14th December 2017
1.00 – 2.30 p.m.

**EXAMINATION FOR THE DEGREES OF M.Sc., M.Sci.
AND Ph.D. Integrated Study**

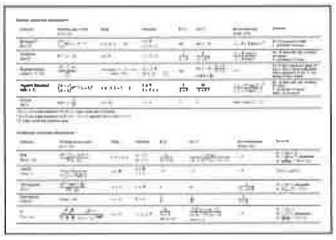
Probability - Level M

**This paper consists of 5 pages and contains 3 questions.
Candidates should attempt all questions.**

Question 1	20 marks
Question 2	20 marks
Question 3	20 marks
Total	60 marks

The following material is made available to you:

Probability formula sheet



"Hand calculators with simple basic functions (log, exp, square root, etc.) may be used in examinations. No calculator which can store or display text or graphics may be used, and any student found using such will be reported to the Clerk of Senate".

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The standard normal distribution function

When Z is a $\mathrm{N}(0,1)$ random variable, this table gives $\Phi(z) = P(Z \le z)$ for values of z from 0.00 to 3.67 in steps of 0.01. When $z < 0$, $\Phi(z)$ can be found from this table using the relationship $\Phi(-z) = 1 - \Phi(z)$.

z	0	1	2	3	4	5	6	7	8	9
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9983	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9999		

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1. A car rental company carries out a major service of a vehicle after it has been rented out 20 times. Suppose the duration (in days) of each rental is an independent random variable and the ith ($i = 1, 2, \dots$) rental has duration

$$X_i \sim \text{Expo}(\lambda) \qquad (\lambda > 0).$$

(i) Prove that the moment-generating function (m.g.f.) of the duration of one single rental, X_i say, is

$$M_{X_i}(t) = \frac{\lambda}{\lambda - t}.$$

As part of your proof, determine and justify the allowed range of t.

[3,1 MARKS]

(ii) Suppose that the company's data analyst were to change to measuring rental durations in weeks, not days. Let Z_i be the duration in weeks of a rental of X_i days. Derive the m.g.f. of Z_i. **[2 MARKS]**

(iii) Let Y be the total duration (in days) of 20 rentals of a car. Derive the m.g.f. $M_Y(t)$ of Y and state the range of t. **[2,1 MARKS]**

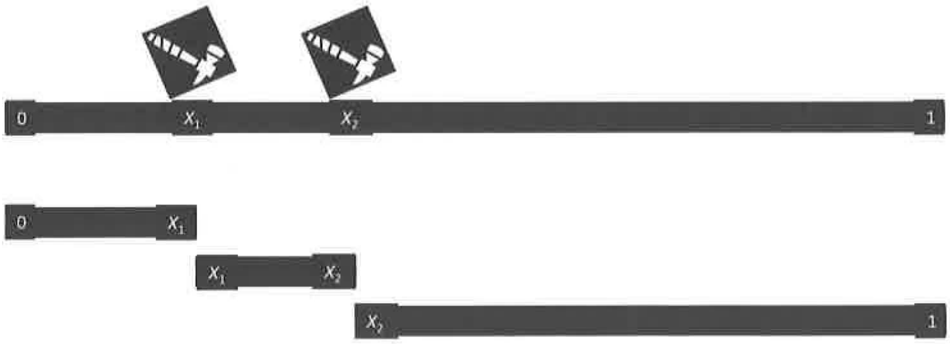
(iv) Using the m.g.f. of Y (or otherwise), derive $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$. **[2,2 MARKS]**

(v) Use the central limit theorem to approximate the distribution function of Y.

[3 MARKS]

(vi) Suppose $\lambda = \frac{1}{3}$. Determine the approximate probability that a car will not have a major service before 90 days. Should one use a continuity correction here? Explain. **[3,1 MARKS]**

2. Suppose a 1-metre ruler is broken in two places. Let X_1 be a random variable describing the location of the break attached to the segment of the ruler containing the zero mark and X_2 the location of the break attached to the end of the ruler marked one:



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(i) Are X_1 and X_2 independent? Explain. **[2 MARKS]**

(ii) Let the random vector $\boldsymbol{X} = (X_1, X_2)$ have joint probability density function

$$f_{\boldsymbol{X}}(x_1, x_2) = \begin{cases} kx_1^{m-1}x_2^{n-1}, & 0 \leq x_1 \leq x_2 \leq 1 \\ 0, & \text{otherwise} \end{cases},$$

where m and n are both positive integers.

a. Prove that $k = m(m+n)$. **[4 MARKS]**

b. Prove that

$$\mathrm{E}(X_1^r X_2^s) = \frac{m(m+n)}{(m+r)(m+n+r+s)},$$

for non-negative integers r and s. **[3 MARKS]**

c. Find $\mathrm{E}(X_1), \mathrm{E}(X_2)$ and $\mathrm{E}(X_1 X_2)$ and, hence, show that

$$\mathrm{Cov}(X_1, X_2) = \frac{m(m+n)}{(m+1)(m+n+1)^2(m+n+2)}.$$

[1,1,1,3 MARKS]

d. Find the marginal p.d.f. of X_1 and hence the conditional p.d.f. of X_2 given $X_1 = x_1$. When $n = 1$, identify this conditional distribution. **[2,2,1 MARKS]**

3. (i) Let $X_1 \sim \mathrm{Expo}(\lambda)$ independent of $X_2 \sim \mathrm{Expo}(\lambda)$ for $\lambda > 0$. Suppose

$$Y_1 = \frac{X_1}{X_1 + X_2} \qquad \text{and} \qquad Y_2 = X_1 + X_2.$$

a. Write down the probability density function (p.d.f.) of $\boldsymbol{X} = (X_1, X_2)$ and its range space. **[2 MARKS]**

b. Derive the p.d.f. of $\boldsymbol{Y} = (Y_1, Y_2)$ and its range space. **[7 MARKS]**

c. What standard distribution is the marginal distribution of Y_1? What standard distribution is the marginal distribution of Y_2? Are Y_1 and Y_2 independent? Explain. **[4 MARKS]**

(ii) E and F are two events in some sample space. E and F are independent. Are E' and F independent? If so, prove it. If not, show that they are not. **[3 MARKS]**

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(iii) Suppose the random vector

$$\boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \mathrm{N}_2\left(\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \right).$$

Let $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$. Find the distribution of $\boldsymbol{Y} = (Y_1, Y_2)$. Are Y_1 and Y_2 independent? **[3,1 MARKS]**

END OF QUESTION PAPER.