability at any $\alpha > 0$, it suffices to count the number of neutral modes with a nonpositive signature (i.e. $\langle L_\alpha \omega_\alpha, \omega_\alpha \rangle \leq 0$). The four types of neutral modes in H^2 are counted separately. In particular, for the counting of non-resonant neutral modes, we introduce a function $f_{\beta,n}(c)$ for $c \notin [U_{\min}, U_{\max}]$ such that $-f_{\beta,n}(c)$ is the n-th eigenvalue of $-\frac{d^2}{dy^2} - \frac{\beta - U''}{U-c}$. An important observation is that for a non-resonant neutral mode (c, α, β, ϕ), the sign $\langle L_\alpha \omega_\alpha, \omega_\alpha \rangle$ is determined by $f'_{\beta,n}(c)$, where $\alpha^2 = f_{\beta,n}(c) > 0$ and $\omega_\alpha = -\phi'' + \alpha^2 \phi$. Therefore, by studying the shape of the graph of $f_{\beta,n}(c)$, we are able to count the non-resonant neutral modes with a nonpositive signature. Combining with the index count for the other three types of neutral modes, we can find the instability boundary in the whole parameter space (α, β). See Subsection 3.3 for more detailed discussions about this approach. In this approach, we avoid the study of the bifurcation of unstable modes near neutral modes, which is particularly tricky for singular and non-resonant neutral modes.

In Section 4, we carry out the above approach in details for the Sinus flow $U(y) = \frac{1+\cos(\pi y)}{2}, y \in (-1,1)$. The sharp stability boundary is found in Theorems 4.2–4.3, which corrects the stability boundary claimed in [30] and [35] based on numerical computations. In particular, neutral limiting modes of all four types are found in the stability boundary we proved. For the linearly stable case, we also count the number of non-resonant neutral modes, near which the existence of nontrivial traveling wave solutions is shown. The instability boundary we obtained is confirmed by more accurate numerical results. See Section 7 for more detailed discussions. Moreover, the Hamiltonian structure of the linearized equation is used to prove the linear inviscid damping in the stable cases with no neutral modes (Theorem 6.1) and in the center space for the unstable case (Theorem 6.2). These results will be useful for the further study of nonlinear dynamics near the shear flows, such as nonlinear inviscid damping (stable case) and the construction of invariant manifolds (unstable case).

This paper is organized as follows. In Section 2, the neutral modes in H^2 are classified for general shear flows. For shear flows in class $\mathcal{K}^+$, we prove uniform H^2 bound for unstable modes, and as a result, the classification of neutral limiting modes is obtained. In Section 3, by writing the linearized fluid equation in a Hamiltonian form, an instability index formula is derived for flows in class $\mathcal{K}^+$. In Section 4, the Sinus flow is studied in details and the sharp stability boundary is obtained in Theorems 4.2–4.3. In Sections 5 and 6, the bifurcation of nontrivial traveling waves and the linear inviscid damping are studied, respectively. Section 7 gives a summary of the results on Sinus flow, including the new numerical evidence and the comparison with previous works.

2 Neutral modes in H^2

In this section, we first study the classification of wave speeds of neutral modes in H^2 for a general shear flow. Then for a flow in class $\mathcal{K}^+$, we prove that any neutral limiting mode is in H^2. As a consequence, we classify all the wave speeds of neutral limiting modes for a flow in class $\mathcal{K}^+$. The neutral limiting modes are important since they give all the possible transition points from instability to stability.

2.1 Classification of neutral modes in H^2

In this subsection, we give a complete description of neutral modes in H^2 for a general shear flow. First, we give the precise definition of neutral modes.

Definition 2.1 *$(c_s, \alpha_s, \beta_s, \phi_s)$ is said to be a neutral mode if $c_s \in \mathbf{R}$, $\alpha_s > 0$, $\beta_s \in \mathbf{R}$, and ϕ_s is a nontrivial solution to the Sturm-Liouville problem*

$$-\phi_s'' + \alpha_s^2 \phi_s - \frac{\beta_s - U''}{U - c_s}\phi_s = 0, \quad on \;\; [y_1, y_2],$$

with the boundary $\phi_s(y_1) = \phi_s(y_2) = 0$. If $\phi_s \in H^2[y_1, y_2]$, we call $(c_s, \alpha_s, \beta_s, \phi_s)$ to be a neutral mode in H^2.

For convenience, we make the following assumption:

Hypothesis 2.1 *Let $U \in C^m([y_1, y_2])$, $m \geq 1$. Assume that for any fixed $y \in [y_1, y_2]$, there exists $1 \leq j_0 \leq m$ such that $U^{(j_0)}(y) \neq 0$.*

Remark 2.1 (i) *If U is analytic and nontrivial on $[y_1, y_2]$, then U satisfies Hypothesis 2.1.*

(ii) *Any flow U in class $\mathcal{K}^+$ satisfies Hypothesis 2.1. Indeed if $0 \in [\min U''$, $\max U'']$, then U satisfies Hypothesis 2.1 by Remark 3.2 in [18]. If $0 \notin [\min U'', \max U'']$, then $U''(y) \neq 0$ for any $y \in [y_1, y_2]$ and Hypothesis 2.1 is again satisfied.*

Assume that U satisfies Hypothesis 2.1. Then for any $c \in [U_{\min}, U_{\max}]$, it is easy to see that $\{U - c = 0\}$ is a finite set, which we denote by

$$\{z_i \mid 1 \leq i \leq k_c,\ z_1 < z_2 < \cdots < z_{k_c}\}.$$

Set $z_0 := y_1$ and $z_{k_c+1} := y_2$.

Lemma 2.1 *Assume that U satisfies Hypothesis 2.1. Let ϕ be a solution of (1.6)–(1.7) with $\alpha > 0$, $\beta \in \mathbf{R}$ and $c \in [U_{\min}, U_{\max}]$. Assume that ϕ is sectionally continuous on (z_j, z_{j+1}), $j = 0, 1, \cdots, k_c$. If $U'(z_{i_0}) \neq 0$ for some $1 \leq i_0 \leq k_c$, then ϕ can not vanish at z_{i_0-1}, z_{i_0} and z_{i_0+1} simultaneously unless it vanishes identically on at least one of the intervals (z_{i_0-1}, z_{i_0}) and (z_{i_0}, z_{i_0+1}).*

Proof. Suppose $\phi(z_j) = 0$ for any $j \in \{i_0 - 1, i_0, i_0 + 1\}$. Assume that ϕ is nontrivial on both intervals (z_{i_0-1}, z_{i_0}) and (z_{i_0}, z_{i_0+1}). By (1.6), we get that

$$[(U - c)\phi' - U'\phi]' = \left[\alpha^2(U - c) - \beta\right]\phi. \tag{2.1}$$

We divide the proof into several cases.

Case 1. $\beta > 0$ and $U'(z_{i_0}) > 0$. Then $U(z) - c < 0$ for any $z \in (z_{i_0-1}, z_{i_0})$.

Let $\tilde{z} \in [z_{i_0-1}, z_{i_0})$ be the nearest zero of ϕ to z_{i_0}. We can assume that $\phi > 0$ on $(\tilde{z}, z_{i_0})$, $\phi'(\tilde{z}) \geq 0$, and $\phi'(z_{i_0}) \leq 0$.

Integrating (2.1) over $(\tilde{z}, z_{i_0})$, we get that

$$-(U(\tilde{z})-c)\phi'(\tilde{z}) = \int_{\tilde{z}}^{z_{i_0}} [\alpha^2(U-c)-\beta]\phi \, dy. \tag{2.2}$$

If $\tilde{z} = z_{i_0-1}$, then the LHS of (2.2) is 0. Since $\beta > 0$ and $U(y) - c < 0, \phi(y) > 0$ for any $y \in (\tilde{z}, z_{i_0})$, the RHS of (2.2) is negative, which is a contradiction.

If $\tilde{z} \neq z_{i_0-1}$, by (2.2) we have

$$-\phi'(\tilde{z}) = \int_{\tilde{z}}^{z_{i_0}} \left(\alpha^2 \frac{U-c}{U(\tilde{z})-c} - \frac{\beta}{U(\tilde{z})-c} \right) \phi \, dz. \tag{2.3}$$

Since $\frac{\beta}{U(\tilde{z})-c} < 0$ and

$$\frac{U(y)-c}{U(\tilde{z})-c} > 0, \;\; \phi(y) > 0, \tag{2.4}$$

for any $y \in (\tilde{z}, z_{i_0})$, the RHS of (2.3) is positive. This is a contradiction since the LHS of (2.3) is nonpositive.

Case 2. $\beta < 0$ and $U'(z_{i_0}) < 0$. Then $U(z) - c > 0$ for any $z \in (z_{i_0-1}, z_{i_0})$.

Let $\tilde{z} \in [z_{i_0-1}, z_{i_0})$ be the nearest zero of ϕ to z_{i_0}. We assume that $\phi > 0$ on $(\tilde{z}, z_{i_0})$, $\phi'(\tilde{z}) \geq 0$, and $\phi'(z_{i_0}) \leq 0$. Integrating (2.1) from $\tilde{z}$ to z_{i_0}, we get (2.2) in this case and the contradiction is obtained by a similar reason as that in Case 1.

Case 3. $\beta > 0$ and $U'(z_{i_0}) < 0$. Then $U(z) - c < 0$ for any $z \in (z_{i_0}, z_{i_0+1})$.

Let $\tilde{z} \in (z_{i_0}, z_{i_0+1}]$ be the nearest zero of ϕ to z_{i_0}. We can assume that $\phi > 0$ on $(z_{i_0}, \tilde{z})$, $\phi'(z_{i_0}) \geq 0$, and $\phi'(\tilde{z}) \leq 0$. Integrating (2.1) from z_{i_0} to $\tilde{z}$, we get (2.2) in this case. If $\tilde{z} = z_{i_0+1}$, then the LHS of (2.2) is 0. Since $\beta > 0$, and $U(y) - c < 0, \phi > 0$ for any $y \in (z_{i_0}, \tilde{z})$, the RHS of (2.2) is positive, which is a contradiction. If $\tilde{z} \neq z_{i_0+1}$, by (2.2) we get (2.3). Since $\frac{\beta}{U(\tilde{z})-c} < 0$ and (2.4) holds for any $y \in (z_{i_0}, \tilde{z})$, the RHS of (2.3) is negative while the LHS of (2.3) is nonnegative. A contradiction again.

Case 4. $\beta < 0$ and $U'(z_{i_0}) > 0$. Then $U(z) - c > 0$ for any $z \in (z_{i_0}, z_{i_0+1})$.

Let $\tilde{z} \in (z_{i_0}, z_{i_0+1}]$ be the nearest zero of ϕ to z_{i_0}. We assume that $\phi > 0$ on $(z_{i_0}, \tilde{z})$, $\phi'(z_{i_0}) \geq 0$, and $\phi'(\tilde{z}) \leq 0$. Integrating (2.1) from z_{i_0} to $\tilde{z}$, we get (2.2) in this case. The contradiction is obtained by a similar reason as that in Case 3.

Thus, ϕ must vanish identically on at least one of the intervals (z_{i_0-1}, z_{i_0}) and (z_{i_0}, z_{i_0+1}). ■

The following lemma is on the uniqueness of the initial value problem for the ODE (1.6) when the initial point is a non-degenerate zero of $U-c$. It will be used later in the classification of neutral modes.

Lemma 2.2 *Let ϕ be a solution of (1.6) with $\alpha > 0$, $\beta \in \mathbf{R}$ and $c \in [U_{\min}, U_{\max}]$. Assume that $y_1 \le x_0 < x_1 < x_2 \le y_2$ satisfy $U(x_1) - c = 0$ and $U - c \neq 0$ on $(x_0, x_1) \cup (x_1, x_2)$. If $U'(x_1) \neq 0$ and $\phi \in C^1(x_0, x_2)$ satisfies the initial conditions $\phi(x_1) = a_1$, $\phi'(x_1) = a_2$ for some $a_1, a_2 \in \mathbf{R}$, then ϕ is unique on the interval (x_0, x_2).*

Proof. We will show that $\phi(x_1) = \phi'(x_1) = 0$ implies $\phi \equiv 0$ on (x_1, x_2), and $\phi \equiv 0$ on (x_0, x_1) can be shown similarly.

Let $v := \phi \in C^1(x_1, x_2)$ and $u := \phi' \in C^0(x_1, x_2)$. Then (1.6) can be written as a first order ODE system

$$\begin{cases} \frac{dv}{dt} = u, \\ \frac{du}{dt} = \alpha^2 v - \frac{\beta - U''}{U-c} v, \end{cases} \tag{2.5}$$

with the initial data $v(x_1) = u(x_1) = 0$. For a fixed $z \in [x_1, x_2)$ and any $s \in [x_1, z]$,

$$|v(s)| \le \int_{x_1}^{s} |u(\tau)| d\tau \le (z - x_1) |u|_{L^\infty}(z) \,, \tag{2.6}$$

where $|u|_{L_\infty}(z) := \sup_{x_1 \le s \le z} |u(s)|$. Thus,

$$|v|_{L^\infty}(z) \le (z - x_1)|u|_{L^\infty}(z).$$

Since $U'(x_1) \neq 0$, there exists $\delta_0 > 0$ such that $U'(s) \neq 0$ for $s \in [x_1, x_1 + \delta_0]$. Let $\delta_1 := \min_{x_1 \le s \le x_1 + \delta_0} |U'(s)|$ and $C_0 := \max_{y_1 \le s \le y_2} |\beta - U''(s)|$. Then for each $z \in [x_1, x_1 + \delta_0]$,

$$\left| \frac{[(\beta - U'')v](z)}{U(z) - c} \right| = \left| (\beta - U'')(z) \frac{v(z) - v(x_1)}{U(z) - U(x_1)} \right| \le \frac{C_0 |u|_{L^\infty}(z)}{\delta_1}, \tag{2.7}$$

and thus by (2.5)–(2.7)

$$\begin{aligned} |u|_{L^\infty}(z) &\le \int_{x_1}^{z} \left(|\alpha^2 v(\tau)| + \left| \frac{[(\beta - U'')v](\tau)}{U(\tau) - c} \right| \right) d\tau \\ &\le \left(\alpha^2 (x_2 - x_1) + \frac{C_0}{\delta_1} \right) \int_{x_1}^{z} |u|_{L^\infty}(\tau) d\tau. \end{aligned}$$

Therefore by Gronwall inequality, we have $u \equiv 0$ and thus $v = \phi \equiv 0$ on $[x_1, x_1 + \delta_0]$. This implies that $\phi \equiv 0$ on (x_1, x_2), since the ODE (1.6) is regular in (x_1, x_2). This completes the proof. ■

The following theorem lists all possible wave speeds of neutral modes in H^2 for a general shear flow.

Theorem 2.1 *Assume that U satisfies Hypothesis 2.1. Let $(\phi_s, \alpha_s, \beta, c_s)$ be a neutral mode with $\alpha_s > 0$ $\beta \in \mathbf{R}$ and $\phi_s \in H^2$. Then the wave speed c_s must be one of the following:*

(i) $c_s = U_\beta$;

(ii) $c_s = U(y_1)$ *or* $c_s = U(y_2)$;

(iii) c_s *is a critical value of* U;

(iv) $c_s \notin [U_{\min}, U_{\max}]$.

Proof. It suffices to show that if $c_s \in [U_{\min}, U_{\max}]$, then one of cases (i)-(iii) is true. Suppose that $c_s \in [U_{\min}, U_{\max}]$ and

$$\{U(z) - c_s = 0\} \cap \{y_1, y_2\} \neq \emptyset.$$

Then $c_s = U(y_1)$ or $c_s = U(y_2)$, that is, case (ii) is true. Otherwise, $c_s \neq U(y_i), i = 1, 2$. Denote the zeros of $U - c_s = 0$ in the interval (y_1, y_2) by $z_1 < \cdots < z_{k_c}$. Let $z_0 := y_1$ and $z_{k_c+1} := y_2$. We consider two cases below.

Case 1. There exists $1 \leq i_0 \leq k_c$ such that $\beta = U''(z_{i_0})$. Since U is in class $\mathcal{K}^+$, this implies that $U(z_{i_0}) - U_\beta = 0$. Thus we have $c_s = U(z_{i_0}) = U_\beta$, i.e. (i) is true.

Case 2. $\beta \neq U''(z_i)$ for any $1 \leq i \leq k_c$. We divide it into two subcases.

Case 2a. $U'(z_i) \neq 0$ for any $1 \leq i \leq k_c$.

In this subcase, we claim that there exists $1 \leq i_1 \leq k_c$ such that $\phi_s(z_{i_1}) \neq 0$. Suppose otherwise, $\phi_s(z_i) = 0$ for any $1 \leq i \leq k_c$. For any fixed $1 \leq i_0 \leq k_c$, by the fact that $U'(z_{i_0}) \neq 0$ and by Lemma 2.1, $\phi_s \equiv 0$ on at least one of the intervals $[z_{i_0-1}, z_{i_0}]$ and $[z_{i_0}, z_{i_0+1}]$. Since $\phi_s \in H^2[y_1, y_2]$, it follows that $\phi_s \in C^1([y_1, y_2])$ and by Lemma 2.2, $\phi \equiv 0$ on $[z_{i_0-1}, z_{i_0+1}]$ and hence on $[y_1, y_2]$. Thus, there exists $1 \leq i_1 \leq k_c$ such that $\phi(z_{i_1}) \neq 0$. Then near z_{i_1},

$$\phi_s'' = \alpha_s^2 \phi_s - \frac{\beta - U''}{U - c_s} \phi_s \notin L^2_{loc}[y_1, y_2],$$

which is a contradiction to $\phi_s \in H^2[y_1, y_2]$.

Case 2b. There exists $1 \le i_2 \le k_c$ such that $U'(z_{i_2}) = 0$. In this subcase, $c_s = U(z_{i_2})$ is a critical value of U. This finishes the proof of Theorem 2.1. $\blacksquare$

Remark 2.2 *For the neutral modes in Theorem 2.1, we call (i) to be regular, (ii)–(iii) to be singular and (iv) to be non-resonant. For $\beta = 0$, it can be shown that only (i) is true, that is, for all neutral modes in H^2, the phase speed must be an inflection value of U (see, for example, Appendix of [20]).*

2.2 Neutral limiting modes for flows in class $\mathcal{K}^+$

In this subsection, we show that for flows in class $\mathcal{K}^+$, unstable solutions of (1.6)–(1.7) have uniform H^2 bound, which implies that any neutral limiting mode must be in H^2 . Consequently, by Theorem 2.1 we get the classification of neutral limiting phase speeds for flows in class $\mathcal{K}^+$.

First, we give the precise definition of neutral limiting modes.

Definition 2.2 *Let $\beta \in (\min U'', \max U'')$. We call $(c_s, \alpha_s, \beta, \phi_s)$ be a neutral limiting mode if $c_s \in \mathbf{R}$, $\alpha_s > 0$ and there exists a sequence of unstable modes $\{(c_k, \alpha_k, \beta, \phi_k)\}$ (with $c_k^i > 0$ and $\|\phi_k\|_{L^2} = 1$) to (1.6)–(1.7) such that $c_k \to c_s, \alpha_k \to \alpha_s$, ϕ_k converges uniformly to ϕ_s on any compact subset of S_0 as $k \to \infty$, ϕ_s'' exists on S_0, and ϕ_s satisfies*

$$(U - c_s)(-\phi_s'' + \alpha_s^2 \phi_s) - (\beta - U'')\phi_s = 0 \tag{2.8}$$

on S_0, where S_0 denotes the complement of the set $\{z \in [y_1, y_2] |\, U(z) - c_s = 0\}$ in the interval $[y_1, y_2]$. Here c_s is called the neutral limiting phase speed and α_s is called the neutral limiting wave number.

The following two identities for the solutions of (1.6)–(1.7) will be used later. (2.9) below was obtained in [29].

Lemma 2.3 *Let ϕ be a solution of (1.6)–(1.7) with $c = c_r + ic_i$ $(c_i > 0)$. Then*

$$\int_{y_1}^{y_2} \frac{(\beta - U'')}{|U - c|^2} |\phi|^2 dy = 0, \tag{2.9}$$

and

$$J_q(\phi) := \int_{y_1}^{y_2} \left[|\phi'|^2 + \alpha^2 |\phi|^2 - \frac{(\beta - U'')(U - q)}{|U - c|^2} |\phi|^2 \right] dy = 0. \tag{2.10}$$

for any $q \in \mathbf{R}$.

Proof. Multiplying (1.6) by the complex conjugate $\bar{\phi}$ and integrating it, by (1.7) we have that

$$\int_{y_1}^{y_2} \left[|\phi'|^2 + \alpha^2|\phi|^2 - \frac{(\beta - U'')(U - c_r + ic_i)}{|U-c|^2}|\phi|^2 \right] dy = 0.$$

Comparing real and imaginary parts of above, we get

$$\int_{y_1}^{y_2} \left[|\phi'|^2 + \alpha^2|\phi|^2 - \frac{(\beta - U'')(U - c_r)}{|U-c|^2}|\phi|^2 \right] dy = 0, \tag{2.11}$$

and

$$\int_{y_1}^{y_2} \frac{(\beta - U'')c_i}{|U-c|^2}|\phi|^2 dy = 0,$$

which implies (2.9) due to $c_i > 0$. Combining (2.9) and (2.11), we get (2.10). ■

In the next lemma, we get a priori estimates of unstable solutions of (1.6)–(1.7), from which the regularity of neutral limiting modes follows.

Lemma 2.4 *Let the flow* U *be in class* $\mathcal{K}^+$ *and* $\beta \in (\min U'', \max U'')$. *If* ϕ *is a solution of (1.6)–(1.7) with* $c = c_r + ic_i$ $(c_i > 0)$, *then*

$$\int_{y_1}^{y_2} (|\phi'|^2 + \alpha^2|\phi|^2) dy < \int_{y_1}^{y_2} K_\beta |\phi|^2 dy, \tag{2.12}$$

$$\int_{y_1}^{y_2} (|\phi''|^2 + 2\alpha^2|\phi'|^2 + \alpha^4|\phi|^2) dy < \|K_\beta\|_\infty \int_{y_1}^{y_2} K_\beta |\phi|^2 dy. \tag{2.13}$$

Proof. Let $q = U_\beta - 2(U_\beta - c_r)$ in (2.10). Then

$$
\begin{aligned}
&\int_{y_1}^{y_2} (|\phi'|^2 + \alpha^2|\phi|^2)dy \\
=&\int_{y_1}^{y_2} \frac{(\beta - U'')[(U-U_\beta) + 2(U_\beta - c_r)]}{|U-c|^2}|\phi|^2 dy \\
=&\int_{y_1}^{y_2} K_\beta \frac{(U-U_\beta)^2 + 2(U-U_\beta)(U_\beta - c_r)}{|U-c|^2}|\phi|^2 dy \\
=&\int_{y_1}^{y_2} K_\beta \frac{(U-c_r)^2 - (U_\beta - c_r)^2}{|U-c|^2}|\phi|^2 dy \\
\le&\int_{y_1}^{y_2} K_\beta \frac{(U-c_r)^2}{|U-c|^2}|\phi|^2 dy \\
<&\int_{y_1}^{y_2} K_\beta |\phi|^2 dy.
\end{aligned}
$$

This completes the proof of (2.12).

Now we show that (2.13) holds. Multiplying (1.6) by $\bar{\phi}''$ and integrating it, by (1.7) one has that

$$
\begin{aligned}
&\int_{y_1}^{y_2} (|\phi''|^2 + \alpha^2|\phi'|^2)dy \\
=&-\alpha^2 \int_{y_1}^{y_2} \frac{\beta - U''}{U-c}|\phi|^2 dy + \int_{y_1}^{y_2} \frac{(\beta - U'')^2|\phi|^2}{|U-c|^2}dy \\
=&-\alpha^2 \int_{y_1}^{y_2} \frac{(\beta - U'')(U - c_r + ic_i)}{|U-c|^2}|\phi|^2 dy + \int_{y_1}^{y_2} \frac{(\beta - U'')^2}{|U-c|^2}|\phi|^2 dy.
\end{aligned}
$$

Taking the real part in the above equality and using Lemma 2.3, we get that

$$
\begin{aligned}
&\int_{y_1}^{y_2} (|\phi''|^2 + \alpha^2|\phi'|^2)dy \\
=&-\alpha^2 \int_{y_1}^{y_2} \frac{(\beta - U'')(U - c_r)}{|U-c|^2}|\phi|^2 dy + \int_{y_1}^{y_2} \frac{(\beta - U'')^2}{|U-c|^2}|\phi|^2 dy \\
=&-\alpha^2 \int_{y_1}^{y_2} (|\phi'|^2 + \alpha^2|\phi|^2)dy + \int_{y_1}^{y_2} \frac{(\beta - U'')^2}{|U-c|^2}|\phi|^2 dy.
\end{aligned}
$$

Thus, by Lemma 2.3 and (2.12) one gets that

$$\begin{aligned}
&\int_{y_1}^{y_2}(|\phi''|^2+2\alpha^2|\phi'|^2+\alpha^4|\phi|^2)dy\\
=&\int_{y_1}^{y_2}\frac{(\beta-U'')^2}{|U-c|^2}|\phi|^2dy\\
=&\int_{y_1}^{y_2}\frac{K_\beta^2(U-U_\beta)^2}{|U-c|^2}|\phi|^2dy\\
\le&\|K_\beta\|_\infty\int_{y_1}^{y_2}\frac{K_\beta(U-U_\beta)^2}{|U-c|^2}|\phi|^2dy\\
=&\|K_\beta\|_\infty\int_{y_1}^{y_2}\frac{(\beta-U'')(U-U_\beta)}{|U-c|^2}|\phi|^2dy\\
=&\|K_\beta\|_\infty\int_{y_1}^{y_2}(|\phi'|^2+\alpha^2|\phi|^2)dy\\
<&\|K_\beta\|_\infty\int_{y_1}^{y_2}K_\beta|\phi|^2dy.
\end{aligned}$$

This completes the proof of (2.13). ■

By Lemma 2.4, unstable modes with the normalized L^2 norm have a uniform H^2 bound, which then implies the H^2 regularity of neutral limiting modes as shown in the next Lemma.

Lemma 2.5 *Let the flow U be in class $\mathcal{K}^+$ and $\beta\in(\min U'',\max U'')$. Suppose that $(\phi_s,\alpha_s,\beta,c_s)$ is a neutral limiting mode. Then $\phi_s\in H^2[y_1,y_2]$.*

Proof. Let $\{(\phi_k,\alpha_k,\beta,c_k)\}_{k=1}^\infty$ be a sequence of unstable modes such that $\{(\phi_k,\alpha_k,\beta,c_k)\}_{k=1}^\infty$ converges to $(\phi_s,\alpha_s,\beta,c_s)$ in the sense of Definition 2.2. Note that ϕ_k has been normalized such that $\|\phi_k\|_{L^2}=1$ for each $k\ge1$. Since U is in class $\mathcal{K}^+$, by Lemma 2.4, there exists a constant $C>0$ such that $\|\phi_k\|_{H^2}\le C$ for all $k\ge1$. Thus there exists a subsequence (to simplify notations, we still use $\{\phi_k\}_{k=1}^\infty$ for this subsequence) and $\phi_0\in H^2[y_1,y_2]$ such that $\{\phi_k\}_{k=1}^\infty$ weakly converges to ϕ_0 in $H^2[y_1,y_2]$ as $k\to\infty$. By the compact embedding of $H^2[y_1,y_2]\hookrightarrow C^1[y_1,y_2]$, up to a subsequence, $\{\phi_k\}_{k=1}^\infty$ strongly converges to ϕ_0 in $C^1[y_1,y_2]$ as $k\to\infty$. Note that for each $k\ge1$, $(\phi_k,\alpha_k,\beta,c_k)$ satisfies the equation

$$-(U-c_k)\phi_k''+\alpha_k^2(U-c_k)\phi_k-(\beta-U'')\phi_k=0$$

with $\phi_k(y_1) = \phi_k(y_2) = 0$. For any test function $\varphi \in H_0^1[y_1, y_2]$, we have

$$\int_{y_1}^{y_2} \left[\phi_k'[(U - c_k)\varphi]' + \alpha_k^2(U - c_k)\phi_k\varphi - (\beta - U'')\phi_k\varphi\right] dy = 0.$$

Letting $k \to \infty$, we obtain

$$\int_{y_1}^{y_2} \left[\phi_0'[(U - c_s)\varphi]' + \alpha_s^2(U - c_s)\phi_0\varphi - (\beta - U'')\phi_0\varphi\right] dy = 0.$$

Thus,

$$\int_{y_1}^{y_2} [-(U - c_s)\phi_0'' + \alpha_s^2(U - c_s)\phi_0 - (\beta - U'')\phi_0]\varphi dy = 0. \tag{2.14}$$

Therefore ϕ_0 is a weak solution of (2.8) and $\|\phi_0\|_{H^2} \leq C$.

It remains to show that $\phi_s = \phi_0$. Recall that S_0 denotes the complement of the set $\{z \in [y_1, y_2] | U(z) - c_s = 0\}$ in the interval $[y_1, y_2]$. Let S_1 be any compact subset in S_0. Since $\min\limits_{y \in S_1}\{|U(y) - c_k|\} > \delta$ for some $\delta > 0$, there exists a constant $C_1 > 0$ independent of k such that $\|\phi_k\|_{H^3(S_1)} \leq C_1$. By the compact embedding of $H^3(S_1) \hookrightarrow C^2(S_1)$, up to a subsequence, we have $\phi_k \to \tilde{\phi}$ in $C^2(S_1)$ for some $\tilde{\phi} \in C^2(S_1)$. Since $\phi_k \to \phi_0$ in $C^1(S_1)$, $\phi_0 \equiv \tilde{\phi}$ on S_1. Let

$$f := -(U - c_s)\phi_0'' + \alpha_s^2(U - c_s)\phi_0 - (\beta - U'')\phi_0.$$

Since $\phi_0 \in C^2(S_1)$, we have $f \in C^0(S_1)$. By the arbitrary choice of the test function φ, (2.14) implies that $f \equiv 0$ on S_1. Therefore, ϕ_0 satisfies the equation (2.8) on S_0 and by Definition 2.2, $\phi_s \equiv \phi_0$ on $[y_1, y_2]$. Thus, $\phi_s \in H^2[y_1, y_2]$. This completes the proof. ■

Combining Remark 2.1 (ii), Theorem 2.1, and Lemma 2.5, we get the classification of neutral limiting modes for flows in class $\mathcal{K}^+$.

Theorem 2.2 *Assume that U is in class $\mathcal{K}^+$. Let $(\phi_s, \alpha_s, \beta, c_s)$ be a neutral limiting mode. Then the neutral limiting phase speed c_s must be one of the following:*

(i) *$c_s = U_\beta$;*

(ii) *$c_s = U(y_1)$ or $c_s = U(y_2)$;*

(iii) *c_s is a critical value of U;*

(iv) $c_s \notin [U_{\min}, U_{\max}]$.

Remark 2.3 *In Theorem IV of [39], Tung showed that for a general C^2 shear flow $U(y)$, the wave speed c_s of any neutral limiting mode $(c_s, \alpha_s, \beta, \phi_s)$ must lie in the range of U. His proof is under the assumption that for the fixed β the dispersion relation $c(\alpha)$ is an analytic function of α near α_s when $c(\alpha_s) = c_s \notin [U_{\min}, U_{\max}]$. However, as suggested in [31], the analytic assumption might not always hold and it is possible that the wave speed of neutral limiting modes could lie outside the range of U. In Theorems 4.2–4.3, we give the sharp stability boundary for the Sinus flow $U(y) = \frac{1+\cos(\pi y)}{2}$, part of which consists of non-resonant neutral modes. This shows that the phase speed of neutral limiting modes can indeed lie beyond the range of U.*

Below we give some explanation why the analytic assumption of $c(\alpha)$ could fail. Assume that $(\phi_s, \alpha_s, \beta, c_s)$ is a neutral mode of (1.6)-(1.7) and $c_s \notin [U_{\min}, U_{\max}]$. From the Rayleigh equation (1.6)-(1.7), the perturbation of the eigenvalue c near c_s appears to be analytic in α when c_s is not in the range of U. However, we should consider the operator associated with the linearized equation (1.5) with the wave number α (β is fixed):

$$B_\alpha \omega := U\omega - (\beta - U'')(-\frac{d^2}{dy^2} + \alpha^2)^{-1}\omega.$$

Then c_s is an isolated eigenvalue of B_{α_s}. Define the Riesz projection operator

$$P_{\alpha_s} := -\frac{1}{2\pi i}\int_\Gamma (B_{\alpha_s} - \zeta)^{-1} d\zeta,$$

where Γ is a circle in $\rho(B_{\alpha_s})$ enclosing c_s and no other spectral points of B_{α_s}. Note that $Range(P_{\alpha_s})$ is the generalized eigenspace of the eigenvalue c_s and $\dim(Range(P_{\alpha_s}))$ is the algebraic multiplicity of c_s. (see Section 6.5, Chapter 3 in [26]). Although the geometric multiplicity of c_s is 1, the algebraic multiplicity of c_s may be larger than 1. In such case, there might be more than one branches of eigenvalues emanating from c_s when we perturb the parameter α in a neighborhood of α_s. As a consequence, the expansion of $c(\alpha) - c(\alpha_s)$ near $\alpha = \alpha_s$ is given by the Puiseux series (see P. 65 of [26]) instead of the power series in the analytic case. This suggests that we can not exclude the possibility that for a neutral limiting mode, c_s is outside the range of U.

Similar to the proof of Theorem 4.1 in [18], we can construct unstable modes near regular neutral modes.

Lemma 2.6 *Let U be in class $\mathcal{K}^+$ and $(\phi_s, \alpha_s, \beta, U_\beta)$ with $\alpha_s > 0$, $\beta \in (\min U'', \max U'')$ satisfy*

$$-\phi_s'' - \frac{\beta - U''}{U - U_\beta}\phi_s = -\phi_s'' - K_\beta \phi_s = -\alpha_s^2 \phi_s,$$

with $\phi_s(y_1) = \phi_s(y_2) = 0$. Then there exists $\varepsilon_0 < 0$ such that if $\varepsilon_0 < \varepsilon < 0$, there is a nontrivial solution ϕ_ε to the equation

$$(U - U_\beta - c(\varepsilon))(\frac{d^2}{dy^2} - \alpha(\varepsilon)^2)\phi_\varepsilon + (\beta - U'')\phi_\varepsilon = 0$$

with $\phi_\varepsilon(y_1) = \phi_\varepsilon(y_2) = 0$. Here $\alpha(\varepsilon) = \sqrt{\varepsilon + \alpha_s^2}$ is the perturbed wave number and $U_\beta + c(\varepsilon)$ is an unstable eigenvalue with $\mathrm{Im}(c(\varepsilon)) > 0$.

The next lemma, which was proved in [29, 39], shows that there is no neutral mode when $\beta \geq 0$ and $c_s \geq U_{\max}$ or $\beta \leq 0$ and $c_s \leq U_{\min}$. This is consistent with Lemma 1.1 for the unstable modes.

Lemma 2.7 *Let $U \in C^2[y_1, y_2]$. When $\beta \geq 0$ and $c_s \geq U_{\max}$ or $\beta \leq 0$ and $c_s \leq U_{\min}$, for any $\alpha > 0$ there does not exist neutral solution to (1.6)-(1.7) with $\phi_s \in H^2$.*

3 Hamiltonian formulation, index formula and instability criteria

In this section, we write the linearized fluid equation for flows in class $\mathcal{K}^+$ in a Hamiltonian form and derive an instability index formula to be used later to find the stability condition. Then we compute the associated quadratic forms for unstable modes and neutral limiting modes. Also, we establish an important identity, by which the sign of the associated quadratic form for a non-resonant neutral mode can be determined. Combining above, we give a new way to study the stability of shear flows in class $\mathcal{K}^+$.

3.1 Hamiltonian formulation and index formula

In this subsection, we write (1.4) as a Hamiltonian system and use the index theory developed in [22] to derive an instability index formula for flows in class $\mathcal{K}^{+}$. Fix $\beta \in (\min U'', \max U'')$. In the traveling frame $(x - U_\beta t, y, t)$, the linearized equation

$$\partial_t \omega + U \partial_x \omega - (\beta - U'') \partial_x \psi = 0$$

becomes

$$\partial_t \omega + (U - U_\beta) \partial_x \omega - (\beta - U'') \partial_x \psi = 0. \tag{3.1}$$

Note that the change of coordinates $(x, y, t) \to (x - U_\beta t, y, t)$ does not affect the stability of the shear flows. Recall that for flows in class $\mathcal{K}^{+}$,

$$K_\beta(y) = \frac{\beta - U''(y)}{U(y) - U_\beta} > 0.$$

Let the x period be $2\pi/\alpha$ for some $\alpha > 0$. Define the non-shear space on the periodic channel $S_{2\pi/\alpha} \times [y_1, y_2]$ by

$$X = \left\{ \omega = \sum_{k \in \mathbf{Z},\ k \neq 0} e^{ik\alpha x} \omega_k\left(y\right),\ \|\omega\|_X^2 = \|\frac{1}{\sqrt{K_\beta}} \omega\|_{L^2}^2 < \infty \right\}. \tag{3.2}$$

Clearly, $X \subset L^2$ and $L^2 = X$ if $\min K_\beta > 0$. The equation (3.1) can be written in a Hamiltonian form

$$\omega_t = -(\beta - U'') \partial_x \left(\frac{\omega}{K_\beta} - \psi \right) = JL\omega, \tag{3.3}$$

where

$$J = -(\beta - U'') \partial_x : X^* \to X, \qquad L = \frac{1}{K_\beta} - \left(-\Delta\right)^{-1} : X \to X^*, \tag{3.4}$$

are anti-self-adjoint and self-adjoint, respectively. Denote $n^{-}\left(L\right)$ $\left(n^{0}\left(L\right)\right)$ to be the number of negative (zero) directions of L on X. Define the operator

$$A_0 = -\Delta - K_\beta : H^2 \to L^2 \tag{3.5}$$

in the channel $S_{2\pi/\alpha} \times [y_1, y_2]$ and

$$\tilde{L}_0 = -\frac{d^2}{dy^2} - K_\beta : H^2\left(y_1, y_2\right) \to L^2\left(y_1, y_2\right), \tag{3.6}$$

with the Dirichlet boundary conditions. Then by Lemma 11.3 in [22], we have

$$n^0(L) = n^0(A_0) = 2\sum_{l\geq 1} n^0\left(\tilde{L}_0 + l^2\alpha^2\right),$$

and

$$n^-(L) = n^-(A_0) = 2\sum_{l\geq 1} n^-\left(\tilde{L}_0 + l^2\alpha^2\right).$$

If $n^-\left(\tilde{L}_0\right) \neq 0$, let $-\alpha_{\max}^2$ be the smallest eigenvalue of $\tilde{L}_0$ and ϕ_0 be the eigenvector. When $\tilde{L}_0 \geq 0$, let $\alpha_{\max} = 0$. Then by the above relations, when $\alpha \geq \alpha_{\max}$, L is non-negative and the stability holds.

Below, we consider the case when $\alpha < \alpha_{\max}$. Let $Y = L^2_{\frac{1}{K_\beta}}(y_1, y_2)$. The space X has an invariant decomposition $X = \cup_{l\in\mathbf{Z},\ l\neq 0} X^l$, where

$$X^l = \left\{e^{i\alpha l x}\omega_l(y),\ \omega_l \in Y\right\}. \tag{3.7}$$

The linearized equation can be studied on each X^l separately. To simplify notations, we only consider $l = \pm 1$ below. On the subspace

$$X_\alpha := \left\{e^{i\alpha x}\omega(y),\ \omega \in Y\right\},$$

the operator JL is reduced to the ODE operator $J_\alpha L_\alpha$ acting on the weighted space Y, where

$$J_\alpha = -i\alpha(\beta - U''), \qquad L_\alpha = \frac{1}{K_\beta} - \left(-\frac{d^2}{dy^2} + \alpha^2\right)^{-1}. \tag{3.8}$$

By the same proof of Lemma 11.3 in [22], we have

$$n^-(L_\alpha) = n^-\left(\tilde{L}_0 + \alpha^2\right),\ n^0(L_\alpha) = n^0\left(\tilde{L}_0 + \alpha^2\right).$$

Since J_α is not a real operator on Y, we define the invariant subspace

$$\begin{aligned} X^\alpha &= X_\alpha \oplus X_{-\alpha} \\ &= \{\cos(\alpha x)\omega_1(y) + \sin(\alpha x)\omega_2(y),\ \omega_1, \omega_2 \in Y\}, \end{aligned}$$

which is isomorphic to the real space $Y \times Y = \left(L^2_{\frac{1}{K_\beta}}(y_1, y_2)\right)^2$. For any

$$\omega = \cos(\alpha x)\omega_1(y) + \sin(\alpha x)\omega_2(y) \in X^\alpha,$$

we have

$$JL\omega = (\cos(\alpha x), \sin(\alpha x))\, J^\alpha L^\alpha \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix},$$

where

$$J^\alpha = \begin{pmatrix} 0 & -\alpha(\beta - U'') \\ \alpha(\beta - U'') & 0 \end{pmatrix}, \quad L^\alpha = \begin{pmatrix} L_\alpha & 0 \\ 0 & L_\alpha \end{pmatrix}.$$

In the above, the operator L_α is defined in (3.8). Thus to study the spectra of JL on X^α, it is equivalent to study the spectra of $J^\alpha L^\alpha$ on $Y \times Y$. We note that

$$\sigma\left(J^\alpha L^\alpha|_{Y\times Y}\right) = \sigma\left(J_\alpha L_\alpha|_Y\right) \cup \sigma\left(J_{-\alpha} L_{-\alpha}|_Y\right), \tag{3.9}$$

and $\sigma\left(J_\alpha L_\alpha|_Y\right)$ is the complex conjugate of $\sigma\left(J_{-\alpha} L_{-\alpha}|_Y\right)$.

By the instability index Theorem 2.3 in [22] for linear Hamiltonian PDEs, we have

$$2\tilde{k}_i^{\leq 0} + 2\tilde{k}_c + \tilde{k}_0^{\leq 0} + \tilde{k}_r = n^-(L^\alpha) = 2n^-(L_\alpha), \tag{3.10}$$

where $n^-(L^\alpha)$ denotes the sum of multiplicities of negative eigenvalues of L^α, $\tilde{k}_r$ is the sum of algebraic multiplicities of positive eigenvalues of $J^\alpha L^\alpha$, $\tilde{k}_c$ is the sum of algebraic multiplicities of eigenvalues of $J^\alpha L^\alpha$ in the first quadrant, $\tilde{k}_i^{\leq 0}$ is the total number of non-positive dimensions of $\langle L^\alpha \cdot, \cdot\rangle$ restricted to the subspaces of generalized eigenvectors of purely imaginary eigenvalues of $J^\alpha L^\alpha$ with positive imaginary parts, and $\tilde{k}_0^{\leq 0}$ is the number of non-positive dimensions of $\langle L^\alpha \cdot, \cdot\rangle$ restricted to the generalized kernel of $J_\alpha L_\alpha$ modulo $\ker L^\alpha$. By the next lemma, we have $\tilde{k}_0^{\leq 0} = 0$, from which it follows that

$$2\tilde{k}_i^{\leq 0} + 2\tilde{k}_c + \tilde{k}_r = 2n^-(L_\alpha). \tag{3.11}$$

Lemma 3.1 *Let E_0 be the generalized zero eigenspace of $J^\alpha L^\alpha$. Then $E_0 = \ker L^\alpha$.*

Proof. It suffices to show that the generalized zero eigenspace of $J_\alpha L_\alpha$ on Y coincides with $\ker L_\alpha$. Suppose there exists $\omega \in Y$ such that

$$J_\alpha L_\alpha \omega = -i\alpha(U - U_\beta)\left(\omega - K_\beta \psi\right) = \tilde{\omega} \in \ker L_\alpha. \tag{3.12}$$

Let $\tilde{\psi} = \left(-\frac{d^2}{dy^2} + \alpha^2\right)^{-1} \tilde{\omega}$. Then

$$\left(-\frac{d^2}{dy^2} + \alpha^2\right)\tilde{\psi} - K_\beta \tilde{\psi} = 0.$$

By Lemmas 2.1–2.2, $\tilde{\psi}$ is not all zero on the set $\{U = U_\beta\}$, which implies the same for $\tilde{\omega} = K_\beta \tilde{\psi}$. Thus (3.12) implies that

$$\omega - K_\beta \psi = \frac{\tilde{\omega}}{-i\alpha(U - U_\beta)} \notin L^2(y_1, y_2).$$

This contradiction shows that the generalized kernel of $J_\alpha L_\alpha$ on Y is the same as $\ker L_\alpha$. ■

Now we state the index formula for $J_\alpha L_\alpha$ on Y. Let k_r be the sum of algebraic multiplicities of positive eigenvalues of $J_\alpha L_\alpha$, k_c be the sum of algebraic multiplicities of eigenvalues of $J_\alpha L_\alpha$ in the first and the forth quadrants, $k_i^{\leq 0}$ be the total number of non-positive dimensions of $\langle L_\alpha \cdot, \cdot \rangle$ restricted to the subspaces of generalized eigenvectors of nonzero purely imaginary eigenvalues of $J_\alpha L_\alpha$. By (3.9), we have the following relation

$$2k_i^{\leq 0} = 2\tilde{k}_i^{\leq 0}, \;\; 2k_c = 2\tilde{k}_c, \;\; 2k_r = \tilde{k}_r. \tag{3.13}$$

Combining (3.11) and (3.13), we get the following index formula for $J_\alpha L_\alpha$.

Theorem 3.1 *Let U be a shear flow in class $\mathcal{K}^+$ and $\beta \in (\min U'', \max U'')$. Then the following index formula holds for the operator $J_\alpha L_\alpha$ on Y:*

$$k_c + k_r + k_i^{\leq 0} = n^-(L_\alpha). \tag{3.14}$$

From the index formula (3.14), the stability of shear flows is reduced to determine $k_i^{\leq 0}$, that is, the number of non-positive dimensions of $\langle L_\alpha \cdot, \cdot \rangle$ restricted to generalized eigenspace of nonzero purely imaginary eigenvalues of $J_\alpha L_\alpha$. This corresponds to consider neutral modes in H^2 of the Rayleigh equation with the wave speed $c_s \neq U_\beta$.

Remark 3.1 *When $\beta = 0$, it can be shown that $k_i^{\leq 0} = 0$ and the index formula (3.14) reduces to $k_c + k_r = n^-(L_\alpha)$ (see [20]). When $\beta \neq 0$, in general we have $k_i^{\leq 0} \neq 0$ as seen from the example of Sinus flow in Section 4.*

3.2 Computation of the quadratic form $\langle L_\alpha \cdot, \cdot \rangle$

In this subsection, we shall compute the quadratic form $\langle L_\alpha \cdot, \cdot \rangle$ for unstable modes and neutral limiting modes. Then combining with the regular Sturm-Liouville theory, we obtain a useful identity (3.25) to determine the sign of the quadratic form $\langle L_\alpha \cdot, \cdot \rangle$ for non-resonant neutral modes.

Lemma 3.2 *Let (c, α, β, ϕ) be a solution of*

$$-\phi'' + \alpha^2\phi - \frac{\beta - U''}{U - c}\phi = 0, \quad \phi(y_1) = \phi(y_2) = 0, \tag{3.15}$$

with $\phi \in H^2, \alpha > 0, \beta \in \mathbf{R}$. Then

(i)
$$\langle L_\alpha \omega, \omega \rangle = (c - U_\beta) \int_{y_1}^{y_2} \frac{(\beta - U'')}{|U - c|^2} |\phi|^2 dy, \tag{3.16}$$

where $\omega(y) = (-\frac{d^2}{dy^2} + \alpha^2)\phi(y)$.

(ii) *If (c, α, β, ϕ) is an unstable mode, then $\langle L_\alpha \omega, \omega \rangle = 0$.*

(iii) *If (c, α, β, ϕ) is a regular or non-resonant neutral limiting mode, then $\langle L_\alpha \omega, \omega \rangle = 0$. If (c, α, β, ϕ) is a singular neutral limiting mode, then $\langle L_\alpha \omega, \omega \rangle \leq 0$.*

Proof. We first show (3.16) holds. By the definition of L_α,

$$\langle L_\alpha \omega, \omega \rangle = \int_{y_1}^{y_2} \left(\frac{\omega}{K_\beta} - \phi \right) \bar{\omega} \, dy.$$

From the equation (3.15), we get that

$$(U - c)\omega = (\beta - U'')\phi.$$

Therefore

$$\frac{\omega}{K_\beta} - \phi = \frac{\omega \, (U - U_\beta)}{(\beta - U'')} - \frac{\omega \, (U - c)}{(\beta - U'')} = \frac{(c - U_\beta)}{(\beta - U'')}\omega,$$

and

$$\begin{aligned} \langle L_\alpha \omega, \omega \rangle &= \int_{y_1}^{y_2} \frac{(c - U_\beta)}{(\beta - U'')} |\omega|^2 dy \\ &= (c - U_\beta) \int_{y_1}^{y_2} \frac{(\beta - U'')}{|U - c|^2} |\phi|^2 dy. \end{aligned}$$

The conclusion (ii) follows from (i) and Lemma 2.3.

Next, we show conclusions in (iii). For a regular neutral limiting mode (c, α, β, ϕ), $\langle L_\alpha \omega, \omega\rangle = 0$ by (i) and the fact that $c = U_\beta$.

For a non-resonant neutral limiting mode (c, α, β, ϕ), let $\{(c_k, \alpha_k, \beta, \phi_k)\}_{k=1}^{\infty}$ be a sequence of unstable modes converging to (c, α, β, ϕ) in the sense of Definition 2.2. By Lemma 2.4, $\|\phi_k\|_{H^2} \leq C$ for some $C > 0$ independent of k. Thus there exists a subsequence, which we still denote by $\{\phi_k\}_{k=1}^{\infty}$, such that $\phi_k \to \phi$ in $C^1[y_1, y_2]$. Noting that $c \notin [U_{\min}, U_{\max}]$, we have

$$\begin{aligned}\langle L_\alpha \omega, \omega\rangle &= (c - U_\beta)\int_{y_1}^{y_2} \frac{(\beta - U'')}{|U - c|^2}|\phi|^2 dy,\\ &= \lim_{k\to\infty}(c_k - U_\beta)\int_{y_1}^{y_2} \frac{(\beta - U'')}{|U - c_k|^2}|\phi_k|^2 dy = 0.\end{aligned}$$

Lastly, we consider a singular neutral limiting mode (c, α, β, ϕ). Let $\{(c_k, \alpha_k, \beta, \phi_k)\}_{k=1}^{\infty}$ be a sequence of unstable modes converging to $(c_s, \alpha_s, \beta, \phi_s)$ in the sense of Definition 2.2. By the uniform bound of $\|\phi_k\|_{H^2}$ and hence the uniform bound of $\|\omega_k\|_{L^2}$ with $\omega_k = (-\frac{d^2}{dy^2} + \alpha_k^2)\phi_k$ for all $k \geq 1$, up to a subsequence, we have that ω_k weakly converges to ω in L^2 and ϕ_k strongly converges to ϕ in H^1. Hence

$$\begin{aligned}\langle L_\alpha \omega, \omega\rangle &= \int_{y_1}^{y_2}\left(\frac{\omega}{K_\beta} - \phi\right)\bar{\omega}\ dy\\ &= \int_{y_1}^{y_2}\frac{|\omega|^2}{K_\beta} - \left(|\phi'|^2 + \alpha^2|\phi|^2\right)dy\\ &\leq \lim_{k\to\infty}\int_{y_1}^{y_2}\frac{|\omega_k|^2}{K_\beta} - \left(|\phi_k'|^2 + \alpha_k^2|\phi_k|^2\right)dy\\ &= \lim_{k\to\infty}\langle L_{\alpha_k}\omega_k, \omega_k\rangle = 0.\end{aligned}$$

This completes the proof of the lemma. ■

The non-resonant neutral mode satisfies the Rayleigh-Kuo equation with Dirichlet boundary condition, which can be considered as a regular Sturm-Liouville problem. Next, we compute the derivative formulas of the n-th eigenvalue of the regular Sturm-Liouville problems with respect to the parameters β and c separately, which will be helpful to determine the sign of quadratic form $\langle L_\alpha \cdot, \cdot\rangle$ for non-resonant neutral modes.

Lemma 3.3 (i) *Fix $c_0 \notin [U_{\min}, U_{\max}]$. Let λ_n $(n \geq 1)$ be the n-th eigenvalue of*

$$-\phi'' - \frac{\beta - U''}{U - c_0}\phi = \lambda_n \phi, \quad \phi(y_1) = \phi(y_2) = 0, \tag{3.17}$$

and $\phi_{n,\beta}$ be the corresponding real-valued eigenfunction with $\|\phi_{n,\beta}\|_{L^2} = 1$. Then λ_n, as a function of $\beta \in \mathbf{R}$, is differentiable and

$$\frac{d\lambda_n}{d\beta} = -\int_{y_1}^{y_2} \frac{1}{U - c_0}\phi_{n,\beta}^2 \, dy. \tag{3.18}$$

(ii) *Fix $\beta_0 \in \mathbf{R}$. Let $c \notin [U_{\min}, U_{\max}]$, λ_n $(n \geq 1)$ be the n-th eigenvalue of*

$$-\phi'' - \frac{\beta_0 - U''}{U - c}\phi = \lambda_n \phi, \quad \phi(y_1) = \phi(y_2) = 0, \tag{3.19}$$

and $\phi_{n,c}$ be the corresponding real-valued eigenfunction with $\|\phi_{n,c}\|_{L^2} = 1$. Then λ_n, as a function of $c \in (-\infty, U_{\min}) \cup (U_{\max}, +\infty)$, is differentiable and

$$\frac{d\lambda_n}{dc} = -\int_{y_1}^{y_2} \frac{\beta_0 - U''}{(U - c)^2}\phi_{n,c}^2 \, dy. \tag{3.20}$$

Proof. We first show that (i) holds. By Theorem 2.1 in [27], λ_n, as a function of $\beta \in \mathbf{R}$, is continuous. $\beta, \beta' \in \mathbf{R}$ and $\phi_{n,\beta}, \phi_{n,\beta'}$ with $\|\phi_{n,\beta}\|_{L^2} = \|\phi_{n,\beta'}\|_{L^2} = 1$ satisfy

$$-\phi_{n,\beta}'' - \frac{\beta - U''}{U - c_0}\phi_{n,\beta} = \lambda_n(\beta)\phi_{n,\beta}, \quad \phi_{n,\beta}(y_1) = \phi_{n,\beta}(y_2) = 0, \tag{3.21}$$

$$-\phi_{n,\beta'}'' - \frac{\beta' - U''}{U - c_0}\phi_{n,\beta'} = \lambda_n(\beta')\phi_{n,\beta'}, \quad \phi_{n,\beta'}(y_1) = \phi_{n,\beta'}(y_2) = 0. \tag{3.22}$$

Note that by Theorem 3.2 in [28], $\phi_{n,\beta'}$ can be chosen such that $\phi_{n,\beta'}$ uniformly converges to $\phi_{n,\beta}$ as $\beta' \to \beta$ on any compact subinterval of (y_1, y_2). Multiplying (3.21) by $\phi_{n,\beta'}$ and subtracting $\phi_{n,\beta}$ times (3.22), then integrating from y_1 to y_2, we get

$$\frac{\lambda_n(\beta) - \lambda_n(\beta')}{\beta - \beta'} \int_{y_1}^{y_2} \phi_{n,\beta}\phi_{n,\beta'} dy = -\int_{y_1}^{y_2} \frac{1}{U - c_0}\phi_{n,\beta}\phi_{n,\beta'} dy.$$

Taking the limit $\beta' \to \beta$ in the above, we prove (3.18).

The formula (3.20) can be proved in a similar way and we skip the details.
■

The following is a straightforward consequence of Lemma 3.3.

Corollary 3.1 (i) *Let $c_0 \in (-\infty, U_{\min})$, λ_n $(n \geq 1)$ be the n-th eigenvalue of (3.17). Then λ_n is a decreasing function of $\beta \in \mathbf{R}$.*

(ii) *Let $c_0 \in (U_{\max}, +\infty)$, λ_n $(n \geq 1)$ be the n-th eigenvalue of (3.17). Then λ_n is an increasing function of $\beta \in \mathbf{R}$.*

(iii) *Let $\beta_0 \in (-\infty, U''_{\min}]$, λ_n $(n \geq 1)$ be the n-th eigenvalue of (3.19). Then λ_n is an increasing function of $c \in (-\infty, U_{\min})$ and $c \in (U_{\max}, +\infty)$, respectively.*

(iv) *Let $\beta_0 \in [U''_{\max}, +\infty)$, λ_n $(n \geq 1)$ be the n-th eigenvalue of (3.19). Then λ_n is a decreasing function of $c \in (-\infty, U_{\min})$ and $c \in (U_{\max}, +\infty)$, respectively.*

Let $\beta \in \mathbf{R}$ and $n_0 \geq 1$. We define a function

$$f_{\beta,n_0} : (-\infty, U_{\min}) \cup (U_{\max}, +\infty) \to \mathbf{R} \tag{3.23}$$

such that $-f_{\beta,n_0}(c)$ is the n_0-th eigenvalue of

$$\mathcal{L}_{\beta,c} := -\frac{d^2}{dy^2} - \frac{\beta - U''}{U - c},\ H_0^1 \cap H^2(y_1, y_2) \to L^2(y_1, y_2). \tag{3.24}$$

Combining (3.16) and (3.20), we get the following identity, by which we may determine the sign of $\langle L_\alpha \cdot, \cdot \rangle$ for a non-resonant neutral mode.

Theorem 3.2 *Let $(c_0, \alpha_0, \beta, \phi_0)$ be a non-resonant neutral mode. Then $\alpha_0^2 = f_{\beta,n_0}(c_0) > 0$ for some $n_0 \geq 1$ and*

$$\langle L_{\alpha_0} \omega_{\alpha_0}, \omega_{\alpha_0} \rangle = (c_0 - U_\beta) \frac{df_{\beta,n_0}}{dc}\Big|_{c=c_0}, \tag{3.25}$$

where $\omega_{\alpha_0} = (-\frac{d^2}{dy^2} + \alpha_0^2)\phi_0$.

3.3 Stability condition by the index formula

In this subsection, we show how to study the stability of a shear flow U in class $\mathcal{K}^+$ by the instability index formula. Fix $\beta \in (\min U'', \max U'')$ and $\alpha > 0$. We shall determine the barotropic stability of the shear flow U by using the index formula (3.14) and the computation of the L_α quadratic form in the following steps.

First, recall that $n^{-}(L_\alpha) = n^{-}\left(\tilde{L}_0 + \alpha^2\right)$, where $\tilde{L}_0$ is defined in (3.6). By the instability index formula (3.14), linear stability at the wave number α is equivalent to the condition $n^{-}(L_\alpha) = k_i^{\leq 0}$. To determine $k_i^{\leq 0}$, we need to study the neutral modes in H^2. By Theorem 2.1, the neutral wave speed c must be one of the following four types: (i) U_β; (ii) $U(y_1)$ or $U(y_2)$; (iii) critical values of U; (iv) outside the range of U. Since $c = U_\beta$ corresponds to the zero eigenvalue of $J_\alpha L_\alpha$ (defined in (3.8)), it has no contribution to $k_i^{\leq 0}$. To find neutral modes of types (ii) and (iii), we need to solve a (possibly) singular eigenvalue problem for the operator $\mathcal{L}_{\beta,c}$ defined in (3.24) with c to be $U(y_1), U(y_2)$ or a critical value of U. For such c, if $-\alpha^2$ is a negative eigenvalue of $\mathcal{L}_{\beta,c}$ with the eigenfunction $\phi \in H^2[y_1, y_2]$, then $\lambda = -i\alpha(c - U_\beta)$ is a nonzero imaginary eigenvalue of $J_\alpha L_\alpha$ with the eigenfunction $\omega = -\phi'' + \alpha^2\phi \in L^2[y_1, y_2]$. Denote $k_i^{-}(\lambda)$ $\left(k_i^{\leq 0}(\lambda)\right)$ to be the number of negative (non-positive) dimensions of $\langle L_\alpha \cdot, \cdot\rangle$ restricted to the generalized eigenspace of λ for $J_\alpha L_\alpha$. Then when $\langle L_\alpha \omega, \omega\rangle \neq 0$, by noting that λ is purely imaginary, one can easily verify that λ is a simple eigenvalue of $J_\alpha L_\alpha$ and

$$k_i^{\leq 0}(\lambda) = k_i^{-}(\lambda) = \begin{cases} 1 & \text{if } \langle L_\alpha \omega, \omega\rangle < 0 \\ 0 & \text{if } \langle L_\alpha \omega, \omega\rangle > 0. \end{cases}$$

When $\langle L_\alpha \omega, \omega\rangle = 0$, we have $k_i^{\leq 0}(\lambda) \geq 1$ and λ might be a multiple eigenvalue of $J_\alpha L_\alpha$.

For the case (iv) of non-resonant neutral modes, $\mathcal{L}_{\beta,c}$ is a regular Sturm-Liouville operator but $c \notin \mathit{Range}(U)$ is not given explicitly. By the definition of $f_{\beta,n}$ in the above subsection, we obtain that for $\alpha > 0$, the number of non-resonant neutral modes is exactly the number of solutions of $f_{\beta,n}(c) = \alpha^2$ for all $n \geq 1$. Let c^* be a solution of $f_{\beta,n_0}(c) = \alpha^2$ for some $n_0 \geq 1$. Then $-f_{\beta,n_0}(c^*) < 0$ is the n_0-th eigenvalue of $\mathcal{L}_{\beta,c^*}$ with the eigenfunction ϕ^*, and correspondingly, $\lambda^* = -i\alpha(c^* - U_\beta)$ is a nonzero imaginary eigenvalue of $J_\alpha L_\alpha$ with the eigenfunction $\omega^* = -\phi^{*\prime\prime} + \alpha^2\phi^*$. If $f'_{\beta,n_0}(c^*) \neq 0$, then by (3.25),

$$\langle L_\alpha \omega^*, \omega^*\rangle = (c^* - U_\beta) f'_{\beta,n_0}(c^*) \neq 0,$$

which implies that λ^* is a simple eigenvalue of $J_\alpha L_\alpha$ with

$$k_i^{-}(\lambda^*) = \begin{cases} 1 & \text{if } (c^* - U_\beta) f'_{\beta,n_0}(c^*) < 0 \\ 0 & \text{if } (c^* - U_\beta) f'_{\beta,n_0}(c^*) > 0. \end{cases}$$

If $f'_{\beta,n_0}(c^*) = 0$, then $\langle L_\alpha \omega^*, \omega^*\rangle = 0$ and λ^* might be a multiple eigenvalue of $J_\alpha L_\alpha$. In this case, we have $k_i^{\leq 0}(\lambda^*) \geq 1$. Note that by Lemma 3.2, only points with $f'_{\beta,n_0}(c^*) = 0$ could be a neutral limiting mode, i.e., possibly lie on the boundary for stability/instability.

The instability index formula and the above computation of the index with non-positive signature provide a useful way to study the stability of shear flows. In the next section, we will analyze the barotropic stability of Sinus flow in details by using this approach.

4 Sharp stability criteria for the Sinus flow

In this section, we consider the barotropic stability of the flow of Sinus profile

$$U(y) = \frac{1+\cos(\pi y)}{2}, \quad y \in [-1,1]. \tag{4.1}$$

Based on the approach in Subsection 3.3, we give the sharp boundary of barotropic stability and instability for the Sinus profile in the parameter space (α, β). The boundary of stability and instability for this flow had been studied by numerical computations in [30, 35]. Our analysis corrects the picture of stability boundary reported in above references. Moreover, the corrected boundary of barotropic stability is confirmed by more accurate numerical results.

Note that $U''(y) = -\frac{\pi^2}{2}\cos(\pi y)$. Hence by the Rayleigh-Kuo criterion, a necessary condition for instability is $\beta \in (-\frac{\pi^2}{2}, \frac{\pi^2}{2})$. Since for any $\beta \in \mathbf{R}$,

$$\frac{\beta - U''}{U - (\frac{1}{2} - \frac{\beta}{\pi^2})} = \pi^2,$$

the Sinus profile belongs to class $\mathcal{K}^+$ with $U_\beta = \frac{1}{2} - \frac{\beta}{\pi^2}$.

4.1 Index formula and neutral modes for the Sinus flow

In this subsection, we derive the stability index formula for the Sinus flow. Then we study the first three types of H^2 neutral modes by the combination of ODE techniques, hypergeometric functions and Sturm-Liouville theory. The study of the fourth type of neutral modes (i.e. non-resonant) is left for the next two subsections, where the sharp stability boundary is then given.

Fix $\alpha > 0$. The linearized equation near the Sinus flow is written in the Hamiltonian form

$$\partial_t \omega = J_\alpha L_\alpha \omega, \ \omega \in L^2[-1,1],$$

where

$$J_\alpha = -i\alpha(\beta - U''), \quad L_\alpha = \frac{1}{\pi^2} - (-\frac{d^2}{dy^2} + \alpha^2)^{-1}. \tag{4.2}$$

Since the Sturm-Liouville problem $-\frac{d^2\phi}{dy^2} = \lambda\phi$ with the boundary conditions $\phi(\pm 1) = 0$ consists of discrete eigenvalues

$$\lambda_k = \frac{k^2\pi^2}{4}, \quad k = 1, 2, \cdots,$$

we have that

$$\sigma(L_\alpha) = \{\frac{1}{\pi^2} - \frac{1}{\frac{k^2\pi^2}{4} + \alpha^2}, \quad k = 1, 2, \cdots\}.$$

By Theorem 2.1, the possible wave speeds of neutral modes in H^2 are

(i) $c_s = U_\beta$;

(ii) $c_s = 0$;

(iii) $c_s = 1$;

(iv) $c_s \notin [U_{\min}, U_{\max}]$.

For any $\beta \in \mathbf{R}$, the neutral mode with $c_s = U_\beta$ satisfies the following regular Sturm-Liouville problem

$$-\phi'' - \pi^2\phi = \lambda\phi, \quad \phi(\pm 1) = 0, \tag{4.3}$$

with $\lambda = -\alpha^2$. It corresponds to the zero eigenvalue of $J_\alpha L_\alpha$. All the eigenvalues of (4.3) are

$$\lambda_k = (\frac{k^2}{4} - 1)\pi^2, \quad k = 1, 2, \cdots.$$

Hence $\lambda_1 = -\alpha_{\max}^2 = -\frac{3\pi^2}{4}$ is the unique negative eigenvalue of the Sturm-Liouville problem (4.3). By Theorem 3.1, we get the following index formula for the Sinus flow.

Theorem 4.1 *Consider the Sinus flow* $U(y) = \frac{1+\cos(\pi y)}{2}$ *and* $\beta \in \mathbf{R}$. *For any* $\alpha \in [\alpha_{\max}, +\infty)$, L_α *is a nonnegative operator and the flow is linearly stable for perturbations of period* $\frac{2\pi}{\alpha}$. *For any* $\alpha \in (0, \alpha_{\max})$, *the index formula*

$$k_c + k_r + k_i^{\leq 0} = 1 \tag{4.4}$$

is satisfied for the eigenvalues of $J_\alpha L_\alpha$, *where* J_α *and* L_α *are defined in (4.2). The indexes* $k_c,\ k_r,\ k_i^{\leq 0}$ *are defined as in Theorem 3.1.*

There is exactly one regular neutral mode corresponding to $c_s = U_\beta$, that is,

$$c_s = U_\beta = \frac{1}{2} - \frac{\beta}{\pi^2},\ \alpha_s = \frac{\sqrt{3}}{2}\pi,\ \beta \in (-\frac{\pi^2}{2}, \frac{\pi^2}{2}),\ \phi_s = \cos(\frac{\pi y}{2}), \tag{4.5}$$

and one antisymmetric mode

$$c_s = U_\beta,\ \alpha_s = 0,\ \beta \in (-\frac{\pi^2}{2}, \frac{\pi^2}{2}),\ \phi_s = \sin(\pi y),$$

which had been found by Kuo [30].

In the rest of this subsection, we study H^2 neutral modes with $c_s = 0$ or 1, which cover both the critical and end point values of U.

By Lemma 2.7, there is no neutral mode when $c_s \leq 0$, $\beta \leq 0$ or $c_s \geq 1$, $\beta \geq 0$.

In [30], Kuo found a family of singular neutral modes for $c_s = 0$ and $\beta \in \left(0, \frac{\pi^2}{2}\right)$:

$$c_s = 0,\ \alpha_s = \pi\sqrt{1-r^2},\ \beta_s = \pi^2(-r^2 + \frac{1}{2}r + \frac{1}{2}),\ \phi_s(y) = \cos^{2r}\left(\frac{\pi y}{2}\right), \tag{4.6}$$

where $r \in \left(\frac{1}{2}, 1\right)$.

Remark 4.1 *Based on numerical results, Kuo in [30] claimed that the above singular neutral curve (4.6) is the stability boundary for* $\beta_s \in \left(0, \frac{\pi^2}{2}\right)$, *that is, the Sinus flow is linearly unstable when* $\alpha \in (\alpha_s, \alpha_{\max})$ *and it is linearly stable when* $\alpha \in (0, \alpha_s]$. *The same stability picture for the Sinus flow also appeared in [35]. By Lemma 2.5, any neutral limiting mode* ϕ_s *must lie in* $H^2[-1,1]$. *However,*

$$\cos^{2r}\left(\frac{\pi y}{2}\right) \notin H^2[-1,1], \quad r \in \left(\frac{1}{2}, \frac{3}{4}\right],$$

and it is shown below that there exists no eigenfunction in $H^2[-1,1]$ for

$$c_s = 0,\ \alpha_s(r) = \pi\sqrt{1-r^2},\ \beta_s(r) = \pi^2(-r^2 + \frac{1}{2}r + \frac{1}{2}), \tag{4.7}$$

when $r \in (\frac{1}{2}, \frac{3}{4}]$. This contradiction implies that (4.6) cannot be a neutral limiting mode when $r \in (\frac{1}{2}, \frac{3}{4}]$, i.e., $\beta_s \in [\frac{5\pi^2}{16}, \frac{\pi^2}{2})$. Therefore, Kuo's claim on the stability boundary is incorrect, at least for the case $\beta_s \in [\frac{5\pi^2}{16}, \frac{\pi^2}{2})$.

To show that there is no eigenfunction in $H^2[-1,1]$ for $(c_s, \alpha_s(r), \beta_s(r))$ in (4.7), we use the following elementary lemma.

Lemma 4.1 *Let φ_1 be a nontrivial solution of the second order ODE*

$$\varphi'' + p\varphi' + q\varphi = 0, \quad \text{on } (y_1, y_2)$$

where p and q are continuous functions on (y_1, y_2). Then the general solution of the ODE is

$$\varphi(y) = \varphi_1(y)[C_1 + C_2 \int_{y_0}^{y} \frac{1}{\varphi_1^2(s)} e^{-\int_{y_0}^{s} p(t)dt} ds],$$

where C_1 and C_2 are arbitrary constants, and $y_0 \in (y_1, y_2)$.

By Lemma 4.1, besides $\cos^{2r}(\frac{\pi y}{2})$ the other independent solution of the equation

$$-\phi'' + \alpha_s^2(r)\phi - \frac{\beta_s(r) - U''}{U}\phi = 0, \quad y \in (-1, 1), \tag{4.8}$$

where $(\alpha_s(r), \beta_s(r))$ with $r \in (\frac{1}{2}, 1]$ are defined in (4.7), is

$$\phi_1(y) = \cos^{2r}\left(\frac{\pi y}{2}\right) \int_0^y \frac{1}{\cos^{4r}(\frac{\pi s}{2})} ds.$$

It is clear that $\phi_1 \notin H^2[-1,1]$ for all $r \in (\frac{1}{2}, 1]$. Therefore, for $(\alpha_s(r), \beta_s(r))$ in (4.7), there is no nontrivial solution of (4.8) in $H^2[-1,1]$ when $r \in (\frac{1}{2}, \frac{3}{4}]$.

To find the correct stability boundary, we use the approach outlined in Subsection 3.3. First, we compute the quadratic form $\langle L_{\alpha_s}\omega_{\alpha_s}, \omega_{\alpha_s}\rangle$ for the neutral mode (4.6), where $\omega_{\alpha_s} = -\phi_s'' + \alpha_s^2\phi_s$. When $r \in (\frac{1}{2}, \frac{3}{4}]$, $\phi_s \notin H^2[-1,1]$

and we get $\langle L_{\alpha_s}\omega_{\alpha_s}, \omega_{\alpha_s}\rangle = +\infty$. When $r \in (\frac{3}{4}, 1]$, by (i) of Lemma 3.2, we have

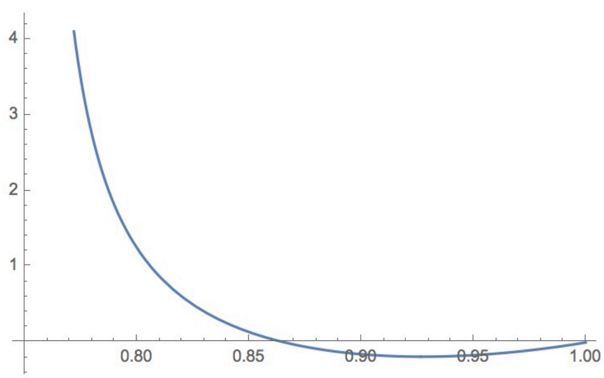
$$
\begin{aligned}
&\langle L_{\alpha_s}\omega_{\alpha_s}, \omega_{\alpha_s}\rangle \\
=&(c_s - U_{\beta_s}) \int_{-1}^{1} \frac{(\beta_s - U'')}{|U - c_s|^2} |\phi_s|^2 dy \\
=&(\frac{\beta_s}{\pi^2} - \frac{1}{2}) \int_{-1}^{1} \frac{\beta_s + \frac{\pi^2}{2}\cos(\pi y)}{\cos^4(\frac{\pi y}{2})} \cos^{4r}\left(\frac{\pi y}{2}\right)\, dy \\
=&(\frac{r}{2} - r^2) \int_{-1}^{1} \left(\frac{\pi^2}{2} + \frac{\pi^2}{2} r - \pi^2 r^2 + \frac{\pi^2}{2}\cos(\pi y)\right) \cos^{4r-4}\left(\frac{\pi y}{2}\right)\, dy.
\end{aligned}
$$

We compute the last integral by Mathematica as follows.

$$\mathtt{Integrate}\left[\left(\frac{r}{2} - r^2\right)\left(\frac{\pi^2}{2} + \frac{\pi^2 r}{2} - \pi^2 r^2 + \frac{\pi^2}{2}\mathtt{Cos}[\pi y]\right)\mathtt{Cos}\left[\frac{\pi y}{2}\right]^{4r-4}, \{y, -1, 1\}\right]$$

$$\mathtt{ConditionalExpression}\left[\frac{\pi^{3/2}\,(-1+r)\,r\,(-1+2r)\,(-3+4r^2)\,\mathtt{Gamma}\left[-\frac{3}{2}+2r\right]}{2\,\mathtt{Gamma}[2r]}, \mathtt{Re}[r] > \frac{3}{4}\right]$$

$$\mathtt{Plot}\left[\%, \left\{r, \frac{3}{4}, 1\right\}\right]$$



$$\mathtt{Solve}[\%449 == 0, r]$$

$$\left\{\{r \to 1\}, \left\{r \to \frac{\sqrt{3}}{2}\right\}\right\}$$

This implies that

$$\langle L_{\alpha_s}\omega_{\alpha_s},\omega_{\alpha_s}\rangle \begin{cases} =+\infty, & r\in(\frac{1}{2},\frac{3}{4}],\\ >0, & r\in(\frac{3}{4},\frac{\sqrt{3}}{2}),\\ =0, & r=\frac{\sqrt{3}}{2} \text{ or } 1,\\ <0, & r\in(\frac{\sqrt{3}}{2},1). \end{cases} \tag{4.9}$$

Hence, it follows from Lemma 3.2 (iii) that (4.6) is not a neutral limiting mode when $r\in(\frac{1}{2},\frac{\sqrt{3}}{2})$, that is, $\beta_s\in(\frac{(\sqrt{3}-1)\pi^2}{4},\frac{\pi^2}{2})$. In particular, for $\beta\in(\frac{(\sqrt{3}-1)\pi^2}{4},\frac{5\pi^2}{16})$, the singular neutral curve (4.6) is not part of the stability boundary even if the singular neutral modes are in H^2.

Next, we show that there is no neutral mode in H^2 for $c_s=1$, the maximum of U. By Lemma 2.7, there is no neutral mode for $c_s=1,\beta\geq 0$. The neutral modes with $c_s=1$ and $\beta\in(-\frac{\pi^2}{2},0)$ is excluded by the following lemma.

Lemma 4.2 *For any $\beta\in(-\frac{\pi^2}{2},0]$ and $\alpha\geq 0$, there is no nontrivial solution ϕ to the equation*

$$-\phi''+\alpha^2\phi-\frac{\beta-U''}{U-1}\phi=0 \tag{4.10}$$

on $(0,1)$ with $\phi(1)=0$ and $\phi(0)$ to be finite. Similarly, there is no nontrivial solution to (4.10) on $(-1,0)$ with $\phi(-1)=0$ and $\phi(0)$ to be finite.

Proof. The proof is by transforming (4.10) into a hypergeometric differential equation and then using hypergeometric functions to express its solutions.

Define

$$z=-\tan^2\left(\frac{\pi y}{2}\right),\quad y\in[0,1). \tag{4.11}$$

Then

$$U(y)=\frac{1}{1-z},\quad U''(y)=-\frac{\pi^2}{2}\left(\frac{1+z}{1-z}\right). \tag{4.12}$$

For $y\in[0,1)$, we inverse the transform (4.11) to get

$$y=\frac{2\arctan\sqrt{-z}}{\pi},\ z\in(-\infty,0].$$

For any solution $\phi(y)$ of (4.10), we define

$$\psi(z):=\phi\left(\frac{2\arctan\sqrt{-z}}{\pi}\right)=\phi(y).$$

Then it follows that

$$\begin{aligned}\phi''(y) =& \frac{d^2}{dy^2}\psi\left(-\tan^2\left(\frac{\pi y}{2}\right)\right)\\ =& \frac{\pi^2(1-z)}{2}\left[\left(-2z+2z^2\right)\psi''(z)+(-1+3z)\psi'(z)\right],\end{aligned}$$

which, together with (4.10) and (4.12), implies that

$$\begin{aligned}0 =& \frac{\pi^2(1-z)}{2}\left[\left(-2z+2z^2\right)\psi''(z)+(-1+3z)\psi'(z)\right]\\ &-\alpha^2\psi(z)+\frac{\beta+\frac{\pi^2}{2}\left(\frac{1+z}{1-z}\right)}{\frac{1}{1-z}-1}\psi(z).\end{aligned} \tag{4.13}$$

The boundary conditions are as follows:

$$\psi \text{ is finite at } z=0 \text{ and } \lim_{z\to-\infty}\psi(z)=0. \tag{4.14}$$

Let

$$F(z) := \frac{\psi(z)}{(1-z)^\gamma},$$

where γ is a constant to be determined. Then

$$\begin{aligned}\psi'(z) &= (1-z)^{\gamma-1}\left[(1-z)F'(z)-\gamma F(z)\right],\\ \psi''(z) &= (1-z)^{\gamma-2}\left[(1-z)^2F''(z)-2\gamma(1-z)F'(z)+\gamma(\gamma-1)F(z)\right].\end{aligned}$$

Hence the equation for $F(z)$ is

$$\begin{aligned}&z(1-z)F''(z)+\frac{1-(3+4\gamma)z}{2}F'(z)\\ &+\left[\frac{1-\frac{\alpha^2}{\pi^2}-\gamma^2}{z-1}-\frac{\frac{\beta}{\pi^2}+\frac{1}{2}}{z}-\gamma\left(\gamma+\frac{1}{2}\right)\right]F(z)=0.\end{aligned} \tag{4.15}$$

To cancel the $\frac{1-\frac{\alpha^2}{\pi^2}-\gamma^2}{z-1}$ term in (4.15), we set $\gamma := \sqrt{1-\frac{\alpha^2}{\pi^2}}$.

Let $G(z) := \frac{F(z)}{z^{\gamma'}}$, where γ' is a constant to be determined. Then

$$\begin{aligned}F'(z) &= z^{\gamma'-1}\left[zG'(z)+\gamma' G(z)\right],\\ F''(z) &= z^{\gamma'-2}\left[z^2G''(z)+2\gamma' zG'(z)+\gamma'\left(\gamma'-1\right)G(z)\right].\end{aligned}$$

Hence the equation for $G(z)$ is

$$z(1-z)G''(z)+\left[\frac{1}{2}+2\gamma'-\left(2\gamma+2\gamma'+\frac{3}{2}\right)z\right]G'(z) \tag{4.16}$$
$$-\left[\frac{\frac{\beta}{\pi^2}+\frac{1}{2}-\gamma'\left(\gamma'-\frac{1}{2}\right)}{z}+(\gamma+\gamma')\left(\gamma+\gamma'+\frac{1}{2}\right)\right]G(z)=0.$$

To cancel the $\frac{\frac{\beta}{\pi^2}+\frac{1}{2}-\gamma'\left(\gamma'-\frac{1}{2}\right)}{z}$ term in (4.16), we set $\gamma':=\frac{1}{4}+\sqrt{\frac{\beta}{\pi^2}+\frac{9}{16}}$. Then (4.16) becomes

$$z(1-z)G''(z)+\left[\frac{1}{2}+2\gamma'-\left(2\gamma+2\gamma'+\frac{3}{2}\right)z\right]G'(z) \tag{4.17}$$
$$-(\gamma+\gamma')\left(\gamma+\gamma'+\frac{1}{2}\right)G(z)=0.$$

The equation (4.17) is exactly the Euler's hypergeometric differential equation

$$z(1-z)G''(z)+\left[c-(a+b+1)z\right]G'(z)-abG(z)=0, \tag{4.18}$$

with $a=\gamma+\gamma', b=\gamma+\gamma'+\frac{1}{2}, c=\frac{1}{2}+2\gamma'$. The Euler's hypergeometric differential equation was first studied by Euler [12]. Its solutions are called the hypergeometric functions or Gaussian functions ${}_2F_1$, introduced by Gauss [13]. We consider two cases.

Case 1. $\gamma\in[0,1]\cup i\mathbf{R}^+$ and $\gamma'\in\left(\frac{1}{2},\frac{3}{4}\right)\cup\left(\frac{3}{4},1\right]$.

The solutions of (4.18) are

$$G_1(z)={}_2F_1\left(\gamma+\gamma',\gamma+\gamma'+\frac{1}{2};\frac{1}{2}+2\gamma';z\right),$$
$$G_2(z)=z^{\frac{1}{2}-2\gamma'}\,{}_2F_1\left(\gamma-\gamma'+\frac{1}{2},\gamma-\gamma'+1;\frac{3}{2}-2\gamma';z\right),$$

where

$${}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},$$

and $(q)_n$ is the Pochhammer symbol defined by:

$$(q)_n=\begin{cases}1, & n=0,\\ q(q+1)\cdots(q+n-1), & n>0.\end{cases}$$

Therefore the two independent solutions of (4.13) are

$$\psi_1(z) = (1-z)^\gamma z^{\gamma'}\ {}_2F_1\left(\gamma+\gamma', \gamma+\gamma'+\frac{1}{2}; \frac{1}{2}+2\gamma'; z\right),$$
$$\psi_2(z) = (1-z)^\gamma z^{\frac{1}{2}-\gamma'}\ {}_2F_1\left(\gamma-\gamma'+\frac{1}{2}, \gamma-\gamma'+1; \frac{3}{2}-2\gamma'; z\right),$$

where $z \in (-\infty, 0)$. Since $\frac{3}{2}-2\gamma'$ is not a non-positive integer, ${}_2F_1$ is analytic at $z=0$ and

$${}_2F_1\left(\gamma-\gamma'+\frac{1}{2}, \gamma-\gamma'+1; \frac{3}{2}-2\gamma'; 0\right) = 1.$$

This implies that ψ_2 must blow up at $z=0$, while ψ_1 stays finite. On the other hand, it is shown in [3] that

$$\lim_{z\to-\infty} \psi_1(z) = \frac{\sqrt{\pi}e^{i\pi\gamma'}\Gamma\left(\frac{1}{2}+2\gamma'\right)}{\Gamma\left(\frac{1}{2}-\gamma+\gamma'\right)\Gamma\left(\frac{1}{2}+\gamma+\gamma'\right)},$$

which is a nonzero constant. Hence no solution of (4.13) satisfies the boundary conditions (4.14).

Case 2. $\gamma \in [0,1] \cup i\mathbf{R}^+$ and $\gamma' = \frac{3}{4}$.

In this case,

$$\psi_1(z) = (1-z)^\gamma z^{\gamma'}\ {}_2F_1\left(\gamma+\gamma', \gamma+\gamma'+\frac{1}{2}; \frac{1}{2}+2\gamma'; z\right)$$

is still a nontrivial solution of (4.13). However, since $\frac{3}{2}-2\gamma'=0$, the function

$${}_2F_1\left(\gamma-\gamma'+\frac{1}{2}, \gamma-\gamma'+1; \frac{3}{2}-2\gamma'; \cdot\right)$$

is not well-defined in this case and so is ψ_2. Thus we have to find another nontrivial solution of (4.13) which is linear independent of ψ_1. By Lemma 4.1, we obtain that

$$\psi_3(z) = \psi_1(z)\int_{-1}^{z} \frac{1}{\psi_1^2(s)} e^{-\int_{-1}^{s}\frac{1}{t}\cdot\frac{1-3t}{\pi^2(1-t)^2}dt} ds$$

is the other desired solution of (4.13). After a simple computation, we deduce that

$$\lim_{z\to 0^-} |\psi_3(z)| = \infty.$$

By the similar argument as in Case 1, ψ_1 does not satisfy the boundary condition (4.14) at $-\infty$.

In conclusion, there is no nontrivial solution to (4.10) on $(0,1)$ with $\phi(1) = 0$ and $\phi(0)$ to be finite. $\blacksquare$

In the next lemma, we show that (4.6) is the only neutral mode for $c_s = 0$.

Lemma 4.3 *Let $\beta \in [0, \frac{\pi^2}{2})$ and $c = 0$. Then there is no neutral mode for $\alpha^2 \geq \frac{3\pi^2}{4}$; and there is exactly one neutral mode (4.6) for $0 \leq \alpha^2 < \frac{3\pi^2}{4}$.*

Proof. Define the new variable $t = \tan(\frac{\pi y}{2})$, $y \in (-1, 1)$. Then $t \in (-\infty, \infty)$ and $y = \frac{2}{\pi}\arctan t$. In this new variable,

$$U(y) = \cos^2(\frac{\pi y}{2}) = \frac{1}{t^2+1}, \quad U''(y) = -\frac{\pi^2}{2}[2\cos^2(\frac{\pi y}{2}) - 1] = -\frac{\pi^2}{2}\frac{1-t^2}{1+t^2}.$$

Let

$$\psi(t) := \phi(\frac{2}{\pi}\arctan t) = \phi(y).$$

Then

$$\begin{aligned}
\phi'(y) &= \frac{\pi}{2}\psi'(t)\left(1+t^2\right), \\
\phi''(y) &= \frac{\pi^2}{4}(1+t^2)^2\psi''(t) + \frac{\pi^2}{2}t(1+t^2)\psi'(t).
\end{aligned}$$

Thus (1.6)–(1.7) with $c = 0$ becomes the following boundary value problem

$$-\frac{\pi^2}{4}[(1+t^2)\psi']' + \frac{\alpha^2}{1+t^2}\psi - \frac{2\beta + \pi^2 + t^2(2\beta - \pi^2)}{2(1+t^2)}\psi = 0, \tag{4.19}$$

and $\lim_{t\to\pm\infty}\psi(t) = 0$.

Note that the neutral wave number and neutral solution in (4.6) with $r \in (\frac{1}{2}, 1]$ are

$$\alpha_s^2(r) = \pi^2(1-r^2) \in [0, \frac{3\pi^2}{4}), \qquad \phi_s(y) = \cos^{2r}(\frac{\pi y}{2}),$$

and

$$\beta_s(r) = \pi^2(-r^2 + \frac{r}{2} + \frac{1}{2}) \in [0, \frac{\pi^2}{2}).$$

In the new variable t, ϕ_s becomes

$$\psi_s(t) = \frac{1}{(1+t^2)^r}.$$

Since ψ_s has no zeros in $(-\infty, \infty)$, $\lambda_1 = -\pi^2(1-r^2)$ is the first eigenvalue of the boundary value problem (4.19). Therefore, there is no neutral mode for $\alpha^2 \geq \frac{3\pi^2}{4}$. For the Sturm-Liouville problem (1.6)–(1.7) with $c = 0$, we can compute the second eigenvalue and the corresponding eigenfunction to be

$$\lambda_2 = \pi^2(r^2 + r - \frac{3}{4}) > 0, \quad \phi_2(y) = \cos^{2r}(\frac{\pi y}{2}) \sin(\frac{\pi y}{2}), \quad r \in (\frac{1}{2}, 1].$$

In the new variable t, ϕ_2 becomes

$$\psi_2(t) = \frac{t}{(1+t^2)^{r+\frac{1}{2}}}.$$

Since ψ_2 has exactly one zero $t = 0$ in $(-\infty, \infty)$, $\lambda_2 = \pi^2(r^2 + r - \frac{3}{4}) > 0$ is the second eigenvalue of the boundary value problem (4.19). Therefore, (4.6) is the only neutral mode for $0 \leq \alpha^2 < \frac{3\pi^2}{4}$. The proof is complete. ∎

4.2 Stability boundary for $\beta \in (0, \frac{\pi^2}{2})$

In this subsection, we find the stability boundary for $(\alpha, \beta) \in \left(0, \frac{\sqrt{3}\pi}{2}\right) \times \left(0, \frac{\pi^2}{2}\right)$. This is done by studying the non-resonant neutral modes left and then using the index formula (4.4).

Fix $\beta_0 \in (0, \frac{\pi^2}{2})$. By Lemma 2.7, there is no non-resonant neutral mode for $c > 1, \beta_0 > 0$ and we can restrict to the case $c < 0$. Recall that $-f_{\beta_0,n}(c)$ is the n-th eigenvalue of $\mathcal{L}_{\beta_0,c}$ defined in (3.24) for $c \in (-\infty, 0)$. In the next lemma, we will show that $f_{\beta_0,n} < 0$ for any $n \geq 2$ and any $c \in (-\infty, 0)$. Therefore, to study the non-resonant neutral modes, we only need to consider $f_{\beta_0,1}$ which will be denoted by f_{β_0} for simplicity.

Lemma 4.4 *Let $\beta_0 \in \left(-\frac{\pi^2}{2}, \frac{\pi^2}{2}\right)$. Then $f_{\beta_0,n}(c) < 0$ for any $n \geq 2$ and any $c \in (-\infty, 0) \cup (1, +\infty)$.*

Proof. We divide the proof into two cases.

Case 1.

Let $c \in (-\infty, 0)$. It suffices to show that $f_{\beta_0,2}(c) < 0$. There exists $\beta_1 \in (\frac{\pi^2}{2}, +\infty)$ such that $c = \frac{1}{2} - \frac{\beta_1}{\pi^2}$. It is easy to see that $\lambda = 0$ and $\phi(y) = \sin(\pi y)$ are the second eigenvalue and the corresponding eigenfunction of the regular Sturm-Liouville problem

$$-\phi'' - \frac{\beta_1 - U''}{U - c}\phi = -\phi'' - \pi^2\phi = \lambda\phi, \ \ \phi(\pm 1) = 0.$$

This implies that $f_{\beta_1,2}(c) = 0$. By Corollary 3.1 (i), $f_{\beta,2}(c)$ is an increasing function of $\beta \in \mathbf{R}$ for fixed c. Since $\beta_0 < \frac{\pi^2}{2} < \beta_1$, we get $f_{\beta_0,2}(c) < 0$.

Case 2.

Let $c \in (1, +\infty)$. The proof is similar to that of Case 1 by using Corollary 3.1 (ii). ■

Our first task is to show that for $\beta_0 = \pi^2(-r_0^2 + \frac{1}{2}r_0 + \frac{1}{2}) \in (0, \frac{\pi^2}{2}]$, where $r_0 \in [\frac{1}{2}, 1)$, the non-resonant wave number $f_{\beta_0}(c)$ converges to the singular neutral wave number $\alpha_s^2 = \pi^2(1 - r_0^2)$ in (4.6) as $c \to 0^-$. To study this singular limit, we need some preliminary results.

Definition 4.1 *Let H be a complex separable Hilbert space and T, T_n $(n \geq 1)$ be self-adjoint operators in H. The sequence $\{T_n\}_{n=1}^{\infty}$ is said to be strong resolvent convergent (SRC) to T in H, if for some $z \in \mathbf{C} \backslash \mathbf{R}$,*

$$(T_n - z)^{-1}f \to (T - z)^{-1}f \ \ \textit{as} \ \ n \to +\infty$$

for all $f \in H$.

The following result gives a sufficient condition for a sequence $\{T_n\}_{n=1}^{\infty}$ to be SRC to T, see Theorem VIII 25 (a) in [36] or Theorem 9.16 (i) in [44].

Lemma 4.5 *Let T_n $(n \geq 1)$ and T be self-adjoint operators in H. Suppose there exists a linear manifold $C(T)$ of H such that*

(i) *$C(T)$ is a core of T,*

(ii) *if $f \in C(T)$, then there exists $n_0 \geq 1$ depending on f, such that $f \in D(T_n)$ for all $n \geq n_0$,*

(iii) *$T_n f \to Tf$ in H for all $f \in C(T)$ as $n \to +\infty$,*

then $\{T_n\}_{n=1}^{\infty}$ is SRC to T in H.

Let T be a closed operator in H. For any closable operator S such that $\bar{S} = T$, its domain $D(S)$ is called a core of T ([26, P.166]).

Definition 4.2 *The sequence $\{T_n\}_{n=1}^{\infty}$ is said to be spectral included for T, if for any $\lambda \in \sigma(T)$, there exists a sequence $\{\lambda_n\}_{n=1}^{\infty}$ with $\lambda_n \in \sigma(T_n)$ $(n \geq 1)$ such that $\lim_{n\to\infty} \lambda_n = \lambda$.*

The following result tells us that the strong resolvent convergence of self-adjoint operators implies the spectral convergence. See Theorem VIII 24 (a) in [36].

Lemma 4.6 *Let T_n $(n \geq 1)$ and T are self-adjoint operators in H. If $\{T_n\}_{n=1}^{\infty}$ is SRC to T in H, then $\{T_n\}_{n=1}^{\infty}$ is spectral included for T.*

The classification of the endpoints of singular Sturm-Liouville equation to be in the limit point and the limit circle cases is given below (See Definition 7.3.1 in [46]).

Definition 4.3 *For a general self-adjoint Sturm-Liouville equation*

$$-(p\varphi')' + q\varphi = \lambda w\varphi, \qquad \text{on } (y_1, y_2), \tag{4.20}$$

the endpoint y_1 is regular if $p(y) > 0, w(y) > 0$ a.e. on (y_1, y_2), $1/p, q \in L^1_{loc}(y_1, y_2)$ and $1/p, q, w \in L^1(y_1, c)$ for some $c \in (y_1, y_2)$. It is singular if it is not regular. It is in the limit circle case if it is singular and for any given $\lambda \in \mathbf{C}$, all the solutions of (4.20) are in $L^2_w(y_1, c)$ for some $c \in (y_1, y_2)$. It is in the limit point case if it is singular and not in the limit circle case.

By Theorem 7.2.2 in [46], the classification of limit circle and limit point cases is independent of $\lambda \in \mathbf{C}$. Definition 4.3 can be applied similarly to the endpoint y_2.

Now, we shall show that for $\beta_0 \in (0, \frac{\pi^2}{2}]$, the wave number of the non-resonant neutral mode with $c < 0$ converges to the wave number of the singular neutral mode (4.6) as $c \to 0^-$.

Proposition 4.1 *Let $\beta_0 = \pi^2(-r_0^2 + \frac{1}{2}r_0 + \frac{1}{2})$, where $r_0 \in [\frac{1}{2}, 1)$. Then*

$$f_{\beta_0}(c) \to \pi^2(1 - r_0^2), \quad \text{as } c \to 0^-. \tag{4.21}$$

Proof. If $r_0 = \frac{1}{2}$ (i.e. $\beta_0 = \frac{\pi^2}{2}$), then the regular neutral mode (4.5) and the singular neutral mode (4.6) coincide. If $r_0 \in (\frac{1}{2}, 1)$, the neutral mode (4.6) is not smooth. Recall that the two linearly independent solutions of the singular Sturm-Liouville equation (4.8) with

$$\alpha_s(r_0) = \pi\sqrt{1 - r_0^2}, \beta_s(r_0) = \beta_0 = \pi^2 \left(-r_0^2 + \frac{1}{2}r_0 + \frac{1}{2} \right), \tag{4.22}$$

are

$$\phi_0(y) = \cos^{2r_0}\left(\frac{\pi y}{2}\right), \quad \phi_1(y) = \cos^{2r_0}\left(\frac{\pi y}{2}\right) \int_0^y \frac{1}{\cos^{4r_0}(\frac{\pi s}{2})} ds,$$

where $r_0 \in \left(\frac{1}{2}, 1\right)$. Hence, if $r_0 \in (\frac{1}{2}, \frac{3}{4})$, then $\phi_0, \phi_1 \in L^2(-1,1)$ and thus by Definition 4.3 the singular Sturm-Liouville equation (4.8) is in the limit circle case at both endpoints -1 and 1; if $r_0 \in [\frac{3}{4}, 1)$, then $\phi_0 \in L^2(-1,1)$, $\phi_1 \notin L^2(-1,1)$ and thus the singular Sturm-Liouville equation (4.8) is in the limit point case at both endpoints -1 and 1.

First, we show that the limit (4.21) holds for $r_0 \in [\frac{3}{4}, 1)$ (i.e. $\beta_0 \in (0, \frac{5}{16}\pi^2]$). Define the operator $\mathcal{L}_{\beta_0}$ in the follow way:

$$\begin{aligned} \mathcal{L}_{\beta_0}\phi &:= -\phi'' - \frac{\beta_0 - U''}{U}\phi, \\ D(\mathcal{L}_{\beta_0}) &:= \{\phi \in L^2(-1,1) : \phi, \phi' \in AC_{loc}(-1,1),\ \mathcal{L}_{\beta_0}\phi \in L^2(-1,1)\}. \end{aligned}$$

By Theorem 5.8 in [45] or Theorem 10.4.1 in [46], $\mathcal{L}_{\beta_0}$ is a self-adjoint operator in $L^2(-1,1)$. For reader's convenience, we sketch the proof that $\mathcal{L}_{\beta_0}$ is self-adjoint. We define the maximal operator $\mathcal{L}_{\beta_0,\max} := \mathcal{L}_{\beta_0}$ and set

$$\begin{aligned} \mathcal{L}'_{\beta_0,\min}\phi &:= -\phi'' - \frac{\beta_0 - U''}{U}\phi, \\ D(\mathcal{L}'_{\beta_0,\min}) &:= \{\phi \in D(\mathcal{L}_{\beta_0}) : \phi \text{ has compact support in } (-1,1)\}. \end{aligned}$$

Here, $\mathcal{L}'_{\beta_0,\min}$ is called the pre-minimal operator, which is closable in $L^2(-1,1)$. The minimal operator $\mathcal{L}_{\beta_0,\min}$ is defined as the closure of $\mathcal{L}'_{\beta_0,\min}$. Note that $\mathcal{L}_{\beta_0,\min}$ is a closed symmetric operator and $\mathcal{L}^*_{\beta_0,\min} = \mathcal{L}_{\beta_0,\max}$. The deficiency indices (d^+, d^-) for $\mathcal{L}_{\beta_0,\min}$ are defined by

$$d^+ = \dim(Range(i - \mathcal{L}_{\beta_0,\min}))^\perp = \dim(\ker(-i - \mathcal{L}_{\beta_0,\max}))$$

and

$$d^- = \dim(Range(-i - \mathcal{L}_{\beta_0,\min}))^\perp = \dim(\ker(i - \mathcal{L}_{\beta_0,\max})).$$

Since (4.8) with $\alpha_s(r_0), \beta_s(r_0)$ given by (4.22) is in the limit point case at both endpoints -1 and 1, we have that $(d^+, d^-) = (0,0)$ for $\mathcal{L}_{\beta_0,\min}$. Then by the theory of self-adjoint extensions of a closed symmetric operator with equal deficiency indices (e.g. Theorem 8.11 in [44]), we deduce that $\mathcal{L}_{\beta_0,\min} = \mathcal{L}_{\beta_0,\max} = \mathcal{L}_{\beta_0}$ is self-adjoint.

From the above discussions, we immediately know, as suggested in [1], that $D(\mathcal{L}'_{\beta_0,\min})$ is a core of $\mathcal{L}_{\beta_0}$. It is obvious that $D(\mathcal{L}'_{\beta_0,\min}) \subset D(\mathcal{L}_{\beta_0,c})$ for any $c<0$, where $\mathcal{L}_{\beta_0,c}$ is defined in (3.24). Furthermore, for any $\phi \in D(\mathcal{L}'_{\beta_0,\min})$, by setting $\mathrm{supp}(\phi) = [a,b] \subset (-1,1)$ we get that

$$
\begin{aligned}
\|\mathcal{L}_{\beta_0,c}\phi - \mathcal{L}_{\beta_0}\phi\|_{L^2(-1,1)} &= \left\| \frac{\beta_0 - U''}{U-c}\phi - \frac{\beta_0 - U''}{U}\phi \right\|_{L^2(a,b)} \\
&= \left\| \frac{(\beta_0 - U'')c}{U(U-c)}\phi \right\|_{L^2(a,b)} \\
&= \int_a^b \frac{(\beta_0 - U'')^2 c^2}{U^2(U-c)^2}\phi^2 \, dy \to 0, \quad as \quad c \to 0^-.
\end{aligned}
$$

Thus, by Lemma 4.5 $\{\mathcal{L}_{\beta_0,c}, c<0\}$ is SRC to $\mathcal{L}_{\beta_0}$ in $L^2(-1,1)$. Then it follows from Lemma 4.6 that $\{\mathcal{L}_{\beta_0,c},\ c<0\}$ is spectral included for $\mathcal{L}_{\beta_0}$. By Definition 4.2, there exists $\lambda_c \in \sigma(\mathcal{L}_{\beta_0,c})$ such that $\lim\limits_{c\to 0^-} \lambda_c = -\pi^2(1-r_0^2) < 0$. Note that $\sigma(\mathcal{L}_{\beta_0,c})$ consists of discrete eigenvalues and is bounded from below. By Lemma 4.4, the second eigenvalue of $\mathcal{L}_{\beta_0,c}$ is positive for any fixed $c<0$. Therefore, for any $c<0$, $\lambda_c<0$ must be the first eigenvalue (i.e. $-f_{\beta_0}(c)$) of $\mathcal{L}_{\beta_0,c}$. Thus, $-\lambda_c = f_{\beta_0}(c) \to \pi^2(1-r_0^2)$ as $c \to 0^-$.

Next, we show that (4.21) holds for $r_0 \in [\frac{1}{2}, \frac{3}{4})$ (i.e. $\beta_0 \in (\frac{5}{16}\pi^2, \frac{\pi^2}{2}]$). Since we have shown in the above that $f_{\hat\beta}(c) \to \pi^2(1-\hat r^2) > 0$ as $c \to 0^-$ for a given $\hat\beta = \pi^2(-\hat r^2 + \frac{1}{2}\hat r + \frac{1}{2}) \in (0, \frac{5\pi^2}{16}]$, where $\hat r \in [\frac{3}{4}, 1)$, there exists $\delta > 0$ such that for any $c \in (-\delta, 0)$, $f_{\hat\beta}(c) > \varepsilon_0$ for some $\varepsilon_0 > 0$. Now fix any $c_2 \in (-\delta, 0)$. By Corollary 3.1 (i), $f_\beta(c_2)$ is an increasing function of $\beta \in \mathbf{R}$. Therefore, we have that $f_{\beta_0}(c_2) > \varepsilon_0$ for $\beta_0 = \pi^2(-r_0^2 + \frac{1}{2}r_0 + \frac{1}{2}) \in (\frac{5\pi^2}{16}, \frac{\pi^2}{2}]$. Note that there exists $\beta_2 \in (\frac{\pi^2}{2}, +\infty)$ such that $c_2 = \frac{1}{2} - \frac{\beta_2}{\pi^2}$. It is clear that $f_{\beta_2}(c_2) = \frac{3\pi^2}{4}$. It follows from Corollary 3.1 (i) again and $\beta_0 \in (\frac{5\pi^2}{16}, \frac{\pi^2}{2}]$ that $f_{\beta_0}(c_2) < \frac{3\pi^2}{4}$. Thus, we have shown that for the fixed $r_0 \in [\frac{1}{2}, \frac{3}{4})$, $0 < \varepsilon_0 < f_{\beta_0}(c) < \frac{3\pi^2}{4}$ for any $c \in (-\delta, 0)$.

For any $c \in (-\delta, 0)$, let ϕ_c be the corresponding real-valued eigenfunction of the eigenvalue $-f_{\beta_0}(c)$ with $\|\phi_c\|_{L^2} = 1$. First, we get a uniform bound of

$\|\phi_c\|_{H^1}$, $c<0$. The eigenfunction ϕ_c satisfies

$$-\phi_c'' + f_{\beta_0}(c)\phi_c - \frac{\beta_0 - U''}{U-c}\phi_c = 0, \tag{4.23}$$

with $\phi_c(\pm 1) = 0$. Multiplying (4.23) by ϕ_c and integrating from -1 to 1, we get that

$$\begin{aligned} \int_{-1}^{1} \left[|\phi_c'|^2 + f_{\beta_0}(c)|\phi_c|^2 \right] dy &= \int_{-1}^{1} \frac{\beta_0 - U''}{U-c} |\phi_c|^2 dy \\ &= \pi^2 \int_{-1}^{1} \frac{U - U_{\beta_0}}{U-c} |\phi_c|^2 \, dy \\ &= \pi^2 \int_{-1}^{1} \frac{U - c + c - U_{\beta_0}}{U-c} |\phi_c|^2 \, dy \\ &\le \pi^2 \|\phi_c\|_{L^2}^2 = \pi^2, \end{aligned} \tag{4.24}$$

since $U - c > 0$ on $y \in (-1,1)$ and $c - U_{\beta_0} = c - (\frac{1}{2} - \frac{\beta_0}{\pi^2}) < 0$. In view of $0 < f_{\beta_0}(c) < \frac{3\pi^2}{4}$ for any $c \in (-\delta, 0)$, we get $\|\phi_c\|_{H^1} \le \pi^2 + 1$. Therefore, for any sequence $\{c_k\}_{k=1}^{\infty} \subset (-\delta, 0)$ satisfying $\lim_{k\to\infty} c_k = 0$, up to a subsequence, we have $\phi_{c_k} \to \phi_0$ weakly in H^1 with $\|\phi_0\|_{L^2} = 1$. By the compact embedding $H^1(-1,1) \hookrightarrow C^0(-1,1)$, we also have $\phi_{c_k} \to \phi_0$ in $C^0(-1,1)$ and thus $\phi_0(\pm 1) = 0$. Let

$$\lim_{k\to\infty} f_{\beta_0}(c_k) = \tilde{\alpha}^2 \in \left[\varepsilon_0, \frac{3\pi^2}{4} \right].$$

We claim that ϕ_0 satisfies the equation

$$-\phi'' + \tilde{\alpha}^2 \phi - \frac{\beta_0 - U''}{U}\phi = 0, \quad \text{on } (-1,1), \tag{4.25}$$

with $\phi(\pm 1) = 0$. Assuming this is true, then by Lemma 4.3 and the fact that $\alpha^2 = \frac{3}{4}\pi^2$ is the only neutral wave number in $[\varepsilon_0, \frac{3}{4}\pi^2]$ for $\beta = \frac{\pi^2}{2}$ and $c = 0$, we have $\tilde{\alpha}^2 = \pi^2(1 - r_0^2)$ and $\phi_0(y) = \cos^{2r_0}(\frac{\pi y}{2})$. Thus, for the fixed $r_0 \in [\frac{1}{2}, \frac{3}{4})$,

$$\lim_{c\to 0^-} f_{\beta_0}(c) = \pi^2(1 - r_0^2).$$

It remains to show that ϕ_0 satisfies (4.25). Take any closed interval $[a,b] \in (-1,1)$. There exists $\delta_0 > 0$ such that $|U - c_k| \ge \delta_0$ on $[a,b]$ for any $k \ge 1$. Since ϕ_{c_k} satisfies the equation

$$-\phi_{c_k}'' + f_{\beta_0}(c_k)\phi_{c_k} - \frac{\beta_0 - U''}{U - c_k}\phi_{c_k} = 0, \tag{4.26}$$

it is easy to get

$$\|\phi_{c_k}\|_{H^3[a,b]} \le C\,\|\phi_{c_k}\|_{H^1[a,b]} \le \hat{C},$$

for constants C and $\hat{C}$ depending on a and b. Thus $\phi_{c_k} \to \phi_0$ in $C^2[a,b]$. Taking the limit $k \to \infty$ in the equation (4.26), we deduce that ϕ_0 satisfies the equation (4.25) on $[a,b]$ and also on $(-1,1)$ since $[a,b] \subset (-1,1)$ is arbitrary. This finishes the proof of the proposition. ∎

We define $f_{\beta_0}(0) = \pi^2(1-r_0)^2$ for any given $\beta_0 \in (0, \frac{\pi^2}{2}]$. Then by Proposition 4.1, $f_{\beta_0}(c)$ is left-continuous at 0.

By Lemma 3.2 (i) and similar to the proof of Lemma 3.3, we have

Lemma 4.7 *Let* $(c_0, \alpha_0, \beta_0, \phi_0)$ *satisfy* $c_0 = 0$, $\alpha_0^2 = f_{\beta_0}(0)$, $\beta_0 \in (0, \frac{\pi^2}{2}]$ *and*

$$-\phi_0'' + \alpha_0^2\phi_0 - \frac{\beta_0 - U''}{U}\phi_0 = 0, \quad \phi_0(\pm 1) = 0.$$

Then

$$\langle L_{\alpha_0}\omega_{\alpha_0}, \omega_{\alpha_0}\rangle = -U_{\beta_0}\int_{-1}^{1} \frac{\beta_0 - U''}{U^2}\phi_0^2 dy = -U_{\beta_0}\frac{df_{\beta_0}}{dc}\Big|_{c=0^-},$$

where $\omega_{\alpha_0} = (-\frac{d^2}{dy^2} + \alpha_0^2)\phi_0$.

The following is a consequence of (4.9) and Lemma 4.7.

Corollary 4.1 *Let* $(c_0, \alpha_0, \beta_0, \phi_0)$ *satisfy* $c_0 = 0$, $\alpha_0^2 = f_{\beta_0}(0)$, $\beta_0 \in (0, \frac{\pi^2}{2}]$ *and*

$$-\phi_0'' + \alpha_0^2\phi_0 - \frac{\beta_0 - U''}{U}\phi_0 = 0, \quad \phi_0(\pm 1) = 0.$$

Then

$$\frac{df_{\beta_0}}{dc}\Big|_{c=0^-} \begin{cases} = -\infty, & r \in (\frac{1}{2}, \frac{3}{4}], \\ < 0, & r \in (\frac{3}{4}, \frac{\sqrt{3}}{2}), \\ = 0, & r = \frac{\sqrt{3}}{2} \text{ or } 1, \\ > 0, & r \in (\frac{\sqrt{3}}{2}, 1). \end{cases} \tag{4.27}$$

Next, we continue to study the function f_{β_0}.

Lemma 4.8 *Let* $\beta_0 \in (0, \frac{\pi^2}{2}]$. *Then there exists* $K > 0$ *such that* $f_{\beta_0}(c) < 0$ *for any* $c < -K$.

Proof. Let ϕ_c be the eigenfunction satisfying

$$-\phi_c'' + f_{\beta_0}(c)\phi_c - \frac{\beta_0 - U''}{U - c}\phi_c = 0, \tag{4.28}$$

with $\phi_c(\pm 1) = 0$, and $\|\phi_c\|_{L^2} = 1$. Note that for any $\phi \in H_0^1(-1,1)$, we have $\|\phi'\|_{L^2} \geq \frac{\pi}{2}\|\phi\|_{L^2}$. Choose $K > 0$ large enough such that for any $c < -K$,

$$\max_{y\in[-1,1]} \left| \frac{\beta_0 - U''(y)}{U(y) - c} \right| < \left(\frac{\pi}{2}\right)^2.$$

Multiplying (4.28) by $\bar{\phi}_c$ and then integrating it from -1 to 1, we have

$$\|\phi_c'\|_{L^2}^2 + f_{\beta_0}(c)\|\phi_c\|_{L^2}^2 - \int_{-1}^{1} \frac{\beta_0 - U''}{U - c}|\phi_c|^2 \, dy = 0.$$

Then when $c < -K$, we have

$$-f_{\beta_0}(c)\|\phi_c\|_{L^2}^2 > \|\phi_c'\|_{L^2}^2 - \left(\frac{\pi}{2}\right)^2 \|\phi_c\|_{L^2}^2 \geq 0,$$

and thus $f_{\beta_0}(c) < 0$. ■

Let $\beta_0 \in (0, \frac{\pi^2}{2}]$. Since $f_{\beta_0} \in C^\infty(-\infty, 0)$, by Lemma 4.8 and (4.21), $f_{\beta_0}(c)$ has zero in $(-\infty, 0)$. Denote c_{β_0} to be the nearest zero of f_{β_0} to 0 in $(-\infty, 0)$. When we consider the restriction of f_{β_0} to $[c_{\beta_0}, 0]$, it is clear that $f_{\beta_0} \in C^\infty[c_{\beta_0}, 0) \cap C^0[c_{\beta_0}, 0]$. Now we are ready to give all stable wave numbers for any fixed $\beta_0 \in (0, \frac{\pi^2}{2})$.

Theorem 4.2 *Let $\beta_0 = \pi^2(-r_0^2 + \frac{1}{2}r_0 + \frac{1}{2}) \in (0, \frac{\pi^2}{2})$, where $r_0 \in (\frac{1}{2}, 1)$, and*

$$\alpha_{\beta_0,\max}^2 := \max_{c\in[c_{\beta_0},0]}\{f_{\beta_0}(c)\}. \tag{4.29}$$

Then $\alpha_{\beta_0,\max}^2 \in \left(0, \frac{3\pi^2}{4}\right)$, and

(i) *if $\beta_0 \in (0, \frac{(\sqrt{3}-1)\pi^2}{4}]$, then $\alpha_{\beta_0,\max}^2 = \pi^2(1 - r_0^2)$; if $\beta_0 \in (\frac{(\sqrt{3}-1)\pi^2}{4}, \frac{\pi^2}{2})$, then $\alpha_{\beta_0,\max}^2 > \pi^2(1 - r_0^2)$;*

(ii) *for any $\alpha^2 \in (\alpha_{\beta_0,\max}^2, \frac{3\pi^2}{4})$, there exist exactly one unstable mode and no neutral mode in H^2;*

(iii) *for any* $\alpha^2 \in (0, \alpha^2_{\beta_0,\max}]$*, there exists no unstable mode;*

(iv) *when* $\beta_0 \in (\frac{(\sqrt{3}-1)\pi^2}{4}, \frac{\pi^2}{2})$ *and* $\alpha^2 \in (\pi^2(1-r_0^2), \alpha^2_{\beta_0,\max})$*, there exist exactly two non-resonant neutral modes with* $c < 0$*; when* β_0 *and* α^2 *satisfy one of the following conditions:*

   (a) $\beta_0 \in (\frac{(\sqrt{3}-1)\pi^2}{4}, \frac{\pi^2}{2})$ *and* $\alpha^2 = \alpha^2_{\beta_0,\max}$*,*

   (b) $\beta_0 \in (\frac{(\sqrt{3}-1)\pi^2}{4}, \frac{\pi^2}{2})$ *and* $\alpha^2 = \pi^2(1-r_0^2)$*,*

   (c) $\beta_0 \in (0, \frac{\pi^2}{2})$ *and* $\alpha^2 \in (0, \pi^2(1-r_0^2))$*,*

there exists exactly one non-resonant neutral mode with $c < 0$*; in all cases above, there exists exactly one non-resonant neutral mode* $(c, \alpha, \beta_0, \phi)$ *such that* $\langle L_\alpha \omega_\alpha, \omega_\alpha \rangle \le 0$*, where* $\omega_\alpha = (-\frac{d^2}{dy^2} + \alpha^2)\phi$*.*

Proof. First, we show that $f^{-1}_{\beta_0}(\alpha^2_{\beta_0,\max})$ is a singleton set. Suppose otherwise, there exist $c_1 \neq c_2 \in [c_{\beta_0}, 0]$ such that $f_{\beta_0}(c_1) = f_{\beta_0}(c_2) = \alpha^2_{\beta_0,\max}$. Then $f'_{\beta_0}(c_1) \ge 0$ and $f'_{\beta_0}(c_2) \ge 0$. It follows from Theorem 3.2 that

$$\langle L_{\alpha_{\beta_0,\max}} \omega^i_{\alpha_{\beta_0,\max}}, \omega^i_{\alpha_{\beta_0,\max}} \rangle \le 0, \quad i = 1, 2,$$

where $\omega^i_{\alpha_{\beta_0,\max}} = (-\frac{d^2}{dy^2} + \alpha^2_{\beta_0,\max})\phi_i$, and ϕ_i satisfies

$$-\phi_i'' + \alpha^2_{\beta_0,\max}\phi_i - \frac{\beta_0 - U''}{U - c_i}\phi_i = 0, \quad \phi_i(\pm 1) = 0, \quad i = 1, 2.$$

Thus, $k_i^{\le 0} \ge 2$ for $\alpha_{\beta_0,\max}$, which contradicts Theorem 4.1. Thus, $f^{-1}_{\beta_0}(\alpha^2_{\beta_0,\max})$ is a singleton set and we denote $f^{-1}_{\beta_0}(\alpha^2_{\beta_0,\max}) = \{c_{\beta_0,\max}\}$.

Next, we show that for any fixed $\alpha^2 \in (0, \alpha^2_{\beta_0,\max})$, $A_{\alpha,\beta_0} := f^{-1}_{\beta_0}(\alpha^2) \cap (c_{\beta_0}, c_{\beta_0,\max})$ is a singleton set. By the continuity of f_{β_0}, $A_{\alpha,\beta_0} \neq \emptyset$ and is a closed set. Let $\hat{c}_1 := \min\limits_{c \in A_{\alpha,\beta_0}} \{c\}, \hat{c}_2 := \max\limits_{c \in A_{\alpha,\beta_0}} \{c\}$. Then for any $c \in (c_{\beta_0}, \hat{c}_1)$, we have $0 = f_{\beta_0}(c_{\beta_0}) < f_{\beta_0}(c) < f_{\beta_0}(\hat{c}_1)$. This implies $f'_{\beta_0}(\hat{c}_1) \ge 0$. Similarly, $f'_{\beta_0}(\hat{c}_2) \ge 0$, since $f_{\beta_0}(\hat{c}_2) < f_{\beta_0}(c) < f_{\beta_0}(c_{\beta_0,\max})$ for all $c \in (\hat{c}_2, c_{\beta_0,\max})$. It again follows from Theorem 3.2 that

$$\langle L_\alpha \omega^i_\alpha, \omega^i_\alpha \rangle \le 0, \quad i = 1, 2,$$

where $\omega_\alpha^i = (-\frac{d^2}{dy^2} + \alpha^2)\hat{\phi}_i$, and $\hat{\phi}_i$ satisfies

$$-\hat{\phi}_i'' + \alpha^2 \hat{\phi}_i - \frac{\beta_0 - U''}{U - \hat{c}_i}\hat{\phi}_i = 0, \quad \hat{\phi}_i(\pm 1) = 0, \quad i = 1, 2.$$

By Theorem 4.1, we must have $\hat{c}_1 = \hat{c}_2$ and hence A_{α,β_0} is a singleton set.

Above arguments also imply that $f'_{\beta_0}(c) \geq 0$ for all $c \in [c_{\beta_0}, c_{\beta_0,\max}]$ and there does not exist a subinterval $I \subset [c_{\beta_0}, c_{\beta_0,\max}]$ such that $f'_{\beta_0}(c) \equiv 0$ on $c \in I$. Thus f_{β_0} is increasing on $[c_{\beta_0}, c_{\beta_0,\max}]$. This means that for any $\alpha^2 \in (0, \alpha^2_{\beta_0,\max}]$, there exists exactly one non-resonant neutral mode $(c, \alpha, \beta_0, \phi)$ with $c \in (c_{\beta_0}, c_{\beta_0,\max}]$ such that $\langle L_\alpha \omega_\alpha, \omega_\alpha \rangle \leq 0$, where $\omega_\alpha = (-\frac{d^2}{dy^2} + \alpha^2)\phi$, and thus $k_i^{\leq 0} \geq 1$. By Theorem 4.1, $k_c + k_r = 0$ and hence there is no unstable mode for any $\alpha^2 \in (0, \alpha^2_{\beta_0,\max}]$. Therefore, (iii) has been proved.

For any fixed $\beta_0 \in (0, \frac{\pi^2}{2})$, we show that $\alpha^2_{\beta_0,\max} < \frac{3\pi^2}{4}$. In fact, by Lemma 2.6 there exists $\delta > 0$ sufficiently small that for any $\alpha^2 \in (\frac{3\pi^2}{4} - \delta, \frac{3\pi^2}{4})$, there exists an unstable mode, that is, $k_c + k_r \geq 1$. Suppose that $\alpha^2_{\beta_0,\max} \geq \frac{3\pi^2}{4}$. Then by the previous arguments, we have that $k_i^{\leq 0} \geq 1$ for any

$$\alpha^2 \in \left(\frac{3\pi^2}{4} - \delta, \frac{3\pi^2}{4} \right) \subset (0, \alpha^2_{\beta_0,\max}).$$

Thus $k_c + k_r + k_i^{\leq 0} \geq 2$ for any $\alpha^2 \in (\frac{3\pi^2}{4} - \delta, \frac{3\pi^2}{4})$, which contradicts the index formula (4.4).

To get the detailed information about $\alpha^2_{\beta_0,\max}$, we consider three cases.

Case 1. $\beta_0 \in [\frac{5\pi^2}{16}, \frac{\pi^2}{2})$, resp. $r_0 \in (\frac{1}{2}, \frac{3}{4}]$.

In this case, by (4.27), $\frac{df_{\beta_0}}{dc}\Big|_{c=0^-} = -\infty$ and thus $c_{\beta_0,\max} < 0$. Then we show that f_{β_0} is decreasing on $(c_{\beta_0,\max}, 0)$. Indeed, we will show that $f'_{\beta_0}(c) < 0$ for $c \in (c_{\beta_0,\max}, 0)$. For any fixed $c \in (c_{\beta_0,\max}, 0)$, $f_{\beta_0}(c) \in (0, \alpha^2_{\beta_0,\max})$ and there exists $\tilde{c} \in (c_{\beta_0}, c_{\beta_0,\max})$ such that $f_{\beta_0}(c) = f_{\beta_0}(\tilde{c}) := \alpha^2$. Note that $f'_{\beta_0}(\tilde{c}) \geq 0$. By Theorem 3.2, we have

$$\langle L_\alpha \omega_\alpha^{\tilde{c}}, \omega_\alpha^{\tilde{c}} \rangle \leq 0,$$

where $\omega_\alpha^{\tilde{c}} = (-\frac{d^2}{dy^2} + \alpha^2)\phi_{\tilde{c}}$, and $\phi_{\tilde{c}}$ satisfies

$$-\phi_{\tilde{c}}'' + \alpha^2 \phi_{\tilde{c}} - \frac{\beta_0 - U''}{U - \tilde{c}}\phi_{\tilde{c}} = 0, \quad \phi_{\tilde{c}}(\pm 1) = 0.$$

It follows from Theorem 4.1 that

$$\langle L_\alpha \omega_\alpha^c, \omega_\alpha^c \rangle > 0,$$

where ω_α^c is defined similarly as $\omega_\alpha^{\tilde{c}}$ (with $\tilde{c}$ replaced by c). This implies $f'_{\beta_0}(c) < 0$ again by Theorem 3.2. The graph of f_{β_0} for Case 1 is single humped, as shown in Figure 1.

Case 2. $\beta_0 \in (\frac{(\sqrt{3}-1)\pi^2}{4}, \frac{5\pi^2}{16})$, resp. $r_0 \in (\frac{3}{4}, \frac{\sqrt{3}}{2})$.

This case is almost the same as Case 1, with the only difference being that $-\infty < \frac{df_{\beta_0}}{dc}\big|_{c=0^-} < 0$. Thus we skip the details. The graph of f_{β_0} for Case 2 (see Figure 2) is very similar to that for Case 1.

Case 3. $\beta_0 \in (0, \frac{(\sqrt{3}-1)\pi^2}{4}]$, resp. $r_0 \in [\frac{\sqrt{3}}{2}, 1)$.

In this case, by (4.27), $\frac{df_{\beta_0}}{dc}\big|_{c=0^-} \geq 0$. We claim that $c_{\beta_0,\max} = 0$. Suppose otherwise, $c_{\beta_0,\max} < 0$ and as a consequence,

$$f_{\beta_0}(c_{\beta_0,\max}) > f_{\beta_0}(0) > 0 = f_{\beta_0}(c_{\beta_0}).$$

Since f_{β_0} is continuous and increasing on $(c_{\beta_0}, c_{\beta_0,\max})$, there exists exactly one $\hat{c} \in (c_{\beta_0}, c_{\beta_0,\max})$ such that $f_{\beta_0}(\hat{c}) = f_{\beta_0}(0) := \hat{\alpha}^2$ and $f'_{\beta_0}(\hat{c}) \geq 0$. Thus, by Theorem 3.2,

$$\langle L_{\hat{\alpha}} \omega_{\hat{\alpha}}^{\hat{c}}, \omega_{\hat{\alpha}}^{\hat{c}} \rangle \leq 0, \quad \langle L_{\hat{\alpha}} \omega_{\hat{\alpha}}^{0}, \omega_{\hat{\alpha}}^{0} \rangle \leq 0,$$

where $\omega_{\hat{\alpha}}^c = (-\frac{d^2}{dy^2} + \hat{\alpha}^2)\phi_c$ $(c = 0, \hat{c})$ and ϕ_c satisfies

$$-\phi_c'' + \hat{\alpha}^2 \phi_c - \frac{\beta_0 - U''}{U - c}\phi_c = 0, \quad \phi_c(\pm 1) = 0.$$

This implies that $k_i^{\leq 0} \geq 2$, a contradiction to (4.4). Thus $c_{\beta_0,\max} = 0$ and $f'_{\beta_0}(c) \geq 0$ for $c \in (c_{\beta_0}, 0)$. The graph of f_{β_0} in this case is monotone increasing, as seen in Figure 3.

(i) follows readily from the properties of the graph of f_{β_0} discussed above.

Then we show that

$$f_{\beta_0}(c) \leq 0, \quad \forall \; c \in (-\infty, c_{\beta_0}). \tag{4.30}$$

Assume that this is true. Then the conclusions in Cases 1–3 imply (iv) holds. For any fixed $\alpha^2 \in (\alpha^2_{\beta_0,\max}, \frac{3\pi^2}{4})$, we now show that there exists no neutral mode in H^2 such that $k_i^{\leq 0} = 1$ and exactly one unstable mode. It follows from Lemma 2.7 that there exists no such neutral mode with $c \geq 1$.

Clearly, there exists no such neutral mode with $c = U_\beta$. There exists no such neutral mode with $c = 0$ by Lemma 4.3. In view of (4.29)–(4.30), there exists no such neutral mode with $c < 0$. Consequently, $k_c + k_r = 1$ by (4.4), and thus there exists exactly one unstable mode. This finishes the proof of (ii).

It remains to prove (4.30). Suppose otherwise, there exists $c_0 < c_{\beta_0}$ such that $f_{\beta_0}(c_0) > 0$. By Lemma 4.8, there exists $c_1 \in (-K, c_0)$ such that $\alpha^2_{\beta_0,\max} > \alpha_0^2 := f_{\beta_0}(c_1) > 0$ and $f'_{\beta_0}(c_1) \geq 0$. From the discussions in Cases 1–3, we know that there exists $c_2 \in (c_{\beta_0}, c_{\beta_0,\max})$ such that $f_{\beta_0}(c_2) = \alpha_0^2$ and $f'_{\beta_0}(c_2) \geq 0$. This is a contradiction due to Theorem 3.2 and (4.4). ■

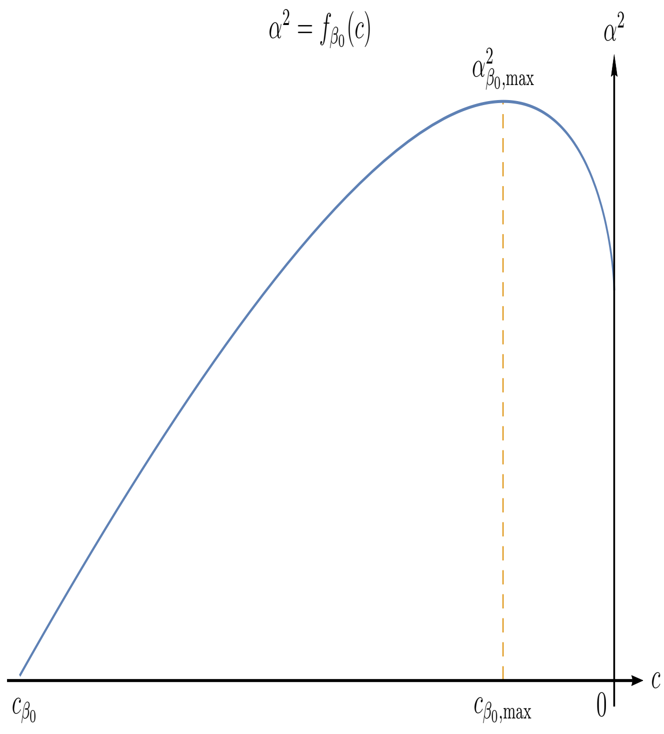


Figure 1.

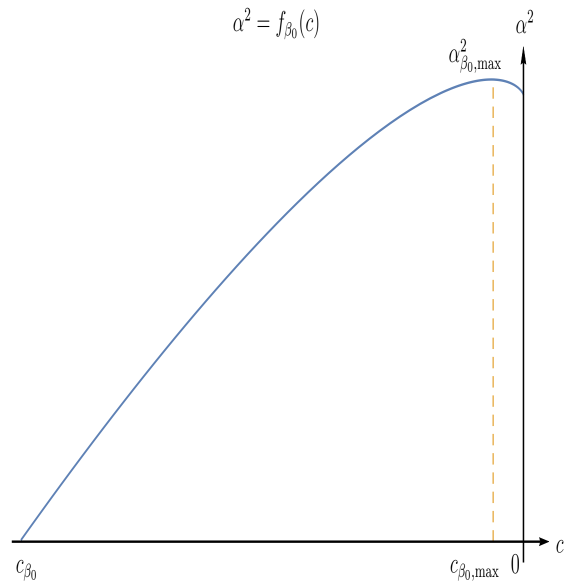


Figure 2.

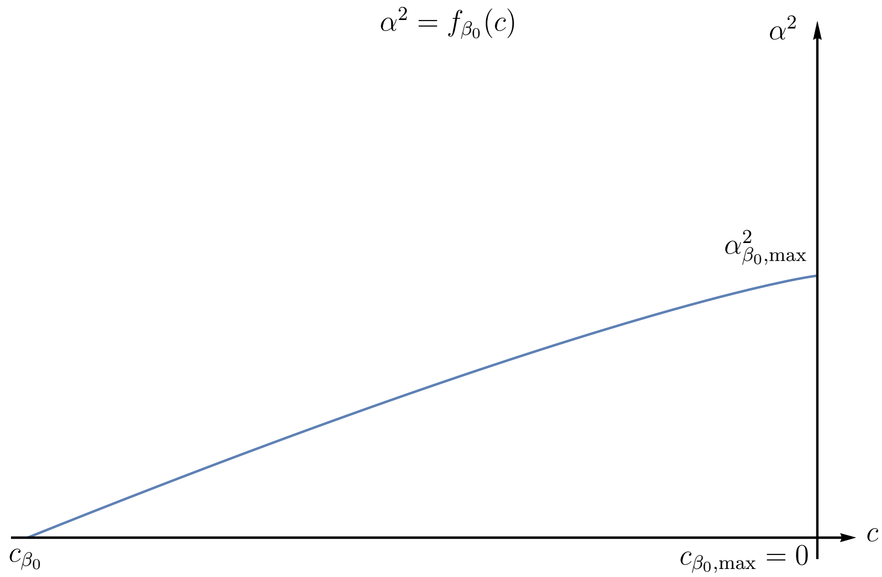


Figure 3.

4.3 Stability boundary for $\beta \in (-\frac{\pi^2}{2}, 0)$

In this subsection, we find the stability boundary for the case $(\alpha, \beta) \in \left(0, \frac{\sqrt{3}\pi}{2}\right) \times \left(-\frac{\pi^2}{2}, 0\right)$. As in the last subsection, we will study the non-resonant neutral modes and then use the index formula (4.4) to find the sharp stability condition. By Lemma 2.7, there is no non-resonant neutral mode for $c < 0$ and we can restrict to the case $c > 1$. By Lemma 4.4, $f_{\beta_0,n}(c) < 0$ for any $n \geq 2$ and $c > 1$, where $f_{\beta_0,n}$ is defined in (3.23). Hence, it suffices to study $f_{\beta_0,1}(c)$ for $c > 1$, which is denoted by $f_{\beta_0}(c)$ below.

The following is a consequence of (4.8) in [39].

Lemma 4.9 *Let $\beta_0 = 0$. Then $f_{\beta_0}(c) < 0$ for all $c \in (1, +\infty)$.*

The next lemma is similar to Lemma 4.8.

Lemma 4.10 *Let $\beta_0 \in [-\frac{\pi^2}{2}, 0)$. Then there exists $K > 1$ such that $f_{\beta_0}(c) < 0$ for any $c > K$.*

The following result gives the properties of f_{β_0} when $\beta_0 = -\frac{\pi^2}{2}$.

Lemma 4.11 *Let $\beta_0 = -\frac{\pi^2}{2}$. Then f_{β_0} is decreasing on $c \in (1, +\infty)$, $f_{\beta_0}(c) \to \frac{3\pi^2}{4}^-$ as $c \to 1^+$, and there exists exactly one $c_* \in (1, +\infty)$ such that $f_{\beta_0}(c_*) = 0$.*

Proof. By Corollary 3.1 (iii), f_{β_0} is decreasing on $c \in (1, +\infty)$.

For any fixed $\hat{c} \in (1, +\infty)$, by Corollary 3.1 (ii), $f_\beta(\hat{c})$ is decreasing on $\beta \in \mathbf{R}$, and there exists $\hat{\beta} \in (-\infty, -\frac{\pi^2}{2})$ such that $\hat{c} = \frac{1}{2} - \frac{\hat{\beta}}{\pi^2}$. Note that $f_{\hat{\beta}}(\hat{c}) = \frac{3\pi^2}{4}$ and $\hat{\beta} < \beta_0$. Thus, we get that $f_{\beta_0}(\hat{c}) < \frac{3\pi^2}{4}$.

We claim that for any fixed $\alpha \in (0, \frac{\sqrt{3}\pi}{2})$, there exists $c_\alpha \in (1, +\infty)$ such that $\alpha^2 = f_{\beta_0}(c_\alpha) > 0$. In fact, by Rayleigh-Kuo criterion there is no unstable mode for $(\alpha, \beta_0 = -\frac{\pi^2}{2})$, which yields that $k_c + k_r = 0$. It follows from Theorem 4.1 that $k_i^{\leq 0} = 1$ and thus there must exist a neutral mode in H^2 with $c = c_\alpha$. Now we show that $c_\alpha \in (1, +\infty)$. By Theorem 2.1, $c_\alpha \notin (0, 1)$. According to Lemmas 2.7 and 4.2, we have that $c_\alpha \notin (-\infty, 0]$ and $c_\alpha \neq 1$. Thus, $c_\alpha \in (1, +\infty)$.

Therefore, there exists $c_1 \in (1, +\infty)$ such that $0 < f_{\beta_0}(c_1) < f_{\beta_0}(c) < \frac{3\pi^2}{4}$ for all $c \in (1, c_1)$. Similar to the proof of (4.24), we get a uniform H^1 bound for the eigenfunctions ϕ_c (normalized by $\|\phi_c\|_{L^2} = 1$) corresponding to the eigenvalues $-f_{\beta_0}(c)$ for all $c \in (1, c_1)$. Thus there exists $\phi_0 \in H^1(-1,1)$ with $\|\phi_0\|_{L^2} = 1$ such that $\phi_c \to \phi_0$ weakly in $H^1(-1,1)$ as $c \to 1^+$. In view of $H^1(-1,1) \hookrightarrow C^0(-1,1)$, $\phi_c \to \phi_0$ in $C^0(-1,1)$ as $c \to 1^+$ and $\phi_0(\pm 1) = 0$. Since $\min_{y\in[a,b]}\{|U(y) - c|\} =: \delta > 0$ for any fixed $[a,b] \subset (-1,0)$ or $(0,1)$, we get the uniform bound of $\|\phi_c\|_{H^3(a,b)}$ and as a consequence $\phi_c \to \phi_0$ in $C^2[a,b]$ as $c \to 1^+$. Let $\hat{\alpha}^2 = \lim_{c\to 1^+} f_{\beta_0}(c)$. Then ϕ_0 satisfies the equation

$$-\phi_0'' + \hat{\alpha}^2\phi_0 - \frac{\beta_0 - U''}{U - 1}\phi_0 = -\phi_0'' + \hat{\alpha}^2\phi_0 - \pi^2\phi_0 = 0 \tag{4.31}$$

on $(-1,0)\cup(0,1)$, and by the continuity of ϕ_0 on $[-1,1]$ (4.31) is also satisfied at $y = 0$. Thus we have $\hat{\alpha}^2 = \frac{3\pi^2}{4}$, that is, $f_{\beta_0}(c) \to \frac{3\pi^2}{4}^-$ as $c \to 1^+$.

Then Lemma 4.10, together with the continuity and monotonicity of f_{β_0}, implies that there exists exactly one $c_* \in (1, +\infty)$ such that $f_{\beta_0}(c_*) = 0$. This finishes the proof of the lemma. ∎

By Lemma 4.11, for $\beta_0 = -\frac{\pi^2}{2}$ and each $\alpha_0 \in \left(0, \frac{\sqrt{3}\pi}{2}\right)$, there exists a non-resonant neutral mode $(c_0, \alpha_0, \beta_0, \phi_0)$ with $c_0 > 1$ such that

$$\langle L_{\alpha_0}\omega_{\alpha_0}, \omega_{\alpha_0}\rangle = (c_0 - 1)\int_{-1}^{1} \frac{(-\frac{\pi^2}{2} - U'')}{|U - c_0|^2}|\phi_0|^2 dy < 0,$$

where $\omega_{\alpha_0} = (-\frac{d^2}{dy^2} + \alpha_0^2)\phi_0$.

For any fixed $c \in (1, +\infty)$, there exists $\hat{\beta}_c \in (-\infty, -\frac{\pi^2}{2})$ such that $c = \frac{1}{2} - \frac{\hat{\beta}_c}{\pi^2}$ and $f_{\hat{\beta}_c}(c) = \frac{3\pi^2}{4}$. By Corollary 3.1 (ii), $f_\beta(c)$ is decreasing in $\beta \in \mathbf{R}$ and by Lemma 4.9, $f_0(c) < 0$. Thus there exists exactly one $\beta_c \in (\hat{\beta}_c, 0)$ such that $f_{\beta_c}(c) = 0$. Define

$$h(c) := \beta_c$$

for each $c \in (1, +\infty)$. Apparently, $h \in C^0(1, +\infty)$ and $h(c_*) = -\frac{\pi^2}{2}$, where c_* is given in Lemma 4.11. We will show that $h(c) \to -\frac{\pi^2}{2}^+$ as $c \to 1^+$ in Lemma 4.13 below. Thus we can define

$$\beta_{\max} = \max_{c\in(1,c_*]}\{h(c)\}. \tag{4.32}$$

Be Lemma 4.13, $\beta_{\max} > -\frac{\pi^2}{2}$. By Lemma 4.9, we know that $\beta_{\max} < 0$ and thus $\beta_{\max} \in \left(-\frac{\pi^2}{2}, 0\right)$. The next lemma shows that $\beta_{\max}$ is the maximal β value for the existence of non-resonant neutral modes with the phase speed $c > 1$.

Lemma 4.12 (i) *If $\beta_0 \in (\beta_{\max}, 0)$, then $f_{\beta_0}(c) < 0$ for any $c \in (1, +\infty)$.*

(ii) *If $\beta_0 = \beta_{\max}$, then $f_{\beta_0}(c) \le 0$ for any $c \in (1, +\infty)$.*

(iii) *If $\beta_0 \in (-\frac{\pi^2}{2}, \beta_{\max})$, then there exists $c_0 \in (1, c_*)$ such that $f_{\beta_0}(c_0) > 0$.*

Proof. Suppose that (i) is not true. Then there exist $\beta_0 \in (\beta_{\max}, 0)$ and $c_1 > 1$ such that $f_{\beta_0}(c_1) \ge 0$. By Lemma 4.10 and the continuity of f_{β_0}, there exists $c_2 > 1$ such that $f_{\beta_0}(c_2) = 0$. By Corollary 3.1 (ii), $f_{\beta_1}(c_2) > 0$ for $\beta_1 = -\frac{\pi^2}{2}$. It follows from Lemma 4.11 that $c_2 < c_*$. This contradicts the definition of $\beta_{\max}$ and thus (i) is true.

Suppose that (ii) is not true. Then there exists $c_3 > 1$ such that $f_{\beta_0}(c_3) > 0$. By Lemma 4.9 and the continuity of $f_\beta(c_3)$ in β, there exists $\beta_2 \in (\beta_{\max}, 0)$ such that $f_{\beta_2}(c_3) = 0$. Since $f_{\beta_1}(c_3) > 0$ for $\beta_1 = -\frac{\pi^2}{2}$, by Lemma 4.11 we have $c_3 < c_*$. This again contradicts the definition of $\beta_{\max}$ and thus (ii) is true.

To show (iii), we note that by the continuity of h and the definition of $\beta_{\max}$, if $\beta_0 \in (-\frac{\pi^2}{2}, \beta_{\max})$, then for any given $\beta_1 \in (\beta_0, \beta_{\max}]$, there exists $c_0 \in (1, c_*)$ such that $f_{\beta_1}(c_0) = 0$. By Corollary 3.1 (ii), we get that $f_{\beta_0}(c_0) > f_{\beta_1}(c_0) = 0$. ■

The behavior of h as $c \to 1^+$ is studied in the next lemma.

Lemma 4.13 $h(c) \to -\frac{\pi^2}{2}^+$ *as $c \to 1^+$.*

Proof. Let $c \in (1, c_*]$ and ϕ_c satisfy

$$-\phi_c'' - \frac{h(c) - U''}{U - c}\phi_c = 0, \quad \phi_c(\pm 1) = 0, \tag{4.33}$$

with $\|\phi_c\|_{L^2} = 1$. Note that $h(c) \in (-\frac{\pi^2}{2}, 0)$ for any $c \in (1, c_*)$ by Corollary 3.1, Lemmas 4.9 and 4.11. By using the facts $U(y) - c < 0$ on $y \in (-1, 1)$ and

$$U_{h(c)} = \frac{1}{2} - \frac{h\,(c)}{\pi^2} \le 1 < c,$$

we can get a uniform bound of $\|\phi_c\|_{H^1}$ for $c \in (1, c_*)$ as in the proof of (4.24). Thus there exists $\phi_1 \in H^1(-1, 1)$ with $\|\phi_1\|_{L^2} = 1$ such that $\phi_c \to \phi_1$ in $C^0[-1, 1]$ as $c \to 1^+$ and $\phi_1(\pm 1) = 0$. Since $\|\phi_1\|_{L^2} = 1$, ϕ_1 is nontrivial on $(0, 1)$ or $(-1, 0)$. We assume that ϕ_1 is nontrivial on $(0, 1)$. For any fixed

$[a, b] \subset (0, 1)$, by (4.33) we can obtain a uniform bound of $\|\phi_c\|_{H^3[a,b]}$ for $c \in (1, c_*)$. Thus $\phi_c \to \phi_1$ in $C^2[a, b]$ as $c \to 1^+$. Let

$$\hat{\beta} = \lim_{c \to 1^+} h(c) \in [-\frac{\pi^2}{2}, 0].$$

Then ϕ_1 satisfies the equation

$$-\phi_1'' - \frac{\hat{\beta} - U''}{U - 1}\phi_1 = 0, \quad \text{on} \quad (0, 1)$$

with $\phi_1(1) = 0$ and $\phi_1(0)$ to be finite. Be Lemma 4.2, we must have $\hat{\beta} = -\frac{\pi^2}{2}$, that is, $h(c) \to -\frac{\pi^2}{2}^+$ as $c \to 1^+$. ■

Next we study the behavior of f_{β_0} as $c \to 1^+$.

Lemma 4.14 *Let* $\beta_0 \in (-\frac{\pi^2}{2}, \beta_{\max}]$. *Then* $f_{\beta_0}(c) < 0$ *as* $c \to 1^+$.

Proof. Suppose otherwise, then there exists a sequence $c_n \to 1^+$ such that $f_{\beta_0}(c_n) \geq 0$. We know that $f_{\beta_0}(c) \leq \frac{3\pi^2}{4}$ for each $c \in (1, +\infty)$. Similar to the proof of (4.24), we can get a uniform bound of $\|\phi_{c_n}\|_{H_1}$, where ϕ_{c_n} is the eigenfunction satisfying $\|\phi_{c_n}\|_{L^2} = 1$ and

$$-\phi_{c_n}'' + f_{\beta_0}(c_n)\phi_{c_n} - \frac{\beta_0 - U''}{U - c_n}\phi_{c_n} = 0, \;\; \phi_{c_n}(\pm 1) = 0.$$

Thus there exists $\phi_0 \in H^1[-1, 1]$ such that $\phi_{c_n} \to \phi_0$ in $C^0[-1, 1]$ as $n \to \infty$, and

$$\|\phi_0\|_{L^2} = 1, \;\; \phi_0(\pm 1) = 0.$$

Let

$$\lambda_0 = \lim_{n \to \infty} f_{\beta_0}(c_n) \in \left[0, \frac{3\pi^2}{4}\right].$$

Then as in the proof of Lemmas 4.11 and 4.13, ϕ_0 satisfies

$$-\phi_0'' + \lambda_0\phi_0 - \frac{\beta_0 - U''}{U - 1}\phi_0 = 0, \;\; \text{on} \;\; (-1, 0) \cup (0, 1)$$

with $\phi_0(\pm 1) = 0$ and $\phi_0(0)$ to be finite. Then by Lemma 4.2, $\phi_0 \equiv 0$ on $(-1, 1)$, which contradicts $\|\phi_0\|_{L^2} = 1$. Hence, $f_{\beta_0}(c) < 0$ as $c \to 1^+$. This finishes the proof of the lemma. ■

The stability boundary for $\beta_0 \in \left(-\frac{\pi^2}{2}, 0\right)$ is given as follows.

Theorem 4.3 (i) *Let $\beta_0 \in [\beta_{\max}, 0)$. Then for any $\alpha^2 \in \left(0, \frac{3\pi^2}{4}\right)$, there exist exactly one unstable mode and no neutral mode in H^2.*

(ii) *Let $\beta_0 \in \left(-\frac{\pi^2}{2}, \beta_{\max}\right)$ and define*

$$\alpha^2_{\beta_0,\max} := \max_{c\in(1,+\infty)} \{f_{\beta_0}(c)\}. \tag{4.34}$$

Then

(iia) $\alpha^2_{\beta_0,\max} \in \left(0, \frac{3\pi^2}{4}\right)$*;*

(iib) *for any $\alpha^2 \in (\alpha^2_{\beta_0,\max}, \frac{3\pi^2}{4})$, there exist exactly one unstable mode and no neutral mode in H^2;*

(iic) *for any $\alpha^2 \in (0, \alpha^2_{\beta_0,\max}]$, there exists no unstable mode;*

(iid) *for any $\alpha^2 \in \left(0, \alpha^2_{\beta_0,\max}\right)$, there exists exactly two non-resonant neutral modes with $c > 1$; for $\alpha^2 = \alpha^2_{\beta_0,\max}$, there exists exactly one non-resonant neutral mode with $c > 1$; in both cases, there exists exactly one non-resonant neutral mode $(c, \alpha, \beta_0, \phi)$ satisfying $\langle L_\alpha \omega_\alpha, \omega_\alpha\rangle \le 0$, where $\omega_\alpha = (-\frac{d^2}{dy^2} + \alpha^2)\phi$.*

Proof. First, we show that (i) is true. By Lemmas 2.7, 4.2 and (i)–(ii) of Lemma 4.12, there is no neutral mode in H^2 for any $\alpha^2 \in \left(0, \frac{3\pi^2}{4}\right)$. Then by the index formula (4.4), there exists exactly one unstable mode.

Now we show that (ii) holds. Similar to the proof of Theorem 4.2, we can show the following conclusions:

(1) $f^{-1}_{\beta_0}(\alpha^2_{\beta_0,\max})$ is a singleton set. Let $f^{-1}_{\beta_0}(\alpha^2_{\beta_0,\max}) := \{c_{\beta_0,\max}\}$. Note that $\alpha^2_{\beta_0,\max} > 0$ by (iii) of Lemma 4.12. By Lemmas 4.10 and 4.14, there exist at least one zero of f_{β_0} in $(1, c_{\beta_0,\max})$ and at least one zero of f_{β_0} in $(c_{\beta_0,\max}, +\infty)$. Let $c_{1,\beta_0} \in (1, c_{\beta_0,\max})$ be the left nearest zero of f_{β_0} to $c_{\beta_0,\max}$ and $c_{2,\beta_0} \in (c_{\beta_0,\max}, +\infty)$ be the right nearest zero of f_{β_0} to $c_{\beta_0,\max}$;

(2) for any $\alpha^2 \in (0, \alpha^2_{\beta_0,\max})$, $A_{\alpha,\beta_0} := f^{-1}_{\beta_0}(\alpha^2) \cap (c_{\beta_0,\max}, c_{2,\beta_0})$ is a singleton set;

(3) for a fixed $\beta_0 \in (-\frac{\pi^2}{2}, \beta_{\max})$, $\alpha^2_{\beta_0,\max} < \frac{3\pi^2}{4}$;

(4) $f_{\beta_0}(c)$ is increasing in $(c_{1,\beta_0}, c_{\beta_0,\max})$, and decreasing in $(c_{\beta_0,\max}, c_{2,\beta_0})$;

(5) $f_{\beta_0}(c) \leq 0$ for all $c \in (1, c_{1,\beta_0}) \cup (c_{2,\beta_0}, +\infty)$.

Then the conclusions in (ii) follows from (1)–(5), Theorem 3.2, and the index formula (4.4).

The graph of the restriction of f_{β_0} to $[c_{1,\beta_0}, c_{2,\beta_0}]$ in this case is given in Figure 4. ■

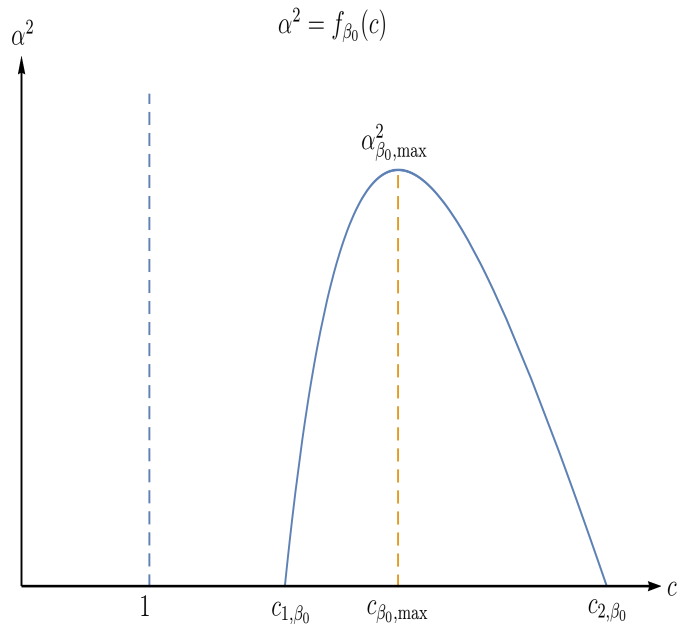


Figure 4.

Remark 4.2 *Let $\beta_0 = 0$. Then for any $\alpha \in (0, \frac{\sqrt{3}\pi}{2})$, there exist exactly one unstable mode and no neutral mode in H^2. See also Theorem 1.2 in [18] or [20] .*

Next, we show the existence of an unstable mode with zero wave number for any $\beta_0 \in (\beta_{\max}, 0)$.

Proposition 4.2 *For any $\beta_0 \in (\beta_{\max}, 0)$, there exists an unstable mode with $\alpha^2 = 0$.*

Proof. By Theorem 4.3, there exists a sequence of unstable modes $\{(c_k,\alpha_k,\beta_0,\phi_k)\}_{k=1}^{\infty}$ with $\|\phi_k\|_{L^2}=1$, $c_k^i=\operatorname{Im}c_k>0$ and $\alpha_k^2\to 0^+$ as $k\to+\infty$. We claim that $\{c_k^i\}_{k=1}^{\infty}$ has a lower bound $\delta>0$. Suppose otherwise, there exists a subsequence $\{(c_{k_j},\alpha_{k_j},\beta_0,\phi_{k_j})\}_{j=1}^{\infty}$ such that $\alpha_{k_j}^2\to 0^+$, $c_{k_j}^r\to c_s$, $c_{k_j}^i\to 0^+$ as $j\to+\infty$ for some $c_s\in\mathbf{R}\cup\pm\infty$. By a similar argument as in the proof of Lemma 4.8, $\left\{c_{k_j}^i\right\}$ is bounded and thus $c_s\in\mathbf{R}$. Since $\beta_0\in(\beta_{\max},0)$, the only choice for c_s is $c_s=U_{\beta_0}$. By Lemma 2.4, there is a uniform H^2 bound for the unstable solutions $\left\{\phi_{k_j}\right\}_{j=1}^{\infty}$. Thus, there exists $\phi_0\in H^2[-1,1]$ such that $\phi_{k_j}\to\phi_0$ in $C^1[-1,1]$ and $\|\phi_0\|_{L^2}=1$. Since ϕ_{k_j} satisfies

$$-\phi_{k_j}''+\alpha_{k_j}^2\phi_{k_j}-\frac{\beta_0-U''}{U-c_{k_j}}\phi_{k_j}=0,\quad \phi_{k_j}(\pm1)=0, \tag{4.35}$$

by passing to the limit $j\to\infty$ in (4.35), we have

$$-\phi_0''-\frac{\beta_0-U''}{U-U_{\beta_0}}\phi_0=-\phi_0''-\pi^2\phi_0=0,\quad \phi_0(\pm1)=0. \tag{4.36}$$

It is clear that $\phi_0(y)=\sin(\pi y)$, $y\in[-1,1]$. It follows from (4.35)–(4.36) that

$$\begin{aligned}\frac{\alpha_{k_j}^2}{c_{k_j}-U_{\beta_0}}\int_{-1}^1\phi_{k_j}\phi_0dy&=\int_{-1}^1\frac{\beta_0-U''}{(U-c_{k_j})(U-U_{\beta_0})}\phi_{k_j}\phi_0dy \\ &=\pi^2\int_{-1}^1\frac{1}{U-c_{k_j}}\phi_{k_j}\phi_0dy.\end{aligned} \tag{4.37}$$

Note that

$$\lim_{j\to\infty}\int_{-1}^1\phi_{k_j}\phi_0dy=\int_{-1}^1\phi_0^2dy, \tag{4.38}$$

and

$$\begin{aligned}&\lim_{j\to\infty}\int_{-1}^1\frac{1}{U-c_{k_j}}\phi_{k_j}\phi_0dy \\ =&\lim_{j\to\infty}\left(\int_{-1}^1\frac{(U-c_{k_j}^r)\phi_{k_j}\phi_0}{(U-c_{k_j}^r)^2+c_{k_j}^{i2}}dy+i\int_{-1}^1\frac{c_{k_j}^i\phi_{k_j}\phi_0}{(U-c_{k_j}^r)^2+c_{k_j}^{i2}}dy\right) \\ =&\mathcal{P}\int_{-1}^1\frac{\phi_0^2}{U-U_{\beta_0}}dy+i\pi\sum_{l=1}^2\left(|U'|^{-1}\phi_0^2\right)\big|_{y=a_l},\end{aligned} \tag{4.39}$$

by using (86) and (88) in [18], where $\mathcal{P}\int_{-1}^{1}$ denotes the Cauchy principal part and a_1, a_2 are the points such that $U(a_1) = U(a_2) = U_{\beta_0}$. Taking the imaginary part of (4.37), we have

$$\operatorname{Im}\frac{\alpha_{k_j}^2 \int_{-1}^{1} \phi_{k_j}\phi_0 \, dy}{\pi^2 \int_{-1}^{1} \frac{1}{U-c_{k_j}}\phi_{k_j}\phi_0 \, dy} = c_{k_j}^i. \tag{4.40}$$

Then for sufficiently large k, by (4.38)–(4.39), the LHS of (4.40) is negative, while the RHS of (4.40) is positive. This contradiction shows that $\{c_k^i\}_{k=1}^{\infty}$ has a lower bound $\delta > 0$.

Now we show the existence of an unstable mode with $\alpha = 0$, by taking the limit of the sequence of unstable modes $\{(c_k, \alpha_k, \beta_0, \phi_k)\}_{k=1}^{\infty}$. Similar to the proof of Lemma 4.8, $\{c_k\}_{k=1}^{\infty}$ is bounded. Thus there exists $c_0 \in \mathbf{C}$ with $\operatorname{Im} c_0 \geq \delta$ such that, up to a subsequence, $c_k \to c_0$ as $k \to \infty$. Since ϕ_k satisfies the equation (4.35) with k_j replaced by k and $\{|U(y) - c_k| : y \in [-1,1]\}$ has a uniform lower bound $\delta > 0$ for all $k \geq 1$, we therefore have a uniform bound of $\|\phi_k\|_{H^3[-1,1]}$. Let $\phi_k \to \tilde{\phi}_0$ in $C^2[-1,1]$. Then $\tilde{\phi}_0$ solves the equation

$$-\tilde{\phi}_0'' + \frac{\beta_0 - U''}{U - c_0}\tilde{\phi}_0 = 0, \quad \text{on} \ \ (-1,1)$$

with $\tilde{\phi}_0(\pm 1) = 0$. Thus $(c_0, 0, \beta_0, \tilde{\phi}_0)$ is an unstable mode. The proof of this proposition is finished. ■

5 Bifurcation of nontrivial steady solutions

In this section, we prove the bifurcation of non-parallel steady flows near the shear flow $(U(y), 0)$ if there exists a non-resonant neutral mode.

Proposition 5.1 *Consider a shear flow $U \in C^3(-1,1)$ and fix $\beta \in \mathbf{R}$. Suppose there is a non-resonant neutral mode $(c_0, \alpha_0, \beta, \phi_0)$ satisfying (1.6)–(1.7) with $c_0 > U_{\max}$ or $c_0 < U_{\min}$, and $\alpha_0 > 0$. Then there exists $\varepsilon_0 > 0$ such that for each $0 < \varepsilon < \varepsilon_0$, there exists a traveling wave solution $\vec{u}_\varepsilon(x - c_0 t, y) = (u_\varepsilon(x - c_0 t, y), v_\varepsilon(x - c_0 t, y))$ to the equation (1.1) with boundary condition (1.2) which has minimal period T_ε in x,*

$$\|\omega_\varepsilon(x,y) - \omega_0(y)\|_{H^2(0,T_\varepsilon)\times(-1,1)} = \varepsilon, \quad \omega_\varepsilon = \operatorname{curl}\vec{u}_\varepsilon, \ \ \omega_0 = -U'(y),$$

and $T_\varepsilon \to \frac{2\pi}{\alpha_0}$ when $\varepsilon \to 0$. Moreover, $u_\varepsilon(x,y) \neq 0$ and v_ε is not identically zero.

Proof. We assume $c_0 > U_{\max}$ and the case $c_0 < U_{\min}$ is similar. The proof is similar to that of Lemma 1 in [21], we give it here for completeness. From the vorticity equation (1.3), it can be seen that $\vec{u}\,(x - c_0 t, y)$ is a solution of (1.1) if and only if

$$\frac{\partial\,(\omega + \beta y, \psi - c_0 y)}{\partial\,(x, y)} = 0$$

and ψ takes constant values on $\{y = \pm 1\}$, where ω and ψ are the vorticity and stream function corresponding to $\vec{u}$, respectively. Let ψ_0 be a stream function associated with the shear flow $(U - c_0, 0)$, i.e., $\psi_0'\,(y) = U\,(y) - c_0$. Since $U - c_0 < 0$, ψ_0 is decreasing on $(-1, 1)$. Therefore we can define a function $f_0 \in C^2(Range(\psi_0))$ such that

$$f_0\,(\psi_0\,(y)) = \omega_0\,(y) + \beta y = -\psi_0''\,(y) + \beta y. \tag{5.1}$$

Thus

$$f_0'\,(\psi_0\,(y)) = \frac{\beta - U''\,(y)}{U\,(y) - c_0} =: \mathcal{K}_{c_0}\,(y)\,.$$

Then we extend f_0 to $f \in C_0^2\,(\mathbf{R})$ such that $f = f_0$ on $Range(\psi_0)$. We construct steady solutions near $(U - c_0, 0)$ by solving the elliptic equation

$$-\Delta\psi + \beta y = f\,(\psi)\,,$$

where $\psi\,(x, y)$ is the stream function and $(u, v) = (\psi_y, -\psi_x)$ is the steady velocity. Let $\xi = \alpha x,\ \psi\,(x, y) = \tilde{\psi}\,(\xi, y)\,,$ where $\tilde{\psi}\,(\xi, y)$ is 2π-periodic in ξ. We use α^2 as the bifurcation parameter. The equation for $\tilde{\psi}\,(\xi, y)$ becomes

$$-\alpha^2 \frac{\partial^2 \tilde{\psi}}{\partial \xi^2} - \frac{\partial^2 \tilde{\psi}}{\partial y^2} + \beta y - f(\tilde{\psi}) = 0, \tag{5.2}$$

with the boundary conditions that $\tilde{\psi}$ takes constant values on $\{y = \pm 1\}$. Define the perturbation of the stream function by

$$\phi\,(\xi, y) = \tilde{\psi}\,(\xi, y) - \psi_0\,(y)\,.$$

Then by using (5.1), we reduce the equation (5.2) to

$$-\alpha^2 \frac{\partial^2 \phi}{\partial \xi^2} - \frac{\partial^2 \phi}{\partial y^2} - \left(f(\phi + \psi_0) - f\,(\psi_0)\right) = 0. \tag{5.3}$$

Define the spaces

$$\begin{aligned} B = \;& \{\phi(\xi, y) \in H^3([0, 2\pi] \times [-1, 1]) : \; \phi(\xi, -1) = \phi(\xi, 1) = 0, \\ & 2\pi\text{-periodic and even in } \xi\} \end{aligned}$$

and

$$C = \left\{\phi(\xi, y) \in H^1([0, 2\pi] \times [-1, 1]) : \; 2\pi\text{-periodic and even in } \xi\right\}.$$

Consider the mapping

$$F \; : B \times \mathbf{R}^+ \to C$$

defined by

$$F(\phi, \alpha^2) = -\alpha^2 \frac{\partial^2 \phi}{\partial \xi^2} - \frac{\partial^2 \phi}{\partial y^2} - (f(\phi + \psi_0) - f(\psi_0)).$$

We study the bifurcation near the trivial solution $\phi = 0$ of the equation $F(\phi, \alpha^2) = 0$ in B, whose solutions give steady flows with x-period $\frac{2\pi}{\alpha}$. The linearized operator of F around $(0, \alpha_0^2)$ has the form

$$\mathcal{G} := F_\phi(0, \alpha_0^2) = -\alpha_0^2 \frac{\partial^2}{\partial \xi^2} - \frac{\partial^2}{\partial y^2} - f'(\psi_0) = -\alpha_0^2 \frac{\partial^2}{\partial \xi^2} - \frac{\partial^2}{\partial y^2} - \mathcal{K}_{c_0}.$$

By our assumption, the operator $-\frac{\partial^2}{\partial y^2} - \mathcal{K}_{c_0} : H_0^1 \cap H^2(-1, 1) \to L^2(-1, 1)$ has a negative eigenvalue $-\alpha_0^2$ with the eigenfunction ϕ_0. Therefore, the kernel of $\mathcal{G} : \; B \to C$ is given by

$$\ker(\mathcal{G}) = \{\phi_0(y) \cos \xi\}.$$

In particular, the dimension of $\ker(\mathcal{G})$ is 1. Since $\mathcal{G}$ is self-adjoint, $\phi_0(y) \cos \xi \notin Range(\mathcal{G})$. Note that $\partial_{\alpha^2} \partial_\phi F(\phi, \alpha^2)$ is continuous and

$$\partial_{\alpha^2} \partial_\phi F(0, \alpha_0^2)(\phi_0(y) \cos \xi) = -\frac{\partial^2}{\partial \xi^2}[\phi_0(y) \cos \xi] = \phi_0(y) \cos \xi \notin Range(\mathcal{G}).$$

By the Crandall-Rabinowitz local bifurcation theorem [5], there exists a local bifurcating curve $(\phi_\gamma, \alpha^2(\gamma))$ of $F(\phi, \alpha^2) = 0$, which intersects the trivial curve $(0, \alpha^2)$ at $\alpha^2 = \alpha_0^2$, such that

$$\phi_\gamma(\xi, y) = \gamma \phi_0(y) \cos \xi + o(\gamma),$$

$\alpha^2(\gamma)$ is a continuous function of γ, and $\alpha^2(0) = \alpha_0^2$. So the stream functions of the perturbed steady flows in (ξ, y) coordinates take the form

$$\psi_\gamma(\xi, y) = \psi_0(y) + \gamma\phi_0(y)\cos\xi + o(\gamma). \tag{5.4}$$

Let the velocity $\vec{u}_\gamma = (u_\gamma, v_\gamma) = (\partial_y\psi_\gamma, -\partial_x\psi_\gamma)$. Then

$$u_\gamma = U(y) - c_0 + \gamma\phi_0'(y)\cos\xi + o(1) \neq 0$$

when γ is small. ∎

By adjusting the traveling speed, we can construct traveling waves near the Sinus flow with the period $\frac{2\pi}{\alpha_0}$.

Theorem 5.1 *Consider the shear flow* $U(y) = \frac{1+\cos(\pi y)}{2}$, $y \in [-1, 1]$. *Then there exist at least one non-resonant neutral mode* $(c_0, \alpha_0, \beta, \phi_0)$ *in the following stable cases:*

(1) $|\beta| > \frac{\pi^2}{2}$ *and* $0 < \alpha_0 < \frac{\sqrt{3}\pi}{2}$;

(2) $-\frac{\pi^2}{2} < \beta < \beta_{\max}$ *and* $0 < \alpha_0 < \alpha_{\beta,\max}$;

(3) $0 < \beta < \frac{\pi^2}{2}$ *and* $0 < \alpha_0 < \alpha_{\beta,\max}$,

where $\beta_{\max} < 0$ *is defined in (4.32),* $\alpha_{\beta,\max}$ *is defined in (4.34) for case (2) and in (4.29) for case (3). In these three cases, there exists* $\varepsilon_0 > 0$ *such that for each* $0 < \varepsilon < \varepsilon_0$*, there exists a traveling wave solution* $\vec{u}_\varepsilon(x - c_\varepsilon t, y) = (u_\varepsilon(x - c_\varepsilon t, y), v_\varepsilon(x - c_\varepsilon t, y))$ *to the equation (1.1) with boundary condition (1.2) which has minimal period* $T_0 = \frac{2\pi}{\alpha_0}$ *in* x,

$$\|\omega_\varepsilon(x, y) - \omega_0(y)\|_{H^2(0,T_0)\times(-1,1)} = \varepsilon, \quad \omega_\varepsilon = \operatorname{curl}\vec{u}_\varepsilon, \ \omega_0 = -U'(y),$$

with $c_\varepsilon \to c_0$ *when* $\varepsilon \to 0$*. Moreover,* $u_\varepsilon(x, y) \neq 0$ *and* v_ε *is not identically zero.*

Proof. First, we show the existence of non-resonant neutral modes in the three cases. For case (1), since the flow is linearly stable, by the index formula (4.4), we have $k_i^{\leq 0} = 1$. Therefore there exists a nonzero imaginary eigenvalue of the operator $J_\alpha L_\alpha$ (defined in (4.2)). This corresponds to a neutral mode $(c_0, \alpha_0, \beta, \phi_0)$ with $\phi_0 \in H^2$ for the Rayleigh-Kuo equation (1.6)–(1.7) with

$\langle L_{\alpha_0}\omega_{\alpha_0}, \omega_{\alpha_0}\rangle \leq 0$, where $\omega_{\alpha_0} = \left(-\frac{d^2}{dy^2} + \alpha_0^2\right)\phi_0$. By Theorem 2.1, c_0 must be one of $0, 1, U_\beta = \frac{1}{2} - \frac{\beta}{\pi^2}$ or $c_0 \notin [0,1]$. Suppose that $\beta > \frac{\pi^2}{2}$. It is easy to see that $c_0 \neq U_\beta$ since $\alpha_0 \in (0, \frac{\sqrt{3}\pi}{2})$. Also we have $c_0 \neq 1$ since otherwise

$$\int_{-1}^{1}\left[|\phi_0'|^2 - \frac{\beta - U''}{U-1}|\phi_0|^2\right]dy = -\alpha_0^2\int_{-1}^{1}|\phi_0|^2\,dy < 0,$$

a contradiction. To show $c_0 \neq 0$, suppose otherwise and by (3.16), we have

$$\langle L_{\alpha_0}\omega_{\alpha_0}, \omega_{\alpha_0}\rangle = -U_\beta\int_{-1}^{1}\frac{(\beta - U'')}{|U|^2}|\phi_0|^2dy > 0,$$

a contradiction to the condition $\langle L_{\alpha_0}\omega_{\alpha_0}, \omega_{\alpha_0}\rangle \leq 0$. Therefore, we must have $c_0 \notin [0,1]$, that is, there exists a non-resonant neutral mode. The proof for the case $\beta < -\frac{\pi^2}{2}$ is similar. For the cases (2)–(3), the existence of non-resonant neutral modes follows from Theorems 4.2–4.3.

Let $(c_0, \alpha_0, \beta, \phi_0)$ be a non-resonant neutral mode. We consider the case $c_0 > 1$ and the case $c_0 < 0$ is similar. Let $I \subset (1,\infty)$ be a small interval centered at c_0. For each $c \in I$, denote $-\alpha(c)^2$ to be the negative eigenvalue near $-\alpha^2(c_0) = -\alpha_0^2$ of the operator

$$-\frac{d^2}{dy^2} - \frac{\beta - U''}{U - c}$$

defined in $H_0^1 \cap H^2(-1,1)$. By Corollary 3.1 and Theorems 4.2–4.3, if we choose $|I|$ to be small enough, then $\alpha(c)$ is strictly monotone on I. Assume that $\alpha(c)$ is increasing on I. Let c_1 and c_2 in I such that $c_1 < c_0 < c_2$. Then

$$\alpha(c_1) < \alpha_0 < \alpha(c_2). \tag{5.5}$$

By Proposition 5.1, for any $c \in (c_1, c_2)$, there exists local bifurcation of non-parallel traveling wave solutions of the equation (1.1) with boundary condition (1.2), near the shear flow $(U, 0)$. More precisely, we can find $r_0 > 0$ (independent of $c \in (c_1, c_2)$) such that for any $0 < r < r_0$, there exists a nontrivial traveling wave solution

$$\vec{u}_{c,r}(x - ct, y) = (u_{c,r}(x - ct, y), v_{c,r}(x - ct, y))$$

with vorticity $\omega_{c,r}$ which has minimum x-period $T_{c,r}$ and

$$\|\omega_{c,r} - \omega_0\|_{H^2(0,T_{c,r})\times(-1,1)} = r.$$

Moreover,

$$\frac{2\pi}{T_{c,r}} \to \alpha(c) \text{ when } r \to 0.$$

By (5.5), when r_0 is chosen to be small enough,

$$T_{c_2,r} < \frac{2\pi}{\alpha_0} < T_{c_1,r} \text{ for any } r \in (0, r_0).$$

Since $T_{c,r}$ is continuous to c, for each $r \in (0, r_0)$, there exists $c^*(r) \in (c_1, c_2)$ such that $T_{c^*(r),r} = \frac{2\pi}{\alpha_0}$. Then the traveling wave solution

$$\vec{u}_r(x - c^*(r)t, y) := \left(u_{c^*(r),r}(x - c^*(r)t, y), v_{c^*(r),r}(x - c^*(r)t, y)\right)$$

with the vorticity $\omega_r := \omega_{c^*(r),r}$ is a nontrivial steady solution of (1.1) satisfying boundary condition (1.2), with minimal x-period $\frac{2\pi}{\alpha_0}$ and

$$\|\omega_r - \omega_0\|_{H^2\left(0,\frac{2\pi}{\alpha_0}\right)\times(-1,1)} = r.$$

This finishes the proof of the theorem. ■

Remark 5.1 *The non-resonant neutral mode does not exist when there is no Coriolis effects (i.e. $\beta = 0$). The traveling waves constructed above are thus purely due to the Coriolis forces, with traveling speeds beyond the range of the basic flow. Their existence indicates that the addition of Coriolis effects can significantly change the dynamics of fluids.*

6 Linear inviscid damping

In this section, we prove the linear inviscid damping by using the Hamiltonian structures of the linearized equation (3.3). First, we show that for the Sinus flow, when $\alpha^2 > \frac{3\pi^2}{4}$ and $|\beta| \le \frac{\pi^2}{2}$, there is no H^2 neutral mode to the Rayleigh-Kuo equation.

Lemma 6.1 *Consider the shear flow $U(y) = \frac{1+\cos(\pi y)}{2}$, $y \in [-1, 1]$. Fix any $\beta_0 \in [-\frac{\pi^2}{2}, \frac{\pi^2}{2}]$.*

(i) *When $\alpha^2 > \frac{3\pi^2}{4}$, there exists no neutral solution $\phi \in H^2[-1, 1]$ to the Rayleigh-Kuo equation (1.6)–(1.7).*

(ii) *When* $\alpha^2 = \frac{3\pi^2}{4}$, *then* $\phi_0(y) = \cos(\frac{\pi y}{2})$ *is the only neutral solution in* $H^2[-1,1]$ *with the neutral phase speed* $c = U_{\beta_0}$.

Proof. Fix $\beta_0 \in [-\frac{\pi^2}{2}, \frac{\pi^2}{2}]$. We divide our proof into two cases.

Case 1. $c \in [0,1]$.

Let $c = U_{\beta_0}$. Since the first eigenvalue of (4.3) is $\lambda_1 = -\frac{3\pi^2}{4}$, there is no neutral solution in H^2 for $\alpha^2 > \frac{3\pi^2}{4}$ and exactly one neutral solution $\phi_0 \in H^2$ for $\alpha^2 = \frac{3\pi^2}{4}$.

Let $c \in (0,1)$ and $c \neq U_{\beta_0}$. By Theorem 2.1, there is no neutral solution in H^2 for $\alpha^2 \geq \frac{3\pi^2}{4}$.

Let $c = 0$. If $\beta_0 \in [-\frac{\pi^2}{2}, 0)$, there is no neutral solution for $\alpha^2 \geq \frac{3\pi^2}{4}$ by Lemma 2.7. If $\beta_0 \in [0, \frac{\pi^2}{2})$, there is no neutral solution in H^2 for $\alpha^2 \geq \frac{3\pi^2}{4}$ by Lemma 4.3. If $\beta_0 = \frac{\pi^2}{2}$, c coincides with U_{β_0} and consequently, there is no neutral solution in H^2 for $\alpha^2 > \frac{3\pi^2}{4}$ and exactly one neutral solution ϕ_0 for $\alpha^2 = \frac{3\pi^2}{4}$.

Let $c = 1$. If $\beta_0 = -\frac{\pi^2}{2}$, c also coincides with U_{β_0} and similarly, there is no neutral solution in H^2 for $\alpha^2 > \frac{3\pi^2}{4}$ and exactly one neutral solution ϕ_0 for $\alpha^2 = \frac{3\pi^2}{4}$. If $\beta_0 \in (-\frac{\pi^2}{2}, 0]$, by Lemma 4.2 there is no neutral solution in $H^2[-1,1]$ for $\alpha^2 \geq \frac{3\pi^2}{4}$. If $\beta_0 \in (0, \frac{\pi^2}{2}]$, by Lemma 2.7 there is no neutral solution for $\alpha^2 \geq \frac{3\pi^2}{4}$.

Case 2. $c \notin [0,1]$.

We consider two subcases.

Case 2a. $c \in (-\infty, 0)$. Let $\beta_1 \in (\frac{\pi^2}{2}, +\infty)$ such that $c = \frac{1}{2} - \frac{\beta_1}{\pi^2}$. Clearly, $-\frac{3\pi^2}{4}$ is the first eigenvalue of $\mathcal{L}_{\beta_1,c}$ defined in (3.24). Denote $-\alpha^2 = -g_c(\beta)$ is the first eigenvalue of $\mathcal{L}_{\beta,c}$. By (i) of Corollary 3.1, $g_c(\beta)$ is increasing on $\beta \in \mathbf{R}$. Since $g_c(\beta_1) = \frac{3\pi^2}{4}$ and $\beta_0 \leq \frac{\pi^2}{2} < \beta_1$, we have $g_c(\beta_0) < \frac{3\pi^2}{4}$. Therefore, for this subcase, there is no neutral solution in H^2 for $\alpha^2 \geq \frac{3\pi^2}{4} > g_c(\beta_0)$.

Case 2b. $c \in (1, +\infty)$. Let $\beta_2 \in (-\infty, -\frac{\pi^2}{2})$ such that $c = \frac{1}{2} - \frac{\beta_2}{\pi^2}$. Again, denote $-\alpha^2 = -g_c(\beta)$ to be the first eigenvalue of $\mathcal{L}_{\beta,c}$. We have $g_c(\beta_2) = \frac{3\pi^2}{4}$ as in Case 2a. By (ii) of Corollary 3.1, $g_c(\beta)$ is decreasing on $\beta \in \mathbf{R}$. Since $\beta_0 \geq -\frac{\pi^2}{2} > \beta_2$, again we have $g_c(\beta_0) < \frac{3\pi^2}{4}$ and there is no neutral solution in H^2 for $\alpha^2 \geq \frac{3\pi^2}{4}$. ■

The above lemma implies that there is no purely imaginary eigenvalues of the linearized Euler operator JL defined in (3.3) when $\alpha^2 > \frac{3\pi^2}{4}$. This implies the following inviscid damping of the velocity fields.

Theorem 6.1 *Consider the linearized equation (3.1) with* $U(y) = \frac{1+\cos(\pi y)}{2}$, $y \in [-1, 1]$.

(i) *For any* $\alpha^2 \in (\frac{3\pi^2}{4}, +\infty)$ *and* $\beta \in [-\frac{\pi^2}{2}, \frac{\pi^2}{2}]$, *we have*

$$\frac{1}{T}\int_0^T \|\vec{u}(t)\|_{L^2}^2 \, dt \to 0, \text{ when } T \to \infty,$$

for any solution $\omega(t) = \operatorname{curl}\vec{u}(t)$ *of (3.1) with* $\omega(0)$ *in the non-shear space* X *defined in (3.2).*

(ii) *For* $\alpha^2 = \frac{3\pi^2}{4}$ *and* $\beta \in [-\frac{\pi^2}{2}, \frac{\pi^2}{2}]$, *we have*

$$\frac{1}{T}\int_0^T \|\vec{u}_1(t)\|_{L^2}^2 \, dt \to 0, \text{ when } T \to \infty,$$

where $\vec{u}_1(t)$ *is the velocity corresponding to the vorticity* $(I - P_1)\omega(t)$ *with* $\omega(0) \in X$. *Here,* P_1 *is the projection of* X *to*

$$\ker L = span\left\{e^{\pm i\frac{\sqrt{3}\pi}{2}x}\cos\left(\frac{\pi y}{2}\right)\right\},$$

and L *is given in (6.1) below.*

Proof. The solution of the linearized equation (3.1) is written as $\omega(t) = e^{tJL}\omega(0)$, where

$$J = -(\beta - U'')\partial_x, \; L = \frac{1}{\pi^2} - (-\Delta)^{-1} \tag{6.1}$$

as in (3.4). First, we note that when $\alpha^2 > \frac{3\pi^2}{4}$, L is positive on X. As a consequence, $[\cdot,\cdot] = \langle L\cdot,\cdot\rangle$ defines an equivalent inner product on X with the L^2 inner product. For any $\omega_1, \omega_2 \in X$, we have

$$\langle LJL\omega_1, \omega_2\rangle = \langle JL\omega_1, L\omega_2\rangle = -\langle L\omega_1, JL\omega_2\rangle,$$

and thus JL is anti-self-adjoint on $(X, [\cdot,\cdot])$. Therefore, the spectrum of JL on $(X, [\cdot,\cdot])$ is on the imaginary axis. Since the operator JL is a compact perturbation of $-(U - U_\beta)\partial_x$, whose spectrum is clearly the whole imaginary axis, it follows from Weyl's Theorem that the continuous spectrum of JL is also the whole imaginary axis. Moreover, by Lemma 6.1, JL has no embedded eigenvalues on the imaginary axis. By applying the RAGE theorem ([6]) to e^{tJL}, we have that

$$\frac{1}{T}\int_0^T \|B\omega\,(t)\|_{L^2}^2\,dt \to 0, \text{ when } T\to\infty$$

for any compact operator B on $L^2\left(S_{2\pi/\alpha}\times[y_1,y_2]\right)$ and for any solution $\omega\,(t)$ of (3.1) with $\omega\,(0)\in X$. The conclusion (i) follows by choosing

$$B\omega = \nabla^\perp\,(-\Delta)^{-1}\,\omega = \vec{u},$$

that is, the mapping operator from vorticity to velocity.

To prove (ii), we define $X_1 = (I-P_1)\,X$. Then $L|_{X_1} > 0$ and $A_1 = (I-P_1)\,JL|_{X_1}$ is anti-self-adjoint on $(X_1,[\cdot,\cdot])$. The operator A_1 has no nonzero purely imaginary eigenvalues. Moreover, the proof of Lemma 3.1 implies that $\ker A_1 = \{0\}$. Therefore, A_1 has purely continuous spectrum in the imaginary axis. The conclusion again follows from the RAGE theorem to e^{tA_1} on X_1. $\blacksquare$

Next, we consider the inviscid damping for the unstable case. By Theorems 4.2–4.3 and Remark 4.2, there exist exactly one unstable mode and no neutral mode in H^2 in the following cases: 1) $\beta_0\in(0,\frac{\pi^2}{2})$ and $\alpha^2\in(\alpha^2_{\beta_0,\max},\frac{3\pi^2}{4})$; 2) $\beta_0\in(\beta_{\max},0]$ and $\alpha^2\in\left(0,\frac{3\pi^2}{4}\right)$; 3) $\beta_0\in(-\frac{\pi^2}{2},\beta_{\max})$ and $\alpha^2\in(\alpha^2_{\beta_0,\max},\frac{3\pi^2}{4})$. Here, $\beta_{\max}$ is defined in (4.32), $\alpha^2_{\beta_0,\max}$ is defined in (4.29) for Case 1 and (4.34) for Case 3. As in the stable case, we consider the linearized equation (3.1) written as Hamiltonian form $\partial_t\omega = JL\omega$ in the non-shear space X, where J and L are defined in (6.1). The space X is defined in (3.2) with α to be an unstable wave number in the above three cases.

Denote $E^s\,(E^u)\subset X$ to be the stable (unstable) eigenspace of JL. Then by Corollary 6.1 in [22], $L|_{E^s\oplus E^u}$ is non-degenerate and

$$n^-\left(L|_{E^s\oplus E^u}\right) = \dim E^s = \dim E^u. \tag{6.2}$$

Define the center space E^c to be the orthogonal (in the inner product $[\cdot,\cdot]$) complement of $E^s\oplus E^u$ in X, that is,

$$E^c = \left\{\omega\in X \mid \langle L\omega,\omega_1\rangle = 0,\ \forall\,\omega_1\in E^s\oplus E^u\right\}. \tag{6.3}$$

Then we get the following results.

Lemma 6.2 *Consider the shear flow* $U(y) = \frac{1+\cos(\pi y)}{2}$, $y \in [-1, 1]$. *Let* α *be an unstable wave number. Then the decomposition* $X = E^s \oplus E^c \oplus E^u$ *is invariant under* JL. *Moreover, we have*

(i)
$$\dim E^s = \dim E^u = n^-(L) . \tag{6.4}$$

(ii) $n^-(L|_{E^c}) = 0$ *and as a consequence,* $L|_{E^c/\ker L} > 0$.

(iii) *The operator* $JL|_{E^c}$ *has no nonzero purely imaginary eigenvalues.*

Proof. The invariance of the decomposition follows from the invariance of $\langle L\cdot, \cdot\rangle$ under JL. To prove (6.4), we note that JL can be decomposed as the operators $J_{l\alpha}L_{l\alpha}$ on the spaces X^l (defined in (3.7)) with the wave number αl, where $0 \neq l \in \mathbf{Z}$. Then

$$\dim E^s = \dim E^u = \sum_l k_u^l,$$

where k_u^l is the number of unstable modes for $J_{l\alpha}L_{l\alpha}$. For each l, when $|\alpha l|$ is an unstable wave number, there is exactly one unstable mode, and thus we have $k_u^l = 1 = n^-(L_{l\alpha})$. If $|\alpha l| \geq \frac{3\pi^2}{4}$, then we also have $k_u^l = 0 = n^-(L_{l\alpha})$. Therefore

$$\dim E^s = \dim E^u = \sum_l k_u^l = \sum_l n^-(L_{l\alpha}) = n^-(L)$$

and (6.4) is proved.

To show (ii), noting that by the definition of E^c, (6.2) and (6.4), we have

$$n^-(L|_{E^c}) = n^-(L) - n^-(L_{E^s \oplus E^u}) = 0,$$

and thus $L|_{E^c/\ker L} > 0$.

For each l, by Theorems 4.2–4.3 and Lemma 6.1, the operator $J_{l\alpha}L_{l\alpha}$ has no neutral mode except for $c = U_\beta$ when $|\alpha l| = \frac{\sqrt{3}\pi}{2}$, which corresponds to nontrivial $\ker L_{l\alpha}$ and $\ker L$. Thus the property (iii) follows. ∎

Since E^c is invariant under JL, we can restrict the linearized equation (3.1) on E^c. The linear inviscid damping still holds true for initial data in E^c. By the same proof of Theorem 6.1, we have the following.

Theorem 6.2 *Consider the linearized equation (3.1) with* $U(y) = \frac{1+\cos(\pi y)}{2}$, $y \in [-1, 1]$. *Let* α *be an unstable wave number.*

(i) *If* $|\alpha l| \neq \frac{\sqrt{3}\pi}{2}$ *for any* $l \in \mathbf{Z}$, *then*

$$\frac{1}{T}\int_0^T \|u(t)\|_{L^2}^2 \, dt \to 0, \text{ when } T \to \infty,$$

for any solution $\omega(t)$ *of (3.1) with* $\omega(0) \in E^c$. *Here,* E^c *is the center space defined in (6.3).*

(ii) *If* $|\alpha l| = \frac{\sqrt{3}\pi}{2}$ *for some* $l \in \mathbf{Z}$, *then*

$$\frac{1}{T}\int_0^T \|u_1(t)\|_{L^2}^2 \, dt \to 0, \text{ when } T \to \infty,$$

where $u_1(t)$ *is the velocity corresponding to the vorticity* $(I - P_1)\omega(t)$ *with* $\omega(0) \in E^c$.

Remark 6.1 *For general flows* U *in class* $\mathcal{K}^+$, *when there is no nonzero imaginary eigenvalue for the linearized operator* JL *(defined in (3.3)), the linear inviscid damping can be shown as in Theorems 6.1–6.2, for* $\omega(0) \in L^2$.

When $\beta = 0$, *the nonexistence of nonzero imaginary eigenvalues and as a consequence the linear damping is true for flows in class* $\mathcal{K}^+$ *(see [20]). Recently, when* $\beta = 0$, *for symmetric and monotone shear flows, more explicit linear decay estimates of the velocity were obtained in [42, 43, 47] for more regular initial data (e.g.* $\omega(0) \in H^1$ *or* H^2*).*

7 Discussions

In this section, we discuss and relate our results for the Sinus flow to the previous work in [30, 35].

The stability picture obtained in Theorems 4.2–4.3 is shown in Figure 5 for the parameters (α, β) below.

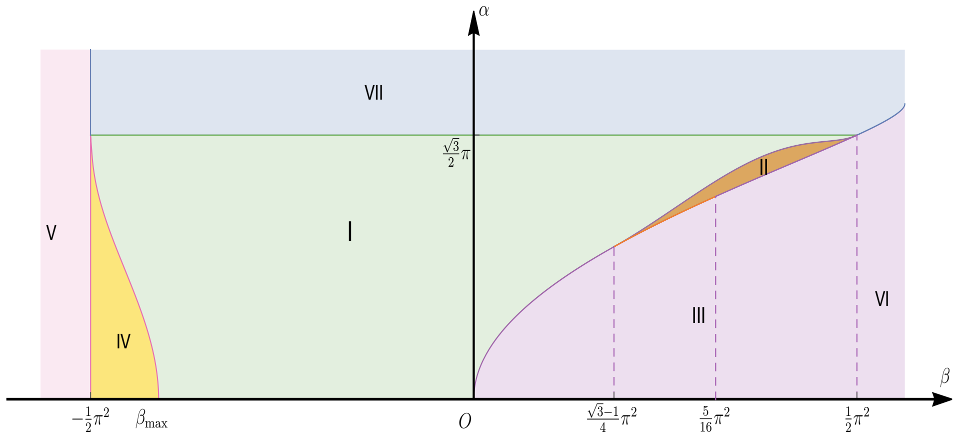


Figure 5.

In Figure 5, the right boundary of region IV (yellow area) is given by the curve

$$\Gamma_1 : \alpha = \alpha_{\beta,\max}, \quad \beta \in \left(-\frac{\pi^2}{2}, \beta_{\max}\right),$$

where $\alpha_{\beta,\max}$ and $\beta_{\max}$ are defined in (4.34) and (4.32), respectively. The upper boundary of region III (purple area) is given by the curve

$$\Gamma_2 : \alpha\left(r\right) = \pi^2(1-r^2), \quad \beta\left(r\right) = \pi^2\left(-r^2 + \frac{1}{2}r + \frac{1}{2}\right) \in \left(0, \frac{\pi^2}{2}\right),$$

where $r \in \left(\frac{1}{2}, 1\right)$ and $\left(\alpha\left(r\right), \beta\left(r\right)\right)$ corresponds to the singular neutral mode (SNM) with $c = 0$ (see (4.6)). Here, the boundary of regions III and VI is $\alpha \in \left(0, \frac{\sqrt{3}\pi}{2}\right)$, $\beta = \frac{\pi^2}{2}$. The upper boundary of region II (orange area) is given by

$$\Gamma_3 : \alpha = \alpha_{\beta,\max}, \quad \beta = \pi^2\left(-r^2 + \frac{1}{2}r + \frac{1}{2}\right) \in \left(\frac{\sqrt{3}-1}{4}\pi^2, \frac{1}{2}\pi^2\right),$$

where $\alpha_{\beta,\max} > \pi^2(1-r_0^2)$ is defined in (4.29).

Only region I (Green area) is the unstable domain, consisting of unstable parameters (α, β) given in Theorems 4.2–4.3. In region I, there exist exactly one unstable mode and no neutral mode in H^2. All other regions in Figure 5 are stable domains, but with different properties on neutral modes. In region

VII (blue area), there is no neutral mode in H^2, see Lemma 6.1. In region V, there exists exactly one non-resonant neutral mode with $c > 1$. In region VI, there exists exactly one non-resonant neutral mode with $c < 0$. In region II, there exist exactly two non-resonant neutral modes with $c < 0$. In region IV (yellow area), there exist exactly two non-resonant neutral modes with $c > 1$. In region III, there exists exactly one non-resonant neutral mode with $c < 0$. The dynamical behavior of the fluid equation (1.3) is quite different in these regions. In region VII, the linear inviscid damping is shown for non-shear perturbations. In regions III, VI and V, the non-resonant neutral mode generates nontrivial traveling waves with the wave number α. In regions II and IV, the two non-resonant neutral modes generate two traveling waves with different speeds. Moreover, for region I, the linear damping is true in the finite codimensional center space. These different behaviors indicate that with the addition of Coriolis effects, the dynamics near the Sinus flow is very rich.

In the work of [30] (see Section A of Chapter VII), based on numerical results, Kuo wrongly claimed that the stability boundary in the rectangular domain $\left[-\frac{\pi^2}{2}, \frac{\pi^2}{2}\right] \times \left[0, \frac{3\pi^2}{4}\right]$ is given by the curve Γ_2 of SNMs. That is, the instability domain in [30] consists of regions I, II, and IV. See (b) in Figure 6 of [30]. The same stability picture can be also found in [35]. This is in contradiction to our results, where regions II and IV lie in the stability domain. The stability boundary Γ_1 for $\beta \in \left(-\frac{\pi^2}{2}, \beta_{\max}\right)$ was not detected in [30]. Moreover, two of the stability boundaries in our results, the right boundary Γ_1 of region IV and upper boundary Γ_3 of region II, are not SNM curves. Instead, they consist of non-resonant neutral modes with $c > 1$ or $c < 0$. The incorrectness of using the SNM curve Γ_2 as the instability boundary can also be seen as follows. By Lemma 3.2, any neutral mode $(c_s, \alpha_s, \beta_s, \phi_s)$ on the stability boundary (i.e. neutral limiting modes) must satisfy the condition $\langle L_{\alpha_s}\omega_{\alpha_s}, \omega_{\alpha_s}\rangle \le 0$, where $\omega_{\alpha_s} = \left(-\frac{d^2}{dy^2} + \alpha_s^2\right)\phi_s$. However, for the singular neutral mode (4.6), we have shown in (4.9) that $\langle L_{\alpha_s}\omega_{\alpha_s}, \omega_{\alpha_s}\rangle > 0$ for $\beta_s \in (\frac{(\sqrt{3}-1)\pi^2}{4}, \frac{5\pi^2}{16})$ and $\langle L_{\alpha_s}\omega_{\alpha_s}, \omega_{\alpha_s}\rangle = +\infty$ for $\beta_s \in [\frac{5\pi^2}{16}, \frac{\pi^2}{2})$. This implies that the SNM curve Γ_2 is not the stability boundary at least for $\beta \in (\frac{(\sqrt{3}-1)\pi^2}{4}, \frac{\pi^2}{2})$.

To confirm our theoretical results, we run the numerical simulations with more accuracy. We find that the difference between the α values in the stability boundary Γ_3 of non-resonant neutral modes and those in the SNM curve Γ_2 is actually very small. More precisely, as shown in the following table, for a fixed

$$\beta_0 = \pi^2\left(-r_0^2 + \frac{1}{2}r_0 + \frac{1}{2}\right) \in \left(\frac{(\sqrt{3}-1)\pi^2}{4}, \frac{\pi^2}{2}\right),\ r_0 \in \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right),$$

the difference between $\alpha^2_{\beta_0,\max}$ and $\pi^2(1-r_0^2)$ is as small as 10^{-5} to 10^{-3}, and the phase speed c for the non-resonant neutral mode on Γ_3 is as small as 10^{-3}. Such small difference partly explained why the true instability boundary was not found by the numerical results in [30].

β_0	$\alpha^2_{\beta_0,\max}$ on Γ_3	$\pi^2(1-r_0^2)$ on Γ_2	difference	c on Γ_3
1.80626	1.57080	1.57080	0	0
2.60650	1.90050	1.90050	0.000004894	-0.00003
2.85444	1.99395	1.99394	0.000014579	-0.00006
3.05645	2.06795	2.06792	0.000029048	-0.00009
3.24603	2.13593	2.13588	0.000049360	-0.00012
3.44449	2.20585	2.20577	0.000078511	-0.00015
3.69853	2.29388	2.29376	0.000126720	-0.00018
4.18261	2.45904	2.45882	0.000222321	-0.00018
4.37126	2.52328	2.52304	0.000233368	-0.00015
4.49531	2.56575	2.56554	0.000219151	-0.00012
4.59739	2.60097	2.60078	0.000188895	-0.00009
4.69034	2.63332	2.63318	0.000144032	-0.00006
4.78396	2.66631	2.66623	0.000083277	-0.00003
4.93480	2.72070	2.7207	0	0

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