Assignment 0 Solutions

1 Linear Algebra

a) Find the inverse of the following matrices:

$$A = \begin{bmatrix} 0.5 & 0 \\ 0 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}$$

Since A is diagonal, we just need to invert each diagonal element. A^{-1} = $\begin{bmatrix} 2 & 0 \\ 0 & 0.25 \end{bmatrix}$

For B and C, we can use the closed form formula for the inverse of a 2-by-2 matrix, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. So $B^{-1} = -\frac{1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}, C^{-1} = -\frac{1}{5} \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$

b)

$$\begin{aligned} BC &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix} && \text{(1a)} \\ &= \begin{bmatrix} 1 \times 2 + 2 \times (-3) & 1 \times (-3) + 2 \times 2 \\ 3 \times 2 + 4 \times (-3) & 3 \times (-3) + 4 \times 2 \end{bmatrix} && \text{(1b)} \\ &= \begin{bmatrix} -4 & 1 \\ -6 & -1 \end{bmatrix} && \text{(1c)} \end{aligned}$$

$$\begin{aligned} CB &= \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} && \text{(2a)} \\ &= \begin{bmatrix} 2 \times 1 + (-3) \times 3 & 2 \times 2 + (-3) \times (4) \\ (-3) \times 1 + 2 \times 4 & -3 \times 2 + 2 \times 4 \end{bmatrix} && \text{(2b)} \\ &= \begin{bmatrix} -7 & -8 \\ 3 & 2 \end{bmatrix} && \text{(2c)} \end{aligned}$$

c) Let $\vec{e}$ and λ respectively be an eigenvalue and eigenvalue of C.

$$\begin{aligned} C\vec{e} &= \lambda \vec{e} && \text{(3a)} \\ C\vec{e} - \lambda I \vec{e} &= 0 && \text{(3b)} \\ (C - \lambda I)\vec{e} &= 0 && \text{(3c)} \end{aligned}$$

This means $C - \lambda I$ is singular (non-invertible) since by assumption $\vec{e} \neq \vec{0}$, and thus $\det(C - \lambda I) = 0$.

$$\det(C - \lambda I) = 0 \tag{4a}$$
$$\det\left(\begin{bmatrix} 2-\lambda & -3 \\ -3 & 2-\lambda \end{bmatrix}\right) = 0 \tag{4b}$$
$$(2-\lambda)^2 - 9 = 0 \tag{4c}$$
$$2 - \lambda = \pm 3 \tag{4d}$$
$$\lambda = 2 \pm 3 = -1, 5 \tag{4e}$$

To obtain the corresponding $\vec{e}$, substitute in the values of λ into $(C - \lambda I)\vec{e} = 0$.

$$\lambda = -1 \Rightarrow \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \vec{e} = 0 \tag{5a}$$
$$\Rightarrow \vec{e} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{5b}$$

$$\lambda = 5 \Rightarrow \begin{bmatrix} -3 & -3 \\ -3 & -3 \end{bmatrix} \vec{e} = 0 \tag{6a}$$
$$\Rightarrow \vec{e} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{6b}$$

2 Calculus

a) First, we expand $f(\vec{x})$.

$$f(\vec{x}) = f(x_1, x_2) \tag{7a}$$
$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2x_1 + 2x_2 \\ 4x_1 \end{bmatrix} \tag{7b}$$
$$= 2x_1^2 + 2x_1x_2 + 4x_1x_2 \tag{7c}$$
$$= 2x_1^2 + 6x_1x_2 \tag{7d}$$

Now, we can take the partial derivatives:

$$\frac{\partial f}{\partial x_1} = 4x_1 + 6x_2 \tag{8a}$$
$$\frac{\partial f}{\partial x_1}(1,2) = 4 + 12 \tag{8b}$$
$$= 16 \tag{8c}$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_1} \tag{9a}$$

$$= \frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial x_1}\right) \tag{9b}$$

$$= 6 \text{ (for any } \vec{x}\text{, including } (3,4)) \tag{9c}$$

$$\tag{9d}$$

b) Gradient: First, compute $\frac{\partial f}{\partial x_2}$.

$$\frac{\partial f}{\partial x_2} = 6x_1 \tag{10a}$$

$$\Rightarrow \frac{\partial f}{\partial \vec{x}} = \begin{bmatrix} 4x_1 + 6x_2 \\ 6x_1 \end{bmatrix} \tag{10b}$$

Alternatively, we can use the fact that $\frac{\partial}{\partial \vec{x}}(\vec{x}A\vec{x}) = (A + A^\top)\vec{x}$.

$$\frac{\partial f}{\partial \vec{x}} = (A + A^\top)\vec{x} \tag{11a}$$

$$= (\begin{bmatrix} 2 & 2 \\ 4 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 2 & 0 \end{bmatrix})\vec{x} \tag{11b}$$

$$= (\begin{bmatrix} 4 & 6 \\ 6 & 0 \end{bmatrix} \vec{x} \tag{11c}$$

$$\tag{11d}$$

Hessian: First, we need to compute $\frac{\partial^2 f}{\partial x_1^2}$ and $\frac{\partial^2 f}{\partial x_2^2}$.

$$\frac{\partial^2 f}{\partial x_1^2} = 4 \tag{12a}$$

$$\frac{\partial^2 f}{\partial x_2^2} = 0 \tag{12b}$$

$$\Rightarrow \frac{\partial^2 f}{\partial \vec{x} \partial \vec{x}^\top} = \begin{bmatrix} 4 & 6 \\ 6 & 0 \end{bmatrix} \tag{12c}$$

Alternatively, we can use the fact that $\frac{\partial K}{\partial \vec{y}} K$ for any vector $\vec{y}$ and matrix K, so $\frac{\partial^2 f}{\partial \vec{x} \partial \vec{x}^\top} = \frac{\partial}{\partial \vec{x}}\left((A + A^\top)\vec{x}\right) = A + A^\top$.

3 Probability

a) For $i \in \{1,2,3,4,5,6\}$, $\text{pmf}_X(i) = P(X = i) = \frac{1}{6}$

$$\text{cdf}_X(i) = P(X \leq i) = \begin{cases} 0, & \text{if } i <= 0 \\ \frac{i}{6}, & \text{if } i \in \{1,2,3,4,5,6\} \\ 1, & \text{otherwise} \end{cases}$$

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$$\begin{aligned} P(X=4|X\text{is even}) &= \frac{P(X=4 \text{ and } X\text{is even})}{P(X\text{is even})} && \text{(13a)} \\ &= \frac{P(X=4)}{P(X\text{is even})} && \text{(13b)} \\ &= \frac{\frac{1}{6}}{\frac{1}{2}} && \text{(13c)} \\ &= \frac{1}{3} && \text{(13d)} \end{aligned}$$

$\text{Var}(X) = E[X^2] - E[X]^2$.

$$\begin{aligned} E[X] &= \sum_{i=1}^{6} \frac{i}{6} && \text{(14a)} \\ &= \frac{7}{2} && \text{(14b)} \end{aligned}$$

$$\begin{aligned} E[X^2] &= \sum_{i=1}^{6} \frac{i^2}{6} && \text{(15a)} \\ &= \frac{91}{6} && \text{(15b)} \end{aligned}$$

$$\begin{aligned} \text{Var}(X) &= E[X^2] - E[X]^2 && \text{(16a)} \\ &= \frac{91}{6} - \frac{49}{4} && \text{(16b)} \\ &= \frac{182}{12} - \frac{147}{12} && \text{(16c)} \\ &= \frac{35}{12} && \text{(16d)} \end{aligned}$$

We could have also used $\text{Var}(X) = E[X - E[X]^2]$ to arrive at the same answer.