Machine Learning

CMPT 726

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Linear Algebra and Calculus Review

Vectors

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$$

Addition

$$\vec{x} + \vec{y} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

Scaling

$$c\vec{x} = \begin{pmatrix} cx_1 \\ cx_2 \\ cx_3 \\ cx_4 \end{pmatrix}$$

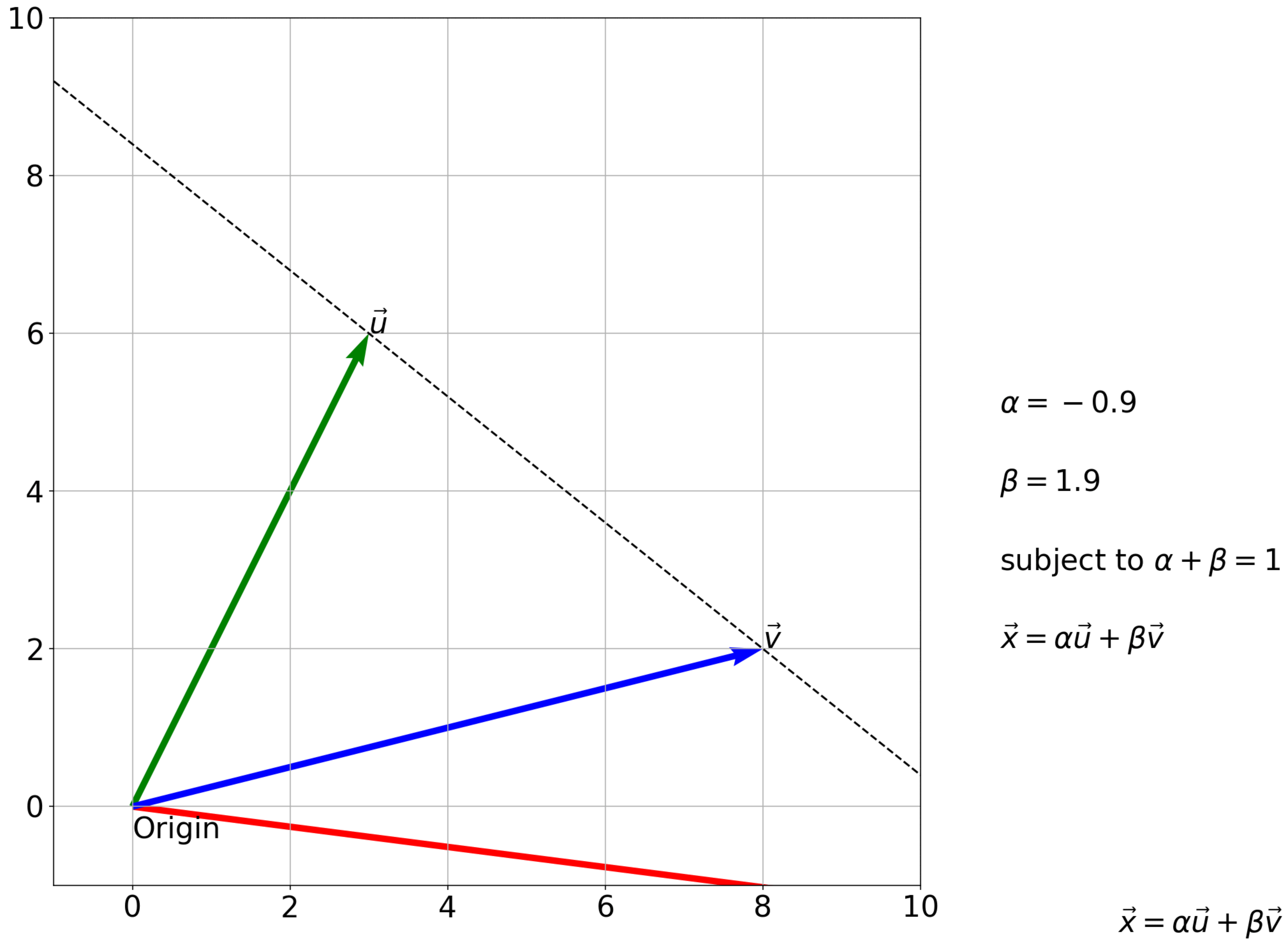
Linear Combinations

$$\vec{x} = \alpha_1 \vec{v}_1 + \cdots + \alpha_n \vec{v}_n = \sum_{i=1}^{N} \alpha_i \vec{v}_i$$

Special Case:

$$\sum_{i=1}^{N} \alpha_i = 1$$

Called “Affine Combination”



Linear Combinations

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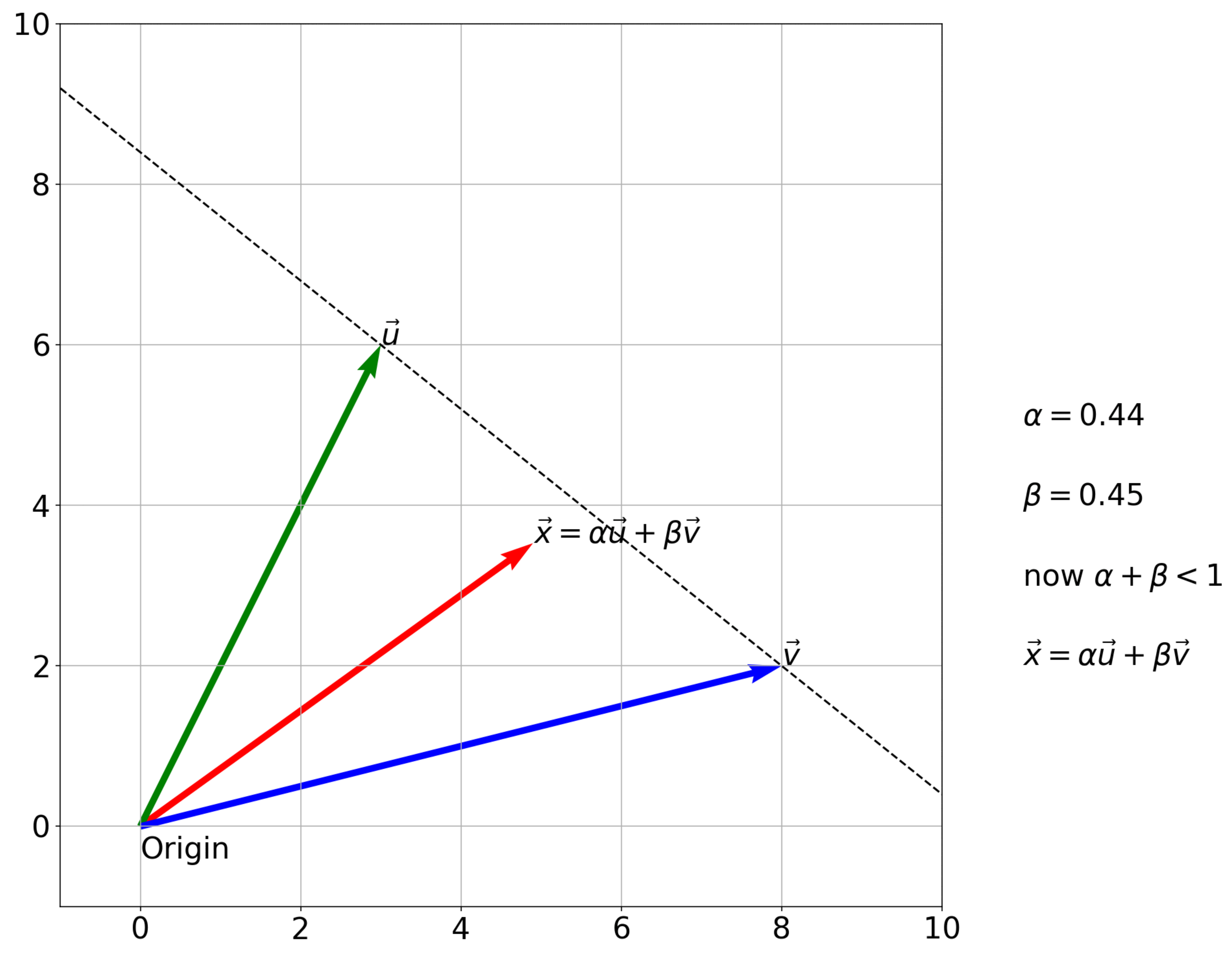
Special Case:

$$\sum_{i=1}^{N} \alpha_i < 1$$



Linear Combinations

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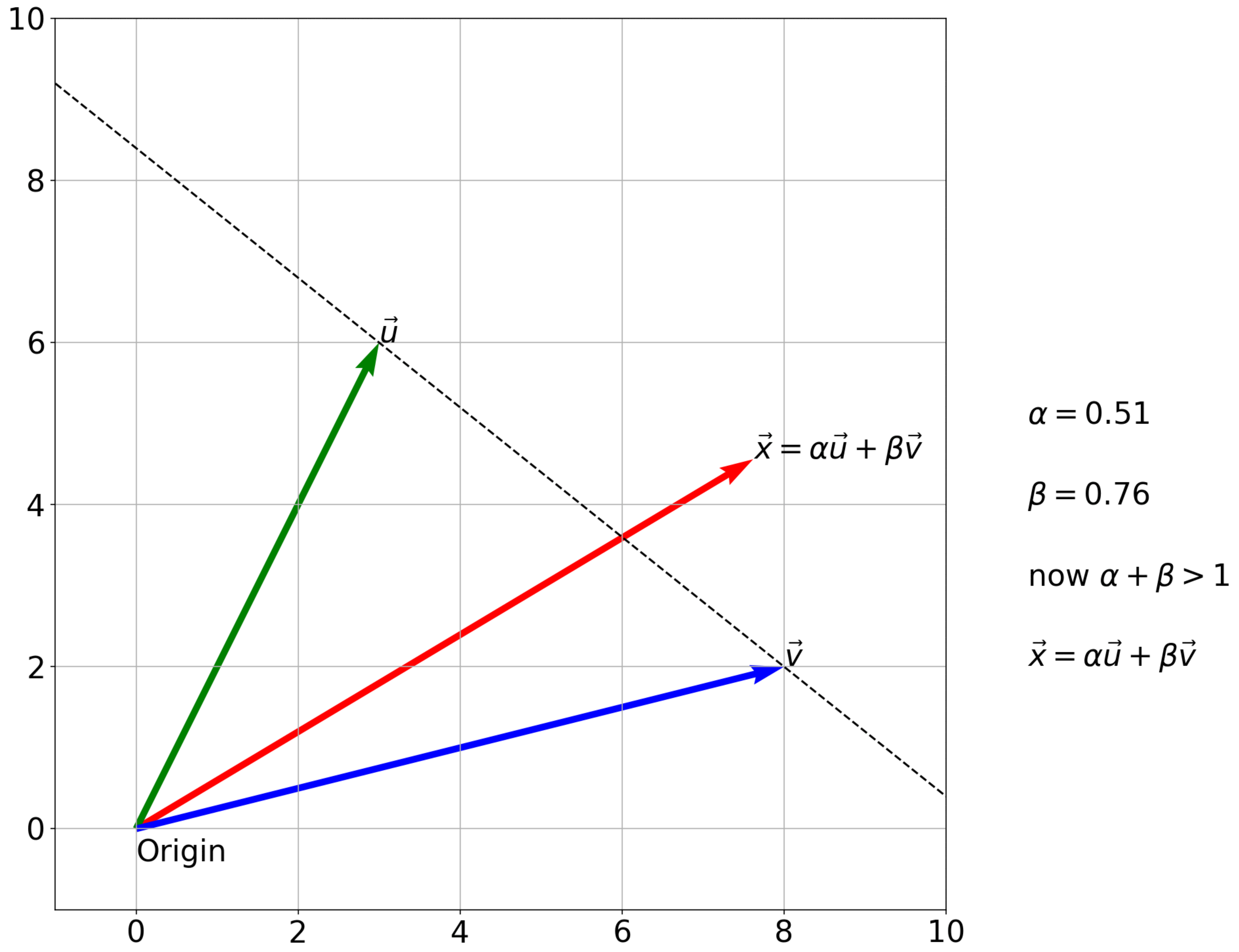
Special Case:

$$\sum_{i=1}^{N} \alpha_i > 1$$



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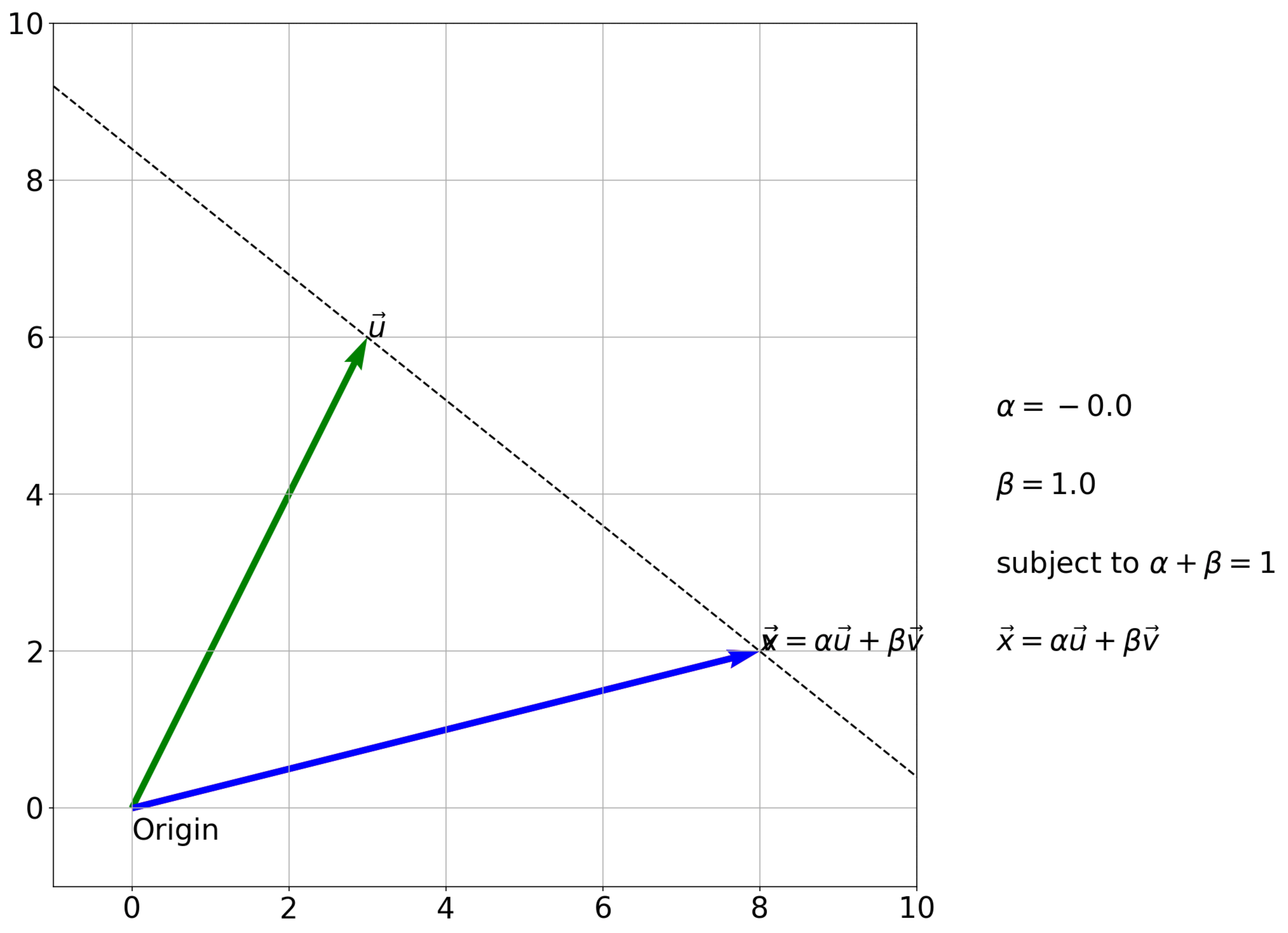
Special Case:

$$\sum_{i=1}^{N} \alpha_i = 1$$

$$\alpha_i \geq 0 \; \forall i$$

Called “Convex Combination”



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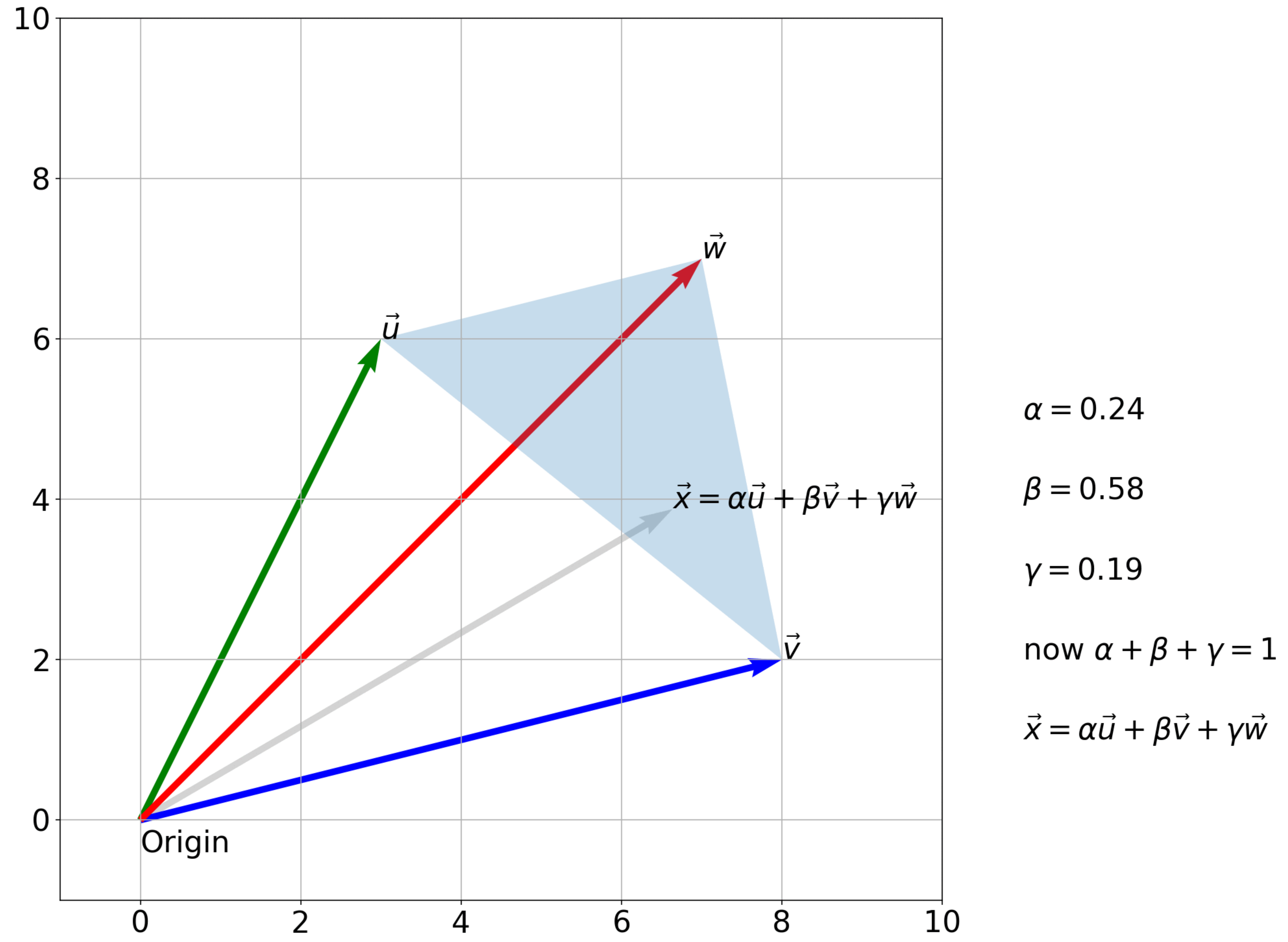


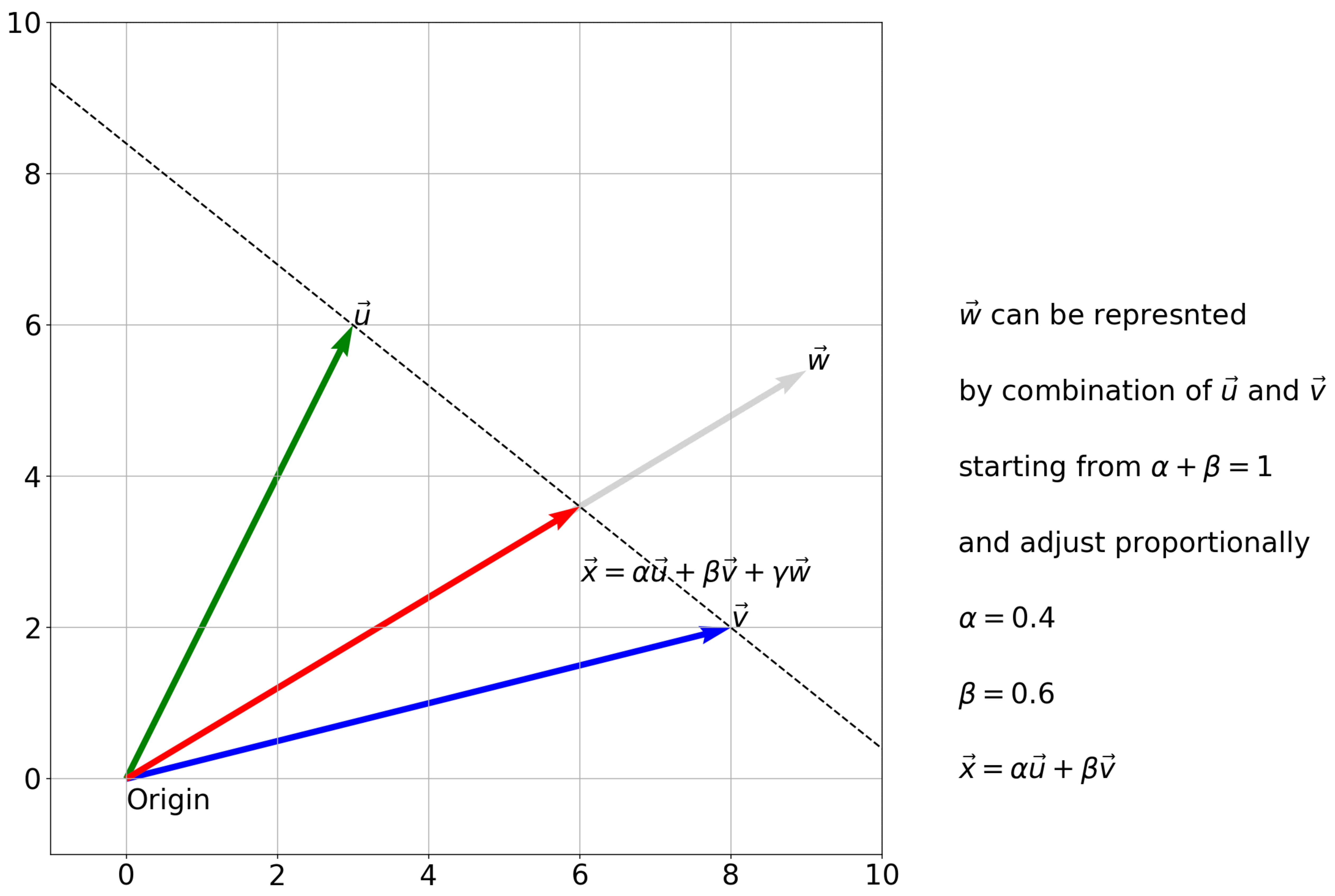
Span and Linear Independence

$$\text{span}\{\vec{v}_1, \dots, \vec{v}_n\} = \left\{ \sum_{i=1}^{N} \alpha_i \vec{v}_i \;\middle|\; \alpha_1, \dots, \alpha_n \in \mathbb{R} \right\}$$

A set of vectors is *linearly independent* if no vector is in the span of the other vectors.

$$\sum_{i=1}^{N} \alpha_i \vec{v}_i = \vec{0} \Longrightarrow \alpha_1, \dots, \alpha_n = 0$$

Otherwise, they are *linearly dependent*.



Inner Products*

$$\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{N} x_i y_i$$

For the special case of real vectors:

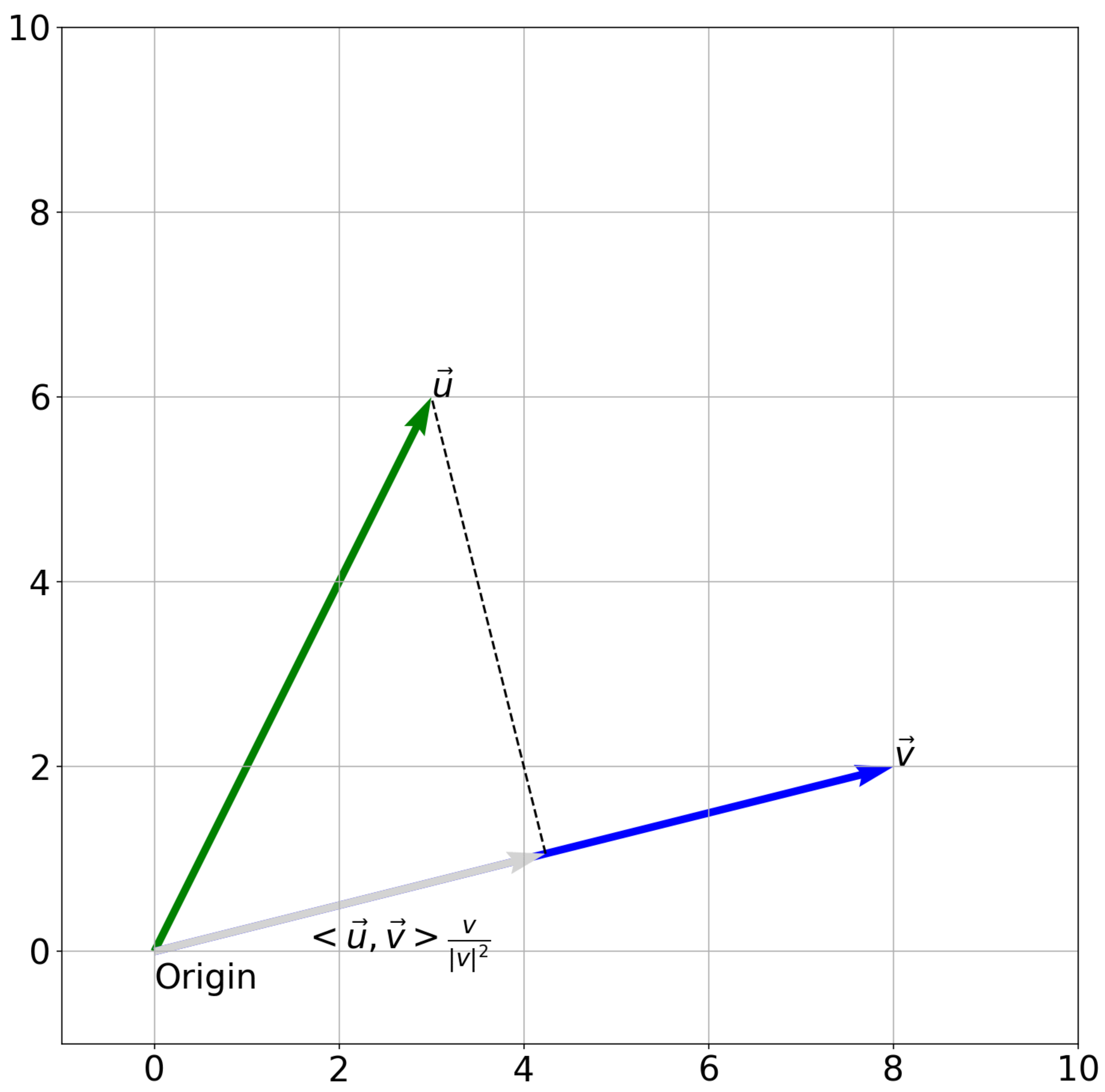
Symmetry: $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$

Bilinearity:

$$\langle \alpha\vec{x} + \beta\vec{y}, \vec{z} \rangle = \alpha\langle \vec{x}, \vec{z} \rangle + \beta\langle \vec{y}, \vec{z} \rangle$$

$$\langle \vec{x}, \alpha\vec{y} + \beta\vec{z} \rangle = \alpha\langle \vec{x}, \vec{y} \rangle + \beta\langle \vec{x}, \vec{z} \rangle$$

Nonnegativity: $\langle \vec{x}, \vec{x} \rangle \geq 0 \;\; \forall \vec{x}$



*Strictly speaking, this is the special case of standard inner products

Norms

Euclidean norm:

$$\|\vec{x}\|_2 = \sqrt{\langle \vec{x}, \vec{x} \rangle} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

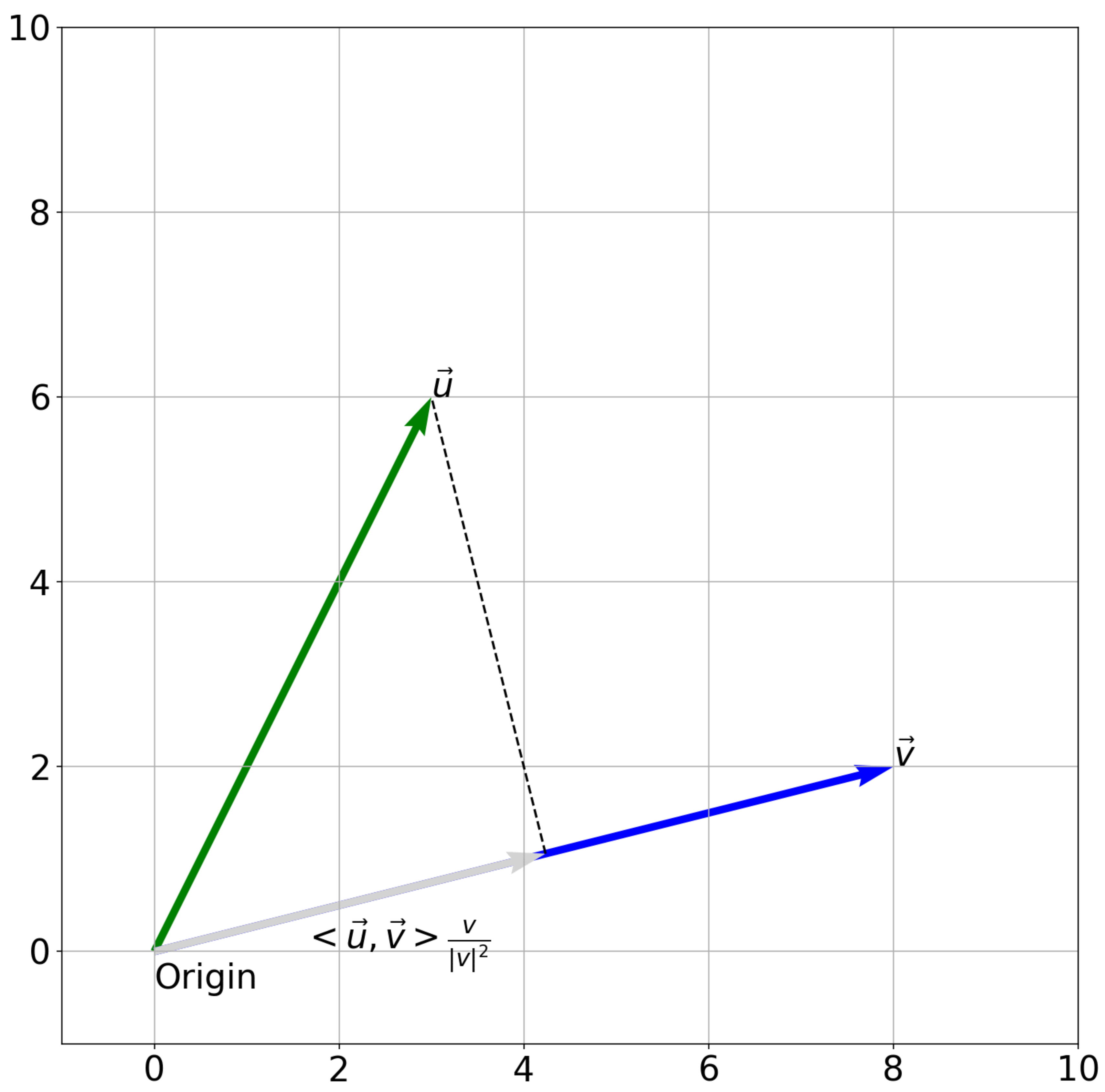
$$\|\vec{x}\|_2 \geq 0 \ \forall \vec{x}$$

Unit vector: vector with norm 1, often referred to as "unit norm"

Normalizing a vector $\vec{x}$: $\dfrac{\vec{x}}{\|\vec{x}\|_2}$

Projecting a vector $\vec{x}$ onto $\vec{y}$:

$$\left\langle \vec{x}, \frac{\vec{y}}{\|\vec{y}\|_2} \right\rangle \frac{\vec{y}}{\|\vec{y}\|_2}$$



Sign of $\langle \vec{x}, \vec{y} \rangle$ is positive if angle is less than 90 degrees, and negative otherwise.

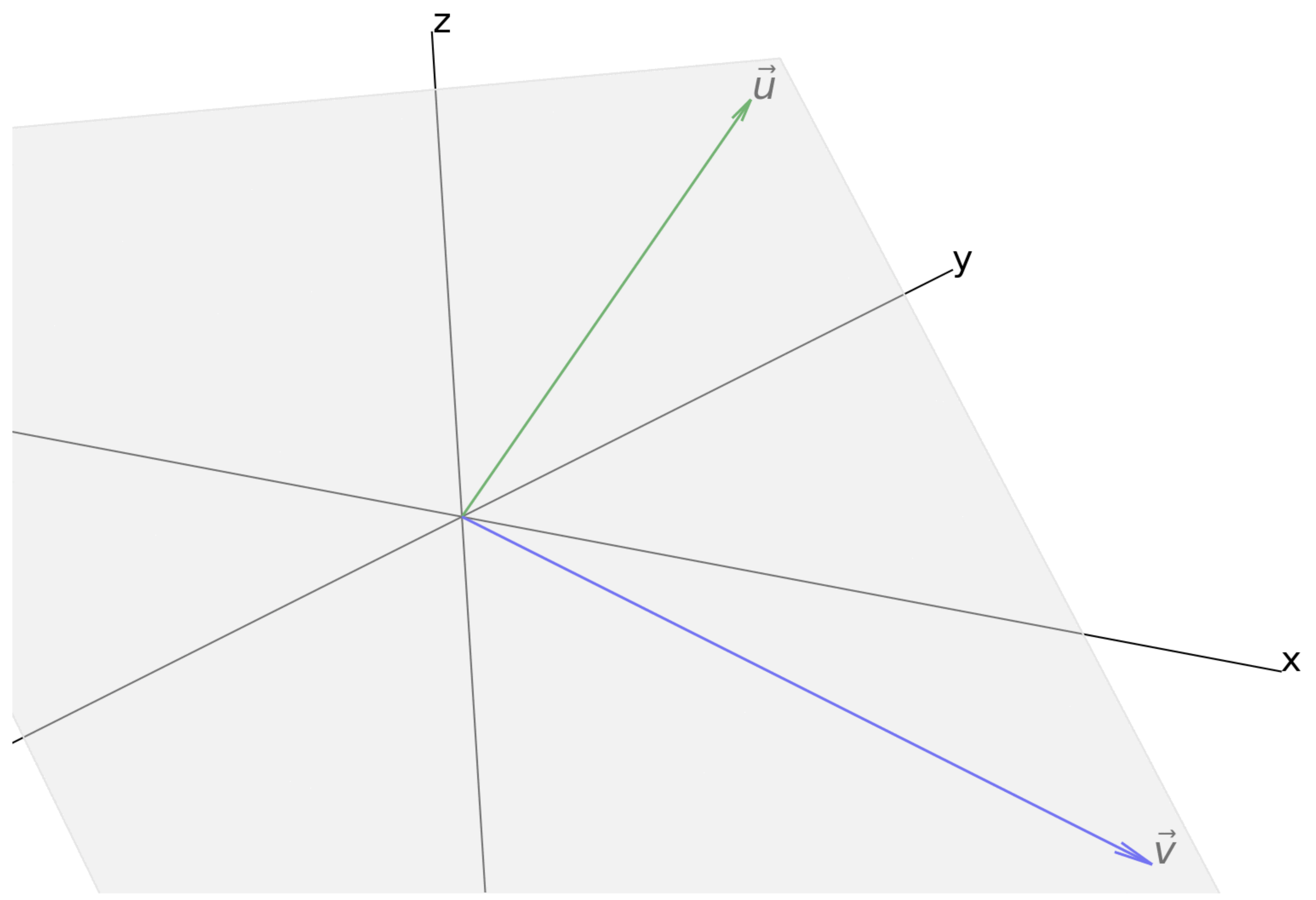
Linear Subspace

A lower-dimensional slice of the vector space that passes through the origin.

Examples:

A plane through the origin in 3D Euclidean space

A line through the origin in 3D Euclidean space

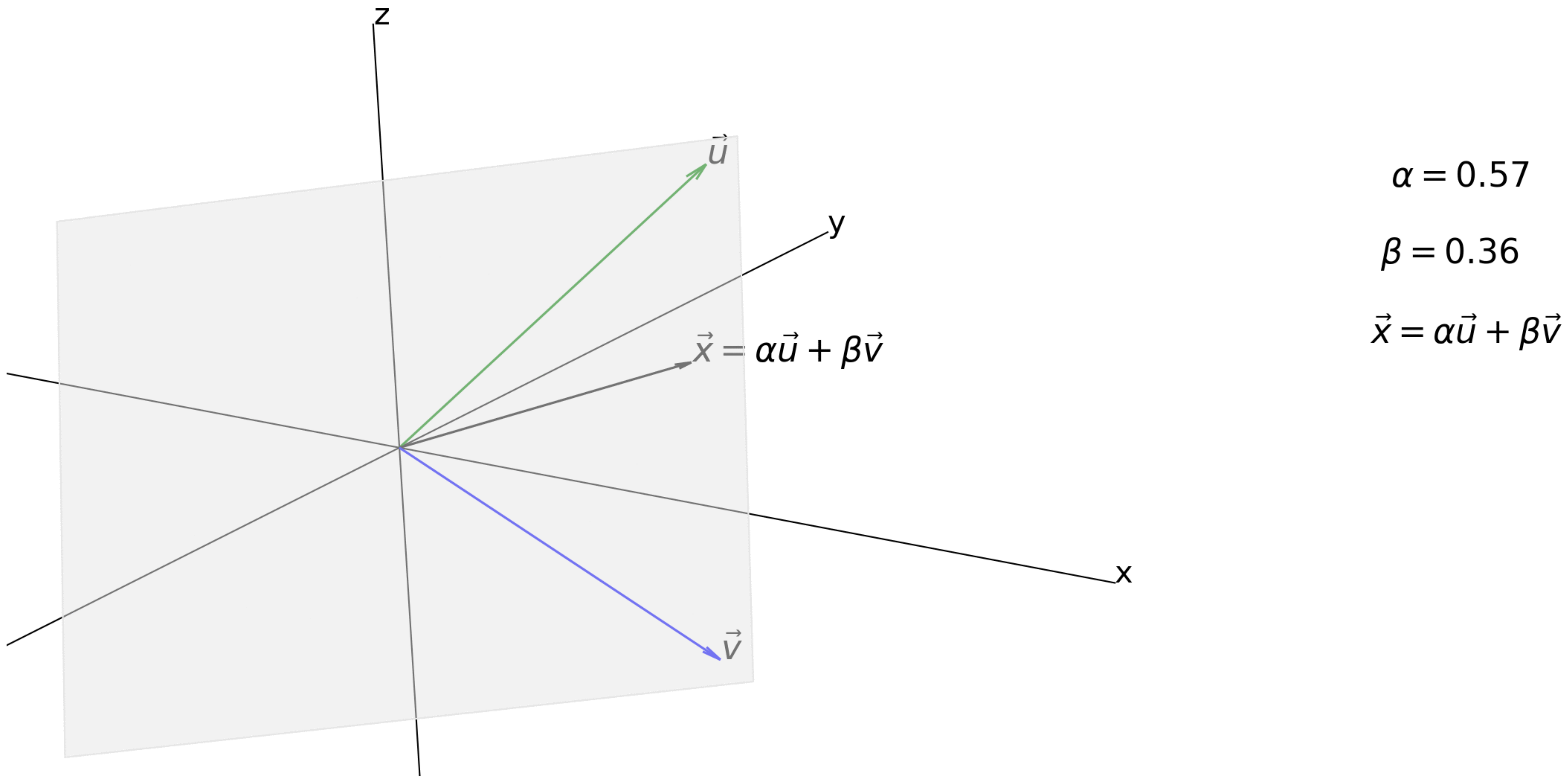


Basis

A linearly independent set of vectors that spans the full (sub)space.

Any vector in the (sub)space can be written as a linear combination of the basis vectors.

$\vec{u}, \vec{v}$ are a basis for the subspace

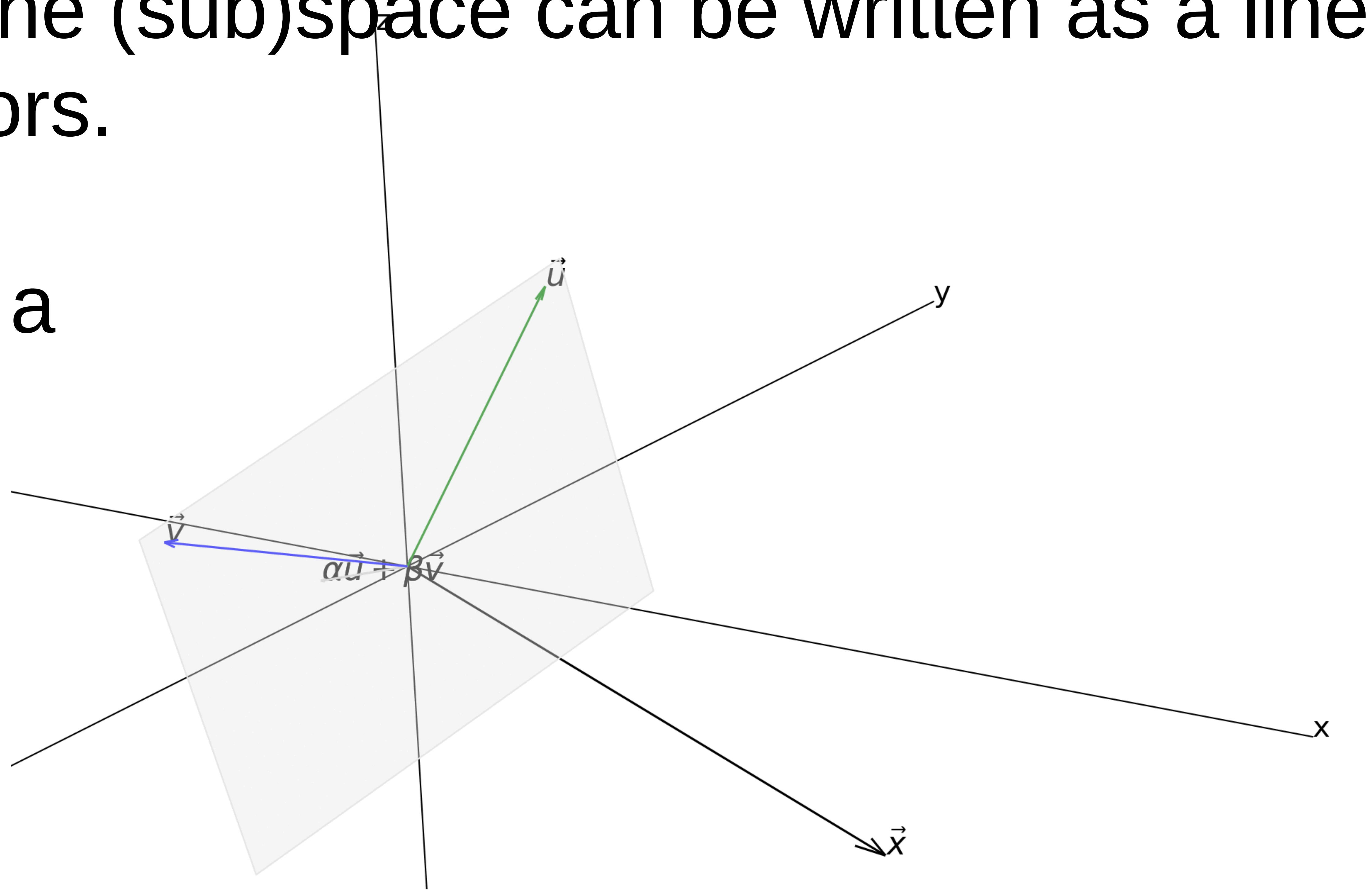


Basis

A linearly independent set of vectors that spans the full (sub)space.

Any vector in the (sub)space can be written as a linear combination of the basis vectors.

$\vec{u}, \vec{v}$ are not a basis for $\mathbb{R}^3$



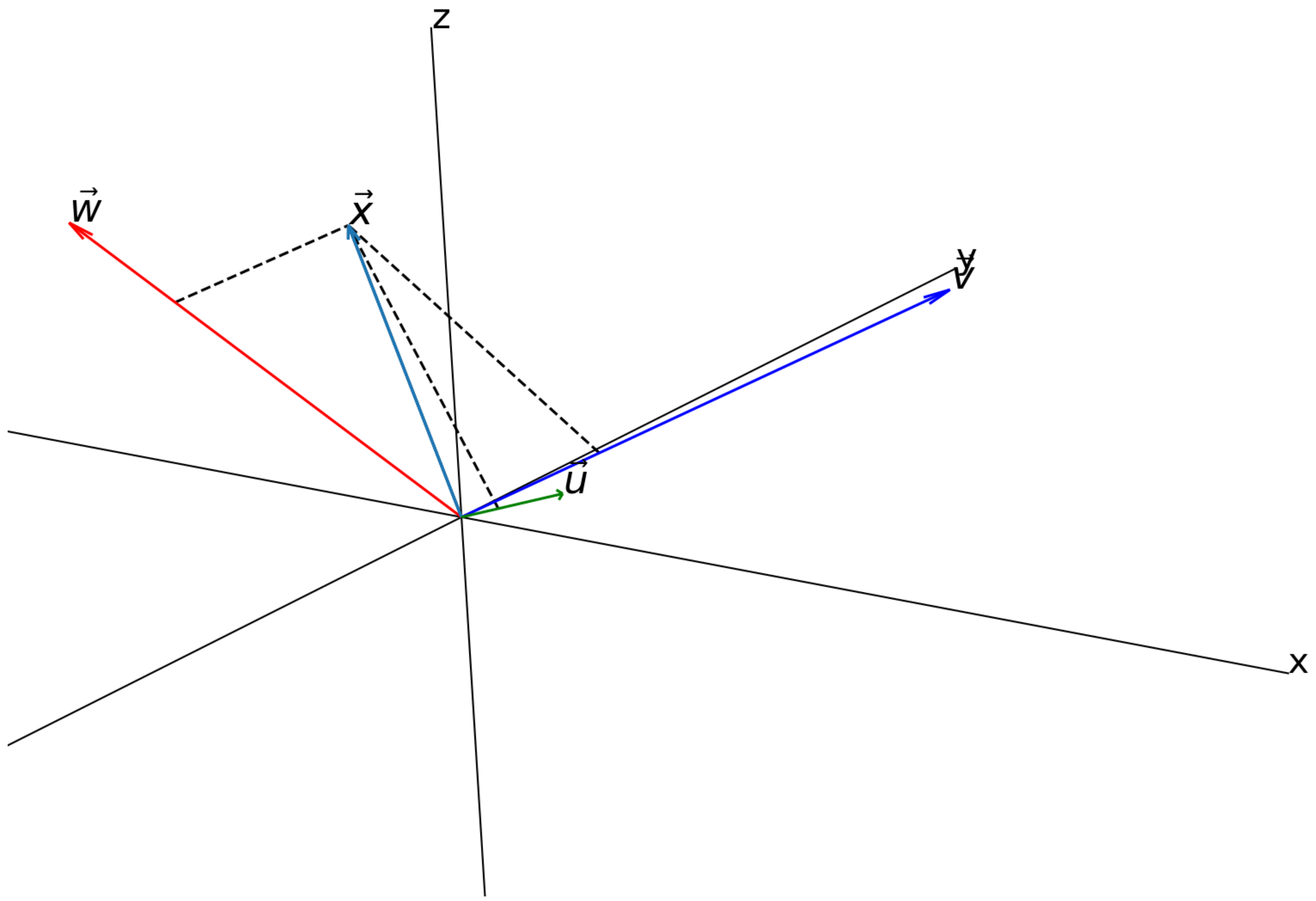
$\nexists \alpha, \beta, s.t.$

$\vec{x} = \alpha\vec{u} + \beta\vec{v}$

Orthogonal Basis

Two vectors $\vec{x}, \vec{y}$ are orthogonal if and only if $\langle \vec{x}, \vec{y} \rangle = 0$.

An orthogonal basis is a basis whose vectors are orthogonal to one another.

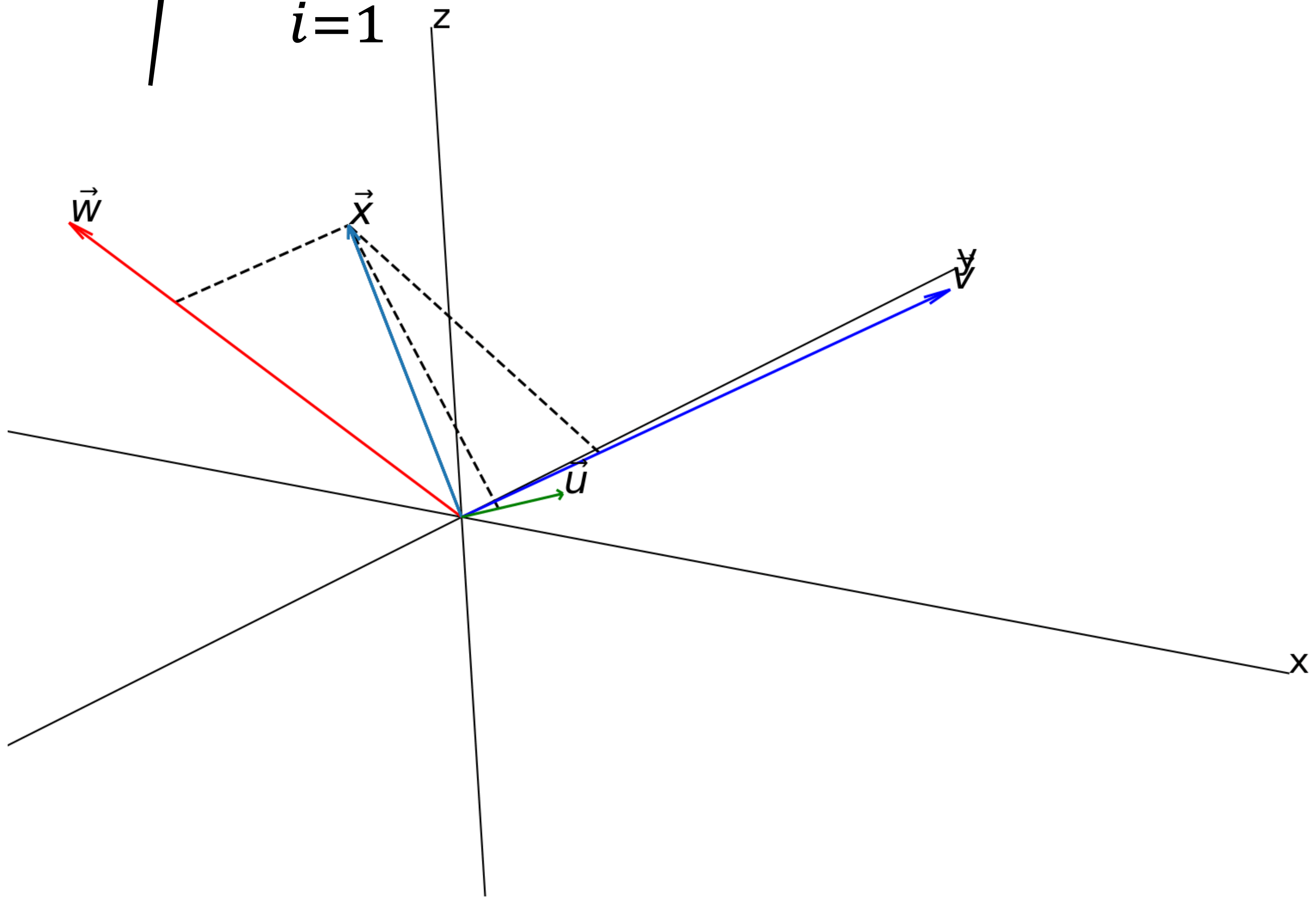


$\vec{x}$ is an arbitray vector

Orthogonal Basis

We like an orthogonal basis because it is easy to compute the coordinates with respect to the basis.

$$\langle \vec{x}, \vec{v}_j \rangle = \left\langle \sum_{i=1}^{N} \alpha_i \vec{v}_i \,, \vec{v}_j \right\rangle = \sum_{i=1}^{n} \alpha_i \langle \vec{v}_i, \vec{v}_j \rangle = \alpha_j \langle \vec{v}_j, \vec{v}_j \rangle = \alpha_j \left\| \vec{v}_j \right\|_2^2 \Longrightarrow \alpha_j = \frac{\langle \vec{x}, \vec{v}_j \rangle}{\left\| \vec{v}_j \right\|_2^2}$$

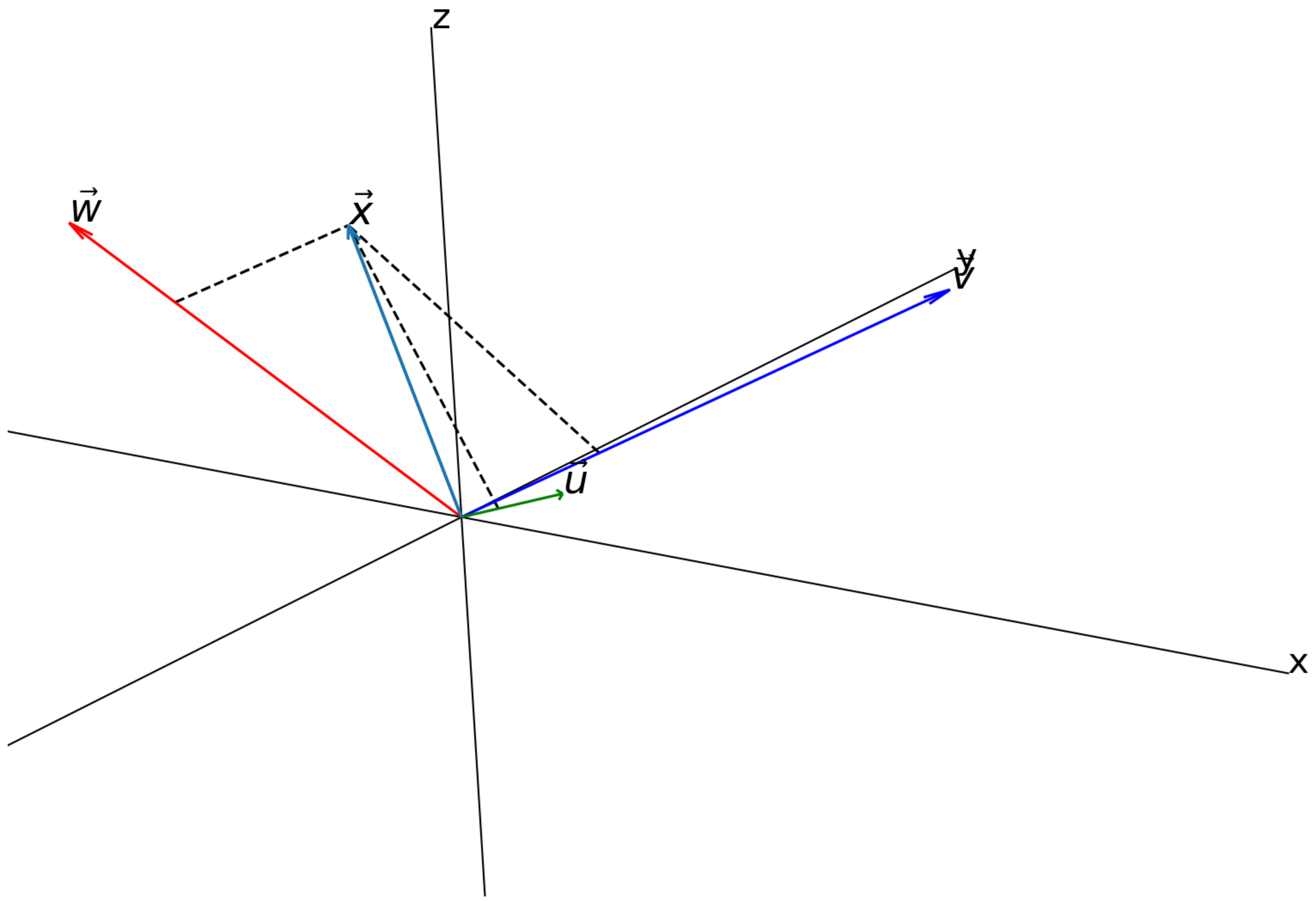


$\vec{x}$ is an arbitray vector

Orthonormal Basis

An orthonormal basis is a special case of an orthogonal basis with unit vectors.

$$\langle \vec{x}, \vec{v}_j \rangle = \left\langle \sum_{i=1}^{n} \alpha_i \vec{v}_i \, , \vec{v}_j \right\rangle = \sum_{i=1}^{n} \alpha_i \langle \vec{v}_i, \vec{v}_j \rangle = \alpha_j \langle \vec{v}_j, \vec{v}_j \rangle = \alpha_j \left\| \vec{v}_j \right\|_2^2 \Longrightarrow \alpha_j = \frac{\langle \vec{x}, \vec{v}_j \rangle}{\left\| \vec{v}_j \right\|_2^2}$$



$$\left\| \vec{v}_j \right\|_2 = 1 \Longrightarrow \alpha_j = \langle \vec{x}, \vec{v}_j \rangle$$

$\vec{x}$ is an arbitray vector

Finding the coordinates gets even easier!

Standard Basis

The standard basis is an orthonormal basis with the following vectors:

$$\begin{pmatrix}1\\0\\0\\\vdots\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\\vdots\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\\vdots\\0\\0\end{pmatrix}, \cdots, \begin{pmatrix}0\\0\\0\\\vdots\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\\vdots\\0\\1\end{pmatrix}$$

Usually denoted as:

$$\vec{e}_1, \vec{e}_2, \vec{e}_3, \dots, \vec{e}_{n-1}, \vec{e}_n$$

Known as the standard basis vectors.

Matrices

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{pmatrix} \in \mathbb{R}^{4\times 3}$$

Short-hand:

$$A = \left(a_{ij}\right) \in \mathbb{R}^{m\times n}$$

$$A + B = \left(a_{ij} + b_{ij}\right)$$

$$cA = \left(c a_{ij}\right)$$

Multiplying a matrix with a vector:

$$\vec{y} = A\vec{x}$$

$$\begin{pmatrix} ?\,? \\ ?\,? \\ ?\,? \\ ?\,? \end{pmatrix} = \begin{pmatrix} 3 & 6 & 9 \\ 1 & 1 & -1 \\ 2 & 0 & 4 \\ -2 & 0 & -4 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

Multiplying a matrix with a matrix:

$$AX = A \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ x_{21} & \cdots & x_{2n} \\ x_{31} & \cdots & x_{3n} \end{pmatrix}$$

$$= A(\vec{x}_{\cdot 1} \quad \cdots \quad \vec{x}_{\cdot n})$$

$$= (A\vec{x}_{\cdot 1} \quad \cdots \quad A\vec{x}_{\cdot n})$$

$(AB)C = A(BC)$, but $AB \neq BA$

Matrices

Transpose: $A^\top$

If $A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{pmatrix}$

then $A^\top = \begin{pmatrix} a_{1,1} & a_{2,1} & a_{3,1} & a_{4,1} \\ a_{1,2} & a_{2,2} & a_{3,2} & a_{4,2} \\ a_{1,3} & a_{2,3} & a_{3,3} & a_{4,3} \end{pmatrix}$

Alternative inner product notation:

$$\vec{x}^\top \vec{y} = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$= \sum_{i=1}^{n} x_i y_i = \langle \vec{x}, \vec{y} \rangle$$

Will treat vectors as column vectors:

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Outer product:

$$\vec{x}\vec{y}^\top = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \begin{pmatrix} y_1 & \cdots & y_n \end{pmatrix}$$

$$= \begin{pmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_n \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_n y_1 & x_n y_2 & \cdots & x_n y_n \end{pmatrix}$$

Vector transpose denotes a row vector:

$$\vec{x}^\top = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}$$

Properties of Transpose:

$$(A^\top)^\top = A$$

$$(A + B)^\top = A^\top + B^\top$$

$$(cA)^\top = cA^\top$$

$$(AB)^\top = B^\top A^\top$$

Matrices

Rank: Number of linearly independent columns, or equivalently the number of linearly independent rows, denoted as $\operatorname{rank}(A)$

$$A = \begin{pmatrix} -1 & 2 \\ 0 & 0 \\ 2 & -4 \end{pmatrix} \quad \operatorname{rank}(A) = 1$$

Rank-deficient

$$B = \begin{pmatrix} -1 & 2 \\ 0 & 1 \\ 2 & -4 \end{pmatrix} \quad \operatorname{rank}(B) = 2$$

Full-rank

Identity Matrix:

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Property: $I\vec{x} = \vec{x}$

Inverse:

A matrix A^{-1} such that $A^{-1}A = I$

Or equivalently,

A matrix A^{-1} such that $AA^{-1} = I$

Not all matrices are invertible! Only those that are square and full-rank are.

Interpreting Matrices

- Each matrix represents a linear transformation, that is, a linear function that maps a vector to a vector.

$$A = (\vec{a}_{\cdot 1} \quad \cdots \quad \vec{a}_{\cdot n})$$

$$A\vec{x} = (\vec{a}_{\cdot 1} \quad \cdots \quad \vec{a}_{\cdot n}) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 \vec{a}_{\cdot 1} + \cdots + x_n \vec{a}_{\cdot n}$$

- This is a linear combination of $\vec{a}_{\cdot 1}, \dots, \vec{a}_{\cdot n}$, where the coefficients are given by $\vec{x}$

  - Before: $\vec{x} = x_1 \vec{e}_1 + \cdots + x_n \vec{e}_n$

  - After: $A\vec{x} = x_1 \vec{a}_{\cdot 1} + \cdots + x_n \vec{a}_{\cdot n}$

Essentially replacing the original basis vectors with the columns of A

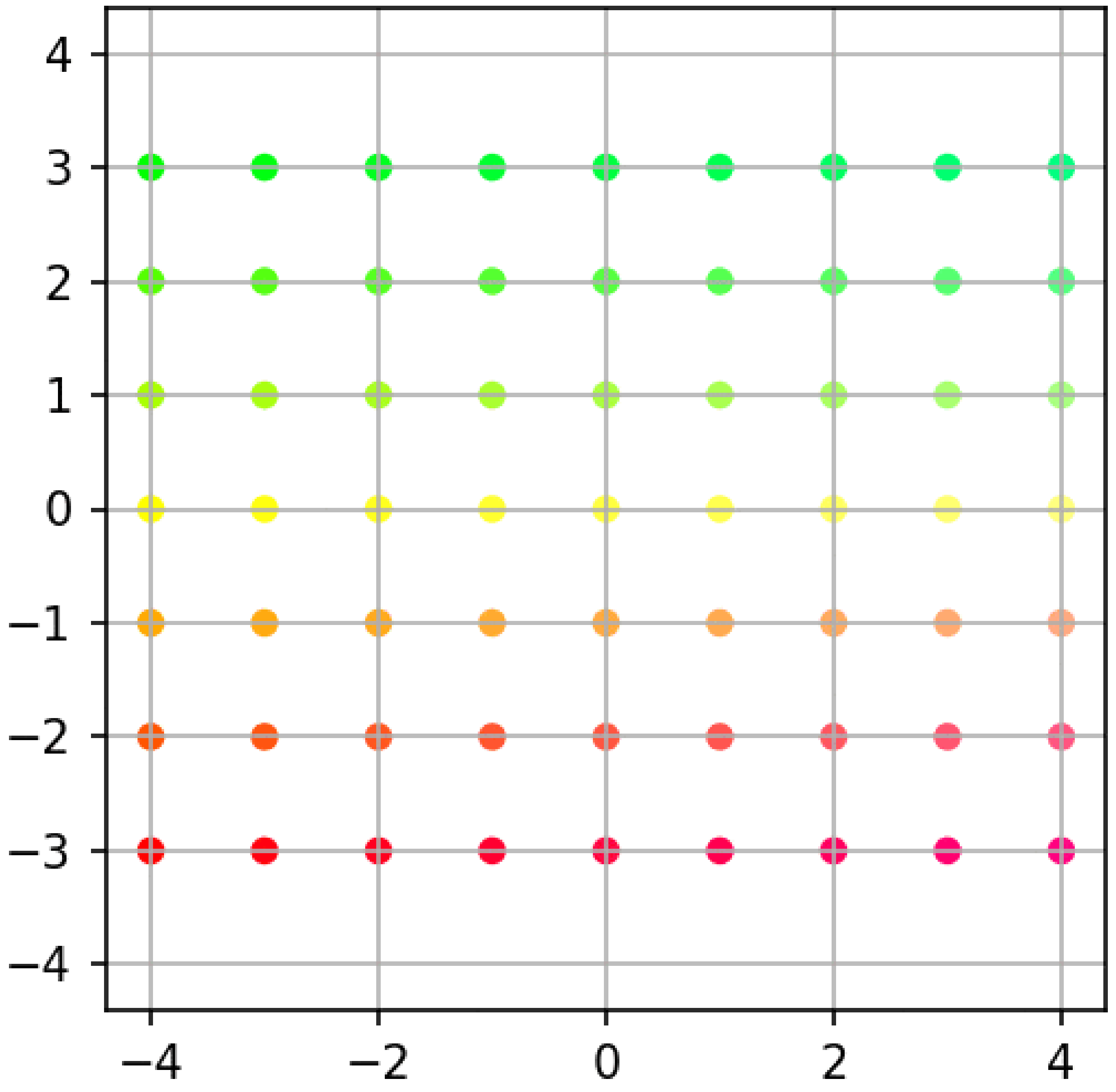
Diagonal Matrices

$$A = \begin{pmatrix} a_{1,1} & 0 \\ 0 & a_{2,2} \end{pmatrix}$$

$$A\vec{x} = A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{1,1}x_1 \\ a_{2,2}x_2 \end{pmatrix}$$

In the case of diagonal matrices, rank is the number of non-zero entries.

Full-rank: $A = \begin{pmatrix} a_{1,1} & 0 \\ 0 & a_{2,2} \end{pmatrix}$



0.9	0.0
0.0	0.5

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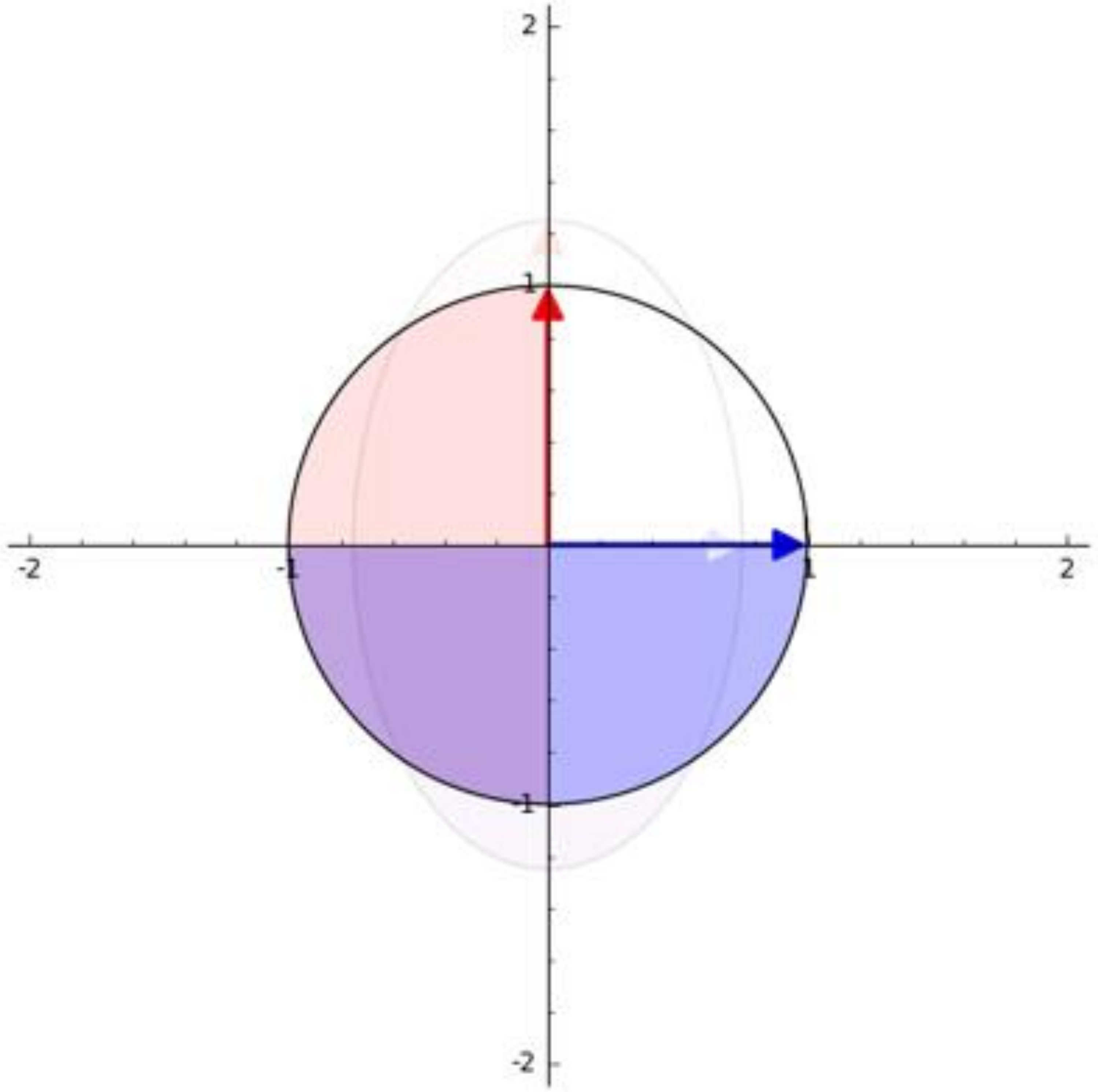

Credit: Ryan Holbrook

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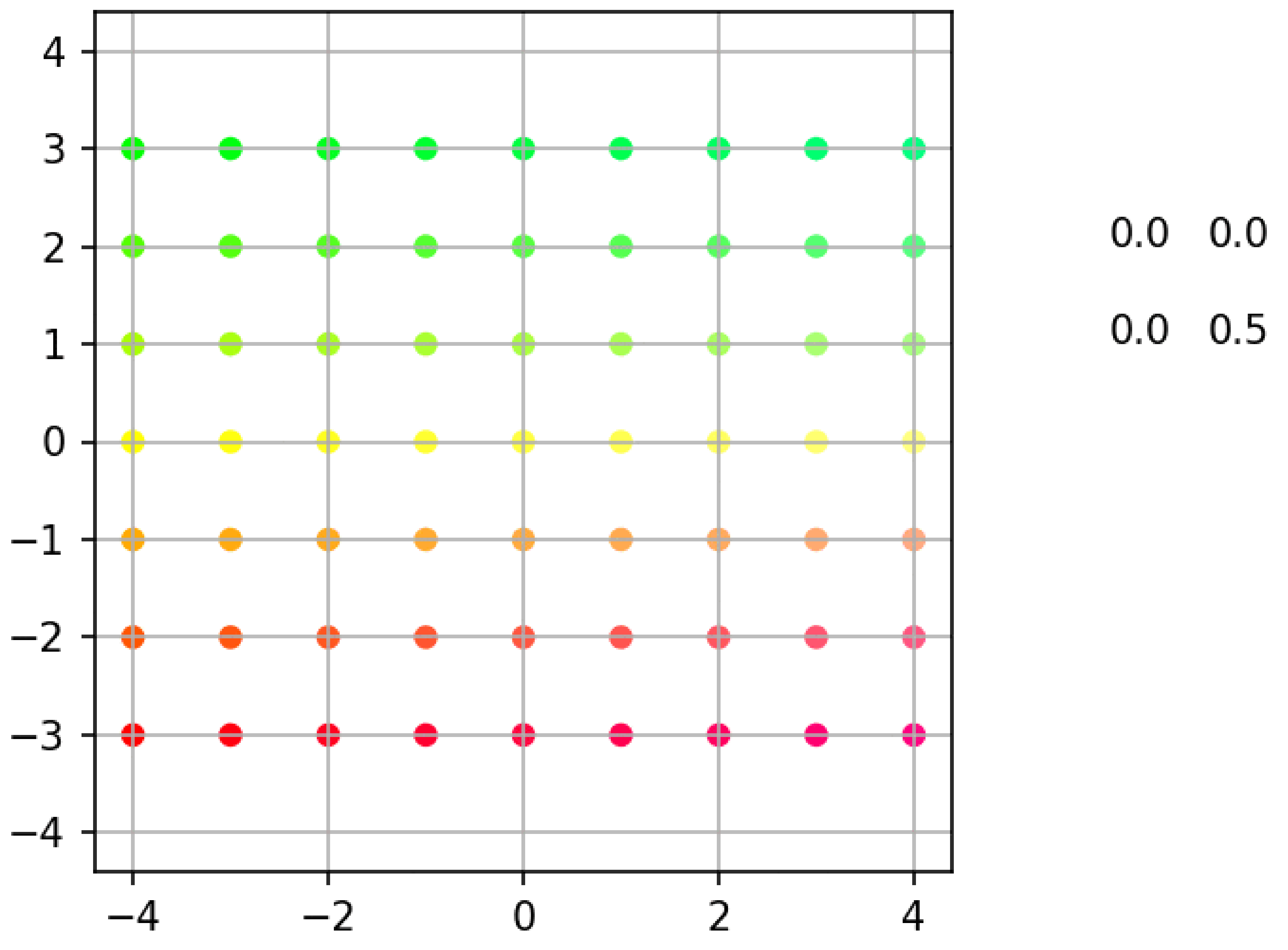
Rank-deficient: $A = \begin{pmatrix} 0 & 0 \\ 0 & a_{2,2} \end{pmatrix}$



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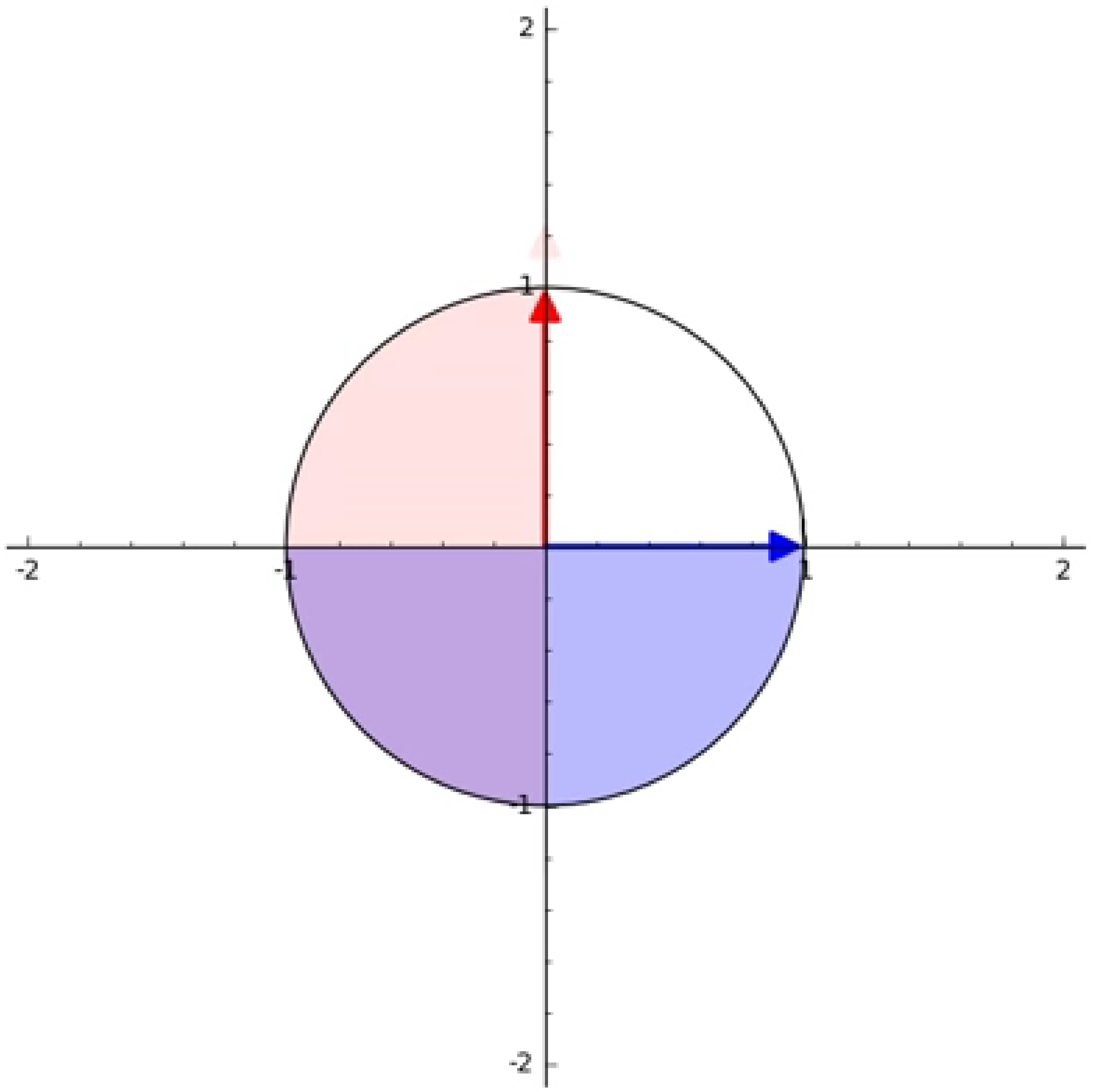


Credit: Ryan Holbrook

Orthogonal Matrices

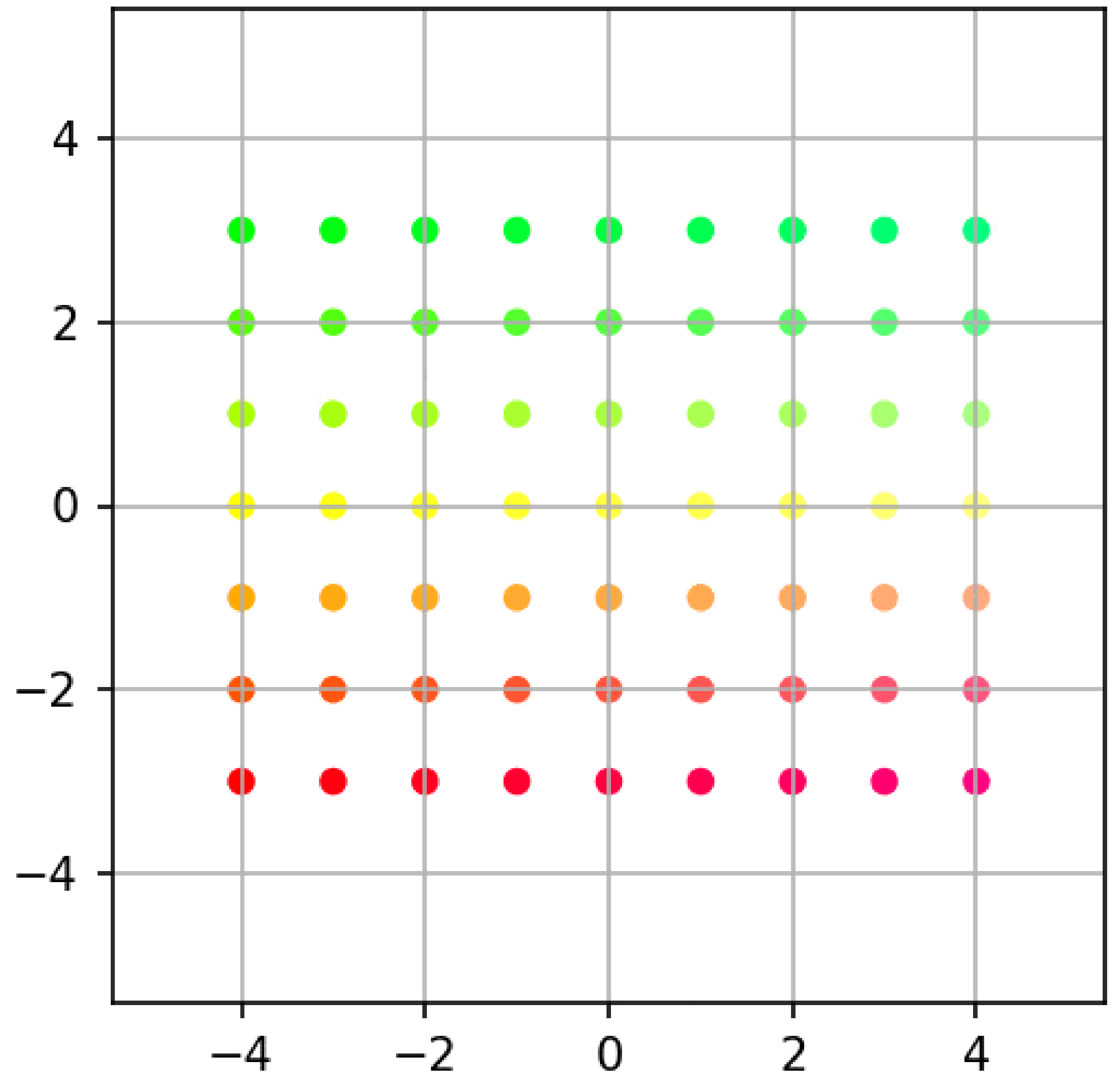
A square matrix A such that $AA^\top = I$ and $A^\top A = I$ (implies $A^{-1} = A^\top$)

$$A^\top A = \begin{pmatrix} \vec{a}_{\cdot 1}^\top \vec{a}_{\cdot 1} & \vec{a}_{\cdot 1}^\top \vec{a}_{\cdot 2} & \cdots & \vec{a}_{\cdot 1}^\top \vec{a}_{\cdot n} \\ \vec{a}_{\cdot 2}^\top \vec{a}_{\cdot 1} & \vec{a}_{\cdot 2}^\top \vec{a}_{\cdot 2} & \cdots & \vec{a}_{\cdot 2}^\top \vec{a}_{\cdot n} \\ \vdots & \vdots & \ddots & \vdots \\ \vec{a}_{\cdot n}^\top \vec{a}_{\cdot 1} & \vec{a}_{\cdot n}^\top \vec{a}_{\cdot 2} & \cdots & \vec{a}_{\cdot n}^\top \vec{a}_{\cdot n} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

The columns of A form an orthonormal basis

Determinant of 1: $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$



$\begin{matrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{matrix}$

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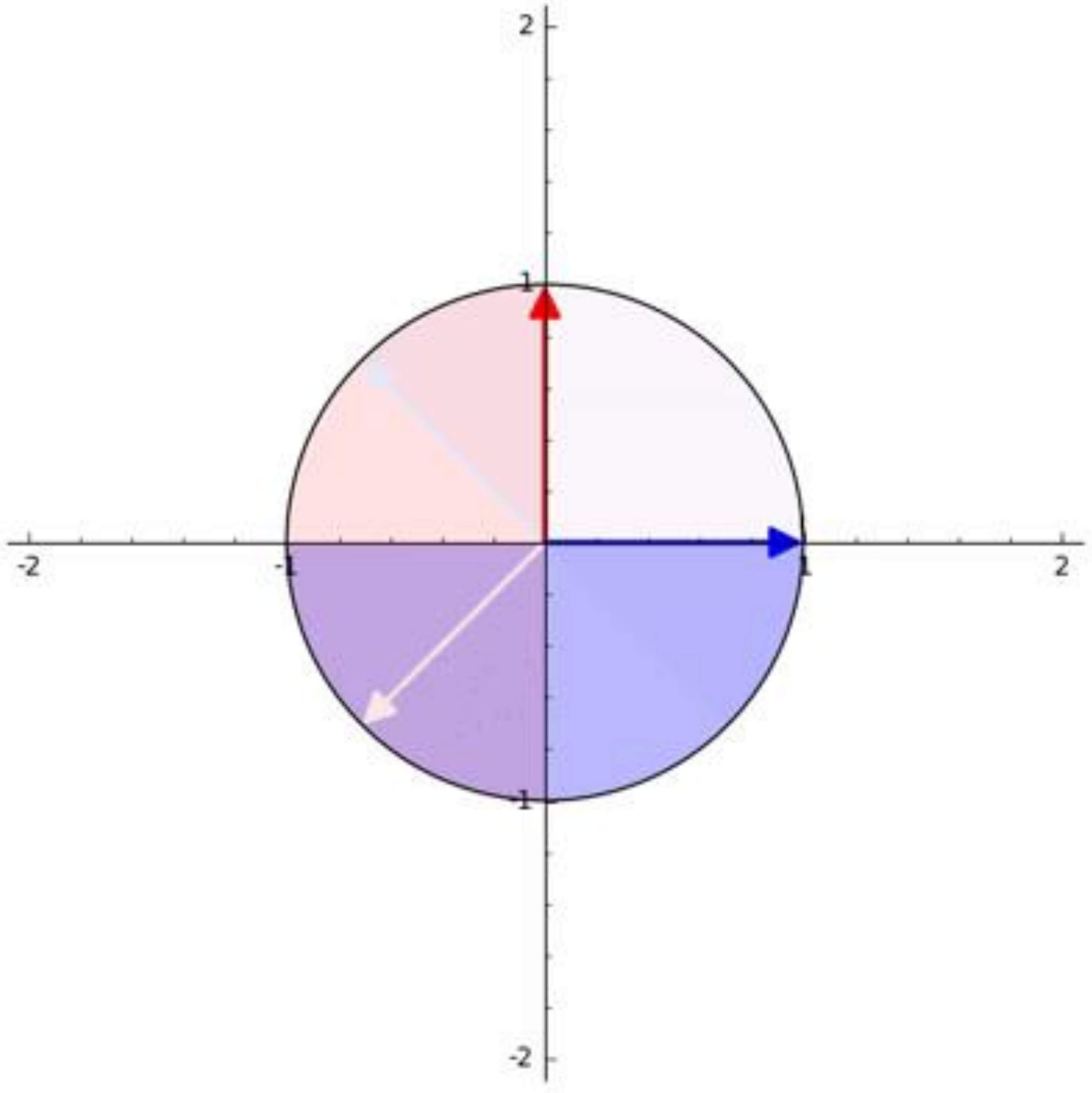

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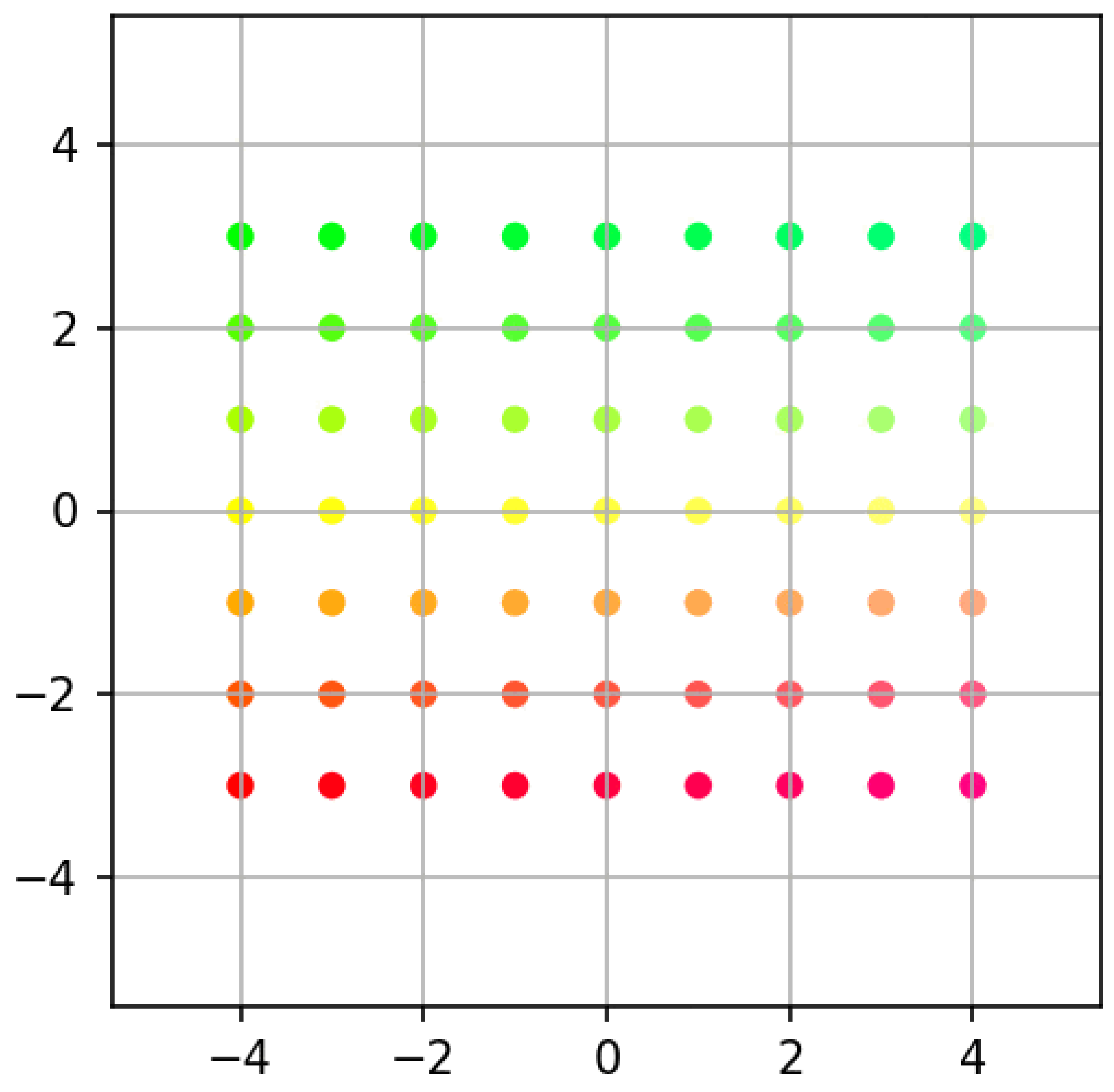
Determinant of -1: $A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$



$$\begin{matrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{matrix}$$

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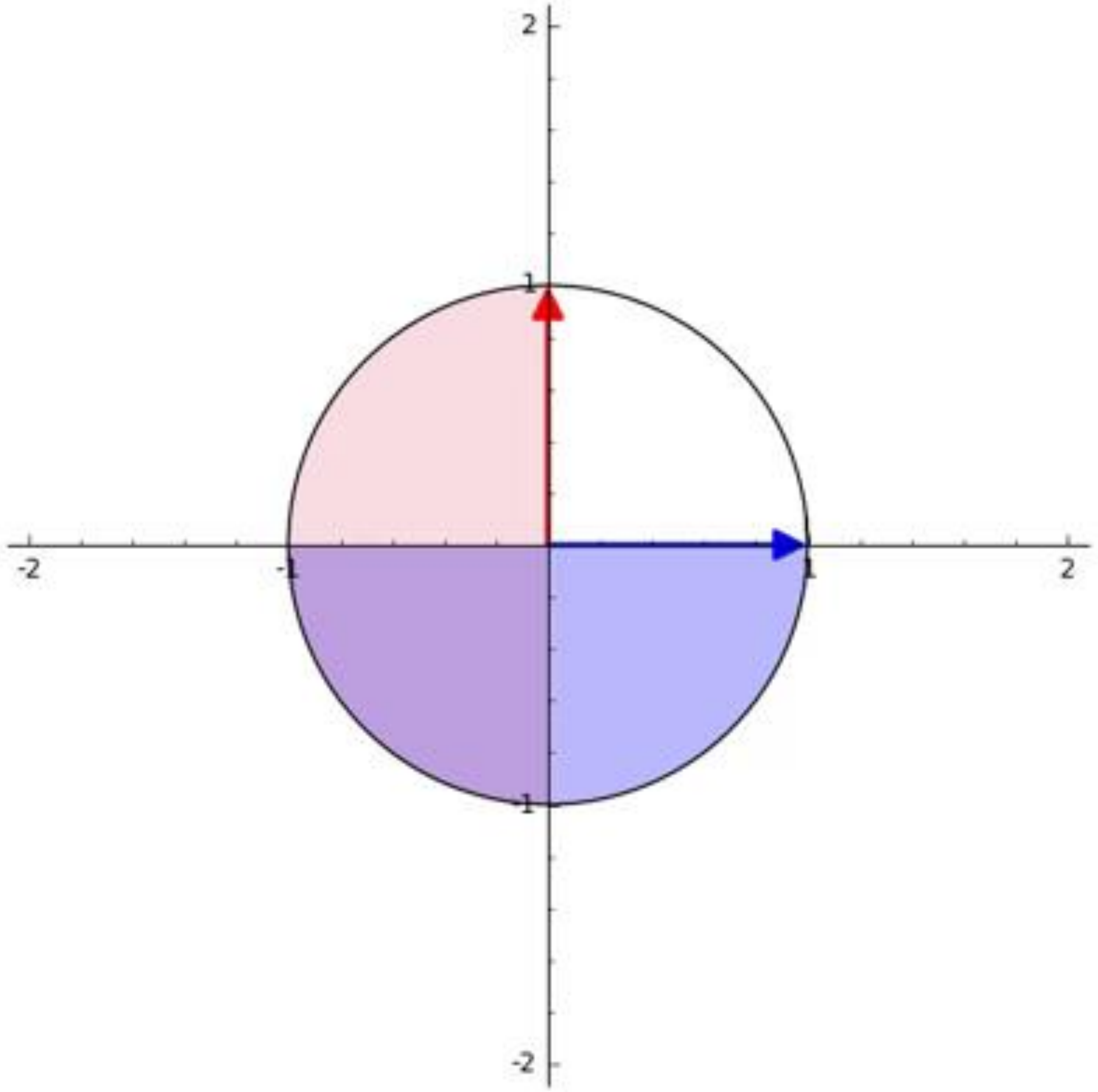
The columns of A form an orthonormal basis

Determinant of -1: $A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$



Credit: Ryan Holbrook

Addendum (Coordinates for Non-Orthogonal Basis)

Suppose $\vec{x} = \sum_{i=1}^{N} \alpha_i \vec{v}_i$. Given $\vec{x}$ and the basis $\{\vec{v}_i\}$, we'd like to determine the coordinates $\{\alpha_i\}$. First, let $v_i = \begin{bmatrix} v_{i1} \\ \vdots \\ v_{iN} \end{bmatrix}$. Then, we can write

$$\begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} v_{11} & \cdots & v_{N1} \\ \vdots & \ddots & \vdots \\ v_{N1} & \cdots & v_{NN} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix}$$

Next, we'd solve the above equation for the unknowns $\{\alpha_i\}$. There are many ways. One (potentially inefficient) way is by finding the inverse:

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix} = \begin{bmatrix} v_{11} & \cdots & v_{N1} \\ \vdots & \ddots & \vdots \\ v_{N1} & \cdots & v_{NN} \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$$