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Linear Inviscid Damping for Couette Flow in Stratified Fluid

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Abstract. We study the inviscid damping of Couette flow with an exponentially stratified density. The optimal decay rates of the velocity field and the density are obtained for general perturbations with minimal regularity. For Boussinesq approximation model, the decay rates we get are consistent with the previous results in the literature. We also study the decay rates for the full Euler equations of stratified fluids, which were not studied before. For both models, the decay rates depend on the Richardson number in a very similar way. Besides, we also study the dispersive decay due to the exponential stratification when there is no shear.

1. Introduction

Couette flow in exponentially stratified fluid is a shear flow $U(y) = Ry$ with the density profile $\rho_0(y) = Ae^{-\beta y}$. The stability of such a flow was first studied by Taylor [21] in the half space by the method of normal modes. He presented a convincing but somewhat incomplete analysis to show that the spectrum of the linearized equation (now called Taylor–Goldstein equation) is quite different when the Richardson number $B^2 = \frac{\beta g}{R^2}$ (g is the gravitational constant) is greater or less than 1/4. He found that there exist infinitely many discrete neutral eigenvalues when $B^2 > \frac{1}{4}$ and no such neutral eigenvalues exist when $B^2 < \frac{1}{4}$. This claim was later proved by Dyson [10] and Dikki [9]. However, Taylor did not provide a clear answer to the problem of stability of Couette flow. From 1950s, there have been lots of work trying to understand the stability of stratified Couette flow, by studying the initial value problem. They include Høiland [15], Eliassen et al. [11], Case [6], Dikki [8], Kuo [16], Hartman [14], Chimonas [7], Brown and Stewartson [4], Farrell and Ioannou [13]. We refer to Section 3.2.3 of the book of Yaglom [23] for a detailed survey of the literature. Most of the papers used the Boussinesq approximation. One exception is Dikki [8], where he proved the Liapunov stability of Couette flow in the half space for the full stratified Euler equations, and for any $B^2 > 0$. We note that for the exponentially stratified fluid (i.e. $\rho_0(y) = Ae^{-\beta y}$), the Boussinesq approximation is valid only when β is small. One interesting result following from the initial value approach is the inviscid damping of velocity fields. Such inviscid damping phenomena was known by Orr [18] in 1907, where the Couette flow in a homogeneous fluid was considered. Orr showed that the horizontal and vertical velocities decay by t^{-1} and t^{-2} respectively. Such damping is not due to the viscosity, but instead is due to the mixing of the vorticity under the Couette flow. In recent years, the inviscid damping phenomena attracted new attention. In [17], Lin and Zeng showed that if we consider initial (vorticity) perturbation in the Sobolev space H^s $\left(s < \frac{3}{2}\right)$ then the nonlinear damping is not true due to the existence of nonparallel steady flows of the form of Kelvin's cats eye near Couette. In [2], Bedrossian and Masmoudi proved the nonlinear inviscid damping for perturbations near Couette in Gevrey class (i.e. almost analytic). The linear inviscid damping for more general shear flows in a homogeneous fluid were also studied in [22,24].

In this paper, our goal is to get the precise estimates of linear decay rates for Couette flow in exponentially stratified fluid, which might be useful in the future study of nonlinear damping. We restrict ourselves to the case in the whole space. The including of the boundary (half space, finite

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channel) causes additional complication, as can be seen from Taylor's results mentioned at the beginning.

Our first result is about the linear decay estimates for solutions of the linearized equations under Boussinesq approximation. Consider the steady shear flow $\boldsymbol{v}_0 = (Ry, 0)$ with an exponentially stratified density profile $\rho_0(y) = Ae^{-\beta y}$, where $R \in \mathbb{R}, A > 0, \beta \geq 0$ are constants. Denote $B^2 = \frac{\beta g}{R^2}$ to be the Richardson number. When β is small, we approximate $\rho_0(y)$ by $A(1 - \beta y)$ and the linearized equations under the Boussinesq approximation (see Sect. 2.1) is

$$(\partial_t + Ry\partial_x)\,\Delta\psi = -\partial_x\left(\frac{\rho}{A}\right)g, \tag{1.1}$$

$$(\partial_t + Ry\partial_x)\left(\frac{\rho}{A}\right) = \beta\partial_x\psi, \tag{1.2}$$

where ψ and $\frac{\rho}{A}$ are the perturbations of stream function and relative density variation.

Theorem 1.1. *Let $\left(\psi(t;x,y), \frac{\rho}{A}(t;x,y)\right)$ be the solution of (1.1)–(1.2) with the initial data*

$$\psi(0;x,y) = \psi^0(x,y), \quad \frac{\rho(0;x,y)}{A} = \rho^0(x,y),$$

where $y \in \mathbb{R}$ and x is periodic with period L. Denote the velocity $\boldsymbol{v} = \nabla^\perp\psi = (v^x, v^y)$. Below, $f \lesssim g$ stands for $f \leq Cg$ for a constant C depending only on R, β, g. We denote $\langle f\rangle := \sqrt{1+f^2}$ and $P_{\neq 0}$ to be the projection to nonzero Fourier modes (in x), that is,

$$P_{\neq 0}f(t;x,y) = f(t;x,y) - \frac{1}{L}\int_0^L f(t;x,y)dx.$$

The following estimates hold true:

(i) If $0 < B^2 < \frac{1}{4}$, let $\nu = \sqrt{\frac{1}{4} - B^2}$, then

$$\|P_{\neq 0}v^x\|_{L^2} \lesssim \langle t\rangle^{-\frac{1}{2}+\nu}\left(\|\psi^0\|_{H_x^1H_y^2} + \|\rho^0\|_{L_x^2H_y^1}\right),$$
$$\|v^y\|_{L^2} \lesssim \langle t\rangle^{-\frac{3}{2}+\nu}\left(\|\psi^0\|_{H_x^1H_y^3} + \|\rho^0\|_{L_x^2H_y^2}\right),$$
$$\|P_{\neq 0}\frac{\rho}{A}\|_{L^2} \lesssim \langle t\rangle^{-\frac{1}{2}+\nu}\left(\|\psi^0\|_{H_x^1H_y^2} + \|\rho^0\|_{L_x^2H_y^1}\right).$$

(ii) If $B^2 > \frac{1}{4}$ then

$$\|P_{\neq 0}v^x\|_{L^2} \lesssim \langle t\rangle^{-\frac{1}{2}}\left(\|\psi^0\|_{H_x^1H_y^2} + \|\rho^0\|_{L_x^2H_y^1}\right),$$
$$\|v^y\|_{L^2} \lesssim \langle t\rangle^{-\frac{3}{2}}\left(\|\psi^0\|_{H_x^1H_y^3} + \|\rho^0\|_{L_x^2H_y^2}\right),$$
$$\|P_{\neq 0}\frac{\rho}{A}\|_{L^2} \lesssim \langle t\rangle^{-\frac{1}{2}}\left(\|\psi^0\|_{H_x^1H_y^2} + \|\rho^0\|_{L_x^2H_y^1}\right).$$

(iii) If $B^2 = \frac{1}{4}$, then

$$\|P_{\neq 0}v^x\|_{L^2} \lesssim \langle t\rangle^{-\frac{1}{2}}\langle\log\langle t\rangle\rangle\left(\|\psi^0\|_{H_x^1H_y^2} + \|\rho^0\|_{L_x^2H_y^1}\right),$$
$$\|v^y\|_{L^2} \lesssim \langle t\rangle^{-\frac{3}{2}}\langle\log\langle t\rangle\rangle\left(\|\psi^0\|_{H_x^1H_y^3} + \|\rho^0\|_{L_x^2H_y^2}\right),$$
$$\|P_{\neq 0}\frac{\rho}{A}\|_{L^2} \lesssim \langle t\rangle^{-\frac{1}{2}}\langle\log\langle t\rangle\rangle\left(\|\psi^0\|_{H_x^1H_y^2} + \|\rho^0\|_{L_x^2H_y^1}\right).$$

(iv) If $B^2 = 0$, i.e., $\beta = 0$, then $\left\|\frac{\rho}{A}\right\|_{L^2}(t) = \|\rho^0\|_{L^2}$ and

$$\|P_{\neq 0}v^x\|_{L^2} \lesssim \|\rho^0\|_{L_x^2H_y^1} + \langle t\rangle^{-1}\|\psi^0\|_{H_x^1H_y^3},$$
$$\|v^y\|_{L^2} \lesssim \langle t\rangle^{-1}\|\rho^0\|_{L_x^2H_y^2} + \langle t\rangle^{-2}\|\psi^0\|_{H_x^1H_y^4}.$$

(v) If $B^2=\infty$, i.e. $R=0$, then $\frac{g}{\beta}\left\|\frac{\rho}{A}\right\|_{L^2}^2+\|\boldsymbol{v}\|_{L^2}^2$ is conserved. The following decay estimates hold true in $L_x^2L_y^\infty$,

$$\|P_{\neq 0}v^x\|_{L_x^2L_y^\infty}\lesssim |t|^{-\frac{1}{3}}\left(\|\psi^0\|_{H_x^{3/2}\left(H_y^{9/2}\cap W_y^{1,1}\right)}+\|\rho^0\|_{H_x^{1/2}\left(H_y^{9/2}\cap W_y^{1,1}\right)}\right),$$

$$\|v^y\|_{L_x^2L_y^\infty}\lesssim |t|^{-\frac{1}{3}}\left(\|\psi^0\|_{H_x^{5/2}\left(H_y^{7/2}\cap L_y^1\right)}+\|\rho^0\|_{H_x^{3/2}\left(H_y^{7/2}\cap L_y^1\right)}\right),$$

$$\|P_{\neq 0}\frac{\rho}{A}\|_{L_x^2L_y^\infty}\lesssim |t|^{-\frac{1}{3}}\left(\|\psi^0\|_{H_x^{5/2}\left(H_y^{9/2}\cap W_y^{1,1}\right)}+\|\rho^0\|_{H_x^{3/2}\left(H_y^{7/2}\cap L_y^1\right)}\right).$$

Theorem 1.1 gives a complete picture of the linear damping for the Couette flow in an exponentially stratified fluid in an infinite channel (i.e. $-\infty<y<+\infty$ and x periodic). More specifically, we obtain optimal decay rates for initial perturbations of minimal regularity. We make some comments to relate our results to the previous works on this problem. When $B^2>\frac{1}{4}$, the decay rates $t^{-\frac{3}{2}}$ for v^y and $t^{-\frac{1}{2}}$ for v^x were obtained by Booker and Bretherton [3] for a special class of solutions, which generalized the earlier results in [19, Chap. 5] for $B^2\gg 1$. In [14], the decay rates as in Theorem 1.1 (i)–(iii) were obtained for special solutions by hypergeometric functions, which are similar to g_1,g_2 defined in (3.4) and (3.5). However, it was not shown in [14] that general solutions can be expressed by these special solutions. Chimonas [7] considered the case $B^2<\frac{1}{4}$ and wrongly claimed that v^y decays at the rate $t^{2\nu-1}$ and v^x grows by $t^{2\nu}$. Later, an error in [7] was pointed out by Brown and Stewartson [4], where they also found the correct decay rates as in Theorem 1.1. In [4], the initial value problem was solved for analytic initial data by taking the Laplace transform in time and then the decay rates were obtained from the asymptotic analysis of the inverse Laplace transform of the solutions. Moreover, it was assumed in [4] that the discrete neutral eigenvalues do no exist, such that there are no poles in the Laplace transform of their solutions. In our analysis, we do not need to assume the nonexistence of discrete neutral eigenvalues, which actually follows as a corollary from the decay estimates in Theorem 1.1 for any $B^2>0$. This contrasts significantly with the case in the half space [9,10,21] or in a finite channel [11], where it was shown that there exist infinitely many discrete neutral eigenvalues when $B^2>\frac{1}{4}$. In Theorem 1.1, the decay rates are optimal with the minimal regularity requirement for the initial data. In particular, when $B^2<\infty$ it suffices to have the initial perturbations of vorticity and density variation $\omega\left(0\right),\rho^0\in H^1$ to get the optimal decay for $\|v^x\|_{L^2}$, and $\omega\left(0\right),\rho^0\in H^2$ to get the optimal decay for $\|v^y\|_{L^2}$. These minimal regularity requirement on the initial data are consistent with the results in [17] for the Couette flow with constant density. Moreover, if $B\to 0+$ (i.e. $\nu\to\frac{1}{2}-$), the decay rates for the horizontal and vertical velocities are $t^{-\frac{1}{2}+\nu}$ and $t^{-\frac{3}{2}+\nu}$ respectively even when $\rho^0=0$, which are almost one order slower than the rates (t^{-1} and t^{-2} respectively) for homogeneous fluids (i.e. $B=0$). This suggests that the stratified effects cannot be ignored even when the steady density is a small deviation of the constant.

The decay rate $t^{-\frac{1}{3}}$ for the case $B^2=\infty$ (i.e. no shear flow) is optimal (see the example at the end of Sect. 6.1). When $(x,y)\in\mathbb{R}^2$, the optimal decay rate was shown to be $t^{-\frac{1}{2}}$ in [12]. We note that the decay mechanisms are very different for the cases of $B^2=\infty$ and $B^2<\infty$. When $B^2<\infty$, the decay of $\|\boldsymbol{v}\|_{L^2}$ is due to the mixing of vorticity caused by the shear motion. When $B^2=\infty$, $\|\boldsymbol{v}\|_{L^2}$ does not decay while the decay of $\|\boldsymbol{v}\|_{L^\infty}$ is due to dispersive effects of the linear waves in a stably stratified fluid.

Most papers on Couette flow used the Boussinesq approximation to analyze the linearized solutions. However, this approximation is valid only when β is small. For β not small, the full Euler equations should be used. In this case, the linearized equations at the Couette flow $(Ry,0)$ with the exponential density profile $\rho_0(y)=Ae^{-\beta y}$ become

$$\beta\left[R\partial_x-\left(\partial_t+Ry\partial_x\right)\partial_y\right]\psi+\left(\partial_t+Ry\partial_x\right)\Delta\psi=-\partial_x\left(\frac{\rho}{\rho_0}\right)g, \tag{1.3}$$

$$\left(\partial_t+Ry\partial_x\right)\left(\frac{\rho}{\rho_0}\right)=\beta\partial_x\psi. \tag{1.4}$$

We obtain similar results on decay estimates in the $e^{-\frac{1}{2}\beta y}$ weighted norms.

Theorem 1.2. *Let* $\left(\psi(t;x,y), \frac{\rho}{\rho_0}(t;x,y)\right)$ *be the solution of* (1.3)–(1.4) *with the initial data*

$$\psi(0;x,y)=\psi^0(x,y), \quad \frac{\rho(0;x,y)}{\rho_0(y)}=\rho^0(x,y),$$

where $y\in\mathbb{R}$ *and* x *is periodic with period* L. *Let* $\boldsymbol{v}=\nabla^{\perp}\psi=(v^x,v^y)$. *The following is true:*

(i) If $0<B^2<\frac{1}{4}$, *let* $\nu=\sqrt{\frac{1}{4}-B^2}$, *then*

$$\begin{aligned}
\|e^{-\frac{1}{2}\beta y}P_{\neq 0}v^x\|_{L^2} &\lesssim \langle t\rangle^{-\frac{1}{2}+\nu}\left(\|e^{-\frac{1}{2}\beta y}\psi^0\|_{H^1_xH^2_y}+\|e^{-\frac{1}{2}\beta y}\rho^0\|_{L^2_xH^1_y}\right),\\
\|e^{-\frac{1}{2}\beta y}v^y\|_{L^2} &\lesssim \langle t\rangle^{-\frac{3}{2}+\nu}\left(\|e^{-\frac{1}{2}\beta y}\psi^0\|_{H^1_xH^3_y}+\|e^{-\frac{1}{2}\beta y}\rho^0\|_{L^2_xH^2_y}\right),\\
\|e^{-\frac{1}{2}\beta y}P_{\neq 0}\rho/\rho_0\|_{L^2} &\lesssim \langle t\rangle^{-\frac{1}{2}+\nu}\left(\|e^{-\frac{1}{2}\beta y}\psi^0\|_{H^1_xH^2_y}+\|e^{-\frac{1}{2}\beta y}\rho^0\|_{L^2_xH^1_y}\right).
\end{aligned}$$

(ii) If $B^2>\frac{1}{4}$ *then*

$$\begin{aligned}
\|e^{-\frac{1}{2}\beta y}P_{\neq 0}v^x\|_{L^2} &\lesssim \langle t\rangle^{-\frac{1}{2}}\left(\|e^{-\frac{1}{2}\beta y}\psi^0\|_{H^1_xH^2_y}+\|e^{-\frac{1}{2}\beta y}\rho^0\|_{L^2_xH^1_y}\right),\\
\|e^{-\frac{1}{2}\beta y}v^y\|_{L^2} &\lesssim \langle t\rangle^{-\frac{3}{2}}\left(\|e^{-\frac{1}{2}\beta y}\psi^0\|_{H^1_xH^3_y}+\|e^{-\frac{1}{2}\beta y}\rho^0\|_{L^2_xH^2_y}\right),\\
\|e^{-\frac{1}{2}\beta y}P_{\neq 0}\rho/\rho_0\|_{L^2} &\lesssim \langle t\rangle^{-\frac{1}{2}}\left(\|e^{-\frac{1}{2}\beta y}\psi^0\|_{H^1_xH^2_y}+\|e^{-\frac{1}{2}\beta y}\rho^0\|_{L^2_xH^1_y}\right).
\end{aligned}$$

(iii) If $B^2=\frac{1}{4}$, *then*

$$\begin{aligned}
\|e^{-\frac{1}{2}\beta y}P_{\neq 0}v^x\|_{L^2} &\lesssim \langle t\rangle^{-\frac{1}{2}}\langle\log\langle t\rangle\rangle\left(\|e^{-\frac{1}{2}\beta y}\psi^0\|_{H^1_xH^2_y}+\|e^{-\frac{1}{2}\beta y}\rho^0\|_{L^2_xH^1_y}\right),\\
\|e^{-\frac{1}{2}\beta y}v^y\|_{L^2} &\lesssim \langle t\rangle^{-\frac{3}{2}}\langle\log\langle t\rangle\rangle\left(\|e^{-\frac{1}{2}\beta y}\psi^0\|_{H^1_xH^3_y}+\|e^{-\frac{1}{2}\beta y}\rho^0\|_{L^2_xH^2_y}\right),\\
\|e^{-\frac{1}{2}\beta y}P_{\neq 0}\rho/\rho_0\|_{L^2} &\lesssim \langle t\rangle^{-\frac{1}{2}}\langle\log\langle t\rangle\rangle\left(\|e^{-\frac{1}{2}\beta y}\psi^0\|_{H^1_xH^2_y}+\|e^{-\frac{1}{2}\beta y}\rho^0\|_{L^2_xH^1_y}\right).
\end{aligned}$$

(iv) If $B^2=0$, *i.e,* $\beta=0$, *then the results are the same as in the Boussinesq case, with* ρ/ρ_0 *replacing* $\frac{\rho}{A}$.

(v) If $B^2=\infty$, *i.e.* $R=0$, *then*

$$\left\|e^{-\frac{1}{2}\beta y}\boldsymbol{v}\right\|^2_{L^2}+\frac{g}{\beta}\left\|e^{-\frac{1}{2}\beta y}\frac{\rho}{\rho_0}\right\|^2_{L^2}$$

is conserved and

$$\begin{aligned}
\|e^{-\frac{1}{2}\beta y}P_{\neq 0}v^x\|_{L^2_xL^\infty_y} \lesssim |t|^{-\frac{1}{3}}\Big(&\|e^{-\frac{1}{2}\beta y}\psi^0\|_{H^{3/2}_x\left(H^{9/2}_y\cap W^{1,1}_y\right)}\\
&+\|e^{-\frac{1}{2}\beta y}\rho^0\|_{H^{1/2}_x\left(H^{9/2}_y\cap W^{1,1}_y\right)}\Big),\\
\|e^{-\frac{1}{2}\beta y}v^y\|_{L^2_xL^\infty_y} \lesssim |t|^{-\frac{1}{3}}\Big(&\|e^{-\frac{1}{2}\beta y}\psi^0\|_{H^{5/2}_x\left(H^{7/2}_y\cap L^1_y\right)}\\
&+\|e^{-\frac{1}{2}\beta y}\rho^0\|_{H^{3/2}_x\left(H^{7/2}_y\cap L^1_y\right)}\Big),\\
\|e^{-\frac{1}{2}\beta y}P_{\neq 0}\rho/\rho_0\|_{L^2_xL^\infty_y} \lesssim |t|^{-\frac{1}{3}}\Big(&\|e^{-\frac{1}{2}\beta y}\psi^0\|_{H^{5/2}_x\left(H^{9/2}_y\cap W^{1,1}_y\right)}\\
&+\|e^{-\frac{1}{2}\beta y}\rho^0\|_{H^{3/2}_x\left(H^{7/2}_y\cap L^1_y\right)}\Big).
\end{aligned}$$

Compared with Theorem 1.1, it is interesting to note that for the $e^{-\frac{1}{2}\beta y}$ weighted v and ρ, the decay rates and the initial regularity requirement for the full equations are exactly the same as in the Boussinesq approximation.

Lastly, we make some comments on the proof. First, we use Fourier transform on the linearized equations in the sheared coordinates and then reduce them to a second order ODE for the stream function. The general solution is expressed by two special solutions of hypergeometric functions. The decay rates and initial regularity are then obtained by using the asymptotic behaviors of hypergeometric functions. In the case of $B^2 = \infty$ (i.e. no shear), the decay rates are obtained by the dispersive estimates and oscillatory integrals.

This paper is organized as follows. In Sect. 2, we derive the linearized equations and give some identities of hypergeometric functions to be used later. In Sect. 3, we solve the linearized equations by hypergeometric functions. In Sects. 4 and 5, we obtain the decay estimates from the solution formula for the case $B^2 < \infty$. In Sect. 6, the dispersive decay estimates are obtained for the case $B^2 = \infty$.

2. Preliminary

2.1. Linearized Euler Equation and Boussinesq Approximation

The equations for two dimensional inviscid incompressible flows in stratified fluids are

$$\rho\left(\partial_t + \boldsymbol{v}\cdot\nabla\right)\boldsymbol{v} + \nabla p = \rho\boldsymbol{g}, \tag{2.1}$$

$$\left(\partial_t + \boldsymbol{v}\cdot\nabla\right)\rho = 0,$$

$$\nabla\cdot\boldsymbol{v} = 0, \tag{2.2}$$

where $(x, y) \in \mathbb{T}\times\mathbb{R}$, $\boldsymbol{v} = (v^x, v^y)$ is the velocity, ρ is the density and $\boldsymbol{g} = (0, -g)$ is the gravitational acceleration directing downward with g being the gravitational constant. The simplest stationary solution is the shear flow, with $\boldsymbol{v}_0 = (U(y), 0)$ and $\rho_0 = \rho_0(y)$. Let $\psi = \psi(t; x, y)$ be the stream function such that $\boldsymbol{v} = \nabla^\perp\psi$. Here $\nabla^\perp = (-\partial_y, \partial_x)$.

We consider the linearized equations near a shear $(\boldsymbol{v}_0, \rho_0)$. Let $\boldsymbol{v} = \nabla^\perp\psi$ and ρ be infinitesimal perturbations of velocity and density. The linearized equations are

$$\rho_0\left[\left(\partial_t + U(y)\partial_x\right)\boldsymbol{v} + \left(v^y\partial_y\right)\boldsymbol{v}_0\right] + \nabla p = \rho\boldsymbol{g}, \tag{2.3}$$

$$\left(\partial_t + U(y)\partial_x\right)\rho + v^y\rho_0'(y) = 0.$$

$$\nabla\cdot\boldsymbol{v} = 0. \tag{2.4}$$

Taking the curl of (2.3), we get

$$\begin{aligned} &-\frac{\rho_0'(y)}{\rho_0}\left[U'(y)\partial_x\psi + \left(\partial_t + U(y)\partial_x\right)\left(-\partial_y\psi\right)\right] \\ &\quad + \left(\partial_t + U(y)\partial_x\right)\Delta\psi - U''(y)\partial_x\psi = -\partial_x\left(\frac{\rho}{\rho_0}\right)g. \end{aligned} \tag{2.5}$$

The Eq. (2.4) can be written as

$$\left(\partial_t + U(y)\partial_x\right)\frac{\rho}{\rho_0} = -\partial_x\psi\frac{\rho_0'(y)}{\rho_0}. \tag{2.6}$$

Consider Couette flow with an exponential density profile, that is, $U(y) = Ry$, $\rho_0(y) = Ae^{-\beta y}$. Then (2.5)–(2.6) become

$$\beta\left[R\partial_x - \left(\partial_t + Ry\partial_x\right)\partial_y\right]\psi + \left(\partial_t + Ry\partial_x\right)\Delta\psi = -\partial_x\left(\frac{\rho}{\rho_0}\right)g, \tag{2.7}$$

$$\left(\partial_t + Ry\partial_x\right)\left(\frac{\rho}{\rho_0}\right) = \beta\partial_x\psi. \tag{2.8}$$

If $R \neq 0$, denote $B^2 = \frac{\beta g}{R^2}$ to be the Richardson number, $T = \frac{R\rho}{\beta \rho_0(y)}$ be the relative density perturbation, $\omega = -\Delta\psi$ be the vorticity perturbation and let $t' = Rt$. Then we have

$$\begin{aligned} -\beta\left[\partial_x - \left(\partial_{t'} + y\partial_x\right)\partial_y\right]\psi + (\partial_{t'} + y\partial_x)\omega &= B^2\partial_x T, \\ (\partial_{t'} + y\partial_x)T &= \partial_x\psi. \end{aligned}$$

For convenience we still use t for t'. Thus the resulting linearized system is

$$-\beta\left[\partial_x - \left(\partial_t + y\partial_x\right)\partial_y\right]\psi + (\partial_t + y\partial_x)\omega = B^2\partial_x T, \tag{2.9}$$

$$(\partial_t + y\partial_x)T = \partial_x\psi, \tag{2.10}$$

$$\omega = -\Delta\psi. \tag{2.11}$$

The system (2.9)–(2.11) is rather complicated. Many authors, including Høiland [15], Case [6], Kuo [16], Hartman [14], Chimonas [7], Brown and Stewartson [4], Farrell and Ioannou [13], chose to consider the Boussinesq approximation, where the variation of density is ignored except for the gravity force term $\rho\boldsymbol{g}$. To apply the Boussinesq approximation, the density perturbation should be relatively small compared with the constant density. Under this approximation, the Euler momentum equation becomes

$$\bar{\rho}\left(\partial_t\, \boldsymbol{v} + \left(\boldsymbol{v}\cdot\nabla\right)\boldsymbol{v}\right) + \nabla p = \rho\boldsymbol{g},$$

where $\bar{\rho}$ is a constant and ρ is the variation of density. The linearized Boussinesq equations near a shear flow $(U(y), 0)$ with the density variation profile $\rho_0(y)$ is

$$\left(\partial_t + U(y)\partial_x\right)\Delta\psi - U''(y)\partial_x\psi = -\partial_x\left(\frac{\rho}{\bar{\rho}}\right)g, \tag{2.12}$$

$$\left(\partial_t + U(y)\partial_x\right)\frac{\rho}{\bar{\rho}} = -\partial_x\psi\frac{\rho_0'}{\bar{\rho}}. \tag{2.13}$$

Compared this with the linearized original equation (2.5), it can be regarded as the case when ρ_0'/ρ_0 is very small, such that the first term of (2.5) is neglected and ρ_0 is taken to be a constant $\bar{\rho}$. For Couette flow $U(y) = Ry$ with the exponential profile $\rho_0 = Ae^{-\beta y}$, to use the Boussinesq approximation, β should be small which implies that $\rho_0 \approx A(1 - \beta y)$. Thus, we consider the linearized Boussinesq equations near Couette flow $(Ry, 0)$ with the linear density variation profile $\rho_0(y) = -A\beta y$ and a constant density background $\bar{\rho} = A$. Then (2.12)–(2.13) become

$$\left(\partial_t + Ry\partial_x\right)\Delta\psi = -\partial_x\left(\frac{\rho}{A}\right)g, \tag{2.14}$$

$$\left(\partial_t + Ry\partial_x\right)\left(\frac{\rho}{A}\right) = \beta\partial_x\psi. \tag{2.15}$$

If $R \neq 0$, denoting $B^2 = \frac{\beta g}{R^2}$, $T = \frac{R\rho}{\beta A}$ and scaling the time t by Rt, then we have

$$(\partial_t + y\partial_x)\omega = B^2\partial_x T, \tag{2.16}$$

$$(\partial_t + y\partial_x)T = \partial_x\psi, \tag{2.17}$$

$$\omega = -\Delta\psi. \tag{2.18}$$

2.2. Sobolev Spaces

Without loss of generality, from now on we assume period length L in x direction is 2π. Define the Fourier transform of $f(x, y)$ $((x, y) \in \mathbb{T}\times\mathbb{R})$, as

$$\hat{f}(k, \eta) = \frac{1}{2\pi}\int_{\mathbb{T}\times\mathbb{R}} e^{-ixk - iy\eta} f(x, y)\mathrm{d}x\mathrm{d}y, \quad (k, \eta) \in \mathbb{Z}\times\mathbb{R}.$$

Fourier inversion formula is

$$f(x, y) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} e^{ixk + iy\eta}\hat{f}(k, \eta)\mathrm{d}x\mathrm{d}y.$$

The Sobolev space $H_x^{s_x}H_y^{s_y}$ is defined to be all functions f in $L^2(\mathbb{T}\times\mathbb{R})$ satisfying

$$\sum_{k\in\mathbb{Z}}(1+k^2)^{s_x}\int_{\mathbb{R}}\left(1+\eta^2\right)^{s_y}\left|\hat{f}(k,\eta)\right|^2\mathrm{d}\eta<+\infty,$$

with the norm

$$\|f\|_{H_x^{s_x}H_y^{s_y}}=\left(\sum_{k\in\mathbb{Z}}(1+k^2)^{s_x}\int_{\mathbb{R}}\left(1+\eta^2\right)^{s_y}\left|\hat{f}(k,\eta)\right|^2\mathrm{d}\eta\right)^{\frac{1}{2}}.$$

Similarly, we define

$$\|f\|_{H_x^{s_x}W_y^{s_y,p}}=\left(\sum_{k\in\mathbb{Z}}(1+k^2)^{s_x}\|\hat{f}(k,y)\|^2_{W_y^{s_y,p}}\right)^{\frac{1}{2}},$$

where $W_y^{s_y,p}$ is the L^p Sobolev space in $\mathbb{R}$ and

$$\hat{f}(k,y)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{T}}e^{-ixk}f(x,y)\mathrm{d}x,\quad k\in\mathbb{Z}.$$

2.3. Hypergeometric Functions

Gaussian hypergeometric function $F(a,b;c;z)$ is defined by the power series

$$F(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$$

for $|z|<1$, where

$$(x)_n=\begin{cases}1 & n=0,\\ x(x+1)\cdots(x+n-1) & n>0.\end{cases}$$

Its value $F(z)$ for $|z|\geq 1$ is defined by the analytic continuation. If $c,z\in\mathbb{R}$, and a,b are complex conjugate, then $F(a,b;c;z)$ is also real. The following lemma is known as Gauss' contiguous relation.

Lemma 2.1. *The derivative of $F(z)=F(a,b;c;z)$ can be expressed as*

$$\begin{aligned}\frac{dF}{dz}&=\frac{ab}{c}F(a+1,b+1;c+1;z)\\&=\frac{c-1}{z}\left(F(a,b;c-1;z)-F(a,b;c;z)\right)\\&=\frac{1}{c(1-z)}\left[(c-a)\left(c-b\right)F(a,b;c+1;z)+c\left(a+b-c\right)F(a,b;c;z)\right].\end{aligned}$$

Hypergeometric functions are related to solutions of Euler's hypergeometric differential equation.

Lemma 2.2. *Assume c is not an integer. Euler's hypergeometric differential equation*

$$z(1-z)f''(z)+\left[c-(a+b+1)z\right]f'(z)-abf(z)=0 \tag{2.19}$$

has two linearly independent solutions

$$\begin{aligned}f_1(z)&=F(a,b;c;z),\\f_2(z)&=z^{1-c}F(1+a-c,1+b-c;2-c;z).\end{aligned}$$

The proof of these two lemmas can be found in pages 57 and 74 of the book [1].

Hypergeometric functions have one branch point at $z=1$, and another at $z=\infty$. The default cut-line connecting these two branch points is chosen as $z>1, z\in\mathbb{R}$. Pfaff transform can relate the value of a hypergeometric functions near $z=1$ to the value of another one near $z=\infty$ in the following way:

$$F(a,b;c;z)=(1-z)^{-b}F\left(c-a,b;c;\frac{z}{z-1}\right), \tag{2.20}$$

$$F(a,b;c;z)=(1-z)^{-b}F\left(c-a,b;c;\frac{z}{z-1}\right). \tag{2.21}$$

By combining these two transforms, we obtain the Euler transform

$$F(a,b;c;z)=(1-z)^{c-a-b}F\left(c-a,c-b;c;z\right). \tag{2.22}$$

When $\mathrm{Re}(c)>\mathrm{Re}(a+b)$ we have the Gauss formula

$$F(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{2.23}$$

When $\mathrm{Re}(c)<\mathrm{Re}(a+b)$, $F(a,b;c;1)$ is infinity.

The following lemma plays an important role in solving the linearized equations in the next Section.

Lemma 2.3. *The Wronskian of the two solutions listed above is*

$$W(z)=f_1(z)f_2'(z)-f_1'(z)f_2(z)=(1-c)z^{-c}(1-z)^{c-1-a-b}.$$

Proof. By Liouville's formula, the Wronskian of Euler's hypergeometric differential equation (2.19) can be written as

$$\begin{aligned} W(z)&=C\exp\left(-\int\frac{c-(a+b+1)z}{z(1-z)}\mathrm{d}z\right)\\ &=C\exp\left(-\log(1-z)(a+b+1-c)-c\log(z)\right)\\ &=Cz^{-c}(1-z)^{c-1-a-b}=Cz^{-c}+O(z^{-c-1}) \end{aligned}$$

To determine the constant C, it is sufficient to calculate the leading order term of $W(z)$ in the power series expansion near $z=0$. By the definition,

$$f_1(0)=1,\quad f_1'(0)=\frac{ab}{c},\quad f_2(z)\sim z^{1-c},\quad f_2'(z)\sim(1-c)z^{-c}$$

when $z\to 0$, so $C=1-c$ and $W(z)=(1-c)z^{-c}(1-z)^{c-1-a-b}$. □

3. Solutions by Hypergeometric Functions

In this section, we apply Fourier transform on the linearized systems (2.16–2.18) based on the Boussinesq approximation and (2.9–2.11) based on full Euler equations respectively. Then we reduce them to a second order ODE in t, and solve it explicitly by using hypergeometric functions. We will study these equations in the sheared coordinates $(z,y)=(x-ty,y)$ and define

$$\begin{aligned} f(t;z,y)&=\omega(t;z+ty,y)=\omega(t;x,y),\\ \phi(t;z,y)&=\psi(t;z+ty,y)=\psi(t;x,y),\\ \tau(t;z,y)&=T(t;z+ty,y)=T(t;x,y). \end{aligned}$$

3.1. Boussinesq Approximation

In the new coordinates (z, y), equations (2.16–2.18) become the following:

$$\begin{aligned}
\partial_t f(t; z, y) &= (\partial_t + y\partial_x)\,\omega(t; x, y) = B^2\partial_x T(t; x, y) = B^2\partial_z\tau(t; z, y),\\
\partial_t\tau(t; z, y) &= (\partial_t + y\partial_x)\,T(t; x, y) = \partial_x\psi(t; x, y) = \partial_z\phi(t; z, y),\\
\left[\partial_{zz} + (\partial_y - t\partial_z)^2\right]\phi(t; z, y) &= \psi_{xx} + \psi_{yy} = -\omega(t; x, y) = -f(t; z, y).
\end{aligned}$$

By the Fourier transform $(z, y) \to (k, \eta)$, we get

$$\begin{aligned}
\hat{f}_t = B^2(ik)\hat{\tau},\quad \hat{\tau}_t = (ik)\hat{\phi},\\
\left[(ik)^2 + (i\eta - ikt)^2\right]\hat{\phi} = -\hat{f}.
\end{aligned} \tag{3.1}$$

Differentiate (3.1) twice with respect to t to get

$$\begin{aligned}
&\left[(ik)^2 + (i\eta - ikt)^2\right]\hat{\phi}_t + 2(i\eta - ikt)(-ik)\hat{\phi} = -\hat{f}_t = -B^2(ik)\hat{\tau},\\
&\left[(ik)^2 + (i\eta - ikt)^2\right]\hat{\phi}_{tt} + 4(i\eta - ikt)(-ik)\hat{\phi}_t + 2(-ik)^2\hat{\phi}\\
&= -\hat{f}_{tt} = -B^2(ik)\hat{\tau}_t = -B^2(ik)^2\hat{\phi}.
\end{aligned} \tag{3.2}$$

For fixed $k \neq 0$ and η, define $s = t - \frac{\eta}{k}$ and $s_0 = -\frac{\eta}{k}$. Then we obtain a second order linear ODE for $\hat{\phi}$

$$(1 + s^2)\hat{\phi}_{tt} + 4s\hat{\phi}_t + (2 + B^2)\hat{\phi} = 0. \tag{3.3}$$

First, we look for special solutions of the form $\hat{\phi}(t; k, \eta) = g(-s^2)$. Let $u = -s^2$, then $\hat{\phi}_t = -2sg'$ and $\hat{\phi}_{tt} = 4s^2g'' - 2g'$. Equation (3.3) becomes

$$u(1 - u)g'' + \left(\frac{1}{2} - \frac{5}{2}u\right)g' - \frac{2 + B^2}{4}g = 0.$$

This is in the form of Euler's hypergeometric differential equation (2.19) with $c = \frac{1}{2}$ and $a, b = \frac{3}{4} \pm \frac{\nu}{2}$, where $\nu = \sqrt{\frac{1}{4} - B^2}$. By Lemma 2.2, it has two linearly independent solutions

$$g_1(s) = F\left(a, b; c; u\right) = F\left(\frac{3}{4} - \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}; \frac{1}{2}; -s^2\right), \tag{3.4}$$

$$g_2(s) = -iu^{1-c}F\left(1 + a - c, 1 + b - c; 2 - c; u\right) = sF\left(\frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}; \frac{3}{2}; -s^2\right). \tag{3.5}$$

Therefore, the general solutions to the Eq. (3.3) can be written as

$$\hat{\phi} = C_1g_1(s) + C_2g_2(s), \tag{3.6}$$

where C_1, C_2 are some constants depending only on (k, η). Note that although a hypergeometric function has a branch point or singularity at $z = 1$, we only need its value at $z = -s^2$ which lies on the negative real axis. Therefore, there is no ambiguity or singularity in (3.6).

The coefficients C_1, C_2 are determined by the initial conditions $\psi(0; x, y)$ and $T(0; x, y)$. Let $\hat{\psi}^0(k, \eta), \hat{T}^0(k, \eta)$ be the Fourier transforms of $\psi(0; x, y)$ and $T(0; x, y)$. First,

$$\hat{\phi}(0; k, \eta) = \hat{\phi}^0(k, \eta) = \hat{\psi}^0(k, \eta),$$

and by Eq. (3.2),

$$\hat{f}_t = k^2(1 + s^2)\hat{\phi}_t + 2k^2s\hat{\phi}.$$

Noticing that when $t = 0$, $s = -\frac{\eta}{k} = s_0$, so we have

$$\begin{aligned}
\hat{\phi}_t(0; k, \eta) &= \frac{\hat{f}_t(0; k, \eta) - 2k^2s_0\hat{\phi}(0; k, \eta)}{k^2(1 + s_0^2)} = \frac{B^2(ik)\hat{\tau}(0; k, \eta) - 2k^2s_0\hat{\phi}(0; k, \eta)}{k^2(1 + s_0^2)}\\
&= \frac{1}{1 + s_0^2}\left(\frac{iB^2}{k}\hat{\tau}^0 - 2s_0\hat{\phi}^0\right) = \frac{1}{1 + s_0^2}\left(\frac{iB^2}{k}\hat{T}^0 - 2s_0\hat{\psi}^0\right).
\end{aligned}$$

Now we have a linear system for (C_1, C_2)

$$C_1 g_1(s_0) + C_2 g_2(s_0) = \hat{\psi}^0,$$
$$C_1 g_1'(s_0) + C_2 g_2'(s_0) = \frac{1}{1+s_0^2}\left(\frac{iB^2}{k}\hat{T}^0 - 2s_0\hat{\psi}^0\right).$$

Therefore, the coefficients are

$$\begin{aligned} C_1(k,\eta) =& \frac{1}{\Delta}\left[g_2'(s_0) + \frac{2s_0}{1+s_0^2}g_2(s_0)\right]\hat{\psi}^0(k,\eta) \\ &+ \frac{1}{\Delta}\left[-\frac{iB^2}{1+s_0^2}g_2(s_0)\right]\frac{\hat{T}^0(k,\eta)}{k}, \end{aligned} \tag{3.7}$$

$$\begin{aligned} C_2(k,\eta) =& \frac{1}{\Delta}\left[-g_1'(s_0) - \frac{2s_0}{1+s_0^2}g_1(s_0)\right]\hat{\psi}^0(k,\eta) \\ &+ \frac{1}{\Delta}\left[\frac{iB^2}{1+s_0^2}g_1(s_0)\right]\frac{\hat{T}^0(k,\eta)}{k}, \end{aligned} \tag{3.8}$$

where by Lemma 2.3

$$\begin{aligned} \Delta &= g_1(s_0)g_2'(s_0) - g_1'(s_0)g_2(s_0) \\ &= -i(-2s_0)\left(1-\frac{1}{2}\right)\left(-s_0^2\right)^{-\frac{1}{2}}\left(1+s_0^2\right)^{-2} = \frac{1}{\left(1+s_0^2\right)^2}, \end{aligned}$$

which is strictly positive for all $s_0 \in \mathbb{R}$.

Thus the solution of (3.3) is given explicitly by

$$\hat{\phi}(t;k,\eta) = C_1(k,\eta)g_1(s) + C_2(k,\eta)g_2(s).$$

As for $\hat{\tau}$, from Eq. (3.2), for $B^2 > 0$ we have

$$\begin{aligned} \hat{\tau}(t;k,\eta) =& -\frac{ik}{B^2}\left((1+s^2)\hat{\phi}_t + 2s\hat{\phi}\right), \\ =& -\frac{ik}{B^2}\left[(1+s^2)\left(C_1(k,\eta)g_1'(s) + C_2(k,\eta)g_2'(s)\right)\right. \\ &\left. +2s\left(C_1(k,\eta)g_1(s) + C_2(k,\eta)g_2(s)\right)\right]. \end{aligned} \tag{3.9}$$

3.2. Full Euler Equations

Now we solve the linearized systems (2.9)–(2.11) based on the full Euler equations. With f, ϕ, τ defined at the beginning of this section, Eqs. (2.9)–(2.11) turn into

$$\begin{aligned} &-\beta\left[\partial_z - \partial_t\left(\partial_y - t\partial_z\right)\right]\phi + \partial_t f = B^2\partial_z\tau, \\ &\partial_t\tau = \partial_z\phi, \quad -\left[\partial_{zz} + \left(\partial_y - t\partial_z\right)^2\right]\phi = f. \end{aligned} \tag{3.10}$$

By the Fourier transform $(z,y) \to (k,\eta)$, (3.10) becomes

$$-\beta\left[ik - \partial_t\left(i\eta - ikt\right)\right]\hat{\phi} + \hat{f}_t = B^2(ik)\hat{\tau}. \tag{3.11}$$

Differentiate above with respect to t, we get

$$-\beta\left[ik\partial_t - \partial_{tt}\left(i\eta - ikt\right)\right]\hat{\phi} + \hat{f}_{tt} = B^2(ik)\hat{\tau}_t.$$

Substituting

$$\hat{\tau}_t = (ik)\hat{\phi}, \quad \hat{f} = -\left[(ik)^2 + (i\eta - ikt)^2\right]\hat{\phi}, \tag{3.12}$$

we have

$$\partial_{tt}\left[k^2 + (\eta - kt)^2 + \beta(i\eta - ikt)\right]\hat{\phi} - \beta(ik)\hat{\phi}_t + B^2k^2\hat{\phi} = 0.$$

Define $\chi = e^{-\frac{1}{2}\beta y}\phi$, then $\hat{\phi}(k,\eta) = \hat{\chi}(k, \eta + \frac{1}{2}i\beta)$ and the above equation implies

$$\partial_{tt}\left[k^2 + \left(\eta - \frac{1}{2}i\beta - kt\right)^2 + \beta\left(i\left(\eta - \frac{1}{2}i\beta\right) - ikt\right)\right]\hat{\chi} \\ - \beta(ik)\hat{\chi}_t + B^2k^2\hat{\chi} = 0,$$

After simplification, we have

$$\partial_{tt}\left[\frac{1}{4}\beta^2 + k^2 + (\eta - kt)^2\right]\hat{\chi} - i\beta k\hat{\chi}_t + B^2k^2\hat{\chi} = 0.$$

For $k \neq 0$, again define $s = t - \frac{\eta}{k}$, $s_0 = -\frac{\eta}{k}$, then

$$\partial_{tt}\left[\left(\frac{1}{4}\beta^2 + k^2 + k^2s^2\right)\hat{\chi}\right] - i\beta k\hat{\chi}_t + B^2k^2\hat{\chi} = 0.$$

Define $m = \sqrt{\frac{1}{4}\beta^2 + k^2}$, $\kappa = \frac{k}{m}$, $\beta_1 = \frac{\beta}{2m}$, then we have

$$\partial_{tt}\left[\left(m^2 + k^2s^2\right)\hat{\chi}\right] - i\beta k\hat{\chi}_t + B^2k^2\hat{\chi} = 0,$$

$$\partial_{tt}\left[\left(1 + \kappa^2s^2\right)\hat{\chi}\right] - 2i\beta_1\kappa\hat{\chi}_t + B^2\kappa^2\hat{\chi} = 0.$$

Set $u = -i\kappa s$, then

$$-\partial_{uu}\left(1 - u^2\right)\hat{\chi} - 2\beta_1\hat{\chi}_u + B^2\hat{\chi} = 0,$$

$$\left(1 - u^2\right)\hat{\chi}_{uu} + (2\beta_1 - 4u)\hat{\chi}_u - (2 + B^2)\hat{\chi} = 0.$$

Define $v = \frac{1-u}{2}$, then

$$v\left(1 - v\right)\hat{\chi}_{vv} + (-\beta_1 + 2 - 4v)\hat{\chi}_v - (2 + B^2)\hat{\chi} = 0, \tag{3.13}$$

which is of the form of Euler's hypergeometric differential equation (2.19) with $c = 2 - \beta_1$ and $a, b = \frac{3}{2} \pm \nu$, where $\nu = \sqrt{\frac{1}{4} - B^2}$. By Lemma 2.2, it has two linear independent solutions,

$$g_3(s) = F\left(\frac{3}{2} - \nu, \frac{3}{2} + \nu; 2 - \beta_1; v\right) = F\left(\frac{3}{2} - \nu, \frac{3}{2} + \nu; 2 - \beta_1; \frac{1 + i\kappa s}{2}\right),$$

$$g_4(s) = \left(\frac{1 + i\kappa s}{2}\right)^{-1+\beta_1} F\left(\frac{1}{2} + \beta_1 - \nu, \frac{1}{2} + \beta_1 + \nu; \beta_1; \frac{1 + i\kappa s}{2}\right)$$

Therefore, the general solution to Eq. (3.13) is

$$\hat{\chi} = C_3g_3(s) + C_4g_4(s),$$

where C_3, C_4 are constants depending only on (k, η). Note that we only need values of g_1, g_2 at $\frac{1}{2} + \frac{\kappa s}{2}i$ $(s \in \mathbb{R})$, that is, on the line $\mathrm{Re}(z) = \frac{1}{2}$. Therefore, the branch point at $z = 1$ will not cause any ambiguity or singularity.

The initial conditions $\psi(0; x, y)$ and $T(0; x, y)$ are used to determine the coefficients C_3, C_4. Denote $\mu = e^{-\frac{1}{2}\beta y}\tau$, $\Psi^0 = e^{-\frac{1}{2}\beta y}\psi^0$, $\Upsilon^0 = e^{-\frac{1}{2}\beta y}T^0$, then

$$\hat{\chi}(0; k, \eta) = \hat{\phi}^0\left(k, \eta - \frac{1}{2}i\beta\right) = \widehat{e^{-\frac{1}{2}\beta y}\psi^0} = \hat{\Psi}^0.$$

By Eqs. (3.11) and (3.12), we have

$$\hat{\phi}_t = \frac{1}{1 + s^2 - \frac{i\beta}{k}s}\left[\left(\frac{2i\beta}{k} - 2s\right)\hat{\phi} + \frac{iB^2}{k}\hat{\tau}\right].$$

Hence

$$\begin{aligned}\hat{\chi}_t(t;k,\eta) &= \hat{\phi}_t\left(t;k,\eta-\frac{1}{2}i\beta\right)\\ &= \frac{1}{1+\left(s+\frac{i\beta}{2k}\right)^2-\frac{i\beta}{k}\left(s+\frac{i\beta}{2k}\right)}\left[\left(\frac{2i\beta}{k}-2s-2\frac{i\beta}{2k}\right)\hat{\chi}+\frac{iB^2}{k}\hat{\mu}\right]\\ &= \frac{1}{1+|\tilde{s}|^2}\left(\frac{iB^2}{k}\hat{\chi}-2\tilde{s}\hat{\mu}\right),\end{aligned}$$

and

$$\hat{\chi}_t(0;k,\eta) = \frac{1}{1+|\tilde{s}_0|^2}\left(\frac{iB^2}{k}\hat{\Upsilon}^0-2\tilde{s}_0\hat{\Psi}^0\right),$$

where $\tilde{s}=s-\frac{i\beta}{2k}, \tilde{s}_0=s_0-\frac{i\beta}{2k}$.

So we have a linear system for (C_3,C_4):

$$\begin{aligned}&C_3g_3(s_0)+C_4g_4(s_0)=\hat{\Psi}^0,\\ &C_3g_3'(s_0)+C_4g_4'(s_0)=\frac{1}{1+|\tilde{s}_0|^2}\left(\frac{iB^2}{k}\hat{\Upsilon}^0-2\tilde{s}_0\hat{\Psi}^0\right),\end{aligned}$$

which gives

$$\begin{aligned}C_3(k,\eta) &= \frac{1}{\Delta}\left[g_4'(s_0)+\frac{2\tilde{s}_0}{1+|\tilde{s}_0|^2}g_4(s_0)\right]\hat{\Psi}^0(k,\eta)\\ &\quad+\frac{1}{\Delta}\left[-\frac{iB^2}{1+|\tilde{s}_0|^2}g_4(s_0)\right]\frac{\hat{\Upsilon}^0(k,\eta)}{k},\\ C_4(k,\eta) &= \frac{1}{\Delta}\left[-g_3'(s_0)-\frac{2\tilde{s}_0}{1+|\tilde{s}_0|^2}g_3(s_0)\right]\hat{\Psi}^0(k,\eta)\\ &\quad+\frac{1}{\Delta}\left[\frac{iB^2}{1+|\tilde{s}_0|^2}g_3(s_0)\right]\frac{\hat{\Upsilon}^0(k,\eta)}{k},\end{aligned}$$

where by Lemma 2.3

$$\begin{aligned}\Delta &= g_3(s_0)g_4'(s_0)-g_3'(s_0)g_4(s_0)\\ &= \frac{\kappa i}{2}(-1+\beta_1)\left(\frac{1}{2}+\frac{\kappa s_0}{2}i\right)^{-2+\beta_1}\left(\frac{1}{2}-\frac{\kappa s_0}{2}i\right)^{-2-\beta_1},\end{aligned}$$

which is never zero, because $|\kappa|,\ \beta_1\in(0,1)$ by definition. Moreover,

$$|\kappa|\geq\frac{1}{\sqrt{\frac{1}{4}\beta^2+1}},\quad 1-\beta_1\geq 1-\frac{\beta/2}{\sqrt{\frac{1}{4}\beta^2+1}}$$

are both uniformly bounded away from zero for all integers $k\neq 0$. Hence

$$|\Delta|^{-1}=\left|\frac{1}{2}+\frac{\kappa s_0}{2}i\right|^4\left|\frac{\kappa}{2}\right|^{-1}(1-\beta_1)^{-1}\lesssim\langle s_0\rangle^4.$$

By Eqs. (3.11) and (3.12), for $B^2>0$ we have

$$\hat{\tau}(t;k,\eta)=-\frac{ik}{B^2}\left[-\frac{2i\beta}{k}\hat{\phi}-\frac{i\beta}{k}s\hat{\phi}_t+(1+s^2)\hat{\phi}_t+2s\hat{\phi}\right],$$

and

$$\begin{aligned}\hat{\mu}(t;k,\eta) =& \hat{\tau}\left(t;k,\eta-\frac{1}{2}i\beta\right)\\ =& -\frac{ik}{B^2}\left[-\frac{2i\beta}{k}\hat{\chi}-\frac{i\beta}{k}\left(s+\frac{i\beta}{2k}\right)\hat{\chi}_t+\left(1+\left(s+\frac{i\beta}{2k}\right)^2\right)\hat{\chi}_t\right.\\ &\left.+2\left(s+\frac{i\beta}{2k}\right)\hat{\chi}\right]\\ =& -\frac{ik}{B^2}\left[\left(1+s^2+\frac{\beta^2}{4k^2}\right)\hat{\chi}_t+2\left(s-\frac{i\beta}{2k}\right)\hat{\chi}\right]\\ =& -\frac{ik}{B^2}\left[\left(1+|\tilde{s}|^2\right)\hat{\chi}_t+2\tilde{s}\hat{\chi}\right].\end{aligned}$$

4. Decay Estimates in the Case of Boussinesq Approximation

In this section, we use the solution formula obtained in the last section to obtain the inviscid decay estimates in Theorem 1.1, for solutions of the linearized equations under Boussinesq approximation.

4.1. The Case $B^2 > 0$ and $B^2 \neq \frac{1}{4}$

By expanding $g_1(s)$, $g_2(s)$, $g_1'(s_0)$, $g_2'(s_0)$ at infinity, we obtain the following asymptotics

$$\begin{aligned}g_1(s) =& \sqrt{\pi}\left[\frac{\Gamma(\nu)}{\Gamma(-\frac{1}{4}+\frac{\nu}{2})\Gamma(\frac{3}{4}+\frac{\nu}{2})}s^{-\frac{3}{2}+\nu}\right.\\ &\left.+\frac{\Gamma(-\nu)}{\Gamma(-\frac{1}{4}-\frac{\nu}{2})\Gamma(\frac{3}{4}-\frac{\nu}{2})}s^{-\frac{3}{2}-\nu}\right]+O\left(|s|^{-\frac{7}{2}+Re(\nu)}\right),\end{aligned}\tag{4.1}$$

$$\begin{aligned}g_2(s) =& \frac{\sqrt{\pi}}{2}\left[\frac{\Gamma(\nu)}{\Gamma(\frac{1}{4}+\frac{\nu}{2})\Gamma(\frac{5}{4}+\frac{\nu}{2})}s^{-\frac{3}{2}+\nu}\right.\\ &\left.+\frac{\Gamma(-\nu)}{\Gamma(\frac{1}{4}-\frac{\nu}{2})\Gamma(\frac{5}{4}-\frac{\nu}{2})}s^{-\frac{3}{2}-\nu}\right]+O\left(|s|^{-\frac{5}{2}+Re(\nu)}\right),\end{aligned}\tag{4.2}$$

$$\begin{aligned}g_1'(s_0) =& 2\sqrt{\pi}\left[\frac{\left(-\frac{3}{4}+\frac{\nu}{2}\right)\Gamma(\nu)}{\Gamma(-\frac{1}{4}+\frac{\nu}{2})\Gamma(\frac{3}{4}+\frac{\nu}{2})}s_0^{-\frac{5}{2}+\nu}\right.\\ &\left.+\frac{\left(-\frac{3}{4}-\frac{\nu}{2}\right)\Gamma(-\nu)}{\Gamma(-\frac{1}{4}-\frac{\nu}{2})\Gamma(\frac{3}{4}-\frac{\nu}{2})}s_0^{-\frac{5}{2}-\nu}\right]+O\left(|s_0|^{-\frac{7}{2}+Re(\nu)}\right),\end{aligned}\tag{4.3}$$

$$\begin{aligned}g_2'(s_0) =& \sqrt{\pi}\left[\frac{\left(-\frac{3}{4}+\frac{\nu}{2}\right)\Gamma(\nu)}{\Gamma(\frac{1}{4}+\frac{\nu}{2})\Gamma(\frac{5}{4}+\frac{\nu}{2})}s_0^{-\frac{5}{2}+\nu}\right.\\ &\left.+\frac{\left(-\frac{3}{4}-\frac{\nu}{2}\right)\Gamma(-\nu)}{\Gamma(\frac{1}{4}-\frac{\nu}{2})\Gamma(\frac{5}{4}-\frac{\nu}{2})}s_0^{-\frac{5}{2}-\nu}\right]+O\left(|s_0|^{-\frac{7}{2}+Re(\nu)}\right).\end{aligned}\tag{4.4}$$

For $B^2 < \frac{1}{4}$ or $> \frac{1}{4}$, ν is real or pure imaginary. We treat these cases separately.

4.1.1. The Case $0 < B^2 < \frac{1}{4}$. In this case ν is a real number between 0 and $\frac{1}{2}$. By using the above asymptotics of $g_1(s), g_2(s)$, we obtain bounds for the coefficients of C_1, C_2 (defined in (3.7), (3.8)). Since

$$\frac{1}{\Delta}\left[g_2'(s_0)+\frac{2s_0}{1+s_0^2}g_2(s_0)\right] \lesssim \langle s_0\rangle^4 \langle s_0\rangle^{-\frac{5}{2}+\nu} = \langle s_0\rangle^{\frac{3}{2}+\nu},$$
$$\frac{1}{\Delta}\left[-\frac{iB^2}{1+s_0^2}g_2(s_0)\right] \lesssim \langle s_0\rangle^4 \langle s_0\rangle^{-\frac{7}{2}+\nu} = \langle s_0\rangle^{\frac{1}{2}+\nu},$$
$$\frac{1}{\Delta}\left[-g_1'(s_0)-\frac{2s_0}{1+s_0^2}g_1(s_0)\right] \lesssim \langle s_0\rangle^4 \langle s_0\rangle^{-\frac{5}{2}+\nu} = \langle s_0\rangle^{\frac{3}{2}+\nu},$$
$$\frac{1}{\Delta}\left[\frac{iB^2}{1+s_0^2}g_1(s_0)\right] \lesssim \langle s_0\rangle^4 \langle s_0\rangle^{-\frac{7}{2}+\nu} = \langle s_0\rangle^{\frac{1}{2}+\nu},$$

and

$$|g_1(s)|, |g_2(s)| \lesssim \langle s\rangle^{-\frac{3}{2}+\nu},$$

so we have

$$|C_1(k,\eta)| \lesssim \langle s_0\rangle^{\frac{3}{2}+\nu}\left(\left|\hat{\psi}^0(k,\eta)\right| + \frac{\left|\hat{T}^0(k,\eta)\right|}{\langle s_0\rangle |k|}\right),$$
$$|C_2(k,\eta)| \lesssim \langle s_0\rangle^{\frac{3}{2}+\nu}\left(\left|\hat{\psi}^0(k,\eta)\right| + \frac{\left|\hat{T}^0(k,\eta)\right|}{\langle s_0\rangle |k|}\right).$$

Therefore

$$\begin{aligned}\left|\hat{\phi}(t;k,\eta)\right| &= |C_1(k,\eta)g_1(s)+C_2(k,\eta)g_2(s)| \\ &\lesssim \langle s\rangle^{-\frac{3}{2}+\nu}\langle s_0\rangle^{\frac{3}{2}+\nu}\left(\left|\hat{\psi}^0(k,\eta)\right| + \frac{\left|\hat{T}^0(k,\eta)\right|}{\langle s_0\rangle |k|}\right).\end{aligned} \tag{4.5}$$

To get the decay estimates in the physical space (x, y) from above, we note that the term $\langle s\rangle^{-\frac{3}{2}+\nu}$ does not decay when $t \approx \frac{\eta}{k}$ (i.e. $s \approx 0$) and as compensation the additional regularity of initial data is needed to ensure the decay. This is made precise in the following lemma.

Lemma 4.1. *Assume that there exists $a > 0$ and $b, c \in \mathbb{R}$ such that*

$$|\hat{g}(t;k,\eta)| \lesssim \langle s\rangle^{-a}\langle s_0\rangle^{b}|k|^{c}\left|\hat{h}(k,\eta)\right|,\ 0 \neq k \in \mathbb{Z}, \eta \in \mathbb{R}, \tag{4.6}$$

then

$$\|P_{\neq 0}g(t)\|_{L^2(\mathbb{T}\times\mathbb{R})} \lesssim \langle t\rangle^{-a}\|h\|_{H_x^c H_y^{b+a}}.$$

Proof. We have

$$\begin{aligned}\int_{\mathbb{R}} |\hat{g}(t;k,\eta)|^2\,\mathrm{d}\eta &= \int_{|s|=\left|t-\frac{\eta}{k}\right|\geq\frac{1}{2}|t|} |\hat{g}(t;k,\eta)|^2\,\mathrm{d}\eta + \int_{\left|t-\frac{\eta}{k}\right|\leq\frac{1}{2}|t|} |\hat{g}(t;k,\eta)|^2\,\mathrm{d}\eta \\ &= I_1 + I_2.\end{aligned}$$

By (4.6), we have

$$I_1 \lesssim \langle t\rangle^{-2a}\int_{\left|t-\frac{\eta}{k}\right|\geq\frac{1}{2}|t|}\langle s_0\rangle^{2b}|k|^{2c}\left|\hat{h}(k,\eta)\right|^2\mathrm{d}\eta.$$

Since $\left|t-\frac{\eta}{k}\right| \leq \frac{1}{2}|t|$ implies $|s_0| = \left|\frac{\eta}{k}\right| \geq \frac{1}{2}|t|$, so

$$I_2 \lesssim \langle t\rangle^{-2a}\int_{\left|t-\frac{\eta}{k}\right|\leq\frac{1}{2}|t|}\langle s_0\rangle^{2b+2a}|k|^{2c}\left|\hat{h}(k,\eta)\right|^2\mathrm{d}\eta.$$

Thus

$$\int_{\mathbb{R}} |\hat{g}(t;k,\eta)|^2 \,\mathrm{d}\eta \lesssim \langle t\rangle^{-2a} \int_{\mathbb{R}} \langle s_0\rangle^{2b+2a} |k|^{2c} \left|\hat{h}(k,\eta)\right|^2 \mathrm{d}\eta,$$

and

$$\begin{aligned}
\|P_{\neq 0} g\,(t)\|^2_{L^2(\mathbb{T}\times\mathbb{R})} &= \sum_{k\neq 0} \int_{\mathbb{R}} |\hat{g}(t;k,\eta)|^2 \,\mathrm{d}\eta \\
&\lesssim \langle t\rangle^{-2a} \sum_{k\neq 0} |k|^{2c} \int_{\mathbb{R}} \langle \eta\rangle^{2b+2a} \left|\hat{h}(k,\eta)\right|^2 \mathrm{d}\eta \\
&\lesssim \langle t\rangle^{-2a} \|h\|^2_{H^c_x H^{b+a}_y}.
\end{aligned}$$

$\square$

Since the velocity perturbation

$$\begin{aligned}
v^x(t;x,y) &= -\,\partial_y \psi(t;x,y) = (-\partial_y + t\partial_z)\phi(t;z,y), \\
v^y(t;x,y) &= \partial_x \psi(t;x,y) = \partial_z \phi(t;z,y),
\end{aligned}$$

so by (4.5), we have

$$\begin{aligned}
|\hat{v}^x\,(t;k,\eta)| &= \left|iks\hat{\phi}(t;k,\eta)\right| \\
&\le \langle s\rangle^{-\frac{1}{2}+\nu} \langle s_0\rangle^{\frac{3}{2}+\nu} \left(|k| \left|\hat{\psi}^0(k,\eta)\right| + \frac{\left|\hat{T}^0(k,\eta)\right|}{\langle s_0\rangle} \right), \\
|\hat{v}^y\,(t;k,\eta)| &= \left|ik\hat{\phi}(t;k,\eta)\right| \\
&\le \langle s\rangle^{-\frac{3}{2}+\nu} \langle s_0\rangle^{\frac{3}{2}+\nu} \left(|k| \left|\hat{\psi}^0(k,\eta)\right| + \frac{\left|\hat{T}^0(k,\eta)\right|}{\langle s_0\rangle} \right).
\end{aligned}$$

From Eq. (3.9) we know

$$\begin{aligned}
|\hat{\tau}(t;k,\eta)| \le& \left|\frac{k}{B^2}\right| \left[(1+s^2)\left|C_1(k,\eta)g_1'(s) + C_2(k,\eta)g_2'(s)\right| \right. \\
&\left. +2|s|\left|C_1(k,\eta)g_1(s) + C_2(k,\eta)g_2(s)\right| \right] \\
\lesssim& \langle s\rangle^{-\frac{1}{2}+\nu} \langle s_0\rangle^{\frac{3}{2}+\nu} \left(|k| \left|\hat{\psi}^0(k,\eta)\right| + \frac{\left|\hat{T}^0(k,\eta)\right|}{\langle s_0\rangle} \right).
\end{aligned}$$

By Lemma 4.1,

$$\begin{aligned}
\|P_{\neq 0} v^x\|_{L^2} &\lesssim \langle t\rangle^{-\frac{1}{2}+\nu} \left(\|\psi^0\|_{H^1_x H^2_y} + \|T^0\|_{L^2_x H^1_y} \right), \\
\|v^y\|_{L^2} &\lesssim \langle t\rangle^{-\frac{3}{2}+\nu} \left(\|\psi^0\|_{H^1_x H^3_y} + \|T^0\|_{L^2_x H^2_y} \right),
\end{aligned}$$

and

$$\|P_{\neq 0} T(t;\cdot,\cdot)\|_{L^2} = \|P_{\neq 0}\tau(t;\cdot,\cdot)\|_{L^2} \lesssim \langle t\rangle^{-\frac{1}{2}+\nu} \left(\|\psi^0\|_{H^1_x H^2_y} + \|T^0\|_{L^2_x H^1_y} \right).$$

4.1.2. The Case $B^2 > \frac{1}{4}$. In this case, $\nu = \sqrt{\frac{1}{4} - B^2}$ is pure imaginary. Then from (4.1–4.4), we have

$$|g_1(s)| \lesssim \langle s\rangle^{-\frac{3}{2}}, \ |g_2(s)| \lesssim \langle s\rangle^{-\frac{3}{2}},$$
$$|g_1'(s_0)| \lesssim \langle s_0\rangle^{-\frac{5}{2}}, \ |g_2'(s_0)| \lesssim \langle s_0\rangle^{-\frac{5}{2}}.$$

By similar calculations,

$$\begin{aligned}
\|P_{\neq 0} v^x\|_{L^2} &\lesssim \langle t\rangle^{-\frac{1}{2}} \left(\|\psi^0\|_{H_x^1 H_y^2} + \|T^0\|_{L_x^2 H_y^1}\right), \\
\|v^y\|_{L^2} &\lesssim \langle t\rangle^{-\frac{3}{2}} \left(\|\psi^0\|_{H_x^1 H_y^3} + \|T^0\|_{L_x^2 H_y^2}\right), \\
\|P_{\neq 0} T\|_{L^2} &\lesssim \langle t\rangle^{-\frac{1}{2}} \left(\|\psi^0\|_{H_x^1 H_y^2} + \|T^0\|_{L_x^2 H_y^1}\right).
\end{aligned}$$

Since T is just ρ/A times a positive constant, this completes the proof of Theorem 1.1 (i)–(ii).

4.2. The Case $B^2 = \frac{1}{4}$

When $B^2 = \frac{1}{4}$, $\nu = 0$, the asymptotic approximations (4.1) and (4.2) no longer hold true, but the following expansions at infinity emerge instead,

$$\begin{aligned}
g_1(s) &= F\left(\frac{3}{4}, \frac{3}{4}; \frac{1}{2}; -s^2\right) \\
&= \frac{2\sqrt{\pi}}{\Gamma\left(-\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)} s^{-\frac{3}{2}} \log(s) - \frac{2\sqrt{\pi}\left(\gamma + \digamma\left(\frac{3}{4}\right) + 2\right)}{\Gamma\left(-\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)} s^{-\frac{3}{2}} + O\left(|s|^{-\frac{7}{2}}\right), \\
g_2(s) &= sF\left(\frac{5}{4}, \frac{5}{4}; \frac{3}{2}; -s^2\right) \\
&= \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)} s^{-\frac{3}{2}} \log(s) - \frac{\sqrt{\pi}\left(\gamma + \digamma\left(\frac{1}{4}\right) + 2\right)}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)} s^{-\frac{3}{2}} + O\left(|s|^{-\frac{7}{2}}\right)
\end{aligned}$$

where γ is the Euler constant, $\digamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function. It can be seen that with the logarithm function, both solutions decay a little bit slower than before.

Similarly, their derivatives also have different asymptotic approximations

$$\begin{aligned}
g_1'(s_0) &= -\frac{9}{4} s_0 F\left(\frac{7}{4}, \frac{7}{4}; \frac{3}{2}; -s_0^2\right) \\
&= -\frac{3\sqrt{\pi}}{\Gamma\left(-\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)} s_0^{-\frac{5}{2}} \log(s_0) \\
&\quad + \frac{3\sqrt{\pi}\left(\gamma + \digamma\left(\frac{3}{4}\right) + \frac{8}{3}\right)}{\Gamma\left(-\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)} s_0^{-\frac{5}{2}} + O\left(|s_0|^{-\frac{7}{2}}\right), \\
g_2'(s_0) &= F\left(\frac{5}{4}, \frac{5}{4}; \frac{3}{2}; -s_0^2\right) - \frac{25}{12} s_0^2 F\left(\frac{9}{4}, \frac{9}{4}; \frac{5}{2}; -s_0^2\right) \\
&= -\frac{3\sqrt{\pi}}{2\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)} s_0^{-\frac{5}{2}} \log(s_0) \\
&\quad + \frac{3\sqrt{\pi}\left(\gamma + \digamma\left(\frac{1}{4}\right) + \frac{8}{3}\right)}{2\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)} s_0^{-\frac{5}{2}} + O\left(|s_0|^{-\frac{7}{2}}\right).
\end{aligned}$$

Therefore, we obtain the following estimates

$$
\begin{aligned}
|g_1(s)| \lesssim \langle s\rangle^{-\frac{3}{2}} \langle \log \langle s\rangle\rangle, \ |g_2(s)| \lesssim \langle s\rangle^{-\frac{3}{2}} \langle \log \langle s\rangle\rangle,\\
|g_1'(s_0)| \lesssim \langle s_0\rangle^{-\frac{5}{2}} \langle \log \langle s_0\rangle\rangle, \ |g_2'(s_0)| \lesssim \langle s_0\rangle^{-\frac{5}{2}} \langle \log \langle s_0\rangle\rangle,
\end{aligned}
$$

and as a result

$$
\begin{aligned}
|C_1(k,\eta)| \lesssim \langle s_0\rangle^{\frac{3}{2}} \langle \log \langle s_0\rangle\rangle \left(\left|\hat{\psi}^0(k,\eta)\right| + \frac{\left|\hat{T}^0(k,\eta)\right|}{\langle s_0\rangle |k|}\right),\\
|C_2(k,\eta)| \lesssim \langle s_0\rangle^{\frac{3}{2}} \langle \log \langle s_0\rangle\rangle \left(\left|\hat{\psi}^0(k,\eta)\right| + \frac{\left|\hat{T}^0(k,\eta)\right|}{\langle s_0\rangle |k|}\right).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\left|\hat{\phi}(t;k,\eta)\right| &= |C_1(k,\eta)g_1(s) + C_2(k,\eta)g_2(s)|\\
&\lesssim \langle s\rangle^{-\frac{3}{2}} \langle s_0\rangle^{\frac{3}{2}} \langle \log \langle s\rangle\rangle \langle \log \langle s_0\rangle\rangle \left(\left|\hat{\psi}^0(k,\eta)\right| + \frac{\left|\hat{T}^0(k,\eta)\right|}{\langle s_0\rangle |k|}\right),
\end{aligned}
$$

from which the estimates of $|\hat{v}^x(t;k,\eta)|$, $|\hat{v}^y(t;k,\eta)|$ and $|\hat{\tau}(t;k,\eta)|$ follow. Then the decay rates of v^x, v^y, T can be obtained similarly as in the proof of Lemma 4.1, so we only sketch it. Notice that for any $a \geq \frac{1}{2}$, the function $h(x) = \frac{\langle x\rangle^a}{\langle \log\langle x\rangle\rangle}$ is increasing for all $x \geq 0$. When $|s| \leq \frac{1}{2}|t|$ (implying $|s_0| \geq \frac{1}{2}|t|$), we have

$$
\begin{aligned}
\langle s\rangle^{-a} \langle s_0\rangle^{\frac{3}{2}} \langle \log \langle s\rangle\rangle \langle \log \langle s_0\rangle\rangle \leq \langle s_0\rangle^{\frac{3}{2}} \langle \log \langle s_0\rangle\rangle \leq \frac{h(s_0)}{h\left(\frac{1}{2}t\right)} \langle s_0\rangle^{\frac{3}{2}} \langle \log \langle s_0\rangle\rangle\\
\lesssim \langle t\rangle^{-a} \langle \log \langle t\rangle\rangle \langle s_0\rangle^{\frac{3}{2}+a}.
\end{aligned}
$$

On the other hand, when $|s| \geq \frac{1}{2}|t|$, we have

$$
\langle s\rangle^{-a} \langle s_0\rangle^{\frac{3}{2}} \langle \log \langle s\rangle\rangle \langle \log \langle s_0\rangle\rangle \lesssim \langle t\rangle^{-a} \langle \log \langle t\rangle\rangle \langle s_0\rangle^{\frac{3}{2}+a},
$$

since $\langle \log \langle s_0\rangle\rangle \leq \langle s_0\rangle^a$. Similar to the proof of Lemma 4.1, we get

$$
\begin{aligned}
\|P_{\neq 0} v^x\|_{L^2} \lesssim \langle t\rangle^{-\frac{1}{2}} \langle \log \langle t\rangle\rangle \left(\|\psi^0\|_{H^1_x H^2_y} + \|T^0\|_{L^2_x H^1_y}\right),\\
\|v^y\|_{L^2} \lesssim \langle t\rangle^{-\frac{3}{2}} \langle \log \langle t\rangle\rangle \left(\|\psi^0\|_{H^1_x H^3_y} + \|T^0\|_{L^2_x H^2_y}\right).
\end{aligned}
$$

and

$$
\|P_{\neq 0} T\|_{L^2} \lesssim \langle t\rangle^{-\frac{1}{2}} \langle \log \langle t\rangle\rangle \left(\|\psi^0\|_{H^1_x H^2_y} + \|T^0\|_{L^2_x H^1_y}\right).
$$

4.3. The Case $B^2 = 0$

When $B^2 = 0$, that is, $\beta = 0$, then by (2.14)–(2.15), we get

$$
\begin{aligned}
(\partial_t + Ry\partial_x)\Delta\psi &= -\,\partial_x \left(\frac{\rho}{A}\right) g,\\
(\partial_t + Ry\partial_x)\left(\frac{\rho}{A}\right) &= 0.
\end{aligned}
$$

For convenience, we let $R = 1$. Again, we define

$$
\begin{aligned}
f(t;z,y) &= \omega(t;z+ty,y) = \omega(t;x,y),\\
\phi(t;z,y) &= \psi(t;z+ty,y) = \psi(t;x,y),\\
\tau(t;z,y) &= \frac{\rho}{A}(t;z+ty,y) = \frac{\rho}{A}(t;x,y).
\end{aligned}
$$

Then

$$
\partial_t f(t;z,y) = g\partial_z \tau(t;z,y),\ \ \partial_t \tau(t;z,y) = 0.
$$

So

$$
\begin{aligned}
\hat{\tau}(t;k,\eta) &= \hat{\tau}(0;k,\eta),\\
\hat{f}\,(t;k,\eta) &= \hat{f}\,(0;k,\eta) + tikg\hat{\tau}(0;k,\eta) = \hat{\omega}^0(k,\eta) + tikg\hat{\rho^0}\,(k,\eta)\,,
\end{aligned}
$$

where $\omega(0;x,y) = \omega^0(x,y)$, $\frac{\rho}{A}(0;x,y) = \rho^0\,(x,y)$. Thus by (3.1), we get

$$
\begin{aligned}
\left|\hat{\phi}(t;k,\eta)\right| &= \frac{1}{k^2\,(1+s^2)}\left|\hat{f}\,(t;\eta,k)\right|\\
&\lesssim \langle s\rangle^{-2}\langle s_0\rangle^2\left|\hat{\psi}^0(k,\eta)\right| + |t|\frac{1}{|k|}\langle s\rangle^{-2}\left|\hat{\rho}^0(k,\eta)\right|.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
|\hat{v}^x\,(t;k,\eta)| &\lesssim \langle s\rangle^{-1}\langle s_0\rangle^2|k|\left|\hat{\psi}^0(k,\eta)\right| + |t|\langle s\rangle^{-1}\left|\hat{\rho}^0(k,\eta)\right|,\\
|\hat{v}^y\,(t;k,\eta)| &\lesssim \langle s\rangle^{-2}\langle s_0\rangle^2|k|\left|\hat{\psi}^0(k,\eta)\right| + |t|\langle s\rangle^{-2}\left|\hat{\rho}^0(k,\eta)\right|.
\end{aligned}
$$

By Lemma 4.1, we get

$$
\begin{aligned}
\|P_{\neq 0}v^x\|_{L^2} &\lesssim \|\rho^0\|_{L^2_x H^1_y} + \langle t\rangle^{-1}\|\psi^0\|_{H^1_x H^3_y},\\
\|v^y\|_{L^2} &\lesssim \langle t\rangle^{-1}\|\rho^0\|_{L^2_x H^2_y} + \langle t\rangle^{-2}\|\psi^0\|_{H^1_x H^4_y}.
\end{aligned}
$$

Also, $\left\|\frac{\rho}{A}\right\|_{L^2}(t) = \left\|\rho^0\right\|$. When $\rho^0 \neq 0$, there is no decay for $\frac{\rho}{A}$ and $P_{\neq 0}v^x$. When $\rho^0 = 0$, we get

$$
\|P_{\neq 0}v^x\|_{L^2} \lesssim \langle t\rangle^{-1}\|\psi^0\|_{H^1_x H^3_y},\ \ \|v^y\|_{L^2} \lesssim \langle t\rangle^{-2}\|\psi^0\|_{H^1_x H^4_y},
$$

which exactly recovers the linear decay results in [17] for the homogeneous fluids.

Remark 4.2. For small $B > 0$, the decay rates for $\|P_{\neq 0}v^x\|_{L^2}$ and $\|v^y\|_{L^2}$ are $t^{-\frac{1}{2}+\nu}$ and $t^{-\frac{3}{2}+\nu}$ respectively even when $\rho^0 = 0$. Hence, if $B \to 0+$ (i.e. $\nu \to \frac{1}{2}-$), surprisingly the decay rates are almost one order slower than the case of homogeneous fluids ($B = 0$). This apparent gap is due to the vanishing of the coefficient of $\langle s\rangle^{-\frac{3}{2}+\nu}$ terms in the asymptotics of hypergeometric functions (4.1)–(4.4).

5. Decay Estimates for the Full Euler Equation

In this section, we prove the decay estimates in Theorem 1.2 for the linearized system of the full Euler equation. The proof is very similar to the Boussinesq case, so we only sketch it.

5.1. The Case $0 < B^2 < \infty$

For each $B^2 > 0$, we can find similar bounds for

$$\hat{\chi} = C_3(k,\eta)g_3(s) + C_4(k,\eta)g_4(s)$$

as in the Boussinesq case. For $B^2 > 0$ and $B^2 \neq \frac{1}{4}$, the asymptotics of g_3, g_4 at $s = \infty$ are

$$\begin{aligned}
g_3(s) &= \left(\frac{1}{2} - \frac{i\kappa s}{2}\right)^{-1-\beta_1} \left[\frac{\Gamma(2-\beta_1)\Gamma(-2\nu)}{\Gamma\left(\frac{3}{2}-\nu\right)\Gamma\left(\frac{1}{2}-\beta_1-\nu\right)}\left(-\frac{i\kappa s}{2}\right)^{-\frac{1}{2}+\beta_1-\nu}\right. \\
&\quad \left. + \frac{\Gamma(2-\beta_1)\Gamma(2\nu)}{\Gamma\left(\frac{3}{2}+\nu\right)\Gamma\left(\frac{1}{2}-\beta_1+\nu\right)}\left(-\frac{i\kappa s}{2}\right)^{-\frac{1}{2}+\beta_1+\nu} + O\left(|s|^{-\frac{3}{2}+Re(\nu)}\right)\right] \\
g_4(s) &= \left(\frac{1}{2} + \frac{i\kappa s}{2}\right)^{-1+\beta_1} \left[\frac{\Gamma(\beta_1)\Gamma(-2\nu)}{\Gamma\left(-\frac{1}{2}-\nu\right)\Gamma\left(\frac{1}{2}+\beta_1-\nu\right)}\left(-\frac{i\kappa s}{2}\right)^{-\frac{1}{2}-\beta_1-\nu}\right. \\
&\quad \left. + \frac{\Gamma(\beta_1)\Gamma(2\nu)}{\Gamma\left(-\frac{1}{2}+\nu\right)\Gamma\left(\frac{1}{2}+\beta_1+\nu\right)}\left(-\frac{i\kappa s}{2}\right)^{-\frac{1}{2}-\beta_1+\nu} + O\left(|s|^{-\frac{3}{2}+Re(\nu)}\right)\right] \\
g_3'(s) &= \left(\frac{i\kappa}{2}\right)\left(\frac{1}{2} - \frac{i\kappa s}{2}\right)^{-\beta_1} \left[\frac{\left(\frac{3}{2}+\nu\right)\Gamma(2-\beta_1)\Gamma(-2\nu)}{\Gamma\left(\frac{3}{2}-\nu\right)\Gamma\left(\frac{1}{2}-\beta_1-\nu\right)}\left(-\frac{i\kappa s}{2}\right)^{-\frac{5}{2}+\beta_1-\nu}\right. \\
&\quad \left. + \frac{\left(\frac{3}{2}-\nu\right)\Gamma(2-\beta_1)\Gamma(2\nu)}{\Gamma\left(\frac{3}{2}+\nu\right)\Gamma\left(\frac{1}{2}-\beta_1+\nu\right)}\left(-\frac{i\kappa s}{2}\right)^{-\frac{5}{2}+\beta_1+\nu} + O\left(|s|^{-\frac{7}{2}+Re(\nu)}\right)\right] \\
g_4'(s) &= \left(\frac{i\kappa}{2}\right)\left(\frac{1}{2} + \frac{i\kappa s}{2}\right)^{\beta_1} \left[\frac{\left(-\frac{3}{2}-\nu\right)\Gamma(2-\beta_1)\Gamma(-2\nu)}{\Gamma\left(-\frac{1}{2}-\nu\right)\Gamma\left(\frac{1}{2}+\beta_1-\nu\right)}\left(\frac{i\kappa s}{2}\right)^{-\frac{5}{2}-\beta_1-\nu}\right. \\
&\quad \left. + \frac{\left(-\frac{3}{2}+\nu\right)\Gamma(2-\beta_1)\Gamma(2\nu)}{\Gamma\left(-\frac{1}{2}+\nu\right)\Gamma\left(\frac{1}{2}+\beta_1+\nu\right)}\left(\frac{i\kappa s}{2}\right)^{-\frac{5}{2}-\beta_1+\nu} + O\left(|s|^{-\frac{7}{2}+Re(\nu)}\right)\right].
\end{aligned}$$

For $B^2 = \frac{1}{4}$, the expansions at $s = \infty$ are

$$\begin{aligned}
g_3(s) &= \left(\frac{1}{2} - \frac{i\kappa s}{2}\right)^{-\beta_1} \\
&\quad \times \left[\frac{2\Gamma(2-\beta)}{\sqrt{\pi}\Gamma\left(\frac{1}{2}-\beta\right)}\left(-\frac{i\kappa s}{2}\right)^{-\frac{3}{2}+\beta_1}\log\left(-\frac{i\kappa s}{2}\right) + O\left(|s|^{-\frac{3}{2}+\beta_1}\right)\right], \\
g_4(s) &= \left(\frac{1}{2} + \frac{i\kappa s}{2}\right)^{\beta_1} \\
&\quad \times \left[\frac{\Gamma(\beta)}{2\sqrt{\pi}\Gamma\left(\frac{1}{2}+\beta\right)}\left(-\frac{i\kappa s}{2}\right)^{-\frac{3}{2}-\beta_1}\log\left(-\frac{i\kappa s}{2}\right) + O\left(|s|^{-\frac{3}{2}-\beta_1}\right)\right], \\
g_3'(s) &= \left(\frac{i\kappa}{2}\right)\left(\frac{1}{2} - \frac{i\kappa s}{2}\right)^{-\beta_1} \\
&\quad \times \left[\frac{3\Gamma(2-\beta)}{\sqrt{\pi}\Gamma\left(\frac{1}{2}-\beta\right)}\left(-\frac{i\kappa s}{2}\right)^{-\frac{5}{2}+\beta_1}\log\left(-\frac{i\kappa s}{2}\right) + O\left(|s|^{-\frac{5}{2}+\beta_1}\right)\right],
\end{aligned}$$

$$g_4'(s) = \left(\frac{i\kappa}{2}\right)\left(\frac{1}{2}+\frac{i\kappa s}{2}\right)^{\beta_1} \times \left[\frac{3\Gamma(\beta)}{4\sqrt{\pi}\Gamma\left(\frac{1}{2}+\beta\right)}\left(-\frac{i\kappa s}{2}\right)^{-\frac{5}{2}-\beta_1}\log\left(-\frac{i\kappa s}{2}\right)+O\left(|s|^{-\frac{5}{2}-\beta_1}\right)\right].$$

Thus, we have the same bounds for $\hat{\chi}$, that is, allowdisplaybreaks

$$|\hat{\chi}(t;k,\eta)| \lesssim \langle s\rangle^{-\frac{3}{2}+\nu}\langle s_0\rangle^{\frac{3}{2}+\nu}\left(\left|\hat{\Psi}^0(k,\eta)\right| + \frac{\left|\hat{\Upsilon}^0(k,\eta)\right|}{\langle s_0\rangle |k|}\right),$$

when $0 < B^2 < \frac{1}{4}$;

$$|\hat{\chi}(t;k,\eta)| \lesssim \langle s\rangle^{-\frac{3}{2}}\langle s_0\rangle^{\frac{3}{2}}\left(\left|\hat{\Psi}^0(k,\eta)\right| + \frac{\left|\hat{\Upsilon}^0(k,\eta)\right|}{\langle s_0\rangle |k|}\right),$$

when $B^2 > \frac{1}{4}$, and

$$|\hat{\chi}(t;k,\eta)| \lesssim \langle s\rangle^{-\frac{3}{2}}\langle s_0\rangle^{\frac{3}{2}}\langle\log\langle s\rangle\rangle\langle\log\langle s_0\rangle\rangle\left(\left|\hat{\Psi}^0(k,\eta)\right| + \frac{\left|\hat{\Upsilon}^0(k,\eta)\right|}{\langle s_0\rangle |k|}\right),$$

when $B^2 = \frac{1}{4}$.

Since

$$\begin{aligned} e^{-\frac{1}{2}\beta y}v^y(t;x,y) = e^{-\frac{1}{2}\beta y}\partial_x\psi(t;x,y) = \partial_x e^{-\frac{1}{2}\beta y}\phi(t;x-ty,y) = \partial_z\chi(t;z,y), \\ e^{-\frac{1}{2}\beta y}v^x(t;x,y) &= e^{-\frac{1}{2}\beta y}\left(-\partial_y\psi(t;x,y)\right) = e^{-\frac{1}{2}\beta y}(-\partial_y + t\partial_z)\phi(t;z,y) \\ &= (-\partial_y + t\partial_z)\left(e^{-\frac{1}{2}\beta y}\phi(t;z,y)\right) - \frac{1}{2}\beta e^{-\frac{1}{2}\beta y}\phi(t;z,y) \\ &= \left(-\partial_y + t\partial_z - \frac{1}{2}\beta\right)\chi(t;x,y), \end{aligned}$$

the decay estimates for $e^{-\frac{1}{2}\beta y}v^x$ and $e^{-\frac{1}{2}\beta y}v^y$ (in Theorem 1.2 (i)–(iii)) can be proved as in the Boussinesq case. The decay of the density variation can be obtained similarly.

5.2. The Case $B^2 = 0$

When $B^2 = 0$, i.e., $\beta = 0$, the linearized equations are exactly the same as the Boussinesq case. Thus all the estimates are the same.

6. Dispersive Decay in the Absence of Shear

The shear plays a crucial role in the inviscid damping. Without a shear, the decay mechanism is totally different. When $B^2 < \infty$, the decay of $\|\boldsymbol{v}\|_{L^2}$ is due to the mixing of vorticity caused by the shear motion. When $B^2 = \infty$, $\|\boldsymbol{v}\|_{L^2}$ does not decay but we have the decay of $\|\boldsymbol{v}\|_{L^\infty}$ due to dispersive effects of the linear waves in a stably stratified fluid.

6.1. Boussinesq Case

When there is no shear, i.e. $R = 0$, $B^2 = \infty$, the equations (2.14–2.15) become

$$\partial_t \Delta\psi = -\partial_x \left(\frac{\rho}{A}\right) g, \qquad \partial_t \left(\frac{\rho}{A}\right) = \beta \partial_x \psi.$$

Denote $T = \frac{\rho}{\beta A}$, then above equations become

$$\Delta\psi_t = -\partial_x T \beta g, \tag{6.1}$$

$$\partial_t T = \partial_x \psi. \tag{6.2}$$

6.1.1. The L^2 Stability. Multiplying (6.1) by ψ and then integrating by parts with (6.2), we get the following invariant

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\beta g \iint T^2 \mathrm{d}x\mathrm{d}y + \iint |\nabla\psi|^2 \,\mathrm{d}x\mathrm{d}y\right) = 0.$$

This shows that in the L^2 norm, the perturbations of velocity and density are Liapunov stable but do not decay. However, below we show that their L^∞ norms decay due to the dispersive effects.

6.1.2. The L^∞ Decay. First, we solve (6.1)–(6.2) by Fourier transforms. Denote $N^2 = \beta g$ to be the squared Brunt-Väisälä frequency. By Fourier transform $(x, y) \to (k, \eta)$,

$$\left((i\eta)^2 + (ik)^2\right) \hat{\psi}_t = -(ik) N^2 \hat{T}, \tag{6.3}$$

$$\hat{T}_t = (ik)\hat{\psi}. \tag{6.4}$$

Combining (6.3)–(6.4), we get

$$\frac{d^2}{dt^2}\hat{\psi} = -\lambda^2 \hat{\psi},$$

where $\lambda^2(k,\eta) = \frac{k^2 N^2}{k^2+\eta^2}$. For $k \neq 0$, its solutions are

$$\hat{\psi}(t) = C_1 e^{i\lambda t} + C_2 e^{-i\lambda t}.$$

By initial conditions,

$$\hat{\psi}(0) = C_1 + C_2 = \hat{\psi}^0, \quad \hat{\psi}'(0) = i\lambda(C_1 - C_2) = \frac{i\lambda^2}{k}\hat{T}^0,$$

thus

$$C_{1,2} = \frac{1}{2}\left(\hat{\psi}^0 \pm \frac{\lambda}{k}\hat{T}^0\right).$$

By (6.3),

$$\hat{T} = -\frac{ik}{\lambda^2}\hat{\psi}_t = \frac{k}{\lambda}\left(C_1 e^{i\lambda t} - C_2 e^{-i\lambda t}\right).$$

To prove the L^∞ decay of solutions, we need two lemmas.

Lemma 6.1. (Van der Corput) *Let $h(x)$ be either convex or concave on $[a, b]$ with $-\infty \leq a < b \leq \infty$. Then*

$$\left|\int_b^a e^{ih(\eta)} \mathrm{d}\eta\right| \leq 2\left(\min_{[a,b]} |h'|\right)^{-1}, \quad \left|\int_b^a e^{ih(\eta)} \mathrm{d}\eta\right| \leq 4\left(\min_{[a,b]} |h''|\right)^{-\frac{1}{2}}. \tag{6.5}$$

Lemma 6.2. *For $\lambda(k,\eta) = \frac{|k|N}{\sqrt{k^2+\eta^2}}$ and n sufficiently large,*

$$\left|\int_{-n}^{n} e^{i(\lambda t + \eta y)} \mathrm{d}\eta\right| \lesssim |k|^{\frac{3}{2}} |Nt|^{-\frac{1}{3}} + |Nt|^{-\frac{1}{2}} |k|^{-\frac{1}{2}} n^{\frac{3}{2}}.$$

Proof. We can assume $N=1$ without loss of generality. Notice that

$$\begin{aligned}\lambda(\eta) =&\frac{1}{\sqrt{1+\left(\frac{\eta}{k}\right)^2}} = \left\langle \frac{\eta}{k}\right\rangle^{-1},\\ \lambda'(\eta) =& -\frac{\eta}{k^2}\left\langle \frac{\eta}{k}\right\rangle^{-3},\\ \lambda''(\eta) =&\frac{2\eta^2-k^2}{k^4}\left\langle \frac{\eta}{k}\right\rangle^{-5},\end{aligned}$$

and $\lambda(\eta)$ has two inflection point, $\eta_{1,2}=\pm\frac{\sqrt{2}}{2}k$. Let $n>\frac{\sqrt{2}}{2}|k|$. Choose $\epsilon>0$ so small that all the Taylor's expansion below are valid in $(\eta_i-\epsilon,\eta_i+\epsilon),\ i=1,2$. Define

$$S_1=(-n,\eta_1-\epsilon)\cup(\eta_1+\epsilon,\eta_2-\epsilon)\cup(\eta_2+\epsilon,n).$$

By (6.5), we have

$$\begin{aligned}\left|\int_{S_1} e^{i(\lambda t+\eta y)}\mathrm{d}\eta\right| &\le 4\left(\min_{[a,b]}|t||\lambda''|\right)^{-\frac{1}{2}}\\ &=4|t|^{-\frac{1}{2}}\left(\frac{2n^2-k^2}{k^4}\left\langle\frac{n}{k}\right\rangle^{-5}\right)^{-\frac{1}{2}}\\ &\lesssim |k|^{-\frac{1}{2}}|t|^{-\frac{1}{2}}n^{\frac{3}{2}},\end{aligned}$$

provided $n=n(\epsilon)$ is sufficiently large. For large t, we can divide $(\eta_1-\epsilon,\eta_1+\epsilon)=\left\{|t|^{-\frac{1}{3}}<|\eta-\eta_1|<\epsilon\right\}\cup\left\{|\eta-\eta_1|\le|t|^{-\frac{1}{3}}\right\}=S_2\cup S_3$, so that

$$\left|\int_{\eta_1-\epsilon}^{\eta_1+\epsilon} e^{i(\lambda t+\eta y)}\mathrm{d}\eta\right|\le 4|t|^{-\frac{1}{2}}\left(\min_{S_2}|\lambda''|\right)^{-\frac{1}{2}}+2|t|^{-\frac{1}{3}}.$$

For $\eta\in S_2$, we have

$$\begin{aligned}|\lambda''(\eta)| &= \frac{\left|2\eta^2-k^2\right|}{k^4}\left\langle\frac{\eta}{k}\right\rangle^{-5}\\ &=\frac{2\left|\eta-\eta_1\right|\left|\eta-\eta_2\right|}{k^4}\left\langle\frac{\eta}{k}\right\rangle^{-5}\\ &>\frac{2\left|\eta-\eta_2\right|}{k^4}\left\langle\frac{\eta}{k}\right\rangle^{-5}|t|^{-\frac{1}{3}}\\ &\gtrsim |k|^{-3}|t|^{-\frac{1}{3}}.\end{aligned}$$

Therefore

$$\left|\int_{\eta_1-\epsilon}^{\eta_1+\epsilon} e^{i(\lambda t+\eta y)}\mathrm{d}\eta\right|\lesssim 4|t|^{-\frac{1}{2}}\left(|k|^{-3}|t|^{-\frac{1}{3}}\right)^{-\frac{1}{2}}+2|t|^{-\frac{1}{3}}\lesssim |k|^{\frac{3}{2}}|t|^{-\frac{1}{3}}.$$

Applying similar estimates to $(\eta_2-\epsilon,\eta_2+\epsilon)$ will complete the proof of this lemma. □

Now we prove the L^∞ decay of the solutions of (6.1)–(6.2). By Fourier inverse transform formula,

$$\begin{aligned}P_{\neq 0}\psi(t;x,y) &= \frac{1}{2\pi}\sum_{k\neq 0}\left(e^{ikx}\int_{-\infty}^{\infty}\hat{\psi}(t)e^{i\eta y}\mathrm{d}\eta\right)\\ &=\frac{1}{2\pi}\sum_{k\neq 0}\left(e^{ikx}\int_{-\infty}^{\infty}\left(C_1(k,\eta)e^{i\lambda t}+C_2(k,\eta)e^{-i\lambda t}\right)e^{i\eta y}\mathrm{d}\eta\right),\end{aligned}$$

where

$$\begin{aligned}&\left|\int_{-\infty}^{\infty} C_1(k,\eta)e^{i\lambda t}e^{i\eta y}\mathrm{d}\eta\right|\\&\quad\le \frac{1}{2}\left|\int_{-\infty}^{\infty}\hat{\psi}^0(k,\eta)e^{i\lambda t}e^{i\eta y}\mathrm{d}\eta\right| + \frac{1}{2|k|}\left|\int_{-\infty}^{\infty}\lambda\hat{T}^0(k,\eta)e^{i\lambda t}e^{i\eta y}\mathrm{d}\eta\right|.\end{aligned}$$

Define

$$\begin{aligned}I(y) &= \int_{-n}^{n} e^{i\lambda(k,\eta)t}\hat{\psi}^0(k,\eta)e^{i\eta y}\mathrm{d}\eta\\&=\sqrt{2\pi}\left(e^{i\lambda(k,\eta)t}\chi_{[-n,n]}\hat{\psi}^0(k,\eta)\right)^{\vee}(y)\\&=\left(e^{i\lambda(k,\eta)t}\chi_{[-n,n]}\right)^{\vee} * \hat{\psi}^0(k,y),\end{aligned}$$

then

$$\begin{aligned}\|I(y)\|_{L^\infty} &\le \left\|\left(e^{i\lambda(k,\eta)t}\chi_{[-n,n]}\right)^{\vee}\right\|_{L^\infty_y}\|\hat{\psi}^0(k,\cdot)\|_{L^1_y}\\&\le \left\|\int_{-n}^{n} e^{i\lambda(k,\eta)t}e^{i\eta y}d\eta\right\|_{L^\infty_y}\|\hat{\psi}^0(k,\cdot)\|_{L^1_y}.\end{aligned}$$

Here, $^\vee$ stands for the inverse Fourier transform. By Lemma 6.2, we have

$$\begin{aligned}&\left|\int_{-\infty}^{\infty}\hat{\psi}^0(k,\eta)e^{i\lambda t}e^{i\eta y}\mathrm{d}\eta\right|\\&\quad\le \int_{|\eta|>n}\left|\hat{\psi}^0(k,\eta)\right|\mathrm{d}\eta + |I(y)|\\&\quad\lesssim \left(\int_{|\eta|>n}\langle\eta\rangle^{-2\alpha}\,\mathrm{d}\eta\right)^{\frac{1}{2}}\|\hat{\psi}^0(k,\cdot)\|_{H^\alpha_y}\\&\qquad+\left(|k|^{\frac{3}{2}}|Nt|^{-\frac{1}{3}} + |k|^{-\frac{1}{2}}|Nt|^{-\frac{1}{2}}n^{\frac{3}{2}}\right)\|\hat{\psi}^0(k,\cdot)\|_{L^1_y}\\&\quad\lesssim\left(n^{-\alpha+\frac{1}{2}} + |k|^{\frac{3}{2}}|t|^{-\frac{1}{3}} + |k|^{-\frac{1}{2}}|t|^{-\frac{1}{2}}n^{\frac{3}{2}}\right)\left(\|\hat{\psi}^0(k,\cdot)\|_{H^\alpha_y} + \|\hat{\psi}^0(k,\cdot)\|_{L^1_y}\right).\end{aligned}$$

Choose $n = |t|^{\frac{1}{2\alpha+2}}$, for $\alpha\in\left(\frac{1}{2},\frac{7}{2}\right]$, we have

$$\left|\int_{-\infty}^{\infty}\hat{\psi}^0(k,\eta)e^{i\lambda t}e^{i\eta y}\mathrm{d}\eta\right| \lesssim |k|^{\frac{3}{2}}|t|^{-\frac{2\alpha-1}{4\alpha+4}}\left(\|\hat{\psi}^0\|_{H^\alpha_y} + \|\hat{\psi}^0\|_{L^1_y}\right).$$

If the initial condition is smooth enough, then

$$\left|\int_{-\infty}^{\infty}\hat{\psi}^0(k,\eta)e^{i\lambda t}e^{i\eta y}\mathrm{d}\eta\right| \lesssim |k|^{\frac{3}{2}}|t|^{-\frac{1}{3}}\left(\|\hat{\psi}^0\|_{H^{7/2}_y} + \|\hat{\psi}^0\|_{L^1_y}\right).$$

Similarly,

$$\left|\int_{-\infty}^{\infty}\lambda\hat{T}^0(k,\eta)e^{i\lambda t}e^{i\eta y}\mathrm{d}\eta\right| \lesssim N|k|^{\frac{3}{2}}|t|^{-\frac{1}{3}}\left(\|\hat{T}^0\|_{H^{7/2}_y} + \|\hat{T}^0\|_{L^1_y}\right).$$

Therefore, we have

$$\begin{aligned}\|P_{\neq 0}\hat{\psi}(t;k,\cdot)\|_{L^\infty_y} \lesssim |t|^{-\frac{1}{3}}\Big(&|k|^{\frac{3}{2}}\|\hat{\psi}^0\|_{H^{7/2}_y} + |k|^{\frac{3}{2}}\|\hat{\psi}^0\|_{L^1_y}\\&+|k|^{\frac{1}{2}}\|\hat{T}^0\|_{H^{7/2}_y} + |k|^{\frac{1}{2}}\|\hat{T}^0\|_{L^1_y}\Big).\end{aligned}$$

Hence the decay in $L^2_xL^\infty_y$ is obtained:

$$\begin{aligned}\|P_{\neq 0}\psi\|_{L^2_xL^\infty_y} &\lesssim |t|^{-\frac{1}{3}}\Big(\|\psi^0\|_{H^{3/2}_xH^{7/2}_y}+\|\psi^0\|_{H^{3/2}_xL^1_y}\\&\quad+\|T^0\|_{H^{1/2}_xH^{7/2}_y}+\|T^0\|_{H^{1/2}_xL^1_y}\Big),\\ \|P_{\neq 0}v^x\|_{L^2_xL^\infty_y} &\lesssim |t|^{-\frac{1}{3}}\Big(\|\psi^0\|_{H^{3/2}_xH^{9/2}_y}+\|\psi^0\|_{H^{3/2}_xW^{1,1}_y}\\&\quad+\|T^0\|_{H^{1/2}_xH^{9/2}_y}+\|T^0\|_{H^{1/2}_xW^{1,1}_y}\Big),\\ \|v^y\|_{L^2_xL^\infty_y} &\lesssim |t|^{-\frac{1}{3}}\Big(\|\psi^0\|_{H^{5/2}_xH^{7/2}_y}+\|\psi^0\|_{H^{5/2}_xL^1_y}\\&\quad+\|T^0\|_{H^{3/2}_xH^{7/2}_y}+\|T^0\|_{H^{3/2}_xL^1_y}\Big).\end{aligned}$$

Similarly, for the density we have

$$\begin{aligned}\|P_{\neq 0}T\|_{L^2_xL^\infty_y} \lesssim& |t|^{-\frac{1}{3}}\Big(\|\psi^0\|_{H^{5/2}_xH^{9/2}_y}+\|\psi^0\|_{H^{5/2}_xW^{1,1}_y}\\&+\|T^0\|_{H^{3/2}_xH^{7/2}_y}+\|T^0\|_{H^{3/2}_xL^1_y}\Big).\end{aligned}$$

Below, we show that the decay rate $|t|^{-\frac{1}{3}}$ obtained above is sharp by constructing an example. Recall that the solution to (6.3)–(6.4) is

$$\hat{\psi}(t;k,\eta)=C_1e^{i\lambda t}+C_2e^{-i\lambda t}.$$

where $k\neq 0$, $\lambda^2(k,\eta)=\frac{k^2N^2}{k^2+\eta^2}$ and $C_{1,2}(k,\eta)$ are determined by $\hat{\psi}^0,\hat{T}^0$. Therefore, for a fixed k, we consider a function of the form

$$\hat{\psi}(t;k,\eta)=f(\eta)e^{i\lambda t},$$

where $f(\eta)$ is to be chosen below. By the Fourier inverse formula

$$\psi(t;k,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(\eta)e^{i\lambda t+i\eta y}\mathrm{d}\eta.$$

We will look at the value of ψ at $y=ct$ where c is a constant to be determined later. Define

$$g(\eta):=\lambda(\eta)+c\eta=\frac{kN}{\sqrt{k^2+\eta^2}}+c\eta.$$

We note that $\eta^*=\frac{k}{\sqrt{2}}$ is one inflection point of $\lambda(\eta)$ (the other one is $-\frac{k}{\sqrt{2}}$). Let $c=\frac{2N}{3\sqrt{3}k}$, then $g''(\eta^*)=\lambda''(\eta^*)=0$,

$$g'(\eta^*)=-\frac{\eta^*N}{k^2}\left\langle\frac{\eta^*}{k}\right\rangle^{-3}+c=-\frac{2N}{3\sqrt{3}k}+c=0,$$

and

$$\begin{aligned}g'''(\eta^*)&=-\frac{N}{k^3}\left\langle\frac{\eta^*}{k}\right\rangle^{-7}\left(-9\frac{\eta^*}{k}+6\left(\frac{\eta^*}{k}\right)^3\right)\\&=\frac{N}{k^3}\frac{16}{27}\sqrt{3}>0\end{aligned}$$

Thus near η^*, we have

$$g(\eta)=g(\eta^*)+\frac{1}{6}g'''(\eta^*)(\eta-\eta^*)^3+o\left((\eta-\eta^*)^3\right), \tag{6.6}$$

and

$$g'(\eta)=\frac{1}{2}g'''(\eta^*)(\eta-\eta^*)^2+o\left((\eta-\eta^*)^2\right). \tag{6.7}$$

Choose $\delta > 0$ small such that (6.6) and (6.7) hold true in $I = (\eta^* - \delta, \eta^* + \delta)$. In particular, $g'(\eta) > 0$ when $\eta \in I$ and $\eta \neq \eta^*$, thus $g(\eta)$ is monotone in I. For a function f with its support in I, letting $u = g(\eta)$ we have

$$\begin{aligned}\hat{\psi}(t; k, ct) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\eta) e^{i\lambda t + i\eta ct} \mathrm{d}\eta \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(g^{-1}(u)) e^{iut} \frac{1}{g'(g^{-1}(u))} \mathrm{d}u.\end{aligned}$$

In the above, $\frac{1}{g'(g^{-1}(u))}$ has singularity at $u^* = g(\eta^*) = \frac{4\sqrt{2}}{3\sqrt{3}} N$. Since

$$u = g(\eta) = u^* + O(\eta - \eta^*)^3, \ \eta \in I,$$

so the order of singularity is

$$\frac{1}{g'(g^{-1}(u))} = O\left(\frac{1}{|\eta - \eta^*|^2}\right) = O\left(\frac{1}{|u - u^*|^{\frac{2}{3}}}\right). \tag{6.8}$$

Choose

$$f(\eta) = \frac{g'(\eta)}{|g(\eta) - u^*|^{\frac{2}{3}}} \chi_I(\eta) = \frac{g'\left(g^{-1}(u)\right)}{|u - u^*|^{\frac{2}{3}}} \chi_I(\eta),$$

which by (6.8) is smooth in its support I. Hence the inverse Fourier transform of f is smooth, and has finite H_y^s norm for arbitrarily $s > 0$. By (6.6),

$$a_- = g(\eta^* - \delta) - g(\eta^*) < 0, \ a_+ = g(\eta^* + \delta) - g(\eta^*) > 0.$$

Therefore, we have

$$\begin{aligned}\hat{\psi}(t; k, ct) &= \frac{1}{2\pi} \int_{g(\eta^* - \delta)}^{g(\eta^* + \delta)} \frac{1}{|u - u^*|^{\frac{2}{3}}} e^{iut} \mathrm{d}u \\ &= \frac{e^{iu^* t}}{2\pi} \int_{a_-}^{a_+} \xi^{-\frac{2}{3}} e^{i\xi t} \mathrm{d}\xi \\ &= \frac{e^{iu^* t}}{2\pi t^{\frac{1}{3}}} \int_{a_- t}^{a_+ t} \xi'^{-\frac{2}{3}} e^{i\xi'} \mathrm{d}\xi',\end{aligned}$$

while

$$\lim_{t \to +\infty} \int_{a_- t}^{a_+ t} \xi'^{-\frac{2}{3}} e^{i\xi'} \mathrm{d}\xi' = \int_{-\infty}^{\infty} x^{-\frac{2}{3}} e^{ix} \mathrm{d}x = \sqrt{3}\Gamma\left(\frac{1}{3}\right).$$

Therefore, $\|\hat{\psi}(t; k, \cdot)\|_{L_y^\infty}$ cannot decay faster than $t^{-\frac{1}{3}}$.

Remark 6.3. The optimal $t^{-\frac{1}{3}}$ decay obatined above for $(x, y) \in \mathbb{T} \times \mathbb{R}$ is essentially for the one dimensional case (in y). By contrast, in [12] the dispersive decay of solutions of (6.1)–(6.2) was shown to be $t^{-\frac{1}{2}}$ for the 2D case, i.e., $(x, y) \in \mathbb{R}^2$. The decay rate in [12] was obtained by the Littlewood-Paley decomposition and stationary phase lemma.

6.2. Original Euler Equation

When there is no shear, i.e. $R = 0$, the original Euler equations (2.7–2.8) become

$$\begin{aligned}-\beta \partial_t \partial_y \psi + \partial_t \Delta \psi &= -\,\partial_x \left(\frac{\rho}{\rho_0}\right) g, \\ \partial_t \left(\frac{\rho}{\rho_0}\right) &= \beta \partial_x \psi.\end{aligned}$$

Likewise, define $T = \frac{\rho}{\beta \rho_0(y)}$, then the equations read

$$(-\beta \partial_y + \Delta)\psi_t = -\partial_x T \beta g, \tag{6.9}$$
$$\partial_t T = \partial_x \psi. \tag{6.10}$$

Let $\Psi = e^{-\frac{1}{2}\beta y}\psi, \Upsilon = e^{-\frac{1}{2}\beta y} T$, then the Eqs. (6.9)–(6.10) become

$$\left(-\frac{1}{4}\beta^2 + \Delta\right)\Psi_t = -N^2 \partial_x \Upsilon, \quad \partial_t \Upsilon = \partial_x \Psi. \tag{6.11}$$

By the Fourier transform $(x, y) \to (k, \eta)$, we have

$$\left(-\frac{1}{4}\beta^2 + (i\eta)^2 + (ik)^2\right)\hat{\Psi}_t = -(ik)N^2 \hat{\Upsilon}, \quad \hat{\Upsilon}_t = (ik)\hat{\Psi}.$$

Therefore,

$$\frac{d^2}{dt^2}\hat{\Psi} = -\lambda^2 \hat{\Psi},$$

where

$$\lambda^2 = \frac{k^2 N^2}{k^2 + \eta^2 + \frac{\beta^2}{4}}.$$

Its solutions are

$$\hat{\Psi}(t) = C_1 e^{i\lambda t} + C_2 e^{-i\lambda t},$$

where

$$C_{1,2} = \frac{1}{2}\left(\hat{\Psi}^0 \pm \frac{\lambda}{k}\hat{\Upsilon}^0\right).$$

Similar to the Boussinesq case, we have the following conservation law for (6.11)

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\iint \left(\frac{1}{4}\beta^2 |\Psi|^2 + |\nabla \Psi|^2 + N^2 |\Upsilon|^2\right) \mathrm{d}x\mathrm{d}y\right).$$

By integration by parts,

$$\begin{aligned}
&\iint \left(\frac{1}{4}\beta^2 |\Psi|^2 + |\nabla \Psi|^2 + N^2 |\Upsilon|^2\right) \mathrm{d}x\mathrm{d}y \\
&\quad = \left\| e^{-\frac{1}{2}\beta y} v^x \right\|_{L^2}^2 + \left\| e^{-\frac{1}{2}\beta y} v^y \right\|_{L^2}^2 + \frac{g}{\beta}\left\| e^{-\frac{1}{2}\beta y} \frac{\rho}{\rho_0} \right\|_{L^2}^2.
\end{aligned}$$

This shows that there is no decay in the L^2 norm for $e^{-\frac{1}{2}\beta y}\boldsymbol{v}$ and $e^{-\frac{1}{2}\beta y}\frac{\rho}{\rho_0}$. For the L^∞ decay, notice that

$$\lambda^2 = \frac{k^2 N^2}{k^2 + \eta^2 + \frac{\beta^2}{4}} = \frac{m^2 (\kappa N)^2}{m^2 + \eta^2}.$$

where $m = \sqrt{\frac{1}{4}\beta^2 + k^2}, \kappa = \frac{k}{m}$. By Lemma 6.2 we have

$$\begin{aligned}
\left| \int_{-n}^{n} e^{i(\lambda t + \eta y)} \mathrm{d}\eta \right| &\lesssim |m|^{\frac{3}{2}} |\kappa N t|^{-\frac{1}{3}} + |\kappa N t|^{-\frac{1}{2}} |m|^{-\frac{1}{2}} n^{\frac{3}{2}} \\
&\simeq |k|^{\frac{3}{2}} |t|^{-\frac{1}{3}} + |t|^{-\frac{1}{2}} |k|^{-\frac{1}{2}} n^{\frac{3}{2}},
\end{aligned}$$

since $\kappa \simeq 1, m \simeq k$. Accordingly, we have

$$\begin{aligned}
\|e^{-\frac{1}{2}\beta y}P_{\neq 0}\psi\|_{L^2_x L^\infty_y} \lesssim |t|^{-\frac{1}{3}} \Big(& \|\Psi^0\|_{H_x^{3/2}H_y^{7/2}} + \|\Psi^0\|_{H_x^{3/2}L_y^1} \\
& + \|\Upsilon^0\|_{H_x^{1/2}H_y^{7/2}} + \|\Upsilon^0\|_{H_x^{1/2}L_y^1}\Big), \\
\|e^{-\frac{1}{2}\beta y}P_{\neq 0}v^x\|_{L^2_x L^\infty_y} \lesssim |t|^{-\frac{1}{3}} \Big(& \|\Psi^0\|_{H_x^{3/2}H_y^{9/2}} + \|\Psi^0\|_{H_x^{3/2}W_y^{1,1}} \\
& + \|\Upsilon^0\|_{H_x^{1/2}H_y^{9/2}} + \|\Upsilon^0\|_{H_x^{1/2}W_y^{1,1}}\Big), \\
\|e^{-\frac{1}{2}\beta y}v^y\|_{L^2_x L^\infty_y} \lesssim |t|^{-\frac{1}{3}} \Big(& \|\Psi^0\|_{H_x^{5/2}H_y^{7/2}} + \|\Psi^0\|_{H_x^{5/2}L_y^1} \\
& + \|\Upsilon^0\|_{H_x^{3/2}H_y^{7/2}} + \|\Upsilon^0\|_{H_x^{3/2}L_y^1}\Big), \\
\|e^{-\frac{1}{2}\beta y}P_{\neq 0}T\|_{L^2_x L^\infty_y} \lesssim |t|^{-\frac{1}{3}} \Big(& \|\Psi^0\|_{H_x^{5/2}H_y^{9/2}} + \|\Psi^0\|_{H_x^{5/2}W_y^{1,1}} \\
& + \|\Upsilon^0\|_{H_x^{3/2}H_y^{7/2}} + \|\Upsilon^0\|_{H_x^{3/2}L_y^1}\Big).
\end{aligned}$$

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Compliance with ethical standards
Conflict of interest The authors declare that they have no conflict of interest.

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