PROBABILITY

Some Useful Mathematical Results

1. Some Important Series

Working with standard discrete distributions requires knowledge and use of some standard results about series.

Distribution	Result
Geometric	sum of Geometric series
Negative Binomial	generalisation of sum of Geometric series
Binomial	Binomial Theorem
Hypergeometric	Hypergeometric Identity
Poisson	Maclaurin expansion of the exponential

These results have been listed below for convenience.

Consider the following **Arithmetic Series**

$$a + (a + r) + (a + 2r) + \ldots$$

The sum of the first n terms of this series is

$$S_n = na + \tfrac{1}{2}n(n-1)r$$

Consider the following **Geometric Series**

$$a + ar + ar^2 + \ldots$$

The sum of the first n terms of this Geometric Series is

$$S_n = a\frac{1-r^n}{1-r} \quad (r \neq 1)$$

Whenever -1 < r < 1, the sum to infinity of the above Geometric Series converges to the limiting value

$$S_\infty = \frac{a}{1-r}$$

The following result is a generalisation of the sum to infinity of a Geometric Series. Whenever -1 < r < 1,

$$\sum_{x=m}^{\infty} \binom{x}{m} r^{x-m} = \frac{1}{(1-r)^{m+1}} \qquad (m = 0, 1, \ldots)$$

The **Binomial Theorem** can be stated as follows. Let n be a non-negative integer, and let a and b be any real numbers. Then

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

The following result is sometimes known as the **Hypergeometric Identity**. Let N, M and n be non-negative integers, with $M \leq N$ and $n \leq N$. Then

$$\sum_{i=0}^{n} \binom{M}{i} \binom{N - M}{n - i} = \binom{N}{n}$$

The **Maclaurin expansion** of the exponential function can be written as follows.

$$e^u = 1 + \frac{u}{1!} + \frac{u^2}{2!} + \frac{u^3}{3!} + \ldots = \sum_{i=0}^{\infty} \frac{u^i}{i!}$$

2. The Gamma and Beta Functions

The **Gamma Function**, $\Gamma(\alpha)$, which is a generalisation of the factorial function, is defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx, \quad \alpha > 0$$

Results **(i)** $\Gamma(\alpha) = (\alpha - 1)\,\Gamma(\alpha - 1)$

(ii) $\Gamma(n) = (n - 1)!, \quad n = 1, 2, \ldots$

(iii) $\Gamma(½) = \sqrt{\pi}$

The **Beta Function**, $\mathbb{B}(\alpha,\beta)$, is defined as follows:

$$\mathbb{B}(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx, \ \alpha > 0, \ \beta > 0$$

Results **(iv)** $\mathbb{B}(\beta,\alpha) = \mathbb{B}(\alpha,\beta)$

(v) $\mathbb{B}(\alpha,\beta) = \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

Proofs (not examinable)

(i) Using integration by parts,

$$\begin{aligned}\Gamma(\alpha) &= \int_0^\infty x^{\alpha-1}e^{-x}dx \\ &= \left[(\alpha-1)x^{\alpha-2}e^{-x}\right]_0^\infty + (\alpha-1)\int_0^\infty x^{\alpha-2}e^{-x}dx \\ &= 0 + (\alpha-1)\Gamma(\alpha-1)\end{aligned}$$

(ii) $$\Gamma(1) = \int_0^\infty e^{-x}dx = \left[-e^{-x}\right]_0^\infty = 1$$

The general result now follows by repeated use of result (i).

(iii) The trick required to evaluate this integral is to first find its square by double integration.

$$\left\{\Gamma\left(\tfrac{1}{2}\right)\right\}^2 = \int_0^\infty \frac{1}{\sqrt{x}}e^{-x}\,dx\int_0^\infty \frac{1}{\sqrt{y}}e^{-y}\,dy = \int_0^\infty\int_0^\infty \frac{1}{\sqrt{xy}}e^{-(x+y)}dx\,dy$$

Now make the following change of variables (to polar co-ordinates):

$x = R^2\cos^2\theta \quad y = R^2\sin^2\theta$

This has Jacobian $4R^3\cos\theta\sin\theta$, so that the integral becomes

$$\int_0^{\pi/2}\int_0^\infty 4R\exp\left(-R^2\right)dR\,d\theta = 2\pi\int_0^\infty R\exp\left(-R^2\right)dR$$

Make the further change of variable $u = R^2$, so that $du = 2R\,dR$, to obtain

$$2\pi\int_0^\infty \frac{1}{2}\exp(-u)\,du = \pi.$$

Finally, taking the square root, it follows that $\Gamma(½) = \sqrt{\pi}$.

(iv) Make the substitution $u = 1 - x$ in $\mathbb{B}(\beta,\alpha)$ to change it to $\mathbb{B}(\alpha,\beta)$.

(v)

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}e^{-(x+y)}\,dx\,dy$$

Make the change of variables $u = x + y$, $v = x$.

This transformation has Jacobian = 1. Noting the restriction $u > v > 0$, the double integral becomes

$$\int_0^\infty\int_0^u v^{\alpha-1}(u-v)^{\beta-1}e^{-u}\,dv\,du$$

$$= \int_0^\infty\left\{\int_0^u\left(\frac{v}{u}\right)^{\alpha-1}\left(1-\frac{v}{u}\right)^{\beta-1}dv\right\}u^{\alpha+\beta-2}e^{-u}\,du$$

$$= \int_0^\infty\left\{\frac{1}{u}\,\mathbb{B}(\alpha,\beta)\right\}u^{\alpha+\beta-2}e^{-u}\,du$$

$$= \mathbb{B}(\alpha,\beta)\Gamma(\alpha+\beta)$$

3. Moments of the Gamma Distribution

The random variable X is said to follow a **Gamma distribution**, $X \sim \text{Ga}(\alpha, \theta)$, for real constants $\alpha > 0$ and $\theta > 0$, if it has probability density function:

$$f_X(x) = \frac{\theta^{\alpha} x^{\alpha-1} e^{-\theta x}}{\Gamma(\alpha)}, \quad x > 0$$

Result $\quad \mathrm{E}(X) = \frac{\alpha}{\theta}, \quad \mathrm{var}(X) = \frac{\alpha}{\theta^2}$

Proof

For $r = 1, 2, \ldots$,

$$\begin{aligned}
\mathrm{E}(X^r) &= \int_0^{\infty} x^r \frac{\theta^{\alpha} x^{\alpha-1} e^{-\theta x}}{\Gamma(\alpha)} dx \\
&= \frac{\theta^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} x^{\alpha+r-1} e^{-\theta x} dx \\
&= \frac{\theta^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} \left(\frac{u}{\theta}\right)^{\alpha+r-1} e^{-u} \frac{du}{\theta} \qquad [\textit{where } u = \theta x] \\
&= \frac{1}{\theta^r} \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)}
\end{aligned}$$

In particular, then,

$$\mathrm{E}(X) = \frac{1}{\theta} . \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} = \frac{1}{\theta} . \frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha)} = \frac{\alpha}{\theta}$$

$$\mathrm{E}\left(X^2\right) = \frac{1}{\theta^2} . \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} = \frac{1}{\theta^2} . \frac{(\alpha+1)\alpha\Gamma(\alpha)}{\Gamma(\alpha)} = \frac{\alpha(\alpha+1)}{\theta^2}$$

$$\mathrm{var}(X) = \frac{\alpha(\alpha+1)}{\theta^2} - \left(\frac{\alpha}{\theta}\right)^2 = \frac{\alpha}{\theta^2}$$

4. Moments of the Beta Distribution

The random variable X is said to follow a **Beta distribution**, $X \sim \mathrm{Be}(\alpha, \beta)$, for real constants $\alpha > 0$ and $\beta > 0$, if it has probability density function:

$$f_X(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\mathbb{B}(\alpha,\beta)}, \qquad 0 < x < 1$$

Result $\quad \mathrm{E}(X) = \dfrac{\alpha}{\alpha+\beta}, \qquad \mathrm{var}(X) = \dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Proof

$$\begin{aligned}
\mathrm{E}(X) &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 x.x^{\alpha-1}(1-x)^{\beta-1}\,dx \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\,\mathbb{B}(\alpha+1,\beta) \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}.\frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} \\
&= \frac{\Gamma(\alpha+\beta)}{(\alpha+\beta)\Gamma(\alpha+\beta)}.\frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha)} \\
&= \frac{\alpha}{(\alpha+\beta)}
\end{aligned}$$

Similarly, $\quad \mathrm{E}(X^2) = \dfrac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$

So

$$\begin{aligned}
\mathrm{var}(X) &= \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} - \left(\frac{\alpha}{(\alpha+\beta)}\right)^2 \\
&= \frac{\alpha(\alpha+1)(\alpha+\beta) - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)} \\
&= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}
\end{aligned}$$