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Barotropic instability of shear flows

Zhiwu Lin[1] | **Jincheng Yang**[2] | **Hao Zhu**[3,4]

[1]School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia

[2]Department of Mathematics, The University of Texas at Austin, Austin, Texas

[3]Department of Mathematics, Nanjing University, Nanjing, P. R. China

[4]Chern Institute of Mathematics, Nankai University, Tianjin, P. R. China

Correspondence
Hao Zhu, Department of Mathematics, Nanjing University, 210093, Nanjing, P. R. China; and Chern Institute of Mathematics, Nankai University, 300071, Tianjin, P. R. China.
Email: haozhu@nankai.edu.cn

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Abstract

We consider barotropic instability of shear flows for incompressible fluids with Coriolis effects. For a class of shear flows, we develop a new method to find the sharp stability conditions. We study the flow with Sinus profile in details and obtain the sharp stability boundary in the whole parameter space, which corrects previous results in the fluid literature. Our new results are confirmed by more accurate numerical computation. The addition of the Coriolis force is found to bring fundamental changes to the stability of shear flows. Moreover, we study dynamical behaviors near the shear flows, including the bifurcation of nontrivial traveling wave solutions and the linear inviscid damping. The first ingredient of our proof is a careful classification of the neutral modes. The second one is to write the linearized fluid equation in a Hamiltonian form and then use an instability index theory for general Hamiltonian partial differential equations. The last one is to study the singular and non-resonant neutral modes using Sturm-Liouville theory and hypergeometric functions.

KEYWORDS

barotropic instability, fluid dynamics, Hamiltonian structure, shear flow

1 | INTRODUCTION

When studying the large-scale motion of ocean and atmosphere, the rotation of the earth may affect the dynamics of the fluids significantly, and therefore, Coriolis effects must be taken into account.[1] In this paper, we study stability and instability of shear flows under Coriolis forces. We consider the fluids in a strip or channel denoted by

$$D = \{(x, y) \mid y \in [y_1, y_2]\},$$

where x is periodic. The fluid motion is modeled by the two-dimensional inviscid incompressible Euler equation with rotation

$$\partial_t \vec{u} + (\vec{u} \cdot \nabla)\vec{u} = -\nabla P - \beta y J \vec{u} \qquad t \in [0, +\infty), (x, y) \in D, \tag{1}$$

where $\vec{u} = (u_1, u_2)$ is the fluid velocity, P is the pressure,

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is the rotation matrix, and β is the Rossby number. Here, the term $-\beta y J \vec{u}$ denotes the Coriolis force under the beta-plane approximation. We assume the incompressible condition $\nabla \cdot \vec{u} = 0$ and the non-permeable boundary condition

$$u_2 = 0 \qquad \text{on } \partial D = \{y = y_1, y_2\}. \tag{2}$$

The vorticity ω is defined as $\omega := \operatorname{curl} \vec{u} = \partial_x u_2 - \partial_y u_1$, and the stream function ψ is introduced such that $\vec{u} = \nabla^{\perp}\psi = (\partial_y \psi, -\partial_x \psi)$. The vorticity form of (1) is

$$\partial_t \omega + (\vec{u} \cdot \nabla)\omega + \beta u_2 = 0, \tag{3}$$

which is also called the quasi-geostrophic equation in geophysical fluids.[1] Consider a shear flow $\vec{u}_0 = (U(y), 0)$, $U \in C^2([y_1, y_2])$, which is a steady solution of (3). The linearized equation of (3) around the shear flow $\vec{u}_0$ is

$$\partial_t \omega + U \partial_x \omega - (\beta - U'')\partial_x \psi = 0. \tag{4}$$

To study the linear instability, it suffices to consider the normal mode solution $\psi(x, y, t) = \phi(y)e^{i\alpha(x-ct)}$, where $\alpha > 0$ is the wave number in the x-direction and $c = c_r + ic_i$ is the complex wave speed. Then (4) is reduced to the Rayleigh-Kuo equation

$$-\phi'' + \alpha^2 \phi - \frac{\beta - U''}{U - c}\phi = 0, \tag{5}$$

with the boundary conditions

$$\phi(y_1) = \phi(y_2) = 0. \tag{6}$$

When $\beta = 0$, (5) becomes the classical Rayleigh equation,[2] which has been studied extensively (cf. Refs. 3–8).

The shear flow U is linear unstable if there exists a nontrivial solution to (5) and (6) with $\operatorname{Im} c > 0$. This so-called barotropic instability is important for the dynamics of atmosphere and oceans. It has been a classical problem in geophysical fluid dynamics[1,9,10] since 1940s. Rossby first recognized the nature of barotropic instability and derived the linearized vorticity equation in Ref. 11. Later, Kuo formulated Equations (5) and (6), and did some early studies in Ref. 9. In particular, he gave a necessary condition for instability that $\beta - U''$ must change sign in the domain $[y_1, y_2]$, which generalized the classical Rayleigh criterion[2] for $\beta = 0$. Pedlosky[12] showed that any unstable wave speed $c = c_r + ic_i$ ($c_i > 0$) must lie in the following semicircle:

$$\left(c_r - (U_{\min} + U_{\max})/2\right)^2 + c_i^2 \le \left((U_{\max} - U_{\min})/2 + |\beta|/2\alpha^2\right)^2,$$

which is a generalization of Howard's semicircle theorem[13] for $\beta = 0$. Here, $U_{\min} = \min U$ and $U_{\max} = \max U$. Additionally, the following characterization for the unstable wave speeds is given in Refs. 9, 14, and 15.

Lemma 1. *If $\beta > 0$, then there are no nontrivial solutions of* (5)-(6) *for $c_r > U_{\max}$; if $\beta < 0$, then there are no nontrivial solutions of* (5)-(6) *for $c_r < U_{\min}$.*

Although there are several necessary conditions as indicated above, there has been very few sufficient conditions for the barotropic instability of shear flows. In the fluid literature, the linear instability was studied for some special shear flows. The barotropic instability of Bickley jet ($U(y) = \operatorname{sech}^2 y$) was studied by numerical computations and asymptotic analysis (cf. Refs. 10, 16–21). Parts of the stability boundary are given analytically or numerically by Lipps[20] and Maslowe[21] for the unbounded Bickley jet. Engevik[18] confirmed and sometimes corrected the results in earlier work. Moreover, he found analytic neutral modes and corresponding neutral curves not known previously, both for the bounded and the unbounded Bickley jet in Ref. 18. He made use of associated Legendre functions, which are related to hypergeometric functions as we used in Appendix B. The stability boundary of hyperbolic-tangent shear flow was studied in Refs. 10, 19, 22. Other references on the barotropic instability include Refs. 23–27. In this paper, we consider the barotropic instability of the following class of shear flows.

Definition 1. The flow U is in class $\mathcal{K}$ if $U \in C^3([y_1, y_2])$, U is not a constant function on $[y_1, y_2]$, and for each $\beta \in \operatorname{Ran}(U'')$, there exists $U_\beta \in \operatorname{Ran}(U)$ such that

$$K_\beta(y) := \frac{\beta - U''(y)}{U(y) - U_\beta}$$

is nonnegative and bounded on $[y_1, y_2]$. Here, Ran means the range of a function. Furthermore, U is said to be in class $\mathcal{K}^+$ if U is in class $\mathcal{K}$ and K_β is positive on $[y_1, y_2]$ for each $\beta \in \operatorname{Ran}(U'')$.

Flows in class $\mathcal{K}^+$ include $U(y) = \sin y,\ \tanh y$, and more generally, any $U(y)$ satisfying the ordinary differential equation (ODE) $U'' = g(U)$ with $g \in C^1(\operatorname{Ran}(U))$ and $g' < 0$ on $\operatorname{Ran}(U)$. One important property for flows in class $\mathcal{K}^+$ is that there is a uniform H^2 bound for the unstable solutions of (5) and (6), see Lemma 5. Neutral modes are the solutions of (5) and (6) with $c \in \mathbf{R}$. In the study of stability of a shear flow $U(y)$, it is often important to locate the neutral modes that are limits of a sequence of unstable modes. These so-called neutral limiting modes determine the boundary from instability to stability. In Theorems 1 and 2, all H^2 neutral modes for a general shear flow, and consequently, all neutral limiting modes for a flow in class $\mathcal{K}^+$, are classified into four types by their phase speed c: (a) $c = U(z)$ such that $\beta = U''(z)$; (b) $c = U(y_1)$ or $U(y_2)$; (c) c is a critical value of U; and (d) c is outside the range of U. Here, the neutral modes of types (b) and (c) might be singular, and type (d) is called nonresonant since the phase speed c causes no interaction with the basic flow $U(y)$. This contrasts greatly with the nonrotating case $\beta = 0$, where it was shown in Ref. 28 that for neutral modes in H^2, c must be an inflection value of U.

In the literature, it is common to look for unstable modes near neutral modes. A useful approach to determine the stability boundary is to study the local bifurcation of unstable modes near all possible neutral limiting wave numbers, and then, combine this information to detect the stability/instability at any wave number. In Ref. 6, this approach was used to show that when $\beta = 0$, any flow $U(y)$ in class $\mathcal{K}^+$ is linearly stable if and only if $\alpha \geq \alpha_{\max}$, where $-\alpha_{\max}^2$ is the principal eigenvalue of the operator $-\frac{d^2}{dy^2} - K_0(y)$. However, when $\beta \neq 0$, there are several difficulties in this approach. First, we need to deal with the subtle perturbation problem near singular neutral modes. Second, for nonresonant neutral modes, the phase speed c is to be determined. Moreover, near these nonresonant neutral modes, the bifurcation of unstable modes is usually nonsmooth (see Remark 1). In some literature (eg, Ref. 29), it was believed that these nonresonant neutral modes are not adjacent to unstable modes. This turns out to be not true from our study of the Sinus flow in Section 4.

In this paper, we develop a new approach to study the barotropic instability of shear flows. First, we write the linearized equation in a Hamiltonian form $\partial_t \omega = JL\omega$, where J is anti-self-adjoint and L is self-adjoint as defined in (15). For a fixed wave number α, by taking the ansatz $\omega = \omega_\alpha(y,t)e^{i\alpha x}$, the linearized equation can be written in a Hamiltonian form $\partial_t \omega_\alpha = J_\alpha L_\alpha \omega_\alpha$, where J_α and L_α are defined in (18). Then by the instability index theorem recently developed in Ref. 30 for general Hamiltonian partial differential equations (PDEs), we get the index formula (23). Similar index formulas have been studied for Hamiltonian systems in the literature, but often J is assumed to have a bounded inverse (eg, Refs. 31–34). In Refs. 35 and 36, the index formulas were studied for Korteweg-de Vries (KdV)-type equations in the whole line, where $J = \partial_x$ does not have bounded inverse. Lin and Zeng generalized these results in Ref. 30, where they allowed J to be any anti-self-dual operator. In our case, J_α indeed has no bounded inverse, so the results of Lin and Zeng apply here. This formula implies that to determine the instability at any $\alpha > 0$, it suffices to count the number of neutral modes with a nonpositive signature (ie, $\langle L_\alpha \omega_\alpha, \omega_\alpha \rangle \le 0$). The four types of neutral modes in H^2 are counted separately. In particular, the counting of nonresonant neutral modes can be reduced to study $\lambda_n(\beta, c)$, the nth eigenvalue of the Sturm-Liouville operator $-\frac{d^2}{dy^2} - \frac{\beta - U''}{U-c}$ for $c \notin \mathrm{Ran}\,(U)$. An important observation is that for a nonresonant neutral mode (c, α, β, ϕ), the sign $\langle L_\alpha \omega_\alpha, \omega_\alpha \rangle$ is determined by $\partial_c \lambda_n(\beta, c)$, where $\alpha^2 = -\lambda_n(\beta, c) > 0$ and $\omega_\alpha = -\phi'' + \alpha^2 \phi$. Therefore, by studying the shape of the graph of $\lambda_n(\beta, c)$, we are able to count the nonresonant neutral modes with a nonpositive signature. Combining with the index count for the other three types of neutral modes, we can find the stability boundary in the whole parameter space (α, β). See Subsection 3.3 for more detailed discussions about this approach. In this approach, we avoid the study of the bifurcation of unstable modes near neutral modes, which is particularly tricky for singular and nonresonant neutral modes.

In Section 4, we study in details the classical Sinus flow

$$U(y) = (1 + \cos(\pi y))/2, \qquad y \in [-1, 1].$$

For the Sinus flow, L_α has at most one negative eigenvalue. Moreover, the singular neutral wave speeds exist only at the endpoints of $\mathrm{Ran}\,(U) = [0, 1]$. The set of all the eigenvalues of corresponding singular Sturm-Liouville operator $-\frac{d^2}{dy^2} - \frac{\beta - U''}{U-c}$ is bounded from below and can be computed by using hypergeometric functions. The spectral continuity of the operators $-\frac{d^2}{dy^2} - \frac{\beta - U''}{U-c}$ can be shown at the end points $c = 0, 1$ by studying the singular limits when $c \to 1^+$ and $c \to 0^-$. Based on these properties and the above approach, we obtain a simple characterization of the stability boundary in the parameter space. By the index formula and the relation of $\partial_c \lambda_1(\beta, c)$ with the $\langle L_\alpha \cdot, \cdot \rangle$ sign for a fixed β, the graph of $\lambda_1^-(\beta, c)$ has at most one hump, more precisely, monotone or single humped, respectively, for negative or positive sign of $\langle L_\alpha \cdot, \cdot \rangle$ at singular neutral modes. Here, λ_1^- is the negative part of λ_1. The lower part of the stability boundary is given exactly by $\sup_{c \in (-\infty, 0] \cup [1, +\infty)} \lambda_1^-(\beta, c)$. In Refs. 10 and 1, it was concluded from numerical computations that the lower stability boundary is the curve of singular neutral modes ($\beta > 0$) and the modes with zero wave number ($\beta < 0$). Our results give a correction to this commonly accepted picture. In fact, only part of the lower stability boundary consists of singular neutral modes with negative $\langle L_\alpha \cdot, \cdot \rangle$ and the modes with zero wave number, while the other part consists of nonresonant neutral modes. The new stability boundary is confirmed by more accurate numerical results. Same results on the stability boundary can be obtained for more general flows similar to Sinus flow. Moreover, we count the exact number of nonresonant neutral modes in each stability region. As we discuss below, this has important implication on the nonlinear dynamics near shear flows.

Lastly, we study some dynamical behaviors near the shear flows. First, the existence of nontrivial traveling wave solutions is shown near shear flows with nonresonant neutral modes. These traveling waves, which have fluid trajectories moving in one direction, do not exist when there is no rotation (ie, $\beta = 0$), and therefore, are purely due to rotating effects. We expect the nonlinear dynamics to be much richer due to the existence of these traveling waves. Second, the Hamiltonian structure of the linearized equation is used to prove the linear inviscid damping for stable shears with no neutral modes (Theorem 8) and in the center space for the unstable shears (Theorem 9). These results are useful for the further study of nonlinear dynamics near the shear flows, such as nonlinear inviscid damping (for stable flows without neutral modes) and the construction of invariant manifolds (for unstable flows).

This paper is organized as follows. In Section 2, we classify all the neutral modes in H^2 for general shear flows. For shear flows in class $\mathcal{K}^+$, by proving a uniform H^2 bound for unstable modes, we obtain a classification of neutral limiting modes. In Section 3, for flows in class $\mathcal{K}^+$, we derive an instability index formula by using the Hamiltonian structure of the linearized fluid equation. Then a general approach is developed to find the stability boundary for flows in class $\mathcal{K}^+$. In Section 4, we find the stability boundary for the Sinus flow in details. In Sections 5 and 6, the bifurcation of nontrivial traveling waves and the linear inviscid damping are studied, respectively. Section 7 contains the summary and discussion of the numerical results for Sinus flow.

2 | NEUTRAL MODES in H^2

In this section, we classify neutral modes in H^2 for general shear flows and neutral limiting modes for flows in class $\mathcal{K}^+$.

2.1 | Classification of neutral modes in H^2

First, we give the definition of neutral modes.

Definition 2. $(c_s, \alpha_s, \beta_s, \phi_s)$ is said to be a neutral mode if $c_s \in \mathbf{R}$, $\alpha_s > 0$, $\beta_s \in \mathbf{R}$, and ϕ_s is a nontrivial solution to

$$-\phi_s'' + \alpha_s^2 \phi_s - \frac{\beta_s - U''}{U - c_s} \phi_s = 0 \ \ on \ \ (y_1, y_2), \ \ and \ \ \phi_s(y_1) = \phi_s(y_2) = 0. \tag{7}$$

If $\phi_s \in H^2(y_1, y_2)$, $(c_s, \alpha_s, \beta_s, \phi_s)$ is said to be a neutral mode in H^2.

If Equation (7) is singular, by a solution ϕ, we mean it solves (7) on $(y_1, y_2)\backslash\{U = c_s\}$. For convenience, we make the following assumption:

Hypothesis 1. *Let $U \in C^2([y_1, y_2])$, and $\{U - c = 0\}$ be a finite set for $c \in (U_{\min}, U_{\max})$.*

Hypothesis 1 is true for generic C^2 flows. Suppose that there exists $c_0 \in (U_{\min}, U_{\max})$ such that $\{U - c_0 = 0\}$ is an infinite set. Then for any accumulation point x_0 of $\{U - c_0 = 0\}$, we have $U^{(n)}(x_0) = 0$ for all $1 \le n \le k$ if $U \in C^k$. This implies that Hypothesis 1 is satisfied for analytic flows and for flows in class $\mathcal{K}^+$. In fact, it is true for any flow $U(y)$ satisfying the second-order ODE $U'' = k(y)g(U)$, where $k > 0$ is bounded and $g \in C^1$ by the uniqueness of ODE solutions.

Assume that U satisfies Hypothesis 1. Then for any $c \in (U_{\min}, U_{\max})$, the set $\{U - c = 0\} \cap (y_1, y_2)$ is nonempty, which we denote by

$$\{z_i \mid 1 \le i \le k_c, \ z_1 < z_2 < \cdots < z_{k_c}\}. \tag{8}$$

Set $z_0 := y_1$ and $z_{k_c+1} := y_2$.

Lemma 2. *Let U satisfy Hypothesis 1 and ϕ solve* (5)-(6) *with $\alpha > 0$, $\beta \in \mathbf{R}$ and $c \in (U_{\min}, U_{\max})$. If there exists $1 \le i_0 \le k_c$ such that $\phi \in H^1(z_{i_0-1},\ z_{i_0+1})$, and $(U(y)-c)(U(z)-c) < 0$ for all $y \in (z_{i_0-1},\ z_{i_0})$ and all $z \in (z_{i_0},\ z_{i_0+1})$, then ϕ cannot vanish at $z_{i_0-1},\ z_{i_0}$ and z_{i_0+1} simultaneously unless it vanishes identically on at least one of the intervals (z_{i_0-1}, z_{i_0}) and (z_{i_0}, z_{i_0+1}).*

Note that $(U(y)-c)(U(z)-c) < 0$ for all $y \in (z_{i_0-1},\ z_{i_0})$ and all $z \in (z_{i_0},\ z_{i_0+1})$ holds true unless $U'(z_{i_0}) = 0$. We refer the readers to Lemma 8 for the cases $c = U_{\min}$ or $c = U_{\max}$. The following lemma will be used to classify neutral modes.

Lemma 3. *Let ϕ be a solution of* (5) *with $\alpha > 0$, $\beta \in \mathbf{R}$ and $c \in \mathrm{Ran}\,(U)$. Assume that $y_1 \le x_0 < x_1 < x_2 \le y_2$ satisfy $U(x_1) - c = 0$ and $U - c \ne 0$ on $(x_0, x_1) \cup (x_1, x_2)$. If $U'(x_1) \ne 0$ and $\phi \in C^1(x_0, x_2)$ satisfies the initial conditions $\phi(x_1) = a_1,\ \phi'(x_1) = a_2$ for some $a_1, a_2 \in \mathbf{R}$, then ϕ is unique on the interval (x_0, x_2).*

We leave the proofs of Lemmas 2 and 3 in Appendix A. Now we classify all the wave speeds of neutral modes in H^2 for a general shear flow.

Theorem 1. *Assume that U satisfies Hypothesis 1. Let $(\phi_s, \alpha_s, \beta, c_s)$ be a neutral mode in H^2. Then the wave speed c_s must be one of the following:*

(i) *there exists $z \in (y_1, y_2)$ such that $c_s = U(z)$ and $\beta = U''(z)$;*

(ii) *$c_s = U(y_1)$ or $c_s = U(y_2)$;*

(iii) *c_s is a critical value of U;*

(iv) *$c_s \notin \mathrm{Ran}\,(U)$.*

Proof. It suffices to show that if $c_s \in \mathrm{Ran}\,(U)$, then one of cases (i)–(iii) is true. Suppose that $c_s \in \mathrm{Ran}\,(U)$ and $\{U - c_s = 0\} \cap \{y_1, y_2\} \ne \emptyset$. Then $c_s = U(y_1)$ or $c_s = U(y_2)$, that is, case (ii) is true. Otherwise, $c_s \ne U(y_i), i = 1, 2$. We consider two cases below.

Case 1. There exists $z_s \in \{U - c_s = 0\}$ such that $\beta = U''(z_s)$. Then (i) is true.
Case 2. $\beta \ne U''(z)$ for all $z \in \{U - c_s = 0\}$. We divide it into two subcases.
Case 2.1. $U'(z) \ne 0$ for all $z \in \{U - c_s = 0\}$. Then $c_s \in (U_{\min}, U_{\max})$.

In this subcase, $\{U - c_s = 0\}$ is nonempty and finite, so we use the notation in (8). We claim that there exists $1 \le i_1 \le k_{c_s}$ such that $\phi_s(z_{i_1}) \ne 0$. Suppose, otherwise, $\phi_s(z_i) = 0$ for any $1 \le i \le k_{c_s}$. For any fixed $1 \le i_0 \le k_{c_s}$, by the fact that $U'(z_{i_0}) \ne 0$ and by Lemma 2, $\phi_s \equiv 0$ on at least one of the intervals $[z_{i_0-1}, z_{i_0}]$ and $[z_{i_0}, z_{i_0+1}]$. Since $\phi_s \in H^2(y_1, y_2)$, it follows that $\phi_s \in C^1([y_1, y_2])$ and by Lemma 3, $\phi \equiv 0$ on $[z_{i_0-1}, z_{i_0+1}]$ and hence on $[y_1, y_2]$. Thus, there exists $1 \le i_1 \le k_{c_s}$ such that $\phi(z_{i_1}) \ne 0$. Then near z_{i_1},

$$\phi_s'' = \alpha_s^2 \phi_s - \frac{\beta - U''}{U - c_s} \phi_s \notin L^2_{\mathrm{loc}}(y_1, y_2),$$

which contradicts $\phi_s \in H^2(y_1, y_2)$.
Case 2.2. There is $z_0 \in \{U = c_s\}$ such that $U'(z_0) = 0$. Then c_s is a critical value of U. ■

For the neutral modes in Theorem 1, we call (i) to be regular, (ii)–(iii) to be singular, and (iv) to be nonresonant. For $\beta = 0$, it is shown in Ref. 28 that only (i) is true, that is, for all neutral modes in H^2, the phase speed must be an inflection value of U.

2.2 | Neutral limiting modes for flows in class $\mathcal{K}^+$

First, we obtain the uniform H^2 bound of unstable solutions for flows in class $\mathcal{K}^+$.

Lemma 4. *Let ϕ be a solution of* (5) *and* (6) *with $c = c_r + ic_i$ ($c_i > 0$). Then*

$$\int_{y_1}^{y_2} \frac{(\beta - U'')}{|U-c|^2}|\phi|^2 dy = 0, \tag{9}$$

and

$$\int_{y_1}^{y_2} \left[|\phi'|^2 + \alpha^2|\phi|^2 - \frac{(\beta - U'')(U-q)}{|U-c|^2}|\phi|^2 \right] dy = 0$$

for any $q \in \mathbf{R}$.

Proof. The proof is similar as that of lemma 3.4 and (25) in Ref. 6. ■

Equation (9) was used in Ref. 9 to show Rayleigh's criterion.

Lemma 5. *Let U be in class $\mathcal{K}^+$ and $\beta \in (\min U'', \max U'')$. If ϕ is a solution of* (5) *and* (6) *with $c = c_r + ic_i$ ($c_i > 0$), then*

$$\int_{y_1}^{y_2} (|\phi'|^2 + \alpha^2|\phi|^2) dy \le \int_{y_1}^{y_2} K_\beta |\phi|^2 dy, \tag{10}$$

$$\int_{y_1}^{y_2} (|\phi''|^2 + 2\alpha^2|\phi'|^2 + \alpha^4|\phi|^2) dy \le \|K_\beta\|_{L^\infty} \int_{y_1}^{y_2} K_\beta |\phi|^2 dy.$$

Proof. The proof is similar as that of lemma 3.7 in Ref. 6. ■

Next, we consider neutral limiting modes defined below.

Definition 3. Let $\beta \in (\min U'', \max U'')$. We call $(c_s, \alpha_s, \beta, \phi_s)$ to be a neutral limiting mode if $c_s \in \mathbf{R}$, $\alpha_s > 0$, and there exists a sequence of unstable modes $\{(c_k, \alpha_k, \beta, \phi_k)\}$ (with $\mathrm{Im}(c_k) = c_k^i > 0$ and $\|\phi_k\|_{L^2} = 1$) to (5) and (6) such that $c_k \to c_s, \alpha_k \to \alpha_s$, ϕ_k converges uniformly to ϕ_s on any compact subset of S_0 as $k \to \infty$, ϕ_s'' exists on S_0, and ϕ_s satisfies

$$(U - c_s)(-\phi_s'' + \alpha_s^2 \phi_s) - (\beta - U'')\phi_s = 0 \tag{11}$$

on S_0, where $S_0 = [y_1, y_2] \backslash \{U = c_s\}$. Here c_s is called the neutral limiting phase speed and α_s is called the neutral limiting wave number.

Then we prove that any neutral limiting mode is in H^2 for flows in class $\mathcal{K}^+$.

Lemma 6. *Let U be in class $\mathcal{K}^+$ and $\beta \in (\min U'', \max U'')$. Suppose that $(\phi_s, \alpha_s, \beta, c_s)$ is a neutral limiting mode. Then $\phi_s \in H^2(y_1, y_2)$.*

Proof. Let $\{(\phi_k, \alpha_k, \beta, c_k)\}$ be a sequence of unstable modes converging to $(\phi_s, \alpha_s, \beta, c_s)$ in the sense of Definition 3. Note that $\|\phi_k\|_{L^2} = 1$. Since U is in class $\mathcal{K}^+$, by Lemma 5, there exists $C > 0$ such that $\|\phi_k\|_{H^2} \le C$ for all $k \ge 1$. Thus, there exists $\phi_0 \in H^2(y_1, y_2)$ such that, up to a subsequence, $\phi_k \rightharpoonup \phi_0$ in $H^2(y_1, y_2)$, $\phi_k \to \phi_0$ in $C^1([y_1, y_2])$ and $\|\phi_0\|_{H^2} \le C$. Let $S_0 = [y_1, y_2] \backslash \{U = c_s\}$. For any compact subset $S_1 \subset S_0$, ϕ_0 solves (11) on S_1 and thus by Definition 3, $\phi_0 \equiv \phi_s \in H^2(y_1, y_2)$. ■

Recall that flows in class $\mathcal{K}^+$ satisfy Hypothesis 1. Combining Theorem 1 and Lemma 6, we get the classification of neutral limiting modes for flows in class $\mathcal{K}^+$.

Theorem 2. *Assume that U is in class $\mathcal{K}^+$. Let $(\phi_s, \alpha_s, \beta, c_s)$ be a neutral limiting mode. Then the neutral limiting phase speed c_s must be one of the following (recall that U_β is defined in Definition 1):*

(i) $c_s = U_\beta$;

(ii) $c_s = U(y_1)$ *or* $c_s = U(y_2)$;

(iii) c_s *is a critical value of* U*; and*

(iv) $c_s \notin \mathrm{Ran}\,(U)$.

Remark 1. In theorem IV of Ref. 29, Tung showed that for a general C^2 shear flow $U(y)$, the phase speed c_s of any neutral limiting mode $(c_s, \alpha_s, \beta, \phi_s)$ must lie in $\mathrm{Ran}\,(U)$. His proof is under the assumption that for fixed β, the dispersion relation $c(\alpha)$ is an analytic function of α near α_s when $c(\alpha_s) = c_s \notin \mathrm{Ran}\,(U)$. However, as suggested in Ref. 21, the analytic assumption might not always hold and it is possible that $c_s \notin \mathrm{Ran}\,(U)$. In Theorem 6, we give the sharp stability boundary for the Sinus flow, part of which consists of nonresonant neutral modes. Thus, the phase speed of neutral limiting modes can indeed lie outside the range of U.

Below we give some explanation why the analytic assumption of $c(\alpha)$ could fail. Assume that $(\phi_s, \alpha_s, \beta, c_s)$ is a neutral mode and $c_s \notin \mathrm{Ran}\,(U)$. From the Rayleigh-Kuo equation (5)-(6), the perturbation of the eigenvalue c near c_s appears to be analytic in α when c_s is not in the range of U. However, we should consider the operator associated with the linearized equation (4) with the wave number α (β is fixed):

$$B_\alpha \omega := U\omega - (\beta - U'')\left(-\frac{d^2}{dy^2} + \alpha^2\right)^{-1} \omega.$$

Then c_s is an isolated eigenvalue of B_{α_s}. Define the Riesz projection operator

$$P_{\alpha_s} := -\frac{1}{2\pi i} \int_\Gamma \left(B_{\alpha_s} - \zeta\right)^{-1} d\zeta,$$

where Γ is a circle in $\rho(B_{\alpha_s})$ enclosing c_s and no other spectral points of B_{α_s}. Note that $\mathrm{Ran}\,(P_{\alpha_s})$ is the generalized eigenspace of the eigenvalue c_s and $\dim(\mathrm{Ran}\,(P_{\alpha_s}))$ is the algebraic multiplicity of c_s (see page 181 in Ref. 37). Although the geometric multiplicity of c_s is 1, the algebraic multiplicity of c_s may be larger than 1. In such case, there might be more than one branches of eigenvalues emanating from c_s when we perturb the parameter α in a neighborhood of α_s. The expansion of $c(\alpha) - c(\alpha_s)$ near $\alpha = \alpha_s$ could be given by the Puiseux series (see page 65 in Ref. 37) instead of the power series in the analytic case. This suggests that we cannot exclude the possibility that for a neutral limiting mode, c_s is outside $\mathrm{Ran}\,(U)$.

Similar to the proof of theorem 4.1 in Ref. 6, we get the existence of unstable modes when the wave number is slightly to the left of a regular neutral wave number.

Lemma 7. *Let U be in class $\mathcal{K}^+$, $\beta \in (\min U'', \max U'')$, and $(c_s, \alpha_s, \beta, \phi_s)$ be a regular neutral mode with $c_s = U_\beta$. Then there exists $\varepsilon_0 < 0$ such that if $\varepsilon_0 < \varepsilon < 0$, there is a nontrivial solution ϕ_ε to the equation*

$$(U - U_\beta - c(\varepsilon))(\phi_\varepsilon'' - \alpha(\varepsilon)^2 \phi_\varepsilon) + (\beta - U'')\phi_\varepsilon = 0$$

with $\phi_\varepsilon(y_1) = \phi_\varepsilon(y_2) = 0$. *Here,* $\alpha(\varepsilon) = \sqrt{\varepsilon + \alpha_s^2}$ *is the perturbed wave number and* $U_\beta + c(\varepsilon)$ *is an unstable wave speed with* $\mathrm{Im}(c(\varepsilon)) > 0$.

The next lemma comes from Refs. 29 and 9.

Lemma 8. *Let* $U \in C^2([y_1, y_2])$. *When* $\beta \geq 0$ *and* $c_s \geq U_{\max}$ *(or* $\beta \leq 0$ *and* $c_s \leq U_{\min}$*), for any* $\alpha > 0$, *there exist no neutral modes in* H^2.

3 | HAMILTONIAN FORMULATION, INDEX FORMULA, AND INSTABILITY CRITERIA

In this section, we first write the linearized fluid equation for flows in class $\mathcal{K}^+$ in a Hamiltonian form and derive an instability index formula. Then we provide a new approach to study the instability of flows in class $\mathcal{K}^+$.

3.1 | Hamiltonian formulation and instability index formula

Since a necessary condition for instability is that $\beta - U''$ must change sign in the domain $[y_1, y_2]$,[9] let us fix $\beta \in (\min U'', \max U'')$. In the traveling frame $(x - U_\beta t, y, t)$, the linearized equation (4) becomes

$$\partial_t \omega + (U - U_\beta)\partial_x \omega - (\beta - U'')\partial_x \psi = 0. \tag{12}$$

Recall that for flows in class $\mathcal{K}^+$, $K_\beta = \frac{\beta - U''}{U - U_\beta} > 0$. Let the x period be $2\pi/\alpha$ for some $\alpha > 0$. Define the nonshear space on the periodic channel $S_{2\pi/\alpha} \times [y_1, y_2]$ by

$$X = \left\{ \omega = \sum_{k \in \mathbf{Z},\ k \neq 0} e^{ik\alpha x}\omega_k(y),\ \|\omega\|_X^2 = \left\| \frac{1}{\sqrt{K_\beta}}\omega \right\|_{L^2}^2 < \infty \right\}. \tag{13}$$

Clearly, $X = L^2$. Equation (12) can be written in a Hamiltonian form

$$\omega_t = -(\beta - U'')\partial_x\big(\omega/K_\beta - \psi\big) = JL\omega, \tag{14}$$

where

$$J = -(\beta - U'')\partial_x : X^* \to X, \quad L = 1/K_\beta - (-\Delta)^{-1} : X \to X^*, \tag{15}$$

are anti-self-adjoint and self-adjoint, respectively. Denote $n^-(L)$ ($n^0(L)$) to be the number of negative (zero) directions of L on X. Define the operator

$$A_0 = -\Delta - K_\beta : H^2 \to L^2$$

and

$$\tilde{L}_0 = -\frac{d^2}{dy^2} - K_\beta : H^2 \cap H_0^1\big(y_1, y_2\big) \to L^2\big(y_1, y_2\big). \tag{16}$$

Then by lemma 11.3 in Ref. 30, we have

$$n^0(L) = n^0\big(A_0\big) = 2\sum_{l\geq 1} n^0\big(\tilde{L}_0 + l^2\alpha^2\big), \quad n^-(L) = n^-\big(A_0\big) = 2\sum_{l\geq 1} n^-\big(\tilde{L}_0 + l^2\alpha^2\big).$$

If $n^-(\tilde{L}_0) \neq 0$, let $-\alpha_{\max}^2$ be the principal eigenvalue of $\tilde{L}_0$ and ϕ_0 be the eigenfunction. When $\tilde{L}_0 \geq 0$, let $\alpha_{\max} = 0$. Then L is nonnegative and the stability holds when $\alpha \geq \alpha_{\max}$.

Let $\alpha < \alpha_{\max}$ and $Y = L^2_{\frac{1}{K_\beta}}(y_1, y_2)$. The space X has an invariant decomposition $X = \bigoplus_{l\in\mathbf{Z},\, l\neq 0} X^l$, where

$$X^l = \left\{e^{i\alpha l x}\omega_l(y),\ \omega_l \in Y\right\}. \tag{17}$$

On the subspace $X_\alpha := \{e^{i\alpha x}\omega(y),\ \omega \in Y\}$, the operator JL is reduced to the ODE operator $J_\alpha L_\alpha$ acting on Y, where

$$J_\alpha = -i\alpha(\beta - U''), \quad L_\alpha = \frac{1}{K_\beta} - \left(-\frac{d^2}{dy^2} + \alpha^2\right)^{-1}. \tag{18}$$

By the same proof of lemma 11.3 in Ref. 30, we have

$$n^-\big(L_\alpha\big) = n^-\big(\tilde{L}_0 + \alpha^2\big),\ n^0\big(L_\alpha\big) = n^0\big(\tilde{L}_0 + \alpha^2\big).$$

Since J_α is not a real operator on Y, we define the invariant subspace

$$X^\alpha = X_\alpha \oplus X_{-\alpha} = \left\{\cos(\alpha x)\omega_1(y) + \sin(\alpha x)\omega_2(y),\ \omega_1, \omega_2 \in Y\right\},$$

which is isomorphic to the real space $Y \times Y$. For any $\omega = \cos(\alpha x)\omega_1(y) + \sin(\alpha x)\omega_2(y) \in X^\alpha$,

$$JL\omega = (\cos(\alpha x), \sin(\alpha x))J^\alpha L^\alpha \begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix},$$

where

$$J^\alpha = \begin{pmatrix} 0 & -\alpha(\beta - U'') \\ \alpha(\beta - U'') & 0 \end{pmatrix}, \quad L^\alpha = \begin{pmatrix} L_\alpha & 0 \\ 0 & L_\alpha \end{pmatrix},$$

and L_α is defined in (18). Thus, to study the spectra of JL on X^α, we study the spectra of $J^\alpha L^\alpha$ on $Y \times Y$. We note that

$$\sigma\big(J^\alpha L^\alpha|_{Y\times Y}\big) = \sigma\big(J_\alpha L_\alpha|_Y\big) \cup \sigma\big(J_{-\alpha}L_{-\alpha}|_Y\big), \tag{19}$$

and $\sigma(J_\alpha L_\alpha|_Y)$ is the complex conjugate of $\sigma(J_{-\alpha}L_{-\alpha}|_Y)$.

By the instability index theorem 2.3 in Ref. 30 for linear Hamiltonian PDEs, we have

$$2\tilde{k}_i^{\leq 0} + 2\tilde{k}_c + \tilde{k}_0^{\leq 0} + \tilde{k}_r = n^-(L^\alpha) = 2n^-(L_\alpha),$$

where $n^-(L^\alpha)$ denotes the sum of multiplicities of negative eigenvalues of L^α, $\tilde{k}_r$ is the sum of algebraic multiplicities of positive eigenvalues of $J^\alpha L^\alpha$, $\tilde{k}_c$ is the sum of algebraic multiplicities of eigenvalues of $J^\alpha L^\alpha$ in the first quadrant, $\tilde{k}_i^{\leq 0}$ is the total number of nonpositive dimensions of $\langle L^\alpha\cdot,\cdot\rangle$ restricted to the generalized eigenspaces of purely imaginary eigenvalues of $J^\alpha L^\alpha$ with positive imaginary parts, and $\tilde{k}_0^{\leq 0}$ is the number of nonpositive dimensions of $\langle L^\alpha \cdot, \cdot \rangle$ restricted to the generalized kernel of $J^\alpha L^\alpha$ modulo ker L^α. By the next lemma, we have $\tilde{k}_0^{\leq 0} = 0$, from which it follows that

$$2\tilde{k}_i^{\leq 0} + 2\tilde{k}_c + \tilde{k}_r = 2n^-(L_\alpha). \tag{20}$$

Lemma 9. *Let E_0 be the generalized zero eigenspace of $J^\alpha L^\alpha$. Then $E_0 = \ker L^\alpha$.*

Proof. It suffices to show that the generalized zero eigenspace of $J_\alpha L_\alpha$ on Y coincides with ker L_α. Suppose there exists $\omega \in Y$ such that

$$J_\alpha L_\alpha \omega = -i\alpha(U - U_\beta)\big(\omega - K_\beta \psi\big) = \tilde{\omega} \in \ker L_\alpha. \tag{21}$$

Let $\tilde{\psi} = (-\frac{d^2}{dy^2} + \alpha^2)^{-1}\tilde{\omega}$. Then $-\tilde{\psi}'' + \alpha^2\tilde{\psi} - K_\beta\tilde{\psi} = 0$. Since $\beta \in (\min U'', \max U'')$, we get by Lemma 2 that $\tilde{\psi}$ is not all zero on $\{U = U_\beta\}$, which implies the same for $\tilde{\omega} = K_\beta\tilde{\psi}$. Thus, (21) gives

$$\omega - K_\beta \psi = \frac{\tilde{\omega}}{-i\alpha(U - U_\beta)} \notin L^2\big(y_1, y_2\big).$$

This contradiction shows that the generalized kernel of $J_\alpha L_\alpha$ on Y is the same as ker L_α. ■

Now we derive the index formula for $J_\alpha L_\alpha$ on Y. Let k_r be the sum of algebraic multiplicities of positive eigenvalues of $J_\alpha L_\alpha$, k_c be the sum of algebraic multiplicities of eigenvalues of $J_\alpha L_\alpha$ in the first and the forth quadrants, and $k_i^{\leq 0}$ be the total number of nonpositive dimensions of $\langle L_\alpha \cdot, \cdot \rangle$ restricted to the generalized eigenspaces of nonzero purely imaginary eigenvalues of $J_\alpha L_\alpha$. By (19), we have the following relation:

$$2k_i^{\leq 0} = 2\tilde{k}_i^{\leq 0}, \;\; 2k_c = 2\tilde{k}_c, \;\; 2k_r = \tilde{k}_r. \tag{22}$$

Combining (20) and (22), we get the following index formula for $J_\alpha L_\alpha$.

Theorem 3. *Let U be in class $\mathcal{K}^+$ and $\beta \in (\min U'', \max U'')$. Then the following index formula holds for the operator $J_\alpha L_\alpha$ on Y:*

$$k_c + k_r + k_i^{\leq 0} = n^-(L_\alpha). \tag{23}$$

Remark 2. When $\beta = 0$, $k_i^{\leq 0} = 0$ and the index formula (23) reduces to $k_c + k_r = n^-(L_\alpha)$ (see Ref. 28). When $\beta \neq 0$, in general, we have $k_i^{\leq 0} \neq 0$ as seen from Sinus flow in Section 4.

From the index formula (23), the stability of shear flows is reduced to determine $k_i^{\leq 0}$. This corresponds to consider neutral modes in H^2 with the wave speed $c_s \neq U_\beta$.

3.2 | Computation of the quadratic form $\langle L_\alpha \cdot, \cdot \rangle$

First, we compute the quadratic form $\langle L_\alpha \cdot, \cdot \rangle$ for unstable modes and neutral limiting modes.

Lemma 10. *Let (c, α, β, ϕ) solve* (5)-(6) *with $\phi \in H^2$, and $\omega = -\phi'' + \alpha^2\phi$. Then*

(i)

$$\langle L_\alpha \omega, \omega \rangle = (c - U_\beta) \int_{y_1}^{y_2} \frac{(\beta - U'')}{|U - c|^2} |\phi|^2 dy. \tag{24}$$

(ii) If (c, α, β, ϕ) is an unstable mode, then $\langle L_\alpha \omega, \omega \rangle = 0$.

(iii) If (c, α, β, ϕ) is a regular or nonresonant neutral limiting mode, then $\langle L_\alpha \omega, \omega \rangle = 0$. If (c, α, β, ϕ) is a singular neutral limiting mode, then $\langle L_\alpha \omega, \omega \rangle \le 0$.

Proof. We first show (24). From (5), we get $(U - c)\omega = (\beta - U'')\phi$. Therefore,

$$\frac{\omega}{K_\beta} - \phi = \frac{(U - U_\beta)\omega}{\beta - U''} - \frac{(U - c)\omega}{\beta - U''} = \frac{(c - U_\beta)\omega}{\beta - U''},$$

and

$$\langle L_\alpha \omega, \omega \rangle = \int_{y_1}^{y_2} \left(\frac{\omega}{K_\beta} - \phi \right) \bar{\omega}\, dy = \int_{y_1}^{y_2} \frac{(c - U_\beta)}{\beta - U''} |\omega|^2 dy = (c - U_\beta) \int_{y_1}^{y_2} \frac{(\beta - U'')}{|U - c|^2} |\phi|^2 dy.$$

The conclusion (ii) follows from (i) and Lemma 4.

Next, we prove (iii). If (c, α, β, ϕ) is regular, then $c = U_\beta$ and $\langle L_\alpha \omega, \omega \rangle = 0$ by (i).

Let $\{(c_k, \alpha_k, \beta, \phi_k)\}$ be a sequence of unstable modes converging to a neutral limiting mode (c, α, β, ϕ) in the sense of Definition 3. By Lemma 5, $\|\phi_k\|_{H^2} \le C$ for $k \ge 1$, and up to a subsequence, $\phi_k \to \phi$ in $C^1([y_1, y_2])$. When (c, α, β, ϕ) is nonresonant, $c \notin \mathrm{Ran}\,(U)$ and thus

$$\langle L_\alpha \omega, \omega \rangle = (c - U_\beta) \int_{y_1}^{y_2} \frac{(\beta - U'')}{|U - c|^2} |\phi|^2 dy = \lim_{k \to \infty} (c_k - U_\beta) \int_{y_1}^{y_2} \frac{(\beta - U'')}{|U - c_k|^2} |\phi_k|^2 dy = 0.$$

When (c, α, β, ϕ) is singular, using the uniform bound of $\|\omega_k\|_{L^2}$ with $\omega_k = -\phi_k'' + \alpha_k^2 \phi_k$, we have, up to a subsequence, $\omega_k \rightharpoonup \omega$ in L^2, and thus,

$$\begin{aligned} \langle L_\alpha \omega, \omega \rangle &= \int_{y_1}^{y_2} (\omega / K_\beta - \phi) \bar{\omega}\, dy = \int_{y_1}^{y_2} \left[|\omega|^2 / K_\beta - (|\phi'|^2 + \alpha^2 |\phi|^2) \right] dy \\ &\le \lim_{k \to \infty} \int_{y_1}^{y_2} \left[|\omega_k|^2 / K_\beta - \left(\left|\phi_k'\right|^2 + \alpha_k^2 |\phi_k|^2 \right) \right] dy = \lim_{k \to \infty} \langle L_{\alpha_k} \omega_k, \omega_k \rangle = 0. \end{aligned}$$

■

Next, we consider nonresonant neutral modes, which naturally correspond to the following regular Sturm-Liouville operators:

$$\mathcal{L}_{\beta,c} = -\frac{d^2}{dy^2} - \frac{\beta - U''}{U - c}, \tag{25}$$

$$D(\mathcal{L}_{\beta,c}) = \{\phi \in L^2(y_1, y_2) : \phi, \phi' \in AC([y_1, y_2]), \mathcal{L}_{\beta,c}\phi \in L^2(y_1, y_2), \phi(y_1) = \phi(y_2) = 0\},$$

where $\beta \in \mathbf{R}$, $c \notin \mathrm{Ran}\,(U)$, and $AC([y_1, y_2])$ is the space of absolutely continuous functions on $[y_1, y_2]$. For $c = U_{\min}$ or $U_{\max}$, $\mathcal{L}_{\beta,c}$ is defined as (25) with $AC([y_1, y_2])$ replaced by $AC_{\mathrm{loc}}([y_1, y_2] \backslash U^{-1}(c))$. The next lemma is to compute the derivative of the nth eigenvalue of $\mathcal{L}_{\beta,c}$ with respect to β and c separately.

Lemma 11. *For $\beta \in \mathbf{R}$ and $c \in (-\infty, U_{\min}) \cup (U_{\max}, +\infty)$, let $\lambda_n(\beta, c)$ $(n \ge 1)$ be the nth eigenvalue of $\mathcal{L}_{\beta,c}$, and $\phi_n^{(\beta,c)}$ be the corresponding eigenfunction with $\|\phi_n^{(\beta,c)}\|_{L^2} = 1$. Then*

$$\frac{\partial \lambda_n}{\partial \beta}(\beta, c) = -\int_{y_1}^{y_2} \frac{1}{U - c} |\phi_n^{(\beta,c)}|^2\, dy, \tag{26}$$

$$\frac{\partial \lambda_n}{\partial c}(\beta, c) = -\int_{y_1}^{y_2} \frac{\beta - U''}{(U-c)^2} |\phi_n^{(\beta,c)}|^2 \, dy. \tag{27}$$

Proof. We first prove (26). By theorem 2.1 in Ref. 38, for a fixed c, λ_n is continuous as a function of $\beta \in \mathbf{R}$. For any $\beta, \tilde{\beta} \in \mathbf{R}$, $\phi = \phi_n^{(\beta,c)}$ and $\tilde{\phi} = \phi_n^{(\tilde{\beta},c)}$ satisfy

$$-\phi'' - \frac{\beta - U''}{U - c}\phi = \lambda_n(\beta, c)\phi, \quad -\tilde{\phi}'' - \frac{\tilde{\beta} - U''}{U - c}\tilde{\phi} = \lambda_n(\tilde{\beta}, c)\tilde{\phi},$$

with $\phi(y_1) = \phi(y_2) = \tilde{\phi}(y_1) = \tilde{\phi}(y_2) = 0$. Thus,

$$\frac{\lambda_n(\beta, c) - \lambda_n(\tilde{\beta}, c)}{\beta - \tilde{\beta}} \int_{y_1}^{y_2} \phi\tilde{\phi} dy = -\int_{y_1}^{y_2} \frac{1}{U - c}\phi\tilde{\phi} dy.$$

Taking the limit $\tilde{\beta} \to \beta$ in the above, we prove (26).

Formula (27) can be proved in a similar way and we skip the details. ■

The following is a straightforward consequence of Lemma 11.

Corollary 1.

(i) For fixed $c_0 \in (-\infty, U_{\min})$, $\lambda_n(\beta, c_0)$ is strictly decreasing for $\beta \in \mathbf{R}$.

(ii) For fixed $c_0 \in (U_{\max}, +\infty)$, $\lambda_n(\beta, c_0)$ is strictly increasing for $\beta \in \mathbf{R}$.

(iii) For fixed $\beta_0 \in (-\infty, U''_{\min}]$, $\lambda_n(\beta_0, c)$ is strictly increasing for $c \in (-\infty, U_{\min})$ and $c \in (U_{\max}, +\infty)$, respectively.

(iv) For fixed $\beta_0 \in [U''_{\max}, +\infty)$, $\lambda_n(\beta_0, c)$ is strictly decreasing for $c \in (-\infty, U_{\min})$ and $c \in (U_{\max}, +\infty)$, respectively.

Now we can determine the sign of $\langle L_\alpha \cdot, \cdot \rangle$ for a nonresonant neutral mode by combining (24) and (27).

Theorem 4. *Let (c, α, β, ϕ) be a nonresonant neutral mode and $\omega_\alpha = -\phi'' + \alpha^2\phi$. Then $\alpha^2 = -\lambda_{n_0}(\beta, c) > 0$ for some $n_0 \geq 1$ and*

$$\langle L_\alpha \omega_\alpha, \omega_\alpha \rangle = -(c - U_\beta)\frac{\partial \lambda_{n_0}}{\partial c}(\beta, c). \tag{28}$$

3.3 | Stability criteria

In this subsection, we give a new method to study the instability of a flow U in class $\mathcal{K}^+$. Fix $\beta \in (\min U'', \max U'')$ and $\alpha > 0$. We determine the barotropic instability of the flow U in the following steps.

Index formula (23) indicates that linear stability at the wave number α is equivalent to the condition $n^-(L_\alpha) = k_i^{\leq 0}$. To determine $k_i^{\leq 0}$, we need to study the neutral modes in H^2. By Theorem 1, the neutral wave speed c must be one of the following four types: (a) U_β; (b) $U(y_1)$ or $U(y_2)$; (c) critical values of U; and (d) outside Ran (U). Since $c = U_\beta$ corresponds to the zero eigenvalue of $J_\alpha L_\alpha$ (defined in (18)), it has no contribution to $k_i^{\leq 0}$. To find neutral modes of types (b) and (c), we need to solve a (possibly) singular eigenvalue problem for the operator $\mathcal{L}_{\beta,c}$ defined in (25) with c to be

$U(y_1), U(y_2)$ or a critical value of U. For such c, if $-\alpha^2$ is a negative eigenvalue of $\mathcal{L}_{\beta,c}$ with the eigenfunction $\phi \in H^2(y_1, y_2)$, then $\lambda = -i\alpha(c - U_\beta)$ is a nonzero and purely imaginary eigenvalue of $J_\alpha L_\alpha$ with the eigenfunction $\omega = -\phi'' + \alpha^2\phi \in L^2(y_1, y_2)$. Denote $k_i^-(\lambda)$ ($k_i^{\le 0}(\lambda)$) to be the number of negative (nonpositive) dimensions of $\langle L_\alpha \cdot, \cdot\rangle$ restricted to the generalized eigenspace of λ for $J_\alpha L_\alpha$. If $\langle L_\alpha\omega, \omega\rangle \neq 0$, then $\lambda \in i\mathbf{R}$ is a simple eigenvalue of $J_\alpha L_\alpha$ and

$$k_i^{\le 0}(\lambda) = k_i^-(\lambda) = \begin{cases} 1 & \text{if } \langle L_\alpha\omega, \omega\rangle < 0, \\ 0 & \text{if } \langle L_\alpha\omega, \omega\rangle > 0. \end{cases}$$

If $\langle L_\alpha\omega, \omega\rangle = 0$, then $k_i^{\le 0}(\lambda) \ge 1$ and λ might be a multiple eigenvalue of $J_\alpha L_\alpha$.

For case (d), $\mathcal{L}_{\beta,c}$ is a regular Sturm-Liouville operator but $c \notin \operatorname{Ran}(U)$ is not given explicitly. For a given $\alpha > 0$, the number of nonresonant neutral modes is exactly the number of solutions of $\lambda_n(\beta, c) = -\alpha^2$ for all $n \ge 1$, where $\lambda_n(\beta, c)$ is given in Lemma 11. Let c^* be a solution of $\lambda_{n_0}(\beta, c) = -\alpha^2$ for some $n_0 \ge 1$. Then $\lambda_{n_0}(\beta, c^*) < 0$ is the n_0th eigenvalue of $\mathcal{L}_{\beta,c^*}$ with the eigenfunction ϕ^*, and correspondingly, $\lambda^* = -i\alpha(c^* - U_\beta)$ is a nonzero and purely imaginary eigenvalue of $J_\alpha L_\alpha$ with the eigenfunction $\omega^* = -\phi^{*\prime\prime} + \alpha^2\phi^*$. If $\partial_c\lambda_{n_0}(\beta, c^*) \neq 0$, then by (28),

$$\langle L_\alpha\omega^*, \omega^*\rangle = -(c^* - U_\beta)\partial_c\lambda_{n_0}(\beta, c^*) \neq 0,$$

which implies that λ^* is a simple eigenvalue of $J_\alpha L_\alpha$ with

$$k_i^-\left(\lambda^*\right) = \begin{cases} 1 & \text{if } (c^* - U_\beta)\partial_c\lambda_{n_0}(\beta, c^*) > 0, \\ 0 & \text{if } (c^* - U_\beta)\partial_c\lambda_{n_0}(\beta, c^*) < 0. \end{cases}$$

If $\partial_c\lambda_{n_0}(\beta, c^*) = 0$, then $\langle L_\alpha\omega^*, \omega^*\rangle = 0$ and λ^* might be a multiple eigenvalue of $J_\alpha L_\alpha$. In this case, we have $k_i^{\le 0}(\lambda^*) \ge 1$. Note that by Lemma 10, only points with $\partial_c\lambda_{n_0}(\beta, c^*) = 0$ could be a neutral limiting mode, ie, possibly be the boundary for stability/instability.

Remark 3. For fixed β, suppose that the operator $\tilde{L}_0$ has at least one negative eigenvalue and recall that the lowest one is denoted by $-\alpha_{\max}^2 < 0$. By (10) and Lemma 7, $\alpha_{\max} > 0$ gives the upper bound for the unstable wave numbers in the sense that linear stability holds when $\alpha \ge \alpha_{\max}$ and there exist unstable modes for α slightly less than $\alpha_{\max}$. When $\beta = 0$, it was shown in Ref. 6 that $(0, \alpha_{\max})$ is exactly the interval of unstable wave numbers. When $\beta \neq 0$, the situation becomes more subtle as seen from the study of Sinus flow in the next section. In particular, when $\beta > 0$, there is always a set of stable wave numbers in $(0, \alpha_{\max})$.

4 | SHARP STABILITY CRITERIA FOR THE SINUS FLOW

In this section, we consider the barotropic instability of the Sinus flow

$$U(y) = (1 + \cos(\pi y))/2, \quad y \in [-1, 1].$$

We will use the approach outlined in Subsection 3.3 to determine the sharp stability boundary for the Sinus flow in the parameter space (α, β). Our results correct the stability boundary given in the classical references Refs. 1 and 10. Moreover, the new stability boundary is confirmed by more accurate numerical results.

To begin with, we confirm that $U \in \mathcal{K}^+$ with $U_\beta = 1/2 - \beta/\pi^2$, because for any $\beta \in \mathbf{R}$,

$$K_\beta = \frac{\beta - U''}{U - (1/2 - \beta/\pi^2)} = \pi^2.$$

Fix $\alpha > 0$, the linearized equation around the Sinus flow is written in the Hamiltonian form

$$\partial_t \omega = J_\alpha L_\alpha \omega,\ \omega \in L^2([-1,1]),$$

where

$$J_\alpha = -i\alpha(\beta - U''), \qquad L_\alpha = \frac{1}{\pi^2} - \left(-\frac{d^2}{dy^2} + \alpha^2\right)^{-1}.$$

Clearly,

$$\sigma(L_\alpha) = \left\{ \frac{1}{\pi^2} - \left(\frac{k^2\pi^2}{4} + \alpha^2\right)^{-1} \right\}_{k=1}^{\infty}.$$

The number of negative eigenvalues of L_α is

$$n^-(L_\alpha) = \begin{cases} 1 & 0 < \alpha < \frac{\sqrt{3}\pi}{2}, \\ 0 & \alpha \geq \frac{\sqrt{3}\pi}{2}. \end{cases}$$

By Theorem 3, we get the following instability index formula for the Sinus flow.

Theorem 5. *For any $\alpha \in [\sqrt{3}\pi/2, +\infty)$, L_α is nonnegative and the flow is linearly stable for perturbations of period $2\pi/\alpha$. For any $\alpha \in (0, \sqrt{3}\pi/2)$, the index formula*

$$k_c + k_r + k_i^{\leq 0} = 1 \tag{29}$$

is satisfied for the eigenvalues of $J_\alpha L_\alpha$.

Rayleigh-Kuo criterion ensures that a necessary condition for instability is $\beta \in (-\pi^2/2, \pi^2/2)$. Therefore, from now on, we should restrict our attention to

$$(\alpha, \beta) \in (0, \sqrt{3}\pi/2) \times (-\pi^2/2, \pi^2/2)$$

for instability. In this case, the index formula (29) implies that $k_i^{\leq 0} \leq 1$, and linear stability holds if and only if $k_i^{\leq 0} = 1$. Thus, the study of linear stability is reduced to the existence of H^2-neutral modes with nonpositive signature.

4.1 | H^2-neutral modes

We start to search for H^2 neutral modes. By Theorem 1, the possible wave speeds of H^2 neutral modes are

(i) $c = U_\beta = 1/2 - \beta/\pi^2$. This corresponds to the zero eigenvalue of $J_\alpha L_\alpha$, and thus, it has no contribution to $k_i^{\leq 0}$.

(ii) $c = U(\pm 1) = 0$. This case is solved explicitly in Appendix B.1.

(iii) $c = U(0) = 1$, where $U'(0) = 0$. This case is also solved explicitly in Appendix B.2.

(iv) $c \in (-\infty, 0) \cup (1, \infty)$. In this case, the nonresonant neutral modes are solutions of the eigenvalue problem

$$\mathcal{L}_{\beta,c}\phi = -\phi'' - \frac{\beta - U''}{U - c}\phi = \lambda\phi, \qquad \phi(\pm 1) = 0 \tag{30}$$

with $\lambda = -\alpha^2 < 0$. Therefore, only negative eigenvalues of $\mathcal{L}_{\beta,c}$ can give rise to nonresonant neutral modes.

Denote $\mathcal{L}_{\beta,\infty} = -\frac{d^2}{dy^2}$ for $c = \pm\infty$. We use two propositions to compute the spectrum of $\mathcal{L}_{\beta,c}$ at $c = U_\beta, 0, 1$ or $\pm\infty$, and show spectral continuity at the boundary. Recall that $\lambda_n(\beta, c)$ is the nth eigenvalue of $\mathcal{L}_{\beta,c}$ for $n \geq 1$. We only consider H^1 eigenfunctions if the equation is singular. Because the proof is rather technical, we leave them in Appendix B.

Proposition 1 (Spectrum at $c = U_\beta, 0, 1, \pm\infty$)**.** *The eigenvalues of $\mathcal{L}_{\beta,c}$ for these special c values are given as follows:*

(i) For $\beta \in \mathbf{R}$, $\lambda_n(\beta, U_\beta) = (\frac{n^2}{4} - 1)\pi^2$, $\phi_n^{(\beta,U_\beta)}(y) = \sin(\frac{n\pi}{2}(y+1))$.

(ii) For $\beta \in \mathbf{R}$, $\lambda_n(\beta, \pm\infty) = \frac{n^2}{4}\pi^2$, $\phi_n^{(\beta,\pm\infty)}(y) = \sin(\frac{n\pi}{2}(y+1))$.

(iii) For $\beta < \frac{9\pi^2}{16}$, $\lambda_n(\beta, 0) = [(\gamma - \frac{1}{2} + \frac{n}{2})^2 - 1]\pi^2$, $\phi_n^{(\beta,0)}(y) = \cos^{2\gamma}(\frac{\pi}{2}y)P_{n-1}(\sin(\frac{\pi}{2}y))$, where $\gamma = \frac{1}{4} + \sqrt{-\frac{\beta}{\pi^2} + \frac{9}{16}}$ and P_{n-1} is a polynomial with order $n-1$.

(iv) For $\beta > -\frac{9\pi^2}{16}$, $\beta \neq -\frac{\pi^2}{2}$, $\lambda_n(\beta, 1) = [(\tilde{\gamma} - \frac{1}{2} + \lceil \frac{n}{2} \rceil)^2 - 1]\pi^2$,

$$\phi_n^{(\beta,1)}(y) = \begin{cases} \operatorname{sign}(y)|\sin(\frac{\pi}{2}y)|^{2\tilde{\gamma}} P_{n-1}(\cos(\frac{\pi}{2}y)) & n \text{ is even,} \\ |\sin(\frac{\pi}{2}y)|^{2\tilde{\gamma}} P_n(\cos(\frac{\pi}{2}y)) & n \text{ is odd,} \end{cases}$$

where $\tilde{\gamma} = \frac{1}{4} + \sqrt{\frac{\beta}{\pi^2} + \frac{9}{16}}$. Note that the case $\beta = -\frac{\pi^2}{2}$ is included in (i).

Here, $\phi_n^{(\beta,c)}$ is the corresponding eigenfunction of $\lambda_n(\beta, c)$. Consequently, the set of all the eigenvalues of $\mathcal{L}_{\beta,c}$ is bounded from below, and have finite multiplicity. For the essential spectrum, we have $\sigma_e(\mathcal{L}_{\beta,0}) = \emptyset$ if $\beta \in (-\infty, \frac{5\pi^2}{16}]$, and $\sigma_e(\mathcal{L}_{\beta,1}) = \emptyset$ if $\beta \in [-\frac{5\pi^2}{16}, +\infty)$.

Proposition 2 (Spectral continuity at boundary)**.** *$\mathcal{L}_{\beta,c}$ is regular for $c \in (-\infty, 0) \cup (1, \infty)$, so $\lambda_n(\beta, c)$ is continuous in these cases. Moreover, we have continuity up to the boundary:*

(i) For $\beta \in \mathbf{R}$, $\lim_{c\to\pm\infty} \lambda_n(\beta, c) = \lambda_n(\beta, \pm\infty)$.

(ii) For $\beta \in (0, \frac{\pi^2}{2})$, $\lim_{c\to 0^-} \lambda_n(\beta, c) = \lambda_n(\beta, 0)$.

(iii) For $\beta \in (-\frac{\pi^2}{2}, 0]$, $\lim_{c\to 1^+} \lambda_n(\beta, c) = \lambda_n(\beta, 1)$.

We put the proof of Proposition 2 in Appendix C. First, we make the following observation.

Lemma 12. *Let $\beta \in (-\pi^2/2, \pi^2/2)$ and $c \in [-\infty, 0] \cup [1, \infty]$. Then $\lambda_n(\beta, c) > (n^2/4 - 1)\pi^2$ for $n \geq 1$. In particular, $\lambda_n(\beta, c) > 0$ for $n \geq 2$.*

Proof. The case for $c = 0, 1, \pm\infty$ is discussed in Proposition 1. If $c \in (-\infty, 0)$, then $(1/2 - c)\pi^2 > \beta$, and by Corollary 1(i), we have $\lambda_n(\beta, c) > \lambda_n((1/2 - c)\pi^2, c) = (n^2/4 - 1)\pi^2$. If $c \in (1, +\infty)$, then $(1/2 - c)\pi^2 < \beta$, and by Corollary 1(ii), we have $\lambda_n(\beta, c) > \lambda_n((1/2 - c)\pi^2, c) = (n^2/4 - 1)\pi^2$. ∎

Since for Sinus flow, there are no neutral modes for $c \in (0, 1)$, so to count the index $k_i^{\leq 0}$, we only need to study the principal eigenvalue $\lambda_1(\beta, c)$ for $c \in (-\infty, 0] \cup [1, +\infty)$. For convenience, we denote $\lambda_\beta(c) := \lambda_1(\beta, c)$. Combining this with index formula and Theorem 4, we have the following simple criterion.

Corollary 2. *Let $(\alpha, \beta) \in (0, \sqrt{3}\pi/2) \times (-\pi^2/2, \pi^2/2)$. If $-\alpha^2 = \lambda_\beta(c) < 0$ and*

$$(c - U_\beta)\lambda_\beta'(c) \geq 0$$

for some $c \in (-\infty, 0] \cup [1, \infty)$, then $k_i^{\leq 0} = 1$ and the flow is linearly stable for perturbations of period $2\pi/\alpha$. Otherwise, $k_i^{\leq 0} = 0$ and the flow is linearly unstable for perturbations of period $2\pi/\alpha$.

To see how the principal eigenvalue $\lambda_\beta(c) = \lambda_1(\beta, c)$ behaves as a function of β and c, we first present the numerical contour plot of $-\lambda_1$ in Figure 1. The slant Γ represents the line $c = U_\beta$ (regular neutral wave speed). The dotted lines Γ_0 and Γ_1 represent $c = 0$ and 1 (singular neutral wave speeds). We denote the open region above Γ and Γ_1 by K_1, and denote the open region below Γ and Γ_0 by K_0. We add two dotted lines $\Gamma_{\pm\infty}$ at infinite far away for $c \to \pm\infty$. By Proposition 2 and Corollary 1, we have

Proposition 3. *λ_1 is analytic in K_1 and K_0 as a function of β and c. For a fixed β, $\lambda_\beta = \lambda_1(\beta, \cdot)$ is continuous in c up to the boundary. For a fixed c, $\lambda_1(\cdot, c)$ is increasing in K_1, decreasing in K_0, and continuous up to Γ.*

4.2 | Sharp stability boundary

As can be seen from Corollary 2, it is important to determine the sign of $\lambda_\beta'(c)$. We first observe the following property of continuous functions.

Lemma 13. *Let $f \in C([a, b])$ be nonnegative, $f(a) = 0$, and f' exists on $\{f > 0\} \cap (a, b)$. Let $M = \max_{[a,b]} f$ and $x^* \in [a, b]$ such that $M = f(x^*)$. Assume that f satisfies:*

$$\text{for each } y \in (0, M], \text{ there is at most one } x \in [a, b] \text{ with } f(x) = y \text{ and } f'(x) \geq 0, \tag{H}$$

where $f'(b)$ means the left derivative of f at b. Then x^* is unique and f is increasing on $[a, x^*]$ and decreasing on $[x^*, b]$. Consequently, if $f(b) > 0$ and $f'(b) \geq 0$, then $x^* = b$ and $M = f(b)$.

Proof. Assume that $M > 0$, otherwise $f \equiv 0$. Let $y \in (0, M]$ and x_1 be the smallest solution to $f(x) = y$. Then $0 = f(a) \leq f(x) < f(x_1) = y$ for $a \leq x < x_1$, and therefore, $f'(x_1) \geq 0$.

If hypothesis (H) is satisfied, then x^* is unique. To see that x_1 is the only solution to $f(x) = y$ in $[a, x^*]$, let $x_2 \leq x^*$ be the biggest solution of $f(x) = y$ in $[a, x^*]$, then $f(x_2) = y < f(x) \leq f(x^*) = M$ for $x_2 < x \leq x^*$. Thus, $f'(x_2) \geq 0$. By hypotheses (H), $x_1 = x_2$, and there are no other solutions in $[a, x^*]$ for $f(x) = y$. This shows that each $y \in (0, M]$ has a unique preimage of f in $[a, x^*]$, and therefore, f is increasing in $[a, x^*]$ due to continuity.

To see that f is decreasing in $[x^*, b]$, we claim that $f'(x) < 0$ for $x \in (x^*, b)$ with $f(x) > 0$. Indeed, let $y = f(x)$, then there exists a unique $x_1 \in [a, x^*]$ with $f(x_1) = y$ and $f'(x_1) \geq 0$. So, by hypothesis (H), $f'(x)$ cannot also be nonnegative. ∎

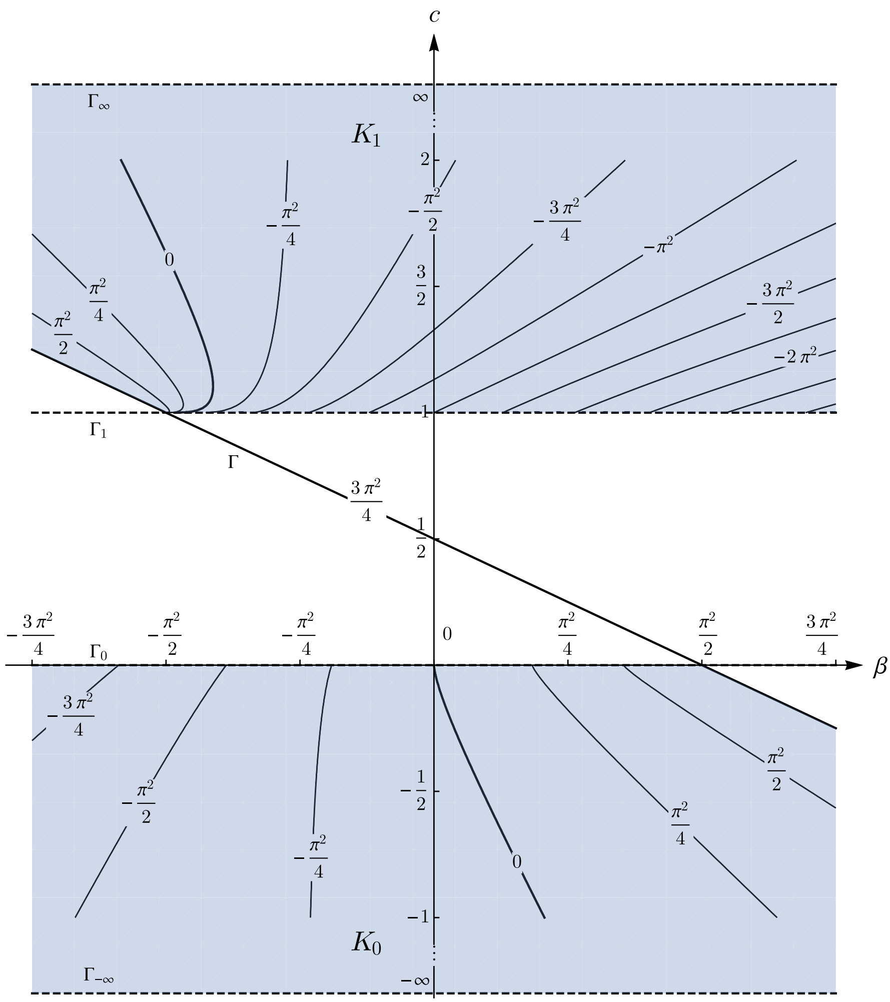

FIGURE 1 Contour plot for $-\lambda_1(\beta, c)$

Proposition 4. *Let $\beta \in (-\pi^2/2, \pi^2/2)$. Then the lower bound of unstable wave numbers for the Sinus flow is given by $\Lambda_\beta := \sup_{c\notin(0,1)} \lambda_\beta^-$, where $\lambda_\beta^-(c) = \max\{-\lambda_\beta(c), 0\}$ is the negative part of $\lambda_\beta(c)$. More precisely, we have $\Lambda_\beta < 3\pi^2/4$ and*

(i) for $\alpha^2 \in (\Lambda_\beta, 3\pi^2/4)$, $k_i^{\leq 0} = 0$ (linear instability);

(ii) for $\alpha^2 \in (0, \Lambda_\beta]$, $k_i^{\leq 0} = 1$ (linear stability).

Proof. $\Lambda_\beta < 3\pi^2/4$ is due to Lemma 12. For any $\alpha^2 \in (\Lambda_\beta, 3\pi^2/4)$, we have $\lambda_\beta(c) \geq -\Lambda_\beta > -\alpha^2$ for $c \notin (0,1)$, and thus, there are no H^2-neutral modes with the wave number α. This implies that $k_i^{\leq 0} = 0$, and the linear instability follows from the index formula (29).

To prove (ii), we assume that $\Lambda_\beta > 0$, since otherwise the conclusion is trivial. Here, we only prove the case $\beta > 0$, as the case $\beta < 0$ can be proven by a similar argument. By Lemma 8, $\lambda_\beta(c) \geq 0$ for $c \geq 1$ (otherwise there exist H^2 neutral modes), and thus $\Lambda_\beta = \sup_{c\leq 0} \lambda_\beta^-$. For a fixed $\beta \in (0, \pi^2/2)$, λ_β^- satisfies the conditions of Lemma 13 in $[-\infty, 0]$. Therefore, for any $\alpha^2 \in (0, \Lambda_\beta]$, there exists $c_1 \in$

$(-\infty, 0]$ with $\lambda_\beta^-(c_1) = \alpha^2$ and $\lambda_\beta^{-\prime}(c_1) \geq 0$, which means that $\lambda_\beta'(c_1) \leq 0$. In addition, we have $c_1 - U_\beta \leq 0$ since $U_\beta \in (0,1)$. By Corollary 2, we obtain that $k_i^{\leq 0} = 1$ and linear stability holds. ∎

By the index formula (29), λ_β^- satisfies hypothesis (H) of Lemma 13. So, the lower bound of unstable wave numbers $\Lambda_\beta = \sup_{c \notin (0,1)} \lambda_\beta^-$ is achieved at exactly one point $c^* \in (-\infty, 0]$ for $\beta \in (0, \pi^2/2)$; and $c^* \in [1, +\infty)$ for $\beta \in (-\pi^2/2, 0)$. Moreover, whether c^* is in the interior $(-\infty, 0) \cup (1, \infty)$ or on the boundary $c^* \in \{0, 1\}$ depends only on the value and derivative at boundary. In particular, Lemma 13 gives following proposition.

Proposition 5. *Assume that $\Lambda_\beta > 0$. For $\beta \in (-\pi^2/2, 0)$,*

(i) if $\lambda_\beta(1) < 0$ and $\lambda_\beta'(1) \geq 0$, then $c^* = 1$, $\Lambda_\beta = -\lambda_\beta(1)$, and λ_β^- is decreasing on $[1, +\infty]$.

(ii) if otherwise, then $c^* \in (1, +\infty)$, and λ_β^- is increasing in $[1, c^*]$ and decreasing on $[c^*, +\infty]$, respectively.

Similarly for $\beta \in (0, \pi^2/2)$,

(iii) if $\lambda_\beta(0) < 0$ and $\lambda_\beta' \leq 0$, then $c^* = 0$, $\Lambda_\beta = -\lambda_\beta(0)$, and λ_β^- is increasing in $[-\infty, 0]$.

(iv) if otherwise, then $c^* \in (-\infty, 0)$, and λ_β^- is increasing in $[-\infty, c^*]$ and decreasing $[c^*, 0]$, respectively.

By Proposition 1, $\lambda_\beta(1) > 0$ and thus $\lambda_\beta^-(1) = 0$ for $\beta \in (-\pi^2/2, 0)$, which means that we are in the case (ii) of Proposition 5. However, for $\beta \in (0, \pi^2/2)$, $\lambda_\beta(0) < 0$, so we need to compute $\lambda_\beta'(0)$. We put this computation in Appendix D and put the result here.

$$\lambda_\beta'(0) \begin{cases} \leq 0 & \beta \in \left(0, \dfrac{\sqrt{3}-1}{4}\pi^2\right], \\ >0 & \beta \in \left(\dfrac{\sqrt{3}-1}{4}\pi^2, \dfrac{5}{16}\pi^2\right), \\ = +\infty & \beta \in \left[\dfrac{5}{16}\pi^2, \dfrac{1}{2}\pi^2\right). \end{cases} \tag{31}$$

Now we are in the position to give the sharp stability boundary.

Theorem 6. *Let $\alpha > 0$ and $\beta \in (-\pi^2/2, \pi^2/2)$. Then the Sinus flow is linearly unstable if and only if $\alpha^2 \in (\Lambda_\beta, 3\pi^2/4)$. The lower bound Λ_β for unstable wave numbers is described as follows: there exist $\beta_- \in (-\pi^2/2, 0)$ and $\beta_+ = (\sqrt{3}-1)\pi^2/4 \in (0, \pi^2/2)$ such that*

(i) for $\beta \in (-\pi^2/2, \beta_-)$, $\Lambda_\beta = \lambda_\beta^-(c^*) > 0$ for some $c^* \in (1, \infty)$, and Λ_β decreases in β;

(ii) for $\beta \in [\beta_-, 0]$, $\Lambda_\beta = 0$;

(iii) for $\beta \in (0, \beta_+]$,

$$\Lambda_\beta = \lambda_\beta^-(0) = \pi^2 \left[1 - \left(\sqrt{-\frac{\beta}{\pi^2} + \frac{9}{16}} + \frac{1}{4} \right)^2 \right] > 0; \tag{32}$$

(iv) for $\beta \in (\beta_+, \pi^2/2)$, $\Lambda_\beta = \lambda_\beta^-(c^*) > \lambda_\beta^-(0)$ for some $c^* \in (-\infty, 0)$.

Proof. The upper bound $3\pi^2/4$ for unstable wave numbers is given in Theorem 5. Now, we consider the lower bound Λ_β for $\beta \in [0, \frac{\pi^2}{2})$:

- for $\beta = 0$, $\Lambda_0 = 0$ since the interval of unstable wave numbers is $(0, \sqrt{3}\pi/2)$ (see Ref. 6);
- for $\beta \in (0, \beta_+]$, since $\lambda_\beta(0) < 0$ and $\lambda'_\beta(0) \le 0$, by Proposition 5, we have $c^* = 0$ and $\Lambda_\beta = -\lambda_\beta(0)$;
- for $\beta \in (\beta_+, \pi^2/2)$, since $\lambda_\beta(0) < 0$ and $\lambda'_\beta(0) > 0$, by Proposition 5, we obtain that $c^* < 0$ and $\Lambda_\beta > -\lambda_\beta(0)$.

For $\beta \in (-\pi^2/2, 0)$, we know that either $\Lambda_\beta = 0$ and the whole $\alpha^2 \in (0, 3\pi^2/4)$ is linearly unstable, or $\Lambda_\beta > 0$ and $\alpha^2 \in (0, \Lambda_\beta]$ is linearly stable. We now show that these two scenarios are separated by some $\beta_- < 0$.

First, we claim that Λ_β is decreasing as a continuous function of $\beta \in (-\pi^2/2, 0)$. Indeed, by Lemma 8, $\lambda_\beta(c) \ge 0$ for $c \le 0$, so $\Lambda_\beta = \max_{c \notin (0,1)} \lambda_\beta^-(c) = (-\inf_{c\in(1,\infty)} \lambda_\beta(c))^+$ because of the continuity of λ_β up to the boundary. Since $\lambda_\beta(c)$ is strictly increasing and continuous in β in region K_1, $\inf_{c\in(1,\infty)} \lambda_\beta(c)$ is continuous and increasing in β. Furthermore, at $\beta = -\pi^2/2$,

$$\lambda_{-\frac{\pi^2}{2}}(1) = -\frac{3}{4}\pi^2 < \lambda_{-\frac{\pi^2}{2}}(c) \qquad \forall\, c > 1$$

by Corollary 1. Thus, $\inf_{c\in(1,\infty)} \lambda_{-\frac{\pi^2}{2}}(c) = -\frac{3}{4}\pi^2 < 0$. At $\beta = 0$,

$$\inf_{c\in(1,\infty)} \lambda_0(c) > 0,$$

because $\lambda_0(c) > 0$ for any $c \in [1, \infty]$ by (4.8) in Ref. 29 and Proposition 1, and λ_0 is continuous on $c \in [1, \infty]$ by Proposition 3. Hence, by continuity and monotonicity of $\inf_{c\in(1,\infty)} \lambda_\beta(c)$, there exists $\beta_- \in (-\pi^2/2, 0)$ such that $0 < \Lambda_\beta < 3\pi^2/4$ for $\beta \in (-\pi^2/2, \beta_-)$, and $\Lambda_\beta = 0$ for $\beta \in [\beta_-, 0)$. ■

Remark 4. (i) By numerical calculation, $\beta_- \approx -0.41224\pi^2 \approx -4.06867$.

(ii) The lower bound Λ_β is strictly decreasing to $\beta \in (-\frac{\pi^2}{2}, \beta_-)$. In fact, for any $-\frac{\pi^2}{2} < \beta_2 < \beta_1 < \beta_-$, there exists $c_1 \in (1, \infty)$ such that $\Lambda_{\beta_1} = -\lambda_{\beta_1}(c_1) > 0$, and by Corollary 1, we have $\lambda_{\beta_2}(c_1) < \lambda_{\beta_1}(c_1)$. This gives $\Lambda_{\beta_2} > \Lambda_{\beta_1}$.

Remark 5. Equation (31) may seem to be counterintuitive, as Figure 1 indicates that the derivative $\lambda'_\beta(0^-)$ should be negative for all $\beta > 0$, different from what we claimed in . However, as we zoom in near the β-axis, numerical results are consistent with (31). Near $c = 0$ and $\beta = 0.1\pi^2$, $0.25\pi^2$, and $0.4\pi^2$, we have the contour plots in Figure 2.

It can be seen that for β near $0.25\pi^2 \in (\frac{\sqrt{3}-1}{4}\pi^2, \frac{5}{16}\pi^2)$, $\lambda'_\beta(c)$ changes sign when c is very close to 0. For β near $0.4\pi^2 \in (\frac{5}{16}\pi^2, \frac{1}{2}\pi^2)$, contours are tangent to β axis, which indicates $\lambda'_\beta(c) = \infty$ as $c \to 0^-$. Therefore, for $\beta > \frac{\sqrt{3}-1}{4}\pi^2$, λ_β^- does not attain its supremum at $c = 0$, but at some $c^* < 0$ that is really close to 0 (with about a distance smaller than 0.001 based on the observation from these plots). This may be the reason why Kuo's lower stability boundary in Ref. 10 is inaccurate for $\beta > \frac{\sqrt{3}-1}{4}\pi^2$.

Remark 6. One should identify Γ_∞ with $\Gamma_{-\infty}$ to have a better understanding about the change of eigenvalues since they correspond to the same regular Sturm-Liouville problem. For instance, one can take inversion $\tilde{c} = \frac{1}{c-1/2}$ so that the domain $c \notin (0, 1)$ becomes $\tilde{c} \in [-2, 2]$. The contour plot will look like the Figure 3.

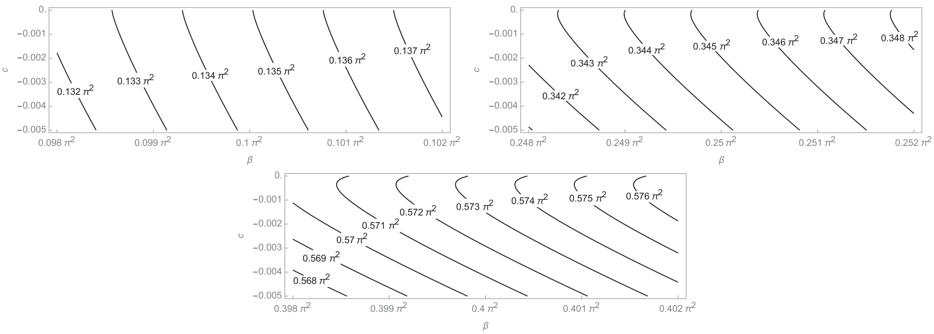

FIGURE 2 Contour plots for $-\lambda_1(\beta, c)$ near $c = 0$

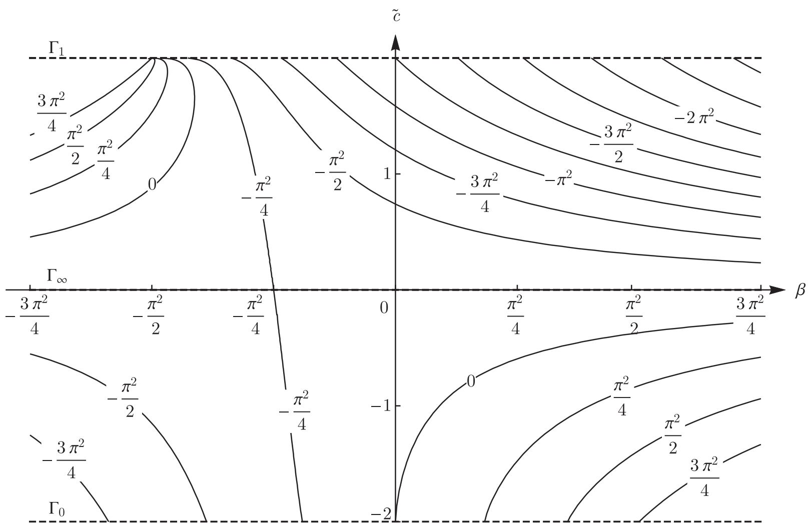

FIGURE 3 Contour plot for $-\lambda_1$ as a function of $(\beta, \tilde{c})$

Remark 7. Consider a general class $\mathcal{K}^+$ flow $U(y)$. For $\beta \in (\min U'', \max U'')$, we define $\sqrt{\Lambda_\beta} \geq 0$ to be the supremum of wave numbers for neutral modes with nonpositive L_α signature. Equivalently, Λ_β is the maximum of $\sup_{c \notin \mathrm{Ran}\,(U)} \lambda_\beta^-(c)$ (defined as in the Sinus flow) and the negative part of eigenvalues of $-\frac{d^2}{dy^2} - \frac{\beta - U''}{U - c}$ ($c = U(y_1), U(y_2)$ or a critical value of U) with nonpositive signature. Then $\sqrt{\Lambda_\beta} < \alpha_{\max}$ (defined in Subsection 3.1) and there is linear instability for $\alpha \in (\sqrt{\Lambda_\beta}, \alpha_{\max})$. Indeed, $\sqrt{\Lambda_\beta} \geq \alpha_{\max}$ would imply that $k_i^{\leq 0} \geq 1$ for $\alpha = \sqrt{\Lambda_\beta}$, which is a contradiction to the index formula (23) and the fact that $n^-(L_\alpha) = 0$ for $\alpha \geq \alpha_{\max}$. The linear instability again follows from the index formula since when $\alpha \in (\sqrt{\Lambda_\beta}, \alpha_{\max})$, we have $k_i^{\leq 0} = 0$ and $n^-(L_\alpha) > 0$.

Moreover, the interval $(\sqrt{\Lambda_\beta}, \alpha_{\max})$ gives the sharp range of unstable wave numbers if the flow shares the properties of Sinus flow. More precisely, this is true for flows satisfying: (a) $n^-(\tilde{L}_0) = 1$ ($\tilde{L}_0$ defined by (16)) so (H) is satisfied; (b) the singular neutral modes only exist with c to be the endpoints of Ran (U); (c) $E(\mathcal{L}_{\beta,U_{\max}})$ and $E(\mathcal{L}_{\beta,U_{\min}})$ are bounded from below, where $E(\mathcal{L}_{\beta,U_{\min}})$ denotes the set of all the eigenvalues of $\mathcal{L}_{\beta,U_{\min}}$; (d) the weak continuity of the principal eigenvalues holds in the sense that $\lim_{c \to U_{\min}^-} \lambda_\beta(c) = \inf E(\mathcal{L}_{\beta,U_{\min}})$ and $\lim_{c \to U_{\max}^+} \lambda_\beta(c) = \inf E(\mathcal{L}_{\beta,U_{\max}})$. The last condition is not required if $\inf E(\mathcal{L}_{\beta,U_{\min}}) > 0$ and $\liminf_{c \to U_{\min}^-} \lambda_\beta(c) > 0$, and similarly for $U_{\max}$.

4.3 | Existence of unstable mode with zero wave number

In this subsection, we show the existence of an unstable mode with zero wave number for any $\beta \in (\beta_-, 0)$.

Proposition 6. *For any $\beta \in (\beta_-, 0)$, there exists an unstable mode with $\alpha = 0$.*

Proof. By Theorem 6, there exists a sequence of unstable modes $\{(c_k, \alpha_k, \beta, \phi_k)\}$ with $\|\phi_k\|_{L^2} = 1$, $c_k^i = \operatorname{Im} c_k > 0$, and $\alpha_k \to 0^+$. We claim that $\{c_k^i\}$ has a lower bound $\delta > 0$. Suppose, otherwise, that there exists a subsequence $\{(c_{k_j}, \alpha_{k_j}, \beta, \phi_{k_j})\}$ such that $\alpha_{k_j} \to 0^+$, $c_{k_j}^r = \operatorname{Re} c_{k_j} \to c_s$, $c_{k_j}^i \to 0^+$ for some $c_s \in \mathbf{R} \cup \{\pm\infty\}$. By Proposition 1(ii), $\{c_{k_j}\}$ is bounded and thus $c_s \in \mathbf{R}$. By Lemma 5, there is a uniform H^2 bound for the unstable solutions $\{\phi_{k_j}\}$. Thus, there exists $\phi_0 \in H^2(-1,1)$ such that $\phi_{k_j} \to \phi_0$ in $C^1([-1,1])$ and $\|\phi_0\|_{L^2} = 1$. Since $\beta \in (\beta_-, 0)$, the only choice for c_s is $c_s = U_\beta$. Similar to the proof of (50)–(53) in Ref. 6, we have

$$c_{k_j}^i = \operatorname{Im} \frac{\alpha_{k_j}^2 \int_{-1}^1 \phi_{k_j} \phi_0 \, dy}{\pi^2 \int_{-1}^1 \frac{1}{U - c_{k_j}} \phi_{k_j} \phi_0 \, dy} < 0 \tag{33}$$

for sufficiently large k. Equation (33) contradicts that $c_{k_j}^i > 0$. Thus, $\{c_k^i\}$ has a lower bound $\delta > 0$.

Now we show the existence of an unstable mode with $\alpha = 0$, by taking the limit of the sequence of unstable modes $\{(c_k, \alpha_k, \beta, \phi_k)\}$. Since $\{c_k\}$ is bounded, there exists $c_0 \in \mathbf{C}$ with $\operatorname{Im} c_0 \geq \delta$ such that, up to a subsequence, $c_k \to c_0$. Since $\{|U(y) - c_k| : y \in [-1,1], k \geq 1\}$ has a uniform lower bound $\delta > 0$, we therefore get a uniform bound of $\|\phi_k\|_{H^3(-1,1)}$. Up to a subsequence, let $\phi_k \to \tilde{\phi}_0$ in $C^2([-1,1])$. Then $\tilde{\phi}_0$ solves the equation

$$-\tilde{\phi}_0'' + \frac{\beta - U''}{U - c_0} \tilde{\phi}_0 = 0, \quad \text{on } (-1,1)$$

with $\tilde{\phi}_0(\pm 1) = 0$. Thus, $(c_0, 0, \beta, \tilde{\phi}_0)$ is an unstable mode. ■

5 | BIFURCATION OF NONTRIVIAL STEADY SOLUTIONS

In this section, we prove the bifurcation of nonparallel steady flows near the shear flow $(U(y), 0)$ if there exists a nonresonant neutral mode.

Proposition 7. *Consider a shear flow $U \in C^3([-1,1])$ and fix $\beta \in \mathbf{R}$. Suppose that there is a nonresonant neutral mode $(c_0, \alpha_0, \beta, \phi_0)$ satisfying (5) and (6) with $c_0 > U_{\max}$ or $c_0 < U_{\min}$, and $\alpha_0 > 0$. Then there exists $\varepsilon_0 > 0$ such that for each $0 < \varepsilon < \varepsilon_0$, there exists a traveling wave solution $\vec{u}_\varepsilon(x - c_0 t, y) = (u_\varepsilon(x - c_0 t, y), v_\varepsilon(x - c_0 t, y))$ to (1)-(2) that has minimal period T_ε in x,*

$$\left\|\omega_\varepsilon(x,y) - \omega_0(y)\right\|_{H^2(0,T_\varepsilon)\times(-1,1)} = \varepsilon, \ \ \omega_\varepsilon = \operatorname{curl} \vec{u}_\varepsilon, \ \omega_0 = -U'(y),$$

and $T_\varepsilon \to 2\pi/\alpha_0$ when $\varepsilon \to 0$. Moreover, $u_\varepsilon(x,y) \neq 0$ and v_ε is not identically zero.

Proof. We assume $c_0 > U_{\max}$ and the case $c_0 < U_{\min}$ is similar. From the vorticity equation (3), it can be seen that $\vec{u}(x - c_0 t, y)$ is a solution of (1) if and only if

$$\frac{\partial\big(\omega + \beta y, \psi - c_0 y\big)}{\partial(x,y)} = 0,$$

and ψ takes constant values on $\{y = \pm 1\}$, where ω and ψ are the vorticity and stream function corresponding to $\vec{u}$, respectively. Let ψ_0 be a stream function associated with the shear flow $(U - c_0, 0)$, ie, $\psi_0'(y) = U(y) - c_0$. Since $U - c_0 < 0$, ψ_0 is decreasing on $(-1, 1)$. Therefore, we can define a function $f_0 \in C^2(\mathrm{Ran}\,(\psi_0))$ such that

$$f_0\big(\psi_0(y)\big) = \omega_0(y) + \beta y = -\psi_0''(y) + \beta y.$$

Thus,

$$f_0'\big(\psi_0(y)\big) = \frac{\beta - U''(y)}{U(y) - c_0} =: \mathcal{K}_{c_0}(y).$$

Then we extend f_0 to $f \in C_0^2(\mathbf{R})$ such that $f = f_0$ on $\mathrm{Ran}\,(\psi_0)$. Using the existence of the nonresonant neutral mode $(c_0, \alpha_0, \beta, \phi_0)$, we can construct steady solutions near $(U - c_0, 0)$ by solving the elliptic equation

$$-\Delta\psi + \beta y = f(\psi),$$

where $\psi(x, y)$ is the stream function and $(u, v) = (\psi_y, -\psi_x)$ is the steady velocity. The construction of steady solutions near $(U - c_0, 0)$ is similar to that near $(U, 0)$ in the proof of lemma 1 in Ref. 39, and thus, we omit its details. ■

By adjusting the traveling speed, we can construct traveling waves near the Sinus flow with the period $2\pi/\alpha_0$.

Theorem 7. *Consider the Sinus flow. Then there exists at least one nonresonant neutral mode $(c_0, \alpha_0, \beta, \phi_0)$ in the following stable cases:*

(i) $|\beta| > \pi^2/2$ *and* $0 < \alpha_0 \leq \sqrt{3}\pi/2$.
(ii) $\beta \in (-\pi^2/2, \beta_-) \cup (0, \pi^2/2)$ *and* $0 < \alpha_0 < \sqrt{\Lambda_\beta}$.

In these two cases, there exists $\varepsilon_0 > 0$ such that for each $0 < \varepsilon < \varepsilon_0$, there exists a traveling wave solution $\vec{u}_\varepsilon(x - c_\varepsilon t, y) = (u_\varepsilon(x - c_\varepsilon t, y), v_\varepsilon(x - c_\varepsilon t, y))$ to (1)-(2), which has minimal period $T_0 = \frac{2\pi}{\alpha_0}$ in x,

$$\big\|\omega_\varepsilon(x, y) - \omega_0(y)\big\|_{H^2(0,T_0)\times(-1,1)} = \varepsilon, \ \ \omega_\varepsilon = \mathrm{curl}\,\vec{u}_\varepsilon, \ \omega_0 = -U'(y),$$

with $c_\varepsilon \to c_0$ when $\varepsilon \to 0$. Moreover, $u_\varepsilon(x, y) \neq 0$ and v_ε is not identically zero.

Proof. First, we show the existence of nonresonant neutral modes in the two cases. For case (i), recall that $\lambda_\beta(c) = \lambda_1(\beta, c)$ is the principal eigenvalue of (30). Note that $\lambda_\beta(U_\beta) = -\frac{3\pi^2}{4}$ and $\lambda_\beta(\pm\infty) = \frac{\pi^2}{4}$. By continuity of λ_β, we have if $\beta > \frac{\pi^2}{2}$, then $[0, \frac{3\pi^2}{4}] \subset \{-\lambda_\beta(c) : c \in (-\infty, U_\beta]\}$, and if $\beta < -\frac{\pi^2}{2}$, then $[0, \frac{3\pi^2}{4}] \subset \{-\lambda_\beta(c) : c \in [U_\beta, +\infty)\}$. Therefore, there exists at least one nonresonant neutral mode for case (i). For case (ii), the existence of nonresonant neutral modes follows from Theorem 6.

Let $(c_0, \alpha_0, \beta, \phi_0)$ be a nonresonant neutral mode. We consider the case $c_0 > 1$ and the case $c_0 < 0$ is similar. Let $I \subset (1, \infty)$ be a small interval centered at c_0. For $c \in I$, $\lambda_\beta(c)$ is the negative eigenvalue of (30) near $\lambda_\beta(c_0) = -\alpha_0^2$. Let $\alpha(c) = \sqrt{-\lambda_\beta(c)}$. By Corollary 1 and Theorem 6, we can choose $|I|$ small enough such that $\alpha(c)$ is strictly monotone on I. Assume that $\alpha(c)$ is increasing on I. Let $c_1, c_2 \in I$ such that $c_1 < c_0 < c_2$. Then

$$\alpha(c_1) < \alpha_0 < \alpha(c_2). \tag{34}$$

By Proposition 7, for any $c \in (c_1, c_2)$, there exists local bifurcation of nonparallel traveling wave solutions of (1)-(2) near the shear flow $(U, 0)$. More precisely, we can find $r_0 > 0$ (independent of $c \in (c_1, c_2)$) such that for any $0 < r < r_0$, there exists a nontrivial traveling wave solution $\vec{u}_{c,r}(x - ct, y) = (u_{c,r}(x - ct, y), v_{c,r}(x - ct, y))$ with vorticity $\omega_{c,r}$ that has minimum x-period $T_{c,r}$ and $\|\omega_{c,r} - \omega_0\|_{H^2(0,T_{c,r})\times(-1,1)} = r$. Moreover, $2\pi/T_{c,r} \to \alpha(c)$as $r \to 0$. By (34), when r_0 is chosen to be small enough, $T_{c_2,r} < 2\pi/\alpha_0 < T_{c_1,r}$ for any $r \in (0, r_0)$. Since $T_{c,r}$ is continuous to c, for each $r \in (0, r_0)$, there exists $c^*(r) \in (c_1, c_2)$, such that $T_{c^*(r),r} = 2\pi/\alpha_0$. Then the traveling wave solution

$$\vec{u}_r\left(x - c^*(r)t, y\right) := \left(u_{c^*(r),r}\left(x - c^*(r)t, y\right), v_{c^*(r),r}\left(x - c^*(r)t, y\right)\right)$$

with the vorticity $\omega_r := \omega_{c^*(r),r}$ is a nontrivial steady solution of (1)-(2) with minimal x-period $2\pi/\alpha_0$ and $\|\omega_r - \omega_0\|_{H^2(0,2\pi/\alpha_0)\times(-1,1)} = r$. ■

Remark 8. The nonresonant neutral mode does not exist when there is no Coriolis effects (ie, $\beta = 0$). The traveling waves constructed above are thus purely due to the Coriolis forces, with traveling speeds beyond the range of the basic flow. Their existence suggests that the long time dynamics near the shear flows is much richer. This indicates that the addition of Coriolis effects can significantly change the dynamics of fluids.

6 | LINEAR INVISCID DAMPING

In this section, we prove the linear inviscid damping using the Hamiltonian structures of the linearized equation (14). First, we show that for the Sinus flow, when $\alpha^2 > 3\pi^2/4$ and $|\beta| \le \pi^2/2$, there are no neutral modes in H^2.

Lemma 14. *Consider the Sinus flow and fix any* $\beta \in [-\pi^2/2, \pi^2/2]$.

(i) For $\alpha^2 > 3\pi^2/4$, *there exist no neutral modes in* H^2.

(ii) For $\alpha^2 = 3\pi^2/4$, $(c = U_\beta, \alpha, \beta, \phi_0(y) = \cos(\pi y/2))$ *is the only neutral mode in* H^2.

Proof. If $c = U_\beta$, then it follows from Proposition 1(i) that there are no neutral modes in H^2 for $\alpha^2 > 3\pi^2/4$ and exactly one neutral solution $\phi_0 \in H^2$ for $\alpha^2 = 3\pi^2/4$. If $c \in [0, 1]$ and $c \neq U_\beta$, then the only neutral mode lies in the singular neutral modes (SNM) curve (32), with wave number $\alpha^2 < 3\pi^2/4$. If $c \notin [0, 1]$, then by Lemma 12, there are no neutral modes for $\alpha^2 \ge 3\pi^2/4$. ■

The above lemma implies that there are no purely imaginary eigenvalues of the linearized Euler operator JL defined in (14) when $\alpha^2 > 3\pi^2/4$. This implies the following inviscid damping of the velocity fields.

Theorem 8. *Consider the linearized equation* (12) *with* U *to be the Sinus flow.*

(i) For any $\alpha^2 \in (3\pi^2/4, +\infty)$ *and* $\beta \in [-\pi^2/2, \pi^2/2]$, *we have*

$$\frac{1}{T}\int_0^T \|\vec{u}(t)\|_{L^2}^2 dt \to 0, \quad \textit{when } T \to \infty,$$

for any solution $\omega(t) = \operatorname{curl} \vec{u}(t)$ *of* (12) *with* $\omega(0)$ *in the nonshear space* X *defined in* (13).

(ii) For $\alpha^2 = 3\pi^2/4$ *and* $\beta \in [-\pi^2/2, \pi^2/2]$, *we have*

$$\frac{1}{T}\int_0^T \left\|\vec{u}_1(t)\right\|_{L^2}^2 dt \to 0, \quad \textit{when } T \to \infty,$$

where $\vec{u}_1(t)$ *is the velocity corresponding to the vorticity* $(I - P_1)\omega(t)$ *with* $\omega(0) \in X$. *Here,* P_1 *is the projection of* X *to*

$$\ker L = \operatorname{Span}\left\{ e^{\pm i \frac{\sqrt{3}\pi}{2} x} \cos(\pi y/2) \right\}.$$

Proof. The solution of the linearized equation (12) is written as $\omega(t) = e^{tJL}\omega(0)$, where

$$J = -(\beta - U'')\partial_x, \; L = 1/\pi^2 - (-\Delta)^{-1} \tag{35}$$

as in (15). First, we note that when $\alpha^2 > 3\pi^2/4$, L is positive on X. Thus, $[\cdot,\cdot] = \langle L\cdot,\cdot\rangle$ defines an equivalent inner product on X with the L^2 inner product. For any $\omega_1, \omega_2 \in X$,

$$\langle LJL\omega_1, \omega_2\rangle = \langle JL\omega_1, L\omega_2\rangle = -\langle L\omega_1, JL\omega_2\rangle,$$

and thus, JL is anti-self-adjoint on $(X, [\cdot,\cdot])$. Then the spectrum of JL on $(X, [\cdot,\cdot])$ is on the imaginary axis. Since JL is a compact perturbation of $-(U - U_\beta)\partial_x$, whose spectrum is clearly the whole imaginary axis, it follows from Weyl's theorem that the continuous spectrum of JL is also the whole imaginary axis. Moreover, by Lemma 14, JL has no embedded eigenvalues on the imaginary axis. Applying the RAGE theorem[40] to e^{tJL}, we have

$$\frac{1}{T}\int_0^T \|B\omega(t)\|_{L^2}^2 dt \to 0, \quad \text{when } T \to \infty$$

for any compact operator B on $L^2(S_{2\pi/\alpha} \times [y_1, y_2])$ and for any solution $\omega(t)$ of (12) with $\omega(0) \in X$. The conclusion (i) follows by choosing $B\omega = \nabla^\perp(-\Delta)^{-1}\omega = \vec{u}$, that is, the mapping operator from vorticity to velocity.

To prove (ii), we define $X_1 = (I - P_1)X$. Then $L|_{X_1} > 0$ and $A_1 = (I - P_1)JL|_{X_1}$ is anti-self-adjoint on $(X_1, [\cdot,\cdot])$. The operator A_1 has no nonzero purely imaginary eigenvalues. Moreover, the proof of Lemma 9 implies that $\ker A_1 = \{0\}$. Therefore, A_1 has purely continuous spectrum in the imaginary axis. The conclusion again follows from the RAGE theorem to e^{tA_1} on X_1. ∎

Next, we consider the inviscid damping for the unstable case. By Theorem 6, there exist exactly one unstable mode and no neutral modes in H^2 when $\beta \in (-\pi^2/2, \pi^2/2)$ and $\alpha^2 \in (\Lambda_\beta, 3\pi^2/4)$. As in the stable case, we consider the linearized equation (12) written as Hamiltonian form $\partial_t\omega = JL\omega$ in the nonshear space X, where J and L are defined in (35). The space X is defined in (13) with α to be an unstable wave number in this case.

Denote $E^s(E^u) \subset X$ to be the stable (unstable) eigenspace of JL. Then by corollary 6.1 in Ref. 30, $L|_{E^s \oplus E^u}$ is nondegenerate and

$$n^-\left(L|_{E^s \oplus E^u}\right) = \dim E^s = \dim E^u. \tag{36}$$

Define the center space E^c to be the orthogonal (in the inner product $[\cdot,\cdot]$) complement of $E^s \oplus E^u$ in X, that is,

$$E^c = \left\{\omega \in X \mid \langle L\omega, \omega_1\rangle = 0,\ \forall\, \omega_1 \in E^s \oplus E^u\right\}. \tag{37}$$

Then we get the following results.

Lemma 15. *Consider the Sinus flow and let α be an unstable wave number. Then the decomposition $X = E^s \oplus E^c \oplus E^u$ is invariant under JL. Moreover, we have*

(i)

$$\dim E^s = \dim E^u = n^-(L). \tag{38}$$

(ii) $n^-(L|_{E^c}) = 0$*, and as a consequence,* $L|_{E^c/\ker L} > 0$*.*

(iii) The operator $JL|_{E^c}$ *has no nonzero purely imaginary eigenvalues.*

Proof. The invariance of the decomposition follows from the invariance of $\langle L\cdot,\cdot\rangle$ under JL. To prove (38), we note that JL can be decomposed as the operators $J_{l\alpha}L_{l\alpha}$ on the spaces X^l (defined in (17)) with the wave number $l\alpha$, where $0 \neq l \in \mathbf{Z}$. Then

$$\dim E^s = \dim E^u = \sum_l k_u^l,$$

where k_u^l is the number of unstable modes for $J_{l\alpha}L_{l\alpha}$. For each l, when $|\alpha l|$ is an unstable wave number, there is exactly one unstable mode, and thus, we have $k_u^l = 1 = n^-(L_{l\alpha})$. If $|\alpha l| \geq \frac{3\pi^2}{4}$, then we also have $k_u^l = 0 = n^-(L_{l\alpha})$. Therefore,

$$\dim E^s = \dim E^u = \sum_l k_u^l = \sum_l n^-\left(L_{l\alpha}\right) = n^-(L)$$

and (38) is proved.

To show (ii), noting that by the definition of E^c, (36) and (38), we have

$$n^-\left(L|_{E^c}\right) = n^-(L) - n^-\left(L_{E^s\oplus E^u}\right) = 0,$$

and thus, $L|_{E^c/\ker L} > 0$.

Finally, we prove (iii). For each l, by Theorem 6 and Lemma 14, $J_{l\alpha}L_{l\alpha}$ has no neutral modes except for $c = U_\beta$ when $|l\alpha| = \sqrt{3}\pi/2$, which corresponds to nontrivial $\ker L_{l\alpha}$ and $\ker L$. ■

Since E^c is invariant under JL, we can restrict the linearized equation (12) on E^c. The linear inviscid damping still holds true for initial data in E^c. By the same proof of Theorem 8, we have the following result.

Theorem 9. *Consider the linearized equation* (12) *with U to be the Sinus flow. Let α be an unstable wave number.*

(i) If $|\alpha l| \neq \sqrt{3}\pi/2$ for any $l \in \mathbf{Z}$, then

$$\frac{1}{T}\int_0^T \|u(t)\|_{L^2}^2 dt \to 0, \quad \text{when } T \to \infty,$$

for any solution $\omega(t)$ of (12) *with $\omega(0) \in E^c$. E^c is the center space defined in* (37).

(ii) If $|\alpha l| = \sqrt{3}\pi/2$ for some $l \in \mathbf{Z}$, then

$$\frac{1}{T}\int_0^T \|u_1(t)\|_{L^2}^2 dt \to 0, \quad \textit{when } T \to \infty,$$

where $u_1(t)$ is the velocity corresponding to the vorticity $(I - P_1)\omega(t)$ with $\omega(0) \in E^c$.

Remark 9. For general flows in class $\mathcal{K}^+$, when there are no nonzero imaginary eigenvalues for the linearized operator JL (defined in (14)), linear damping can be shown as in Theorems 8 and 9 for $\omega(0) \in L^2$.

For $\beta = 0$, nonexistence of nonzero imaginary eigenvalues and linear damping is true for flows in class $\mathcal{K}^+$.[28] Explicit decay estimates of the velocity were obtained for monotone and symmetric flows in Refs. 41–43 with more regular initial data (eg, $\omega(0) \in H^1$ or H^2).

For $\beta \neq 0$, linear damping was shown for a class of flows, and explicit decay estimates of the velocity were obtained for monotone flows in Ref. 44 under certain conditions.

7 | CONCLUSIONS FOR THE SINUS FLOW

In this section, we summarize our results for the Sinus flow and compare them with the previous work in Refs. 1 and 10. The stability picture obtained in Theorem 6 is shown in Figure 4 for the parameters (α, β).

In Figure 4, the right boundary of region IV is given by the curve

$$\Gamma_1 = \left\{ \left(\beta, \sqrt{\Lambda_\beta}\right) : \beta \in (-\pi^2/2, \beta_-) \right\}.$$

Recall that $\beta_- \approx -0.41224\pi^2$. The upper boundary of region III is given by the curve

$$\Gamma_2 = \left\{ \left(\pi^2(-\gamma^2 + \gamma/2 + 1/2), \pi\sqrt{1-\gamma^2}\right) : \gamma \in (1/2, 1) \right\}.$$

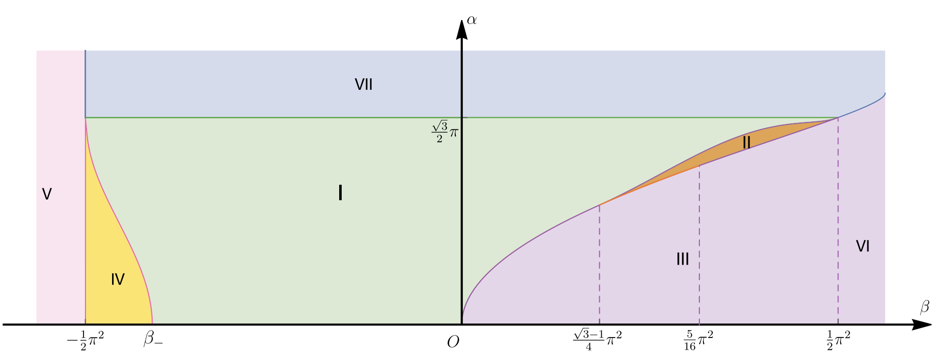

FIGURE 4 Stability picture for Sinus flow $U(y) = \frac{1+\cos(\pi y)}{2}$, $y \in [-1, 1]$

TABLE 1 Stability regions

Region	Color	Stability	Neutral modes	Wave speed c
I	Green	Unstable	0	$\mathrm{Im}(c) > 0$
II	Orange	Stable	2	$c < 0$
III	Purple	Stable	1	$c < 0$
IV	Yellow	Stable	2	$c > 1$
V	Pink	Stable	≥ 1	$c > 1$
VI	Purple	Stable	≥ 1	$c < 0$
VII	Blue	Stable	0	

This corresponds to the SNM curve with $c = 0$ (see (32)). Here, III and VI are divided by $\beta = \pi^2/2$. The upper boundary of region II (orange area) is given by

$$\Gamma_3 = \left\{ \left(\beta, \sqrt{\Lambda_\beta}\right) : \beta \in (\beta_+, \pi^2/2) \right\}.$$

Recall that $\beta_+ = (\sqrt{3} - 1)\pi^2/4$. This curve has been exaggerated in Figure 4 because it is too close to Γ_2 (See Table 2).

Only region I (green area) is the unstable domain, as indicated in Theorem 6. In region I, there exists exactly one unstable mode. Information on neutral modes in different regions is given in Table 1. More discussion on the number of nonresonant neutral modes in regions V and VI is under investigation. Dynamics near the Sinus flow is quite different in these regions. In region VII, linear inviscid damping is shown for nonshear perturbations. In regions III-V, nontrivial traveling waves are constructed using the nonresonant neutral mode. In particular, two traveling waves with different speeds exist near the Sinus flow for (α, β) in regions II and IV. Moreover, for region I, linear inviscid damping is true in the finite codimensional center space. These different behaviors indicate that with the addition of Coriolis effects, the dynamics near the Sinus flow is very rich.

In the work of Ref. 10 (see section A of chapter VII), based on numerical results, Kuo claimed that the stability boundary in the rectangular domain $(\beta, \alpha^2) \in (-\pi^2/2, \pi^2/2) \times (0, 3\pi^2/4)$ is given by the curve Γ_2 of SNMs, that is, the instability domain in Ref. 10 consists of regions I, II, and IV. See (b) in figure 6 of Ref. 10. The same stability picture can also be found in Ref. 1. The reason of incorrectness using the SNM curve Γ_2 as the stability boundary can be seen in Remark 5. Our results in Theorem 6 correct the stability picture. More precisely, the stability boundary in the rectangular domain is $\sqrt{\Lambda_\beta}$ with $\beta \in (-\pi^2/2, \pi^2/2)$, and regions II and IV actually lie in the stability domain. The stability boundary Γ_1 for $\beta \in (-\pi^2/2, \beta_-)$ was not detected in Ref. 10. Moreover, two curves of stability boundary in our results, the right boundary Γ_1 of region IV and upper boundary Γ_3 of region II, are not SNM curves. Instead, they correspond to nonresonant neutral modes with $c > 1$ or $c < 0$. As pointed out in Remark 1, this seemingly contradiction with theorem IV of Tung[29] is due to the failure of analytic assumption of the dispersion relation $c(\alpha)$. The phenomenon that nonresonant neutral curves serve as stability boundary also happened in the study of sinuous modes of the unbounded Bickley jet. In fact, based on perturbation methods, Engevik proved that $c_r > 1$ for curve (b) of figure 3 in Ref. 18, and therefore, it lies outside the range of Bickley jet $U(y) = \mathrm{sech}^2(y)$.

To confirm our theoretical results in Theorem 6, we run the numerical simulations with more accuracy for $\beta \in ((\sqrt{3} - 1)\pi^2/4, \pi^2/2)$. We find that the difference between the α values in the stability boundary $\sqrt{\Lambda_\beta}$ and those in the SNM curve (32) is actually very small. More precisely, as shown in Table 2, for a fixed β, the difference between $\sqrt{\Lambda_\beta}$ and $\pi\sqrt{1 - \gamma^2}$ is as small as 10^{-5}

TABLE 2 Difference between $\sqrt{\Lambda_\beta}$ and $\pi\sqrt{1-\gamma^2}$

β	$\sqrt{\Lambda_\beta}$	$\pi\sqrt{1-\gamma^2}$	Difference	c^*
1.80626	1.57080	1.57080	0	0
2.60650	1.90050	1.90050	0.000004894	−0.00003
2.85444	1.99395	1.99394	0.000014579	−0.00006
3.05645	2.06795	2.06792	0.000029048	−0.00009
3.24603	2.13593	2.13588	0.000049360	−0.00012
3.44449	2.20585	2.20577	0.000078511	−0.00015
3.69853	2.29388	2.29376	0.000126720	−0.00018
4.18261	2.45904	2.45882	0.000222321	−0.00018
4.37126	2.52328	2.52304	0.000233368	−0.00015
4.49531	2.56575	2.56554	0.000219151	−0.00012
4.59739	2.60097	2.60078	0.000188895	−0.00009
4.69034	2.63332	2.63318	0.000144032	−0.00006
4.78396	2.66631	2.66623	0.000083277	−0.00003
4.93480	2.72070	2.72070	0	0

to 10^{-3}, and the phase speed $c^* \in (-\infty, 0)$ such that $\Lambda_\beta = \lambda_\beta^-(c^*)$ is as small as 10^{-3}. Such small differences partly explained why the true stability boundary was not found by the numerical results in Ref. 10. Our numerical simulations for Figures 1–4 are referred to the website https://web.ma.utexas.edu/users/jcyang/blog/2019/11/19/Barotropic-Instability.

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CONFLICT OF INTEREST

The authors have no conflict of interest to declare.

ORCID

Zhiwu Lin https://orcid.org/0000-0002-7342-1805
Jincheng Yang https://orcid.org/0000-0002-3581-9425
Hao Zhu https://orcid.org/0000-0002-9875-4964

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APPENDIX A: PROOF of LEMMAS 2-3

Proof of Lemma 2. Suppose $\phi(z_j) = 0$ for $j = i_0 - 1, i_0, i_0 + 1$. If $\beta \geq 0$ and $U - c < 0$ on (z_{i_0-1}, z_{i_0}), or $\beta < 0$ and $U - c > 0$ on (z_{i_0-1}, z_{i_0}), we define $x_1 = z_{i_0-1}$ and $x_2 = z_{i_0}$. If $\beta \geq 0$ and $U - c < 0$ on (z_{i_0}, z_{i_0+1}), or $\beta < 0$ and $U - c > 0$ on (z_{i_0}, z_{i_0+1}), we define $x_1 = z_{i_0}$ and $x_2 = z_{i_0+1}$. Then we get

$$\int_{x_1}^{x_2} \left(|\phi'|^2 + \alpha^2 |\phi|^2 - \frac{\beta - U''}{U - c} |\phi|^2 \right) dy = 0.$$

Noting that $|\phi(y)|^2 \leq \|\phi'\|^2_{L^2(x_1,y)}(y - x_1)$ and $U(y) - c = U'(\xi_y)(y - x_1)$, we have $\frac{U'(y)|\phi(y)|^2}{U(y)-c} \to 0$ as $y \to x_1^+$, where $\xi_y \in (x_1, y)$. Similarly, $\frac{U'(y)|\phi(y)|^2}{U(y)-c} \to 0$ as $y \to x_2^-$. Thus, by integration by parts, we obtain

$$\int_{x_1}^{x_2} \left| \phi' - \frac{U'\phi}{U - c} \right|^2 dy + \int_{x_1}^{x_2} \left(\alpha^2 - \frac{\beta}{U - c} \right) |\phi|^2 dy = 0.$$

Note that $\alpha^2 - \beta/(U - c) > 0$ on (x_1, x_2) in all cases. Thus, $\phi \equiv 0$ on (x_1, x_2). ∎

Proof of Lemma 3. It suffices to show that $\phi(x_1) = \phi'(x_1) = 0$ implies $\phi \equiv 0$ on (x_1, x_2), and similarly $\phi \equiv 0$ on (x_0, x_1). Let $v := \phi \in C^1([x_1, x_2))$ and $u := \phi' \in C^0([x_1, x_2))$. Then (5) becomes

$$\begin{cases} \dfrac{dv}{dt} = u, \\ \dfrac{du}{dt} = \alpha^2 v - \dfrac{\beta - U''}{U - c} v, \end{cases} \tag{A.1}$$

with the initial data $v(x_1) = u(x_1) = 0$. For a fixed $z \in [x_1, x_2)$ and any $s \in [x_1, z]$,

$$|v(s)| \leq \int_{x_1}^{s} |u(\tau)| d\tau \leq (z - x_1)|u|_{L^\infty}(z), \tag{A.2}$$

where $|u|_{L_\infty}(z) := \sup_{x_1 \leq s \leq z} |u(s)|$. Thus, $|v|_{L^\infty}(z) \leq (z - x_1)|u|_{L^\infty}(z)$. Since $U'(x_1) \neq 0$, there exist $\delta_0, \delta_1 > 0$ such that $|U'(s)| > \delta_1$ for $s \in [x_1, x_1 + \delta_0]$. Choose $C_0 > 0$ such that $|\beta - U''(s)| < C_0$ for $s \in [y_1, y_2]$. Then for $z \in [x_1, x_1 + \delta_0]$,

$$\left| \frac{[(\beta - U'')v](z)}{U(z) - c} \right| = \left| (\beta - U'')(z) \frac{v(z) - v(x_1)}{U(z) - U(x_1)} \right| \leq \frac{C_0 |u|_{L^\infty}(z)}{\delta_1}, \tag{A.3}$$

and thus, by (A.1)-(A.3),

$$|u|_{L^\infty}(z) \leq \int_{x_1}^{z} \left(|\alpha^2 v(\tau)| + \left| \frac{[(\beta - U'')v](\tau)}{U(\tau) - c} \right| \right) d\tau \leq \left(\alpha^2 (x_2 - x_1) + \frac{C_0}{\delta_1} \right) \int_{x_1}^{z} |u|_{L^\infty}(\tau) d\tau.$$

Therefore, by Gronwall inequality, we have $u \equiv 0$ and thus $v = \phi \equiv 0$ on $[x_1, x_1 + \delta_0]$. This implies that $\phi \equiv 0$ on (x_1, x_2), since the ODE (5) is regular in (x_1, x_2). ■

APPENDIX B: PROOF OF PROPOSITION 1

Proof. The computation of eigenvalues in (i) and (ii) is straightforward. Equation (30) is singular when $c = 0$ or $c = 1$. We solve all the eigenvalues by transforming (30) into two hypergeometric equations as follows. ■

B.1 Case $c = 0$

Equation (30) becomes

$$-\phi'' - \frac{\beta + \frac{\pi^2}{2}\cos(\pi y)}{\cos^2\left(\frac{\pi y}{2}\right)} \phi = \lambda \phi, \quad \phi(\pm 1) = 0. \tag{B.1}$$

We make a change of variable for $y \in (0, 1)$. Set $z = \cos^2(\frac{\pi y}{2}) \in (0, 1)$ so $y = \frac{2 \arccos \sqrt{z}}{\pi}$. Define

$$\psi(z) := \phi\left(\frac{2 \arccos \sqrt{z}}{\pi} \right) = \phi(y).$$

Then the equation for ψ is

$$\frac{\pi^2}{2} \left[\left(-2z + 2z^2 \right) \psi''(z) + (-1 + 2z) \psi'(z) \right] - \frac{\beta + \frac{\pi^2}{2}(2z - 1)}{z} \psi(z) = \lambda \psi(z). \tag{B.2}$$

Suppose $\psi(z) = z^\gamma G(z)$ for some $G(z)$ where γ is a constant to be determined. Then the equation for $G(z)$ is

$$z(1 - z) G''(z) + \left[\frac{1}{2} + 2\gamma - (1 + 2\gamma) z \right] G'(z)$$

$$-\left[\frac{-\frac{\beta}{\pi^2}+\frac{1}{2}-\gamma\left(\gamma-\frac{1}{2}\right)}{z}+\gamma^2-\left(1+\frac{\lambda}{\pi^2}\right)\right]G(z)=0. \tag{B.3}$$

Set $\gamma=\frac{1}{4}+\sqrt{-\frac{\beta}{\pi^2}+\frac{9}{16}}$, so for $\beta<\frac{9\pi^2}{16}$, $\gamma>\frac{1}{4}$. Denote $r=\sqrt{1+\frac{\lambda}{\pi^2}}$, which could be purely imaginary, but nonnegative. Equation (B.3) is simplified to

$$z(1-z)G''(z)+\left[\frac{1}{2}+2\gamma-(1+2\gamma)z\right]G'(z)-(\gamma+r)(\gamma-r)G(z)=0. \tag{B.4}$$

It is the Euler's hypergeometric differential equation

$$z(1-z)G''(z)+[c-(a+b+1)z]G'(z)-ab\,G(z)=0$$

for $a=\gamma-r, b=\gamma+r, c=\frac{1}{2}+2\gamma$. A nontrivial solution of (B.4) is

$$G_1(z)=F\left(\gamma-r,\gamma+r;\frac{1}{2}+2\gamma;z\right). \tag{B.5}$$

Here, F is the hypergeometric function (eg, Ref. 45) defined by

$$F(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \tag{B.6}$$

$$(q)_n=1 \text{ if } n=0,\ (q)_n=q(q+1)\cdots(q+n-1) \text{ if } n>0. \tag{B.7}$$

F is analytic in $z\in(0,1)$ if we choose the branch cut to be $\{z>1\}$. The corresponding solution of (B.2) is

$$\psi_1(z)=z^{\gamma}F\left(\gamma-r,\gamma+r;\frac{1}{2}+2\gamma;z\right). \tag{B.8}$$

The other linearly independent solution to ψ_1 is

$$\psi_2(z)=\psi_1(z)\int_{\frac{1}{2}}^{z}\psi_1^{-2}(s)e^{-\int_{\frac{1}{2}}^{s}\frac{1}{t}\cdot\frac{1-2t}{2(1-t)}dt}ds.$$

If $\beta<\frac{\pi^2}{2}$, $\gamma>\frac{1}{2}$, direct computation deduces $\lim_{z\to 0^-}|\psi_2(z)|=\infty$, while $\lim_{z\to 0^-}\psi_1(z)=0$. Boundary condition in (B.1) converts to $\psi(0)=0$, so the only possible solution to (B.1) is

$$\phi(y)=\begin{cases}c_1\psi_1\left(\cos^2\left(\frac{\pi y}{2}\right)\right) & y\in(0,1),\\ c_2\psi_1\left(\cos^2\left(\frac{\pi y}{2}\right)\right) & y\in(-1,0).\end{cases}$$

Series expansion of ψ_1 near $y=0$ gives

$$\psi_1\left(\cos^2\left(\frac{\pi y}{2}\right)\right)=\frac{\pi^{1/2}\Gamma\left(2\gamma+\frac{1}{2}\right)}{\Gamma\left(\gamma-r+\frac{1}{2}\right)\Gamma\left(\gamma+r+\frac{1}{2}\right)}-\frac{\pi^{3/2}\Gamma\left(2\gamma+\frac{1}{2}\right)}{\Gamma(\gamma-r)\Gamma(\gamma+r)}|y|+O\left(y^2\right).$$

Since (B.1) is regular and ϕ is smooth at $y = 0$, we infer that the constant term and $|y|$ term cannot be nonzero simultaneously. Note that $\gamma > \frac{1}{4}$ is real, and r is nonnegative or purely imaginary. Since $\Gamma(z)$ only has poles at nonpositive integers, either $\gamma - r + \frac{1}{2}$ or $\gamma - r$ equals to a nonpositive integer, so $r = \gamma + \frac{n-1}{2} > \frac{1}{4}$ for a positive integer n. Therefore, $\lambda_n(\beta, 0) = \pi^2(r^2 - 1) = \pi^2((\gamma + \frac{n-1}{2})^2 - 1)$. Inserting (B.6)-(B.7) into (B.5), direct computation gives $\phi_n^{(\beta,0)} = \cos^{2\gamma}(\frac{\pi}{2}y)P_{n-1}(\sin(\frac{\pi}{2}y))$ for some polynomial P_{n-1} of degree $n-1$.

The case $\beta = \frac{\pi^2}{2}$ is included in (i) because $U_\beta = 0$, and the results agree. If $\frac{\pi^2}{2} < \beta < \frac{9\pi^2}{16}$, then $\frac{1}{4} < \gamma < \frac{1}{2}$ and $\frac{1}{2} + 2\gamma$ is not an integer. So, the other linearly independent solution to (B.4) is

$$G_2(z) = z^{\frac{1}{2}-2\gamma} F\Big(\frac{1}{2} - \gamma - r, \frac{1}{2} - \gamma + r; \frac{3}{2} - 2\gamma; z\Big).$$

This corresponds to

$$\psi_2(z) = z^{\frac{1}{2}-\gamma} F\Big(\frac{1}{2} - \gamma - r, \frac{1}{2} - \gamma + r; \frac{3}{2} - 2\gamma; z\Big).$$

So, $\lim_{z\to 0} \psi_2(z) = 0$, and both ψ_1 and ψ_2 satisfy the boundary condition. However, the leading order term of $\psi_2(\cos^2(\frac{\pi y}{2}))$ as $y \to \pm 1$ will be $|\frac{\pi}{2}(y \mp 1)|^{1-2\gamma}$. Hence, this solution cannot belong to H^1.

B.2 Case $c = 1$

Equation (30) is

$$-\phi'' - \frac{\beta + \frac{\pi^2}{2}\cos(\pi y)}{-\sin^2\left(\frac{\pi y}{2}\right)}\phi = \lambda\phi, \quad \phi(\pm 1) = 0. \tag{B.9}$$

We make a change of variable for $y \in (0,1)$. Set $z = \sin^2(\frac{\pi y}{2}) \in (0,1)$ so $y = \frac{2\arcsin\sqrt{z}}{\pi}$. Define

$$\psi(z) := \phi\left(\frac{2\arcsin\sqrt{z}}{\pi}\right) = \phi(y).$$

Then the equation for ψ is

$$\frac{\pi^2}{2}\left[\left(-2z + 2z^2\right)\psi''(z) + (-1 + 2z)\psi'(z)\right] - \frac{-\beta + \frac{\pi^2}{2}(2z-1)}{z}\psi(z) = \lambda\psi(z). \tag{B.10}$$

This is almost the same as Equation (B.2), only with $-\beta$ in the position of β. For $-\beta < \frac{9\pi^2}{16}$, we set $\tilde{\gamma} = \frac{1}{4} + \sqrt{\frac{\beta}{\pi^2} + \frac{9}{16}} > \frac{1}{4}$ and again denote $r = \sqrt{1 + \frac{\lambda}{\pi^2}}$.

If $\beta > -\frac{\pi^2}{2}$, then the only bounded solution to (B.9) is

$$\phi(y) = \begin{cases} c_1\psi_1\left(\sin^2\left(\frac{\pi y}{2}\right)\right) & y \in (0,1), \\ c_2\psi_1\left(\sin^2\left(\frac{\pi y}{2}\right)\right) & y \in (-1,0), \end{cases}$$

where $\psi_1(z)$ is defined as (B.8) with γ replaced by $\tilde{\gamma}$.

To satisfy the boundary condition, $\phi(1) = c_1\psi(1) = 0$, plugging in $y = 1$, we have

$$\psi_1\left(\sin^2\left(\frac{\pi}{2}\right)\right) = \psi_1(1) = F\left(\tilde{\gamma} - r, \tilde{\gamma} + r; \frac{1}{2} + 2\tilde{\gamma}; 1\right) = \frac{\pi^{1/2}\Gamma\left(2\tilde{\gamma} + \frac{1}{2}\right)}{\Gamma\left(\tilde{\gamma} - r + \frac{1}{2}\right)\Gamma\left(\tilde{\gamma} + r + \frac{1}{2}\right)}.$$

Therefore, nontrivial eigenfunction exists if and only if $\tilde{\gamma} - r + \frac{1}{2}$ equals to a nonpositive integer $-m$, so $r = \tilde{\gamma} + \frac{1}{2} + m > \frac{3}{4}$ for a nonnegative integer m. However, since $\psi_1(\sin^2(\frac{\pi y}{2})) = O(y^{2\tilde{\gamma}}) = o(y^{\frac{1}{2}})$ as $y \to 0$, there are two linearly independent eigenfunctions in H^1, that is, $\phi(y) = \psi_1(\sin^2(\frac{\pi y}{2}))$ and $\phi(y) = \text{sign}(y)\psi_1(\sin^2(\frac{\pi y}{2}))$. Therefore, $\lambda_n(\beta, 1) = \pi^2(r^2 - 1) = \pi^2((\tilde{\gamma} - \frac{1}{2} + \lceil \frac{n}{2} \rceil)^2 - 1)$.

The case $\beta = -\frac{\pi^2}{2}$ is again included in (i) because $U_\beta = 1$. If $-\frac{9\pi^2}{16} < \beta < -\frac{\pi^2}{2}$, $\frac{1}{4} < \tilde{\gamma} < \frac{1}{2}$, $\frac{1}{2} + 2\tilde{\gamma}$ is not an integer, so the other linearly independent solution to (B.10) is

$$\psi_2(z) = z^{\frac{1}{2}-\tilde{\gamma}} F\left(\frac{1}{2} - \tilde{\gamma} - r, \frac{1}{2} - \tilde{\gamma} + r; \frac{3}{2} - 2\tilde{\gamma}; z\right).$$

Similar as in the case of $c = 0$, by looking at the expansion near $y = 0$, $\psi_2(\sin^2(\frac{\pi y}{2}))$ cannot be in H^1.

B.3 Essential spectrum

Finally, we show that $\sigma_e(\mathcal{L}_{\beta,0}) = \emptyset$ if $\beta \in (-\infty, \frac{5\pi^2}{16}]$. Since the two linearly independent solutions of (B.1) with $\lambda_1(\beta, 0) = \pi^2(\gamma^2 - 1)$ are

$$\phi_0(y) := \phi_1^{(\beta,0)}(y) = \cos^{2\gamma}(\pi y/2), \quad \phi_1(y) = \cos^{2\gamma}(\pi y/2)\int_0^y \cos^{-4\gamma}(\pi s/2)ds.$$

Then $\phi_0 \in L^2(-1, 1)$, and $\phi_1 \notin L^2(-1, 1)$. The eigenvalue problem (B.1) is in the limit point cases at ± 1. In the limit point cases, we get by remark 10.8.1 in Ref. 46 that the starting point of the essential spectrum, ie, $\sigma_0 = \inf \sigma_e(\mathcal{L}_{\beta,0})$, is exactly the oscillation point of (B.1). More precisely, (B.1) is nonoscillatory for $\lambda < \sigma_0$ and (B.1) is oscillatory for $\lambda > \sigma_0$. Since $\lambda_n(\beta, 0) \to \infty$ as $n \to \infty$ and $\phi_n^{(\beta,0)} = \cos^{2\gamma}(\frac{\pi}{2}y)P_{n-1}(\sin(\frac{\pi}{2}y))$ has finite zeros on $(-1, 1)$, we get $\sigma_0 = \infty$ and thus $\sigma_e(\mathcal{L}_{\beta,0}) = \emptyset$ for $\beta \in (-\infty, \frac{5\pi^2}{16}]$. The proof of $\sigma_e(\mathcal{L}_{\beta,1}) = \emptyset$ when $\beta \in [-\frac{5\pi^2}{16}, +\infty)$ is similar by considering the half interval $y \in (0, 1)$.

APPENDIX C: PROOF of PROPOSITION 2

Proof. We discuss these three cases separately. The case $c \to \pm\infty$ is relatively straightforward. By theorem 2.1 in Ref. 38 and Proposition 1(ii), the conclusion follows from

$$\left\|(\beta - U'')/(U - c)\right\|_{L^1(-1,1)} \to 0 \qquad \text{as } c \to \pm\infty.$$

Next, we consider the finite endpoints $c = 0, 1$. Our method is based on regular approximations of singular Sturm-Liouville problems.

Let T be a self-adjoint operator in a Hilbert space H. Recall that for any closable operator S such that $\bar{S} = T$, its domain $D(S)$ is called a core of T. The sequence of self-adjoint operators $\{T_j\}_{j=1}^{\infty}$ is said to be spectral included for T, if for any $\lambda \in \sigma(T)$, there exists a sequence $\{\lambda(T_j)\}_{j=1}^{\infty}$ with $\lambda(T_j) \in \sigma(T_j)$ $(j \geq 1)$ such that $\lim_{j\to\infty} \lambda(T_j) = \lambda$. ■

C.1 Case $c \to 0^-$

Fix $\beta \in (0, \pi^2/2)$ and $n \geq 1$. We shall show that

$$\lambda_n(\beta, c) \to \lambda_n(\beta, 0) \qquad \text{as } c \to 0^-. \tag{C.1}$$

We begin to show that the limit (C.1) holds for $\beta \in (0, 5\pi^2/16]$. Define

$$\mathcal{L}'_{\beta,\min}\phi = \mathcal{L}_{\beta,0}\phi,\ D(\mathcal{L}'_{\beta,\min}) = \{\phi \in D(\mathcal{L}_{\beta,0}) : \phi \text{ has compact support in } (-1,1)\}.$$

We denote the closure of $\mathcal{L}'_{\beta,\min}$ by $\mathcal{L}_{\beta,\min}$. Since (B.1) is in the limit point cases at ± 1, we infer from theorem 10.4.1 and remark 10.4.2 in Ref. 46 that $\mathcal{L}_{\beta,\min} = \mathcal{L}_{\beta,0}$ and it is a self-adjoint operator on $L^2(-1,1)$. Then $D(\mathcal{L}'_{\beta,\min})$ is a core of $\mathcal{L}_{\beta,0}$. It is obvious that $D(\mathcal{L}'_{\beta,\min}) \subset D(\mathcal{L}_{\beta,c})$ for any $c < 0$, where $\mathcal{L}_{\beta,c}$ is defined in (25). Furthermore, for any $\phi \in D(\mathcal{L}'_{\beta,\min})$, by setting $\text{supp}(\phi) = [a,b] \subset (-1,1)$, we get

$$\begin{aligned}\|\mathcal{L}_{\beta,c}\phi - \mathcal{L}_{\beta,0}\phi\|_{L^2(-1,1)} &= \left\| \frac{\beta - U''}{U - c}\phi - \frac{\beta - U''}{U}\phi \right\|_{L^2(a,b)} = \left\| \frac{(\beta - U'')c}{U(U-c)}\phi \right\|_{L^2(a,b)} \\ &= \int_a^b \frac{\left(\beta - U''\right)^2 c^2}{U^2(U-c)^2}\phi^2\, dy \to 0 \qquad \text{as } c \to 0^-.\end{aligned}$$

Thus, by theorem VIII 25 (a) in Ref. 47 or theorem 9.16 (i) in Ref. 48 we have, $\{\mathcal{L}_{\beta,c}, c < 0\}$ is strongly resolvent convergent to $\mathcal{L}_{\beta,0}$ in $L^2(-1,1)$. Then it follows from theorem VIII 24 (a) in Ref. 47 that $\{\mathcal{L}_{\beta,c},\ c < 0\}$ is spectral included for $\mathcal{L}_{\beta,0}$. We then show that (C.1) holds by induction. Note that $\lambda_n(\beta,0) \in ((n^2/4 - 1)\pi^2, ((n+1)^2/4 - 1)\pi^2)$. Since $\lambda_1(\beta,0) < 0$ and $\lambda_2(\beta,c) > 0$ for all $c < 0$ by Lemma 12, we have $\lim_{c\to 0^-} \lambda_1(\beta,c) = \lambda_1(\beta,0)$. Suppose $\lim_{c\to 0^-} \lambda_n(\beta,c) = \lambda_n(\beta,0)$. Since $\lambda_n(\beta,0) \in ((n^2/4 - 1)\pi^2, ((n+1)^2/4 - 1)\pi^2)$ and $\lambda_{n+2}(\beta,c) > ((n+2)^2/4 - 1)\pi^2$ for all $c < 0$ by Lemma 12, we have $\lim_{c\to 0^-} \lambda_{n+1}(\beta,c) = \lambda_{n+1}(\beta,0)$.

Next, we show that (C.1) holds for $\beta \in (5\pi^2/16, \pi^2/2)$. The above conclusion, Corollary 1(i), and Lemma 12 ensure that for any given $n \geq 1$, there exists $\delta > 0$ such that $\lambda_n(\beta,c) \in ((n^2/4 - 1)\pi^2, ((n+1)^2/4 - 1)\pi^2)$ for any $c \in (-\delta, 0)$.

Let $c \in (-\delta, 0)$, $\phi_{n,c} := \phi_n^{(\beta,c)}$ and recall that $\|\phi_{n,c}\|_{L^2} = 1$. We get by integration by parts that

$$\begin{aligned}\int_{-1}^1 \left|\phi'_{n,c}\right|^2 dy &= \int_{-1}^1 \left[\frac{\beta - U''}{U - c} + \lambda_n(\beta,c)\right] |\phi_{n,c}|^2 dy \\ &= \pi^2 \int_{-1}^1 \left[\frac{U - c + c - U_\beta}{U - c} + \frac{\lambda_n(\beta,c)}{\pi^2}\right] |\phi_{n,c}|^2\, dy \leq \frac{(n+1)^2\pi^2}{4},\end{aligned} \tag{C.2}$$

since $U - c > 0$ on $y \in (-1,1)$ and $c - U_\beta = c - (1/2 - \beta/\pi^2) < 0$. Hence, we get $\|\phi_{n,c}\|_{H^1}^2 \leq (n+1)^2\pi^2/4 + 1$. Therefore, up to a subsequence, we have $\phi_{n,c} \rightharpoonup \tilde{\phi}_{n,0}$ in H^1 and $\phi_{n,c} \to \tilde{\phi}_{n,0}$ in $C^0([-1,1])$ for some $\tilde{\phi}_{n,0} \in H^1$. Moreover, $\|\tilde{\phi}_{n,0}\|_{L^2} = 1$ and $\tilde{\phi}_{n,0}(\pm 1) = 0$. Up to a subsequence, let

$$\lim_{c\to 0^-} \lambda_n(\beta,c) = \tilde{\lambda}_n(\beta,0) \in [(n^2/4 - 1)\pi^2, ((n+1)^2/4 - 1)\pi^2].$$

We claim that $\tilde{\phi}_{n,0}$ solves

$$-\phi'' - \frac{\beta - U''}{U}\phi = \tilde{\lambda}_n(\beta,0)\phi \ \text{ on } (-1,1), \tag{C.3}$$

with $\phi(\pm 1) = 0$. Assuming that this is true, then $\tilde{\lambda}_n(\beta, 0) = \lambda_n(\beta, 0)$, which is the unique eigenvalue in $[(n^2/4-1)\pi^2, ((n+1)^2/4-1)\pi^2]$. This proves (C.1).

It remains to show that $\tilde{\phi}_{n,0}$ satisfies (C.3). Take any closed interval $[a, b] \in (-1, 1)$. There exists $\delta_0 > 0$ such that $|U - c| \geq \delta_0$ on $[a, b]$ for any $c \in (-\delta, 0)$. Since $\phi_{n,c}$ solves the regular equation (B.1) on $[a, b]$, we get a uniform bound for $\|\phi_{n,c}\|_{H^3[a,b]}$. Thus, up to a subsequence, $\phi_{n,c} \to \tilde{\phi}_{n,0}$ in $C^2([a, b])$. Taking the limit $c \to 0^-$ in Equation (30), we deduce that $\tilde{\phi}_{n,0}$ solves Equation (C.3) on $[a, b]$, and also on $(-1, 1)$ since $[a, b] \subset (-1, 1)$ is arbitrary. This finishes the proof of (C.1).

C.2 Case $c \to 1^+$

Fix $\beta \in (-\pi^2/2, 0]$ and $n \geq 1$. We want to show

$$\lambda_n(\beta, c) \to \lambda_n(\beta, 1) \qquad \text{as } c \to 1^+.$$

For any $c \geq 1$, we obverse that the nth eigenvalue $\mu_n(\beta, c)$ of

$$-\psi'' - \frac{\beta - U''}{U - c}\psi = \mu_n(\beta, c)\psi, \quad \psi(0) = \psi(1) = 0$$

with eigenfunction $\psi_{n,c}$ is exactly the $2n$th eigenvalue $\lambda_{2n}(\beta, c)$ of (30) with eigenfunction $\phi_{n,c}$, which is defined by $\phi_{n,c}(y) = \psi_{n,c}(y)$ when $y \in [0, 1)$ and $\phi_{n,c}(y) = -\psi_{n,c}(-y)$ when $y \in (-1, 0)$. Noticing that $\mu_n(\beta, 1) \in (((2n)^2/4 - 1)\pi^2, ((2n + 1)^2/4 - 1)\pi^2]$ and $\mu_n(\beta, c) > ((2n)^2/4 - 1)\pi^2$ for all $c > 1$, and similar to the proof of (C.1), we get $\mu_n(\beta, c) \to \mu_n(\beta, 1)$ as $c \to 1^+$, which gives $\lambda_{2n}(\beta, c) \to \lambda_{2n}(\beta, 1)$ as $c \to 1^+$. This, together with Lemma 12, yields that for any given $n \geq 1$, there exist $\kappa, \nu > 0$ such that $((2n - 1)^2/4 - 1)\pi^2 < \lambda_{2n-1}(\beta, c) < ((2n + 1)^2/4 - 1)\pi^2 + \kappa$ for all $c \in (1, 1 + \nu)$. Using this bound for $\lambda_{2n-1}(\beta, c)$ and similar to the proof of (C.2), we get a uniform bound for $\|\phi_{2n-1,c}\|_{H_1}$, $c \in (1, 1 + \nu)$. Thus, there exists $\tilde{\phi}_{2n-1,1} \in H^1(-1, 1)$ such that, up to a subsequence, $\phi_{2n-1,c} \to \tilde{\phi}_{2n-1,1}$ in $C^0([-1, 1])$, $\|\tilde{\phi}_{2n-1,1}\|_{L^2} = 1$, and $\tilde{\phi}_{2n-1,1}(\pm 1) = 0$. Up to a subsequence, let

$$\tilde{\lambda}_{2n-1}(\beta, 1) = \lim_{c \to 1^+} \lambda_{2n-1}(\beta, c) \in [((2n - 1)^2/4 - 1)\pi^2, ((2n + 1)^2/4 - 1)\pi^2 + \kappa].$$

Then, as in the proof that $\tilde{\phi}_{n,0}$ solves (C.3), we get $\tilde{\phi}_{2n-1,1}$ solves

$$-\phi'' - \frac{\beta - U''}{U - 1}\phi = \tilde{\lambda}_{2n-1}(\beta, 1)\phi \qquad \text{on } (-1, 0) \cup (0, 1)$$

with $\phi(\pm 1) = 0$ and $\phi(0)$ to be finite. We observe $\lambda_{2n-1}(\beta, 1) = \lambda_{2n}(\beta, 1)$ are the only eigenvalues in the interval $[((2n - 1)^2/4 - 1)\pi^2, ((2n + 2)^2/4 - 1)\pi^2]$. Therefore, $\tilde{\lambda}_{2n-1}(\beta, 1) = \lambda_{2n-1}(\beta, 1)$.

APPENDIX D: COMPUTATION of $\lambda_\beta'(0)$

The eigenfunction for $\lambda_\beta(0)$ is $\phi(y) = \frac{1}{\sqrt{C_{4\gamma}}}\cos^{2\gamma}(\frac{\pi y}{2})$, where

$$C_s = \int_{-1}^{1} \cos^s\left(\frac{\pi y}{2}\right) dy = \begin{cases} \frac{2}{\sqrt{\pi}} \frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)} & s > -1, \\ +\infty & s \leq -1, \end{cases}$$

by Beta function $B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}=2\int_0^{\frac{\pi}{2}}\sin^{2p-1}(x)\cos^{2q-1}(x)dx$, $p,q>0$. Therefore,

$$\lambda_\beta'(0)=-\int_{-1}^{1}\frac{\beta-U''}{U^2}\phi^2\,dy=-\frac{1}{C_{4\gamma}}\int_{-1}^{1}\left(\beta-\pi^2\left(\cos^2\left(\frac{\pi y}{2}\right)-\frac{1}{2}\right)\right)\cos^{4\gamma-4}\left(\frac{\pi y}{2}\right)dy$$
$$=\frac{1}{C_{4\gamma}}\left[\left(-\beta+\frac{\pi^2}{2}\right)C_{4\gamma-4}-\pi^2C_{4\gamma-2}\right].$$

If $\frac{1}{2}<\gamma\le\frac{3}{4}$, then $C_{4\gamma-4}=+\infty$ and $\lambda_\beta'(0)=+\infty$. If $\gamma>\frac{3}{4}$, then

$$C_{4\gamma}=\frac{2}{\sqrt{\pi}}\frac{\Gamma\left(2\gamma+\frac{1}{2}\right)}{\Gamma(2\gamma+1)}=\frac{2}{\sqrt{\pi}}\frac{\Gamma\left(2\gamma-\frac{1}{2}\right)}{\Gamma(2\gamma)}\frac{2\gamma-\frac{1}{2}}{2\gamma}=\frac{2\gamma-\frac{1}{2}}{2\gamma}C_{4\gamma-2}=\frac{2\gamma-\frac{3}{2}}{2\gamma-1}\frac{2\gamma-\frac{1}{2}}{2\gamma}C_{4\gamma-4}.$$

Therefore,

$$\lambda_\beta'(0)=\frac{(2\gamma-1)(2\gamma)}{\left(2\gamma-\frac{1}{2}\right)\left(2\gamma-\frac{3}{2}\right)}\pi^2\left(\gamma^2-\frac{1}{2}\gamma-\frac{2\gamma-\frac{3}{2}}{2\gamma-1}\right)=\frac{\pi^2\gamma(\gamma-1)\left(\gamma^2-\frac{3}{4}\right)}{\left(\gamma-\frac{1}{4}\right)\left(\gamma-\frac{3}{4}\right)}.$$

This finishes the proof of (31).