Vorticity estimates for the 3D incompressible Navier–Stokes equation

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Consider the 3D Navier–Stokes equation

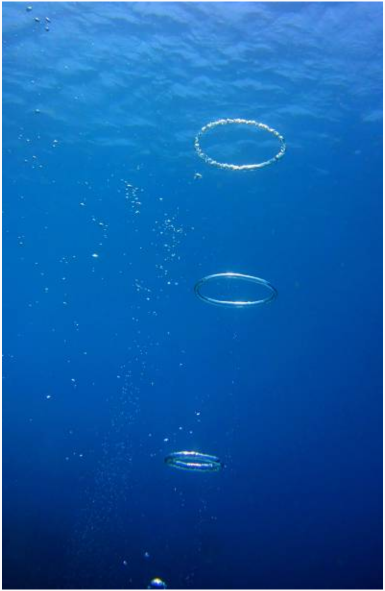
$$\begin{aligned} &\partial_t u + u \cdot \nabla u + \nabla p = \Delta u, \quad \operatorname{div} u = 0 && \text{in } (0,T) \times \Omega \\ &u = 0 && \text{on } (0,T) \times \partial\Omega \\ &u\big|_{t=0} = u_0 && \text{in } \Omega \end{aligned}$$

The vorticity $\omega = \operatorname{curl} u$ is governed by the vorticity equation

$$\begin{aligned} &\partial_t \omega + u \cdot \nabla \omega - \omega \cdot \nabla u = \Delta \omega, \quad \operatorname{div} \omega = 0 && \text{in } (0,T) \times \Omega \\ &u = \operatorname{curl}^{-1} \omega && \text{Biot–Savart law} \end{aligned}$$

We want to study a priori estimates of u, ω and higher derivatives, and trace on a Lipschitz submanifold $\Gamma \subset \Omega$.

Introduction



Leray (1934) and Hopf (1951) established the global existence of weak solutions: for divergence-free $u_0 \in L^2(\Omega)$, there exists

$$u \in C_w(0,T;L^2(\Omega)) \cap L^2(0,T;\dot{H}^1_0(\Omega))$$

with energy inequality

$$\frac{1}{2}\|u\|^2_{L^2(\Omega)}(t) + \int_0^t \|\nabla u\|^2_{L^2(\Omega)}(s)\,\mathrm{d}s \le \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}$$

Suitable weak solution

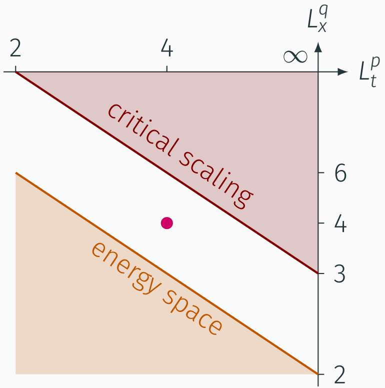
Scheffer (1976), Caffarelli–Kohn–Nirenberg (1982): existence of suitable weak solutions:

$$\partial_t \frac{|u|^2}{2} + \operatorname{div}\left(u\left(\frac{|u|^2}{2} + p\right)\right) + |\nabla u|^2 \leq \Delta \frac{|u|^2}{2}$$

Albritton–Brué–Colombo (2021): nonunique forced Leray solutions

When is a weak solution regular and unique?

Solutions in anisotropic Lebesgue space $u \in L^p(0, T; L^q(\Omega))$



Scaling: $u_\varepsilon(t, x) = \varepsilon u(\varepsilon^2 t, \varepsilon x)$

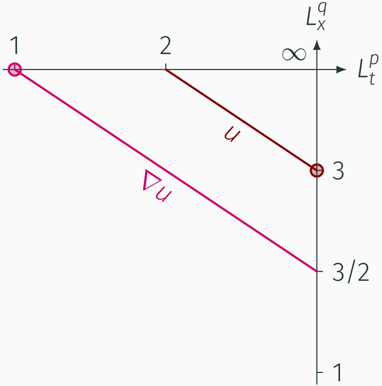
- below energy space: existence of suitable weak solution (Scheffer 1976, CKN 1982)

$$2/p + 3/q = 3/2$$

- above critical scaling: uniqueness and full regularity (Ladyzhenskaya–Prodi–Serrin 1960s)

$$2/p + 3/q = 1$$

- energy conservation: $L^4_t L^4_x$

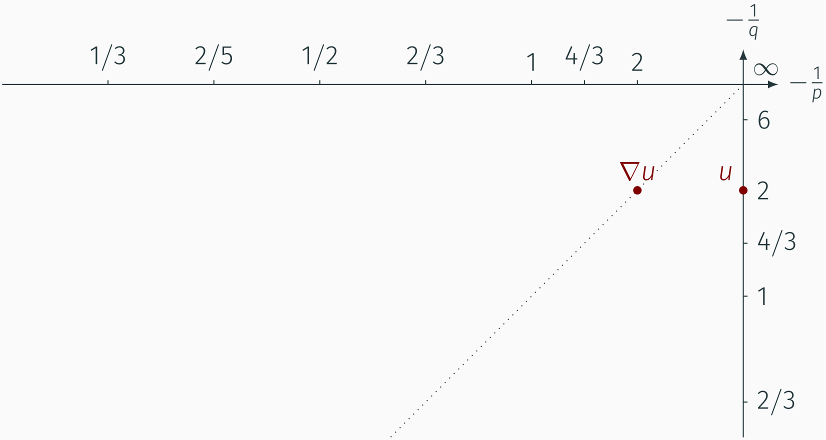


- Ladyzhenskaya–Prodi–Serrin: $u \in L^p_t L^q_x$ with

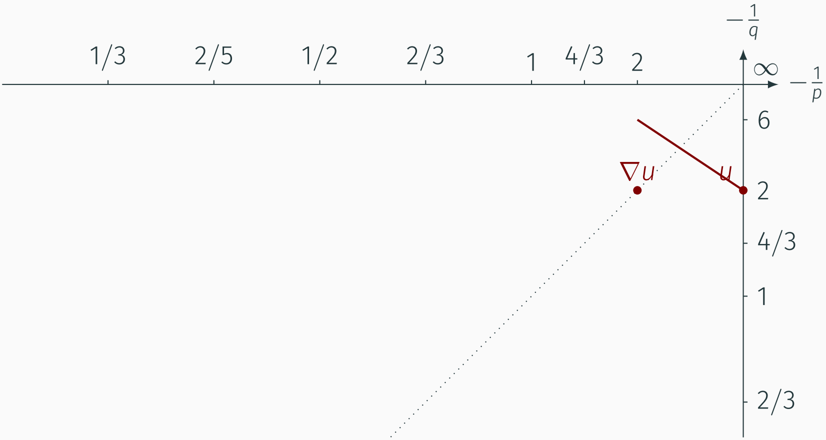
$$2/p + 3/q = 1$$

- Escauriaza–Serëgin–Šverák: $u \in L^\infty_t L^3_x$
- Beal–Kato–Majda: $\omega \in L^1_t L^\infty_x$
- Veiga: $\nabla u \in L^p_t L^q_x$ with

$$2/p + 3/q = 2$$

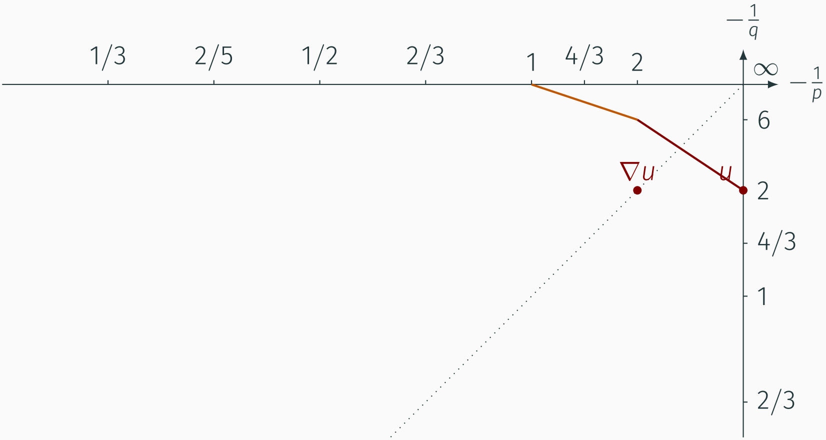


- Leray (1934), Hopf (1951): $u \in L_t^\infty L_x^2$, $\nabla u \in L_{t,x}^2$



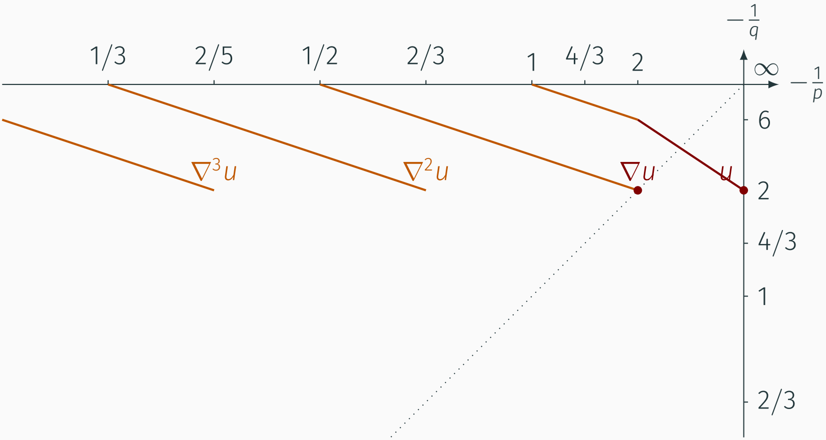
- Leray (1934), Hopf (1951): $u \in L_t^\infty L_x^2$, $\nabla u \in L_{t,x}^2$, $u \in L_t^2 L_x^6$

A priori estimates

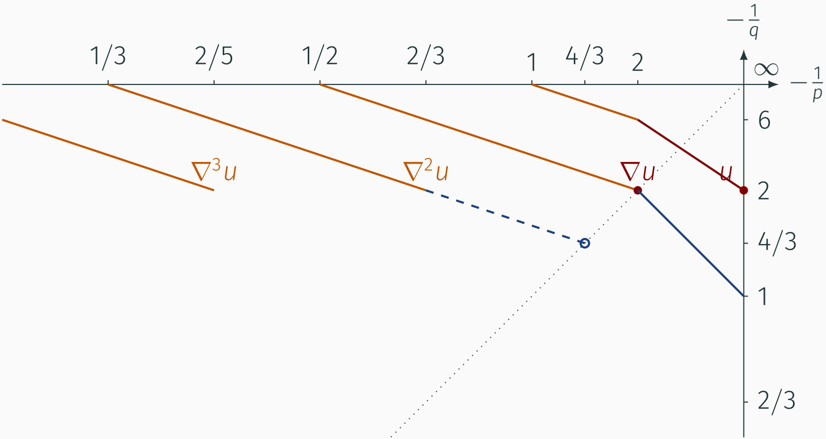


- Foiaş–Guillopé–Temam (1981), Duff (1990): $u \in L^1_t L^\infty_x$

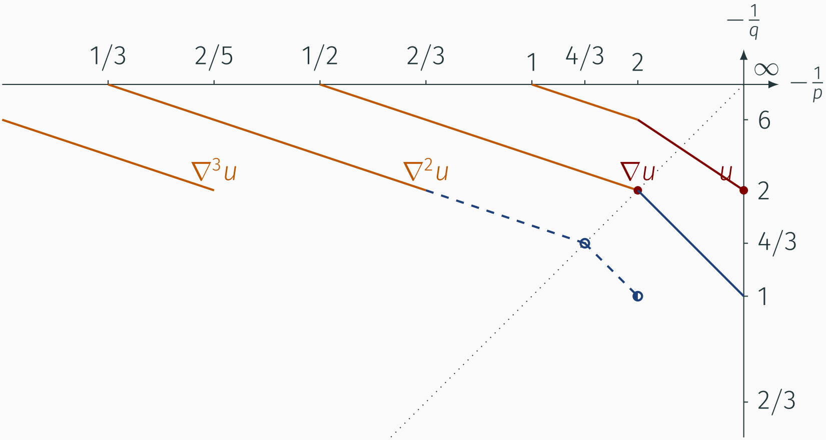
A priori estimates



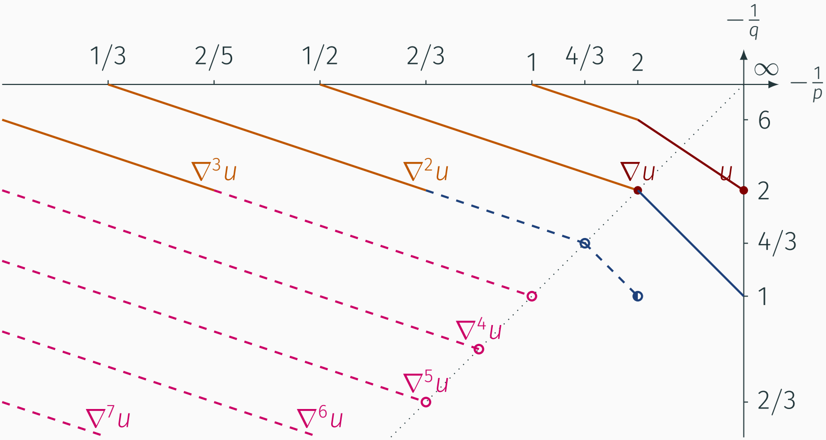
- Foiaş–Guillopé–Temam (1981), Duff (1990): $\nabla^n u \in L_t^{\frac{1}{n+1}} L_x^\infty \cap L_t^{\frac{2}{2n-1}} L_x^2$



- Constantin (1990), Lions (1996): $\nabla u \in L^\infty_t L^1_x$, $\nabla^2 u \in L^{\frac{4}{3},\infty}_{t,x}$

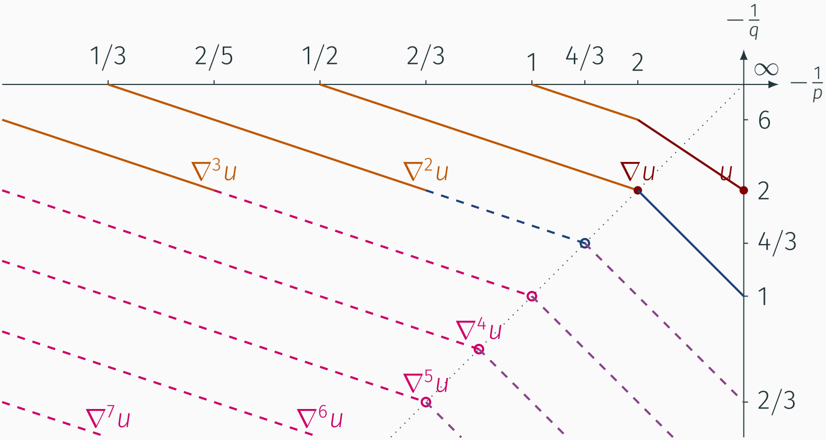


- Constantin (1990), Lions (1996): $\nabla u \in L_t^\infty L_x^1$, $\nabla^2 u \in L_{t,x}^{\frac{4}{3},\infty} \cap L_t^{2^-} L_x^1$



- Vasseur (2010), Choi–Vasseur (2014): $\nabla^n u \in L_t^{\frac{4}{n+1},\infty} L_x^{\frac{4}{n+1},\infty}$
- Vasseur–Y. (2021): $\nabla^n \omega \in L_{t,x}^{\frac{4}{n+2},\frac{4}{n+2}+}$

A priori estimates



- Y. (2023): $\nabla^n u \in L_t^\infty L_x^{\frac{2}{n+1},\infty}$

Theorem (Y., 2023)

Let $T \in (0, \infty]$, $\Omega \subset \mathbb{R}^3$ be Lipschitz, and $\Gamma \subset \Omega$ be d-dimensional Lipschitz graph. There exists $C > 0$ depending on Lipschitzness and universal constants $c_n > 0$ such that the following is true for

$$\mathscr{W}(t,x) := \sum_{n=0}^{\infty} c_n |\nabla^n \omega(t,x)|^{\frac{1}{n+2}}$$

(a) *For any $0 \le d \le 3$, it holds that*

$$\left\| \mathscr{W} \mathbf{1}_{\{\mathscr{W} > r_*^{-1}\}} \right\|_{L^{d+1,\infty}((0,T)\times\Gamma)}^{d+1} \le C \|\nabla u\|_{L^2((0,T)\times\Omega)}^2$$

(b) *If $2 \le d \le 3$ then for every $t \in (0, T)$ it holds that*

$$\left\| \mathscr{W}(t) \mathbf{1}_{\{\mathscr{W}(t) > r_*^{-1}(t)\}} \right\|_{L^{d-1,\infty}(\Gamma)}^{d-1} \le C \|\nabla u\|_{L^2((0,T)\times\Omega)}^2$$

r_* is the parabolic distance to the parabolic boundary $\partial_{\mathcal{P}}((0,T) \times \Omega)$

The proof relies on the blow-up method and ε-regularity.

Theorem (Vasseur–Y., 2021)

If $\int_{-1}^{0}\int_{B_1}|\nabla u|^2\,dx\,dt \le \eta$ *and* $⨍_{B_1} u(t)\,dx = 0$ *then* $|\nabla^n\omega(0,0)| \le C_n$.

Navier–Stokes is invariant under the scaling $u_\varepsilon(t,x) = \varepsilon u(\varepsilon^2 t, \varepsilon x)$, so

Corollary

If $⨍_{Q_\varepsilon(t,x)}|\nabla u|^2\,dx\,dt \le \eta\varepsilon^{-4}$ *then* $|\nabla^n\omega(t,x)| \le C_n\varepsilon^{-n-2}$.

If we choose a sufficiently small scale $\varepsilon(t,x)$, the above holds, then $\mathscr{W}(t,x) = \sum_{n=0}^{\infty} 2^{-n}|\nabla^n\omega(t,x)/C_n|^{\frac{1}{n+2}} \le \varepsilon(t,x)^{-1}$ **pointwise**.

Question: how small should $\varepsilon(t,x)$ be?

Theorem (A generic theorem of the blow-up method)

$f = |\nabla u|^2$ is a "pivot quantity", $g = |\nabla^n \omega|$ is a "controlled quantity"

$$\fint_{Q_\rho(t,x)} f \le \rho^{-\alpha} \qquad \Longrightarrow \qquad g(t,x) \le \rho^{-\beta}$$

For $f \in L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}^D)$, $\alpha > 0$, define **scale operator** $\mathscr{S}_\alpha(f)(t,x) \in [0,\infty]$:

$$\mathscr{S}_\alpha(f)(t,x) := \inf_{0<\rho<\infty} \left\{ \rho : \fint_{Q_\rho(t,x)} f > \rho^{-\alpha} \right\}$$

It selects the *largest scale* below which ε-regularity is applicable.

Average operator $\mathscr{A}_\alpha(f)(t,x) = \fint_{Q_{\mathscr{S}_\alpha(f)(t,x)}(t,x)} f = \mathscr{S}_\alpha(f)(t,x)^{-\alpha}$

Then $g \le \mathscr{A}_\alpha(f)^{\frac{\beta}{\alpha}}$ **pointwise**.

Theorem

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^D)$, $\Gamma \subset \mathbb{R}^D$, $\dim(\Gamma) = d$. *Suppose* $\alpha > D - d$.

1. *If* $f \in L^1(\mathbb{R} \times \mathbb{R}^D)$, *then*

$$\left\| (\mathscr{A}_\alpha f)^{1-\frac{D-d}{\alpha}} \right\|_{L^{1,\infty}_{t,x}(\mathbb{R}\times\Gamma)} \leq C(\alpha, D, d, L)\|f\|_{L^1(\mathbb{R}\times\mathbb{R}^D)}$$

2. *If* $f \in L^p(\mathbb{R} \times \mathbb{R}^D)$ *for some* $p \in (1, \infty]$, *then*

$$\left\| (\mathscr{A}_\alpha f)^{1-\frac{D-d}{p\alpha}} \right\|_{L^p_{t,x}(\mathbb{R}\times\Gamma)} \leq C(p, \alpha, D, d, L)\|f\|_{L^p(\mathbb{R}\times\mathbb{R}^D)}$$

Theorem

Let $f \in L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}^D)$, $\Gamma \subset \mathbb{R}^D$, $\dim(\Gamma) = d$. *Suppose* $\alpha > D - d + 2$.

3. *If* $f \in L^1(\mathbb{R} \times \mathbb{R}^D)$, *then for every* $t \in \mathbb{R}$,

$$\left\| [\mathscr{A}_\alpha(f)(t)]^{1 - \frac{D-d+2}{\alpha}} \right\|_{L^{1,\infty}(\Gamma)} \leq C(\alpha, D, d, L) \|f\|_{L^1(\mathbb{R} \times \mathbb{R}^D)}$$

4. *If* $f \in L^p(\mathbb{R} \times \mathbb{R}^D)$ *for some* $p \in (1, \infty]$, *then for every* $t \in \mathbb{R}$,

$$\left\| [\mathscr{A}_\alpha(f)(t)]^{1 - \frac{D-d+2}{p\alpha}} \right\|_{L^p(\Gamma)} \leq C(p, \alpha, D, d, L) \|f\|_{L^p(\mathbb{R} \times \mathbb{R}^D)}$$

Proposition

Let $0 < p_1, q_1, p_2, q_2 < \infty$, $1 \le q_1 < p_1$, $\alpha q_2 > d$. *Define*

$$\frac{1}{r_1} = \alpha - \frac{2}{p_1} - \frac{D}{q_1}, \qquad \frac{1}{r_2} = \alpha - \frac{2}{p_2} - \frac{d}{q_2}.$$

Let $f \in L_t^{p_1} L_x^{q_1}(\mathbb{R} \times \mathbb{R}^D)$. *If* $0 < \frac{r_2}{r_1} < \frac{p_2}{p_1} \wedge \frac{q_2}{q_1}$, *then*

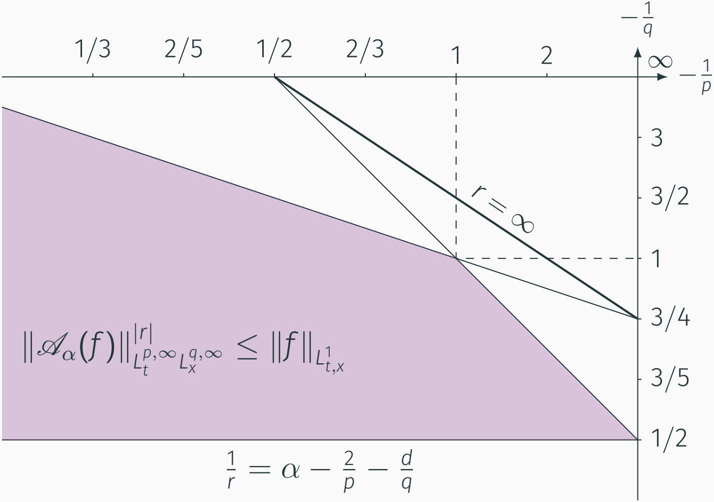
$$\|\mathscr{A}_\alpha(f)\|^{r_2/r_1}_{L_t^{p_2} L_x^{q_2}(\mathbb{R}\times\Gamma)} \le C \|f\|_{L_t^{p_1} L_x^{q_1}(\mathbb{R}\times\mathbb{R}^D)}.$$

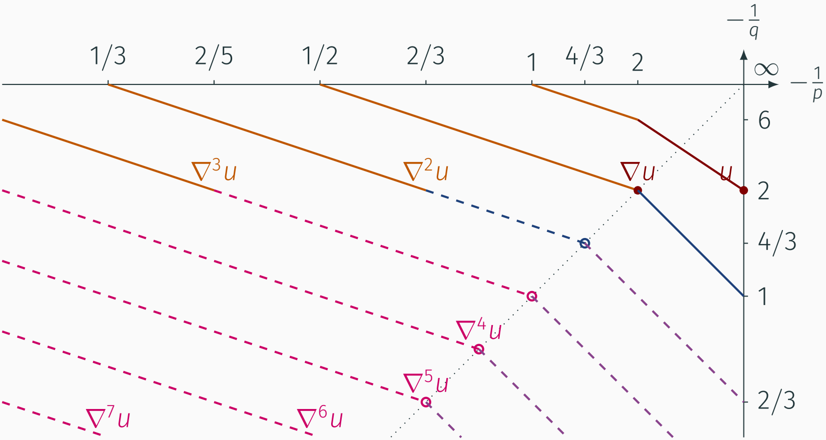
If $r_1 = r_2 = \infty$, $0 < \delta < \frac{p_2}{p_1} \wedge \frac{q_2}{q_1}$, *then*

$$\|\mathscr{A}_\alpha(f)\|^{\delta}_{L_t^{p_2} L_x^{q_2}(\mathbb{R}\times\Gamma)} \le C \|f\|_{L_t^{p_1} L_x^{q_1}(\mathbb{R}\times\mathbb{R}^D)}.$$

Proof of the anisotropic estimate

$D = d = 3$, $\Gamma = \Omega$, $\alpha = 4$, $f = |\nabla u|^2$, $g = |\nabla^n \omega|^{\frac{4}{n+2}} \leq \mathscr{A}_\alpha(f)$.




$-\frac{1}{q}$
1/3
2/5
1/2
2/3
1
4/3
2
∞
$-\frac{1}{p}$
6
2
4/3
1
2/3
$\nabla^3 u$
$\nabla^2 u$
∇u
u
$\nabla^4 u$
$\nabla^5 u$
$\nabla^7 u$
$\nabla^6 u$

Remarks

Some minor issues:

1. blow-up argument misses a remainder term depending on the distance from the parabolic boundary
2. local theorem requires zero mean velocity condition
3. ω can be replaced by ∇u when $\Omega = \mathbb{R}^3$ or $\mathbb{T}^3$
4. Γ can change in time: $\Gamma_t \subset \Omega$
5. a priori estimate also works for suitable weak solutions over the regular set

Thank you for your listening!

https://arxiv.org/abs/2308.09350