SOME EXAM DATE
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EXAMINATION FOR THE DEGREES OF XXXX

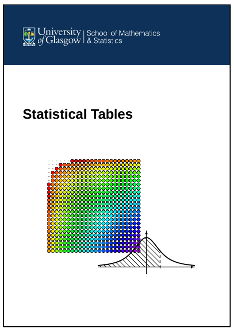
STATISTICS
Spatial Statistics 4H

This paper consists of 4 pages and contains 3 questions.
Candidates should attempt all questions.

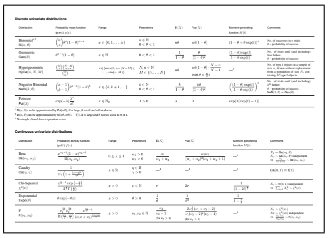
Question 1	20 marks
Question 2	20 marks
Question 3	20 marks
Total	60 marks

The following material is made available to you:

Statistical tables*



Probability formula sheet



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NOTE: Candidates should attempt all questions.

1. (a) An environmental scientist is studying yearly average nitrogen dioxide (NO_2) concentrations, which is a non-negative measure of air pollution. The scientist has made measurements at 100 locations $\mathbf{z} = (z(\boldsymbol{s}_1), \ldots, z(\boldsymbol{s}_{100}))$ across Glasgow, and their only goal in analysing these data is to produce a map of NO_2 predictions at 1 kilometre intervals across the city.

   i. The scientist is worried that as the concentrations are non-negative and skewed to the right that a Gaussian geostatistical model would be inappropriate. What advice would you give her to overcome this problem? **[2 MARKS]**

   ii. Describe briefly how the scientist would assess the data for the presence of residual spatial autocorrelation? **[2 MARKS]**

   iii. The scientist is considering two different geostatistical models for her data, and decides to choose the one that minimises the Bayesian Information Criterion (BIC). Why is this not a good criteria to use to select the best model given the goal of her analysis? **[2 MARKS]**

   iv. Briefly describe an alternative approach to how she should choose the best model given the goal of her analysis. Briefly describe how this approach works, and define 3 criteria she could use to assess the predictive fit of each model considered. **[5 MARKS]**

   (b) Consider the zero-mean geostatistical process $\{Z(\mathbf{s})|\mathbf{s} \in \mathcal{D}\}$ with a weakly stationary and isotropic covariance function given by

$$C(h) = \begin{cases} \xi^2(1+\rho h)\exp(-\rho h), & h > 0 \\ \nu^2 + \xi^2, & \text{h=0.} \end{cases}$$

   i. Compute the semi-variogram for the geostatistical process $\{Z(\mathbf{s})|\mathbf{s} \in \mathcal{D}\}$. **[3 MARKS]**

   ii. What are the nugget, sill and partial sill for this covariance model? Justify your answer. **[3 MARKS]**

   iii. Would the slightly altered covariance function defined below be a good model for spatial data for $\phi > 0$? Justify your answer. **[3 MARKS]**

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$$C(h)\begin{cases} \xi^2(1+\rho h)\exp(-\rho h)+\phi, & h>0 \\ \nu^2+\xi^2+\phi, & \text{h=0.} \end{cases}$$

2. (a) Consider an areal unit process $\mathbf{Z} = (Z(\mathbf{s}_1), \ldots, Z(\mathbf{s}_n))$ relating to n unevenly sized areal units with centroids (central points) $(\mathbf{s}_1, \ldots, \mathbf{s}_n)$. One could model these data as a geostatistical process, where the central points $(\mathbf{s}_1, \ldots, \mathbf{s}_n)$ represent the spatial locations of the areal units. This would allow a geostatistical covariance function to be specified for this process, such as the exponential model given by $\text{Covariance}(Z(\mathbf{s}_i), Z(\mathbf{s}_j)) = \sigma^2 \exp(-||\mathbf{s}_i - \mathbf{s}_j||/\phi)$ where $||.||$ denotes Euclidean distance. Is this likely to be a good representation of spatial correlation for the areal unit process described above? Justify your answer. **[3 MARKS]**

   (b) Suppose the areal unit process $\mathbf{Z} = (Z(\mathbf{s}_1), \ldots, Z(\mathbf{s}_n))$ follows a zero-mean multivariate Gaussian distribution. A geostatistical style model would be parameterised via the covariance $\mathbf{\Sigma} = \text{Covariance}(\mathbf{Z})$, while an areal unit style model would be parameterised via the precision (inverse of the covariance) $\boldsymbol{Q} = \text{Precision}(\mathbf{Z})$. Which representation is faster computationally for evaluating the multivariate Gaussian data likelihood? Justify your answer. **[2 MARKS]**

   (c) Consider a simple 1 dimensional areal unit process with 4 regions ordered as $[A|B|C|D]$, with a corresponding neighbourhood matrix

$$\mathbf{W} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

   so that the only neighbour pairs are (A, B), (B, C) and (C, D). Then suppose that $Z(A) = 6$, $Z(B) = 5$, $Z(C) = 4$ and $Z(D) = 3$. Compute Geary's C statistic and describe what it tells you about the presence/absence of spatial correlation in these data. **[4 MARKS]**

   (d) Consider an areal unit process $\mathbf{Z} = (Z(\mathbf{s}_1), \ldots, Z(\mathbf{s}_n))$ with a corresponding binary $n \times n$ neighbourhood matrix $\mathbf{W}$, where $w_{ij} = 1$ if areas (i, j) are spatial neighbours (share a common border) and $w_{ij} = 0$ otherwise. A conditional autoregressive (CAR) model for $\mathbf{Z}$ has the general form $\mathbf{Z} \sim \text{N}(\mathbf{0}, \tau^2\mathbf{Q}(\mathbf{W})^{-1})$, where τ^2 is a variance parameter and $\mathbf{Q}(\mathbf{W})$ is a precision matrix based on the neighbourhood matrix $\mathbf{W}$.

      i. Define $\mathbf{Q}(\mathbf{W})$ mathematically for the *intrinsic CAR (ICAR)* model and write down formulae for: (i) the diagonal element Q_{ii}; and (ii) the off-diagonal element Q_{ij} where $i \neq j$. **[3 MARKS]**

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ii. From part i. above what does this tell you about the partial (conditional) correlations between $(Z(\mathbf{s}_i), Z(\mathbf{s}_j))$ imposed by the model if: (a) $w_{ij} = 1$; and (b) $w_{ij} = 0$? **[3 MARKS]**

iii. Describe two limitations of the *intrinsic CAR* model. **[2 MARKS]**

iv. Briefly describe a model that overcomes the limitations outlined in iii. above and write down its full conditional distribution $Z(\mathbf{s}_i)|\mathbf{Z}(-\mathbf{s}_i)$. **[3 MARKS]**

3. (a) Consider a spatial point process $Z = \{Z(A)|A \subset D\}$ defined on a spatial domain D.

i. Write down the modelling assumptions for an inhomogeneous Poisson process (IPP) with first order intensity function $\lambda(\mathbf{s})$. **[2 MARKS]**

ii. Write down the general form for a log-linear parametric model for $\lambda(\mathbf{s})$ ensuring you define all the quantities you specify. In practice, name one drawback of fitting such as model. **[3 MARKS]**

(b) Consider a spatial point process $Z = \{Z(A)|A \subset D\}$, where the domain D is a unit square whose four corners have coordinates $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$. The first order intensity function for this process is given by $\lambda(\mathbf{s}) = s_1 + s_2 - s_1 s_2$, where the location $\mathbf{s} = (s_1, s_2)$. Thus, across the domain D both $s_1, s_2 \in [0, 1]$.

i. Compute the first order intensity function at (a) $\mathbf{s} = (0, 0.5)$, (b) $\mathbf{s} = (0.5, 0)$, and (c) $\mathbf{s} = (0.5, 0.5)$. Hence or otherwise briefly describe the spatial pattern in the first order intensity function across the domain D. **[4 MARKS]**

ii. Consider a region $A \subset D$, write down the formula for the expected number of points that occurred in A, $\mathbb{E}[Z(A)]$. **[2 MARKS]**

iii. Let A denote the rectangle in the domain D defined by all points $\mathbf{s} = (s_1, s_2)$ such that $s_1 \in [0, 1]$ and $s_2 \in [0.4, 0.5]$. That is, the four corners of A are $(0, 0.4)$, $(0, 0.5)$, $(1, 0.4)$, $(1, 0.5)$. Compute the expected number of points that occurred in A. **[3 MARKS]**

iv. Now suppose that the second order intensity function for this process is $\lambda_2(\mathbf{s}, \mathbf{t}) = 1$ for all $(\mathbf{s}, \mathbf{t}) \in D$. Compute the pair correlation function $\rho(\mathbf{s}, \mathbf{t})$ for points $(\mathbf{s}, \mathbf{t}) \in D$. **[3 MARKS]**

v. Are there any points $(\mathbf{s}, \mathbf{t}) \in D$ at which the pair correlation function computed in iv. is not a real number? If so what aspect of either the first or second order intensity functions is causing this problem. **[3 MARKS]**

Total: 60 MARKS

END OF QUESTION PAPER.