Probability Level M (STATS5024)
Second Set of Tutorial Problems on Chapter 3

2020–2021

Make sure you try at least the starred (*) questions before the next tutorial.

50 (*) The random variable X_1 has a $\mathrm{Bi}(n, \theta)$ distribution. Given that $X_1 = x_1$, the discrete random variable X_2 has a $\mathrm{Bi}(n - x_1, \phi)$ distribution, where $0 < \phi < 1 - \theta$. Find $E(X_2)$ and $\mathrm{Var}(X_2)$.

51 On a certain factory production line, k electronic components are manufactured each working day (where $k \geqslant 1$). The number of defective items from a randomly-selected day's production, X_2, is a $\mathrm{Bi}(k, X_1)$ random variable, where X_1 varies from day to day in accordance with the Beta distribution:

$$f_1(x_1) = \tfrac{(m-1)!(n-1)!}{(m+n-1)!} x_1^{m-1}(1 - x_1)^{n-1}, \quad 0 < x_1 < 1,$$

where m and n are positive integers. Find $E(X_2)$ and $\mathrm{Var}(X_2)$.

52 (*) A board game is played with an unbiased, six-sided die whose sides are marked $1, 2, \ldots, 6$. If a player scores a 6, then he rolls the die again; the player continues to roll the die until he scores a value other than a 6. The score for the player's turn is the total of the scores on the die all the times he rolls it. Find the expected value and variance of the total score on one turn.

53 Suppose that X_1 and X_2 are continuous random variables with joint probability density function $f_{12}(x_1, x_2)$. Let $g(X_1)$, $h(X_2)$ be real-valued functions of X_1, X_2 (respectively).

(a) Prove that:

$$E\{h(X_2)\} = E\{E[h(X_2)|X_1]\}$$

$$E\{g(X_1)X_2\} = E\{g(X_1)E(X_2|X_1)\}$$

(b) Suppose that the regression of X_2 on X_1 is linear, that is $E(X_2|X_1) = \alpha + \beta X_1$. Find expressions for $E(X_2)$ and $E(X_1X_2)$ in terms of $E(X_1)$ and $E(X_1^2)$. Hence express β in terms of $\mathrm{Var}(X_1)$, $\mathrm{Var}(X_2)$ and ρ_{12}. Find an expression for α.

(c) Suppose that, in addition to the regression of X_2 on X_1 being linear, the conditional variance of X_2 given $X_1 = x_1$ is constant, that is $\mathrm{Var}(X_2|X_1) = \sigma^2$ for all values of X_1. Show that:

$$\sigma^2 = (1 - \rho_{12}^2)\mathrm{Var}(X_2).$$

54 (*) Suppose that X_1 and X_2 are continuous random variables with the following joint probability density function

$$f_{12}(x_1, x_2) = \begin{cases} \theta^2 e^{-\theta x_2}, & x_2 > x_1 > 0 \\ 0, & \text{otherwise} \end{cases}$$

(a) Obtain the joint probability density function of X_1 and $X_2 - X_1$.

(b) Using the Factorisation Theorem, or otherwise, show that X_1 and $X_2 - X_1$ are independent random variables and write down their marginal probability density functions. Identify these distributions.

55 (*) Suppose that X_1 and X_2 are independent random variables, each with the $U(0, 1)$ distribution.

(a) Write down the joint range space and joint probability density function of X_1 and X_2.

(b) Define the random variables Y_1 and Y_2 by $Y_1 = X_1$ and $Y_2 = X_2 - X_1$. Show that Y_1 and Y_2 have joint range space $R_Y = \{(y_1, y_2) : 0 < y_1 < 1, -y_1 < y_2 < 1 - y_1\}$. Find the joint probability density function of Y_1 and Y_2.

(c) Now find the marginal probability density function of Y_2.
[Hint: Consider separately the cases $-1 < y_2 \leqslant 0$ and $0 < y_2 < 1$.]

56 Suppose that $X_1 \sim N(0, 1)$ independently of $X_2 \sim \chi^2(\nu)$, for some $\nu > 0$. Find the joint probability density function Y_1 and Y_2, where:

$$Y_1 = \frac{X_1}{\sqrt{X_2/\nu}}, \; Y_2 = \sqrt{X_2/\nu}.$$

Hence show that $\frac{X_1}{\sqrt{X_2/\nu}}$ has the $t(\nu)$ distribution.

Here, the $\chi^2(\nu)$ distribution has the probability density function

$$f(x) = \frac{x^{\frac{1}{2}\nu - 1} \exp(-\frac{x}{2})}{2^{\frac{1}{2}\nu} \Gamma(\frac{1}{2}\nu)},$$

for $0 < x_2$, and the $t(\nu)$ distribution has the probability density function

$$f(y) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}\nu)} \cdot \frac{\nu^{\frac{1}{2}\nu}}{(y^2+\nu)^{\frac{1}{2}(\nu+1)}}.$$

57 (*) Suppose that X_1, X_2 and X_3 are independent random variables, and that each X_i has a $\mathrm{Ga}(\alpha_i, \theta)$ distribution. Define the random variables:

$$Y_1 = X_1 + X_2 + X_3,\ Y_2 = \tfrac{X_2+X_3}{X_1+X_2+X_3},\ Y_3 = \tfrac{X_3}{X_2+X_3}.$$

Derive the joint probability density function of Y_1, Y_2 and Y_3, and show that these random variables are all independent. Write down the marginal probability density functions of Y_1, Y_2 and Y_3 and identify these special distributions.

58 Let $\mathbf{X}$ be a discrete random vector with $\mathbf{X} = (X_1, X_2) \sim \mathrm{Mu}(3, \frac{1}{2}, \frac{1}{3})$.

(a) Write down the joint range space of $\mathbf{X}$.

(b) Draw up a table of values of the joint probability mass function, $p_{12}(x_1, x_2)$, on this range space.

(c) Obtain the marginal probability mass functions of X_1 and X_2. Recognise each of these as a $\mathrm{Bi}(n, \theta)$ distribution, for values of n and θ that you should specify.

59 Let $\mathbf{X}$ be a discrete random vector with $\mathbf{X} = (X_1, X_2, X_3) \sim \mathrm{Mu}(2, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$.

(a) Write down the joint range space of $\mathbf{X}$.

(b) Draw up a table of values of the joint probability mass function, $p_{123}(x_1, x_2, x_3)$, on this range space.

(c) Obtain the marginal probability mass functions of X_1, X_2 and X_3. Recognise each of these as a $\mathrm{Bi}(n, \theta)$ distribution, for values of n and θ that you should specify.

(d) Obtain the joint marginal probability mass function of (X_1, X_2), and identify it as a particular Multinomial distribution.

60 Let $\mathbf{X}$ be a discrete random vector with $\mathbf{X} = (X_1, X_2, \dots, X_p) \sim \mathrm{Mu}(n, \theta_1, \theta_2, \dots, \theta_p)$.

(a) Write down the marginal probability mass function of (X_1, X_2). Hence find $E(X_1X_2)$ and prove that $\mathrm{Cov}(X_1, X_2) = -n\theta_1\theta_2$. Deduce the correlation ρ_{12}.

(b) Now write down $\mathrm{Cov}(\mathbf{X})$ and $\rho(\mathbf{X})$.

61 Let $\mathbf{X}$ be a discrete random vector with $\mathbf{X} = (X_1, X_2, \dots, X_p) \sim \mathrm{Mu}(n, \theta_1, \theta_2, \dots, \theta_p)$. Prove that, conditional on $X_p = x_p$,

$$(X_1, X_2, \dots, X_{p-1}) \sim \mathrm{Mu}(n - x_p, \tfrac{\theta_1}{1-\theta_p}, \tfrac{\theta_2}{1-\theta_p}, \dots, \tfrac{\theta_{p-1}}{1-\theta_p})$$

Thinking about how the Multinomial distribution arises in practice, give a heuristic argument to justify this result.

62 Let $\mathbf{X}$ be a discrete random vector with $\mathbf{X} = (X_1, X_2, \dots, X_p) \sim \mathrm{Mu}(n, \theta_1, \theta_2, \dots, \theta_p)$.

(a) Find the conditional marginal probability mass function of X_1 given that $X_2 = x_2, X_3 = x_3, \ldots$ and $X_p = x_p$.

(b) Recognise this as a Binomial distribution with parameter values that you should specify.

(c) Thinking about how the Multinomial distribution arises in practice, give a heuristic argument to justify this result.

63 (*) Let (X_1, X_2) be a continuous random vector with

$$(X_1, X_2) \sim N_2\left(\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \right).$$

(a) Identify the marginal distributions of X_1 and X_2. Write down $E(X_1)$, $\text{Var}(X_1)$, $E(X_2)$, $\text{Var}(X_2)$, $\text{Cov}(X_1, X_2)$ and $\rho(X_1, X_2)$.

(b) Identify the conditional distribution of X_2 given $X_1 = x_1$. Write down $E(X_2|x_1)$, $\text{Var}(X_2|x_1)$. What do you notice about the conditional variance of X_2 given X_1?

(c) Show that the Laws of Iterated Expectation and Variance give the correct values of $E(X_2)$ and $\text{Var}(X_2)$ in this case.

(d) Identify the distribution of: (i) $X_1 + X_2$; (ii) $X_1 - X_2$; (iii) $X_2 - X_1$; (iv) $3X_1 - 2X_2 + 1$.

64 Let (X_1, X_2) be a continuous random vector with

$$(X_1, X_2) \sim N_2\left(\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \right).$$

Write out the probability density functions $f_{12}(x_1, x_2)$, $f_1(x_1)$, $f_2(x_2)$. Hence show that X_1 and X_2 are independent. [Note: In the MVN case, $\rho_{12} = 0$ implies that X_1 and X_2 are independent but this is not generally true for other distributions.]

65 Let (X_1, X_2, X_3) be a continuous random vector with

$$(X_1, X_2, X_3) \sim N_3\left(\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & \frac{1}{2} \\ 1 & 2 & 1 \\ \frac{1}{2} & 1 & 2 \end{pmatrix} \right).$$

(a) Identify the marginal distributions of X_1, X_2 and X_3.

(b) Find the correlation matrix ρ_X.

(c) Obtain the conditional distribution of (X_1, X_2) given $X_3 = x_3$.

(d) Obtain the conditional distribution of X_1 given $X_2 = x_2$ and $X_3 = x_3$.

(e) Find $E(X_1 + X_2 + X_3)$ and $\text{Var}(X_1 + X_2 + X_3)$.

66 Let (X_1, X_2, X_3) be a continuous random vector with

$$(X_1, X_2, X_3) \sim N_3\left(\left(\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 & 2 \\ 1 & 4 & 3 \\ 2 & 3 & 9 \end{smallmatrix}\right) \right).$$

(a) Find the correlation matrix ρ_X.

(b) Let $Y_1 = X_1 + X_2 - X_3$ and $Y_2 = 2X_1 - X_2 + X_3$. Obtain the joint distribution of (Y_1, Y_2).

(c) Obtain the conditional distribution of Y_1 given $Y_2 = y_2$.

67 Let (X_1, X_2, X_3, X_4) be a continuous random vector with

$$(X_1, X_2, X_3, X_4) \sim N_4\left(\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 9 & 3 \\ 0 & 0 & 3 & 4 \end{pmatrix}\right).$$

(a) Write down the marginal distributions of (X_1, X_2) and (X_3, X_4).

(b) Obtain the conditional distribution of (X_3, X_4) given $X_1 = x_1$ and $X_2 = x_2$.

(c) What evidence is there that (X_1, X_2) and (X_3, X_4) are independent?

Numerical answers:

52. $\frac{21}{5}$, $\frac{266}{25}$

58. (c) $\mathrm{Bi}(3, \frac{1}{2})$, $\mathrm{Bi}(3, \frac{1}{3})$

59. (c) all are $\mathrm{Bi}(2, \frac{1}{4})$; (d) $\mathrm{Mu}(2, \frac{1}{4}, \frac{1}{4})$

63. (a) $\mathrm{N}(0, 1)$, $\mathrm{N}(0, 1)$, $-\frac{1}{2}$, $-\frac{1}{2}$; (d) $\mathrm{N}(0, 1)$, $\mathrm{N}(0, 3)$, $\mathrm{N}(0, 3)$, $\mathrm{N}(1, 19)$

65. (a) $\mathrm{N}(-1, 2)$, $\mathrm{N}(0, 2)$, $\mathrm{N}(1, 2)$; (c) $\mathrm{N}_2\left(\begin{pmatrix} \frac{\frac{1}{4}(x_3-5)}{4} \\ \frac{x_3-2}{2} \end{pmatrix}, \begin{pmatrix} \frac{7}{8} & \frac{3}{4} \\ \frac{3}{8} & \frac{3}{2} \end{pmatrix}\right)$; (d) $\mathrm{N}(\frac{x_2}{2}, \frac{3}{2})$; (e) 0, 11

66. (b) $\mathrm{N}_2\left(\begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 6 & -6 \\ -6 & 15 \end{pmatrix}\right)$; (c) Normal with $\mathrm{E}(Y_1|y_2) = \frac{6}{5} - \frac{2y_2}{5}$, $\mathrm{Var}(Y_1|y_2) = \frac{54}{15}$