University of Glasgow

12th December 2019
1.00 – 2.30 p.m.

EXAMINATION FOR THE DEGREES OF M.Sc.
AND Ph.D. Integrated Study

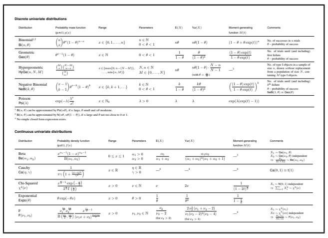
Probability Level M (STATS 5024)

This paper consists of 6 pages and contains 6 questions.
Candidates should attempt all questions.

Question 1	11 marks
Question 2	12 marks
Question 3	9 marks
Question 4	9 marks
Question 5	6 marks
Question 6	13 marks
Total	60 marks

The following material is made available to you:

Probability formula sheet



"An electronic calculator may be used provided that it is allowed under the School of Mathematics and Statistics Calculator Policy. A copy of this policy has been distributed to the class prior to the exam and is also available via the invigilator."

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The standard normal distribution function

When Z is a $\mathrm{N}(0,1)$ random variable, this table gives $\Phi(z) = P(Z \le z)$ for values of z from 0.00 to 3.67 in steps of 0.01. When $z < 0$, $\Phi(z)$ can be found from this table using the relationship $\Phi(-z) = 1 - \Phi(z)$.

z	0	1	2	3	4	5	6	7	8	9
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9983	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9999		

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1. Suppose the countable number of events $E_1, E_2, \ldots$ form a partition of a sample space S.

   (a) Explain, in mathematical notation or in words, what this means. **[2 MARKS]**

   (b) State and prove the Law of Total Probability for the probability of another event F, justifying the steps. **[1,3 MARKS]**

   (c) If the collection of events forming the partition were instead *uncountable*, why would your proof in part (b) fail? **[1 MARK]**

   The probabilities of three designs of (biased) coins (C_1, C_2 and C_3) landing heads up are 0.445, 0.495 and 0.55, respectively. A bag is filled with 90 coins of the first design (C_1), 10 coins of C_2 and 100 coins of C_3. The bag is well shaken.

   (d) Calculate the probability that, if you randomly choose one coin from the bag and toss it, the outcome is heads. **[3 MARKS]**

   (e) The outcome does turn out to be heads. Based on this, what now is the probability that the coin is of design C_1? **[1 MARK]**

2. The independent random variables $X_i \sim \mathrm{Bi}(n_i, \theta)$ for $i = 1, \ldots, N$.

   (a) Prove that the moment-generating function (m.g.f.) $M_{X_i}(t)$ of X_i satisfies

   $$M_{X_i}(t) = [\theta \mathrm{e}^t + 1 - \theta]^{n_i}.$$

   For which values of t does it converge? **[3,1 MARKS]**

   (b) Use m.g.f.s to show that

   $$Y = \sum_{i=1}^{N} X_i \sim \mathrm{Bi}\left(\sum_{i=1}^{N} n_i, \theta\right).$$

   [3 MARKS]

   Suppose a whisky manufacturer bottles 100 bottles of Strathkelvin Single Malt every day. There is a 5% chance that any bottle will be stolen.

   (c) Using the central limit theorem and assuming that the bottles are stolen independently, find an approximation to the probability that in one week (of seven days), more than 45 bottles will be stolen. **[4 MARKS]**

   (d) The bottles are actually stored in cases of 12 bottles. What implication might this have for the analysis? **[1 MARK]**

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3. The two-dimensional random vector $\boldsymbol{X} = (X_1, X_2)^{\mathrm{T}}$ has a multivariate normal distribution:

$$\boldsymbol{X} \sim \mathrm{N}_2(\boldsymbol{\mu}, \Sigma).$$

In addition $X_1 \sim \mathrm{N}(1, 4)$, $X_2 \sim \mathrm{N}(2, 9)$ and the correlation between X_1 and X_2 is $-\frac{1}{2}$.

(a) Based on this information, what are $\boldsymbol{\mu}$ and Σ? **[3 MARKS]**

(b) Let $Y_1 = 2X_1 + X_2 + 2$ and $Y_2 = X_1 - 2X_2 + 1$. What is the distribution of $\boldsymbol{Y} = (Y_1, Y_2)^{\mathrm{T}}$? **[4 MARKS]**

(c) What is the correlation between Y_1 and Y_2? **[1 MARK]**

(d) Suppose that

$$Z = \frac{Y_2 + 2}{\sqrt{52}}.$$

What is the distribution of Z? **[1 MARK]**

4. Suppose the random variables $Z_1, Z_2, \ldots, Z_N$ are independent and

$$Z_i \sim \mathrm{Bi}(m, \theta) \qquad (i = 1, \ldots, N).$$

Let

$$Y = \sum_{i=1}^{N} Z_i.$$

(a) If N is fixed (i.e., not a random variable), what are $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$? **[2 MARKS]**

(b) State the laws of iterated expectation and variance. **[2 MARKS]**

(c) If N now becomes a random variable with $N \sim \mathrm{Poi}(\lambda)$, write down $\mathrm{E}(Y|N = n)$ and $\mathrm{Var}(Y|N = n)$ and hence find $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$. **[1,4 MARKS]**

5. The random variable X has a uniform distribution on the interval $[0, a]$, where $a > 0$:

$$X \sim \mathrm{U}(0, a).$$

(a) Show that the moment-generating function (m.g.f.) of X is

$$M_X(t) = \frac{1}{at}(\mathrm{e}^{at} - 1) \qquad (t \neq 0).$$

[3 MARKS]

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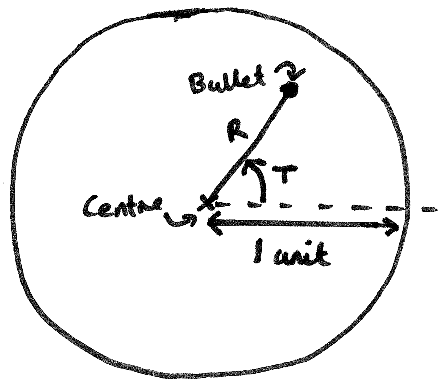
(b) Notice that $M_X(0)$ cannot be evaluated from the expression in (a). Given the definition of the m.g.f., what value must $M_X(0)$ take? **[1 MARK]**

(c) Let $Y = bX$, where $b > 0$. What is the distribution of Y? **[2 MARKS]**

6. A shooting target is a circular disk of radius 1 unit. Your lecturer fires a gun, aiming for the centre of the target. Suppose that the bullet hits the target a distance R units from the centre and that the line joining the point of impact is at an angle of T radians to the horizontal, as shown:



Suppose these two quantities are modelled by independent random variables, where $T \sim \mathrm{U}(0, 2\pi)$ and the probability density function (p.d.f.) of R is:

$$f_R(r) = \begin{cases} kr & 0 \le r \le 1; \\ 0 & r > 1. \end{cases}$$

(a) Find k and hence write down the joint p.d.f. of the random vector $\boldsymbol{W} = (T, R)^{\mathrm{T}}$. **[1,1 MARKS]**

Consider the transformation

$$X = R\cos T$$
$$Y = R\sin T.$$

(b) What is the range space $R_{\boldsymbol{Z}}$ of the random vector $\boldsymbol{Z} = (X, Y)^{\mathrm{T}}$? **[1 MARK]**

(c) Show that the absolute value of the Jacobian determinant $\det\left(\frac{\partial \boldsymbol{W}}{\partial \boldsymbol{Z}}\right)$ is $1/R$. [You may find some, all or none of the following trigonometric identities useful: $\sin^2 t + \cos^2 t = 1$; $\frac{\mathrm{d}}{\mathrm{d}t}\sin t = \cos t$; $\frac{\mathrm{d}}{\mathrm{d}t}\cos t = -\sin t$; $\frac{\mathrm{d}}{\mathrm{d}t}\tan^{-1} t = (1+t^2)^{-1}$.] **[3 MARKS]**

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(d) Hence, show that the joint p.d.f. of $\boldsymbol{Z}$ satisfies

$$f_{\boldsymbol{Z}}(x,y) = \pi^{-1} \qquad (x,y) \in R_{\boldsymbol{Z}}.$$

What does this mean for the shooting ability of your lecturer? **[2,1 MARKS]**

(e) Show that the marginal p.d.f. of the random variable X is

$$f_X(x) = \frac{2}{\pi}\sqrt{1-x^2} \qquad (-1 \leq x \leq 1).$$

What is $f_Y(y)$? Are X and Y independent? Explain. **[2,1,1 MARKS]**

END OF QUESTION PAPER.