1 Exponential Functions | Inverse Functions

Exponential functions are $f(x) = b^x$, where $b > 0$ is a real number. When:

1. $b > 1$: $f(x)$ is increasing;
2. $b = 1$: $f(x) = 1$;
3. $b < 1$: $f(x)$ is decreasing.

Remark 1 $b^{-x} = \frac{1}{b^x} = \left(\frac{1}{b}\right)^x$*, and the graph of* $y = b^x$ *and the graph of* $y = b^{-x}$ *are symmetric about* y*-axis.*

Definition 2 (Natural exponential function) *Natural exponential function is the exponential function whose tangent has slope 1 at* $x = 0$*. The base of natural exponential function is defined to be* $e \approx 2.718$*.*

Definition 3 (One-to-one function) *A function* f *is called* ***one-to-one*** *or* ***injective*** *if it never takes on the same value twice, that is*

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2).$$

Equivalently,

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

.

Proposition 4 (Horizontal line test) *A function is one-to-one iff ("iff" is the abbreviation for if and only if) no horizontal line intersect its graph twice.*

Definition 5 (Inverse function) *Let* f *be a one-to-one function with domain* A *and range* B*. Then its* ***inverse function*** f^{-1} *has domain* B *and range* A*, and is defined by*

$$f^{-1}(y) = x \Leftrightarrow f(x) = y$$

for any y *in* B*.*

Remark 6 (Cancellation equations)

$$\begin{gathered} f^{-1}(f(x)) = x \text{ for every } x \text{ in } A, \\ f\left(f^{-1}(x)\right) = x \text{ for every } x \text{ in } B, \end{gathered}$$

Note that $f^{-1}(x)$ *is not the same as* $\frac{1}{f(x)}$*!*

2 Logarithm Functions

Logarithm functions are the inverse function of an exponential function whose base is not 1. Denote by $f(x) = \log_b(x)$. Its relation with exponential function can be expressed as

$$\log_b x = y \Leftrightarrow b^y = x.$$

By cancellation laws,

$$\log_b (b^x) = x, \text{ for } x \in \mathbb{R},$$

$$b^{\log_b x} = x, \text{ for } x > 0.$$

Proposition 7 (Laws of Logarithms) *If x and y are positive numbers then*

1. $\log_b(xy) = \log_b x + \log_b y$,
2. $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$,
3. $\log_b (x^r) = r \log_b x$.

Definition 8 (Natural Logarithms) *We denote $\ln x = \log_e x$. Then the followings are true:*

$$\ln x = y \Leftrightarrow e^y = x,$$

$$\ln (e^x) = x,$$

$$e^{\ln x} = x,$$

$$\ln e = 1.$$

Proposition 9 (Change of Base Formular) *For any positive numver b, c $(b, c \neq 1)$, we have*

$$\log_b x = \frac{\log_c x}{\log_c b},$$

$$\log_b x = \frac{\ln x}{\ln b}.$$

3 Inverse Trigonometric Functions

Domains and ranges of the inverse functions of trigonometric functions are listed as below:

	Domain	Range
$\sin^{-1}$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\cos^{-1}$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}$	$\mathbb{R}$	$(-\frac{\pi}{2}, \frac{\pi}{2})$

Others are not very commonly used.

The trigonometric function at common angles ($0 \sim \frac{\pi}{2}$)

	$0(0°)$	$\frac{\pi}{6}(30°)$	$\frac{\pi}{4}(45°)$	$\frac{\pi}{3}(60°)$	$\frac{\pi}{2}(90°)$
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	1
tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	does not exist

How to obtain other special values:

$$
\begin{array}{cccc}
\sin\left(x-\frac{\pi}{2}\right) = -\cos(x) & \sin\left(x+\frac{\pi}{2}\right) = \cos(x) & \sin(x+\pi) = -\sin(x) & \sin(x+2\pi) = \sin(x) \\
\cos\left(x-\frac{\pi}{2}\right) = \sin(x) & \cos\left(x+\frac{\pi}{2}\right) = -\sin(x) & \cos(x+\pi) = -\cos(x) & \cos(x+2\pi) = \cos(x) \\
\tan\left(x-\frac{\pi}{2}\right) = -\cot(x) & \tan\left(x-\frac{\pi}{2}\right) = -\cot(x) & \tan(x+\pi) = \tan(x) & \tan(x+2\pi) = \tan(x)
\end{array}
$$

You can also choose to use symmetry property to do the job:

$$
\begin{aligned}
\sin(x) &= \sin(\pi - x) \\
\sin(-x) &= -\sin(x) \\
\cos(x) &= \cos(-x) \\
\cos(\pi - x) &= -\cos(x) \\
\sin(x) &= \cos\left(\frac{\pi}{2} - x\right) \\
\tan(x) &= \frac{1}{\cot(x)} \\
\tan(-x) &= -\tan(x) = \tan(\pi - x)
\end{aligned}
$$

Of course, memorize the unit circle is also one way to be familiar with trigonometric functions at common angles.

4 Limit

Intuitively speaking, we write

$$\lim_{x \to a} f(x) = L$$

or

$$f(x) \to L \text{ as } x \to a$$

and say "the limit of $f(x)$ as x approches a equals L", if $f(x)$ and L can be made arbitrarily close if x is sufficiently close to a.

We write

$$\lim_{x \to a^-} f(x) = L$$

or

$$f(x) \to L \text{ as } x \to a^-$$

and say "the limit of $f(x)$ as x approches a from the left side equals L", or "the left-hand limit of $f(x)$ as x approches a equals L", if $f(x)$ and L can be made arbitrarily close if x is sufficiently close to a from the left.

We write

$$\lim_{x \to a^+} f(x) = L$$

or

$$f(x) \to L \text{ as } x \to a^+$$

and say "the limit of $f(x)$ as x approches a from the right side equals L", or "the right-hand limit of $f(x)$ as x approches a equals L", if $f(x)$ and L can be made arbitrarily close if x is sufficiently close to a from the right.

Proposition 10

$\lim_{x \to a} f(x) = L$ *iff* $\lim_{x \to a^-} f(x) = L$ *and* $\lim_{x \to a^+} f(x) = L$.

Intuitively speaking, we write

$$\lim_{x \to a} f(x) = \infty$$

or

$$f(x) \to \infty \text{ as } x \to a$$

and say "$f(x)$ as x approches a grows without upper bound", if $f(x)$ can be arbitrarily large if x is sufficiently close to a.

Remark 11 *Sometimes we also read "the limit of $f(x)$ as x approches a is infinity", but this does not mean the limit exists!*

We write

$$\lim_{x \to a} f(x) = -\infty$$

or

$$f(x) \to -\infty \text{ as } x \to a$$

and say "$f(x)$ as x approches a grows without lower bound", if $f(x)$ can be arbitrarily large negative if x is sufficiently close to a.

Definition 12 (Vertical asymptote) *Vertical line $x = a$ is called a* ***vertical asymptote*** *of the curve $y = f(x)$ if*

$$\lim_{x \to a^-} f(x) = \infty \ or \ -\infty$$

or

$$\lim_{x \to a^+} f(x) = \infty \ or \ -\infty.$$

Some examples of infinity are:

$$\begin{aligned}
&\lim_{x \to a^+} (x-a)^{-r} = \infty, \text{ for } r > 0\\
&\lim_{x \to 0^+} \ln x = -\infty,\\
&\lim_{x \to \frac{\pi}{2}^-} \tan x = \infty.
\end{aligned}$$

5 Limit Laws

Suppose $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. c is a constant. Then

$$\lim_{x\to a}[f(x)\pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x),$$
$$\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x),$$
$$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x)\lim_{x\to a} g(x),$$
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \text{ if } \lim_{x\to a} g(x)\neq 0,$$
$$\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n, n\in\mathbb{N},$$
$$\lim_{x\to a}[f(x)]^{\frac{1}{n}} = \left[\lim_{x\to a} f(x)\right]^{\frac{1}{n}}, n \text{ is odd},$$
$$\lim_{x\to a}[f(x)]^r = \left[\lim_{x\to a} f(x)\right]^r, \text{ where } r\in\mathbb{R}, f(x)>0,$$

If f is a polynomial or a rational function, then $\lim_{x\to a} f(x) = f(a)$.

Proposition 13 *If $f(x) = g(x)$ for all $x \neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ provided one of the limits exists.*

Theorem 14 (Comparison)

If $f(x) \leq g(x)$ for all x near a (except possibly at a), and both limits of f and of g as $x \to a$ exists, then $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$. Notice that $\leq$ cannot be replaced by $<$.

If $f(x) \leq g(x)$ for all x near a (except possibly at a), and $\lim_{x\to a} f(x) = \infty$, then $\lim_{x\to a} g(x) = \infty$.

Theorem 15 (The Squeeze Theorem) *If $f(x) \leq g(x) \leq h(x)$ for all x near a (except possibly at a), and*

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L,$$

then

$$\lim_{x\to a} g(x) = L.$$

6 Continuity

Definition 16 (Continuity) *A function f is **continuous at** a if*

$$\lim_{x \to a} f(x) = f(a),$$

which implies that

1. *f is defined at $x = a$,*
2. *$\lim_{x \to a} f(x)$ exists, and*
3. *this limit equals to $f(a)$.*

*A function f is **left/right continuous** at a if*

$$\lim_{x \to a^{-/+}} f(x) = f(a).$$

*A function f is **continuous in** (a, b) if f is continuous at x for all $x \in (a, b)$. A function f is continuous in $[a, b]$ if*

1. *f is continuous in (a, b),*
2. *f is right continuous at a,*
3. *f is left continuous at b.*

The intuitive idea of a function which is continuous in an interval, is that the graph of the function can be drawn without lifting the pen. The following theorem guarentees that almost every function we know is continuous on their domain.

Theorem 17 *The following functions are continuous at every point in the domain:*

1. *polynomial / rational functions / root functions*
2. *trignometric functions / inverse trignometric functions*
3. *exponential functions / logarithm functions*

Sum, difference, product, quotients, composition, of the continuous functions are also continuous on their domains. So a lot of well-defined functions are continuous as we expect. The following theorem tells us that if we know some functions are continuous, we can use them to compute limit easier.

Theorem 18 *(Limit commutes with continuous function) If f is continuous at $lim_{x \to a} g(x)$, or even better continuous in all its domain, then*

$$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)).$$

This theorem tells us that we can commute limit symbol with basically anything we know that is continuous, for example,

1. $\lim_{x\to a}[f(x)]^r = [\lim_{x\to a} f(x)]^r$.
2. $\lim_{x\to a} b^{f(x)} = b^{\lim_{x\to a} f(x)}$.
3. $\lim_{x\to a}\sin f(x) = \sin[\lim_{x\to a} f(x)]$.

whenever functions are calculated inside their domain.

Since continuous functions are drawn without lifting the pen, connecting $f(a)$ and $f(b)$ must pass all their intermediate values, which gives

Theorem 19 (IVT) *If f is continuous in $[a,b]$, then for any N between $f(a)$ and $f(b)$, there exists c (might be more than one) between a and b such that $f(c) = N$.*

7 Limits at Infinity

Definition 20 *If $f(x)$ can be made arbitrarily close to L for sufficiently large x, then*

$$\lim_{x\to\infty} f(x) = L.$$

If $f(x)$ can be made arbitrarily close to L for sufficiently large negative x, then

$$\lim_{x\to-\infty} f(x) = L.$$

*If either of these two cases happen, then we call $y = L$ the **horizontal asymptote** for the graph of f.*

8 Derivative

Definition 21 (Dirivative / Differentiability) *f is differentiable at a if*

$$f'(a) := \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{\Delta x\to 0}\frac{\Delta f(x)_{(near\ f(a))}}{\Delta x}.$$

We call so defined $f'(a)$ the derivative of f at a.

9 Midterm Review

Covers: Chapter 1.4-2.7 Definitions:
exponential function, half-life, base of natural logarithm,
one-to-one function, inverse function,
limit, one-side limit, infinite limit, vertical asymptote, horizontal asymptote, limit at infinity.
continuity, differentiability, derivative.

How To Solve...

1. Comparing / Computing / Simplifying exponential expressions:

   (a) If there is natural log or natural exponential in the expression: Take natural logarithm and change everything into e-base using change-base formular.

   (b) Otherwise, using prime decomposition.

   Example 1.4
   $$2^x = 16^y, 16^x = 8^y.$$

   Example 1.13
   $$f(x) = 3^{8 \log_3 e \ln x}$$

2. Find inverse function of $f(x)$.

   (a) Find domain / range.

   (b) $x \leftarrow f^{-1}(x), f(x) \leftarrow x$.

   (c) Solve $f^{-1}(x)$ in terms of .

   Or

   (a) Find domain / range.

   (b) Write $y = f(x)$.

   (c) Solve x in terms of y.

   (d) Exchange x and y.

   (e) Replace y by $f^{-1}(x)$.

   Example 1.18
   $$f(x) = 4 - e^{2x}$$

3. Interprating from a graph. Identifying

   Discontinuity: Removable / Finite jump / Infinity / Vibrating...
   Not Differentiable: Forming an angle, either acute, right or obtuse.
   Derivative is positive / negative / increasing / decreasing.

4. Calculate limit / derivative:

(a) Limit commutes with continuous function / continuous operator.

(b) Quotient of polynomials when $x \to 0/\infty/-\infty$: Divide leading order term / Factorization.

Order when $x \to \infty$: exponential with positive index (larger index dominate smaller index) $>$ polynomial (higher degree dominate lower degree $>$ logarithm function $>$ finite function

(c) Difference / quotient of square roots: Factorization, Rationalization.

(d) Difference of logarithm: combine.

(e) Functions between two other functions which has the same limit at a point / Continuous function which vanishes at zero times a vibrating function: Squeeze theorem.

(f) Functions that are defined piecewisely: Use definition!

Example 3.22

$$\lim_{x \to 3\pi} \sin(x + 6\sin x)$$

Example 4.3

$$\lim_{x \to -\infty} \frac{\sqrt{x^2 + 6x}}{6x + 1}$$

Example 4.4

$$\lim_{x \to \infty} \left(\sqrt{4x^2 + 5} - 2x\right)$$

Example 4.9

$$\lim_{x \to \infty} \frac{e^x + 3e^{-x}}{3e^x + 2e^{-x}}$$

Example 4.18

$$f(x) = \sqrt{x + 3}, f'(1) = ?$$

5. Intermediate value theorem.

10 Differentiation Rules

1. $\frac{d}{dx}(c) = 0.$
2. $\frac{d}{dx}(x) = 1.$
3. $\frac{d}{dx}(x^2) = 2x.$
4. $\frac{d}{dx}(x^r) = rx^{r-1}, r \neq 0.$
5. $\frac{d}{dx}(e^x) = e^x.$
6. $\frac{d}{dx}(a^x) = a^x \ln a.$
7. $\frac{d}{dx}(\sin x) = \cos x.$
8. $\frac{d}{dx}(\cos x) = -\sin x.$
9. $\frac{d}{dx}(\tan x) = \sec^2 x.$
10. $\frac{d}{dx}[cf(x)] = cf'(x).$
11. $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x).$
12. $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$
13. $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$
14. $\frac{d}{dx}[f \circ g(x)] = f'(g(x))g'(x).$
15. $\frac{d}{dx}[f^r(x)] = rf^{r-1}(x)f'(x).$