2561551 P1

1. (a) $b=5$ ✓

$k=4$ ✓

$p=5$ ✓

$r=4$ ✓

$N=bk=rp=20$ ✓

$\lambda=3$ ✓

3/3

(b) Blocked ANOVA table

Source	df	Sum of squares	means squares
Block	4	31.00	7.75 ✓
Treatments	4	41.933	10.48325 ✓
Residual	11	14.817	1.347 ✓
Total	19	87.75 ✓	

4/4

$\Rightarrow RSS(\text{null}) = 31+56.75 = 87.75$

$\Rightarrow RSS(\text{mean}+\text{block}) = 56.75$

$RSS(\text{mean}+\text{Treatment}+\text{block}) = 14.817$

$RSS(\text{null}) - RSS(\text{mean}+\text{block}) = 31$

(c) $F = \frac{10.48325}{1.347} = 7.78267 > qf(0.95,4,11) = 3.35669$ ✓ ✓

$\Rightarrow$ reject the null hypothesis ✓ and conclude that there is a difference between lifetime of all batteries.

3/3

(d) $t_{ij} = \frac{a^T\hat{\beta}}{\sqrt{a^T\text{var}(\hat{\beta})a}}$, where $a^T\text{Var}(\hat{\beta})a = \frac{2k\hat{\sigma}^2}{\lambda \cdot p} = \frac{2\times 4\times 1.347}{3\times 5} = 0.7184$

$\Rightarrow t_{ij} \approx 4.26$ ✓

Critical value $= \frac{1}{\sqrt{2}} q(0.95,5,11) \approx 3.23$ ✓

$\because 4.26 > 3.23$ ✓

$\Rightarrow$ the difference between battery A and E is statistically significant ✓

4/4

$\boxed{14/14}$ Well done!

2561551 P2

2\. (a) (i)

x_1	x_2	x_3	x_4	x_1x_2	$x_2x_3x_4$	$x_1x_2x_3x_4$
−	−	−	−	+	−	+
−	−	+	+	+	−	+
−	+	−	+	−	−	+
−	+	+	−	−	−	+
+	−	−	+	−	+	+
+	−	+	−	−	+	+
+	+	−	−	+	+	+
+	+	+	+	+ ✓	+ ✓	+ ✓

3/3

(ii)

$n=4$ factors, $n_f=8$ runs, $2^{4-1}=8 \Rightarrow q=1$ defining words

$\Rightarrow$ defining words is $V_1=1234$

$\Rightarrow$ defining relation is $\boxed{I=1234}$ ✓

$\Rightarrow$ resolution IV ✓

$\boxed{1=234}$ ✓

$\boxed{12=34}$ ✓

$\boxed{123=4}$ ✓

5/5

(iii)

$B_1=234=1$ ✓

$\Rightarrow$ main effect x_1 ✓ cannot be estimated

2/2

(b) (i)

for d_1: $I=1234=3467=1267$ ✓

for d_2: $I=12567=234567=134$ ✓

for d_3: $I=34567=12345=1267$ ✓

resolution: d_1: IV ✓

d_2: III* ✓

d_3: IV ✓

wordlength pattern: $W(d_1)=(0,3,0,0,0)$ ✓

$W(d_2)=(1,0,1,1,0)$ ✓

$W(d_3)=(0,1,2,0,0)$ ✓

3/3

2561551 P3

2. (b) (ii) Resolution IV is preferred over resolution III, as with resolution IV you have no main effects aliased with other main effects or two-factor interactions. −1

I will recommend Design d_2. because Since resolution V does not appear, resolution III is better than resolution IV. Since d_2 is resolution III with no words of length four, it is resolution III*, hence no aliasing between two-factor interactions. //

−1 what about minimum aberration? 0/2

3. $\boxed{13/15}$

(a) $n_c = 2$ ✓

$n_f = 4$ ✓

$n_a = 6$ ✓ 2/2

(b) $m = 3$

$\alpha = \sqrt{m} = \boxed{\sqrt{3}}$ ✓ 1/1

(c) $N = N_f + N_c + N_a = 12$

$\alpha = \frac{\sqrt{n_f \times N} - n_f}{2} = \frac{\sqrt{4 \times 12} - 4}{2} = \boxed{2\sqrt{3} - 2}$ ✗ −0.5

$\alpha = \left(\frac{\sqrt{n_f \times N} - n_f}{2}\right)^{\frac{1}{2}} = \sqrt{2\sqrt{3} - 2}$ 0.5/1

(d) $x_s = -\frac{1}{2} B^{-1} b$ ✓

$b = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 0.33 \\ 0.48 \\ 0.52 \end{pmatrix}$ ✓ $B = \begin{pmatrix} \hat\beta_{11} & \frac{1}{2}\hat\beta_{12} & \frac{1}{2}\hat\beta_{13} \\ \frac{1}{2}\hat\beta_{21} & \hat\beta_{22} & \frac{1}{2}\hat\beta_{23} \\ \frac{1}{2}\hat\beta_{31} & \frac{1}{2}\hat\beta_{32} & \hat\beta_{33} \end{pmatrix} = \begin{pmatrix} 0.57 & \frac{0.7}{2} & \frac{-0.23}{2} \\ \frac{0.7}{2} & -1.3 & \frac{0.12}{2} \\ \frac{-0.23}{2} & \frac{0.12}{2} & 0.17 \end{pmatrix}$ ✓

by given formula, $B^{-1} = \begin{pmatrix} 1.6456 & 0.4885 & 0.9415 \\ 0.4885 & -0.6131 & 0.5455 \\ 0.9415 & 0.5455 & 6.3267 \end{pmatrix}$ ✓

$\Rightarrow X_s = -\frac{1}{2} B^{-1} b = (-0.63308, -0.07496, -1.93121)^T$ ✓ 6/6

$Y_s = \hat\beta_0 - \frac{1}{4} b^T B^{-1} b) = 9.675437$

$\boxed{9.5/10}$

2561551 p4

4. (a) Yes.
Equal allocation //

0.5/2

(b) $W = \begin{pmatrix} \frac{1}{4} & & & \\ & \frac{1}{4} & & \\ & & \frac{1}{4} & \\ & & & \frac{1}{4} \end{pmatrix}$ $f = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $X = \begin{pmatrix} f(-1,-1) \\ f(-1,1) \\ f(1,-1) \\ f(1,1) \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -1 & 1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix}$

5/5

17.5/19

$M(\xi) = X^T W X = \begin{pmatrix} -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & & & \\ & \frac{1}{4} & & \\ & & \frac{1}{4} & \\ & & & \frac{1}{4} \end{pmatrix} \begin{pmatrix} -1 & -1 \\ -1 & 1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -0.25 & -0.25 & 0.25 & 0.25 \\ -0.25 & 0.25 & -0.25 & 0.25 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ -1 & 1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ //

(c) $V(x, \xi) = f^T M^{-1}(\xi) f$

$= (x_1 \ x_2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1 \ x_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \cdot x_1 + x_2 \cdot x_2 = x_1^2 + x_2^2$ //

2/2

(d) $G\text{-eff}(\xi) = \frac{p}{\varphi_G(\xi)} = \frac{p}{\max V(x,\xi)} = \frac{2}{\max V(x,\xi)}$

$\because x_1 \in [-1,1], x_2 \in [-1,1]$

$V(x,\xi) = x_1^2 + x_2^2$

let $\frac{\partial V}{\partial x_1} = 2x_1 = 0$ $\frac{\partial V}{\partial x_2} = 2x_2 = 0$

$\Rightarrow x_1 = 0, x_2 = 0$ is an extreme point b/c $V''_{x_1x_1} = 2, V''_{x_2x_2} = 2, V''_{x_1x_2} = 0, AC - B^2 > 0$

$V(0,0) = 0$

$V(1,-1) = V(-1,1) = V(1,1) = V(-1,-1) = 2$

$\Rightarrow \max(V, \xi) = 2$

$\Rightarrow G\text{-eff}(\xi) = \frac{2}{2} = 1$

Since ξ is G-optimal

By GET, ξ is D-optimal □

3/3

4.(e) ~~~~

~~to prove~~. the design i find is $\xi' = \begin{pmatrix} (-1,1) & (-1,-1) \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

$\Rightarrow M(\xi') = X^T W X = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \\ & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$

$= \begin{pmatrix} -0.5 & -0.5 \\ 0.5 & -0.5 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow$ Since $M(\xi') = M(\xi)$

$\Rightarrow Var(\xi') = f^T M^{-1}(\xi') f = x_1^2 + x_2^2$

$\Rightarrow G\text{-eff}(\xi') = \frac{p}{\max V(x, \xi')} = \frac{2}{2} = 1$

$\Rightarrow \xi'$ is G-optimal, by G E T, also D-optimal

$\Rightarrow$ proved that $\boxed{\xi' = \begin{pmatrix} (-1,1) & (-1,-1) \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}}$ is the model the problem want.

NO, I would not recommend!

because it is not a full factorial design, the problem of OFAT can cause loss of information.

2561551 P6

$\frac{6.5}{10}$

5.(a)

Since $z_i \sim \text{Bernulli}(\frac{n}{N})$

By formular sheet $E(z_i) = \frac{n}{N}$

$E(\bar{Y}) = E(\frac{1}{n}\sum_{n=1}^{N} y_i z_i)$

Since z_i's are independent

$\Rightarrow E(\bar{Y}) = E(\frac{1}{n}\sum_{n=1}^{N} y_i z_i) = \frac{1}{n}E(\sum_{i=1}^{N} y_i z_i)$

$= \frac{1}{n}E(y_1 z_1 + y_2 z_2 + \cdots + y_N z_N)$

$= \frac{1}{n}\cdot\frac{n}{N}E(y_1 + \cdots y_N)$

$= \frac{1}{N}\cdot E(\sum_{i=1}^{N} y_n)$

$= \frac{1}{N}\cdot\sum_{i=1}^{N} y_n = \mu$

$E(Z_i) =$

$\frac{3}{5}$

$\Rightarrow E(\hat{P}) = E(N\bar{Y}) = N\cdot E(\bar{Y}) = N\cdot\mu = \sum_{i=1}^{N} y_i = P$

$\because E(\hat{P}) = \hat{P} \Rightarrow \hat{P}$ is unbiased estimator for P // Q.E.D.

(b) (i) $\hat{P} = N\cdot\bar{Y} = 1000\times 3 = 3000$ ✓ $\frac{2}{2}$

(ii) ~~$V(\hat{P}) = V(N\bar{Y}) = N^2\cdot V(\bar{Y}) = N^2\cdot(1-\frac{n}{N})\frac{s^2}{n}$~~

~~$\Rightarrow V(\hat{P}) = N(N-n)\frac{s^2}{n} = 1000(1000-250)\times\frac{1.2}{250}$~~

~~$= 4320$~~

$\hat{V}(\bar{Y}) = (1-\frac{n}{N})\frac{s^2}{n} = (1-\frac{250}{1000})\times\frac{1.\cancel{2}}{250} = 0.00432$

$\frac{1.5}{3}$

$\Rightarrow$ 95% CI for population mean is

$3 \pm 1.96\times\sqrt{0.00432} = (2.87, 3.13)$ //

✓

p7

6. (a) $p_1=0.03,\ p_2=0.3,\ p_3=0.2$

$$\pi_1 = 1-(1-p_1)^3 = 0.087327$$

$$\pi_2 = 1-(1-p_2)^3 = 0.657$$

$$\pi_3 = 1-(1-p_3)^3 = 0.488$$

$$\hat{T}_\pi = \sum_{i=1}^{n} \frac{y_i}{\pi_i} = \frac{2}{0.087327} + \frac{7}{0.657} + \frac{10}{0.488} = 54.05$$

9/12

2/2

(b) $$\widehat{V(\hat{T}_\pi)} = \sum_{i=1}^{V}\left(\frac{1-\pi_i}{\pi_i^2}\right) y_i^2 + \sum_{i=1}^{V}\sum_{i\neq j}\left(\frac{\pi_{ij}}{\pi_i\pi_j}-1\right)\frac{y_iy_j}{\pi_{ij}}$$

$\pi_1 = 0.087327,\ \pi_2 = 0.657,\ \pi_3 = 0.488$

π_{12} = P(unit 1 included) + P(unit 2 included) − P(unit 1,2 included same time)

$= 0.087327 + 0.657 - (1 - P(\text{unit 1 and 2 not included}))$

$= 0.087327 + 0.657 - \{1 - [1-(0.03+0.3)]^3\}$

$= 0.087327 + 0.657 - 0.699237 = 0.04509$

P(unit 2, 3 included same time) $= 1-(1-(0.3+0.2))^3 = 0.875$

P(unit 1, 3 included same time) $= 1-(1-(0.03+0.2))^3 = 0.543467$

$\Rightarrow \pi_{23} = \pi_2 + \pi_3 - P(2,3 \text{ same time}) = 0.27$

$\Rightarrow \pi_{13} = \pi_1 + \pi_3 - P(1,3 \text{ same time}) = 0.03186$

4/7

$$\Rightarrow \widehat{Var}(\hat{T}_\pi) = \frac{1-\pi_1}{\pi_1^2}y_1^2 + \frac{1-\pi_2}{\pi_2^2}y_2^2 + \frac{1-\pi_3}{\pi_3^2}y_3^2 + \left(\frac{\pi_{12}}{\pi_1\pi_2}-1\right)\frac{y_1y_2}{\pi_{12}} + \left(\frac{\pi_{13}}{\pi_1\pi_3}-1\right)\frac{y_1y_3}{\pi_{13}} + \left(\frac{\pi_{23}}{\pi_2\pi_3}-1\right)\frac{y_2y_3}{\pi_{23}}$$

$$= \boxed{466.8084}$$

Calculator

1

256C5571 p8

6. (c)

$$\hat{\tau}_g = N\hat{\mu}_g = N \frac{\sum_{i=1}^{v} \frac{y_i}{\pi_i}}{\sum_{i=1}^{v} \frac{1}{\pi_i}} = 10 \times \frac{\frac{y_1}{\pi_1} + \frac{y_2}{\pi_2} + \frac{y_3}{\pi_3}}{\frac{1}{\pi_1} + \frac{1}{\pi_2} + \frac{1}{\pi_3}}$$

$$= \boxed{35.97860056}$$

3/3