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Vorticity interior trace estimates and higher derivative estimates via blow-up method

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Abstract

We derive several nonlinear a priori trace estimates for the 3D incompressible Navier–Stokes equation, which extend the current picture of higher derivative estimates in the mixed norm. The main ingredient is the blow-up method and a novel averaging operator, which could apply to PDEs with scaling invariance and ε-regularity, possibly with a drift.

MSC: 76D05; 35Q30

Keywords: Navier–Stokes equation; Trace estimates; Higher derivatives; Blow-up technique

1. Introduction

This paper aims to provide a family of a priori trace estimates and higher regularity estimates for the vorticity of the three-dimensional incompressible Navier–Stokes equation in a general Lipschitz domain Ω. For some $T \in (0, \infty]$, denote

$$\Omega_T = (0, T) \times \Omega.$$

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Let $u : \Omega_T \to \mathbb{R}^3$ and $P : \Omega_T \to \mathbb{R}$ be a classical solution to the Navier–Stokes equation with no-slip boundary condition:

$$\partial_t u + u \cdot \nabla u + \nabla P = \nu \Delta u, \quad \operatorname{div} u = 0 \qquad \text{in } (0, T) \times \Omega, \tag{1}$$

$$u = 0 \qquad \text{on } (0, T) \times \partial\Omega. \tag{2}$$

(2) is dropped in the absence of a boundary. By rescaling, we renormalize the equation with a unit kinematic viscosity $\nu = 1$. We study a priori bounds on the norm of derivatives of vorticity $\omega = \operatorname{curl} u$ over a lower-dimensional set $\Gamma_t \subset \Omega$, which is allowed to change in time.

This work is motivated by the study of vortex sheet, vortex filament/vortex ring, and point vortex type solutions to the Euler equation in dimensions 2 and 3. Note that the vorticity of solutions to (1) satisfies

$$\partial_t \omega + u \cdot \nabla \omega - \omega \cdot \nabla u = \nu \Delta \omega, \quad \operatorname{div} \omega = 0 \qquad \text{in } (0, T) \times \Omega.$$

For the Euler equation, due to the absence of viscosity $\nu = 0$, vorticity is not dissipated but only transported and stretched. More specifically, if at $t = 0$ vorticity is supported over some lower dimensional manifold Γ_0, then at time t, the vorticity should still be supported on some manifold Γ_t that is transported and stretched from Γ_0 by the velocity field u, which in turn can be recovered from ω via Biot–Savart law.

This type of solution is not possible for the Navier–Stokes equation due to the dissipation. Nevertheless, we want to understand, for a solution of finite energy, how much vorticity can concentrate on lower dimensional manifolds.

1.1. Main results

Throughout the article, we assume $\partial\Omega$ is uniformly Lipschitz and Γ_t is a Lipschitz graph for each $t \in (0, T)$. We define $\Omega_T = (0, T) \times \Omega$ and the space-time graph to be

$$\Gamma_T = \{(t, x) : t \in (0, T), x \in \Gamma_t\}.$$

$r_* : (0, T) \times \Omega \to \mathbb{R}$ is a function defined by (5) which characterizes the parabolic distance to the parabolic boundary. The precise setups are given in Section 2. The main theorem is the following a priori estimate on the vorticity.

Theorem 1.1. *Let $T \in (0, \infty]$, $\Omega \subset \mathbb{R}^3$ satisfy Assumption 2.1, and $\{\Gamma_t\}_{t\in(0,T)}$ be d-dimensional submanifolds satisfying Assumption 2.2 with Lipschitz constant L. There exist universal constants $C_n > 0$ for each $n \geq 0$ and a constant $C_L > 0$ depending on L such that the following is true. Let u be a classical solution to the incompressible Navier–Stokes equation (1) with no-slip boundary condition (2). Denote $\omega = \operatorname{curl} u$. There exists a measurable function $s_1 : (0, T) \times \Omega \to [0, \infty]$ with the following properties:*

$$|\nabla^n \omega(t, x)| \leq C_n s_1(t, x)^{-n-2}, \qquad \forall (t, x) \in \Omega_T, n \geq 0.$$

(a) *For any* $0 \le d \le 3$*, it holds that*

$$\left\| s_1^{-1} \mathbf{1}_{\{s_1<r_*\}} \right\|_{L^{d+1,\infty}(\Gamma_T)}^{d+1} \le C_L \|\nabla u\|_{L^2(\Omega_T)}^2.$$

(b) *If* $2 \le d \le 3$ *then for every* $t \in (0,T)$ *it holds that*

$$\left\| s_1^{-1}(t) \mathbf{1}_{\{s_1<r_*\}} \right\|_{L^{d-1,\infty}(\Gamma_t)}^{d-1} \le C_L \|\nabla u\|_{L^2(\Omega_T)}^2.$$

The theorem shows that $\nabla^n \omega$ is locally controlled in the weak $L^{\frac{d+1}{n+2}}$ space. Since $d \le 3$, the Lebesgue index $\frac{d+1}{n+2}$ is too small to pass to the limit and construct weak solutions in most cases, with two notable exceptions.

1. When $d=3$ and $n=1$, we get local $L^{\frac{4}{3},\infty}$ estimate for the vorticity gradient $\nabla\omega$, which has been proven in several works. Constantin [14] constructed suitable weak solutions in $\Omega = \mathbb{T}^3$ with $\nabla^2 u \in L^{\frac{4}{3+\varepsilon}}$ for any $\varepsilon > 0$, which was improved by Lions [27] to weak space $L^{\frac{4}{3},\infty}$, and local in space for bounded domains. For the case $\Omega = \mathbb{R}^3$, Vasseur [35] obtained local integrability of $\nabla^2 u \in L^{\frac{4}{3+\varepsilon}}_{\mathrm{loc}}$ using the blow-up method, then Choi and Vasseur [12] improved it to iterative local weak norm $L^{\frac{4}{3},\infty}(t_0,T; L^{\frac{4}{3},\infty}_{\mathrm{loc}}(\mathbb{R}^3))$ for $t_0 > 0$. They also extended this to fractional higher derivatives $(-\Delta)^{\frac{\alpha}{2}} \nabla^n u \in L^{p,\infty}(t_0,T; L^{p,\infty}_{\mathrm{loc}}(\mathbb{R}^3))$ with $p = \frac{4}{n+\alpha+1}$. Recently, Vasseur and Yang [36] improved the local integrability to spacetime Lorentz norm $\nabla^2 u \in L^{\frac{4}{3},q}_{\mathrm{loc}}$ for any $q > \frac{4}{3}$, and they also obtained the higher regularity for the vorticity $\nabla^n \omega \in L^{\frac{4}{n+2},q}_{\mathrm{loc}}$ for any $q > \frac{4}{n+2}$ and $n \ge 0$ for classical solutions.
2. When $d=2$ and $n=0$, we obtain local $L^{\frac{3}{2},\infty}$ interior trace of the vorticity. A weaker version was obtained by Vasseur and Yang [37,38] on the boundary trace $\partial\Omega$ for an averaged vorticity $\tilde{\omega}$ instead of ω itself. They used the blow-up method on the boundary to obtain this estimate and studied the inviscid limit problem.

These two estimates can thus be extended to suitable weak solutions. Recall that suitable weak solutions defined in [7] refer to a divergence-free vector field $u \in C_{\mathrm{w}}(0,T; L^2(\Omega)) \cap L^2(0,T; \dot{H}^1_0(\Omega))$ and a scalar field $P \in L^{\frac{5}{4}}(\Omega_T)$, that satisfy (1) and the following local energy inequalities in distributional sense:

$$\partial_t \frac{|u|^2}{2} + \operatorname{div}\left(u\left(\frac{|u|^2}{2} + P\right)\right) + |\nabla u|^2 \le \Delta \frac{|u|^2}{2}.$$

For such suitable solutions, as well as more general Leray–Hopf weak solutions, the energy dissipation is controlled by the initial kinetic energy:

$$\frac{1}{2}\|u(T)\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega_T)}^2 \le \frac{1}{2}\|u(0)\|_{L^2(\Omega)}^2.$$

Therefore, the above and following estimates are a priori in the sense that they only rely on the kinetic energy of the initial data $u(0)$.

Corollary 1.2. *Let $T > 0$, let $\Omega \subset \mathbb{R}^3$ be a bounded set satisfying Assumption 2.1. For any suitable weak solution u to (1)-(2), it holds that*

$$\left\| \nabla\omega \mathbf{1}_{\left\{|\nabla\omega| > Cr_*^{-3}\right\}} \right\|_{L^{\frac{4}{3},\infty}(\Omega_T)}^{\frac{4}{3}} \leq C_L \|\nabla u\|_{L^2(\Omega_T)}^2.$$

Moreover, if $\{\Gamma_t\}_{t\in(0,T)}$ satisfy Assumption 2.2 with $d = 2$, then

$$\left\| \omega \mathbf{1}_{\left\{|\omega| > Cr_*^{-2}\right\}} \right\|_{L^{\frac{3}{2},\infty}(\Gamma_T)}^{\frac{3}{2}} \leq C_L \|\nabla u\|_{L^2(\Omega_T)}^2.$$

Although previous a priori estimates only apply to the vorticity, we can control the higher derivatives for the symmetric part of ∇u as well using the Hessian of the pressure.

Theorem 1.3. *Under the same setting as in Theorem 1.1, there exists a measurable function $s_2 : (0, T) \times \Omega \to [0, \infty]$ with the following properties:*

$$|\nabla^n u(t, x)| \leq C_n s_2(t, x)^{-n-1}, \qquad \forall (t, x) \in \Omega_T, n \geq 1.$$

(a) *For any $0 \leq d \leq 3$, it holds that*

$$\left\| s_2^{-1} \mathbf{1}_{\{s_2 < r_*\}} \right\|_{L^{d+1,\infty}(\Gamma_T)}^{d+1} \leq C_L \left(\|\nabla u\|_{L^2(\Omega_T)}^2 + \left\| \nabla^2 P \right\|_{L^1(\Omega_T)} \right).$$

(b) *If $2 \leq d \leq 3$ then for every $t \in (0, T)$ it holds that*

$$\left\| s_2^{-1}(t) \mathbf{1}_{\{s_2 < r_*\}} \right\|_{L^{d-1,\infty}(\Gamma_t)}^{d-1} \leq C_L \left(\|\nabla u\|_{L^2(\Omega_T)}^2 + \left\| \nabla^2 P \right\|_{L^1(\Omega_T)} \right).$$

When $\Omega = \mathbb{R}^3$ or $\mathbb{T}^3$, the L^1 norm of $\nabla^2 P$ is controlled by L^2 norm of ∇u due to compensated compactness (see Remark 6.8), so the right-hand side is bounded again by the initial kinetic energy. When Ω has a nontrivial C^2 boundary, we can still control $\nabla^n u$ away from the parabolic boundary of Ω_T as in Proposition 6.9.

The main idea of both theorems is inspired by the blow-up method developed by Vasseur in [35]. The key is to find the appropriate "scale function" s. At each (t, x), we blow-up the equation to the scale $\rho = s(t, x)$, then use existing ε-regularity theorems for suitable weak solutions to obtain full regularity in the ρ-neighborhood. By scaling, any higher derivative is controlled by the appropriate negative power of the scale. In [35], the scale is a constant, whereas in [12] the scale can be viewed as a function of time. Now by allowing scale to depend on both time and space, we can obtain finer estimates here and below, with considerably more efforts in quantifying the scale function. In fact, our results also apply to suitable weak solutions on the regular set, and the singular set coincides with $\{s = 0\}$.

1.2. Anisotropic norm estimates and regularity criteria

The trace estimates mentioned above can also provide mixed norms of derivatives of vorticity via interpolation. First, note that taking trace to $d = 0$ yields an L^∞ estimate in x: $\nabla^n \omega$ is locally in $L_t^{\frac{1}{n+2},\infty} L_x^\infty$, but this is slightly weaker than the current knowledge of a priori estimate for higher derivatives. It was proven in [19,16] that $\partial_t^r \nabla^n u$ is $L_t^{\frac{1}{2r+n+1}} L_x^\infty$ and $L_t^{\frac{2}{4r+2n-1}} L_x^2$ in strong norms up to the parabolic boundary. However, their proofs are based on different strategies, and the dependence on the initial energy is less clear.

By interpolating the spatial trace $L_t^{1,\infty} L_x^\infty$, the isotropic norm $L_{t,x}^{4,\infty}$, and the temporal trace $L_t^\infty L_x^{2,\infty}$, we get the following picture of the anisotropic integrability. We remark that even though the isotropic Lorentz norm $L_{t,x}^{4,\infty}$ and the nested Lorentz norm $L_t^{4,\infty} L_x^{4,\infty}$ are not equivalent and not comparable, interpolation is still possible (see Appendix B). For simplicity, we restrict our attention to $\Omega = \mathbb{T}^3$. The case of a bounded domain is studied in Proposition 6.9.

Corollary 1.4. *When $\Omega = \mathbb{T}^3$, we conclude the following a priori bounds: for $0 < p < q \le \infty$ with $\frac{1}{p} + \frac{3}{q} = n + 1$, $n \ge 1$, $0 < t < T$, it holds that*

$$\|\nabla^n u\|_{L^{p,\infty}(t_0,T;L^{q,\infty}(\mathbb{T}^3))} \le C\left(\|\nabla u\|_{L^2((0,T)\times\mathbb{T}^3)}^{\frac{2}{p}} + (T - t_0)^{\frac{1}{p}} \max\{t_0^{-\frac{n+1}{2}}, 1\} \right).$$

Here $C = C(L, p, q, n)$. For $0 < q < p \le \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{n+1}{2}$, $n \ge 1$, it holds that

$$\|\nabla^n u\|_{L^{p,\infty}(t_0,T;L^{q}(\mathbb{T}^3))} \le C\left(\|\nabla u\|_{L^2((0,T)\times\mathbb{T}^3)}^{\frac{2}{q}} + (T - t_0)^{\frac{1}{q}} \max\{t_0^{-\frac{n+1}{2}}, 1\} \right).$$

In addition, for $p = q = \frac{4}{n+1}$,

$$\|\nabla^n u\|_{L^{\frac{4}{n+1},\infty}((t_0,T)\times\mathbb{T}^3)} \le C\left(\|\nabla u\|_{L^2((0,T)\times\mathbb{T}^3)}^{\frac{n+1}{2}} + (T - t_0)^{\frac{n+1}{4}} \max\{t_0^{-\frac{n+1}{2}}, 1\} \right).$$

These results are plotted in Fig. 1. Solid lines represent the strong norm $L_t^p L_x^q$ connecting $q = \infty$ and $q = 2$, which are due to [19] and [16], whereas $\nabla u \in L_t^\infty L_x^1$ is due to [14] and [27]. Dashed lines represent the weak norm $L_{t,\mathrm{loc}}^{p,\infty} L_x^{q,\infty}$ if $p < q$, and $L_{t,\mathrm{loc}}^{p,\infty} L_x^q$ if $p > q$. Note that it passes through $\nabla^2 u \in L_{t,\mathrm{loc}}^{2,\infty} L_x^1$, which strengthens the result of Lions [27] which showed $L_t^p L_x^1$ for $p < 2$. It also passes through $\nabla^2 u \in L_{t,\mathrm{loc}}^{1,\infty} L_x^{\frac{3}{2},\infty}$, which is weaker than the result of Lions [27] which showed $L_t^{1,\infty} L_x^{\frac{3}{2},1}$. The diagonal dotted line $p = q$ represents the isotropic Lorentz space $L_{t,x}^{p,\infty}$.

Unfortunately, none of the anisotropic a priori estimates could reach any regularity criteria due to the essence of the blow-up method. Indeed, since the energy norm is supercritical, all the a priori higher derivative estimates acquired via scaling are supercritical as well. However, all of the current energy criteria are either critical or subcritical. Critical norms are invariant under the scaling of Navier–Stokes equation:

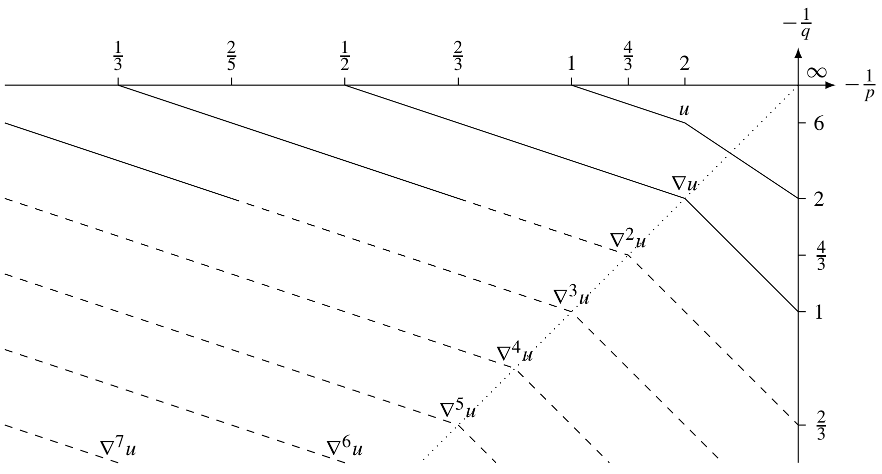

Fig. 1. Higher derivatives in mixed norm $L_t^p L_x^q$ or weak in $\Omega = \mathbb{T}^3$.

$$u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x), \qquad P_\lambda(t,x) = \lambda^2 P(\lambda^2 t, \lambda x).$$

Many of them also serve as regularity criteria: a weak solution is regular as long as one of these criteria norms is bounded. For instance, the Ladyženskaya–Prodi–Serrin criteria [25,31–33,18] assert whenever u is in $L_t^p L_x^q$ with $\frac{2}{p} + \frac{3}{q} = 1$, $2 \le p < \infty$, then u is regular. The limit case $L_t^\infty L_x^3$ was shown by Escauriaza–Serëgin–Šverák [17]. There are some logarithmic improvements or in weak Lebesgue spaces [10,5,6,28,30]. The regularity criteria on velocity was extended to regularity criteria on velocity gradients $\nabla u \in L_t^p L_x^q$ with $\frac{2}{p} + \frac{3}{q} = 2$ by Veiga [3]. This also includes a special case, the Beal–Kato–Majda criterion [2], i.e., $\|\omega\|_{L_t^1 L_x^\infty}$, which was originally proposed as a criterion for the Euler equation. It is possible to require certain norms only on some components of the velocity or velocity gradient [8,9,1,11], or in the direction of the velocity or vorticity [34,29,4,40,21]. Below we give a comparison among some of these criteria in 3D periodic domain.

Proposition 1.5. *Let $\Omega = \mathbb{T}^3$, $T > 0$, and let $u : [0,T) \times \mathbb{T}^3 \to \mathbb{R}^3$ be a classical solution to (1) with possible blowup at terminal time T. Let p, q, p', q' satisfy*

$$\frac{2}{p} + \frac{3}{q} = \frac{2}{p'} + \frac{3}{q'} = 1, \qquad 3 < q \le p < \infty, 2 \le p' \le \infty, 3 \le q' \le \infty.$$

Then there exists $\gamma = \gamma(p,p') > 0$, such that for any $t \in (0,T)$ there exists $C = C(t,T,p,p')$ such that

$$\|u\|_{L^{p'}(t,T;L^{q'}(\mathbb{T}^3))} + \|\nabla u\|^{\frac{1}{2}}_{L^{\frac{p'}{2}}(t,T;L^{\frac{q'}{2}}(\mathbb{T}^3))} \le C\left(\|u\|^{\gamma}_{L^p(0,T;L^q(\mathbb{T}^3))} + 1\right). \tag{3}$$

In addition, if $\int_{\mathbb{T}^3} u(0)\,\mathrm{d}x = 0$, and $3 < q \le \frac{15}{4} < 10 \le p < \infty$ or $4 < q \le p < 8$, then

$$\|u\|_{L^{p'}(t,T;L^{q'}(\mathbb{T}^3))} + \|\nabla u\|^{\frac{1}{2}}_{L^{\frac{p'}{2}}(t,T;L^{\frac{q'}{2}}(\mathbb{T}^3))} \le C\left(\|\nabla u\|^{\gamma}_{L^{\frac{p}{2}}(0,T;L^{\frac{q}{2}}(\mathbb{T}^3))} + 1\right). \tag{4}$$

If any Ladyženskaya–Prodi–Serrin norm or Veiga norm blow up at T^*, then $\|\nabla u\|_{L_t^p L_x^q}$ also blows up for p, q in the range $2 \le q \le p \le 4$ and $\frac{3}{2} < q \le \frac{15}{8} < 5 \le p < \infty$, and $\|u\|_{L_t^p L_x^q}$ blows up for p, q in the range $3 < q \le p < \infty$. Of course, all blow-up criteria should blow up at the same time. The purpose of this proposition is to provide a quantitative comparison of blow-up rates among different norms.

This paper is structured as the following. In Section 2, we introduce the notations and assumptions. We introduce the blow-up technique and the averaging operator in $\mathbb{R}^D$ in Section 3, the time-dependent case in Section 4, and we include drift in Section 5. We conclude in Section 6 with the proofs of the main results. Some technical lemmas and auxiliary results are deferred to Appendix A and B.

2. Preliminary

We will adopt the following assumptions on Ω and Γ_t.

Assumption 2.1. Suppose $D \ge 0$, $\Omega \subset \mathbb{R}^D$ is open and nonempty, and $\partial\Omega$ is uniformly Lipschitz. That is, there exist $r_0 > 0$, $L > 0$ such that for every $y \in \partial\Omega$, $\partial\Omega \cap B_{r_0}(y)$ is an L-Lipschitz graph. Alternatively, we could also have the periodic cube $\Omega = \mathbb{T}^D$; in this case, we denote $r_0 = 1$ to be the period of the set.

Assumption 2.2. Let $\Gamma_t \subset \Omega$ be a time-dependent L-Lipschitz graph for every $t \in (0, T)$ with dimension $0 \le d \le D$. More precisely, for every $t \in (0, T)$, up to choosing an orthonormal basis, there is a Lipschitz function $g_t : U_t \subset \mathbb{R}^d \to \mathbb{R}^{D-d}$ such that $\Gamma_t = \mathrm{Graph}(g_t)$. Note that $\Gamma_t \equiv \Omega$ is a 0-Lipschitz graph with $d = D$.

We use meas(A) or $|A|$ to denote the Lebesgue measure of a set $A \subset \mathbb{R}^D$ or $\mathbb{R}^{1+D}$. The symbol $\mathscr{H}^d$ denotes the d-dimensional Hausdorff measure. Throughout the article, we equip Γ_t with the Hausdorff measure $\mu_t = \mathscr{H}^d \llcorner \Gamma_t$. For any μ_t-measurable function f, $0 < q < \infty$, define

$$\|f\|_{L^q(\Gamma_t)} = \left(\int_{\Gamma_t} |f|^q \,\mathrm{d}\mu_t \right)^{\frac{1}{q}},$$

and $\|f\|_{L^\infty(\Gamma_t)}$ denotes the μ_t-ess sup of $|f|$. For $0 < q_1, q_2 < \infty$, define the Lorentz norm

$$\|f\|_{L^{q_1,q_2}(\Gamma_t)} = \left(q_1 \int_0^\infty \mu_t(\{|f| > \lambda\})^{\frac{q_2}{q_1}} \lambda^{q_2-1} \,\mathrm{d}\lambda \right)^{\frac{1}{q_2}}.$$

When $q_1 < q_2 = \infty$, define the weak norm

$$\|f\|_{L^{q_1,\infty}(\Gamma_t)} = \sup_{\lambda>0} \lambda \mu_t(\{|f| > \lambda\})^{\frac{1}{q_1}}.$$

When $q_1 = \infty$, L^{∞,q_2} coincides with L^∞ norm. When $q_1 = q_2$, L^{q_1,q_2} coincides with L^{q_1} norm.

We introduce the measure μ_T on Γ_T by

$$\mathrm{d}\mu_T = \mathrm{d}\mu_t\,\mathrm{d}t.$$

That is, for a set $A \subset \Gamma_T$, its μ_T measure is defined by

$$\mu_T(A) = \int_0^T \mu_t(A_t)\,\mathrm{d}t, \qquad A_t = \{x \in \Gamma_t : (t,x) \in A\}.$$

For any μ_T-measurable function f, $0 < p < \infty$, define

$$\|f\|_{L^p(\Gamma_T)} = \left(\int_{\Gamma_T} |f|^p \,\mathrm{d}\mu_T \right)^{\frac{1}{q}},$$

and $\|f\|_{L^\infty(\Gamma_T)}$ denotes the μ_T-ess sup of $|f|$. For $0 < p, q \le \infty$, define

$$\|f\|_{L_t^p L_x^q(\Gamma_T)} = \begin{cases} \left(\int_0^T \|f(t)\|_{L^q(\Gamma_t)}^p \,\mathrm{d}t \right)^{\frac{1}{p}} & p < \infty \\ \operatorname{ess\,sup}_{t\in(0,T)} \|f(t)\|_{L^q(\Gamma_t)} & p = \infty\,. \end{cases}$$

The Lorentz norms $L^{p_1,p_2}(\Gamma_T)$ and $L_t^{p_1,p_2} L_x^{q_1,q_2}(\Gamma_T)$ are defined similarly. In particular, the anisotropic weak norm is defined as the following.

Definition 2.3. Let $0 < p, q < \infty$. We define the anisotropic (nested) weak Lebesgue space $L_t^{p,\infty} L_x^{q,\infty}(\Gamma_T)$ by the space of all measurable functions $f : \Gamma_T \to \mathbb{R}$ such that

$$\|f\|_{L_t^{p,\infty} L_x^{q,\infty}(\Gamma_T)} = \sup_{\lambda>0} \lambda \left| \left\{ t \in (0,T) : \|f(t)\|_{L^{q,\infty}(\Gamma_t)} > \lambda \right\} \right|^{\frac{1}{p}} < \infty.$$

In Appendix B, we see that the anisotropic weak Lebesgue space $L_t^{p,\infty} L_x^{p,\infty}(\Gamma_T)$ is not equivalent to the weak Lebesgue space $L_{t,x}^{p,\infty}(\Gamma_T)$ even in the most simple setting, contrary to the strong Lebesgue spaces.

Similar to the strong norm, we have the following well-known interpolation and convolution theorems for the weak norm. Proofs can be found in [20].

Lemma 2.4. *Let* $0 < p_0, p_1, q_0, q_1, r_0, r_1, s_0, s_1 \le \infty$, *and* $0 < \theta < 1$. *Then we have*

$$\|f\|_{L_t^{p_\theta,r_\theta} L_x^{q_\theta,s_\theta}(\Gamma_T)} \le \|f\|_{L_t^{p_0,r_0} L_x^{q_0,s_0}(\Gamma_T)}^{1-\theta} \|f\|_{L_t^{p_1,r_1} L_x^{q_1,s_1}(\Gamma_T)}^{\theta}$$

where

$$p_\theta = (1-\theta)p_0 + \theta p_1, \qquad q_\theta = (1-\theta)q_0 + \theta q_1,$$
$$r_\theta = (1-\theta)r_0 + \theta r_1, \qquad s_\theta = (1-\theta)s_0 + \theta s_1.$$

Lemma 2.5. *Let* $1 < p, q, r < \infty$ *satisfy* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$, $f \in L^p(\mathbb{R})$ *and* $g \in L^{q,\infty}(\mathbb{R})$. *Then*

$$\|f * g\|_{L^r} \le C_{p,q} \|f\|_{L^p} \|g\|_{L^{q,\infty}}.$$

Moreover, in case $f \in L^1(\mathbb{R})$, *we have*

$$\|f * g\|_{L^{q,\infty}} \le C_q \|f\|_{L^1} \|g\|_{L^{q,\infty}}.$$

For $(t, x) \in \mathbb{R}^{1+D}$, define the parabolic distance by

$$\mathrm{dist}_{\mathcal{P}}((t_1, x_1), (t_2, x_2)) = \Big(|t_1 - t_2| + |x_1 - x_2|^2\Big)^{\frac{1}{2}},$$

and denote the parabolic boundary of Ω_T as

$$\partial_{\mathcal{P}} \Omega_T = \{0\} \times \Omega \cup [0, T) \times \partial\Omega.$$

We say a function f is essentially locally bounded at (t, x) if there exist $\delta, M > 0$ such that $|f(s, y)| \le M$ for almost every $(s, y) \in \mathbb{R}^{1+D}$ with $\mathrm{dist}_{\mathcal{P}}((t, x), (s, y)) < \delta$.

Recall the following partial regularity result of Caffarelli, Kohn, and Nirenberg [7].

Theorem 2.6. *Let u be a suitable weak solution to* (1) *in* Ω_T. *The singular set* $\mathrm{Sing}(u)$ *denotes the set where u is not essentially locally bounded. Then the one-dimensional parabolic Hausdorff measure* $\mathscr{P}^1(\mathrm{Sing}(u)) = 0$.

The estimates in this paper are over the complement of the singular set, i.e. the regular set $\mathrm{Reg}(u)$.

We introduce a distance function $r_* : \Omega_T \to \mathbb{R}$, which characterizes the parabolic distance to the parabolic boundary ($a \wedge b$ stands for $\min\{a, b\}$):

$$r_*(t, x) := \frac{\mathrm{dist}_{\mathcal{P}}((t, x), \partial_{\mathcal{P}} \Omega_T) \wedge r_0}{L + 4} = \frac{1}{L + 4} \min\Big\{\sqrt{t}, \mathrm{dist}(x, \partial\Omega), r_0\Big\}. \tag{5}$$

3. Blow-up technique and trace estimate: space case

In the blow-up technique, we deal with an equation with scaling invariance, which contains the following situation. At the unit scale, two quantities f and g are related by the following one-scale, quantitative ε-regularity theorem:

Statement 3.1 *(Template theorem).* There exist two constants $\eta, C > 0$, such that

$$\int_{B_1} f \,\mathrm{d}x \le \eta \qquad \text{implies} \qquad \|g\|_{L^\infty(B_{\frac{1}{2}})} \le C.$$

In the above statement, $f \geq 0$ is called a "pivot quantity", and g is called the "controlled quantity". For instance, suppose $u : \Omega \to \mathbb{R}$ solves the elliptic equation

$$\operatorname{div}(A\nabla u) = 0 \qquad \text{in } \Omega \subset \mathbb{R}^D$$

where $A : \Omega \to \mathcal{M}^{D\times D}_{\mathrm{sym}}(\mathbb{R})$ is a uniformly elliptic matrix with $0 < \lambda\,\mathrm{Id} \leq A \leq \Lambda\,\mathrm{Id}$. Then the first De Giorgi lemma provides Statement 3.1 with $f = |u|^2$ and $g = u$. There are also corresponding variants in the time-dependent setting, with ball B_1 replaced by cylinder $Q_1 = (-1, 0] \times B_1$. One example is the ε-regularity theorem for the Navier–Stokes in [7] with $f = |u|^3 + |p|^{\frac{3}{2}}$ and $g = u$.

Next, we rescale the above statement to ε-scale near $x \in \Omega$, using the scaling of the equation. Below $\alpha \geq 0$ and β are two real numbers.

Statement 3.2 *(Rescaling of the template theorem).* There exist two universal constants $\eta, C > 0$, such that for any $\varepsilon > 0$ sufficiently small, it holds that

$$\fint_{B_\varepsilon(x)} f\,\mathrm{d}x \leq \eta\varepsilon^{-\alpha} \qquad \text{implies} \qquad |g(x)| \leq \|g\|_{L^\infty(B_{\frac{\varepsilon}{2}}(x))} \leq C\varepsilon^{-\beta}.$$

This can be used to prove the partial regularity of quantity g. Over the set at which g is large, ε is small, so we cover it by $5B_{\varepsilon_i}(x_i)$ using a pairwise disjoint family of balls $B_{\varepsilon_i}(x_i)$. Since $\int_{B_{\varepsilon_i}} f\,\mathrm{d}x \sim \varepsilon_i^{D-\alpha}$ is summable, the singular set $\{g = \infty\}$ has Hausdorff dimension at most $D - \alpha$.

Since the singular set has codimension α, intuitively it should be possible to take the trace of g on a subset with codimension strictly less than α. We will show that this is indeed the case.

3.1. *Scale operator and averaging operator*

In this subsection, we define the scale operator, which finds the threshold of scales below which the ε-regularity theorem can be applied.

Definition 3.3. Let $f : \mathbb{R}^D \to [0, \infty]$ be a Lebesgue measurable function. Denote

$$f_\rho(x) := \fint_{B_\rho(x)} f(y)\,\mathrm{d}y, \qquad 0 < \rho < \infty.$$

For $\alpha > 0$, we define the **scale operator** $\mathscr{S}_\alpha[f] : \mathbb{R}^D \to [0, \infty]$ by

$$\mathscr{S}_\alpha[f](x) := \inf_{0<\rho<\infty} \{\rho : f_\rho(x) > \rho^{-\alpha}\}, \qquad x \in \mathbb{R}^D. \tag{6}$$

We define the **averaging operator** $\mathscr{A}_\alpha[f] : \mathbb{R}^D \to [0, \infty]$ by

$$\mathscr{A}_\alpha[f](x) := \mathscr{S}_\alpha[f](x)^{-\alpha}, \qquad x \in \mathbb{R}^D. \tag{7}$$

We define the **singular set** and **regular set** as

$$\mathrm{Sing}_\alpha(f) = \left\{x \in \mathbb{R}^D : \mathscr{S}_\alpha[f](x) = 0\right\}, \qquad \mathrm{Reg}_\alpha(f) = \left\{x \in \mathbb{R}^D : \mathscr{S}_\alpha[f](x) > 0\right\}.$$

Note that $B_\rho(x)$ depends continuously on ρ and x in the following sense. We omit the proof since it is a standard exercise.

Lemma 3.4. *Fix $\rho > 0$ and $x \in \mathbb{R}^D$.*

1. *(Inner regularity) For any compact subset $K \subset\subset B_\rho(x)$, $B_\sigma(y)$ is also a superset of K provided (σ, y) is sufficiently close to (ρ, x).*
2. *(Outer regularity) For any open superset $O \supset\supset \bar{B}_\rho(x)$, $B_\sigma(y)$ is also a subset of O provided (σ, y) is sufficiently close to (ρ, x).*

Because $B_\sigma(y)$ depends on (σ, y) continuously, we have the following (semi-) continuity on $f_\sigma(y)$.

Lemma 3.5. *The map $(\rho, x) \mapsto f_\rho(x)$ is lower semicontinuous in $\mathbb{R}_+ \times \mathbb{R}^D$. Moreover, if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{1+D})$, then this map is continuous in $\mathbb{R}_+ \times \mathbb{R}^D$.*

Proof. Note that

$$f_\rho(x) = \frac{1}{|B_\rho(x)|} \int\limits_{B_\rho(x)} f,$$

where $|B_\rho(x)| = \rho^D |B_1|$ is continuous in $\rho \in \mathbb{R}_+$. To show that the integral is lower semicontinuous, we fix $\rho > 0$ and $x \in \mathbb{R}^D$. For any compact subset $K \subset\subset B_\rho(x)$, Lemma 3.4 implies

$$\liminf_{(\sigma,y)\to(\rho,x)} f_\sigma(y) = \frac{1}{|B_\rho|} \liminf_{(\sigma,y)\to(\rho,x)} \int\limits_{B_\sigma(y)} f \geq \frac{1}{|B_\rho|} \int\limits_K f.$$

This is true for any compact subset K, so taking a sequence $K_n \uparrow B_\rho(x)$ yields $\int_{K_n} f \to \int_{B_\rho(x)} f = |B_\rho| f_\rho(x)$ and completes the proof of lower semicontinuity.

To show continuity when f is locally integrable, we may take an open superset O of the closure of $B_\rho(x)$. Again by Lemma 3.4, we obtain

$$\limsup_{(\sigma,y)\to(\rho,x)} f_\sigma(y) = \frac{1}{|B_\rho|} \limsup_{(\sigma,y)\to(\rho,x)} \int\limits_{B_\sigma(y)} f \leq \frac{1}{|B_\rho|} \int\limits_O f.$$

This is true for any superset O of the closure of $B_\rho(x)$, so by taking a sequence of $O_n \downarrow B_\rho(x)$ completes the proof of upper semicontinuity, provided that f is locally integrable. □

Moreover, we can also show the semicontinuity of $\mathscr{S}_\alpha[f]$ and $\mathscr{A}_\alpha[f]$.

Lemma 3.6. *$\mathscr{S}_\alpha[f]$ is upper semicontinuous, and $\mathscr{A}_\alpha[f]$ is lower semicontinuous.*

Proof. Fix $x \in \mathbb{R}^D$, we want to show that

$$\limsup_{y \to x} \mathscr{S}_\alpha[f](y) \le \mathscr{S}_\alpha[f](x).$$

If $\mathscr{S}_\alpha[f](x) = +\infty$, then there is nothing to prove. If $\mathscr{S}_\alpha[f](x) < +\infty$, then according to (6), for any $\varepsilon > 0$, there exists $M \in [\mathscr{S}_\alpha[f](x), \mathscr{S}_\alpha[f](x) + \varepsilon]$ such that

$$f_M(x) > M^{-\alpha}.$$

By the lower semicontinuity of f_M, there exists $\delta > 0$ such that

$$f_M(y) > M^{-\alpha}, \qquad \forall y \in B_\delta(x).$$

By definition, $\mathscr{S}_\alpha[f](y) \le M \le \mathscr{S}_\alpha[f](x) + \varepsilon$. This shows

$$\limsup_{y \to x} \mathscr{S}_\alpha[f](y) \le \mathscr{S}_\alpha[f](x) + \varepsilon$$

for any $\varepsilon > 0$, thus $\mathscr{S}_\alpha[f]$ is upper semicontinuous, and correspondingly $\mathscr{A}_\alpha[f] = (\mathscr{S}_\alpha[f])^{-\alpha}$ is lower semicontinuous. □

Now we can justify the name “averaging operator”: $\mathscr{A}_\alpha[f]$ is indeed the average over a ball.

Lemma 3.7. *Suppose $f \in L^1_{\mathrm{loc}}(\mathbb{R}^D)$, $\alpha > 0$, $x \in \mathbb{R}^D$. Denote $s = \mathscr{S}_\alpha[f](x)$. If $0 < s < \infty$, then $\mathscr{A}_\alpha[f](x) = f_s(x) = \fint_{B_s(x)} f$.*

Proof. On the one hand, from the definition (6), we have $f_\rho(x) \le \rho^{-\alpha}$ for any $\rho < s$. By the lower semicontinuity of $f_\rho(x)$ in ρ, we have $f_s(x) \le s^{-\alpha} = \mathscr{A}_\alpha[f](x)$. On the other hand, if $f_s(x) < s^{-\alpha}$ then the continuity of $f_\rho(x)$ and $\rho^{-\alpha}$ in ρ would imply that $f_\rho(x) < \rho^{-\alpha}$ for ρ sufficiently close to s, violating definition (6). Therefore $f_s(x) = s^{-\alpha} = \mathscr{A}_\alpha[f](x)$. □

Notice that $\mathscr{A}_\alpha$ is a nonlinear operator, where $\alpha > 0$ is related to nonlinearity. The following property is a simple consequence of Jensen inequality.

Lemma 3.8. *Let $f : \mathbb{R}^D \to [0, \infty]$ be a Lebesgue measurable function. Let $p \in [1, \infty)$, then*

$$\mathscr{S}_\alpha[f] \ge \mathscr{S}_{p\alpha}[f^p], \qquad \mathscr{A}_\alpha[f] \le \mathscr{A}_{p\alpha}[f^p]^{\frac{1}{p}}.$$

Proof. Fix $x \in \mathbb{R}^D$, $\rho \in (0, \infty)$. If $f_\rho(x) > \rho^{-\alpha}$ then

$$\fint_{B_\rho(x)} f^p \ge \left(\fint_{B_\rho(x)} f \right)^p > \rho^{-p\alpha}.$$

Hence

$$\left\{\rho \in (0,\infty) : f_\rho(x) > \rho^{-\alpha}\right\} \subset \left\{\rho \in (0,\infty) : (f^p)_\rho(x) > \rho^{-p\alpha}\right\}.$$

Thus the infimum of the left is no less than the infimum of the right, which by definition (6) says $\mathscr{S}_\alpha[f](x) \geq \mathscr{S}_{p\alpha}[f^p](x)$. The second inequality is a direct consequence from the definition (7). $\square$

We conclude this subsection by showing $\mathscr{A}_\alpha$ is quasiconvex.

Lemma 3.9. *Fix $\alpha > 0$, for any $f, g \in L^1_{\mathrm{loc}}(\mathbb{R}^D)$, $\lambda \in [0,1]$, it holds that*

$$\mathscr{A}_\alpha[(1-\lambda)f + \lambda g](x) \leq \max\{\mathscr{A}_\alpha[f](x), \mathscr{A}_\alpha(g)(x)\}, \qquad \forall x \in \mathbb{R}^D.$$

Proof. Denote $h = (1-\lambda)f + \lambda g$, and take $x \in \mathbb{R}^D$. If $\mathscr{S}_\alpha[h](x) = \infty$, then $\mathscr{A}_\alpha[h](x) = 0$ and there is nothing to prove. Otherwise, there exists a sequence of $\rho_n > \mathscr{S}_\alpha[h](x)$ and $\rho_n \to \mathscr{S}_\alpha[h](x)$ such that

$$\rho_n^{-\alpha} < h_{\rho_n}(x) = (1-\lambda) f_{\rho_n}(x) + \lambda g_{\rho_n}(x).$$

Therefore, at least one of $f_{\rho_n}(x)$ and $g_{\rho_n}(x)$ is greater than $\rho_n^{-\alpha}$. Suppose, up to a subsequence, that $f_{\rho_n}(x) > \rho_n^{-\alpha}$. Then $\mathscr{S}_\alpha[f](x) \leq \rho_n$ for any n, and taking $n \to \infty$ we find $\mathscr{S}_\alpha[f](x) \leq \mathscr{S}_\alpha[h](x)$. Consequently

$$\mathscr{A}_\alpha[h](x) = \mathscr{S}_\alpha[h](x)^{-\alpha} \leq \mathscr{S}_\alpha[f](x)^{-\alpha} = \mathscr{A}_\alpha[f](x).$$

The case $g_{\rho_n}(x) > \rho_n^{-\alpha}$ would result in $\mathscr{A}_\alpha[h](x) \leq \mathscr{A}_\alpha[g](x)$, so we conclude $\mathscr{A}_\alpha[h]$ is smaller than the max of $\mathscr{A}_\alpha[f]$ and $\mathscr{A}_\alpha[g]$. $\square$

3.2. Partial regularity and trace estimate

Recall in Definition 3.3, we decomposed $\mathbb{R}^D = \mathrm{Sing}_\alpha(f) \cup \mathrm{Reg}_\alpha(f)$. First, we show that for a locally L^p function, the singular set has codimension at least $p\alpha$.

Proposition 3.10. *Suppose $f \in L^p_{\mathrm{loc}}(\mathbb{R}^D)$ is a nonnegative function with $p \in [1,\infty)$. If $0 < p\alpha < D$ then the $(D - p\alpha)$-dimensional Hausdorff measure of $\mathrm{Sing}_\alpha(f)$ is zero. If $p\alpha \geq D$ then $\mathrm{Sing}_\alpha(f)$ is empty.*

Proof. We first prove it for $p = 1$.

Define $\beta = \max\{D - \alpha, 0\}$. Let us assume $f \in L^1(\mathbb{R}^D)$ first. We will later see that local integrability is sufficient. We first fix a small number $\delta \in (0,1)$. For any $x \in \mathrm{Sing}_\alpha(f)$, $\mathscr{S}_\alpha[f](x) = 0$. By (6), there exists $\rho_x < \frac{1}{5}\delta$ such that

$$\fint_{B_{\rho_x}(x)} f \,\mathrm{d}x > \rho_x^{-\alpha} \qquad \Longrightarrow \qquad \frac{1}{|B_1|} \int_{B_{\rho_x}(x)} f \,\mathrm{d}x > \rho_x^{D-\alpha} \geq \rho_x^{\beta}.$$

The last inequality holds when $D-\alpha<0$ because $\rho_x<1$. Now $\left\{B_{\rho_x}(x)\right\}_{x\in \mathrm{Sing}_\alpha(f)}$ forms an open cover of $\mathrm{Sing}_\alpha(f)$. Using Vitali covering lemma, we can select an at most countable disjoint subfamily $\left\{B_{\rho_i}(x_i)\right\}_i$ such that

$$\mathrm{Sing}_\alpha(f)\subset\bigcup_i B_{5\rho_i}(x_i).$$

Since $5\rho_i<\delta$, we can bound

$$\begin{aligned}\mathscr{H}^\beta_\delta(\mathrm{Sing}_\alpha(f)) &:= \inf\left\{\sum_j r_j^\beta : \mathrm{Sing}_\alpha(f)\subset\bigcup_j B_{r_j}(x_j)\right\}\\ &\le \sum_i (5\rho_i)^\beta\\ &\le C\sum_i \int\limits_{B_{\rho_i}(x_i)} f\,\mathrm{d}x\\ &= C\int\limits_{\bigcup_i B_{\rho_i}(x_i)} f\,\mathrm{d}x\\ &\le C\int\limits_{\mathbb{R}^D} f\,\mathrm{d}x.\end{aligned}$$

It is finite, so

$$\left|\bigcup_i B_{\rho_i}(x_i)\right| = |B_1|\sum_i \rho_i^D \le C\delta^{D-\beta}\sum_i \rho_i^\beta \le C\delta^{D-\beta}\int\limits_{\mathbb{R}^D} f\,\mathrm{d}x\to 0$$

as $\delta\to 0$. Therefore

$$\mathscr{H}^\beta(\mathrm{Sing}_\alpha(f)) = \lim_{\delta\to 0}\mathscr{H}^\beta_\delta(\mathrm{Sing}_\alpha(f)) \le \lim_{\delta\to 0} C\int\limits_{\bigcup_i B_{\rho_i}(x_i)} f\,\mathrm{d}x = 0.$$

In particular, when $\alpha\ge D$ and $\beta=0$, we have $\mathrm{Sing}_\alpha(f)=\varnothing$.

If f is merely locally integrable but not $L^1(\mathbb{R}^D)$, we can pick any compact set $K\subset\mathbb{R}^D$ and apply the above argument to prove $\mathscr{H}^\beta(\mathrm{Sing}_\alpha(f)\cap K)=0$. Since K is arbitrary, we conclude $\mathscr{H}^\beta(\mathrm{Sing}_\alpha(f))=0$.

For the general case $p>1$, we apply the above argument to $\tilde f = f^p\in L^1_{\mathrm{loc}}$ and $\tilde\alpha = p\alpha>0$. If $0<\tilde\alpha<D$, then the $(D-\tilde\alpha)$-dimensional Hausdorff measure of $\mathrm{Sing}_{\tilde\alpha}(\tilde f)$ is zero. If $\tilde\alpha\ge D$ then $\mathrm{Sing}_{\tilde\alpha}(\tilde f)=\varnothing$. By Lemma 3.8, we know that

$$\mathscr{S}_\alpha[f]\ge\mathscr{S}_{\tilde\alpha}[\tilde f]\implies \mathrm{Sing}_\alpha(f)\subset\mathrm{Sing}_{\tilde\alpha}(\tilde f).$$

This completes the proof. □

Assume $\Gamma \subset \mathbb{R}^D$ is a d-dimensional Lipschitz graph. To establish trace estimates, we need to measure the level sets of $\mathscr{S}_\alpha[f]$.

Lemma 3.11. *Let $\alpha > 0$, and $f \in L^1(\mathbb{R}^D)$ is nonnegative. Denote*

$$A(\rho) := \left\{x' \in \Gamma : \rho \le \mathscr{S}_\alpha[f](x') < 2\rho\right\}, \qquad \rho > 0.$$

Then

$$\mathscr{H}^d(A(\rho)) \le C(D, d, L)\rho^{-D+d+\alpha} \int\limits_{\mathbb{R}^D} f \, \mathrm{d}x.$$

Proof. Recall that up to choosing a coordinate, Γ is the graph of some L-Lipschitz function $g : U \subset \mathbb{R}^d \to \mathbb{R}^{D-d}$. Define $\phi : U \times \mathbb{R}^{D-d} \to \mathbb{R}^D$ by

$$\phi(x', z) = (x', g(x') + z), \qquad x' \in U, z \in \mathbb{R}^{D-d},$$

then

$$A(\rho) = \phi(U(\rho) \times \{0\}), \text{ for some } U(\rho) \subset U.$$

Since g is Lipschitz, the Hausdorff measure is bounded by

$$\mathscr{H}^d(A(\rho)) \le \|\phi\|_{\mathrm{Lip}}^d \mathscr{L}^d(U(\rho)) \le (1 + \|g\|_{\mathrm{Lip}}^2)^{\frac{d}{2}} \mathscr{L}^d(U(\rho)), \tag{8}$$

where $\mathscr{L}^d$ is the Lebesgue outer measure of dimension d. Recall that $\|g\|_{\mathrm{Lip}} \le L$. Next, note that it is clear from the definition of ϕ that

$$\phi(U(\rho) \times B_\rho(0)) \subset \mathcal{U}_\rho(A(\rho)),$$

where $B_\rho(0) \subset \mathbb{R}^{D-d}$ and $\mathcal{U}_\rho(A(\rho))$ is the ρ-tubular neighborhood of $A(\rho)$. As $\det \nabla\phi = 1$, ϕ is measure preserving, so the measure of U equals to

$$\mathscr{L}^d(U(\rho)) = \frac{\mathscr{L}^D(\phi(U(\rho) \times B_\rho(0)))}{\mathscr{L}^{D-d}(B_\rho(0))} \le \frac{\mathscr{L}^D(\mathcal{U}_\rho(A(\rho)))}{c_{D-d}\,\rho^{D-d}}. \tag{9}$$

Here $c_{D-d} = \mathscr{L}^{D-d}(B_1)$ is the measure of a $(D-d)$-dimensional unit ball.

For any $x' \in A(\rho)$, by definition (6), there exists $\rho_{x'} \in [\rho, 2\rho)$ such that

$$f_{\rho_{x'}}(x') > \rho_{x'}^{-\alpha} \qquad \Longrightarrow \qquad \rho_{x'}^{D-\alpha} < \int\limits_{B_{\rho_{x'}}(x')} f \, \mathrm{d}x.$$

Note that $\rho_{x'} \ge \rho$ implies $\mathcal{U}_\rho(A(\rho))$ has an open cover

$$\mathcal{U}_\rho(A(\rho)) \subset \bigcup_{x' \in A(\rho)} B_{\rho_{x'}}(x).$$

By Vitali's covering lemma, we can find a disjoint subcollection $\left\{B_{\rho_i}(x_i')\right\}_i$ such that $\mathcal{U}_\rho(A(\rho)) \subset \bigcup_i B_{5\rho_i}(x_i')$. So

$$\mathscr{L}^D(\mathcal{U}_\rho(A(\rho))) \le \sum_i \mathscr{L}^D(B_{5\rho_i}(x_i)) \le |B_1| 5^D \sum_i \rho_i^D \le C\rho^\alpha \sum_i \rho_i^{D-\alpha}. \tag{10}$$

Finally, since $B_{\rho_i}(x_i')$ are pairwise disjoint, we have

$$\sum_i \rho_i^{D-\alpha} \le \sum_i \int_{B_{\rho_i}(x_i')} f \, \mathrm{d}x \le \int_{\mathbb{R}^D} f \, \mathrm{d}x. \tag{11}$$

Combine (8)-(11) finishes the proof of the lemma. $\square$

With this, now we can prove the trace estimate for the averaging operator.

Theorem 3.12. *Let $f \in L^p_{\mathrm{loc}}(\mathbb{R}^D)$ be a nonnegative function with some $p \in [1, \infty]$. Let $\alpha > 0$ satisfy $p\alpha > D - d$.*

(a) *$\mathscr{A}_\alpha[f]$ is $(\mathscr{H}^d \llcorner \Gamma)$-measurable, and*

$$\mathscr{H}^d(\{\mathscr{A}_\alpha[f] = 0\} \cap \Gamma) = 0.$$

(b) *If $p = 1$, $f \in L^1(\mathbb{R}^D)$, then*

$$\left\| (\mathscr{A}_\alpha[f])^{1-\frac{D-d}{\alpha}} \right\|_{L^{1,\infty}(\Gamma)} \le C(\alpha, D, d, L) \|f\|_{L^1(\mathbb{R}^D)}.$$

(c) *If $p > 1$, $f \in L^p(\mathbb{R}^D)$, then*

$$\left\| (\mathscr{A}_\alpha[f])^{1-\frac{D-d}{p\alpha}} \right\|_{L^p(\Gamma)} \le C(p, \alpha, D, d, L) \|f\|_{L^p(\mathbb{R}^D)}.$$

Proof. Notice that for $\rho > 0$, $x \in \mathbb{R}^D$,

$$(2\rho)^{-\alpha} < \mathscr{A}_\alpha[f](x) \le \rho^{-\alpha} \iff \rho \le \mathscr{S}_\alpha[f](x) < 2\rho.$$

This means

$$\left\{(2\rho)^{-\alpha} < \mathscr{A}_\alpha[f] \le \rho^{-\alpha}\right\} \cap \Gamma = A(\rho),$$

where $A(\rho)$ is defined in Lemma 3.11.

(a) The measurability of $\mathscr{A}_\alpha[f]$ follows from semicontinuity. By Proposition 3.10, we know the singular set $\mathrm{Sing}_\alpha(f) = \{\mathscr{A}_\alpha[f] = \infty\}$ has Hausdorff dimension at most $D - p\alpha$, so its d-dimensional measure is zero.

(b) Fix $\lambda > 0$, and pick $\rho_0 = \lambda^{-\frac{1}{\alpha}}$, $\rho_k = 2^{-k}\rho_0$. Then

$$\begin{aligned}\mathscr{H}^d(\{\mathscr{A}_\alpha[f] > \lambda\} \cap \Gamma) &= \sum_{k=0}^{\infty} \mathscr{H}^d\left(\left\{\rho_k^{-\alpha} < \mathscr{A}_\alpha[f] \le \rho_{k+1}^{-\alpha}\right\} \cap \Gamma\right)\\ &= \sum_{k=0}^{\infty} \mathscr{H}^d(A(\rho_{k+1}))\\ &\le C\sum_{k=0}^{\infty} \rho_{k+1}^{-D+d+\alpha}\|f\|_{L^1(\mathbb{R}^D)}\\ &\le C\rho_0^{\alpha-D+d}\|f\|_{L^1(\mathbb{R}^D)} = \frac{C}{\lambda^{1-\frac{D-d}{\alpha}}}\|f\|_{L^1(\mathbb{R}^D)}.\end{aligned}$$

The summation is a converging geometric series since the index $\alpha - D + d$ is positive. We thus conclude that $\left\|(\mathscr{A}_\alpha[f])^{1-\frac{D-d}{\alpha}}\right\|_{L^{1,\infty}(\Gamma)} \le C\|f\|_{L^1(\mathbb{R}^D)}$.

(c) We prove the case $p = \infty$ first. Fix any $x \in \mathbb{R}^D$, we note from the definition of f_ρ that $f_\rho(x) \le \|f\|_{L^\infty(\mathbb{R}^D)}$ for any $\rho > 0$. Therefore

$$f_\rho(x) < \rho^{-\alpha}, \qquad \forall \rho < \|f\|_{L^\infty(\mathbb{R}^D)}^{-\frac{1}{\alpha}}.$$

From the definition (6) we know

$$\mathscr{S}_\alpha[f](x) \ge \|f\|_{L^\infty(\mathbb{R}^D)}^{-\frac{1}{\alpha}} > 0,$$

and correspondingly $\mathscr{A}_\alpha[f](x) \le \|f\|_{L^\infty(\mathbb{R}^D)}$.

To show the boundedness for $p \in (1, \infty)$, we first assume in addition that $\alpha > D - d$. For any $\lambda > 0$ we define

$$f_1 = 2f\mathbf{1}_{\left\{f \le \frac{\lambda}{2}\right\}}, \qquad f_2 = 2f\mathbf{1}_{\left\{f > \frac{\lambda}{2}\right\}}.$$

Then $f = \frac{1}{2}(f_1 + f_2)$. By the quasiconvexity Lemma 3.9, we conclude that

$$\{\mathscr{A}_\alpha[f] > \lambda\} \subset \{\mathscr{A}_\alpha[f_1] > \lambda\} \cup \{\mathscr{A}_\alpha[f_2] > \lambda\} = \{\mathscr{A}_\alpha[f_2] > \lambda\},$$

as $\mathscr{A}_\alpha[f_1] \le \|f_1\|_{L^\infty(\mathbb{R}^D)} \le \lambda$. Thus

$$\begin{aligned}\mathscr{H}^d(\{\mathscr{A}_\alpha[f] > \lambda\} \cap \Gamma) &\le \mathscr{H}^d(\{\mathscr{A}_\alpha[f_2] > \lambda\} \cap \Gamma)\\ &\le \frac{C}{\lambda^{1-\frac{D-d}{\alpha}}}\|f_2\|_{L^1(\mathbb{R}^D)}.\end{aligned}$$

Here we used part (b) on f_2. And the remaining is analogous to the classical setting:

$$\begin{aligned}
\left\|(\mathscr{A}_\alpha[f])^{1-\frac{D-d}{p\alpha}}\right\|_{L^p(\Gamma)}^p &= \int_\Gamma (\mathscr{A}_\alpha[f])^{p-\frac{D-d}{\alpha}}\, \mathrm{d}\mathscr{H}^d \\
&= C\int_0^\infty \lambda^{p-\frac{D-d}{\alpha}-1}\mathscr{H}^d(\{\mathscr{A}_\alpha[f]>\lambda\}\cap\Gamma)\,\mathrm{d}\lambda \\
&\le C\int_0^\infty \lambda^{p-2}\int_{\mathbb{R}^D} f_2\,\mathrm{d}x\,\mathrm{d}\lambda \\
&= C\int_{\mathbb{R}^D}\int_0^\infty f\mathbf{1}_{\left\{f>\frac{\lambda}{2}\right\}}\lambda^{p-2}\,\mathrm{d}\lambda\,\mathrm{d}x \\
&\le C\int_{\mathbb{R}^D} f^p\,\mathrm{d}x.
\end{aligned}$$

We now use Lemma 3.8 to remove the extra assumption $\alpha > D-d$. Since $p\alpha > D-d$, we can fix $p' < p$ such that $\tilde{\alpha} := p'\alpha > D-d$ and $\tilde{p} := p/p' > 1$. Define $\tilde{f} = f^{p'}$, then $\tilde{f} \in L^{\tilde{p}}(\mathbb{R}^D)$. Apply the above proof for $\tilde{f}$, $\tilde{\alpha}$ and $\tilde{p}$ gives

$$\left\|(\mathscr{A}_{\tilde{\alpha}}[\tilde{f}])^{1-\frac{D-d}{\tilde{p}\tilde{\alpha}}}\right\|_{L^{\tilde{p}}(\Gamma)} \le C\|\tilde{f}\|_{L^{\tilde{p}}(\mathbb{R}^D)} = \|f\|_{L^p}^{p'}.$$

By Lemma 3.8, we know that $\mathscr{A}_\alpha f \le (\mathscr{A}_{\tilde{\alpha}}[\tilde{f}])^{\frac{1}{p'}}$, so

$$\left\|(\mathscr{A}_\alpha[f])^{1-\frac{D-d}{p\alpha}}\right\|_{L^p(\Gamma)} \le \left\|(\mathscr{A}_{\tilde{\alpha}}[\tilde{f}])^{\frac{1}{p'}(1-\frac{D-d}{p\alpha})}\right\|_{L^p(\Gamma)} = \left\|(\mathscr{A}_{\tilde{\alpha}}[\tilde{f}])^{1-\frac{D-d}{\tilde{p}\tilde{\alpha}}}\right\|_{L^{\tilde{p}}(\Gamma)}^{\frac{1}{p'}} \le \|f\|_{L^p}.$$

This completes the proof of the theorem. □

When $D=d$, $\Gamma=\mathbb{R}^D$, Theorem 3.12 simply says

$$\|\mathscr{A}_\alpha[f]\|_{L^{1,\infty}(\mathbb{R}^D)} \le C\|f\|_{L^1(\mathbb{R}^D)}, \qquad \|\mathscr{A}_\alpha[f]\|_{L^p(\mathbb{R}^D)} \le C(p)\|f\|_{L^p(\mathbb{R}^D)}.$$

This can also follow directly from the fact that $\mathscr{A}_\alpha[f] \le \mathcal{M}f$ pointwise by Lemma 3.7, where $\mathcal{M}f(x) = \sup_{\rho>0} f_\rho(x)$ is the Hardy-Littlewood maximal function of f.

(c) is false for $p=1$. Indeed, unlike the maximal operator, when $f = |B_1|\delta_0$ is a Dirac measure, $\alpha \in (0,d)$, we can verify that

$$f_\rho(x) = \rho^{-d}\mathbf{1}_{\{|x|<\rho\}}, \qquad \mathscr{S}_\alpha[f](x) = |x| + \infty\mathbf{1}_{\{|x|>1\}}.$$

So when $f \approx |B_1|\delta_0$ is an L^1 function, $\mathscr{A}_\alpha[f](x) = \mathscr{S}_\alpha[f]^{-\alpha}(x) \approx |x|^{-\alpha}\mathbf{1}_{\{|x|<1\}}$ is still integrable. However, in Appendix A we construct an example where f is a bounded Cantor-style measure and $\mathscr{A}_\alpha[f]$ is $L^{1,\infty}(\mathbb{R})$ but not $L^{1,q}(\mathbb{R})$ for any $q < \infty$.

When $D > d$, Theorem 3.12 cannot hold for the maximal function anymore, as the maximal function does not have a trace on hypersurfaces.

4. Blow-up technique and trace estimate: spacetime case

We want to extend the theory in Section 3 to the spacetime setting. Due to our particular interest in equations with heat diffusion, we will focus on the parabolic scaling between time and space. That is, we define a cylinder $Q_\rho(t,x)$ to be the following:

$$Q_\rho(t,x) = \Big\{(s,y) : s \in (t-\rho^2, t],\, y \in B_\rho(x)\Big\}. \tag{12}$$

We remark that other scalings are possible and similar results can be obtained. $Q_\rho(t,x)$ will replace the role of $B_\rho(x)$ in the space setting. To be precise, we define the following.

Definition 4.1. Let $f : \mathbb{R} \times \mathbb{R}^D \to [0,\infty]$ be a Lebesgue measurable function. Denote

$$f_\rho(t,x) := \fint_{Q_\rho(t,x)} f(s,y)\,\mathrm{d}y\,\mathrm{d}s, \qquad 0 < \rho < \infty.$$

For $\alpha > 0$, we define the **scale operator** $\mathscr{S}_\alpha[f] : \mathbb{R} \times \mathbb{R}^D \to [0,\infty]$ by

$$\mathscr{S}_\alpha[f](t,x) := \inf_{0<\rho<\infty}\big\{\rho : f_\rho(t,x) > \rho^{-\alpha}\big\}, \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^D. \tag{13}$$

We define the **averaging operator** $\mathscr{A}_\alpha[f] : \mathbb{R} \times \mathbb{R}^D \to [0,\infty]$ by

$$\mathscr{A}_\alpha[f](t,x) := \mathscr{S}_\alpha[f](t,x)^{-\alpha}, \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^D. \tag{14}$$

We define the **singular set** and **regular set** as

$$\mathrm{Sing}_\alpha(f) = \Big\{(t,x) \in \mathbb{R} \times \mathbb{R}^D : \mathscr{S}_\alpha[f](t,x) = 0\Big\},$$
$$\mathrm{Reg}_\alpha(f) = \Big\{(t,x) \in \mathbb{R} \times \mathbb{R}^D : \mathscr{S}_\alpha[f](t,x) > 0\Big\}.$$

For simplicity we used the same notation as in the last section, but its meaning should be detected from the context based on the domain of f.

Clearly, Lemma 3.4 holds for cylinders as well. Thus we can parallel the semicontinuity, continuity, nonlinearity, and quasiconvexity results in Section 3. The proof is identical to Lemma 3.5-3.9, so we omit it here.

Lemma 4.2. *Let $\alpha > 0$ and let $f : \mathbb{R} \times \mathbb{R}^D \to [0,\infty]$ be measurable. Then $\mathscr{S}_\alpha[f]$ is upper semicontinuous, and $\mathscr{A}_\alpha[f]$ is lower semicontinuous. For any $p \in [1,\infty)$, it holds that*

$$\mathscr{S}_\alpha[f] \geq \mathscr{S}_{p\alpha}[f^p], \qquad \mathscr{A}_\alpha[f] \leq \mathscr{A}_{p\alpha}[f^p]^{\frac{1}{p}}.$$

Moreover, if $f \in L^1_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^D)$*, then*

$$\mathscr{A}_\alpha[f](t,x) = f_{\mathscr{S}_\alpha[f](t,x)}(t,x), \qquad \text{when } 0 < \mathscr{S}_\alpha[f](t,x) < \infty.$$

For any nonnegative $f, g \in L^1_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^D)$, $\lambda \in [0,1]$, *it holds that*

$$\mathscr{A}_\alpha[(1-\lambda)f + \lambda g](t,x) \leq \max\{\mathscr{A}_\alpha[f](t,x), \mathscr{A}_\alpha[g](t,x)\}, \qquad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^D.$$

4.1. *Partial regularity and trace estimate*

The partial regularity theory for spacetime should be modified in order to fit the parabolic scaling. For $\beta \geq 0$, we introduce parabolic Hausdorff measure as

$$\mathscr{P}^\beta(A) := \lim_{\delta \to 0} \mathscr{P}^\beta_\delta(A) = \lim_{\delta \to 0} \inf \left\{ \sum_j r_j^\beta : A \subset \bigcup_j Q_{r_j}(t_j, x_j) \right\}, \qquad A \subset \mathbb{R} \times \mathbb{R}^D,$$

where infimum is taken over all possible covering of the set A by cylinders. Accordingly, we obtain the following partial regularity theorem.

Proposition 4.3. *Suppose* $f \in L^p_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^D)$ *is nonnegative for some* $p \in [1,\infty)$. *If* $0 < p\alpha < D + 2$ *then the* $(D+2-p\alpha)$*-dimensional parabolic Hausdorff measure of* $\mathrm{Sing}_\alpha(f)$ *is zero:*

$$\mathscr{P}^{D+2-p\alpha}(\mathrm{Sing}_\alpha(f)) = 0.$$

If $p\alpha \geq D+2$ *then* $\mathrm{Sing}_\alpha(f)$ *is empty.*

The proof is identical to the proof of Proposition 3.10, except $|Q_r| = Cr^{D+2}$ comparing to $|B_r| = Cr^D$.

It becomes slightly more delicate to estimate the trace in the parabolic setting. Recall that Γ_T, which satisfies Assumption 2.2, is not necessarily a manifold in $\mathbb{R} \times \mathbb{R}^D$, but can be discontinuous in time. The purpose of allowing this generality will be clear when we would like to replace L^∞ norms in x by taking traces, but let us defer this discussion to the next subsection.

First, we revise Lemma 3.11 to the following time-dependent form.

Lemma 4.4. *Let* $t \in \mathbb{R}$, $\alpha > 0$, $p \in [1,\infty)$, *and* $f \in L^p_{\mathrm{loc},t} L^p_x(\mathbb{R}^{1+D})$. *Denote*

$$A_t(\rho) := \left\{ x' \in \Gamma_t : \rho \leq \mathscr{S}_\alpha[f](t,x') < 2\rho \right\}, \qquad \rho > 0.$$

Then for every $t \in (0,T)$, $\rho > 0$,

$$\mathscr{H}^d(A_t(\rho)) \leq C\rho^{-D+d-2+p\alpha} \int_{t-4\rho^2}^{t} \int_{\mathbb{R}^D} f(s,x)^p \,\mathrm{d}x\,\mathrm{d}s.$$

Proof. For any $x' \in A_t(\rho)$, by definition (13), there exists $\rho_{x'} \in [\rho, 2\rho)$ such that

$$f_{\rho_{x'}}(t,x') > \rho_{x'}^{-\alpha} \qquad \Longrightarrow \qquad \rho_{x'}^{D+2-p\alpha} < \int\limits_{Q_{\rho_{x'}}(t,x')} f(s,y)^p \, \mathrm{d}y \, \mathrm{d}s.$$

Here we used Jensen's inequality as in Lemma 3.8. Same as Lemma 3.11, Lipschitzness of Γ_t implies

$$\mathscr{H}^d(A_t(\rho)) \le (1 + \|g_t\|_{\mathrm{Lip}}^2)^{\frac{d}{2}} \frac{\mathscr{L}^D(\mathcal{U}_\rho(A_t(\rho)))}{c_{D-d}\rho^{D-d}}. \tag{15}$$

Again, $\mathcal{U}_\rho(A_t(\rho))$ is covered by $\bigcup_{x' \in A_t(\rho)} B_{\rho_{x'}}(x')$, so Vitali covering lemma gives a disjoint subcollection $\left\{B_{\rho_i}(x_i')\right\}_i$ with

$$\mathscr{L}^D(\mathcal{U}_\rho(A_t(\rho))) \le |B_1| 5^D \sum_i \rho_i^D = C\rho^{p\alpha-2} \sum_i \rho_i^{D+2-p\alpha}. \tag{16}$$

Finally, since $B_{\rho_i}(x_i')$ are pairwise disjoint, $Q_{\rho_i}(t,x_i')$ are also pairwise disjoint, so

$$\sum_i \rho_i^{D+2-p\alpha} \le C \sum_i \int\limits_{Q_{\rho_i}(t,x_i')} f(s,y)^p \, \mathrm{d}y \, \mathrm{d}s \le C \int\limits_{t-4\rho^2}^{t} \int\limits_{\mathbb{R}^D} f^p \, \mathrm{d}x \, \mathrm{d}t. \tag{17}$$

The last inequality used the fact that each $Q_{\rho_i}(t,x_i')$ has at most a timespan of $4\rho^2$. Combining (15)-(17) finishes the proof of the lemma. □

Using a similar idea, we can also introduce the following finer version of partial regularity comparing to Proposition 4.3, which deals with anisotropic norms on f and separates dimensions of singularity in space and time.

Proposition 4.5. *Let $\alpha > 0$, and $f \in L^p_{\mathrm{loc},t} L^q_{\mathrm{loc},x}(\mathbb{R}^{1+D})$, $1 \le q \le p < \infty$. Denote the singular section by*

$$S_t := \left\{x' \in \Gamma_t : (t,x') \in \mathrm{Sing}_\alpha(f)\right\}, \qquad t \in \mathbb{R},$$

and denote the set of singular time by

$$\mathcal{T} = \left\{t \in \mathbb{R} : \mathscr{H}^d(S_t) > 0\right\}.$$

(a) *If $\frac{D-d}{q} < \alpha \le \frac{2}{p} + \frac{D-d}{q}$, then $\mathcal{T}$ has Hausdorff dimension no greater than*

$$\dim_{\mathscr{H}}(\mathcal{T}) \le 1 - \frac{p}{2}\left(\alpha - \frac{D-d}{q}\right).$$

(b) *If $\alpha > \frac{2}{p} + \frac{D-d}{q}$, then $\mathcal{T} = \varnothing$.*

Proof. The proposition is equivalent to the following statement: for any $\beta \in [0,1]$ with $\frac{2\beta}{p} + \frac{D-d}{q} < \alpha$, it holds that $\mathscr{H}^d(S_t) = 0$ for $\mathscr{H}^{1-\beta}$-a.e. $t \in \mathbb{R}$.

Same as in the proof of Proposition 3.10, we can assume $f \in L_t^p L_x^q(\mathbb{R}^{1+D})$ without loss of generality. For any $\eta > 0$, we define

$$\mathcal{T}(\eta) = \left\{ t \in \mathbb{R} : \limsup_{\rho \to 0} \rho^{2\beta} \fint_{t-(2\rho)^2}^{t} \|f(s)\|_{L^q(\mathbb{R}^D)}^p \, \mathrm{d}s \geq \eta \right\}.$$

We claim that $\mathcal{T} \subset \mathcal{T}(\eta)$ for any $\eta > 0$. Indeed, if $t \notin \mathcal{T}(\eta)$, then for $\varepsilon > 0$ sufficiently small, for all $\rho \in (0,\varepsilon)$, it holds that

$$\rho^{2\beta} \fint_{t-(2\rho)^2}^{t} \|f(s)\|_{L^q(\mathbb{R}^D)}^p \, \mathrm{d}s < \eta.$$

If $x' \in S_t$, there exists $\rho_{x'} \in (0,\varepsilon)$ such that $f_{\rho_{x'}}(t,x') > \rho_{x'}^{-\alpha}$. Denote

$$S_t(\rho) = \left\{ x' \in S_t : \rho \leq \rho_{x'} < 2\rho \right\}, \qquad \rho > 0.$$

Note that in the proof of Lemma 4.4, the only feature we need from $A_t(\rho)$ is to be able to find a $\rho_{x'} \in [\rho, 2\rho)$ with $f_{\rho_{x'}}(t,x') > \rho_{x'}^{-\alpha}$. The same proof yields

$$\begin{aligned}
\mathscr{H}^d(S_t(\rho)) &\leq C\rho^{-D+d-2+q\alpha} \int_{(t-(2\rho)^2,t)\times\mathbb{R}^D} f(s,y)^q \, \mathrm{d}y \, \mathrm{d}s \\
&= C\rho^{-D+d+q\alpha} \fint_{t-(2\rho)^2}^{t} \|f(s)\|_{L^q(\mathbb{R}^D)}^q \, \mathrm{d}s \\
&\leq C\rho^{-D+d+q\alpha} \left(\fint_{t-(2\rho)^2}^{t} \|f(s)\|_{L^q(\mathbb{R}^D)}^p \, \mathrm{d}s \right)^{\frac{q}{p}} \\
&\leq C\rho^{-D+d+q\alpha} (\eta\rho^{-2\beta})^{\frac{q}{p}} \int_{t-(2\rho)^2}^{t} \|f(s)\|_{L^q(\mathbb{R}^D)}^p \, \mathrm{d}s \\
&\leq C\eta^{\frac{q}{p}} \rho^{-D+d+q(\alpha-2\beta/p)} \int_{t-(2\rho)^2}^{t} \|f(s)\|_{L^q(\mathbb{R}^D)}^p \, \mathrm{d}s.
\end{aligned}$$

Recall that $-D + d + q(\alpha - 2\beta/p) > 0$. Since $\left\{ S_t(2^{-k}\varepsilon) \right\}_{k\geq 1}$ forms a partition of S_t, we can control the measure of S_t by

$$\mathscr{H}^d(S_t) \le \sum_{k=1}^{\infty} \mathscr{H}^d(S_t(2^{-k}\varepsilon)) \le C\eta^{\frac{q}{p}}\varepsilon^{-D+d+q(\alpha-2\beta/p)}.$$

This holds for all ε sufficiently small, so $\mathscr{H}^d(S_t) = 0$, $t \notin \mathcal{T}$.

It remains to show that $\mathcal{T}$ has $\mathscr{H}^{1-\beta}$-measure zero. Fix $\eta > 0$ and $\varepsilon > 0$. For every $t \in \mathcal{T}(\eta)$, there exists $\rho_t > 0$ such that $20\rho_t^2 < \varepsilon$ and

$$\rho_t^{2\beta} \fint_{t-(2\rho_t)^2}^{t} \|f(s)\|^p_{L^q(\mathbb{R}^D)}\, \mathrm{d}s \ge \eta.$$

Intervals $\left\{(t-(2\rho_t)^2, t+(2\rho_t)^2)\right\}_{t\in\mathcal{T}(\eta)}$ form an open cover of $\mathcal{T}(\eta)$, so we can find a disjoint subcollection $\left\{(t_i-(2\rho_i)^2, t_i+(2\rho_i)^2)\right\}_i$ by Vitali covering lemma such that

$$\mathcal{T} \subset \mathcal{T}(\eta) \subset \bigcup_i (t_i - 5(2\rho_i)^2, t_i + 5(2\rho_i)^2).$$

So the $\mathscr{H}^{1-\beta}$-Hausdorff measure of $\mathcal{T}$ is bounded by

$$\begin{aligned}
\mathscr{H}^{1-\beta}(\mathcal{T}) \le C\sum_i \rho_i^{2(1-\beta)} &\le \frac{C}{\eta}\int_{t_i-(2\rho_i)^2}^{t_i} \|f(s)\|^p_{L^q(\mathbb{R}^D)}\, \mathrm{d}s \\
&\le \frac{C}{\eta}\int_{\mathcal{T}^*(\eta)} \|f(s)\|^p_{L^q(\mathbb{R}^D)}\, \mathrm{d}s \\
&\le \frac{C}{\eta}\|f\|^p_{L^p_tL^q_x(\mathbb{R}^{1+D})}.
\end{aligned}$$

The constant C comes from the covering lemma and is independent of η. Since this is true for arbitrary large η, we must have $\mathscr{H}^{1-\beta}(\mathcal{T}) = 0$. $\square$

Now we are ready to show the main theorem for trace estimates. We first work on isotropic norms. We show that when $p\alpha > D-d$ we can take trace in space, and if in addition $p\alpha > D-d+2$ then we can take trace in both space and time.

Theorem 4.6. *Let $f \in L^p_{\mathrm{loc}}(\mathbb{R}^{1+D})$ be a nonnegative function with some $p \in [1,\infty]$. Let $\alpha > 0$ satisfy $p\alpha > D-d$. Then*

(a) *$\mathscr{A}_\alpha[f]$ is μ_T-measurable, and*

$$\mu_T(\{\mathscr{A}_\alpha[f] = \infty\}) = 0.$$

(b) *If $p = 1$, $f \in L^1(\mathbb{R}^{1+D})$, then*

$$\left\|(\mathscr{A}_\alpha[f])^{1-\frac{D-d}{\alpha}}\right\|_{L^{1,\infty}_{t,x}(\Gamma_T)} \le C(\alpha, D, d, L)\|f\|_{L^1(\mathbb{R}^{1+D})}.$$

(c) *If $p > 1$, $f \in L^p(\mathbb{R}^{1+D})$, then*

$$\left\| (\mathscr{A}_\alpha[f])^{1-\frac{D-d}{p\alpha}} \right\|_{L^p_{t,x}(\Gamma_T)} \le C(p,\alpha,D,d,L)\|f\|_{L^p(\mathbb{R}^{1+D})}.$$

Moreover, if $p\alpha > D-d+2$, then

(d) *For every $t \in (0,T)$, $\mathscr{A}_\alpha[f](t,\cdot)$ is μ_t-measurable, and*

$$\mu_t(\{\mathscr{A}_\alpha[f](t,\cdot) = \infty\}) = 0.$$

(e) *If $f \in L^1(\mathbb{R}^{1+D})$, then for every $t \in (0,T)$,*

$$\left\| [\mathscr{A}_\alpha[f](t)]^{1-\frac{D-d+2}{\alpha}} \right\|_{L^{1,\infty}(\Gamma_t)} \le C(\alpha,D,d,L)\|f\|_{L^1(\mathbb{R}^{1+D})}.$$

(f) *If $f \in L^p(\mathbb{R}^{1+D})$ for some $p \in (1,\infty]$, then for every $t \in (0,T)$,*

$$\left\| [\mathscr{A}_\alpha[f](t)]^{1-\frac{D-d+2}{p\alpha}} \right\|_{L^p(\Gamma_t)} \le C(p,\alpha,D,d,L)\|f\|_{L^p(\mathbb{R}^{1+D})}.$$

Proof. Notice that for $\rho > 0$, $(t,x) \in \mathbb{R}^{1+D}$,

$$\begin{gathered}(2\rho)^{-\alpha} < \mathscr{A}_\alpha[f](t,x) \le \rho^{-\alpha} \iff \rho \le \mathscr{S}_\alpha[f](t,x) < 2\rho, \\ \mathscr{A}_\alpha[f](t,x) = \infty \iff \mathscr{S}_\alpha[f](t,x) = 0.\end{gathered}$$

Thus

$$\begin{gathered}\left\{(2\rho)^{-\alpha} < \mathscr{A}_\alpha[f](t) \le \rho^{-\alpha}\right\} \cap \Gamma = A_t(\rho), \\ \{\mathscr{A}_\alpha[f](t) = \infty\} \cap \Gamma = S_t,\end{gathered}$$

where $A_t(\rho)$ is defined in Lemma 4.4, and S_t is defined in Lemma 4.5.

(a) The measurability of $\mathscr{A}_\alpha[f]$ comes from semicontinuity. Lemma 4.5 (a) implies $\mu_t(S_t) = 0$ for $\mathscr{L}^1$-almost every $t \in [0,1]$. Therefore

$$\mu_T(\{\mathscr{A}_\alpha[f] = \infty\}) = \int_0^T \mu_t(S_t)\,\mathrm{d}t = 0.$$

(b) Fix $\lambda > 0$, and pick $\rho_0 = \lambda^{-\frac{1}{\alpha}}$, $\rho_k = 2^{-k}\rho_0$. Then

$$\mu_T(\{\mathscr{A}_\alpha[f] > \lambda\}) = \sum_{k=0}^{\infty} \mu_T\left(\left\{\rho_k^{-\alpha} < \mathscr{A}_\alpha[f] \le \rho_{k+1}^{-\alpha}\right\}\right)$$

$$\le \sum_{k=0}^{\infty} \int_0^T \mu_t(A_t(\rho_{k+1}))\,\mathrm{d}t.$$

By Lemma 4.4, we know that

$$\int_0^T \mu_t(A_t(\rho_{k+1}))\,\mathrm{d}t = C\rho_{k+1}^{-D+d-2+\alpha} \int_0^T \int_{t-(2\rho_{k+1})^2}^{t} \int_{\mathbb{R}^D} f(s,y)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}t$$
$$\le C\rho_{k+1}^{-D+d+\alpha} \int_{-\infty}^{T} \int_{\mathbb{R}^D} f(s,y)\,\mathrm{d}y\,\mathrm{d}s.$$

Therefore, we conclude

$$\mu_T(\{\mathscr{A}_\alpha[f] > \lambda\}) \le C \sum_{k=0}^{\infty} \rho_{k+1}^{-D+d+\alpha} \|f\|_{L^1(\mathbb{R}^{1+D})}$$
$$\le C\rho_0^{\alpha-D+d} \|f\|_{L^1(\mathbb{R}^{1+D})} = \frac{C}{\lambda^{1-\frac{D-d}{\alpha}}} \|f\|_{L^1(\mathbb{R}^{1+D})}.$$

Here we use $\alpha > D - d$ to compute the summation. We thus conclude that $\left\| (\mathscr{A}_\alpha[f])^{1-\frac{D-d}{\alpha}} \right\|_{L^{1,\infty}(\Gamma_T)} \le C\|f\|_{L^1(\mathbb{R}^{1+D})}$.

(c) Since we again have semicontinuity and quasiconvexity as the spatial case, we can prove it using an identical Marcinkovich interpolation proof as in Theorem 3.12 (c). We repeat it here for completeness.
For $p = \infty$, we fix any $x \in \mathbb{R}^D$, then

$$f_\rho(t,x) < \|f\|_{L^\infty(\mathbb{R}^{1+D})} \le \rho^{-\alpha}, \qquad \forall \rho < \|f\|_{L^\infty(\mathbb{R}^{1+D})}^{-\frac{1}{\alpha}}.$$

From the definition (13) we know

$$\mathscr{S}_\alpha[f](t,x) \ge \|f\|_{L^\infty(\mathbb{R}^{1+D})}^{-\frac{1}{\alpha}} > 0,$$

and correspondingly $\mathscr{A}_\alpha[f](t,x) \le \|f\|_{L^\infty(\mathbb{R}^{1+D})}$.
To show the boundedness for $p \in (1,\infty)$, we assume in addition that $\alpha > D - d$. This requirement can be lifted in the end by making a nonlinear transform using Lemma 3.8 as in the proof of Theorem 3.12 (c).
For any $\lambda > 0$ we define

$$f_1 = 2f\mathbf{1}_{\left\{f \le \frac{\lambda}{2}\right\}}, \qquad f_2 = 2f\mathbf{1}_{\left\{f > \frac{\lambda}{2}\right\}}.$$

Then $f = \frac{1}{2}(f_1 + f_2)$. By the quasiconvexity in Lemma 4.2, we conclude that

$$\{\mathscr{A}_\alpha[f] > \lambda\} \subset \{\mathscr{A}_\alpha[f_1] > \lambda\} \cup \{\mathscr{A}_\alpha[f_2] > \lambda\} = \{\mathscr{A}_\alpha[f_2] > \lambda\},$$

as $\mathscr{A}_\alpha[f_1] \le \|f_1\|_{L^\infty(\mathbb{R}^{1+D})} \le \lambda$. Thus

$$\mu_T(\{\mathscr{A}_\alpha[f] > \lambda\}) \le \mu_T(\{\mathscr{A}_\alpha[f_2] > \lambda\}) \le \frac{C}{\lambda^{1-\frac{D-d}{\alpha}}}\|f_2\|_{L^1(\mathbb{R}^{1+D})}.$$

Here we used part (b) on f_2. And the remaining is analogous to the classical setting:

$$\begin{aligned}
\left\|(\mathscr{A}_\alpha[f])^{1-\frac{D-d}{p\alpha}}\right\|_{L^p(\Gamma_T)}^p &= \int\limits_{\Gamma_T} (\mathscr{A}_\alpha[f])^{p-\frac{D-d}{\alpha}}\,\mathrm{d}\mu_T \\
&= C\int\limits_0^\infty \lambda^{p-\frac{D-d}{\alpha}-1}\mu_T(\{\mathscr{A}_\alpha[f] > \lambda\})\,\mathrm{d}\lambda \\
&\le C\int\limits_0^\infty \lambda^{p-2}\int\limits_{\mathbb{R}^{1+D}} f_2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\lambda \\
&= C\int\limits_{\mathbb{R}^{1+D}}\int\limits_0^\infty f\mathbf{1}_{\left\{f>\frac{\lambda}{2}\right\}}\lambda^{p-2}\,\mathrm{d}\lambda\,\mathrm{d}x\,\mathrm{d}t \\
&\le C\int\limits_{\mathbb{R}^{1+D}} f^p\,\mathrm{d}x.
\end{aligned}$$

(d) Again, the measurability comes from semicontinuity. Lemma 4.5 (b) implies $\mu_t(S_t) = 0$ for every $t \in (0, T)$.

(e) The proof is analogous to part (b). Fix $\lambda > 0$, and pick $\rho_0 = \lambda^{-\frac{1}{\alpha}}$, $\rho_k = 2^{-k}\rho_0$. Then

$$\begin{aligned}
\mu_t(\{\mathscr{A}_\alpha[f](t) > \lambda\}) &= \sum_{k=0}^\infty \mu_t\left(\left\{\rho_k^{-\alpha} < \mathscr{A}_\alpha[f](t) \le \rho_{k+1}^{-\alpha}\right\}\right) \\
&\le \sum_{k=0}^\infty \mu_t(A_t(\rho_{k+1})) \\
&\le C\sum_{k=0}^\infty \rho_{k+1}^{-D+d-2+\alpha}\int\limits_{t-(2\rho_{k+1})^2}^{t}\int\limits_{\mathbb{R}^D} f(s,y)\,\mathrm{d}y\,\mathrm{d}s \\
&\le C\rho_0^{\alpha-D+d-2}\|f\|_{L^1(\mathbb{R}^{1+D})} \\
&= \frac{1}{\lambda^{1-\frac{D-d+2}{\alpha}}}\|f\|_{L^1(\mathbb{R}^{1+D})}.
\end{aligned}$$

Here we used $\alpha > D - d + 2$ to compute the summation.

(f) The proof is identical to part (c) except the codimension is now $D-d+2$ instead of $D-d$. $\square$

4.2. Anisotropic estimates

By interpolation, we can extend the estimates in Theorem 4.6 to the following anisotropic bounds.

Proposition 4.7. *Let $f \in L^p(\mathbb{R}^{1+D})$, $p \geq 1$. If $p\alpha > D-d+2$, then for $0<\gamma\leq 1$*

$$\left\| (\mathscr{A}_\alpha[f])^{1-\frac{D-d+2\gamma}{\alpha}} \right\|_{L_t^{\frac{1}{1-\gamma},\infty} L_x^{1,\infty}(\Gamma_T)} \leq C(\gamma,\alpha,D,d,L)\|f\|_{L^1(\mathbb{R}^{1+D})}, \qquad p=1. \tag{18}$$

$$\left\| (\mathscr{A}_\alpha[f])^{1-\frac{D-d+2\gamma}{p\alpha}} \right\|_{L_t^{\frac{p}{1-\gamma}} L_x^{p}(\Gamma_T)} \leq C(\gamma,p,\alpha,D,d,L)\|f\|_{L^p(\mathbb{R}^{1+D})}, \qquad p>1. \tag{19}$$

If $p\alpha > D$, then for $0\leq\beta<1$

$$\left\| (\mathscr{A}_\alpha[f])^{1-\frac{D-\beta d}{\alpha}} \right\|_{L_t^{1,\infty} L_x^{\frac{1}{\beta},\infty}(\Gamma_T)} \leq C(\gamma,\alpha,D,d,L)\|f\|_{L^1(\mathbb{R}^{1+D})}, \qquad p=1. \tag{20}$$

$$\left\| (\mathscr{A}_\alpha[f])^{1-\frac{D-\beta d}{p\alpha}} \right\|_{L_t^{p} L_x^{\frac{p}{\beta}}(\Gamma_T)} \leq C(\gamma,p,\alpha,D,d,L)\|f\|_{L^p(\mathbb{R}^{1+D})}, \qquad p>1. \tag{21}$$

If $p\alpha > D+2$, then for $0<\gamma\leq 1$, $0\leq\beta\leq 1$

$$\left\| (\mathscr{A}_\alpha[f])^{1-\frac{D-\beta d+2\gamma}{\alpha}} \right\|_{L_t^{\frac{1}{1-\gamma},\infty} L_x^{\frac{1}{\beta},\infty}(\Gamma_T)} \leq C(\gamma,\beta,\alpha,D,d,L)\|f\|_{L^1(\mathbb{R}^{1+D})},\ p=1. \tag{22}$$

$$\left\| (\mathscr{A}_\alpha[f])^{1-\frac{D-\beta d+2\gamma}{p\alpha}} \right\|_{L_t^{\frac{p}{1-\gamma}} L_x^{\frac{p}{\beta}}(\Gamma_T)} \leq C(\gamma,\beta,p,\alpha,D,d,L)\|f\|_{L^p(\mathbb{R}^{1+D})},\ p>1. \tag{23}$$

Proof. The case $\gamma=1$ and $\beta=0$ can be derived from Theorem 4.6 directly. Indeed, (18) and (19) with $\gamma=1$ are direct consequences of (e) and (f). Moreover, (20) and (21) with $\beta=0$ are direct consequences of (b) and (c) with $d=0$: we can choose Γ_t to be a singleton $\{x(t)\}$ such that $\mathscr{A}_\alpha[f](t,x(t)) > \|\mathscr{A}_\alpha[f](t,\cdot)\|_{L^\infty(\mathbb{R}^D)} - \varepsilon$ for arbitrary small ε (recall that there is no continuity requirement on the map $t\mapsto\Gamma_t$). Existence of such a choice is guaranteed by the measurable selection theorem.

We easily derive (19) and (21) for $\gamma,\beta\in(0,1)$ by interpolation. Indeed, when $\gamma=0$ and $\beta=1$, both degenerates to Theorem 4.6 (c).

To derive (18) and (20) for $\gamma,\beta\in(0,1)$, we can also use interpolation. As explained in Appendix B, iterated weak norm and joint weak norm are not comparable. However, interpolation is possible between them. Denote $g=(\mathscr{A}_\alpha[f])^{1-\frac{D-d}{\alpha}}$. Since $\alpha > D-d$ in both cases, Theorem 4.6 (b) shows

$$\|g\|_{L_{t,x}^{1,\infty}(\Gamma_T)} \leq C\|f\|_{L^1(\mathbb{R}^{1+D})}.$$

(18) with $\gamma=1$ and (20) with $\beta=0$ translate to

$$\|g\|_{L_t^\infty L_x^{\frac{\alpha-D+d-2}{\alpha-D+d},\infty}(\Gamma_T)} \le C\|f\|_{L^1(\mathbb{R}^{1+D})}^{\frac{\alpha-D+d}{\alpha-D+d-2}},$$

$$\|g\|_{L_t^{\frac{\alpha-D}{\alpha-D+d},\infty} L_x^\infty(\Gamma_T)} \le C\|f\|_{L^1(\mathbb{R}^{1+D})}^{\frac{\alpha-D+d}{\alpha-D}}.$$

We now interpolate using Lemma B.3. Here $p_0 = \frac{\alpha-D}{\alpha-D+d}$, $q_0 = \frac{\alpha-D+d-2}{\alpha-D+d}$. For some p, q satisfying (B.1) to be determined, we have

$$\begin{aligned}\|g\|_{L_t^{p,\infty} L_x^{q,\infty}(\Gamma_T)} &\le C\|f\|_{L^1(\mathbb{R}^{1+D})}^{\frac{1}{p}} \|f\|_{L^1(\mathbb{R}^{1+D})}^{\frac{\alpha-D+d}{\alpha-D+d-2}(1-\frac{1}{p})} \\ &= C\|f\|_{L^1(\mathbb{R}^{1+D})}^{\frac{1}{p}+\frac{1}{q_0}(1-\frac{1}{p})} = C\|f\|_{L^1(\mathbb{R}^{1+D})}^{\frac{1}{q}}.\end{aligned}$$

Therefore,

$$\left\|g^q\right\|_{L_t^{\frac{p}{q},\infty} L_x^{1,\infty}(\Gamma_T)} \le C\|f\|_{L^1(\mathbb{R}^{1+D})}.$$

Set $q = \frac{\alpha-D+d-2\gamma}{\alpha-D+d}$, then $g^q = (\mathscr{A}_\alpha[f])^{1-\frac{D-d+2\gamma}{\alpha}}$, and from (B.1) we know

$$\frac{1-q_0}{p} + \frac{q_0}{q} = 1 \implies \frac{p}{q} = \frac{1-q_0}{q-q_0} = \frac{1}{1-\gamma}.$$

Similarly, for some p, q satisfying (B.2) to be determined, we have

$$\begin{aligned}\|g\|_{L_t^{p,\infty} L_x^{q,\infty}(\Gamma_T)} &\le C\|f\|_{L^1(\mathbb{R}^{1+D})}^{\frac{1}{q}} \|f\|_{L^1(\mathbb{R}^{1+D})}^{\frac{\alpha-D+d}{\alpha-D}(1-\frac{1}{q})} \\ &= C\|f\|_{L^1(\mathbb{R}^{1+D})}^{\frac{1}{q}+\frac{1}{p_0}(1-\frac{1}{q})} = C\|f\|_{L^1(\mathbb{R}^{1+D})}^{\frac{1}{p}}.\end{aligned}$$

Therefore,

$$\left\|g^p\right\|_{L_t^{1,\infty} L_x^{\frac{q}{p},\infty}(\Gamma_T)} \le C\|f\|_{L^1(\mathbb{R}^{1+D})}.$$

Set $p = \frac{\alpha-D+(1-\gamma)d}{\alpha-D+d}$, then $g^p = (\mathscr{A}_\alpha[f])^{1-\frac{D-(1-\gamma)d}{\alpha}}$, and from (B.2) we know

$$\frac{p_0}{p} + \frac{1-p_0}{q} = 1 \implies \frac{q}{p} = \frac{1-p_0}{p-p_0} = \frac{1}{\beta}.$$

This proves (18) and (20).

Finally, (22) is an interpolation of (18) and (18) with $d = 0$, while (23) is an interpolation of (19) and (19) with $d = 0$, $\square$

(18) can be slightly improved by the following proposition to (24), where we achieve strong spatial integrability instead of weak integrability, and enlarge the range of α. We could also work with anisotropic norms on f.

Proposition 4.8. *Let $0<\gamma<\theta<1\le p\le\infty$ satisfy $p\alpha>D-d+2\gamma$. Then it holds that*

$$\left\|(\mathscr{A}_\alpha[f])^{1-\frac{D-d+2\gamma}{p\alpha}}\right\|_{L_t^{\frac{p}{1-\gamma},\infty}L_x^p(\Gamma_T)}\le C(p,\gamma,\alpha,D,d,L)\|f\|_{L^p(\mathbb{R}^{1+D})},\tag{24}$$

$$\left\|(\mathscr{A}_\alpha[f])^{1-\frac{D-d+2\gamma}{p\alpha}}\right\|_{L_t^{\frac{p}{\theta-\gamma}}L_x^p(\Gamma_T)}\le C(\theta,p,\gamma,\alpha,D,d,L)\|f\|_{L_t^{\frac{p}{\theta}}L_x^p(\mathbb{R}^{1+D})},\tag{25}$$

$$\left\|(\mathscr{A}_\alpha[f])^{1-\frac{D-d+2\gamma}{p\alpha}}\right\|_{L_t^{\infty}L_x^{p,\infty}(\Gamma_T)}\le C(p,\gamma,\alpha,D,d,L)\|f\|_{L_t^{\frac{p}{\gamma}}L_x^p(\mathbb{R}^{1+D})}.\tag{26}$$

(26) *also holds for $\gamma=0,1$. In addition, if $p\alpha>D+2\gamma$, then for $\beta\in[0,1]$ it holds that*

$$\left\|(\mathscr{A}_\alpha[f])^{1-\frac{D-\beta d+2\gamma}{p\alpha}}\right\|_{L_t^{\frac{p}{1-\gamma},\infty}L_x^{\frac{p}{\beta}}(\Gamma_T)}\le C(p,\gamma,\alpha,D,d,L)\|f\|_{L^p(\mathbb{R}^{1+D})},\tag{27}$$

$$\left\|(\mathscr{A}_\alpha[f])^{1-\frac{D-\beta d+2\gamma}{p\alpha}}\right\|_{L_t^{\frac{p}{\theta-\gamma}}L_x^{\frac{p}{\beta}}(\Gamma_T)}\le C(\theta,p,\gamma,\alpha,D,d,L)\|f\|_{L_t^{\frac{p}{\theta}}L_x^p(\mathbb{R}^{1+D})},\tag{28}$$

$$\left\|(\mathscr{A}_\alpha[f])^{1-\frac{D-\beta d+2\gamma}{p\alpha}}\right\|_{L_t^{\infty}L_x^{\frac{p}{\beta},\infty}(\Gamma_T)}\le C(p,\gamma,\alpha,D,d,L)\|f\|_{L_t^{\frac{p}{\gamma}}L_x^p(\mathbb{R}^{1+D})}.\tag{29}$$

(29) *also holds for $\gamma=0,1$.*

Proof. The case $p=\infty$ reduces to Theorem 4.6 (c), so we assume $p<\infty$ from now on. We will prove (24)-(26) first, and prove (27)-(29) using interpolation.
Proof of (24)-(26). Denote $q=p(1-\frac{D-d+2\gamma}{p\alpha})>0$. Then

$$\begin{aligned}\int_{\Gamma_t}(\mathscr{A}_\alpha[f])^q(t,x)\,\mathrm{d}\mu_t&=\int_0^\infty q\lambda^{q-1}\mu_t(\{\mathscr{A}_\alpha[f]>\lambda\})\,\mathrm{d}\lambda\\&=2^{-\alpha q}\int_0^\infty q\lambda^{q-1}\mu_t(\{\mathscr{A}_\alpha[f]>2^{-\alpha}\lambda\})\,\mathrm{d}\lambda.\end{aligned}$$

Taking the difference of two integrals yields (recall $A_t(\rho)$ defined in Lemma 4.4)

$$\begin{aligned}\int_{\Gamma_t}(\mathscr{A}_\alpha[f])^q(t,x)\mu_t(\mathrm{d}x)&=\frac{1}{2^{\alpha q}-1}\int_0^\infty q\lambda^{q-1}\mu_t(\{2^{-\alpha}\lambda<\mathscr{A}_\alpha[f]\le\lambda\})\,\mathrm{d}\lambda\\&\le\frac{1}{2^{\alpha q}-1}\int_0^\infty q\lambda^{q-1}\mu_t(A_t(\lambda^{-\frac{1}{\alpha}}))\,\mathrm{d}\lambda\\&\le C\int_0^\infty\lambda^{q-1}\lambda^{\frac{D-d+2-p\alpha}{\alpha}}\int_{t-\lambda^{-\frac{2}{\alpha}}}^{t}\int_{\mathbb{R}^D}f(s,x)^p\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}\lambda\end{aligned}$$

$$
\begin{aligned}
&= C\int_0^\infty \int_0^{s^{-\frac{\alpha}{2}}} \lambda^{\frac{2}{\alpha}(1-\gamma)-1}\,\mathrm{d}\lambda \int_{\mathbb{R}^D} f(t-s,x)^p\,\mathrm{d}x\,\mathrm{d}s \\
&= \frac{C\alpha}{2(1-\gamma)} \int_0^\infty s^{-1+\gamma}\|f(t-s)\|^p_{L^p(\mathbb{R}^D)}\,\mathrm{d}s.
\end{aligned}
$$

Since $t \mapsto t^{-1+\gamma}$ is in $L^{\frac{1}{1-\gamma},\infty}(\mathbb{R}_+)$, the generalized Young's convolution inequality in Lemma 2.5 directly implies

$$
\begin{aligned}
\left\|(\mathscr{A}_\alpha[f])^q\right\|_{L_t^{\frac{1}{1-\gamma},\infty} L^1_x(\Gamma_T)} &\le C\|f\|^p_{L^p_t L^p_x(\mathbb{R}^{1+D})}, \\
\left\|(\mathscr{A}_\alpha[f])^q\right\|_{L_t^{\frac{1}{\theta-\gamma}} L^1_x(\Gamma_T)} &\le C\|f\|^p_{L_t^{\frac{p}{\theta}} L^p_x(\mathbb{R}^{1+D})}.
\end{aligned}
$$

This completes the proof for (24)-(25).

To prove (26), note that for any $\lambda > 0$,

$$
\begin{aligned}
\mu_t(\{2^{-\alpha}\lambda < \mathscr{A}_\alpha[f](t) \le \lambda\}) &\le C\lambda^{\frac{D-d+2-p\alpha}{\alpha}} \int_{t-\lambda^{-\frac{2}{\alpha}}}^t \int_{\mathbb{R}^D} f(s,x)^p\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}\lambda \\
&= C\lambda^{\frac{D-d-p\alpha}{\alpha}} \fint_{t-\lambda^{-\frac{2}{\alpha}}}^t \|f(s)\|^p_{L^p(\mathbb{R}^D)}\,\mathrm{d}s.
\end{aligned}
$$

If $\gamma \in (0,1]$, then by Jensen's inequality,

$$
\begin{aligned}
\mu_t(\{2^{-\alpha}\lambda < \mathscr{A}_\alpha[f](t) \le \lambda\}) &\le C\lambda^{\frac{D-d-p\alpha}{\alpha}} \left(\fint_{t-\lambda^{-\frac{2}{\alpha}}}^t \|f(s)\|^{\frac{p}{\gamma}}_{L^p(\mathbb{R}^D)}\,\mathrm{d}s \right)^\gamma \\
&= C\lambda^{\frac{D-d+2\gamma-p\alpha}{\alpha}} \left(\int_{t-\lambda^{-\frac{2}{\alpha}}}^t \|f(s)\|^{\frac{p}{\gamma}}_{L^p(\mathbb{R}^D)}\,\mathrm{d}s \right)^\gamma \\
&\le C\lambda^{-q}\|f\|^p_{L_t^{\frac{p}{\gamma}} L^p_x(\mathbb{R}^{1+D})}.
\end{aligned}
$$

If $\gamma = 0$, then

$$
\begin{aligned}
\mu_t(\{2^{-\alpha}\lambda < \mathscr{A}_\alpha[f](t) \le \lambda\}) &\le C\lambda^{\frac{D-d-p\alpha}{\alpha}} \|f\|^p_{L^\infty_t L^p_x(\mathbb{R}^{1+D})} \\
&= C\lambda^{-q}\|f\|^p_{L_t^{\frac{p}{\gamma}} L^p_x(\mathbb{R}^{1+D})}.
\end{aligned}
$$

Therefore, for every $t \in \mathbb{R}$,

$$\|\mathscr{A}_\alpha[f](t)\|^q_{L^{q,\infty}(\Gamma_t)} \le C\|f\|^p_{L_t^{\frac{p}{\gamma}} L_x^p(\mathbb{R}^{1+D})},$$

$$\left\|\mathscr{A}_\alpha[f](t)^{\frac{q}{p}}\right\|_{L^{p,\infty}(\Gamma_t)} \le C\|f\|_{L_t^{\frac{p}{\gamma}} L_x^p(\mathbb{R}^{1+D})}.$$

Proof of (27)-(29)**.** If $p\alpha > D + 2\gamma$, by setting $d = 0$ in (25), we have

$$\left\|(\mathscr{A}_\alpha[f])^{1-\frac{D+2\gamma}{p\alpha}}\right\|_{L_t^{\frac{p}{\theta-\gamma}} L_x^\infty(\Gamma_T)} \le C(\theta, p, \gamma, \alpha, D, d, L)\|f\|_{L_t^{\frac{p}{\theta}} L_x^p(\mathbb{R}^{1+D})}.$$

This is proven using the same strategy for (20)-(21). (28) is an interpolation between this and (25). (27) and (29) can also be proven by interpolation. □

We summarize Theorem 4.6, Proposition 4.7, and Proposition 4.8 in the following unified form.

Corollary 4.9. *Let* $0 < p_1, q_1, p_2, q_2 \le \infty$ *and* $\alpha > 0$. *Define* $r_1, r_2 \in \mathbb{R} \setminus \{0\} \cup \{\infty\}$ *by*

$$\frac{1}{r_1} = \alpha - \frac{2}{p_1} - \frac{D}{q_1}, \qquad \frac{1}{r_2} = \alpha - \frac{2}{p_2} - \frac{d}{q_2}.$$

Suppose $r_2 = \lambda r_1$ *for some* $\lambda \in (0, \infty)$, $p_2 \ge \lambda p_1$, $q_2 \ge \lambda q_1$, $1 \le q_1 \le p_1$, *and* $f \in L_t^{p_1} L_x^{q_1}(\mathbb{R}^{1+D})$.

(A) *Suppose* $p_1 = q_1 = p \in [1, \infty]$, *and assume* $\frac{1}{r_1} + \frac{2}{p_1} \cdot \mathbf{1}_{\{p_2=\lambda p_1\}} + \frac{d}{q_1} \cdot \mathbf{1}_{\{q_2=\lambda q_1\}} > 0$.
  (a) *If* $p = 1$, *then*

$$\|\mathscr{A}_\alpha[f]\|^\lambda_{L_{t,x}^{\lambda,\infty}(\Gamma_T)} \le C\|f\|_{L^1_{t,x}(\mathbb{R}^{1+D})} \qquad \text{if } p_2 = q_2 = \lambda,$$

$$\|\mathscr{A}_\alpha[f]\|^\lambda_{L_t^{p_2,\infty} L_x^{q_2,\infty}(\Gamma_T)} \le C\|f\|_{L^1_{t,x}(\mathbb{R}^{1+D})} \qquad \text{if } p_2 > \lambda \text{ or } q_2 > \lambda.$$

  (b) *If* $p > 1$, *then*

$$\|\mathscr{A}_\alpha[f]\|^\lambda_{L_t^{p_2} L_x^{q_2}(\Gamma_T)} \le C\|f\|_{L^p_{t,x}(\mathbb{R}^{1+D})}.$$

(B) *Suppose* $\lambda p_1 < p_2 < \infty$. *When* $\lambda q_1 < q_2$, *we further require* $\alpha > \frac{d}{q_2}$.
  (a) *If* $p_1 = q_1$, *then*

$$\|\mathscr{A}_\alpha[f]\|^\lambda_{L_t^{p_2,\infty} L_x^{q_2}(\Gamma_T)} \le C\|f\|_{L_t^{p_1} L_x^{q_1}(\mathbb{R}^{1+D})}.$$

  (b) *If* $p_1 > q_1$, *then*

$$\|\mathscr{A}_\alpha[f]\|^\lambda_{L_t^{p_2} L_x^{q_2}(\Gamma_T)} \le C\|f\|_{L_t^{p_1} L_x^{q_1}(\mathbb{R}^{1+D})}.$$

(C) *Suppose $p_2=\infty$. When $\lambda q_1<q_2$, we further require $\alpha>\frac{d}{q_2}$. Then*

$$\|\mathscr{A}_\alpha[f]\|^\lambda_{L_t^\infty L_x^{q_2,\infty}(\Gamma_T)}\le C\|f\|_{L_t^{p_1}L_x^{q_1}(\mathbb{R}^{1+D})}.$$

In all of the above, the constant $C=C(p_1,q_1,p_2,q_2,\alpha,D,d,L)$.

Proof. When $q_1=\infty$, we have $p_1=p_2=q_2=\infty$, $r_1=r_2=\alpha^{-1}$ so $\lambda=1$. (Ab) and (C) now are simply Theorem 4.6 (c) with $p=\infty$. We now focus on the case $q_1<\infty$. (A)-(B) are translations of Proposition 4.7-4.8 under the following change of parameters:

$$\begin{cases} p_1=\frac{p}{\theta}\\ q_1=p\\ p_2=\lambda\cdot\frac{p}{\theta-\gamma}\\ q_2=\lambda\cdot\frac{p}{\beta}\end{cases}\quad\Longleftarrow\quad\begin{cases} p=q_1\\ \theta=\frac{q_1}{p_1}\\ \beta=\lambda\cdot\frac{q_1}{q_2}\\ \gamma=\frac{q_1}{p_1}-\lambda\cdot\frac{q_1}{p_2}\end{cases}$$

with λ satisfying $r_2=\lambda r_1$. Note that

$$\begin{aligned}\alpha\left(1-\frac{D-\beta d+2\gamma}{p\alpha}\right)&=\alpha-\frac{D}{p}+\frac{\beta d}{p}-\frac{2\gamma}{p}=\alpha-\frac{D}{q_1}+\frac{\lambda d}{q_2}-2\left(\frac{1}{p_1}-\frac{\lambda}{p_2}\right)\\&=\alpha-\frac{D}{q_1}-\frac{2}{p_1}+\lambda\left(\frac{d}{q_2}+\frac{2}{p_2}\right)=\frac{1}{r_1}+\lambda\left(\alpha-\frac{1}{r_2}\right)=\lambda\alpha.\end{aligned}$$

So

$$\lambda=1-\frac{D-\beta d+2\gamma}{p\alpha}.$$

First, we verify that p,θ,β,γ are within the range prescribed by Proposition 4.7-4.8 provided $q_1<\infty$.

- $p=q_1\in[1,\infty]$.
- $\theta=\frac{q_1}{p_1}\in[0,1]$ because $q_1\le p_1<\infty$. $\theta=1$ iff $p_1=q_1$. $\theta=0$ iff $p_1=\infty$.
- $\beta=\lambda\cdot\frac{q_1}{q_2}\in[0,1]$ because $q_2\ge\lambda q_1$ and $q_1<\infty$. $\beta=0$ iff $q_2=\infty$. $\beta=1$ iff $q_2=\lambda q_1$.
- $\gamma>0$ if $p_2>\lambda p_1$ because
  - when $p_2<\infty$, $\gamma=\frac{q_1}{p_1p_2}(p_2-\lambda p_1)>0$.
  - when $p_2=\infty$, $\gamma=\frac{q_1}{p_1}>0$.
- $\gamma=0$ if $p_2=\lambda p_1$.
- $\gamma\le\frac{q_1}{p_1}=\theta$. Equality holds if and only if $p_2=\infty$.

We delay the verification of assumptions on parameters to the end of the proof.

(A) This will be proven using Theorem 4.6 and Proposition 4.7. We separate four cases.
   (1) When $p_2=\lambda p_1$ and $q_2=\lambda q_1$, we have $\gamma=0$ and $\beta=1$, thus $\lambda=1-\frac{D-d}{p\alpha}$. When $p_1=q_1=1$, by Theorem 4.6 (b), we have

$$\left\| (\mathscr{A}_\alpha[f])^{1-\frac{D-d}{\alpha}} \right\|_{L^{1,\infty}_{t,x}(\Gamma_T)} \le C\|f\|_{L^1(\mathbb{R}^{1+D})}$$
$$\Longrightarrow \|\mathscr{A}_\alpha[f]\|^{\lambda}_{L^{\lambda,\infty}_{t,x}(\Gamma_T)} \le C\|f\|_{L^1(\mathbb{R}^{1+D})}.$$

When $p_1 = q_1 > 1$, by Theorem 4.6 (c), we have

$$\left\| (\mathscr{A}_\alpha[f])^{1-\frac{D-d}{p\alpha}} \right\|_{L^{p}_{t,x}(\Gamma_T)} \le C\|f\|_{L^p(\mathbb{R}^{1+D})}$$
$$\Longrightarrow \|\mathscr{A}_\alpha[f]\|^{\lambda}_{L^{p_2}_t L^{q_2}_x(\Gamma_T)} \le C\|f\|_{L^p(\mathbb{R}^{1+D})}.$$

(2) When $p_2 > \lambda p_1$ and $q_2 = \lambda q_1$, we have $0 < \gamma \le 1$ and $\beta = 1$, thus $\lambda = 1 - \frac{D+2\gamma}{p\alpha}$. When $p_1 = q_1 = 1$, by Proposition 4.7 (18), we have

$$\left\| (\mathscr{A}_\alpha[f])^{1-\frac{D-d+2\gamma}{\alpha}} \right\|_{L_t^{\frac{1}{1-\gamma},\infty} L_x^{1,\infty}(\Gamma_T)} \le C\|f\|_{L^1(\mathbb{R}^{1+D})},$$
$$\Longrightarrow \|\mathscr{A}_\alpha[f]\|^{\lambda}_{L^{p_2,\infty}_t L^{q_2,\infty}_x(\Gamma_T)} \le C\|f\|_{L^1(\mathbb{R}^{1+D})}.$$

When $p_1 = q_1 > 1$, by Proposition 4.7 (19), we have

$$\left\| (\mathscr{A}_\alpha[f])^{1-\frac{D-d+2\gamma}{p\alpha}} \right\|_{L_t^{\frac{p}{1-\gamma}} L_x^{p}(\Gamma_T)} \le C\|f\|_{L^p(\mathbb{R}^{1+D})}.$$
$$\Longrightarrow \|\mathscr{A}_\alpha[f]\|^{\lambda}_{L^{p_2}_t L^{q_2}_x(\Gamma_T)} \le C\|f\|_{L^p(\mathbb{R}^{1+D})}.$$

(3) When $p_2 = \lambda p_1$ and $q_2 > \lambda q_1$, we have $\gamma = 0$ and $0 \le \beta < 1$, thus $\lambda = 1 - \frac{D-\beta d}{p\alpha}$. We use Proposition 4.7 (20)-(21).

(4) When $p_2 > \lambda p_1$ and $q_2 > \lambda q_1$, we have $0 < \gamma \le 1$ and $0 \le \beta < 1$. We use Proposition 4.7 (22)-(23).

(B) $\lambda p_1 < p_2 < \infty$ implies $0 < \gamma < \theta$, so we have verified $0 < \gamma < \theta \le 1 \le p < \infty$. If $\lambda q_1 < q_2$, then $\beta \in [0, 1)$, we have

(a) If $p_1 = q_1$, then $\theta = 1$. By (27), we have

$$\left\| (\mathscr{A}_\alpha[f])^{1-\frac{D-\beta d+2\gamma}{p\alpha}} \right\|_{L_t^{\frac{p}{1-\gamma},\infty} L_x^{\frac{p}{\beta}}(\Gamma_T)} \le C\|f\|_{L^p_t L^p_x(\mathbb{R}^{1+D})}$$
$$\Longrightarrow \left\| (\mathscr{A}_\alpha[f])^{\lambda} \right\|_{L_t^{\frac{p}{1-\gamma},\infty} L_x^{\frac{p}{\beta}}(\Gamma_T)} \le C\|f\|_{L^{p_1}_t L^{q_1}_x(\mathbb{R}^{1+D})}$$
$$\Longrightarrow \|\mathscr{A}_\alpha[f]\|^{\lambda}_{L^{p_2,\infty}_t L^{q_2}_x(\Gamma_T)} \le C\|f\|_{L^{p_1}_t L^{q_1}_x(\mathbb{R}^{1+D})}.$$

(b) If $p_1 > q_1$, then $\theta < 1$. By (28), we have

$$\left\| (\mathscr{A}_\alpha[f])^{1-\frac{D-\beta d+2\gamma}{p\alpha}} \right\|_{L_t^{\frac{p}{\theta-\gamma}} L_x^{\frac{p}{\beta}}(\Gamma_T)} \le C\|f\|_{L_t^{\frac{p}{\theta}} L^p_x(\mathbb{R}^{1+D})}$$
$$\Longrightarrow \left\| (\mathscr{A}_\alpha[f])^{\lambda} \right\|_{L_t^{\frac{p}{\theta-\gamma}} L_x^{\frac{p}{\beta}}(\Gamma_T)} \le C\|f\|_{L^{p_1}_t L^{q_1}_x(\mathbb{R}^{1+D})}$$

$$\Longrightarrow \|\mathscr{A}_\alpha[f]\|^{\lambda}_{L_t^{p_2}L_x^{q_2}(\Gamma_T)} \le C\|f\|_{L_t^{p_1}L_x^{q_1}(\mathbb{R}^{1+D})}.$$

If $\lambda q_1 = q_2$, $\beta = 1$, we use (24)-(25) instead of (27)-(28).

(C) When $p_2 = \infty$, we use (26) if $\lambda q_1 = q_2$ and (29) if $\lambda q_1 < q_2$. Now $\gamma = \theta$ is allowed to take value in [0, 1].

The requirements of Theorem 4.6 and Proposition 4.7 can be summarized to

$$p\alpha > D - d \cdot \mathbf{1}_{\{q_2=\lambda q_1\}} + 2 \cdot \mathbf{1}_{\{p_2>\lambda p_1\}},$$

which is equivalent to $\frac{p}{r_1} + d \cdot \mathbf{1}_{\{q_2=\lambda q_1\}} + 2 \cdot \mathbf{1}_{\{p_2=\lambda p_1\}} > 0$ using

$$\frac{p}{r_1} = p\alpha - 2 - D$$

when $p_1 = q_1 = p$.

The requirement $p\alpha > D - d + 2\gamma$ of Proposition 4.8 can be simplified as the following:

$$\begin{aligned}
&\alpha - \frac{D}{q_1} + \frac{d}{q_1} > \frac{2\gamma}{q_1} = \frac{2}{p_1} - \lambda\frac{2}{p_2}\\
\Longleftrightarrow\ &\frac{1}{r_1} + \frac{d}{q_1} > -\lambda\frac{2}{p_2} = \lambda\left(\frac{1}{r_2} - \alpha + \frac{d}{q_2}\right)\\
\Longleftrightarrow\ &\frac{d}{q_1} + \lambda\left(\alpha - \frac{d}{q_2}\right) > 0\\
\Longleftrightarrow\ &\lambda\alpha + d\left(\frac{1}{q_1} - \frac{\lambda}{q_2}\right) > 0.
\end{aligned}$$

This always hold since $\lambda\alpha > 0$, and $q_2 \ge \lambda q_1$.

Similarly, the requirement $p\alpha > D + 2\gamma$ of Proposition 4.8 can be simplified as $\lambda\left(\alpha - \frac{d}{q_2}\right) > 0$, that is, $\alpha \ge \frac{d}{q_2}$. $\square$

Remark 4.10. Recall that $\mathscr{A}_\alpha[f] = (\mathscr{S}_\alpha[f])^{-\alpha}$. The above corollary provides a bound on the inverse scale function, for instance (Bb) implies

$$\left\|(\mathscr{S}_\alpha[f])^{-1}\right\|^{\lambda\alpha}_{L_t^{p_2\alpha}L_x^{q_2\alpha}(\Gamma_T)} \le C\|f\|_{L_t^{p_1}L_x^{q_1}(\mathbb{R}^{1+D})}.$$

5. Blow-up technique and trace estimate: Lagrangian case

5.1. Adapting the drift and boundary

For equations with a transport term, such as transport equations, active scalar equations, kinetic equations, or fluid equations, additional difficulties appear when the pivot quantity f does not control the drift b. It might be more convenient to consider the skewed cylinders that “follow the trajectories”, and interpret the cylinders in the Lagrangian coordinate. The author studied these objects in [39], but similar construction has been introduced by Isett et al. [24,22,23,15] in the study of the Euler equation. Given a locally integrable drift $b : \mathbb{R} \times \mathbb{R}^D \to \mathbb{R}^D$, we fix $\varphi \in C_c^\infty(\mathbb{R}^D)$ to be a nonnegative smooth mollifier supported in B_1. For any $\rho > 0$, define $\varphi_\rho(x) = \rho^{-D}\varphi(x/\rho)$. Let $b_\rho(t) = b(t) * \varphi_\rho$ be the spatially mollified drift, and the flow X_ρ is defined to be the unique solution to the following ODE:

$$\begin{cases} \dot{X}_\rho(s;t,x) = b_\rho(s, X_\rho(s;t,x)) \\ X_\rho(t;t,x) = x. \end{cases} \tag{30}$$

Suppose $b \in L^p(\Omega_T)$ for some $p \in [1,\infty]$, then we can make a zero extension so that $b \in L^p(\mathbb{R} \times \mathbb{R}^D)$, thus the ODE can have a unique solution for all time. We will assume this from now on. Define for $(t,x) \in \mathbb{R}^{1+D}$ the skewed parabolic cylinder

$$Q_\rho(t,x) := \left\{(s, X_\rho(s;t,x) + y) : s \in (t-\rho^2, t], y \in B_\rho\right\}. \tag{31}$$

Although (31) is duplicate notation with (12), in this section we will always assume an implied background drift b and **use (31) to define $Q_\rho(t,x)$ from now on**. In the previous section, the standard parabolic cylinder $(t-\rho^2, t] \times B_\rho(x)$ can be regarded as a skewed parabolic cylinder $Q_\rho(t,x)$ when the underlying drift is $b = 0$.

Definition 5.1 *([39], Definition 2).* We say $Q_\rho(t,x)$ is **admissible** if

$$\fint_{Q_\rho(t,x)} \mathcal{M}((\nabla b)(s))(y)\,\mathrm{d}y\,\mathrm{d}s \le \eta_0 \rho^{-2}. \tag{32}$$

Here $\eta_0 > 0$ is a small universal constant depending only on d and L that will be specified later, and recall $\mathcal{M}$ is the maximal function defined by

$$(\mathcal{M}f)(x) = \sup_{r>0} \fint_{B_r(x)} |f(y)|\,\mathrm{d}y, \qquad f \in L^1_{\mathrm{loc}}(\mathbb{R}^D), x \in \mathbb{R}^D.$$

If there is no background flow (when $b = 0$), then $Q_\rho(t,x) = (t-\rho^2, t] \times B_\rho(x)$ is just a standard parabolic cylinder, and it is admissible. This admissibility ensures a covering lemma for the skewed cylinders.

Lemma 5.2 *([39], Proposition 12). There exists a universal constant C depending on the dimension D such that the following holds. Let $\left\{Q_{\varepsilon_\alpha}(t^\alpha, x^\alpha)\right\}_\alpha$ be a collection of admissible cylinders with uniformly bounded radii. Then there exists a pairwise disjoint subcollection $\{Q_{\varepsilon_{\alpha_j}}(t^{\alpha_j}, x^{\alpha_j})\}_j$ such that*

$$\sum_j |Q_{\varepsilon_{\alpha_j}}(t^{\alpha_j}, x^{\alpha_j})| \ge \frac{1}{C}\left|\bigcup_\alpha Q_{\varepsilon_\alpha}(t^\alpha, x^\alpha)\right|.$$

Now for a measurable function $f : \mathbb{R}^{1+D} \to [0, \infty]$, we define f_ρ for $\rho > 0$, define the scale function $\mathscr{S}_\alpha[f]$ and the averaging function $\mathscr{A}_\alpha[f]$ in the same way as Definition 4.1, except we now use skewed parabolic cylinders. We do not repeat the definition here.

In order to extend results from the previous section to the setting with drift, we verify that the inner and outer regularity still hold for the skewed parabolic cylinders.

Lemma 5.3. *Fix $\rho > 0$ and $(t, x) \in \mathbb{R}^{1+D}$.*

1. *(Inner regularity) For any compact subset $K \subset\subset Q_\rho(t, x)$, $Q_\sigma(s, y)$ is also a superset of K provided (σ, s, y) is sufficiently close to (ρ, t, x).*
2. *(Outer regularity) For any open superset $O \supset\supset \bar{Q}_\rho(t, x)$, $Q_\sigma(s, y)$ is also a subset of O provided (σ, s, y) is sufficiently close to (ρ, t, x).*

Since the drift is locally integrable, this is a simple ODE exercise and we omit the proof. This implies the continuity and semicontinuity for the scale function and average function. In other words, Lemma 4.2 still holds in our now generalized definition.

Below we give two important examples of the usage of the scale operator, and introduce a capped scale operator $\mathscr{S}_\alpha^\wedge$.

Definition 5.4. Recall $\eta_0 > 0$ defined in (32) is the admissibility threshold. Define

$$r_{\text{int}}(t, x) = \mathscr{S}_2[\infty \mathbf{1}_{\Omega_T^c}](t, x) = \inf\{\rho > 0 : Q_\rho(t, x) \not\subseteq \Omega_T\},$$

$$r_{\text{adm}}(t, x) = \mathscr{S}_2\left[\frac{1}{\eta_0}\mathcal{M}(\nabla b)\right](t, x) = \inf\{\rho > 0 : Q_\rho(t, x) \text{ is not admissible}\}.$$

Denote $\bar{r} = r_{\text{adm}} \wedge r_{\text{int}}$. For $f \in L^1_{\text{loc}}(\mathbb{R}^{1+d})$, $\alpha > 0$, define $\mathscr{S}_\alpha^\wedge[f] = \mathscr{S}_\alpha[f] \wedge \bar{r}$.

From the definition, we see that for any $\rho \le r_{\text{adm}}(t, x)$, $Q_\rho(t, x)$ is admissible; for any $\rho \le r_{\text{int}}(t, x)$, $Q_\rho(t, x) \subseteq \Omega_T$. Therefore, $\rho \le \mathscr{S}_\alpha^\wedge[f](t, x) < \infty$ ensures that the cube $Q_\rho(t, x)$ is an admissible cube, fully contained in Ω_T, with $f_\rho(t, x) \le \rho^{-\alpha}$. The benefits of using the capped scale operator $\mathscr{S}_\alpha^\wedge$ are two folds: the cut-off r_{int} makes sure that the epsilon regularity theorem is applicable in $Q_\rho(t, x)$ in the presence of the boundary, whereas the cut-off r_{adm} enables the covering lemma and harmonic analysis which will be introduced in the next subsection. Moreover, note that $\mathscr{S}_\alpha^\wedge[f]$ is also upper semicontinuous since it is the minimum of three upper semicontinuous functions.

Provided $f \in L^1_{\text{loc}}$, let us make the following observations. Given $(t, x) \in \mathbb{R}^{1+D}$, denote $s(t, x) = \mathscr{S}_\alpha^\wedge[f](t, x)$, then

(a) if $s(t, x) = 0$, it means either
 - (a$_1$) $\bar{r}(t, x) = 0$.
 - (a$_2$) $\bar{r}(t, x) > 0$ and $\mathscr{S}_\alpha[f](t, x) = 0$. There exists a sequence of $\varepsilon_i \to 0$ such that $f_{\varepsilon_i}(t, x) \ge \varepsilon_i^{-\alpha}$, so $f_0(t, x) = \limsup_{\varepsilon \to 0} f_\varepsilon(t, x) = +\infty$.

(b) if $s(t, x) > 0$, then $\bar{r}(t, x) > 0$ and $f_\rho(t, x) \le \rho^{-\alpha}$ for every $\rho \le s(t, x)$. In particular, $f_{s(t,x)}(t, x) \le s(t, x)^{-\alpha}$. More precisely:
 - (b$_1$) if $s(t, x) = \bar{r}(t, x)$, then $f_{s(t,x)}(t, x) = f_{\bar{r}(t,x)}(t, x) \le \bar{r}(t, x)^{-\alpha} = s(t, x)^{-\alpha}$.

(b$_2$) if $s(t,x) < \bar{r}(t,x)$, then $f_{s(t,x)}(t,x) = s(t,x)^{-\alpha} > \bar{r}(t,x)^{-\alpha}$ by continuity of $f_\rho(t,x)$ in ρ.

Note that (b) also holds when $s(t,x) = \bar{r}(t,x) = +\infty$. We divide the singular set and regular set by

$$\begin{aligned}
\mathrm{Sing}_\alpha^{=}(f) &= \Big\{(t,x) \in \mathbb{R}^{1+D} : s(t,x) = 0 = \bar{r}(t,x)\Big\},\\
\mathrm{Sing}_\alpha^{<}(f) &= \Big\{(t,x) \in \mathbb{R}^{1+D} : s(t,x) = 0 < \bar{r}(t,x)\Big\},\\
\mathrm{Reg}_\alpha^{=}(f) &= \Big\{(t,x) \in \mathbb{R}^{1+D} : 0 < s(t,x) = \bar{r}(t,x)\Big\},\\
\mathrm{Reg}_\alpha^{<}(f) &= \Big\{(t,x) \in \mathbb{R}^{1+D} : 0 < s(t,x) < \bar{r}(t,x)\Big\}.
\end{aligned}$$

They form a partition of $\mathbb{R}^{1+D}$. Note that $\mathrm{Sing}_\alpha(f) = \mathrm{Sing}_\alpha^{<}(f) \cup \mathrm{Sing}_\alpha^{=}(f)$, and $\mathrm{Reg}_\alpha(f) = \mathrm{Reg}_\alpha^{<}(f) \cup \mathrm{Reg}_\alpha^{=}(f)$, so they are consistent with the definition of singular sets and regular sets from the previous section.

To conclude this subsection, we remark that the interior threshold r_{int} can be replaced by the parabolic distance in Ω_T. This will be accomplished in Lemma 5.7, but we need some auxiliary results first.

Lemma 5.5. *Suppose $c_2 > c_1 > 0$ satisfy*

$$\|\varphi\|_{L^\infty}(c_2+2)^D \eta_0 < \log c_2 - \log c_1.$$

Let $Q_\varepsilon(t,x)$ be admissible, and let $t_* \in (t-\varepsilon^2, t)$. If $x_* \in \mathbb{R}^D$ satisfies

$$|x_* - X_\varepsilon(t_*; t, x)| < c_1\varepsilon,$$

then

$$|X_\varepsilon(s; t_*, x_*) - X_\varepsilon(s; t, x)| < c_2\varepsilon, \qquad \text{for all } s \in (t-\varepsilon^2, t].$$

Recall that X_ε is defined in (30).

Proof. The proof is the same as Lemma 5 of [39] which is a special case of $c_1 = 1$ and $c_2 = 2$. We prove it here for the sake of completeness.

For simplicity, denote the two trajectories by $\gamma(s) = X_\varepsilon(s; t, x)$ and $\gamma_*(s) = X_\varepsilon(s; t_*, x_*)$. Then $|\gamma_*(t_*) - \gamma(t_*)| < c_1\varepsilon$. We have

$$\begin{aligned}
\left|\frac{\mathrm{d}}{\mathrm{d}s}\log|\gamma(s)-\gamma_*(s)|\right| &= \frac{1}{|\gamma(s)-\gamma_*(s)|}\left|\frac{\mathrm{d}}{\mathrm{d}s}|\gamma(s)-\gamma_*(s)|\right|\\
&\le \frac{1}{|\gamma(s)-\gamma_*(s)|}|\dot{\gamma}(s)-\dot{\gamma}_*(s)|\\
&= \frac{1}{|\gamma(s)-\gamma_*(s)|}|b_\varepsilon(s,\gamma(s)) - b_\varepsilon(s,\gamma_*(s))|
\end{aligned}$$

$$\leq |\nabla b_\varepsilon(s,\xi)|$$

for some ξ between $\gamma(s)$ and $\gamma_*(s)$. Then provided $|\gamma(s)-\gamma_*(s)| \leq c_2\varepsilon$ at time s, we would have $|\gamma(s)-\xi| \leq c_2\varepsilon$ and

$$|\nabla b_\varepsilon(s,\xi)| \leq \int\limits_{B_\varepsilon(\xi)} |\nabla b(x)||\varphi(x-\xi)|\,\mathrm{d}x \leq \|\varphi\|_{L^\infty} \int\limits_{B_{(c_2+2)\varepsilon}(y)} |\nabla b(x)|\,\mathrm{d}x$$

for any $y \in B_\varepsilon(\gamma(s))$. Hence

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}\log|\gamma(s)-\gamma_*(s)|\right| \leq \|\varphi\|_{L^\infty}(c_2+2)^D \int\limits_{B_\varepsilon(\gamma(s))} \mathcal{M}(\nabla b)(x)\,\mathrm{d}x.$$

We now prove the lemma using a continuity argument. Suppose there exists $t_0 \in (t-\varepsilon^2, t_*)$ such that $|\gamma(s)-\gamma_*(s)| \leq c_2\varepsilon$ for all $t_0 < s < t_*$, and equality is reached at $s = t_0$. Then

$$\begin{aligned}
\log c_2 - \log c_1 &= \log(c_2\varepsilon) - \log(c_1\varepsilon) \\
&\leq \log|\gamma(t_0)-\gamma_*(t_0)| - \log|\gamma(t_*)-\gamma_*(t_*)| \\
&\leq \|\varphi\|_{L^\infty}(c_2+2)^D \int\limits_{t_0}^{t_*} \int\limits_{B_\varepsilon(\gamma(s))} \mathcal{M}(\nabla b)(x)\,\mathrm{d}x \\
&\leq \|\varphi\|_{L^\infty}(c_2+2)^D \eta_0.
\end{aligned}$$

Hence we reach a contradiction. The same argument holds for $t_0 \in (t_*, t)$. By continuity, $|\gamma(s) - \gamma_*(s)|$ cannot reach $c_2\varepsilon$ at any $s \in (t-\varepsilon^2, t]$. $\square$

Corollary 5.6. *Suppose Ω satisfies Assumption 2.1 with constant L and r_0. Let $(t,x) \in \Omega_T$, and suppose $Q_\varepsilon(t,x)$ is admissible with $(L+4)\varepsilon < r_0$. If*

$$\mathrm{dist}_{\mathcal{P}}(Q_\varepsilon(t,x), \partial_{\mathcal{P}}\Omega_T) < \varepsilon,$$

then

$$\mathrm{dist}_{\mathcal{P}}((t,x), \partial_{\mathcal{P}}\Omega_T) \leq (L+4)\varepsilon.$$

Proof. We argue by contradiction and assume for some $(s,y) \in Q_\varepsilon(t,x)$,

$$\begin{aligned}
&\mathrm{dist}_{\mathcal{P}}((s,y), \partial_{\mathcal{P}}\Omega_T) < \varepsilon, \\
&\mathrm{dist}_{\mathcal{P}}((t,x), \partial_{\mathcal{P}}\Omega_T) > (L+4)\varepsilon.
\end{aligned}$$

Since $(t,x) \in \Omega_T$, it is clear that

$$\mathrm{dist}_{\mathcal{P}}((t,x), \partial_{\mathcal{P}}\Omega_T) = \min\left\{\sqrt{t}, \mathrm{dist}(x, \partial\Omega)\right\},$$

so $\sqrt{t} > (L+4)\varepsilon$ and $\mathrm{dist}(x, \partial\Omega) > (L+4)\varepsilon$. Since $(s, y) \in Q_\varepsilon(t, x)$, we know that

$$0 < 9\varepsilon^2 < ((L+4)^2 - 1)\varepsilon^2 < t - \varepsilon^2 < s \le t < T.$$

As $s > 9\varepsilon^2$, $\mathrm{dist}_{\mathcal{P}}((s, y), \{0\} \times \Omega) > 3\varepsilon$, so it must hold that

$$\varepsilon > \mathrm{dist}_{\mathcal{P}}((s, y), \partial_{\mathcal{P}}\Omega_T) = \mathrm{dist}_{\mathcal{P}}((s, y), [0, T) \times \partial\Omega) = \mathrm{dist}(y, \partial\Omega).$$

Moreover, since $|y - X_\varepsilon(s; t, x)| < \varepsilon$, triangular inequality implies

$$\mathrm{dist}(X_\varepsilon(s; t, x), \Omega^c) < 2\varepsilon.$$

Together with $\mathrm{dist}(X_\varepsilon(t; t, x), \Omega^c) = \mathrm{dist}(x, \Omega^c) > (L+4)\varepsilon > 2\varepsilon$, by continuity, we know there exists $t_* \in (s, t)$ such that

$$\mathrm{dist}(X_\varepsilon(t_*; t, x), \Omega^c) = 2\varepsilon.$$

Therefore, we can find $y_* \in \partial\Omega$ such that $|y_* - X_\varepsilon(t_*; t, x)| = 2\varepsilon$. Since $\partial\Omega \cap B_{r_0}(y_*)$ is an L-Lipschitz graph, we can find a ball

$$B_\varepsilon(x_*) \subset B_{r_0}(y_*) \setminus \Omega$$

with $|x_* - y_*| = (L+1)\varepsilon$, and

$$|x_* - X_\varepsilon(t_*; t, x)| \le |x_* - y_*| + |y_* - X_\varepsilon(t_*; t, x)| \le (L+3)\varepsilon.$$

By Lemma 5.5, provided η_0 is sufficiently small depending on L, we have

$$|X_\varepsilon(t; t_*, x_*) - X_\varepsilon(t; t, x)| \le (L+4)\varepsilon.$$

Since $B_\varepsilon(x_*)$ is disjoint from Ω, the drift is zero, so $X_\varepsilon(t; t_*, x_*) = x_*$. Thus $|x_* - x| \le (L+4)\varepsilon$. Because $x^* \notin \Omega$, we know that $\mathrm{dist}(x, \partial\Omega) \le (L+4)\varepsilon$, which is a contradiction. $\square$

Lemma 5.7. *For any $(t, x) \in \Omega_T$, denote*

$$r_*(t, x) = \frac{\mathrm{dist}_{\mathcal{P}}((t, x), \partial_{\mathcal{P}}\Omega_T)}{L+4} \wedge r_0.$$

Then it holds

$$r_{\mathrm{int}}(t, x) \wedge r_{\mathrm{adm}}(t, x) \ge r_*(t, x) \wedge r_{\mathrm{adm}}(t, x).$$

Proof. If $r_{\mathrm{adm}}(t, x) \le r_{\mathrm{int}}(t, x)$, then there is nothing to prove. If $r_{\mathrm{adm}}(t, x) > r_{\mathrm{int}}(t, x)$, then $Q_{r_{\mathrm{int}}(t,x)}(t, x)$ is an admissible cylinder that touches the parabolic boundary $\partial_{\mathcal{P}}\Omega_T$. Corollary 5.6 then implies $\mathrm{dist}_{\mathcal{P}}((t, x), \partial_{\mathcal{P}}\Omega_T) \le (L+4)r_{\mathrm{int}}(t, x)$. $\square$

5.2. The cutoff averaging operator

We define the cutoff averaging operator $\mathscr{A}_\alpha^\wedge[f]$ by the following.

Definition 5.8. For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{1+D})$, $(t,x) \in \mathbb{R}^{1+D}$, define

$$\mathscr{A}_\alpha^\wedge[f](t,x) := f_{\mathscr{S}_\alpha^\wedge[f](t,x)}(t,x),$$
$$\mathscr{A}_\alpha^<[f](t,x) := \mathscr{A}_\alpha^\wedge[f](t,x)\mathbf{1}_{\mathrm{Sing}_\alpha^<(f)\cup\mathrm{Reg}_\alpha^<(f)}(t,x) = \mathscr{S}_\alpha[f](t,x)^{-\alpha}\mathbf{1}_{\{\mathscr{S}_\alpha[f](t,x)<\bar r(t,x)\}},$$
$$\mathscr{A}_\alpha^=[f](t,x) := \mathscr{A}_\alpha^\wedge[f](t,x)\mathbf{1}_{\mathrm{Sing}_\alpha^=(f)\cup\mathrm{Reg}_\alpha^=(f)}(t,x) = f_{\bar r(t,x)}(t,x)\mathbf{1}_{\{\mathscr{S}_\alpha[f](t,x)=\bar r(t,x)\}}.$$

Remark 5.9. $\mathscr{A}_\alpha^<[f]$ and $\mathscr{A}_\alpha^=[f]$ are disjointly supported, and $\mathscr{A}_\alpha^\wedge[f] = \mathscr{A}_\alpha^<[f] + \mathscr{A}_\alpha^=[f]$. By observation (b), we note that

- $\mathscr{A}_\alpha^<[f](t,x) \le \mathscr{A}_\alpha^\wedge[f](t,x) \le \mathscr{S}_\alpha[f](t,x)^{-\alpha}$ for every (t,x).
- $\mathscr{A}_\alpha^=[f](t,x) = \mathscr{A}_\alpha^\wedge[f](t,x) \le \bar r(t,x)^{-\alpha}$ if $(t,x) \in \mathrm{Reg}_\alpha^=(f)$.
- $\mathscr{A}_\alpha^<[f](t,x) = \mathscr{A}_\alpha^\wedge[f](t,x) > \bar r(t,x)^{-\alpha}$ if $(t,x) \in \mathrm{Reg}_\alpha^<(f)$.

Moreover, the partition of $\mathbb{R}^{1+d}$ can now be written as

$$\begin{aligned}
\mathrm{Sing}_\alpha^<(f) &= \{\mathscr{A}_\alpha^<[f] = +\infty\},\\
\mathrm{Sing}_\alpha^=(f) &= \{\mathscr{A}_\alpha^=[f] = +\infty\},\\
\mathrm{Reg}_\alpha^<(f) &= \{0 < \mathscr{A}_\alpha^<[f] < +\infty\},\\
\mathrm{Reg}_\alpha^=(f) &= \{0 < \mathscr{A}_\alpha^=[f] < +\infty\} \cup \{\mathscr{A}_\alpha^<[f] = \mathscr{A}_\alpha^=[f] = 0\}.
\end{aligned}$$

Since $\mathscr{S}_\alpha^\wedge[f]$ is upper semicontinuous, we know that $\mathscr{A}_\alpha^\wedge[f]$, $\mathscr{A}_\alpha^<[f]$ and $\mathscr{A}_\alpha^=[f]$ are Borel measurable. Note that this average is pointwise bounded from above by the Q-maximal function:

$$\mathcal{M}_Q f(t,x) := \left\{ \sup_{r>0} \fint_{Q_r(t,x)} f \,\mathrm{d}y\,\mathrm{d}s : Q_r(t,x) \text{ is admissible} \right\},$$

the averaging operator $\mathscr{A}_\alpha^\wedge$ inherits the boundedness of the maximal function $\mathcal{M}_Q$: weak type-(1, 1) and strong type-(p,p) for $1 < p \le \infty$ [39, theorem 1].

Proposition 5.10. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{1+D})$. Suppose $b \in L^p(0,T;W_0^{1,p}(\Omega))$ is divergence free, and $\mathcal{M}(\nabla b) \in L^p(\Omega_T)$ for some $1 \le p \le \infty$. Then*

(1) $\mathrm{meas}(\{\mathscr{A}_\alpha^\wedge[f] = \infty\}) = 0$.
(2) *If $f \in L^1(\mathbb{R}^{1+D})$, then $\mathscr{A}_\alpha^\wedge[f] \in L^{1,\infty}(\mathbb{R}^{1+D})$ with estimate*

$$\|\mathscr{A}_\alpha^\wedge[f]\|_{L^{1,\infty}(\mathbb{R}^{1+D})} \le C\|f\|_{L^1(\mathbb{R}^{1+D})}.$$

(3) *If* $f \in L^q(\mathbb{R}^{1+D})$ *for some* $q \in (1, \infty]$, *then* $\mathscr{A}_\alpha^\wedge[f] \in L^q(\mathbb{R}^{1+D})$ *with*

$$\left\|\mathscr{A}_\alpha^\wedge[f]\right\|_{L^q(\mathbb{R}^{1+D})} \le C(p)\|f\|_{L^q(\mathbb{R}^{1+D})}.$$

Next, we explore the finer structure of the averaging operator $\mathscr{A}_\alpha$. We note that $\mathscr{A}_\alpha^\wedge$ is also quasiconvex in the following sense.

Lemma 5.11. *Fix* $\alpha > 0$. *For any nonnegative functions* $f, g \in L^1_{\mathrm{loc}}(\mathbb{R}^{1+D})$, $\lambda \in [0,1]$, *it holds that*

$$\mathscr{A}_\alpha[(1-\lambda)f + \lambda g](t,x) \le \max\{\mathscr{A}_\alpha[f](t,x), \mathscr{A}_\alpha[g](t,x)\}, \qquad \forall (t,x) \in \mathbb{R}^{1+D}.$$

Proof. Denote $h = (1-\lambda)f + \lambda g$, and take $(t,x) \in \mathbb{R}^{1+D}$. We separate the following cases.

(a) If $(t,x) \in \mathrm{Sing}_\alpha(f) \cup \mathrm{Sing}_\alpha(g)$, then the right hand side is $+\infty$ and there is nothing to prove. Otherwise, $(t,x) \in \mathrm{Reg}_\alpha(f) \cap \mathrm{Reg}_\alpha(g)$, $\bar{r}(t,x) > 0$, so $(t,x) \notin \mathrm{Sing}_\alpha^=(h)$.

(b) If $(t,x) \in \mathrm{Sing}_\alpha^<(h) \cup \mathrm{Reg}_\alpha^<(h)$, then there exists a sequence of $\rho_n > 0$ within $\mathscr{S}_\alpha[h](t,x) < \rho_n < \bar{r}(t,x)$ and $\rho_n \to \mathscr{S}_\alpha[h](t,x)$ such that

$$\rho_n^{-\alpha} < h_{\rho_n}(t,x) = (1-\lambda) f_{\rho_n}(t,x) + \lambda g_{\rho_n}(t,x).$$

Therefore, at least one of $f_{\rho_n}(t,x)$ and $g_{\rho_n}(t,x)$ is greater than $\rho_n^{-\alpha}$. Suppose, up to a subsequence, that $f_{\rho_n}(t,x) > \rho_n^{-\alpha}$. Then $(t,x) \in \mathrm{Reg}_\alpha^<(f)$, and $0 < \mathscr{S}_\alpha[f](t,x) \le \rho_n$. Taking $n \to \infty$ we find $\mathscr{S}_\alpha[f](t,x) \le \mathscr{S}_\alpha[h](t,x)$, and consequently

$$\mathscr{A}_\alpha^\wedge[h](t,x) = \mathscr{S}_\alpha[h](t,x)^{-\alpha} \le \mathscr{S}_\alpha[f](t,x)^{-\alpha} = \mathscr{A}_\alpha^\wedge[f](t,x).$$

(c) If $(t,x) \in \mathrm{Reg}_\alpha^=(h) \cap \mathrm{Reg}_\alpha^<(f)$, then

$$\mathscr{A}_\alpha^\wedge[h](t,x) \le \bar{r}(t,x)^{-\alpha} < \mathscr{A}_\alpha^\wedge[f](t,x).$$

Similarly, if $(t,x) \in \mathrm{Reg}_\alpha^=(h) \cap \mathrm{Reg}_\alpha^<(g)$, then $\mathscr{A}_\alpha^\wedge[h](t,x) < \mathscr{A}_\alpha^\wedge[g](t,x)$.

(d) If $(t,x) \in \mathrm{Reg}_\alpha^=(h) \cap \mathrm{Reg}_\alpha^=(f) \cap \mathrm{Reg}_\alpha^=(g)$, then

$$\begin{aligned}\mathscr{A}_\alpha^\wedge[h](t,x) &= \fint_{Q_{\bar{r}(t,x)}(t,x)} h \,\mathrm{d}y\,\mathrm{d}s \\ &= (1-\lambda) \fint_{Q_{\bar{r}(t,x)}(t,x)} f \,\mathrm{d}y\,\mathrm{d}s + \lambda \fint_{Q_{\bar{r}(t,x)}(t,x)} g \,\mathrm{d}y\,\mathrm{d}s \\ &= (1-\lambda)\mathscr{A}_\alpha^\wedge[f](t,x) + \lambda \mathscr{A}_\alpha^\wedge[g](t,x).\end{aligned}$$

These are all the possible cases, so it always holds that $\mathscr{A}_\alpha^\wedge[h] \le \max\left\{\mathscr{A}_\alpha^\wedge[f], \mathscr{A}_\alpha^\wedge[g]\right\}$. $\square$

Finally, we remark that $\bar{r} \ge r_*$ in $\mathrm{Reg}_\alpha^=(f)$ if f is dominates the drift gradient.

Lemma 5.12. *Suppose $\alpha \geq 2$, $\mathcal{M}(\nabla b) \in L^1_{\mathrm{loc}}(\Omega_T)$, and $f \geq [\frac{1}{\eta_0}\mathcal{M}(\nabla b)]^{\frac{\alpha}{2}}$ in Ω_T. Then $\bar{r}(t,x) \geq r_*(t,x)$ for any $(t,x) \in \mathrm{Reg}^{=}_{\alpha}(f)$. Moreover, if f is bounded in a neighborhood of (t,x), then $(t,x) \in \mathrm{Reg}_{\alpha}(f)$.*

Proof. First, we use Jensen Lemma 3.8:

$$r_{\mathrm{adm}} = \mathscr{S}_2\left[\frac{\mathcal{M}(\nabla b)}{\eta_0}\right] \geq \mathscr{S}_{\frac{\alpha}{2}\cdot 2}\left[\left(\frac{1}{\eta_0}\mathcal{M}(\nabla b)\right)^{\frac{\alpha}{2}}\right] \geq \mathscr{S}_{\alpha}[f].$$

By Lemma 5.7, we know that

$$\bar{r}(t,x) \geq r_{\mathrm{adm}}(t,x) \wedge r_*(t,x).$$

Moreover, for any $(t,x) \in \mathrm{Sing}^{=}_{\alpha}(f) \cup \mathrm{Reg}^{=}_{\alpha}(f)$ it holds that

$$\mathscr{S}_{\alpha}[f](t,x) \geq \mathscr{S}^{\wedge}_{\alpha}[f](t,x) = \bar{r}(t,x).$$

Combining the above three inequalities, we find that $\bar{r}(t,x) \geq r_*(t,x)$ for any $(t,x) \in \mathrm{Reg}^{=}_{\alpha}(f)$.

Next, suppose f is bounded in a neighborhood $U \subset \Omega_T$ of (t,x). Then $\mathcal{M}(\nabla b)$ is also bounded in U. For ε sufficiently small, $Q_\varepsilon(t,x) \subset U$ and $⨍_{Q_\varepsilon(t,x)} \mathcal{M}(\nabla b) \leq \eta_0 \varepsilon^{-2}$ imply $Q_\varepsilon(t,x)$ is admissible and $\bar{r}(t,x) > 0$. This shows $(t,x) \notin \mathrm{Sing}^{=}_{\alpha}(f)$. Moreover, $⨍_{Q_\varepsilon(t,x)} f \leq \varepsilon^{-\alpha}$ for sufficiently small ε, so $(t,x) \notin \mathrm{Sing}^{<}_{\alpha}(f)$. Combined, they prove $(t,x) \in \mathrm{Reg}_{\alpha}(f)$. $\square$

In the next subsection, we will show some trace estimates on $\mathscr{A}^{<}_{\alpha}$, which extends results from Section 4.

5.3. Trace estimate for the averaging operator

To establish trace estimates, we need to measure the level sets of $\mathscr{S}_{\alpha}[f]$. We start with the section of a regular set.

Lemma 5.13. *Let $t \in \mathbb{R}$, $\alpha > 0$, $p \in [1,\infty)$, and $f \in L^p_{\mathrm{loc},t}L^p_x(\mathbb{R}^{1+D})$. Denote*

$$A_t(\rho) := \left\{x' \in \Gamma_t : \rho \leq \mathscr{S}_{\alpha}[f](t,x') < 2\rho \wedge r_{\mathrm{adm}}(t,x')\right\}, \qquad \rho > 0.$$

Then for every $t \in (0,T)$, $\rho > 0$,

$$\mathscr{H}^d(A_t(\rho)) \leq C\rho^{-D+d-2+p\alpha} \int_{t-4\rho^2}^{t} \int_{\mathbb{R}^D} f(s,x)^p \,\mathrm{d}x\,\mathrm{d}s.$$

Proof. We can borrow almost entirely the proof of Lemma 4.4, with one exception that pairwise disjoint $B_{\rho_i}(x_i')$ do not guarantee disjoint $Q_{\rho_i}(t,x_i')$. We will patch this up by the following alternative argument instead. For simplicity, suppose $p = 1$ and $f \in L^1_{t,x}(\mathbb{R}^{1+D})$.

Same as Lemma 4.4, we know

$$\mathscr{H}^d(A_t(\rho)) \leq (1+\|g_t\|_{\mathrm{Lip}}^2)^{\frac{d}{2}} \frac{\mathscr{L}^D(\mathcal{U}_\rho(A_t(\rho)))}{c_{D-d}\rho^{D-d}},$$

with $\mathcal{U}_\rho(A_t(\rho))$ covered by $\bigcup_{x'\in A_t(\rho)} B_{\rho_{x'}}(x')$ for some $\rho' \in [\rho, 2\rho \wedge r_{\mathrm{adm}}(t,x'))$ satisfying

$$f_{\rho_{x'}}(t,x') > \rho_{x'}^{-\alpha}.$$

Now we choose a bigger covering:

$$\mathcal{U}_\rho(A_t(\rho)) \subset \bigcup_{x'\in A_t(\rho)} B_{9\rho_{x'}}(x').$$

By Vitali's covering lemma, we can find a disjoint subcollection $\left\{B_{9\rho_i}(x_i)\right\}_i$ with

$$\mathscr{L}^D(\mathcal{U}_\rho(A_t(\rho))) \leq C\sum_i \mathscr{L}^D(B_{9\rho_i}(x_i)) \leq C\rho^{-2}\sum_i \mathscr{L}^{D+1}(Q_{\rho_i}(t,x_i)).$$

Here $C = C(d)$ depends only on the dimension. Since $\rho_i \leq r_{\mathrm{adm}}(x_i)$, $Q_{\rho_i}(t,x_i)$ is an admissible cylinder. By [39, Proposition 7], we know $B_{9\rho_i}(x_i) \cap B_{9\rho_j}(x_j) = \varnothing$ implies $Q_{\rho_i}(t,x_i)$ and $Q_{\rho_j}(t,x_j)$ are disjoint. Hence $\left\{Q_{\rho_i}(t,x_i)\right\}_i$ are also pairwise disjoint. Therefore, the total volume is

$$\begin{aligned}
\sum_i \mathscr{L}^{D+1}(Q_{\rho_i}(t,x_i)) &= \sum_i \frac{1}{f_{\rho_i}(t,x_i)} \int_{Q_{\rho_i}(t,x_i)} f(s,y)\,\mathrm{d}y\,\mathrm{d}s \\
&\leq \sum_i \rho_i^\alpha \int_{Q_{\rho_i}(t,x_i)} f(s,y)\,\mathrm{d}y\,\mathrm{d}s \\
&\leq \sum_i (2\rho)^\alpha \int_{Q_{\rho_i}(t,x_i)} f(s,y)\,\mathrm{d}y\,\mathrm{d}s \\
&\leq (2\rho)^\alpha \int_{(t-(2\rho)^2,t)\times\mathbb{R}^D} f(s,y)\,\mathrm{d}y\,\mathrm{d}s.
\end{aligned}$$

This is because $Q_{\rho_i}(t,x_i) \subset (t-(2\rho)^2,t)\times\mathbb{R}^D$. Combine these estimates, we have

$$\mu_t(A_t(\rho)) \leq C\rho^{-D+d-2+\alpha} \int_{(t-(2\rho)^2,t)\times\mathbb{R}^D} f(s,y)\,\mathrm{d}y\,\mathrm{d}s.$$

This finishes the proof of the lemma. $\square$

Next, we measure the singular set. We obtain the same estimate on the dimension of the singular set as in Proposition 4.5, and anisotropic estimates as in Corollary 4.9.

Proposition 5.14. *Let $\alpha > 0$, and $f \in L^p_{\mathrm{loc},t} L^q_{\mathrm{loc},x}(\mathbb{R}^{1+D})$, $1 \le q \le p < \infty$. Denote the singular section by*

$$S_t := \Big\{x' \in \Gamma_t : (t, x') \in \mathrm{Sing}_\alpha(f), r_{\mathrm{adm}}(t, x') > 0\Big\}, \qquad t \in \mathbb{R},$$

and denote the set of singular time by

$$\mathcal{T} = \Big\{t \in \mathbb{R} : \mathscr{H}^d(S_t) > 0\Big\}.$$

(a) *If $\frac{D-d}{q} < \alpha \le \frac{2}{p} + \frac{D-d}{q}$, then $\mathcal{T}$ has Hausdorff dimension no greater than*

$$\dim_{\mathscr{H}}(\mathcal{T}) \le 1 - \frac{p}{2}\left(\alpha - \frac{D-d}{q}\right).$$

(b) *If $\alpha > \frac{2}{p} + \frac{D-d}{q}$, then $\mathcal{T} = \varnothing$.*

Proof. If $r_{\mathrm{adm}}(t, x') > 0$, then for every $\varepsilon > 0$, there exists $\rho = \rho_{x'} < \varepsilon$ such that $f_\rho(t, x') > \rho^{-\alpha}$ and $Q_\rho(t, x')$ is admissible, so the covering lemma applies. The rest of the proof is identical to that of Proposition 4.5. □

Theorem 5.15. *Let $0 < p_1, q_1, p_2, q_2 \le \infty$ and $\alpha > 0$. Define $r_1, r_2 \in \mathbb{R} \setminus \{0\} \cup \{\infty\}$ by*

$$\frac{1}{r_1} = \alpha - \frac{2}{p_1} - \frac{D}{q_1}, \qquad \frac{1}{r_2} = \alpha - \frac{2}{p_2} - \frac{d}{q_2}.$$

Suppose $r_2 = \lambda r_1$ for some $\lambda \in (0, \infty)$, $p_2 \ge \lambda p_1$, $q_2 \ge \lambda q_1$, $1 \le q_1 \le p_1$, and $f \in L^{p_1}_t L^{q_1}_x(\Omega_T)$.

(A) *Suppose $p_1 = q_1 = p \in [1, \infty]$, and assume $\frac{1}{r_1} + \frac{2}{p_1} \cdot \mathbf{1}_{\{p_2 = \lambda p_1\}} + \frac{d}{q_1} \cdot \mathbf{1}_{\{q_2 = \lambda q_1\}} > 0$.*
   (a) *If $p = 1$, then*

$$\left\|\mathscr{A}^<_\alpha[f]\right\|^\lambda_{L^{\lambda,\infty}_{t,x}(\Gamma_T)} \le C\|f\|_{L^1_{t,x}(\Omega_T)} \qquad \text{if } p_2 = q_2 = \lambda,$$

$$\left\|\mathscr{A}^<_\alpha[f]\right\|^\lambda_{L^{p_2,\infty}_t L^{q_2,\infty}_x(\Gamma_T)} \le C\|f\|_{L^1_{t,x}(\Omega_T)} \qquad \text{if } p_2 > \lambda \text{ or } q_2 > \lambda.$$

   (b) *If $p > 1$, then*

$$\left\|\mathscr{A}^<_\alpha[f]\right\|^\lambda_{L^{p_2}_t L^{q_2}_x(\Gamma_T)} \le C\|f\|_{L^p_{t,x}(\Omega_T)}.$$

(B) *Suppose $\lambda p_1 < p_2 < \infty$. When $\lambda q_1 < q_2$, we further require $\alpha > \frac{d}{q_2}$.*
   (a) *If $p_1 = q_1$, then*

$$\left\|\mathscr{A}^<_\alpha[f]\right\|^\lambda_{L^{p_2,\infty}_t L^{q_2}_x(\Gamma_T)} \le C\|f\|_{L^{p_1}_t L^{q_1}_x(\Omega_T)}.$$

   (b) *If $p_1 > q_1$, then*

$$\left\|\mathscr{A}^<_\alpha[f]\right\|^\lambda_{L^{p_2}_t L^{q_2}_x(\Gamma_T)} \le C\|f\|_{L^{p_1}_t L^{q_1}_x(\Omega_T)}.$$

(C) *Suppose* $p_2 = \infty$*. When* $\lambda q_1 < q_2$*, we further require* $\alpha > \frac{d}{q_2}$*. Then*

$$\left\| \mathscr{A}_\alpha^<[f] \right\|_{L_t^\infty L_x^{q_2,\infty}(\Gamma_T)}^\lambda \leq C \| f \|_{L_t^{p_1} L_x^{q_1}(\Omega_T)}.$$

In all of the above, the constant $C = C(p_1, q_1, p_2, q_2, \alpha, D, d, L)$*.*

Proof. Recall that f is extended to be zero outside Ω_T. The measurability of $\mathscr{A}_\alpha[f]$ comes from semicontinuity. Since $\bar{r}$ is also semicontinuous, sets $\mathrm{Sing}_\alpha^<(f)$ and $\mathrm{Reg}_\alpha^<(f)$ are also measurable. Same as before, in both cases $q_1\alpha > D - d$. Lemma 5.14 implies $\mu_t(S_t) = 0$ for $\mathscr{L}^1$-almost every $t \in [0, 1]$. Therefore

$$\mu_T(\{\mathscr{A}_\alpha^<[f] = \infty\}) \leq \int_0^T \mu_t(S_t)\, \mathrm{d}t = 0.$$

Notice that for $\rho > 0$, $(t, x) \in \Gamma_T$,

$$\begin{gathered}(2\rho)^{-\alpha} < \mathscr{A}_\alpha^<[f](t, x) \leq \rho^{-\alpha} \implies \rho \leq \mathscr{S}_\alpha[f](t, x) < 2\rho \wedge r_{\mathrm{adm}}(t, x) \implies x \in A_t(\rho), \\ \mathscr{A}_\alpha^<[f](t, x) = \infty \implies \mathscr{S}_\alpha[f](t, x) = 0 < \bar{r}(t, x) \leq r_{\mathrm{adm}}(t, x) \implies x \in S_t,\end{gathered}$$

where $A_t(\rho)$ is defined in Lemma 5.13, and S_t is defined in Lemma 5.14. All the estimates are based on these two lemmas, same as in Corollary 4.9 from the previous section, so we omit the proofs. □

6. Trace of vorticity

In this section, we apply the blow-up method on vorticity and higher derivative estimates. First, we recall some local estimates for the Navier–Stokes equation. Then we apply the averaging operator to prove the main results.

6.1. *ε-regularity theory for the Navier–Stokes equation*

Lemma 6.1 *(Caffarelli, Kohn, and Nirenberg, [7]; Lin [26]). There exists* $\epsilon_0 > 0$ *such that if* u, P *is a suitable weak solution to* (1) *in* $(-4, 0) \times B_2$*, satisfying*

$$\int_{-4}^0 \int_{B_2} |u|^3 + |P|^{\frac{3}{2}}\, \mathrm{d}x\, \mathrm{d}t \leq \epsilon_0,$$

then for $n \geq 0$, $\|\nabla^n u\|_{L^\infty((-1,0)\times B_1)} \leq C_n$.

By scaling, we obtain the following corollary.

Corollary 6.2. *Let* u *be a suitable weak solution to* (1) *in* Ω_T*. For* $\varepsilon > 0$ *and* $(t, x) \in \Omega_T$*, define* $Q_\varepsilon(t, x)$ *as in* (12)*. If* $Q_\varepsilon(t, x) \subset \Omega_T$ *and*

$$\fint_{Q_\varepsilon(t,x)} |u|^3 + |P|^{\frac{3}{2}}\,\mathrm{d}x\,\mathrm{d}t \le \epsilon_0 \varepsilon^{-3},$$

then $|\nabla^n u(t,x)| \le C_n \varepsilon^{-n-1}$.

The downside of this estimate is that the scaling of u is worse than the scaling of ∇u. If we want to have the same scaling as $\nabla u \in L^2_{t,x}$, u needs to have $L^4_{t,x}$ integrability. This is not known for suitable weak solutions. Therefore, we use the following lemma instead.

Lemma 6.3 *(Choi and Vasseur, [12]). There exists $\bar\eta$ and a sequence of constants C_n with the following property. If u, P is a classical solution to* (1) *in* $(-4,0) \times B_2$*, satisfying*

$$\int_{B_2} \varphi(x) u(t,x)\,\mathrm{d}x = 0, \qquad \textit{for all } t \in (-2,0),$$

and

$$\int_{-4}^{0}\int_{B_2} |\mathcal{M}(\nabla u)|^2 + |\nabla^2 P|\,\mathrm{d}x\,\mathrm{d}t \le \bar\eta,$$

then for $n \ge 1$, $\|\nabla^n u\|_{L^\infty((-\frac{1}{9},0)\times B_{\frac{1}{3}})} \le C_n$.

Proof. This is the first part of Proposition 2.2 of [12] with $r = 0$ (see also Remark 2.7). Although the theorem stated in (18) that the domain of the solution is $\Omega = \mathbb{R}^3$, the integer-order derivatives part of the proof is based on De Giorgi iteration and hence is purely local. Therefore, it suffices to require u to solve the Navier–Stokes equation in $(-2,0) \times B_2$. □

By scaling, we have the following corollary at the ε scale.

Corollary 6.4. *Let u be a classical solution to* (1) *in Ω_T. For $\varepsilon > 0$ and $(t,x) \in \Omega_T$, define $Q_\varepsilon(t,x) \subset \Omega_T$ as in* (30)-(31) *with $b = u$. If $Q_\varepsilon(t,x) \subset \Omega_T$ and*

$$\fint_{Q_\varepsilon(t,x)} |\mathcal{M}(\nabla u)|^2 + |\nabla^2 P|\,\mathrm{d}x\,\mathrm{d}t \le \bar\eta \varepsilon^{-4},$$

then $|\nabla^n u(t,x)| \le C_n \varepsilon^{-n-1}$.

Proof. The proof of the corollary uses the Galilean invariance of the Navier–Stokes equation. See [39, Corollary 18] for instance. Given a solution (u,P) to (1) in $Q_\varepsilon(t,x)$ for some $(t,x) \in \Omega_T$, for $s \in (-1,0]$ and $y \in B_1(0) \subset \mathbb{R}^3$ define

$$r(s) = t + \varepsilon^2 s,$$

$$z(s) = X_\varepsilon(r(s); t, x),$$

$$v(s,y)=\varepsilon u(r(s),z(s)+\varepsilon y)-\varepsilon u_\varepsilon(r(s),z(s)),$$
$$p(s,y)=\varepsilon^2 P(r(s),z(s)+\varepsilon y)+\varepsilon y\partial_s[u_\varepsilon(r(s),z(s))].$$

Recall that X_ε is defined by (30). (v,p) then solves (1) in $(-1,0)\times B_1$, and

$$\fint_{(-1,0)\times B_1} |\mathcal{M}(\nabla v)|^2+|\nabla^2 p|\,\mathrm{d}x\,\mathrm{d}t=\varepsilon^4\fint_{Q_\varepsilon(t,x)} |\mathcal{M}(\nabla u)|^2+|\nabla^2 P|\,\mathrm{d}x\,\mathrm{d}t.$$

Also for any $n\geq 1$, $\nabla^n u(t,x)=\varepsilon^{-n-1}\nabla^n v(0,0)$. $\square$

Here we need to use the second derivative of the pressure. If $\Omega=\mathbb{R}^3$ or $\mathbb{T}^3$, then $\|\nabla^2 P\|_{L^1_t\mathcal{H}^1_x}\leq C\|\nabla u\|^2_{L^2_{t,x}}$ in Hardy space due to compensated compactness [13,27,35]. However, if $\partial\Omega\neq\varnothing$, we can separate $P=P_0+P_1$ with $\|\nabla^2 P_1\|_{L^1_t\mathcal{H}^1_x}\leq C\|\nabla u\|^2_{L^2_{t,x}}$, $P_1=0$ on $\partial\Omega$, and P_0 is harmonic in Ω.

Next, we remove the dependence on the pressure. We will have a similar result for the control of vorticity.

Lemma 6.5 *(Vasseur and Yang, [36]). Let $\frac{11}{6}<p<2$, $\frac{4-2p}{3-p}<\theta\leq\frac{14p-24}{13p-18}$. There exist universal constants $\eta>0$ and $C_n>0$ for $n\geq 0$, such that if a suitable weak solution u satisfies*

$$\left(\fint_{Q_\varepsilon(t,x)} |\nabla u|^p\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1-\theta}{p}}\left(\fint_{Q_\varepsilon(t,x)} |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{\theta}{2}}\leq\eta\varepsilon^{-2},\tag{33}$$

then (t,x) is a regular point, and $|\nabla^n\omega(t,x)|\leq C_n\varepsilon^{-n-2}$.

Proof. Pick $\nu=\frac{1}{2}(1-\frac{1}{\theta})$. According to [36], there exists $\eta_2,\eta_3>0$, such that for any $\delta\leq\eta_2$, if u is a suitable weak solution to (1) in $Q_\varepsilon(t,x)$ satisfying

$$\delta^{-2\nu}\left(\fint_{Q_\varepsilon(t,x)} |\nabla u|^p\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{2}{p}}+\delta\fint_{Q_\varepsilon(t,x)} |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t\leq\eta_3\varepsilon^{-4},\tag{34}$$

then $|\nabla^n\omega(t,x)|\leq C_n\varepsilon^{-n-2}$. Now we prove (33) implies this condition with a suitable choice of η.

Define η_4,η_5 by

$$\left(\fint_{Q_\varepsilon(t,x)} |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}=\eta_4\varepsilon^{-2},\qquad\left(\fint_{Q_\varepsilon(t,x)} |\nabla u|^p\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p}}=\eta_5\varepsilon^{-2}.$$

Then the assumption implies $\eta_4^{1-\theta}\eta_5^{\theta} \le \eta$. Moreover, since $p < 2$, Jensen's inequality implies $\eta_4 \ge \eta_5$, and thus $\eta_5 \le \eta$. Define $\delta = \min\left\{\eta_2, \eta_4^{-2\theta}\eta_5^{2\theta}\right\}$, then

$$\delta^{-2\nu}\left(\fint_{Q_\varepsilon(t,x)} |\nabla u|^p \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{2}{p}} + \delta \fint_{Q_\varepsilon(t,x)} |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}t = \delta^{-2\nu}\eta_5^2\varepsilon^{-4} + \delta\eta_4^2\varepsilon^{-4}.$$

If $\delta = \eta_4^{-2\theta}\eta_5^{2\theta}$, then

$$\delta^{-2\nu}\eta_5^2 + \delta\eta_4^2 = 2\eta_4^{2-2\theta}\eta_5^{2\theta} \le 2\eta^2.$$

Here we used that $\delta^{-2\nu} = \eta_4^{2-2\theta}\eta_5^{2\theta-2}$. If $\delta = \eta_2 < \eta_4^{-2\theta}\eta_5^{2\theta}$, then $\eta_4 < \eta_5\eta_2^{-1/2\theta}$, so

$$\delta^{-2\nu}\eta_5^2 + \delta\eta_4^2 \le \eta_2^{-2\nu}\eta_5^2 + \eta_2\eta_5^2\eta_2^{-1/\theta} = 2\eta_2^{-2\nu}\eta^2.$$

Therefore, if we choose $\eta = (2+2\eta_2^{-2\nu})^{-1/2}\eta_3^{1/2}$, then for both possible values of δ, we always have $\delta^{-2\nu}\eta_5^2 + \delta\eta_4^2 \le \eta_3$. Therefore (34) is satisfied and $|\nabla^n\omega(t,x)| \le C_n\varepsilon^{-n-2}$. □

Using Jensen's inequality, we can deduce the following corollary.

Corollary 6.6. *There exists $\eta > 0$, such that for any suitable weak solution u to the Navier–Stokes equation in Ω_T, if*

$$\fint_{Q_\varepsilon(t,x)} |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}t \le \eta\varepsilon^{-4},$$

then

$$\left|\nabla^n\omega(t,x)\right| \le C_n\varepsilon^{-n-2}.$$

6.2. *Proof of the main results*

Now we prove the vorticity trace estimates for the Navier–Stokes equation. The previous subsection shows the following pointwise estimate on the regular part.

Lemma 6.7. *Let (u, P) be a suitable weak solution to (1) in Ω_T, and suppose u satisfies no-slip boundary condition (2) if the boundary is non-empty. Define $f_1, f_2, f_3 : \Omega_T \to \mathbb{R}$ by*

$$f_1 = \frac{\mathcal{M}(\nabla u)^2}{\eta}, \qquad f_2 = \frac{\mathcal{M}(\nabla u)^2 + |\nabla^2 P|}{\bar{\eta}}, \qquad f_3 = \frac{|u|^3 + |P|^{\frac{3}{2}}}{\epsilon_0}.$$

Choose drift $b = u$ and define $s_1 = \mathscr{S}_4^{\wedge}[f_1]$, $s_2 = \mathscr{S}_4^{\wedge}[f_2]$. Choose drift $b = 0$ and define $s_3 = \mathscr{S}_3^{\wedge}[f_3]$. Then there exists $C_n > 0$ for every $n \ge 0$, such that for any regular *point $(t,x) \in \Omega_T$, we have*

$$|\nabla^n \omega(t,x)| \le C_n s_1(t,x)^{-n-2} \qquad n \ge 0,$$
$$|\nabla^n u(t,x)| \le C_n s_2(t,x)^{-n-1} \qquad n \ge 1,$$
$$|\nabla^n u(t,x)| \le C_n s_3(t,x)^{-n-1} \qquad n \ge 0.$$

Moreover,

$$s_i(t,x)^{-1} \le \mathscr{A}_4^<[f_i](t,x)^{\frac{1}{4}} \vee r_*(t,x)^{-1}, \qquad i = 1,2$$
$$s_3(t,x)^{-1} \le \mathscr{A}_3^<[f_3](t,x)^{\frac{1}{3}} \vee r_*(t,x)^{-1}.$$

Proof. Fix a regular point (t,x), and let $\rho = s_1(t,x)$. Since $\rho = \mathscr{S}_4^\wedge[f_1](t,x) \le \mathscr{S}_4[f_1]$, by definition we know that

$$\mathscr{A}_4^\wedge[f_1](t,x) = \fint_{Q_\rho(t,x)} \frac{\mathcal{M}(\nabla u)^2}{\eta} \le \rho^{-4}.$$

Since $\mathcal{M}(\nabla u) \ge |\nabla u|$, Corollary 6.6 implies that $|\nabla^n \omega(t,x)| \le C_n \rho^{-n-2}$.

Note that the conditions in Lemma 5.12 are satisfied with $\alpha = 4$, $b = u$, and $f = f_1$. Since (t,x) is a regular point, $(t,x) \in \mathrm{Reg}_4^<(f_1) \cup \mathrm{Reg}_4^=(f_1)$. If $(t,x) \in \mathrm{Reg}_4^=(f_1)$, then $\rho = \bar{r}(t,x)$. By Lemma 5.12, we have $\rho \ge r_*(t,x)$ and thus $s_1(t,x)^{-1} = \rho^{-1} \le r_*^{-1}(t,x)$. If $(t,x) \in \mathrm{Reg}_4^<(f_1)$, then $\rho = \mathscr{S}_4[f_1](t,x) = \mathscr{A}_4^<[f_1](t,x)^{-\frac{1}{4}}$. This completes the proof for s_1, and the proofs for s_2, s_3 are similar. $\square$

With this lemma, we prove the main results listed in Section 1.

Proof of Theorem 1.1. Since u is a classical solution, the singular set is empty, and any derivative of the solution is locally bounded everywhere. Note that $\mathcal{M}(\nabla u)$ is also locally bounded.

By Lemma 6.7, we know that $s_1^{-1}\mathbf{1}_{\{s_1 < r_*\}} \le \mathscr{A}_4^<[f_1]^{\frac{1}{4}}$. Apply Theorem 5.15 (Aa) with $D = d = 3$, $f = f_1 = \frac{1}{\eta}\mathcal{M}(\nabla u)^2$, $\alpha = 4$, $p_1 = q_1 = 1$, $p_2 = q_2 = \frac{d+1}{4}$, we have $r_1 = -1$, $r_2 = -\frac{d+1}{4}$, $\lambda = \frac{d+1}{4}$, and

$$\left\| \mathscr{A}_4^<[f_1] \right\|_{L^{\frac{d+1}{4},\infty}(\Gamma_T)}^{\frac{d+1}{4}} \le C \left\| \frac{1}{\eta}\mathcal{M}(\nabla u)^2 \right\|_{L^1(\Omega_T)} \le \frac{C}{\eta}\|\nabla u\|_{L^2(\Omega_T)}^2.$$

Apply Theorem 5.15 (Aa) again with $d \ge 2$, $p_1 = q_1 = 1$, $p_2 = \infty$, $q_2 = \frac{d-1}{4}$, we have $r_1 = -1$, $r_2 = -\frac{d-1}{4}$, $\lambda = \frac{d-1}{4}$, and

$$\left\| \mathscr{A}_4^<[f_1] \right\|_{L_t^\infty L^{\frac{d-1}{4},\infty}(\Gamma_T)}^{\frac{d-1}{4}} \le C \left\| \frac{1}{\eta}\mathcal{M}(\nabla u)^2 \right\|_{L^1(\Omega_T)} \le \frac{C}{\eta}\|\nabla u\|_{L^2(\Omega_T)}^2.$$

The theorem is thus proven by using $s_1^{-1}\mathbf{1}_{\{s_1 < r_*\}} \le \mathscr{A}_4^<[f_1]^{\frac{1}{4}}$. $\square$

Proof of Corollary 1.2. For a classical solution, the first estimate is a direct consequence of Theorem 1.1 (a) with $\Gamma_t = \Omega$ and $D = d = 3$, $|\nabla\omega| \le C s_1^{-3}$, so

$$\left\| \nabla\omega \mathbf{1}_{\left\{|\nabla\omega| > C r_*^{-3}\right\}} \right\|_{L^{\frac{4}{3},\infty}(\Omega_T)}^{\frac{4}{3}} \le \left\| s_1^{-3} \mathbf{1}_{\{s_1 < r_*\}} \right\|_{L^{\frac{4}{3},\infty}(\Omega_T)}^{\frac{4}{3}} \le C \|\nabla u\|_{L^2(\Omega_T)}^2.$$

And the second estimate is the case $d = 2$ with $|\omega| \le C s_1^{-2}$:

$$\left\| \omega \mathbf{1}_{\left\{|\nabla\omega| > C r_*^{-2}\right\}} \right\|_{L^{\frac{3}{2},\infty}(\Gamma_T)}^{\frac{3}{2}} \le \left\| s_1^{-3} \mathbf{1}_{\{s_1 < r_*\}} \right\|_{L^{\frac{4}{3},\infty}(\Omega_T)}^{\frac{4}{3}} \le C \|\nabla u\|_{L^2(\Omega_T)}^2.$$

For a suitable weak solution, we have the same bounds on the regular set, which is the complement of the singular set $\mathrm{Sing}(u)$ defined in Theorem 2.6.

Note that the Navier–Stokes equation (1) can be understood as an evolutionary Stokes equation with a forcing term $u \cdot \nabla u$, which can be bounded by

$$\begin{aligned}\|u \cdot \nabla u\|_{L^{\frac{5}{4}}(\Omega_T)} &\le \|u\|_{L^{\frac{10}{3}}(\Omega_T)} \|\nabla u\|_{L^2(\Omega_T)} \\ &\le \|u\|_{L_t^\infty L_x^2(\Omega_T)}^{\frac{2}{5}} \|u\|_{L_t^2 L_x^6(\Omega_T)}^{\frac{3}{5}} \|\nabla u\|_{L^2(\Omega_T)} \lesssim \|u_0\|_{L^2(\Omega)}^2.\end{aligned}$$

As $u, \nabla u, P \in L^{\frac{5}{4}}_{\mathrm{loc}}(\Omega_T)$, by linear Stokes theory, we know that $\nabla^2 u \in L^{\frac{5}{4}}_{\mathrm{loc}}(\Omega_T)$ and $\nabla P \in L^{\frac{5}{4}}_{\mathrm{loc}}(\Omega_T)$. Therefore $\nabla\omega$ is a measurable function in Ω_T, and thus for almost every $t \in (0, T)$, the trace of $\omega(t)$ on a codimension 1 hypersurface Γ_t is a well-defined function. Since the singular set $\mathrm{Sing}(u)$ is a $\mathscr{P}^1$-nullset, it vanishes at almost every $t \in (0, T)$, so $\mathrm{Sing}(u)$ is also a μ_T-nullset.

As Lemma 6.7 holds over the regular set, and the singular set is negligible, we finish the proof. □

Proof of Theorem 1.3. The proof is analogous to Theorem 1.1 and Corollary 1.2, where we use the estimate on s_2 in Lemma 6.7 instead of s_1. □

Proof of Corollary 1.4. This proof is also analogous to Theorem 1.1. We apply Theorem 5.15 (Aa) with $D = d = 3$, $\Gamma_t = \Omega$, $\alpha = 4$, $f = f_2 = \frac{1}{\eta_0}(\mathcal{M}(\nabla u)^2 + |\nabla^2 P|)$, $p_1 = q_1 = 1$, we have $r_1 = -1$. Moreover, for $0 < p_2 < q_2 \le \infty$ with $\frac{1}{p_2} + \frac{3}{q_2} = 4$, we have

$$\frac{1}{r_2} = \alpha - \frac{2}{p_2} - \frac{d}{q_2} = 4 - \frac{2}{p} - \frac{3}{q} = -\frac{1}{p_2}.$$

So $\lambda = p_2 = \lambda p_1$ and $q_2 > p_2 = \lambda q_1$. By Theorem 5.15 (Aa), we have

$$\|\mathscr{A}_\alpha^<[f_2]\|_{L_t^{p_2,\infty} L_x^{q_2,\infty}(\Omega_T)}^{p_2} \le C \|f_2\|_{L^1(\Omega_T)} \le \frac{C}{\eta_0} \|\nabla u\|_{L^2(\Omega_T)}^2.$$

Here in a torus, $\nabla^2 P$ in L^1 can be controlled by ∇u in L^2 (see Remark 6.8 below). Therefore, we have

$$\left\| s_2^{-1}\mathbf{1}_{\{s_2<r_*\}} \right\|^{4p_2}_{L_t^{4p_2,\infty}L_x^{4q_2,\infty}(\Omega_T)} \leq \left\| \mathscr{A}_\alpha^<[f_2] \right\|^{p_2}_{L_t^{p_2}L_x^{q_2}(\Omega_T)} \leq \frac{C}{\eta_0}\|\nabla u\|^2_{L^2(\Omega_T)}.$$

Moreover, when $\Omega = \mathbb{T}^3$, $r_*(t,x) = \min\left\{1, \sqrt{t}\right\}$. So

$$\left\| r_*^{-1} \right\|^{4p_2}_{L^{4p_2,\infty}(t_0,T;L^{4q_2,\infty}(\mathbb{T}^3))} \leq \int_{t_0}^{T} t^{-2p_2} \vee 1 \,\mathrm{d}t \leq (T-t_0)\max\{t_0^{-2p_2}, 1\}.$$

Combined, we have proven that

$$\left\| s_2^{-1} \right\|^{4p_2}_{L^{4p_2,\infty}(t_0,T;L^{4q_2,\infty}(\mathbb{T}^3))} \leq C\Big(\|\nabla u\|^2_{L^2(\Omega_T)} + (T-t_0)\max\{t_0^{-2p_2}, 1\} \Big).$$

The theorem is proven by $|\nabla^n u| \leq C_n s_2^{-n-1}$ and setting $p = \frac{4p_2}{n+1}$, $q = \frac{4q_2}{n+1}$.

Next, we apply Theorem 5.15 (Ba) with $\frac{1}{p_2} + \frac{1}{q_2} = 2$. We have

$$\frac{1}{r_2} = \alpha - \frac{2}{p_2} - \frac{d}{q_2} = 2\Big(2 - \frac{1}{p}\Big) - \frac{3}{q} = -\frac{1}{q_2}.$$

So $\lambda = q_2 = \lambda q_1$ and $p_2 > q_2 = \lambda p_1$. By Theorem 5.15 (Ba), we have

$$\left\| \mathscr{A}_\alpha^<[f_2] \right\|^{q_2}_{L_t^{p_2,\infty}L_x^{q_2}(\Omega_T)} \leq C\|f_2\|_{L^1(\Omega_T)} \leq \frac{C}{\eta_0}\|\nabla u\|^2_{L^2(\Omega_T)}.$$

Similarly, by bounding r_*^{-1}, we have

$$\left\| s_2^{-1} \right\|^{4q_2}_{L^{4p_2,\infty}(t_0,T;L^{4q_2}(\mathbb{T}^3))} \leq C\Big(\|\nabla u\|^2_{L^2(\Omega_T)} + (T-t_0)\max\{t_0^{-2q_2}, 1\} \Big).$$

The theorem is proven by setting $p = \frac{4p_2}{n+1}$, $q = \frac{4q_2}{n+1}$.

Finally, we apply Theorem 5.15 (Aa) to prove the last statement. The proof is similar so we omit the details. $\square$

Remark 6.8. By taking the divergence of (1), we know that

$$-\Delta P = \operatorname{div}(u\cdot\nabla u) = \operatorname{div}((\nabla u)^\top)\cdot u + (\nabla u)^\top : \nabla u.$$

Note that $\operatorname{div}((\nabla u)^\top) = 0$, $\operatorname{curl}\nabla u = 0$. When $\Omega = \mathbb{R}^3$ or $\mathbb{T}^3$, by the div-curl lemma [13], we know that

$$\|\Delta P(t)\|_{\mathcal{H}^1(\Omega)} \leq C\|\nabla u(t)\|^2_{L^2(\Omega)},$$

where $\mathcal{H}^1$ stands for the Hardy $\mathcal{H}^p$ space with $p = 1$. Since the Riesz transform is bounded on $\mathcal{H}^1$, we obtain

$$\left\|\nabla^2 P\right\|_{L^1(\Omega_T)} \le C\left\|\nabla^2 P\right\|_{L^1(0,T;\mathcal{H}^1(\Omega))} \le C\|\nabla u\|^2_{L^2(\Omega_T)}.$$

See also [27, section 3.2].

Corollary 1.4 applies to the case when $\Omega = \mathbb{T}^3$ has no boundary. In fact, if we only wish to bound the vorticity and its derivatives, similar bounds hold in any domain Ω of finite measure because f_1 requires nothing from the pressure. When there is a boundary, if we want to bound $\nabla^n u$ we still need to control the pressure term. In this case, we can use the following proposition to handle the pressure term by a splitting scheme.

Proposition 6.9. *Let $\Omega \subset \mathbb{R}^3$ be a bounded set with C^2 boundary satisfying Assumption 2.1. Let u be a classical solution to (1)-(2) in $(0,T)\times\Omega$, with initial kinetic energy $\|u(0)\|_{L^2(\Omega)} \le E$ for some $E \ge 0$. For any $\delta \in (0, r_0)$, denote*

$$\Gamma_T^\delta = \{(t,x) \in \Gamma_T : \mathrm{dist}_{\mathcal{P}}((t,x), \partial_{\mathcal{P}}\Omega_T) > \delta\}.$$

Then for any $K \subset \Gamma_T^\delta$, for every $n \ge 1$, the following holds. For any $0 < p < q \le \infty$ with $\frac{1}{p} + \frac{d}{q} = n+1$:

$$\left\|\nabla^n u\right\|_{L_t^{p,\infty} L_x^{q,\infty}(K)} \le C(\Omega, T, K, \delta, E, p, q)^{n+1}.$$

For any $0 < q < p \le \infty$ with $\frac{2}{p} + \frac{d-1}{q} = n+1$:

$$\left\|\nabla^n u\right\|_{L_t^{p,\infty} L_x^{q}(K)} \le C(\Omega, T, K, \delta, E, p, q)^{n+1}.$$

When $p = q = \frac{d+1}{n+1}$:

$$\left\|\nabla^n u\right\|_{L^{\frac{d+1}{n+1},\infty}(K)} \le C(\Omega, T, K, \delta, E)^{n+1}.$$

Proof. We only sketch how to control the pressure term using the idea of Lions [27, section 3.3]. We can split $P = P_0 + P_1$, $\Delta P_0 = 0$, $\fint_\Omega P_0(t)\,\mathrm{d}x = 0$ for a.e. $t \in (0,T)$, and $P_1 = 0$ on $\partial\Omega$. We have

$$\|\nabla P_1\|_{L^{\frac{5}{4}}((\delta^2,T)\times\Omega)} + \|\nabla P_0\|_{L^{\frac{5}{4}}((\delta^2,T)\times\Omega)} \le C(\|u\cdot\nabla u\|_{L^{\frac{5}{4}}(\Omega_T)} + 1).$$

Using Riesz transform on ΔP_1 and harmonicity on P_0, we have

$$\begin{aligned}
\left\|\nabla^2 P_1\right\|_{L^1(0,T;\mathcal{H}^1(\Omega))} &\le C\|\nabla u\|^2_{L^2(\Omega_T)},\\
\left\|\nabla^2 P_0\right\|_{L^1(\delta^2,T;L^1(\Omega^\varepsilon))} &\le C(\|u\cdot\nabla u\|_{L^{\frac{5}{4}}(\Omega_T)} + 1), \qquad k \ge 0.
\end{aligned}$$

Here $\Omega^\varepsilon = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \varepsilon\}$ for some $\varepsilon > 0$ to be chosen. Note that both $\|\nabla u\|^2_{L^2}$ and $\|u\cdot\nabla u\|_{L^{\frac{5}{4}}}$ are dominated by $\|u_0\|^2_{L^2}$ (see the Proof of Theorem 1.1). Therefore, we have

$$\int_{\delta^2}^{T}\int_{\Omega^\varepsilon} \mathcal{M}(\nabla u)^2 + |\nabla^2 P|\,\mathrm{d}x\,\mathrm{d}t \le C(\|u_0\|^2_{L^2(\Omega)} + 1).$$

Take $(t,x) \in K$, then $\operatorname{dist}(x,\partial\Omega) \ge \delta$. Suppose $s_2(t,x) = \rho < \varepsilon$. Recall that $Q_\rho(t,x)$ are defined by (30)-(31) with $b = u$. Since at any time $\|u(t)\|_{L^2(\Omega)} \le \|u_0\|_{L^2(\Omega)} \le E$, we have

$$|b_\rho|(t,x) \le \|u(t)\|_{L^2} |B_\rho|^{-\frac{1}{2}} \le C\rho^{-\frac{3}{2}}.$$

In a timespan of length ρ^2,

$$|X_\rho(s;t,x) - x| \le C\rho^{-\frac{3}{2}}\rho^2 \le C\rho^{\frac{1}{2}}.$$

This shows that $\operatorname{dist}(y,\partial\Omega) \ge \delta - C\varepsilon^{\frac{1}{2}} - \varepsilon > 2\varepsilon$ for any $(s,y) \in Q_\rho(t,x)$, if we choose ε sufficiently small. Thus, $Q_{s_2(t,x)}(t,x) \subset\subset \Omega_\varepsilon$ for every $(t,x) \in K$, so the value of $s_2 = \mathscr{S}^{\wedge}_\alpha[f_2]$ inside Ω^δ is not affected by the value of $f_2 = \mathcal{M}(\nabla u)^2 + |\nabla^2 P|$ outside Ω^ε.

Same as in Corollary 1.4, for $0 < p_2 < q_2 \le \infty$ with $\frac{1}{p_2} + \frac{d}{q_2} = 4$ we have

$$\frac{1}{r_2} = \alpha - \frac{2}{p_2} - \frac{d}{q_2} = 4 - \frac{2}{p_2} - \frac{d}{q_2} = -\frac{1}{p_2}.$$

Thus

$$\left\| s_2^{-1}\mathbf{1}_{\{s_2<r_*\}}\right\|^{4p_2}_{L_t^{4p_2,\infty}L_x^{4q_2,\infty}(K)} \le \left\|\mathscr{A}^{<}_\alpha[f_2]\right\|^{p_2}_{L_t^{p_2}L_x^{q_2}((0,T)\times\Omega^\varepsilon)} \le C.$$

Because r_* is bounded from below in $(\delta^2, T)\times\Omega^\delta$, we have

$$\left\| s_2^{-1}\right\|^{4p_2}_{L_t^{4p_2,\infty}L_x^{4q_2,\infty}(K)} \le C + \left\| r_*^{-1}\right\|^{4p_2}_{L_t^{4p_2,\infty}L_x^{4q_2,\infty}(K)} \le C.$$

By $|\nabla^n u| \le s_2^{-1}$, we finish the proof for the first estimate.

For $0 < q_2 < p_2 \le \infty$ with $\frac{2}{p_2} + \frac{d-1}{q_2} = 4$. We have

$$\frac{1}{r_2} = \alpha - \frac{2}{p_2} - \frac{d}{q_2} = \frac{d-1}{q_2} - \frac{d}{q} = -\frac{1}{q_2}.$$

Similarly, we have

$$\left\| s_2^{-1}\right\|^{4p_2}_{L_t^{4p_2,\infty}L_x^{4q_2}(K)} \le C + \left\| r_*^{-1}\right\|^{4p_2}_{L_t^{4p_2,\infty}L_x^{4q_2,\infty}(K)} \le C.$$

By $|\nabla^n u| \le s_2^{-1}$, we finish the proof for the second estimate. The third is similar so we do not repeat here. $\square$

Proof of Proposition 1.5. First, we want to apply Theorem 5.15 on f_3 defined in Lemma 6.7, with $D = d = 3$, $\Gamma_t = \Omega = \mathbb{T}^3$, $\alpha = 3$, $p_1 = p/3$, $q_1 = q/3$, $p_2 = p'/3$, $q_2 = q'/3$, we have

$$\frac{1}{r_1} = 3 - \frac{2}{p_1} - \frac{3}{q_1} = 3\left(1 - \frac{2}{p} - \frac{3}{q}\right) = 0$$

and similarly $\frac{1}{r_2} = 0$. Since $r_1 = r_2 = \infty$, $r_2 = \lambda r_1$ holds for any $\lambda > 0$. Note that $q \le p < \infty$ implies $q_1 \le p_1 < \infty$. We follow the following steps:

Step 1: case $p = q = 5$ for (3).

In this case, $p_1 = q_1 = \frac{5}{3}$. By choosing λ small, we can make sure $q_2 > \lambda q_1$ and $p_2 > \lambda p_1$. Using Theorem 5.15 (Ab), we have

$$\left\| \mathscr{A}_\alpha^<[f_3] \right\|^{\lambda}_{L_t^p L_x^q(\Omega_T)} \le C \| f_3 \|_{L^{\frac{5}{3}}(\Omega_T)}.$$

Here

$$\| f_3 \|_{L^{\frac{5}{3}}(\Omega_T)} = C \|u\|^3_{L^5((0,T)\times\mathbb{T}^3)} + C \|P\|^{\frac{3}{2}}_{L^{\frac{5}{2}}((0,T)\times\mathbb{T}^3)} \le C \|u\|^3_{L^5((0,T)\times\mathbb{T}^3)}.$$

Here we used $P = (-\Delta)^{-1} \operatorname{div}\operatorname{div}(u \otimes u)$, and $(-\Delta)^{-1} \operatorname{div}\operatorname{div}$ is a Riesz transform, so $\|P\|_{L^{\frac{5}{2}}} \le \|u \otimes u\|_{L^{\frac{5}{2}}} \le \|u\|^2_{L^5}$. Hence Lemma 6.7 implies

$$\left\| s_3^{-1} \mathbf{1}_{\{s_3 < r_*\}} \right\|^{3\lambda}_{L_t^{3p_2} L_x^{3q_2}(\Omega_T)} \le C \|u\|^3_{L^5((0,T)\times\mathbb{T}^3)}.$$

Since r_* is bounded from below in $(t_0, T) \times \mathbb{T}^3$, we have

$$\left\| s_3^{-1} \right\|_{L_t^{p'} L_x^{q'}((t_0,T)\times\mathbb{T}^3)} \le C \left(\|u\|^{\frac{1}{\lambda}}_{L^5((0,T)\times\mathbb{T}^3)} + 1 \right).$$

Because Lemma 6.7 also implies $|u| \le s_3^{-1}$ and $|\nabla u| \le s_3^{-2}$, we conclude

$$\|u\|_{L_t^{p'} L_x^{q'}((t_0,T)\times\mathbb{T}^3)} + \|\nabla u\|^{\frac{1}{2}}_{L_t^{\frac{p'}{2}} L_x^{\frac{q'}{2}}((t_0,T)\times\mathbb{T}^3)} \le C\left(\|u\|^{\frac{1}{\lambda}}_{L^5((0,T)\times\mathbb{T}^3)} + 1 \right).$$

We can pick $\gamma = \frac{1}{\lambda}$. This proves (3) for $p = q = 5$.

Step 2: case $3 < q < p < \infty$ for (3).

Given Step 1, it suffices to work with $p' = q' = 5$. Again, choose λ sufficiently small so that $q_2 = \frac{5}{3} > \lambda q_1$ and $p_2 = \frac{5}{3} > \lambda p_1$. Note that $\alpha = 3 > \frac{3}{q_2} = \frac{9}{5}$. Using Theorem 5.15 (Bb), we have

$$\left\| \mathscr{A}_\alpha^<[f_3] \right\|^{\lambda}_{L_{t,x}^{\frac{5}{3}}(\Omega_T)} \le C \| f_3 \|_{L_t^{p_1} L_x^{q_1}(\Omega_T)}.$$

Similar as Step 1, we have $\|f_3\|_{L_t^{p_1} L_x^{q_1}} \le \|u\|^3_{L_t^p L_x^q}$. So

$$\|u\|_{L^5_{t,x}((\frac{t_0}{2},T)\times\mathbb{T}^3)}+\|\nabla u\|^{\frac{1}{2}}_{L^{\frac{5}{2}}_{t,x}((t_0,T)\times\mathbb{T}^3)}\le C\Big(\|u\|^{\frac{1}{\lambda}}_{L^5((0,T)\times\mathbb{T}^3)}+1\Big).$$

Combined with Step 1, we prove (3) for $3<q<p<\infty$.

Step 3: case $3<q\le\frac{15}{4}<10\le p<\infty$ **for (4).**

Recall the conservation of momentum: $\int_{\mathbb{T}^3}u(t)\,\mathrm{d}x=\int_{\mathbb{T}^3}u_0\,\mathrm{d}x=0$. By Sobolev embedding in $\mathbb{T}^3$,

$$\|u\|_{L_t^{\frac{p}{2}}L_x^{q'}((0,T)\times\mathbb{T}^3)}\le C\|\nabla u\|_{L_t^{\frac{p}{2}}L_x^{\frac{q}{2}}((0,T)\times\mathbb{T}^3)}.$$

Here $q'=\frac{3\frac{q}{2}}{3-\frac{q}{2}}=\frac{3q}{6-q}$, so

$$\frac{2}{\frac{p}{2}}+\frac{3}{q'}=\frac{4}{p}+\frac{3(6-q)}{3q}=\frac{4}{p}+\frac{6}{q}-1=1.$$

Moreover, $\frac{p}{2}\ge5$ and $q'\le5$. By Step 1 and Step 2, (3) implies (4) in this range.

Next, we work with f_2 defined in Lemma 6.7 instead of f_3, with $D=d=3$, $\Gamma_t=\Omega=\mathbb{T}^3$, $\alpha=4$, $p_1=p/4$, $p_2=p/4$, $q_1=q/4$, $q_2=q/4$.

Step 4: case $q=p=5$ **for (4).**

We let $p'=10$ and $q'=\frac{15}{4}$. Choose $\lambda=\frac{3}{4}$. Then $p_1=q_1=\frac{5}{4}$, $p_2=\frac{5}{2}$, $q_2=\frac{15}{16}$, $p_2>\lambda p_1$ and $q_2=\lambda q_1$. Using Theorem 5.15 (Ab), we have

$$\begin{aligned}\left\|\mathscr{A}_\alpha^<[f_2]\right\|^{\frac{3}{4}}_{L_t^{\frac{5}{2}}L_x^{\frac{15}{16}}(\Omega_T)}&\le C\|f_2\|_{L^{\frac{5}{4}}_{t,x}(\Omega_T)}=C\|\nabla u\|^2_{L^{\frac{5}{2}}_{t,x}(\Omega_T)}+C\left\|\nabla^2P\right\|_{L^{\frac{5}{4}}_{t,x}(\Omega_T)}\\&\le C\|\nabla u\|^2_{L^{\frac{5}{2}}_{t,x}(\Omega_T)}.\end{aligned}$$

Here we used Remark 6.8 to bound $\left\|\nabla^2P\right\|_{L^{\frac{5}{4}}}$ by $\|\nabla u\|^2_{L^{\frac{5}{2}}}$ using boundedness of Riesz transform. Similar to Step 1,

$$\|\nabla u\|^{\frac{3}{2}}_{L_t^5L_x^{\frac{8}{5}}((\frac{t_0}{4},T)\times\mathbb{T}^3)}\le C\Big(\|\nabla u\|^2_{L^{\frac{5}{2}}_{t,x}((0,T)\times\Omega)}+1\Big).$$

This combined with Step 3 proves (4) for $p=q=5$.

Step 5: case $4<q<p<8$ **for (4).**

We let $p'=q'=5$. Choose $\lambda=\frac{5}{8}$. Then $p_2=q_2=\frac{5}{4}$, $2>p_1>q_1>1$, and $p_2>\lambda p_1>\lambda q_1$. Note $\alpha=4>\frac{d}{q_2}=\frac{12}{5}$. Using Theorem 5.15 (Bb), we have

$$\begin{aligned}\left\|\mathscr{A}_\alpha^<[f_2]\right\|^{\frac{5}{8}}_{L_t^{\frac{5}{4}}L_x^{\frac{5}{4}}(\Omega_T)}&\le C\|f_2\|_{L_t^{p_1}L_x^{q_1}}=C\|\nabla u\|^2_{L_t^{2p_1}L_x^{2q_1}}+C\left\|\nabla^2P\right\|_{L_t^{p_1}L_x^{q_1}}\\&\le\|\nabla u\|^2_{L_t^{2p_1}L_x^{2q_1}}.\end{aligned}$$

Similarly, we can control

Fig. A.2. Construction of a Cantor-type measure.

$$\|\nabla u\|^{\frac{5}{4}}_{L^{\frac{5}{2}}_{t,x}((\frac{t_0}{8},T)\times\mathbb{T}^3)} \leq C\left(\|\nabla u\|^2_{L_t^{\frac{p}{2}} L_x^{\frac{q}{2}}((0,T)\times\mathbb{T}^3)} + 1\right).$$

This combined with Step 4 proves (4) for $4 < q < p < 8$. We have now proved all the claims in Proposition 1.5. $\square$

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Appendix A. Unboundedness of averaging operator in L^1

We claim that the averaging operator $\mathscr{A}_\alpha$ is not bounded in $L^1(\mathbb{R})$ for $\alpha = \frac{1}{2}$, by constructing a sequence of functions f_k as the following. For $k \geq 1$, define

$$S_k = \left\{ \sum_{j=1}^{k} a_j 4^{-j} : a_j = 0 \text{ or } 2 \right\}.$$

S_k contains fractions whose quaternary expression contains only 0 and 2. Define $A_0 = [0, 1)$, and for $k \geq 1$ define

$$A_k = \bigcup_{x \in S_k} [x, x + 4^{-k}).$$

We draw the definition of A_k in Fig. A.2. Clearly, $|A_k| = 2^{-k}$. Finally, we define

$$f_k = 2^{\frac{3}{2}} \cdot 2^k \mathbf{1}_{A_k}.$$

So that f_k are uniformly bounded in $L^1(\mathbb{R})$.

Now, for every $k \geq 1$, $x \in A_{k-1} \setminus A_k$, $j \geq k$, we know that $x \in [x^* + 4^{-k}, x^* + 2 \cdot 4^{-k})$ for some $x^* \in S_k$. Then

$$\int_{B_{2\cdot 4^{-k}}(x)} f_j \geq \int_{x^*}^{x^*+4^{-k}} f_j = 2^{\frac{3}{2}} \cdot 2^k \cdot 4^{-k} = 2^{\frac{3}{2}} \cdot 2^{-k}.$$

Thus

$$\fint_{B_{2\cdot 4^{-k}}(x)} f_j \geq 2^{-\frac{1}{2}} \cdot 2^k = (2 \cdot 4^{-k})^{-\frac{1}{2}}.$$

By definition, $\mathscr{S}_\alpha[f_j](x) \leq 2 \cdot 4^{-k}$, and $\mathscr{A}_\alpha[f_j](x) \geq (2 \cdot 4^{-k})^{-\frac{1}{2}} = 2^{-\frac{1}{2}} \cdot 2^k$. So

$$\left|\left\{\mathscr{A}_\alpha[f_j](x) \geq 2^{-\frac{1}{2}} \cdot 2^k\right\}\right| \geq |A_{k-1} \setminus A_k| = 2^{-k}, \qquad \forall k \leq j.$$

Hence $\left\{\mathscr{A}_\alpha[f_j]\right\}_j$ is not uniformly bounded in $L^{1,q}(\mathbb{R})$ for any $q < \infty$. In particular, letting $f_j \to \infty$ we obtain a Cantor measure ν such that for any $k \geq 1$, $x^* \in S_k$,

$$\nu([x^*, x^* + 4^{-k})) = 2^{\frac{3}{2}} \cdot 2^{-k}, \qquad \nu([x^* + 4^{-k}, x^* + 2 \cdot 4^{-k})) = 0.$$

Then $|\nu| = 2^{\frac{3}{2}}$ is a finite measure, but $\mathscr{A}_\alpha(\nu)$ is not in $L^{1,q}(\mathbb{R})$ for any $q < \infty$.

Appendix B. Lorentz norm

We claim that the quasinorms $L_t^{1,\infty} L_x^{1,\infty}$ and $L_{t,x}^{1,\infty}$ are not comparable.

Lemma B.1. *Let $\Omega = (0, 1)$. For any $\varepsilon > 0$, there exist a pair of functions $u_1, u_2 \in L^1_{\mathrm{loc}}((0, 1) \times \Omega)$ with*

$$\|u_1\|_{L^{1,\infty}(0,1;L^{1,\infty}(\Omega))} = \|u_2\|_{L^{1,\infty}(0,1;L^{1,\infty}(\Omega))} = 1,$$

but $\|u_1\|_{L^{1,\infty}((0,1)\times\Omega)} \leq \varepsilon$ and $\|u_2\|_{L^{1,\infty}((0,1)\times\Omega)} = +\infty$.

Proof. Define

$$u_1(t, x) = e^{\frac{t}{\varepsilon}} \mathbf{1}_{[0, e^{-\frac{t}{\varepsilon}}]}(x), \qquad u_2(t, x) = \frac{1}{tx}.$$

Then for any $t \in (0, 1)$, $\|u_1(t)\|_{L^{1,\infty}(\Omega)} = 1$ and $\|u_2(t)\|_{L^{1,\infty}(\Omega)} = \frac{1}{t}$. They are both $L^{1,\infty}$ functions in t.

Note that for any $\alpha > 0$,

$$\{u_1 > \alpha\} = \left\{(t, x) \in (0, 1) \times \Omega : t > \varepsilon \log_+ \alpha, 0 < x < e^{-\frac{t}{\varepsilon}}\right\}$$

$$\implies |\{u_1 > \alpha\}| = \int_{\varepsilon \log_+ \alpha}^{1} e^{-\frac{t}{\varepsilon}} \, \mathrm{d}t = \varepsilon \int_{\log_+ \alpha}^{\frac{1}{\varepsilon}} e^{-s} \, \mathrm{d}s < \frac{\varepsilon}{\alpha},$$

hence $\|u_1\|_{L^{1,\infty}((0,1)\times\Omega)} \leq \varepsilon$. On the other hand, for any $\alpha > 1$,

$$\{u_2 > \alpha\} = \left\{(t, x) \in (0, 1) \times \Omega : 0 < t < \frac{1}{\alpha} \text{ or } 0 < x < \frac{1}{t\alpha}\right\}$$

$$\implies |\{u_2 > \alpha\}| = \frac{1}{\alpha} + \int_{\frac{1}{\alpha}}^{1} \frac{1}{t\alpha} \, \mathrm{d}t = \frac{1}{\alpha}(1 + \log \alpha)$$

hence $\|u_2\|_{L^{1,\infty}((0,1)\times\Omega)} = +\infty$. $\square$

Remark B.2. The example u_1 indeed satisfies $\|u_1(t)\|_{L^{1,q}(\Omega)} \equiv 1$ for any $q \in [1, \infty]$. Moreover, by interpolation, we have $\|u_1\|_{L^{1,q}((0,1)\times\Omega)} \le \varepsilon^{\frac{p-1}{p}}$. Hence

$$L^{1,q}_{t,x} \not\hookrightarrow L^{1,q}_t L^{1,q}_x$$

for any $q > 1$.

Although they are not equivalent, we show below that it is still possible to interpolate between an isotropic norm and a repeated norm.

Lemma B.3. *Let $T \in (0, \infty]$, and let Ω be a measurable space. Suppose μ_t is a measure on Ω for every $t > 0$, and we define μ_T by $\mathrm{d}\mu_T = \mathrm{d}t\, \mathrm{d}\mu_t$ as before.*

(a) *If $f \in L^\infty_t L^{q_0,\infty}_x$ and $f \in L^{1,\infty}_{t,x}$ for some $q_0 \in (0, 1)$, then $f \in L^{p,\infty}_t L^{q,\infty}_x$ with*

$$\frac{1-q_0}{p} + \frac{q_0}{q} = 1, \qquad 1 < p < \infty, q_0 < q < 1. \tag{B.1}$$

Moreover,

$$\|f\|_{L^{p,\infty}_t L^{q,\infty}_x} \le C(p, q, q_0)\|f\|^{\frac{1}{p}}_{L^{1,\infty}_{t,x}} \|f\|^{1-\frac{1}{p}}_{L^\infty_t L^{q_0,\infty}_x}.$$

(b) *If $f \in L^{p_0,\infty}_t L^\infty_x$ and $f \in L^{1,\infty}_{t,x}$ for some $p_0 \in (0, 1)$, then $f \in L^{p,\infty}_t L^{q,\infty}_x$ with*

$$\frac{p_0}{p} + \frac{1-p_0}{q} = 1, \qquad p_0 < p < 1, 1 < q < \infty. \tag{B.2}$$

Moreover,

$$\|f\|_{L^{p,\infty}_t L^{q,\infty}_x} \le C(p, q, p_0)\|f\|^{\frac{1}{q}}_{L^{1,\infty}_{t,x}} \|f\|^{1-\frac{1}{q}}_{L^{p_0,\infty}_t L^\infty_x}.$$

Proof. Define $S(t, \alpha) = \mu_t(\{x \in \Omega : f(t, x) > \alpha\})$, and define $\alpha_k = 2^k$. Then

$$\|f(t)\|^q_{L^{q,\infty}(\Omega,\mu_t)} \approx \sup_k \alpha_k^q S(t, \alpha_k).$$

Fix $\beta > 0$, and denote

$$B := \big\{t \in (0, T) : \|f(t)\|_{L^{q,\infty}(\Omega,\mu_t)} > \beta\big\}.$$

For every $t \in B$, there exists $k \in \mathbb{Z}$ such that $\alpha_k^q S(t, \alpha_k) > \beta^q$. Hence, we can partition $B = \cup_k B_k$ into a sequence of pairwise disjoint sets $B_k \subset B$, such that

$$\alpha_k^q S(t, \alpha_k) > \beta^q, \qquad \forall t \in B_k. \tag{B.3}$$

Moreover, if $\|f\|_{L^{1,\infty}_{t,x}} = K < \infty$, then for any $\alpha > 0$,

$$\frac{K}{\alpha} > \mu_T(\{f > \alpha\}) = \int_0^T S(t,\alpha)\,\mathrm{d}\alpha \geq \sum_{k\in\mathbb{Z}} \int_{B_k} S(t,\alpha)\,\mathrm{d}t.$$

In particular, for any $k \in \mathbb{Z}$, we have

$$\frac{K}{\alpha_k} > \int_{B_k} S(t,\alpha)\,\mathrm{d}t > |B_k|\beta^q \alpha_k^{-q}. \tag{B.4}$$

(a) Without loss of generality assume $\|f\|_{L^\infty_t L^{q_0,\infty}_x} = 1$, thus for any $t \in (0,T)$ and any $\alpha > 0$, we have

$$\alpha^{q_0} S(t,\alpha) \leq 1.$$

In particular, choose $t \in B_k$ and $\alpha = \alpha_k$, (B.3) yields

$$1 \geq \alpha_k^{q_0} S(t,\alpha_k) > \beta^q \alpha_k^{q_0-q}.$$

So $\alpha_k > \beta^{\frac{q}{q-q_0}}$, hence $k > k_* := \frac{q}{q-q_0}\log_2\beta$. Moreover, (B.4) yields

$$|B| = \sum_{k>k_*} |B_k| \leq \sum_{k>k_*} K\alpha_k^{q-1}\beta^{-q} \leq K a_{k_*}^{q-1}\beta^{-q} = K\beta^{\frac{q(q-1)}{q-q_0}-q} = K\beta^{-p}.$$

(b) Without loss of generality assume $\|f\|_{L^{p_0,\infty}_t L^\infty_x} = 1$. Note that (B.3) implies $S(t,\alpha_k) > 0$, hence $\|f(t)\|_{L^\infty(\Omega,\mu_t)} \geq \alpha_k$. Therefore, for any $k \in \mathbb{Z}$ we have

$$\alpha_k^{p_0}|B_k| < 1.$$

Combined with (B.4), we have

$$|B_k| \leq \min\left\{\alpha_k^{-p_0}, K\alpha_k^{q-1}\beta^{-q}\right\}, \qquad \forall k \in \mathbb{Z}.$$

Hence for some k_* to be determined, we have

$$\begin{aligned} |B| = \sum_{k\in\mathbb{Z}} |B_k| = \sum_{k\leq k_*} |B_k| + \sum_{k>k_*} |B_k| &\leq \sum_{k\leq k_*} K\alpha_k^{q-1}\beta^{-q} + \sum_{k>k_*} \alpha_k^{-p_0} \\ &\leq K\alpha_{k_*}^{q-1}\beta^{-q} + \alpha_{k_*}^{-p_0}. \end{aligned}$$

By selecting $k_* := \frac{1}{p_0+q-1}\log_2\left(\frac{\beta^q}{K}\right)$, we have

$$|B| \leq \left(\frac{\beta^q}{K}\right)^{-\frac{p_0}{p_0+q-1}} = K^{\frac{p}{q}} \beta^{-p}. \quad \square$$

Data availability

No data was used for the research described in the article.

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