2501551

P1

1. (a) $\because E, F$ are independant

$\Rightarrow P(E)\cdot P(F) = P(F\cap E)$

By disjoint axiom, $F = (F\cap E)\cup(F\cap E')$

$\Rightarrow P(F) = P(F\cap E) + P(F\cap E')$

$= P(F)P(E) + P(F\cap E')$

$\Rightarrow P(F\cap E') = P(F) - P(F)P(E)$

$= (1-P(E))\cdot P(F)$

$= P(F)P(E')$

$\Rightarrow$ E' and F are independent

Similarly, by symmetry, F' and E are independant

$\Rightarrow P(E'\cap F) = P(E')\cdot P(F)$

$P(F'\cap E) = P(F')\cdot P(E)$

$\Rightarrow P(F'\cap E') = 1 - P(E\cap F')$

$= 1 - P(F') - P(E)$

$= P(F')\cdot P(E')$

$\Rightarrow$ E' and F' are independent

By symetry, E' and G' are independent

G' and F' are independent

$\Leftrightarrow$ E', F', G' are pair-wise independent

Q.E.D. $\square$

$\wedge$ 1/3

(b) let A = "the outcome of coin is head"

$\because$ coins are randomly chosen.

$\Rightarrow P(C_1) = P(C_2) = P(C_3) = P(C_4) = \frac{1}{4}$

$\Rightarrow P(A|C_1) = \theta_1 = 0.47$

$P(A|C_2) = \theta_2 = 0.51$

$P(A|C_3) = \theta_3 = 0.49$

$P(A|C_4) = \theta_4 = 0.53$

$\Rightarrow P(A) = \sum_{i=1}^{4} P(A|C_i)\cdot P(C_i)$

$= 0.47\times\frac{1}{4} + 0.51\times\frac{1}{4} + 0.49\times\frac{1}{4} + 0.53\times\frac{1}{4}$

$= 0.5$ ✓ 3/3

the probability of randomly chose one of the coins and toss it, the outcome is heads is $\boxed{0.5}$ ✓

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P2

1. (c)

By Bayes Theorm,

$$P(C_1|A) = \frac{P(C_1)\cdot P(A|C_1)}{P(A)}$$

$$= \frac{0.25 \times 0.47}{0.5}$$

$$= \frac{47}{200} = \boxed{0.235}$$

If outcome is heads, the probability of coin was C_1 is 0.235

1/1

(d) By part (b)

Let $P(A) = \frac{1}{4}\theta_1 + \frac{1}{4}\times 0.51 + \frac{1}{4}\times 0.48 + \frac{1}{4}\times 0.53 > 0.55$

$$\Rightarrow \frac{1}{4}\theta_1 > 0.1675$$

$$\theta_1 > \boxed{0.67}$$

$\Rightarrow$ I would have to change θ_1, s.t. $\theta_1 \in (0.67, 1)$ to ensure the probability of outcome 'heads' become more than 0.55 //

3/3

Q1: 8/10

2561551

P3

2. (a)

$\int_0^c x\,dx + \int_c^2 (-\frac{1}{2}x^2 + x)\,dx = 1$

$[\frac{1}{2}x^2]_0^c + [-\frac{1}{6}x^3]_c^2 + [\frac{1}{2}x^2]_c^2 = 1$

$\cancel{\frac{1}{2}c^2} + \frac{c^3-8}{6} + 2 - \cancel{\frac{c^2}{2}} = 1$

$\frac{c^3-8}{6} = -1$

$c^3 = 2$

$c = \sqrt[3]{2}$

✓ 4/4

(b) $E(X) = \int_0^c x \cdot x\,dx + \int_c^2 x(-\frac{1}{2}x^2 + x)\,dx$

$= [\frac{x^3}{3}]_0^c + [-\frac{x^4}{8}]_c^2 + [\frac{x^3}{3}]_c^2$

$= \frac{c^3}{3} + \frac{c^4}{8} - \frac{2^4}{8} + \frac{2^3}{3} - \frac{c^3}{3}$

$= \frac{2}{3} + \frac{c^4}{8} = \boxed{\frac{2}{3} + \frac{2^{\frac{4}{3}}}{8}}$

$E(X^2) = \int_0^c x^2 \cdot x\,dx + \int_c^2 x^2(-\frac{1}{2}x^2 + x)\,dx$

$= [\frac{x^4}{4}]_0^c + (-\frac{1}{2} \times [\frac{x^5}{5}]_c^2) + [\frac{x^4}{4}]_c^2$

$= \frac{4}{5} + \frac{1}{10}c^5$

$= \boxed{\frac{4}{5} + \frac{1}{10} \times 2^{\frac{5}{3}}}$

$Var(X) = E(X^2) - (E(X))^2$

$= \frac{4}{5} + \frac{1}{10}c^5 - (\frac{2}{3} + \frac{c^4}{8})^2$

$= -\frac{c^4}{6} + \frac{c^5}{10} - \frac{c^8}{64} + \frac{16}{45}$

$= -\frac{2^{\frac{4}{3}}}{6} + \frac{2^{\frac{5}{3}}}{10} - \frac{2^{\frac{8}{3}}}{64} + \frac{16}{45}$ //

✓ 3/3

2561551

P4

2.(C)

$$F_X(x) = \begin{cases} \frac{1}{2}x^2, & 0 \leq x \leq \sqrt[3]{2} \\ -\frac{1}{6}x^3 + \frac{x^2}{2}, & \sqrt[3]{2} < x \leq 2 \\ 0, & \text{elsewhere} \end{cases}$$

X

$\because \int_0^{2^{\frac{1}{3}}} x\,dx = \left[\frac{x^2}{2}\right]_0^{2^{\frac{1}{3}}} = 2^{-\frac{1}{3}} \approx \underline{0.79 > 0.75}$

$\Rightarrow \int_0^{Q_{25}} x\,dx = 0.25$

$\left[\frac{1}{2}x^2\right]_0^{Q_{25}} = 0.25$

$\boxed{Q_{25} = \sqrt{\frac{1}{2}}}$

Let $\int_0^{Q_{50}} x\,dx = 0.5$

$\left[\frac{1}{2}x^2\right]_0^{Q_{50}} = 0.5$

$\Rightarrow \boxed{Q_{50} = 1}$

Let $\int_0^{Q_{75}} x\,dx = 0.75$

$\boxed{Q_{75} = \sqrt{\frac{3}{2}}}$

25th, 50th, 75th percentile are $\sqrt{0.5}$, 1, $\sqrt{1.5}$ //

✓ 3/5

Q2: 10/12

2561551

~~P5~~

3. (a) $\because X \sim Exp(\theta)$

$$\Rightarrow f_X(x) = \begin{cases} \theta \exp(-\theta x), & x>0 \\ 0, & \text{others} \end{cases}$$

$$\Rightarrow M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx} \cdot \theta \cdot \exp(-\theta x)\,dx$$

$$= \int_0^\infty \theta \cdot \exp -(\theta - t) x\,dx$$

$$\frac{\theta}{t-\theta} \cdot \left[\exp -(\theta-t)x\right]_0^\infty$$

1° if $\theta - t < 0$, i.e. $\theta < t \Rightarrow -(\theta - t) > 0$

$-x(\theta - t) \to \infty \; (x \to \infty)$

$\Rightarrow M_X(t)$ does not exist

2° $\theta - t > 0 \Leftrightarrow \theta > t$

$$M_X(t) = \frac{\theta}{t-\theta}(0-1)$$

$$= \frac{\theta}{\theta - t} \quad (t<\theta) \; //$$

✓ 4/4

(b) $E(X) = M'_X(0)$, $E(X^2) = M''_X(0)$

$$M'_X(t) = \frac{\theta}{(t-\theta)^2}$$

$$M'_X(0) = \frac{\theta}{\theta^2} = \boxed{\frac{1}{\theta}} = E(X)$$

$$M''_X(t) = \frac{2\theta}{(\theta - t)^3}$$

$$M''_X(0) = \frac{2\theta}{\theta^3} = \boxed{\frac{2}{\theta^2}} = E(X^2)$$

$$Var(X) = E(X^2) - (E(X))^2 = \frac{2}{\theta^2} - \frac{1}{\theta} = \boxed{\frac{1}{\theta^2}} \; //$$

✓ 3/3

2561551

P8

3. (c)

$\because f_X(x) = \lambda e^{-\lambda x} \ (\lambda > 0)$

$F_X(x) = \int \theta e^{-\lambda x} dx = 1 - e^{-\theta x} \ (\theta > 0)$

$\because Y = (x+1)^2 \Leftrightarrow x = -1 \pm \sqrt{Y}$

$\Leftrightarrow F_Y(y) = 1 - e^{-\theta(-1 \pm \sqrt{Y})}$

$= 1 - e^{\theta \mp \sqrt{Y}}$

$\because \theta > 0 \Rightarrow$ since we want $e^{\theta \mp \sqrt{Y}}$ converge

$\Rightarrow F_Y(y)$ can only equal to $1 - e^{\theta - \sqrt{Y}}$, satisfy $\theta - \sqrt{Y} < 0$

$Y > \theta^2 \Leftrightarrow Y \in [\theta^2, +\infty)$

$\Rightarrow F_Y(y) = 1 - e^{\theta - \sqrt{Y}}$

$f_Y(y) = F'_Y(y) = \frac{e^{\theta - \sqrt{y}}}{2\sqrt{y}} \ (y \in [\theta^2, +\infty))$ //

X 0/3

Q3: 7/10

2561551

P7

4. (a) $\because X_1 \sim N(1, 0.2), X_2 \sim N(2, 0.8)$

$\Rightarrow \mu = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$Cov(X_1, X_2) = Corr(X_1, X_2) \cdot \sqrt{Var(X_1) \cdot Var(X_2)}$

3/3

$= 0.5 \times \sqrt{0.2 \times 0.8}$

$= 0.2$

$\Rightarrow \Sigma = \begin{pmatrix} 0.2 & 0.2 \\ 0.2 & 0.8 \end{pmatrix}$

(b) Let $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, where $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\Rightarrow Y \sim N_2(\mu_2, \Sigma_2)$

$\Rightarrow \mu_2 = A\mu + B = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 10 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 13 \end{pmatrix}$

$\Sigma_2 = A \cdot \Sigma A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 0.2 & 0.2 \\ 0.2 & 0.8 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

4/4

$= \begin{pmatrix} 0.8 & 2.6 \\ 1.2 & 3.6 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$= \begin{pmatrix} 8.6 & 12 \\ 12 & 16.8 \end{pmatrix}$

$Y \sim N_2\left(\begin{bmatrix} 8 \\ 13 \end{bmatrix}, \begin{bmatrix} 8.6 & 12 \\ 12 & 16.8 \end{bmatrix}\right)$ //

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P8

4.(c) $corr(Y_1, Y_2) = \frac{cov(Y_1, Y_2)}{\sqrt{Var(Y_1) Var(Y_2)}} = \frac{12}{\sqrt{8.6 \times 16.8}} \approx 0.998$ ✓

1/1

(d) $\because Y_1 \sim N(8, 8.6)$

1/2 $Z = \frac{Y_1 - E(Y_1)}{\sqrt{6}} \overset{?}{=} \frac{Y_1 - 8}{\sqrt{8.6}} \sim N(0,1)$

$\frac{1}{\sqrt{8.6}} Y_1 + \zeta = \frac{Y_1 - 8}{\sqrt{8.6}}$

$\cancel{Y_1} + \sqrt{8.6}\zeta = \cancel{Y_1} - 8$

$\boxed{\zeta = \frac{-8}{\sqrt{8.6}}}$ //

5.(a) $\because X_i \sim Geo(\theta)$

2/2

$\Rightarrow f_{X_i}(x_i) = \theta^{x_i - 1}(1-\theta)$, $E(X_i) = \frac{1}{1-\theta}$, $Var(X_i) = \frac{\theta}{(1-\theta)^2}$ (by formula sheet)

$E(Y) = \sum_{i=1}^{N} E(X_i) = N \cdot \frac{1}{1-\theta} = \boxed{\frac{N}{1-\theta}}$ ✓

Since $X_1, \cdots X_N$ are independent

$\Rightarrow Var(Y) = \sum_{i=1}^{N} Var(X_i) = \boxed{\frac{N\theta}{(1-\theta)^2}}$ ✓

(b) by part (a), $E(Y|N=n) = \boxed{\frac{N}{1-\theta}}$ ✗ $\because N \sim NeBi(k, \theta)$

$Var(Y|N=n) = \boxed{\frac{N\theta}{(1-\theta)^2}}$ ✗ $\Rightarrow E(N) = \frac{k}{1-\theta}$, $Var(N) = \frac{k\theta}{(1-\theta)^2}$ (by formula sheet)

$E(Y) = E(E(Y|N\cancel{=n})) = E\left(\frac{N}{1-\theta}\right)$

$= \frac{1}{1-\theta} \cdot E(N)$

2/4

$= \frac{1}{1-\theta} \cdot \frac{k}{1-\theta}$

$= \boxed{\frac{k}{(1-\theta)^2}}$ ✓

(see next page)

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~~P9~~ P9

$$Var(Y) = E(Var(Y|N)) + Var(E(Y|N))$$
$$= E\left(\frac{N\theta}{(1-\theta)^2}\right) + Var\left(\frac{N}{1-\theta}\right)$$
$$= E(N) \cdot \frac{\theta}{(1-\theta)^2} + \frac{1}{(1-\theta)^2} \cdot Var(N)$$
$$= \frac{k}{1-\theta} \cdot \frac{\theta}{(1-\theta)^2} + \frac{1}{(1-\theta)^2} \cdot \frac{k\theta}{(1-\theta)^2}$$ ✓
$$= \frac{k\theta}{(1-\theta)^3} + \frac{k\theta}{(1-\theta)^4}$$
$$= \frac{(1-\theta)k\theta + k\theta}{(1-\theta)^4} = \boxed{\frac{2k\theta - k\theta^2}{(1-\theta)^4}}$$ //

6. (a)

Let X be the score of dice once, X_i ($i=1,2,\cdots 40000$ be ith time you die a dice)

X	1	2	3	4	5	6
$P(X)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$\Rightarrow E(X) = \frac{1}{6}(1+2+3+4+5+6) = \frac{7}{2}$$ ✓
$$E(X^2) = \frac{1}{6}(1^2+2^2+3^2+4^2+5^2+6^2) = \frac{91}{6}$$
$$Var(X) = E(X^2) - (E(X))^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$ ✓

By CLT, if $X_1 \cdots X_n$ i.i.d, w/ $E(X_i)=\mu$, $Var(X_i)=\sigma^2$, for sufficient large n,

$$Z_n = \frac{\sum_{1}^{n} X_i - n\mu}{\sqrt{n\sigma^2}} \sim N(0,1)$$

i.e. $\sum_{i=1}^{N} X_i \sim N(n\mu, n\sigma^2)$

$$\Rightarrow \text{Let } Y = \sum_{i=1}^{40000} X_i \sim N\left(40000 \times \frac{7}{2}, 40000 \times \frac{35}{12}\right) = N\left(140000, \frac{350000}{3}\right)$$

$$P(Y \geq 141000) = P\left(Y > 141000 - \frac{1}{2}\right) \text{ (continuous correction)}$$
$$= P\left(Z > \frac{41000 - \frac{1}{2} - 140000}{\sqrt{\frac{350000}{3}}}\right)$$
$$= 1 - P(Z \leq 2.926)$$ ✓
$$= 1 - \Phi(2.926)$$
$$= 1 - 0.9983 = \boxed{0.0017}$$

5/5

2561551

P10

6. (b) let X_i the random variable of whether ith solar cell fails to work.

let $Y = \sum_{i=1}^{N} X_i$

$Y = \sum_{i=1}^{N} X_i \sim Bi(N, 0.2)$, where $N \geq 20$, $N\theta \geq 5$ and $N(1-\theta) \geq 5$

By CLT, $\Rightarrow Y = \sum_{i=1}^{N} X_i \sim N(N\theta, N\theta(1-\theta)) = N(0.8N, 0.16N)$

Let $P(Y \geq 100) = P(Y > 100 - \frac{1}{2})$ (continuous correction)

Let $P(Y \geq 100) = P(Z > \frac{99.5 - 0.8N}{\sqrt{0.16N}}) \leq 0.005$

2/7

$\Rightarrow 1 - P(Z \leq \frac{99.5 - 0.8N}{\sqrt{0.16N}}) \leq 0.005$

$\Phi\left(\frac{99.5 - 0.8N}{\sqrt{0.16N}}\right) \leq 0.995$

$\Leftrightarrow \frac{99.5 - 0.8N}{\sqrt{0.16N}} = 2.575$

$N = 10$ ✗