IFT6390 - Homework 2

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1 Linear and non-linear regularized regression (50 pts)

1.1 Linear Regression

Lets consider a regression problem for which we have a training dataset D_n with n samples (input, target):

$$D_n = \{(x^{(1)}, t^{(1)}), ..., (x^{(n)}, t^{(n)}\}$$

with $x^{(i)} \in R^d$, and $t^{(i)} \in R$.

The linear regression assumes a parametrized form for the function f which predicts the value of the target from a new data point x. (More precisely, it seeks to predict the expectation of the target variable conditioned on the input variable f(x) $\simeq$ E[t—x].)

The parametrization is a linear transformation of the input, or more precisely an affine transformation.

$$f(x) = w^T x + b$$

1. Precise this models set of parameters θ, as well as the nature and dimensionality of each of them.

Solution

The paramters are:

- the weight w of dimension: d x 1
- the bias b of dimension: 1 x 1

2. The loss function typically used for linear regression is the quadratic loss:

$$L((x, t), f) = (f(x) - t)^2$$

We are now defining the **empirical risk** $\hat{R}$ on the set D_n as the **sum** of the losses on this set (instead of the average of the losses as it is sometimes defined). Give the precise mathematical formula of this risk.

Solution

$$\hat{R}(f_\theta, D_n) = \sum_{i=1}^{n} L(f_\theta(x^{(i)}), t^{(i)}) = \sum_{i=1}^{n} (w^T x^{(i)} + b - t^{(i)})^2$$

3. Following the principle of Empirical Risk Minimization (ERM), we are going to seek the parameters which yield the smallest quadratic loss. Write a mathematical formulation of this minimization problem.

Solution

$$\theta^* = \arg\min_\theta \hat{R}(f_\theta, D_n) = \arg\min_\theta \sum_{i=1}^{n} L(f_\theta(x^{(i)}), t^{(i)}) = \arg\min_\theta \sum_{i=1}^{n} (w^T x^{(i)} + b - t^{(i)})^2$$

4. A general algorithm for solving this optimization problem is gradient descent. Give a formula for the gradient of the empirical risk with respect to each parameter.

Solution

$$\frac{\partial \hat{R}}{\partial \theta} = \sum_{i=1}^{n} \frac{\partial}{\partial \theta} L(f_\theta(x^{(i)}), t^{(i)}) = \sum_{i=1}^{n} \frac{\partial}{\partial \theta} (w^T x^{(i)} + b - t^{(i)})^2 = \sum_{i=1}^{n} 2(w^T x^{(i)} + b - t) \times \frac{\partial}{\partial \theta} (w^T x^{(i)} + b - t^{(i)})$$

where:

$$\frac{\partial}{\partial w} (w^T x^{(i)} + b - t^{(i)}) = x^{(i)}$$

$$\frac{\partial}{\partial b} (w^T x^{(i)} + b - t^{(i)}) = 1$$

combining it all, we get:

$$\frac{\partial \hat{R}}{\partial \theta} = \sum_{i=1}^{n} 2(w^T x^{(i)} + b - t) \times [1, x_1^{(i)}), \ldots, x_d^{(i)}]$$

5. Define the error of the model on a single point (x, t) by f(x)t. Explain in English the relationship between the empirical risk gradient and the errors on the training set.

Solution

The errors on individual point can be large depending on the dataset. The empirical risk gradient is a method to get the lowest *total* error over the entire dataset by fine-tuning the model iteratively.

1.2 Ridge Regression

Instead of $\hat{R}$, we will now consider a **regularized empirical risk**: $\widetilde{R} = \hat{R} + \lambda\mathcal{L}(\theta)$. Here $\mathcal{L}$ takes the parameters θ and returns a scalar penalty. This penalty is smaller for parameters for which we have an a priori preference. The scalar λ is an **hyperparameter** that controls how much we favor minimizing the empirical risk versus this penalty. Note that we find the unregularized empirical risk when $\lambda = 0$.

We will consider a regularization called Ridge, or weight decay that penalizes the squared norm (l^2 norm) of the weights (but not the bias): $\mathcal{L}(\theta) = ||w||^2 = \sum_{k=1}^{d} w_k^2$. We want to minimize $\widetilde{R}$ rather than $\hat{R}$.

1. Express the gradient of $\widetilde{R}$. How does it differ from the unregularized empirical risk gradient?

Solution

$$\widetilde{R} = \hat{R} + \lambda L(\theta)$$

$$\frac{\partial\widetilde{R}}{\partial\theta} = \frac{\partial\hat{R}}{\partial\theta} + \frac{\partial}{\partial\theta}(\lambda\mathcal{L}(\theta))$$

$$= \frac{\partial\hat{R}}{\partial\theta} + \frac{\partial}{\partial\theta}\lambda\sum_{k=1}^{d} w_k^2$$

$$= \frac{\partial\hat{R}}{\partial\theta} + \lambda\frac{\partial}{\partial\theta}\sum_{k=1}^{d} w_k^2$$

where

$$\frac{\partial}{\partial w_k}\sum_{k=1}^{d} w_k^2 = 2w_k \implies \frac{\partial}{\partial w}\sum_{k=1}^{d} w_k^2 = 2w$$

$$\frac{\partial}{\partial b}\sum_{k=1}^{d} w_k^2 = 0$$

Combining it all:

$$\frac{\partial\widetilde{R}}{\partial\theta} = \sum_{i=1}^{n} 2(w^T x^{(i)} + b - t) \times [1, x_1^{(i)}, \ldots, x_d^{(i)}] + 2\lambda w$$

The gradient of $\widetilde{R}$ accounts for the regularization term $\lambda\mathcal{L}(\theta)$ which increases with w and thus penalizes large weight vectors by a factor λ. Unregularized empirical risk gradient is the situation where $\lambda = 0$.

2. Write down a detailed pseudocode for the training algorithms that finds the optimal parameters minimizing $\widetilde{R}$ by gradient descent. To keep it simple, use a constant step-size η.

Solution

- We initialize the parameters randomly
- We update them iteratively following the gradient:

$$\theta \leftarrow \theta - \eta \frac{\partial \hat{R}}{\partial \theta}$$

$$\frac{\partial \hat{R}}{\partial \theta} = \sum_{i=1}^{n} \frac{\partial}{\partial \theta} L(f_\theta(x^{(i)}), t^{(i)}) + \lambda \frac{\partial}{\partial \theta} L(\theta)$$

def ridgeRegressionTrain(traindata, λ, η, max_iter):
    t $\leftarrow$ last column of traindata
    n $\leftarrow$ number of rows in traindata
    d $\leftarrow$ number of columns in traindata
    init θ
    for j in range(max_iter):
        b $\leftarrow \theta_0$
        w $\leftarrow \theta_1, \ldots, \theta_d$
        $\frac{\partial \widetilde{R}}{\partial \theta} \leftarrow \sum_{i=1}^{n} 2(w^T x^{(i)} + b - t) \times [1, x_1^{(i)}, \ldots, x_d^{(i)}] + 2\lambda w$
        $\theta \leftarrow \theta \eta \frac{\partial \widetilde{R}}{\partial \theta}$
    return θ

3. There happens to be an analytical solution to the minimization problem coming from linear regression (regularized or not). $\underline{Assuming\ no\ bias\ (meaning\ b = 0)}$, find a matrix formulation for the empirical risk and its gradient, with the matrix $\mathbf{X} = \begin{pmatrix} x_1^{(1)} & \cdots & x_d^{(1)} \\ \vdots & \ddots & \vdots \\ x_1^{(n)} & \cdots & x_d^{(n)} \end{pmatrix}$ and the vector $\mathbf{t} = \begin{pmatrix} t^{(1)} \\ \vdots \\ t^{(n)} \end{pmatrix}$.

Solution

$\hat{R} = (Xw - t)^T (Xw - t)$
$\frac{\partial \hat{R}}{\partial w} = \frac{\partial}{\partial_w}(w^T X^T X w - w^T X^T t - t^T X w + t^T t) = \frac{\partial}{\partial_w} tr(w^T X^T X w - w^T X^T t - t^T X w + t^T t) =$
$\frac{\partial}{\partial_w}(tr(w^T X^T X w) - 2tr(t^T X w)) = \frac{\partial}{\partial_w}(X^T X w + X^T X w - 2X^T t) = 2X^T X w - 2X^T t$

4. Derive a matrix formulation of the analytical solution to the ridge regression minimization problem by expressing that the gradient is null at the optimum. What happens when n $<$ d and λ = 0?

Solution

$$\frac{\partial \hat{R}_\lambda(f_\theta, D_n)}{\partial \theta} = \sum_{i=1}^{n} \frac{\partial}{\partial \theta} L(f_\theta(x^{(i)}), t^{(i)}) + \lambda \frac{\partial}{\partial \theta} L(\theta) = 0$$

Following the matrix notation:

$$\sum_{i=1}^{n} \frac{\partial}{\partial_\theta} L(f_\theta(x^{(i)}, t^{(i)}) = \frac{\partial}{\partial_\theta}(X\theta - t)^2$$

$$= \frac{\partial}{\partial_\theta}(X\theta - t)^T(X\theta - t)$$

$$= \frac{\partial}{\partial_\theta}(\theta^T X^T X\theta - \theta^T X^T t - t^T X\theta + t^T t)$$

$$= \frac{\partial}{\partial_\theta} tr(\theta^T X^T X\theta - \theta^T X^T t - t^T X\theta + t^T t)$$

$$= \frac{\partial}{\partial_\theta}(tr(\theta^T X^T X\theta) - 2tr(t^T X\theta))$$

$$= \frac{\partial}{\partial_\theta}(X^T X\theta + X^T X\theta - 2X^T t)$$

$$= 2X^T X\theta - 2X^T t$$

$$L(\theta) = \sum_{j=1}^{d} \theta_j^2$$

$$\frac{\partial}{\partial_\theta} L(\theta) = 2\sum_{j=1}^{d} \theta_j$$

So, we can get

$$\frac{\partial \hat{R}_\lambda(f_\theta, D_n)}{\partial\theta} = \sum_{i=1}^{n} \frac{\partial}{\partial\theta} L(f_\theta(x^{(i)}), t^{(i)}) + \lambda \frac{\partial}{\partial\theta} L(\theta)$$

$$= 2X^T X\theta - 2X^T t + 2\lambda I\theta = 0$$

Then we can get $\theta = (X^T X + \lambda I)^{-1} X^T t$

X = $\begin{pmatrix} x_1^{(1)} & \cdots & x_d^{(1)} \\ \vdots & \ddots & \vdots \\ x_1^{(n)} & \cdots & x_d^{(n)} \end{pmatrix}$, **t** = $\begin{pmatrix} t^{(1)} \\ \vdots \\ t^{(n)} \end{pmatrix}$ and the **I** = $\begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$

(**Remark:** $X = n \times d; \theta = d \times 1; t = n \times 1; I = d \times d$)

When N < d and λ = 0, $\theta = (X^T X)^{-1} X^T t$, $X^T X$ shows that the matrix is singular, cannot perform inverse. Then we can't get θ.

1.3 Regression with a fixed non-linear pre-processing

We can make a non-linear regression algorithm by first passing the data through a fixed non-linear filter: a function ϕ(x) that maps x non-linearly to a higher dimensional $\widetilde{x}$.

For instance, if x ∈ R is one dimensional, we can use the polynomial transformation:

$$\widetilde{x} = \phi_{poly^k}(x) = \begin{pmatrix} x \\ x^2 \\ \vdots \\ x_k \end{pmatrix}$$

We can then train a regression, not on the $(x^{(i)}, t^{(i)})$ from the initial training set D_n, but on the transformed data $(\phi(x^{(i)}), t^{(i)})$. This training finds the parameters of an affine transformation f

To predict the target for a new training point x, you wont use f(x) but $\widetilde{f}(x) = f(\phi(x))$.

1. Write the detailed expression of $\widetilde{f}(x)$ when x is one-dimensional (univariate) and we use $\phi = \phi_{poly^k}$.

Solution

$$\widetilde{f}(x) = f(\phi_{poly^k}(x)) = w^T\widetilde{x} + b = (\sum_{i=1}^{k} w_i \cdot x^i) + b$$

2. Give a detailed explanation of the parameters and their dimensions.

Solution

As we known, x ∈ R is one dimensional, after using the polynomial transformation, $\phi_{poly^k}(x)$, that would get k dimension. For each dimension, w would be calculated by gradient descent. So we know the parameters are:

- the weight vector w of dimension: $k \times 1$
- the bias b of dimension: 1×1

3. In dimension $d \geq 2$, a polynomial transformation should include not only the individual variable exponents x_i^j , for powers $j \leq k$, and variables $i \leq d$, but also all the interaction terms of order k and less between several variables (e.g. terms like $x_i^{j_1} x_l^{j_2}$, for $j_1 + j_2 \leq k$ and variables i, $l \leq d$). For d = 2, write down as a function of each of the 2 components of x the transformations $\phi_{poly^1}(x)$, $\phi_{poly^2}(x)$, and $\phi_{poly^3}(x)$.

Solution

$$\phi_{poly^1}(x) = x_1 + x_2$$
$$\phi_{poly^2}(x) = x_1 + x_2 + x_1^2 + x_1x_2 + x_2^2$$
$$\phi_{poly^3}(x) = x_1 + x_2 + x_1^2 + x_1x_2 + x_2^2 + x_1^3 + x_1^2x_2 + x_1x_2^2 + x_2^3$$

4. What is the dimensionality of $\phi_{poly^k}(x)$, as a function of d and k?

Solution

According to Multinomial theorem and "stars and bars" as combinatorial theorems,

$(x_1 + x_2 + ... + x_d)^k = C_{d-1}^{k+d-1} = \frac{(k+d-1)!}{(d-1)!k!}$

So the $dim(\phi_{poly^k}) = dim(\phi_{poly^{k-1}}) + \frac{(k+d-1)!}{(d-1)!k!} = \frac{(d+k)!}{d!k!} - 1$

IFT6390 HW2-Q2

By Philippe Helal (20133703) and Jinfang Luo (20111308)

In [1]:
```
%pylab inline
import numpy as np
import pylab
```

```
Populating the interactive namespace from numpy and matplotlib
```

1. Implement in python the ridge regression with gradient descent. We will call this algorithm regression_gradient. Note that we now have parameters w and b we want to learn on the training set, as well an hyper-parameter to control the capacity of our model: λ. There are also hyper-parameters for the optimization: the step-size η, and potentially the number of steps.

In [2]:
```
class RidgeRegression:
    def __init__(self):
        self.data = []

    def regression_gradient(self, train_data, lam, eta, max_iter):

        iteration = 0
        n = train_data.shape[0]
        d = train_data.shape[1]
        t = train_data[:, -1]

        self.theta = np.ones(d)

        # Add a column of ones to the inputs.
        # The column will be multiplied by the bias
        x0 = np.ones(n)
        X0 = np.reshape(x0, (n, 1))
        X1 = train_data[:,:-1]
        X = np.c_[X0, X1]

        for i in range(max_iter):
            regularization = 2 * lam * self.theta
            f_of_x = np.dot(X, self.theta)
            gradient = 2 * np.dot((f_of_x - t),X)
            self.theta = self.theta - eta*(regularization + gradient)

        self.bias = self.theta[0]
        self.weight = self.theta[1:]
        return self.weight, self.bias
```

2. Consider the function h(x) = sin(x) + 0.3x − 1. Draw a dataset D_n of pairs (x, h(x)) with n = 15 points where x is drawn uniformly at random in the interval [−5, 5]. Make sure to use the same set D_n for all the plots below.

In [3]:
```
def fsin(v1):
    return np.sin(v1)+0.3*(v1)-1

def fpre(v2, w, b):
    return w*v2+b
```

In [4]:
```
X_train = np.random.uniform(-5,5,15)
Y_train = fsin(X_train)

XX = np.linspace(-10, 10, 50)
YY = fsin(XX)

data = np.vstack((X_train, Y_train))
data_set = np.transpose(data)
```

3. With λ = 0, train your model on D_n with the algorithm regression_gradient(). Then plot on the interval [−10, 10]: the points from the training set D_n, the curve h(x), and the curve of the function learned by your model using gradient descent. Make a clean legend. Remark: The solution you found with gradient descent should converge to the straight line that is closer from the n points (and also to the analytical solution). Be ready to adjust your step-size (small enough) and number of iterations (large enough) to reach this result.

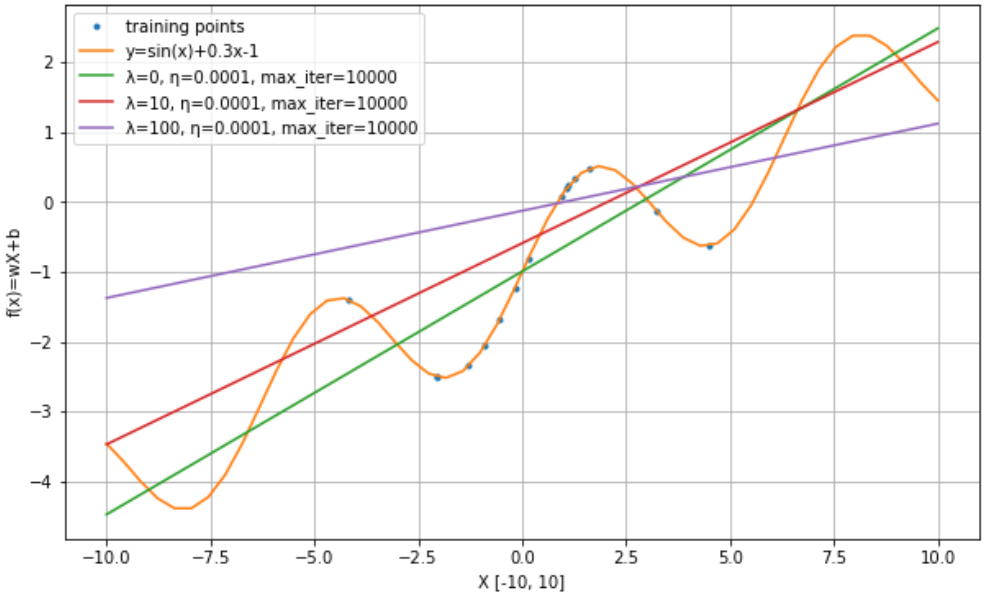
4. on the same graph, add the predictions you get for intermediate value of λ, and for a large value of λ. Your plot should include the value of λ in the legend. It should illustrate qualitatively what happens when λ increases.

In [5]:
```
# regression_gradient(train_data, lam, eta, max_iter):
model = RidgeRegression()
w1, b1 = model.regression_gradient(data_set, 0, 0.001, 10)
w2, b2 = model.regression_gradient(data_set, 0, 0.0001, 1000)
w3, b3 = model.regression_gradient(data_set, 0, 0.0001, 10000)
w4, b4 = model.regression_gradient(data_set, 0, 0.000001, 150000)
w5, b5 = model.regression_gradient(data_set, 0, 0.000001, 100000)
# What do these numbers represent? -> [350.82144017111142, 82.047822709361668,
52.046680055674877, 52.04667952617298, 52.046680065815941]

y1 = fpre(XX,w1,b1)
y2 = fpre(XX,w2,b2)
y3 = fpre(XX,w3,b3)
y4 = fpre(XX,w4,b4)
y5 = fpre(XX,w5,b5)

# find combination that minimizes loss
losses = []
for y in (y1, y2, y3, y4, y5):
    losses.append(np.sum((y - YY)**2))
print(losses)
```

```
[160.31412173600876, 24.69788687536965, 24.605731065642324, 24.53091126865892,
24.70070016520962]
```

We can see that after reaching a small-enough eta & a large enough max_iter, the losses converge. We will use y3 for our prediction

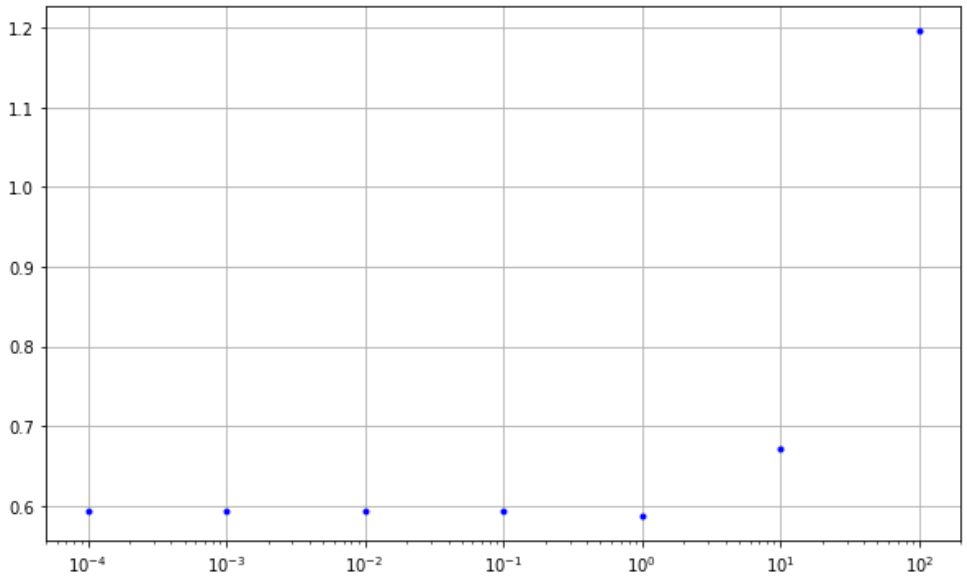
In [6]:
```
med_lambda = 10
w6, b6 = model.regression_gradient(data_set, 10, 0.0001, 10000)
large_lambda = 100
w7, b7 = model.regression_gradient(data_set, 100, 0.0001, 10000)

y6 = fpre(XX,w6,b6)
y7 = fpre(XX,w7,b7)

pylab.figure(figsize=(10,6))
pylab.plot(X_train, Y_train, '.')
pylab.plot(XX, YY, '-')
pylab.plot(XX, y3)
pylab.plot(XX, y6)
pylab.plot(XX, y7)
pylab.xlabel('X [-10, 10]')
pylab.ylabel('f(x)=wX+b')
pylab.grid()
pylab.legend(('training points', 'y=sin(x)+0.3x-1','λ=0, η=0.0001, max_iter=100
00','λ=10, η=0.0001, max_iter=10000','λ=100, η=0.0001, max_iter=10000'))
pylab.show()
```



5. Draw another dataset D_{test} of 100 points by following the same procedure as D_n. Train your linear model on D_n for λ taking values in [0.0001, 0.001, 0.01, 0.1, 1, 10, 100]. For each value of λ, measure the average quadratic loss on D_{test}. Report these values on a graph with λ on the x-axis and the loss value on the y-axis.

In [7]:
```
def floss(v3, w, b, t3):
    """Returns the mean quadratic loss"""
    loss_value = np.mean(((w*v3+b)-t3)**2)
    return loss_value
```

In [8]:
```
X_test = np.random.uniform(-5,5,100)
Y_test = fsin(X_test)

w7, b7 = model.regression_gradient(data_set, 0.0001, 0.001, 10000)
w8, b8 = model.regression_gradient(data_set, 0.001, 0.001, 10000)
w9, b9 = model.regression_gradient(data_set, 0.01, 0.001, 10000)
w10, b10 = model.regression_gradient(data_set, 0.1, 0.001, 10000)
w11, b11 = model.regression_gradient(data_set, 1, 0.001, 10000)
w12, b12 = model.regression_gradient(data_set, 10, 0.001, 10000)
w13, b13 = model.regression_gradient(data_set, 100, 0.001, 10000)

y7 = floss(X_test, w7, b7, Y_test)

y8 = floss(X_test, w8, b8, Y_test)
y9 = floss(X_test, w9, b9, Y_test)
y10 = floss(X_test, w10, b10, Y_test)
y11 = floss(X_test, w11, b11, Y_test)
y12 = floss(X_test, w12, b12, Y_test)
y13 = floss(X_test, w13, b13, Y_test)

pylab.figure(figsize=(10,6))

lamb = (0.0001, 0.001, 0.01, 0.1, 1, 10, 100)
los = (y7, y8, y9, y10, y11, y12, y13)

pylab.plot(lamb, los, '.', color = 'b' )
pylab.xscale('log')
pylab.grid()
pylab.show()
```

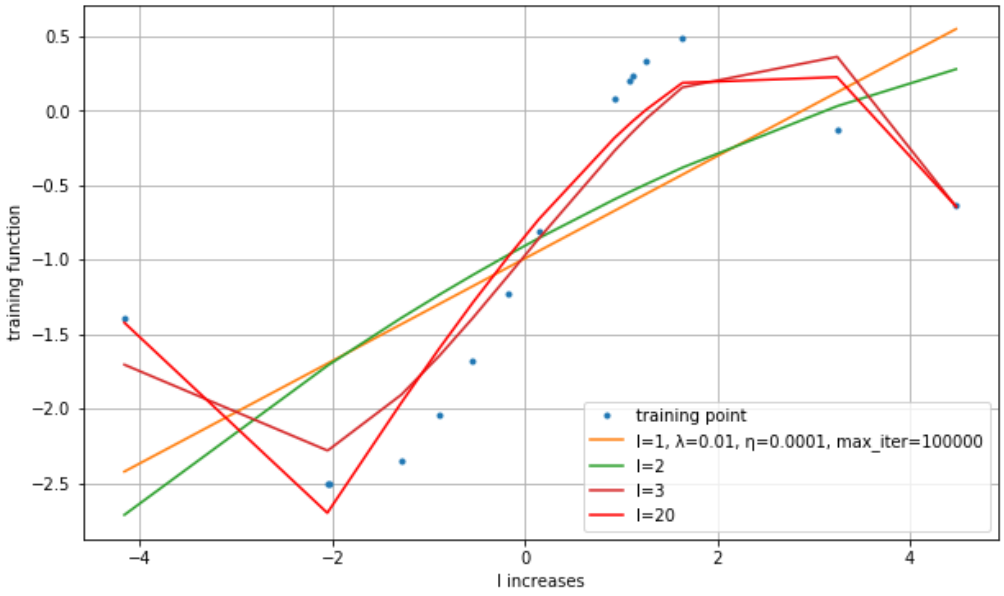


6. Use the technique studied in problem 1.3 above to learn a non-linear function of x. Specifically, use Ridge regression with the fixed preprocessing ϕ_{poly^l} described above to get a polynomial regression of order l. Apply this technique with λ = 0.01 and different values of l. Plot a graph similar to question 2.2 with all the prediction functions you got. Don't plot too many functions to keep it readable and precise the value of l in the legend.

In [9]:
```
def polynomial(X, l):
    XP = np.zeros((X.shape[0], l))
    for i in range(0, l):
        XP[:,i] = X**(i+1)
    mu = np.zeros((1, l))
    xmin = np.zeros((1, l))
    xmax = np.zeros((1, l))
    mu = np.mean(XP, axis = 0)
    xmin = np.min(XP, axis = 0)
    xmax = np.max(XP, axis =0)

    XP_norm = (XP - mu)/(xmax - xmin)
    return XP_norm
```

In [10]:
```
Xl1 = polynomial(X_train, 1)
X_poly1 = np.c_[Xl1, data_set[:,-1]]

Xl2 = polynomial(X_train, 2)
X_poly2 = np.c_[Xl2, data_set[:,-1]]

Xl3 = polynomial(X_train, 3)
X_poly3 = np.c_[Xl3, data_set[:,-1]]

Xl4 = polynomial(X_train, 4)
X_poly4 = np.c_[Xl4, data_set[:,-1]]

Xl5 = polynomial(X_train, 5)
X_poly5 = np.c_[Xl5, data_set[:,-1]]

Xl20 = polynomial(X_train, 20)
X_poly20 = np.c_[Xl20, data_set[:,-1]]

model3 = RidgeRegression()
ww1, bb1 = model3.regression_gradient(X_poly1, 0.01, 0.0001, 100000)
ww2, bb2 = model3.regression_gradient(X_poly2, 0.01, 0.0001, 100000)
ww3, bb3 = model3.regression_gradient(X_poly3, 0.01, 0.0001, 100000)
#ww4, bb4 = model3.regression_gradient(X_poly4, 0.01, 0.0001, 100000)
#ww5, bb5 = model3.regression_gradient(X_poly5, 0.01, 0.0001, 100000)
ww20, bb20 = model3.regression_gradient(X_poly20, 0.01, 0.0001, 100000)

zz1 = np.dot(Xl1,ww1)+bb1
zz2 = np.dot(Xl2,ww2)+bb2
zz3 = np.dot(Xl3,ww3)+bb3
#z4 = np.dot(XX4,ww4)+bb4
#z5 = np.dot(XX5,ww5)+bb5
zz20 = np.dot(Xl20,ww20)+bb20

XX1 = polynomial(X_test, 1)
XX2 = polynomial(X_test, 2)
XX3 = polynomial(X_test, 3)
#XX4 = polynomial(X_test, 4)
#XX5 = polynomial(X_test, 5)
XX20 = polynomial(X_test, 20)

z1 = np.dot(XX1,ww1)+bb1
z2 = np.dot(XX2,ww2)+bb2
z3 = np.dot(XX3,ww3)+bb3
#z4 = np.dot(XX4,ww4)+bb4
#z5 = np.dot(XX5,ww5)+bb5
z20 = np.dot(XX20,ww20)+bb20
```

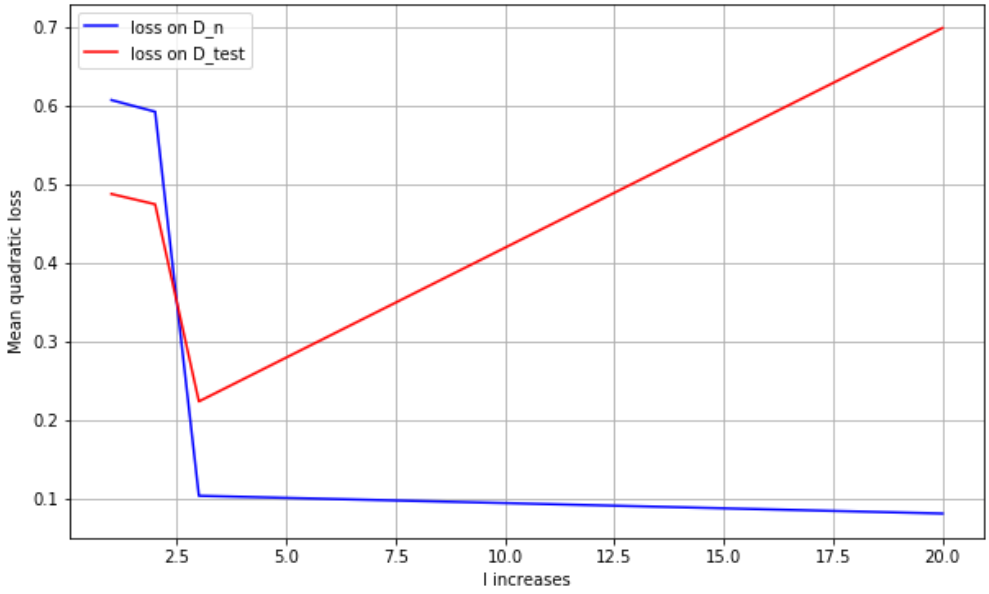
In [11]:
```
pylab.figure(figsize=(10,6))
pylab.plot(X_train, Y_train, '.')
#pylab.plot(XX, YY, '-')
pylab.plot(np.sort(X_train),zz1[np.argsort(X_train)])
pylab.plot(np.sort(X_train),zz2[np.argsort(X_train)])
pylab.plot(np.sort(X_train),zz3[np.argsort(X_train)])
#pylab.plot(np.sort(X_test),z4[np.argsort(X_test)])
#pylab.plot(np.sort(X_test),z5[np.argsort(X_test)])
pylab.plot(np.sort(X_train),zz20[np.argsort(X_train)],color='r')

pylab.grid()
pylab.legend(('training point','l=1, λ=0.01, η=0.0001, max_iter=100000', 'l=2',
'l=3','l=20'))
pylab.xlabel('l increases')
pylab.ylabel('training function')
pylab.show()
```



7. Comment on what happens when l increases. What happens to the empirical risk (loss on D_n), and to the true risk (loss on D_{test})?

In [12]:
```
def floss_train(train_set, w, b, t):
    loss_value = (1/15)*(((np.sum((w*train_set), axis = 1)-t)+b)**2)
    return loss_value

def floss_test(test_set, w, b, t):
    loss_value = (1/100)*(((np.sum((w*test_set), axis = 1)-t)+b)**2)
    return loss_value
```

In [13]:
```
tl1 = np.sum(floss_train(Xl1, ww1, bb1, Y_train))
tl2 = np.sum(floss_train(Xl2, ww2, bb2, Y_train))
tl3 = np.sum(floss_train(Xl3, ww3, bb3, Y_train))
#tl4 = np.sum(floss_train(Xl4, ww4, bb4, Y))
#tl5 = np.sum(floss_train(Xl5, ww5, bb5, Y))
tl20 = np.sum(floss_train(Xl20, ww20, bb20, Y_train))

lt = (1, 2, 3, 20)
tl = (tl1, tl2, tl3, tl20)

Xh1 = polynomial(X_test, 1)
Xh2 = polynomial(X_test, 2)
Xh3 = polynomial(X_test, 3)
#Xh4 = polynomial(X_test, 4)
#Xh5 = polynomial(X_test, 5)
Xh20 = polynomial(X_test, 20)

th1 = np.sum(floss_test(Xh1, ww1, bb1, Y_test))
th2 = np.sum(floss_test(Xh2, ww2, bb2, Y_test))
th3 = np.sum(floss_test(Xh3, ww3, bb3, Y_test))
#th4 = np.sum(floss_test(Xh4, ww4, bb4, Y_test))
#th5 = np.sum(floss_test(Xh5, ww5, bb5, Y_test))
th20 = np.sum(floss_test(Xh20, ww20, bb20, Y_test))

th = (th1, th2, th3, th20)

pylab.figure(figsize=(10,6))
pylab.plot(lt, tl,'b')
pylab.plot(lt, th,'r')
pylab.xlabel('l increases')
pylab.ylabel('Mean quadratic loss')
pylab.grid()
pylab.legend(('loss on D_n', 'loss on D_test'))
pylab.show()
```



We can see that while the loss on the training data decreases as we increase the degree of our polynomial, the loss on the test data increases due to overfitting the training data

In [ ]: