University of Glasgow

Thursday, 30th April 2015
2.00 pm – 3.30 pm

EXAMINATION FOR THE DEGREES OF M.A., M.SCI AND B.SC. (SCIENCE)

Probability

Hand calculators with simple basic functions (log, exp, square root, etc.) may be used in examinations. No calculator which can store or display text or graphics may be used, and any student found using such will be reported to the Clerk of Senate".

Note: **Candidates should attempt THREE out of the FOUR questions. If more than three questions are attempted please indicate which questions should be marked; otherwise, the first three questions will be graded.**

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The Standard Normal Distribution Function

When Z is a N(0,1) random variable, this table gives $\Phi(z) = P(Z \leq z)$ for values of z from 0.00 to 3.67 in steps of 0.01.

When $z < 0$, $\Phi(z)$ can be found from this table using the relationship $\Phi(-z) = 1-\Phi(z)$.

z	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9983	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9999		

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1. The continuous random variables *X* and *Y* have the joint probability density function

$$f(x,y)=\begin{cases}24xy, & 0\le x\le 1,\ 0\le y\le 1-x,\\ 0, & \text{otherwise.}\end{cases}$$

(i) Without carrying out any calculations, explain why *X* and *Y* cannot be independent.

[2 MARKS]

(ii) Show that, for non-negative integers, *r* and *s*,

$$\mathrm{E}(X^rY^s)=\frac{24(r+1)!(s+1)!}{(r+s+4)!}.$$

Using this formula, find the marginal expected values and variances of *X* and *Y*, and the covariance and correlation between *X* and *Y*.

[10 MARKS]

(iii) For any value w, such that $0 \le w \le 1$, find $\mathrm{P}(X + Y \le w)$. Hence write down the cumulative distribution function and probability density function of the random variable $W = X + Y$.

[4 MARKS]

(iv) By recognising *W* as a Beta random variable, write down E(*W*) and var(*W*). Write down expressions that would allow E(*W*) and var(*W*) to be found from the results for *X* and *Y* that were obtained in part *(ii)*.

[4 MARKS]

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2. **(a)** Suppose that the discrete random variables X and Y are independent with $X \sim \text{Poi}(\theta)$ and $Y \sim \text{Poi}(\lambda)$, where $\theta > 0$ and $\lambda > 0$. Let $Z = X + Y$.

(i) Explain why, for any z in R_Z,

$$P(Z = z) = \sum_{x=0}^{z} P(X = x)P(Y = z - x).$$

Using this result, show that the random variable Z also has a Poisson distribution and state its parameter.

[8 MARKS]

(ii) Suppose that Z is found to have the value z, for some z in R_Z. Obtain the conditional probability mass function, $P(X = x \mid Z = z)$. Identify this conditional probability distribution and state its parameters.

[6 MARKS]

(b) Suppose that the continuous random variable T has the Exponential distribution with expected value $\lambda > 0$. Given that $T = t$, for any possible value t, the discrete random variable S has the Poisson distribution with expected value t. Find $E(S)$ and $\text{var}(S)$.

[6 MARKS]

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3. **(a)** Suppose that the continuous random vector, $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \mathrm{N}_2\left(\begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 25 & -15 \\ -15 & 36 \end{pmatrix}\right)$.

(i) Find the correlation between X_1 and X_2. **[2 MARKS]**

(ii) Let $Y_1 = 2X_1 - 3X_2$. Find $\mathrm{E}(Y_1)$ and $\mathrm{var}(Y_1)$. **[3 MARKS]**

(iii) Let $Y_2 = X_1 + 4X_2$. Find the covariance and correlation between Y_1 and Y_2.
[4 MARKS]

(b) In a certain town there are three taxi companies that operate (respectively) 8 black taxis, 6 red taxis and 6 white taxis. The taxi rank at the railway station has space for 5 taxis and, the next time I pass it when it is full, I intend to record the colours of the 5 taxis sitting there. Let X be the number of black taxis I record and Y the number of red taxis I record.

(i) Briefly justify the following joint probability mass function for X and Y. (You may assume that every taxi in the town is equally likely to be at the station.)

$$\mathrm{P}(X = x, Y = y) = \binom{8}{x}\binom{6}{y}\binom{6}{5-x-y} \bigg/ \binom{20}{5},$$
$$x = 0,1,\ldots,5;\; y = 0,1,\ldots,5-x.$$

[4 MARKS]

(ii) Write down the marginal probability mass function of X. Hence find the probability that at least one of the taxis is black.
[4 MARKS]

(iii) Find the conditional probability mass function of Y given that $X = x$ (for $x = 0, 1, \ldots, 5$). Use it to find the probability that, given exactly 3 of the taxis are black, the remaining 2 taxis are both white.
[3 MARKS]

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4. The random variables X_0, X_1, X_2, … are the prices (in £), at consecutive time points 0, 1, 2, …, of a particular share in a company listed on the London Stock Exchange. Time point 0 is today and X_0 is observed to take the value 1. A simple model states that, at every future time point, the share price will either increase from its previous value by a factor of $(1 + c)$ or decrease by a factor of $(1 - c)$, where $0 < c < 1$. So, for some probability θ $(0 < \theta < 1)$,

$$P(X_{n+1} = x_n(1+c) \mid X_n = x_n) = \theta$$

and

$$P(X_{n+1} = x_n(1-c) \mid X_n = x_n) = 1-\theta.$$

(i) Write down the probability mass function of X_2. Hence find the probability mass function of $Y_2 = \log_e X_2$.

[3 MARKS]

(ii) Let the random variable $Y_n = \log_e(X_n)$, $n = 1, 2, \ldots$. Show that Y_n can be expressed in the form $a + bW_n$, where W_n is a Binomial random variable. State the constants a and b, and the parameters of the distribution of W_n. Deduce $E(Y_n)$ and $\mathrm{var}(Y_n)$.

[8 MARKS]

(iii) Referring to the Central Limit Theorem, explain why the random variable W_n can be approximated by a Normal distribution when n is large enough. Deduce that Y_n is also approximately Normally distributed, for large enough values of n.

[4 MARKS]

(iv) If the random variable U has a $N(\mu, \sigma^2)$ distribution, then it has moment-generating function

$$M_U(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right).$$

Use this moment-generating function to obtain the expected value and variance of $\exp(U)$ in terms of μ and σ^2. Use this general result to deduce the expected value and variance of the random variable X_n.

[5 MARKS]

/END OF QUESTION PAPER.