Wednesday, 18th May 2016
2.00 pm – 3.30 pm

EXAMINATION FOR THE DEGREE OF M.SC. (SCIENCE)

Probability (Level M)

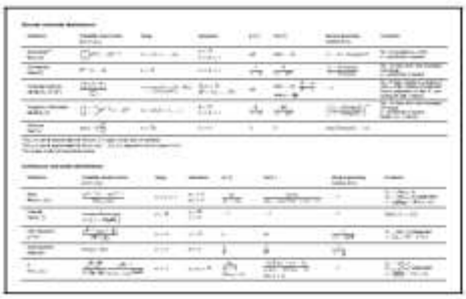
Hand calculators with simple basic functions (log, exp, square root, etc.) may be used in examinations. No calculator which can store or display text or graphics may be used, and any student found using such will be reported to the Clerk of Senate".

This paper consists of 5 pages and contains 4 questions. Candidates should attempt THREE out of the FOUR questions. If more than three questions are attempted please indicate which questions should be marked; otherwise, the first three questions will be graded.

Question 1	20 marks
Question 2	20 marks
Question 3	20 marks
Question 4	20 marks

The following material is made available to you:

Probability formula sheet



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The Standard Normal Distribution Function

When Z is a N(0,1) random variable, this table gives $\Phi(z) = P(Z \leq z)$ for values of z from 0.00 to 3.67 in steps of 0.01.

When $z < 0$, $\Phi(z)$ can be found from this table using the relationship $\Phi(-z) = 1 - \Phi(z)$.

z	0	1	2	3	4	5	6	7	8	9
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9983	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9999		

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1. **(a)** The discrete random variable U is equally likely to take any of the values 1, 2, …, k (where k is a positive integer). Show that $\mathrm{E}(U) = \dfrac{k+1}{2}$ and $\mathrm{var}(U) = \dfrac{k^2-1}{12}$.

[You may use without proof the result that $1^2 + 2^2 + \cdots + k^2 = \dfrac{k(k+1)(2k+1)}{6}$.]

[5 MARKS]

(b) X and Y are independent random variables, each with the Geo(θ) distribution for some θ in the range $0 < \theta < 1$.

(i) Let $S = X + Y$. Explain why

$$\mathrm{P}(S = s) = \sum_{x=1}^{s} \mathrm{P}(X = x)\,\mathrm{P}(Y = s - x), \quad s = 2, 3, 4, \cdots.$$

Using this result, write down the probability mass function of the random variable, S. What distribution is this?

[8 MARKS]

(ii) Find the conditional probability mass function, $\mathrm{P}(X = x \mid S = s)$, when s is an integer greater than or equal to 2. Using the results proved in part (a), or otherwise, write down expressions for $\mathrm{E}(X \mid S = s)$ and $\mathrm{var}(X \mid S = s)$.

[7 MARKS]

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2. **(a)** Suppose that the continuous random variable U has the Ga(α, θ) distribution for parameters $\alpha > 0$ and $\theta > 0$. Show that $\mathrm{E}(U) = \dfrac{\alpha}{\theta}$ and $\mathrm{var}(U) = \dfrac{\alpha}{\theta^2}$.

[6 MARKS]

(b) Suppose that the continuous random variables X and Y have joint probability density function

$$f(x, y) = \begin{cases} 4x^2 e^{-(x+y)}, & 0 < x < y, \\ 0, & \text{otherwise.} \end{cases}$$

(i) Derive the marginal probability density function of X. Using the results proved in part (a), or otherwise, obtain $\mathrm{E}(X)$ and $\mathrm{var}(X)$. **[5 MARKS]**

(ii) Show that the conditional probability density function of Y given $X = x$ is

$$f(y \mid x) = e^{-(y-x)},$$

on a range space you should specify. Use it to find $\mathrm{E}(Y \mid X)$.

[6 MARKS]

(iii) Hence find $\mathrm{E}(Y)$. **[3 MARKS]**

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3. **(a)** The continuous random variables X_1 and X_2 jointly have a bivariate Normal distribution. X_1 has expected value 50 and standard deviation 8. X_2 has expected value 45 and standard deviation 10. The correlation between X_1 and X_2 is –0.25.

In different physical units, the same quantities can be recorded as

$$Y_1 = 0.555(X_1 - 32) \text{ and } Y_2 = 0.447X_2.$$

Let $\underline{Y}$ denote the random vector $(Y_1, Y_2)^{\mathrm{T}}$. Find $\mathrm{E}(\underline{Y}), \mathrm{cov}(\underline{Y})$ and the correlation between Y_1 and Y_2.

[6 MARKS]

(b) A competitor is to shoot at a target, with centre point O. X and Y are, respectively, the horizontal and vertical displacements (in cm) from O to the point where a bullet fired by this competitor hits the target. R and Q are defined by

$$X = R\cos Q, \quad Y = R\sin Q.$$

Here, $R > 0$ is the distance (in cm) from O to the point where a bullet hits the target and Q (in the range 0 to 2π radians) is the angle from the horizontal axis to the ray through O on which the bullet lies, measuring counter-clockwise.

(i) Suppose that X and Y are independent $\mathrm{N}(0, \sigma^2)$ random variables. Find the joint probability density function of R and Q.

[10 MARKS]

(ii) Referring to the Factorisation Theorem, or otherwise, show that R and Q are independent. Find the marginal probability density functions of R and Q.

[4 MARKS]

4. (i) Suppose that the discrete random variable X has the $\mathrm{Bi}(m, \theta)$ distribution, where m is a positive integer and $0 < \theta < 1$. Show that X has moment generating function

$$M_X(t) = \left(1 - \theta + \theta e^t\right)^m.$$

Use this moment-generating function to find the expected value and variance of X.

[9 MARKS]

(ii) $X_1, X_2, \ldots, X_n$ are independent random variables, each with the $\mathrm{Bi}(1, \theta)$ distribution. Use moment generating functions to show that $S = X_1 + X_2 + \ldots + X_n$ is a $\mathrm{Bi}(n, \theta)$ random variable. **[5 MARKS]**

(iii) State the Central Limit Theorem. Use it, along with the results from parts (i) and (ii), to show how the $\mathrm{Bi}(n, \theta)$ distribution can be approximated by a Normal distribution for large enough n. State clearly the parameters of this Normal distribution.

[6 MARKS]

/END OF QUESTION PAPER.