Advanced Algorithms

Lecture 11
Approximation Algorithms for NP-Complete Problems

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ILO of Lecture 11

- Approximation algorithms
  - to understand the concepts of **approximation ratio** and **approximation algorithm**;
  - to understand the examples of approximation algorithms for the problems of vertex-cover and traveling-salesman.

Agenda

- P, NP, and NP-complete
- Approximation ratio, approximation algorithm, and approximation scheme
- Approximation algorithm for vertex-cover
- Approximation algorithm for traveling-salesman

P, NP, NP-complete

- P
  - Problems that are solvable in polynomial time.
- NP
  - Problems that are “verifiable” in polynomial time.
  - Any problem in P is also in NP.
    - $P \subseteq NP$
- NP-complete
  - A problem is in NP, and is as hard as any problem in NP.
  - No polynomial-time algorithm has yet been discovered.
  - No one is able to prove that no polynomial-time algorithm can exist for any one of them.
- Whether P=NP
  - It is unknown.
  - Most computer scientists believe that $P \neq NP$.

Example

- Subset sum problem
- Given a set of integers, is there a non-empty subset whose sum is *x*, e.g., 0?
  - Assume that the set has n integers in total.
  - Consider set {−3, −2, 1, 5, 8}

  - NP?
    - Yes, given any subset, you can verify if its sum is *x* in linear time O(n).
    - {1, 5, 8}
    - {−3, −2, 1, 5, 8}

  - P?
    - No, in the worst case, in order to identify a non-empty subset whose sum is x, we need to enumerate all 2^n possible subsets, thus having exponential runtime.

P, NP, NP-complete

Problems	Verifiable in Polynomial time	Solvable in polynomial time
P	Yes	Yes
NP	Yes	Yes or Unknown
NP-Complete	Yes	Unknown

Coping with NP-complete problems

- Many interesting and important problems are NP-complete.
- Do we always surrender if we have an NP-complete problem?
- NO! We have some ways to deal with an NP-complete problem.
    - If the actual inputs are small, an algorithm with exponential running time may be acceptable.
    - Come up approaches to find *near-optimal* solutions in polynomial time.
        - Approximation algorithm.
    - Use *heuristics* to speed up exponential running time.
        - Next lecture.

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Approximation ratios

- Suppose that
  - we are working on an *optimization* problem with input size n;
  - each solution has a cost value, and we want to identify the optimal solution, i.e., the one with the minimum or maximum possible cost;
  - optimal solution is C*, returned by an **exact** algorithm that runs in **exponential** time;
  - approximate solution is C, returned by an **approximation** algorithm that runs in **polynomial** time.
- Maximization problem:
  - $0 < C \leq C^*$, $\frac{C*}{C}$ gives a factor.
  - E.g., C*=100, C=90. $\frac{C*}{C} = \frac{10}{9}$
- Minimization problem:
  - $0 < C^* \leq C$, $\frac{C}{C*}$ gives a factor.
  - E.g., C*=100, C=110. $\frac{C}{C*} = \frac{11}{10}$

Approximation ratios

- A ρ(n)-approximation algorithm has an approximation ratio ρ(n), if, for any input size of n, it satisfies
  - $\max(\frac{C}{C*}, \frac{C*}{C}) \leq \rho(n)$.
  - It means that C is within a factor of ρ(n) of the optimal cost C*.
  - It provides a guarantee on the performance of an approximation algorithm.
    - Consider a 1.2-approximation algorithm with optimal cost C*=100.
    - For a minimization problem, the algorithm returns a value that is no larger than 100*1.2=120.
    - For a maximization problem, the algorithm returns a value that is no smaller than $\frac{100}{1.2}$=83.3.
- Approximation ratio is never smaller than 1.
  - $\frac{C}{C*} \leq 1$ implies $\frac{C*}{C} \geq 1$
- 1-approximation algorithm produces the optimal solution.

Approximation scheme

- An **approximation scheme** for an optimization problem is an **approximation algorithm** that takes as input
  - not only an instance of the problem,
  - but also a value $\varepsilon > 0$;
- Then, the scheme is a $(1+\varepsilon)$-approximation algorithm.
- Polynomial-time approximation scheme
  - For any fixed $\varepsilon > 0$, the scheme runs in polynomial time of n.
  - E.g., $O(n^{2/\varepsilon})$.
  - As ε decreases, the run time goes up quickly.
- Fully polynomial-time approximation scheme
  - The scheme runs in polynomial time of n and $1/\varepsilon$.
  - E.g., $O((1/\varepsilon)^2 n^3)$.
- When ε is small, $1/\varepsilon$ is thus big
  - Better approximation ratio, and thus longer run time.

Exam 2018

5. Take a careful look at the following statements and decide if they are correct.

5.1 (*2 points*) Consider an approximation algorithm with approximation ratio 1.1 for solving a NP-complete problem P. Assume that P is a **maximization** problem and its optimal solution is 100. Then, the approximation algorithm may return a value 105.

☐ **1)** Correct ☐ **2)** Wrong

5.2 (*2 points*) Consider an approximation algorithm with approximation ratio 2 for solving a NP-complete problem P. Assume that P is a **minimization** problem and its optimal solution is 100. Then, the approximation algorithm may return a value 201.

☐ **1)** Correct ☐ **2)** Wrong

Exam 2018

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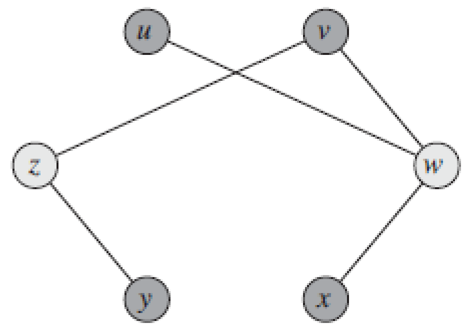
The vertex-cover problem

- Given an undirected graph G=(V, E)
- A vertex cover of G is a ***subset of vertices*** V’ ⊆ V, s.t.,
  - For each (u, v)∈ E, we have u ∈ V’ or v ∈ V’ or both.

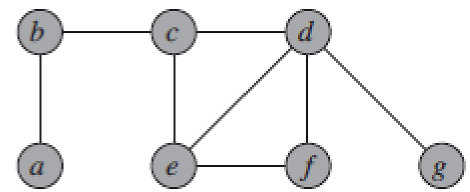
  - V_1’= {u, v, w, x, y, z}
  - V_2’= {w, z}
  - V_3’= {u, v, y, x}

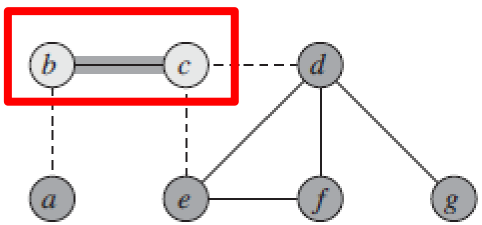


- The size of a vertex cover is the number of vertices in it.
  - Sizes of V_1’, V_2’, and V_3’ are 6, 2, and 4, respectively.
- Vertex-cover problem: find a vertex cover of minimum size.

Approximation algorithm

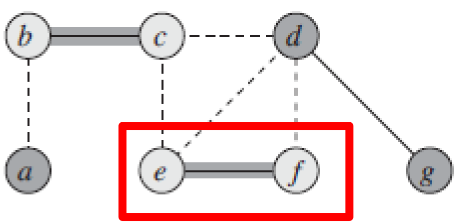
```
APPROX-VERTEX-COVER(G)
C = ∅
E' = G.E
while E' ≠ ∅
    let (u, v) be an arbitrary edge of E'
    C = C ∪ {u, v}
    remove from E' every edge incident on either u or v
return C
```





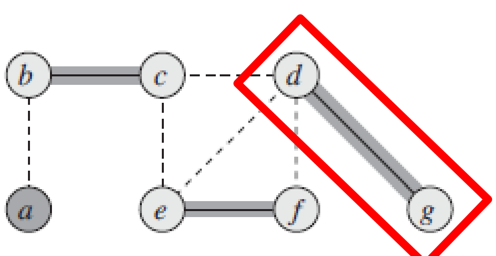
C={b, c}

Run time: O(V+E)



C={b, c, e, f}

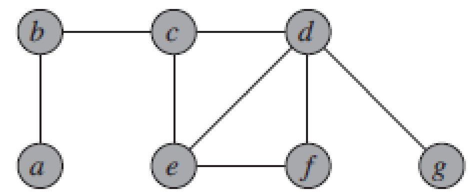
C*={b, d, e}



C={b, c, e, f, d, g}

Approximation ratio

- C*={b, d, e}, C={b, c, e, f, d, g}



- $\frac{C}{C*}$ =2
- What if we are lucky (i.e., having a lucky order in line 4), can we get a better solution or even exact solution?
  - Visit (d, e) first. Then (b, c)
  - C={d, e, b, c}
  - $\frac{C}{C*} = \frac{4}{3}$, better than 2.

APPROX-VERTEX-COVER(G)

1 $C = \emptyset$
2 $E' = G.E$
3 **while** $E' \neq \emptyset$
4   let (u, v) be an arbitrary edge of E'
5     $C = C \cup \{u, v\}$
6   remove from E' every edge incident on either u or v
7 **return** C

- What is the approximation ratio then?
  - Observing from the two examples, it should be at least 2.
  - Then, we need to prove the ratio.

Approximation ratio

- Let A denote the set of edges that line 4 picked.
  - $A \subseteq E$
  - Note that C is a set of vertices and A is a set of edges.
- To cover the edges in A, any vertex cover, including C*, must include at least one endpoint of each edge in A.
  - This is due to the definition of a vertex cover: a vertex cover contains at least one vertex of each edge.
  - The optimal vertex cover C* should also include at least one endpoint of each edge in A.

APPROX-VERTEX-COVER(G)

1 $C = \emptyset$
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3 **while** $E' \neq \emptyset$
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5 $C = C \cup \{u, v\}$
6 remove from E' every edge incident on either u or v
7 **return** C

Approximation ratio

- No two edges in A share an endpoint
  - Once an edge is picked in line 4 and is added into A, all the edges that share the edge's endpoints are deleted from E' in line 6.
- Thus, we have the lower bound |C*|≥|A|.

```
APPROX-VERTEX-COVER(G)
1  C = ∅
2  E' = G.E
3  while E' ≠ ∅
4      let (u, v) be an arbitrary edge of E'
5      C = C ∪ {u, v}
6      remove from E' every edge incident on either u or v
7  return C
```

Approximation ratio

- When line 4 picks an edge, both endpoints of the edge are added into C.
  - We have |C|=2|A|
- Considering the lower bound |C*|≥|A|, we have
  - |C|=2|A|≤2|C*|
  - Approximation ratio: $\frac{|C|}{|C*|} \leq 2$
- Conclusion: we have a 2-approximation algorithm.

APPROX-VERTEX-COVER(G)

```
1  C = Ø
2  E' = G.E
3  while E' ≠ Ø
4      let (u, v) be an arbitrary edge of E'
5      C = C ∪ {u, v}
6      remove from E' every edge incident on either u or v
7  return C
```

Reflection on approximation ratio proof

- How can we possibly prove the approximation ratio without even knowing the size of an optimal solution?

- Instead of knowing the exact size of an optimal solution, we rely on a lower bound on the size of an optimal solution.
  - Vertex-cover problem: $|C^*| \geq |A|$
- Next, we consider the relationship between the result returned by an approximation and the lower bound.
  - $|C|=2|A|$

- This is a common ***methodology*** used in approximation ratio proof.

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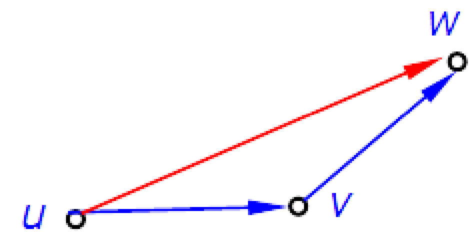
Traveling-salesman problem (TSP)

- Given
  - A list of cities and the distance between each pair of cities.
- Compute
  - the shortest *path that visits each city exactly once and returns to the origin city* or shortest *simple cycle with all vertices*?
- Given a **complete** undirected graph G=(V, E)
  - Every pair of vertices is connected by an edge.
  - Each vertex has *V*-1 edges to all remaining vertices.
- For each edge (u, v)∈ E, it has a nonnegative integer cost c(u, v), e.g., the Euclidean distance.
- Identify a Hamiltonian cycle of G with minimum cost.
  - A Hamiltonian cycle is a simple cycle that contains each vertex in V.
  - A simple cycle is a path $(v_0, v_1, v_2, \ldots, v_k)$ where $v_0=v_k$ and $v_1, v_2, \ldots, v_k$ are distinct.

Simplified TSP

- The cost function c satisfies triangle inequality.
- For all $u, v, w \in V$:
  - $c(u, w) \le c(u, v) + c(v, w)$



- These are natural simplifications
  - *Vertices* – points in the plane.
  - *Cost of an edge* – Euclidean distance between the two vertices of the edge.
- General TSP
  - Without the triangle inequality assumption.

A simpler but similar problem

- What is a tree?
  - A connected undirected graph without cycles.
- Is Hamiltonian cycle a tree?
  - No, because it is a cycle.
- Can we change a Hamiltonian cycle to a tree?
  - Yes, by deleting an edge to break the cycle.
- Can we start finding a tree with minimum total cost?
  - How about finding the minimum spanning tree (MST)?
  - Minimum spanning tree
    - A tree (connected, undirected graph without cycles)
    - Contains all vertices in V.
    - It has the minimum total cost.
- Convert the MST tree into a Hamiltonian cycle.
  - This Hamiltonian cycle may not be the optimal cycle with minimal cost but we can prove an approximation ratio for it.

Approximation algorithm

APPROX-TSP-TOUR(G, c)

1 select a vertex $r \in G.V$ to be a "root" vertex
2 compute a minimum spanning tree T for G from root r
    using MST-PRIM(G, c, r)
3 let H be a list of vertices, ordered according to when they are first visited
    in a preorder tree walk of T
4 **return** the hamiltonian cycle H

- Choose a vertex, say vertex r, as root.
- Compute a MST from the chosen root r.
- Preorder tree work on the MST.
  - Visits each vertex before visiting its children.

Prim's algorithm

```
MST-Prim(G,r)
01 for each vertex u ∈ G.V()
02     u.setkey(∞)
03     u.setparent(NIL)
04 r.setkey(0)
05 Q.init(G.V())    // Q is a priority queue ADT
06 while not Q.isEmpty()
07     u ← Q.extractMin()    // making u part of T
08     for each v ∈ u.adjacent() do
09         if v ∈ Q and G.w(u,v) < v.key() then
10             v.setkey(G.w(u,v))
11             Q.modifyKey(v)
12             v.setparent(u)
```

Initialize all vertices: Θ(|V|)

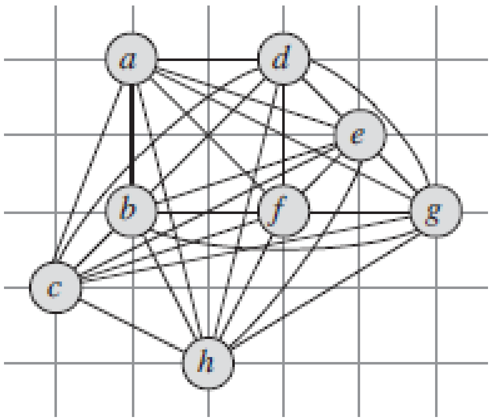
Initialize a priority queue with |V| elements. O(|V|)

While loop: |V| times
Line 7: Each Q.extractMin() takes O(lg|V|), in total O(|V|lg|V|).
Line 8: For loop: given a vertex, it iterates on all its adjacent vertices.
Together with the while loop, in total it iterates |E|.
Each iteration it calls Q.modifyKey, which takes O(lg|V|).
In total, O(|E|lg|V| + |V|lg|V|) = O(|E|lg|V|).

Prim's algorithm runs in polynomial time.

Example

- Choose vertex a as the root
- Identify the MST from a.



a	b	c	d	e	f	g	h
0/-	∞	∞	∞	∞	∞	∞	∞

b	d	c	e	f	g	h
$2/a$	$2/\mathrm{a}$	$\ldots$	...	...	...	...

c	d	h	e	f	g
$\sqrt{2}/b$	$2/\mathrm{a}$	$\sqrt{5}/b$	...	...	...

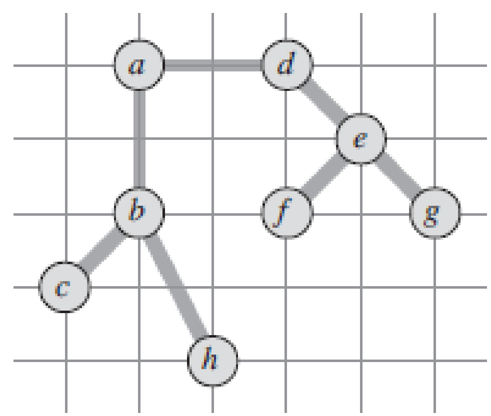
d	h	f	g	e
$2/\mathrm{a}$	$\sqrt{5}/b$	...	...	...

e	f	h	g
$\sqrt{2}/\mathrm{d}$	2/d	$\sqrt{5}/b$	...

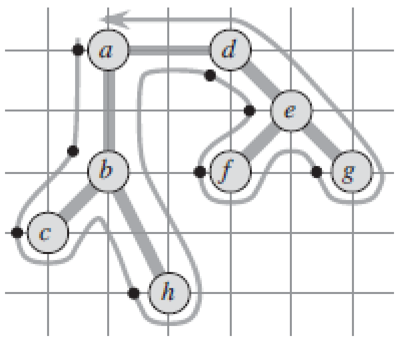
f	g	h
$\sqrt{2}/\mathrm{e}$	$\sqrt{2}/\mathrm{e}$	$\sqrt{5}/b$

g	h
$\sqrt{2}/\mathrm{e}$	$\sqrt{5}/b$

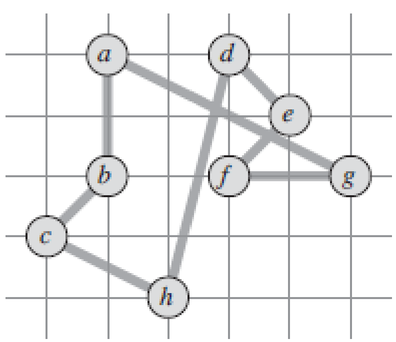
h
$\sqrt{5}/b$



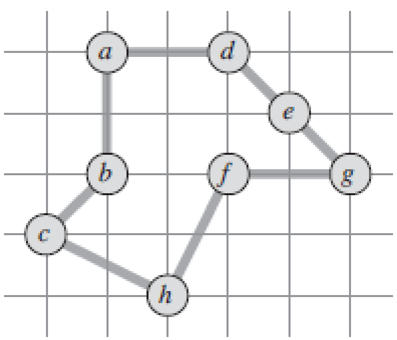
Pre-order tree walk on the MST



- a, b, c, h, d, e, f, g
- Add the root a to the end, so we have <a, b, c, h, d, e, f, g, a>



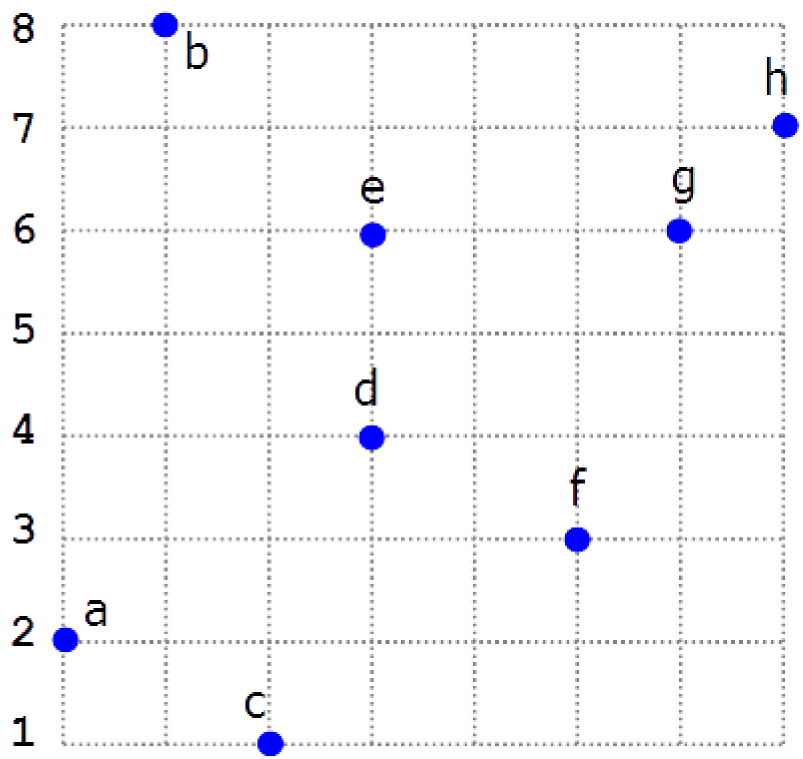
C=19.074



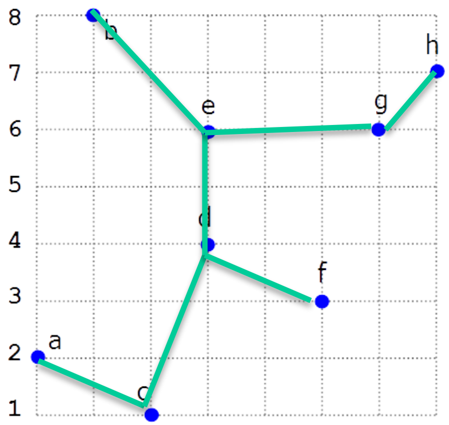
C*=14.715

Mini quiz (also on Moodle)

- Compute an approximate TSP tour:
  - Use vertex *a* as the starting vertex
  - When there is a choice (in Prim's and the pre-order tree walk), choose the alphabetically “smaller” vertex.



Example



a	b	c	d	e	f	g	h
0/-	∞	∞	∞	∞	∞	∞	∞

c	d	e	f	b	g	h
$\sqrt{5}/a$	$\sqrt{13}/\mathrm{a}$	5/a	$\sqrt{26}/\mathrm{a}$	$\sqrt{37}/\mathrm{a}$	...	...

d	f	e	b	g	h
$\sqrt{10}/\mathrm{c}$	$\sqrt{13}/\mathrm{c}$	5/a	$\sqrt{37}/\mathrm{a}$	$\dots$	...

e	f	b	g	h
$2/\mathrm{d}$	$\sqrt{5}/\mathrm{d}$	$\sqrt{20}/\mathrm{d}$	$\dots$	...

f	b	g	h
$\sqrt{5}/\mathrm{d}$	$\sqrt{8}/\mathrm{e}$	3/e	...

b	g	h
$\sqrt{8}/\mathrm{e}$	3/e	...

g	h
3/e	...

h
$1/g$

Pre-order tree walk:
a, c, d, e, b, g, h, f

Approximate result:
a, c, d, e, b, g, h, f, a

Approximation ratio

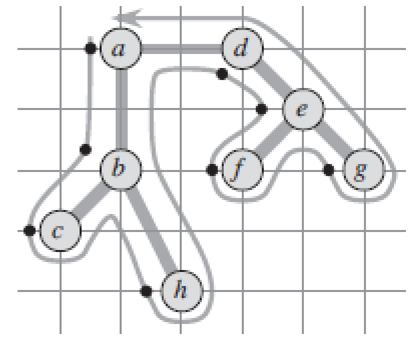
- Recall the methodology for proving approximation ratio.
  - Based on the lower bound of optimal result.

- Let H* denote an optimal cycle and H denote the cycle identified by our approximation algorithm.
- Let T be a minimum spanning tree.
- By deleting any edge from H*, we will get a spanning tree.

- Recall that each edge has a non-negative cost.
- Thus, we have $c(T) \leq c(H^*)$.
  - This is the lower bound.
- What is the relationship between c(H) and the lower bound c(T)?
  - To this end, we introduce a new concept called **full walk**.

Approximation ratio

- A ***full walk*** of a tree T lists the vertices when they are first visited and also whenever they are returned to after a visit to a sub-tree.



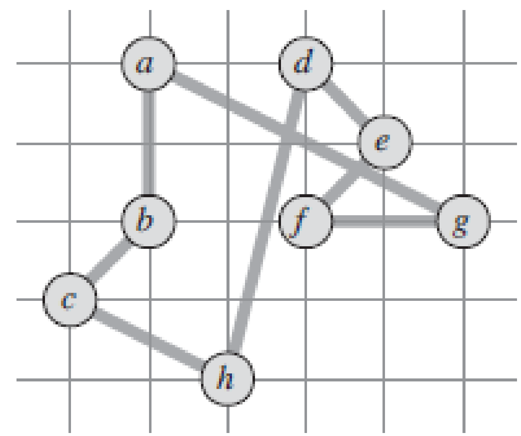
- Full walk, denoted as W:
  - a, b, c, b, h, b, a, d, e, f, e, g, e, d, a.
  - <a, b>, <b, c>, <c, b>, <b, h>, <h, b>, <b, a>, <a, d>, <d, e>, <e, f>, <f, e>, <e, g>, <g, e>, <e, d>, <d, a>.
  - c(W)=2c(T), as it contains every edge in the MST T twice.
- Consider the lower bound $c(T) \leq c(H^*)$, we have
  - $\mathbf{c(W) \leq 2c(H^*)}$
- What is the relationship between the full walk W and the approximated cycle H?
  - c(W) and c(H)?

Approximation ratio

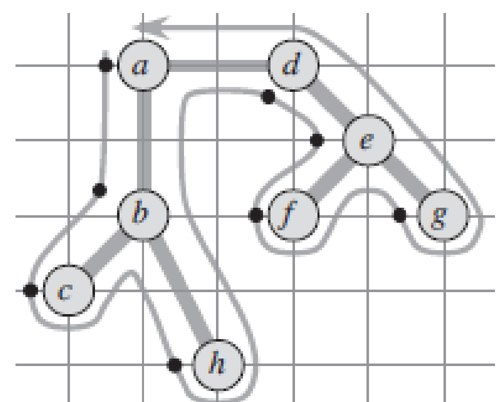
- Pre-order walk: a, b, c, h, d, e, f, g
  - H=<a, b>, <b, c>, $\boxed{\text{<c, h>}}$, <h, d>, <d, e>, <e, f>, <f, g>, <g, a>
  - W=<a, b>, <b, c>, $\boxed{\text{<c, b>, <b, h>}}$, <h, b>, <b, a>, <a, d>, <d, e>, <e, f>, <f, e>, <e, g>, <g, e>, <e, d>, <d, a>.

H	W
<a, b>	<a, b>
<b, c>	<b, c>
<c, h>	<c, b>, <b, h>
<h, d>	<h, b>, <b, a>, <a, d>
<d, e>	<d, e>
<e, f>	<e, f>
<f, g>	<f, e>, <e, g>
<g, a>	<g, e>, <e, d>, <d, a>

H



W



Due to triangle inequality, we have c(H)≤c(W). Thus, c(H) ≤ c(W) ≤ 2c(H*)

Approximation ratio

- From $c(H) \leq c(W) \leq 2c(H^*)$, we have
- $\dfrac{c(H)}{c(H*)} \leq 2$
- Thus, approximation ratio is 2.

- This means that the approximate cycle will never have more than twice distance of the optimal cycle.

No efficient ρ-approximation

- Do all NP-complete problems have polynomial ρ-approximation algorithms (where ρ is a constant)?
  - No!

- Next, we will prove that the general TSP problem cannot have a polynomial *ρ*-approximation algorithm, unless P=NP.

- In the general TSP problem, we drop the assumption that the cost function c satisfies the triangle inequality.
  - E.g., use travel times as costs, but not Euclidean distances.

Proof sketch

- Given a graph G=(V, E), a ***Hamiltonian cycle*** of G is a simple cycle that contains each vertex in V.
- What is the Hamiltonian-cycle problem?
  - A decision problem: does a graph G have a Hamiltonian cycle?
  - It is a NP-complete problem, Theorem 34.13.
  - Solving it in polynomial time implies P=NP, Theorem 34.4.
- Proof by *contradiction*:
  - Since the Hamiltonian cycle problem is NP-complete, no polynomial time algorithms exist unless P=NP.
  - If there exists a ***polynomial*** ρ-approximation algorithm A for solving general TSP, we are also able to use A to solve the Hamiltonian-cycle problem (How? see the comining slides).
  - Recall that A is polynomial. This means that we use A to solve the Hamiltonian cycle problem in ***polynomial*** time.
  - This is a contradiction, unless P=NP. In other words, if P$\neq$NP, this is a contradiction.

Hamiltonian cycle problem to General TSP

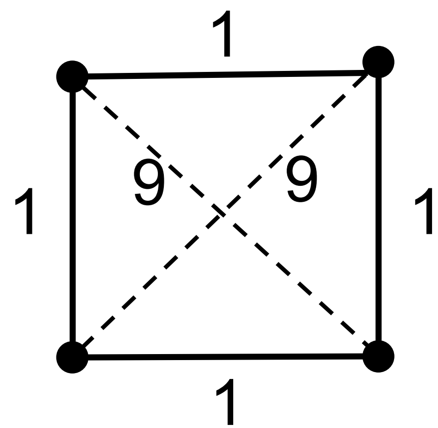
- Suppose we have a polynomial time, approximation algorithm A with approximation ratio ρ for general TSP.
  - Assume ρ is an integer.
- We now show how to use A to solve the Hamiltonian cycle problem.
  - Given a graph G=(V, E), whether or not there is a Hamiltonian cycle in G.
- We turn G into a ***complete*** graph G'=(V, E')
  - Assign an integer cost to each edge in E'.

$$c(u,v)=\begin{cases} 1 & \text{if } (u,v)\in E \\ \rho|V|+1 & \text{otherwise} \end{cases}$$

  - For example, assuming that we have ρ=2, then we have the edge weights of 2|V|+1 for all newly added edges in E'.
- Now we consider a general TSP on G' with cost function c.

Hamiltonian cycle problem to General TSP

- Assume ρ=2. |V|=4.



$$c(u,v)=\begin{cases}1 & \text{if } (u,v)\in E \\ \rho|V|+1 & \text{otherwise}\end{cases}$$

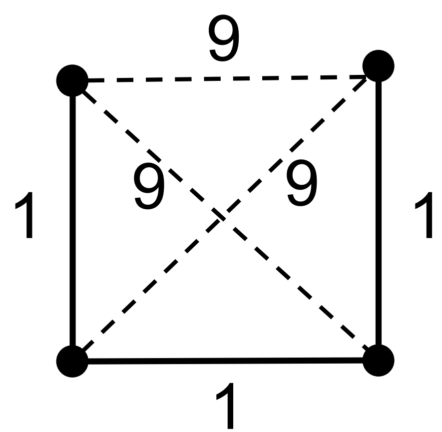
The weights do not satisfy triangle-inequality anymore.

- If the original graph G has a Hamiltonian cycle H*
  - Each edge should have cost 1 and in total |V| edges. Thus, H*'s cost is |V|, i.e., C(H*)=|V|.
    - This example: C(H*)=4.
  - If we use the ρ-approximation algorithm A, it will return a cycle H with cost at most ρ|V|, i.e., C(H)≤ ρC(H*)=ρ|V|.
    - This example: C(H) ≤8.

Hamiltonian cycle problem to General TSP

- Assume ρ=2. |V|=4.

$$c(u,v)=\begin{cases}1 & \text{if } (u,v)\in E\\ \rho|V|+1 & \text{otherwise}\end{cases}$$



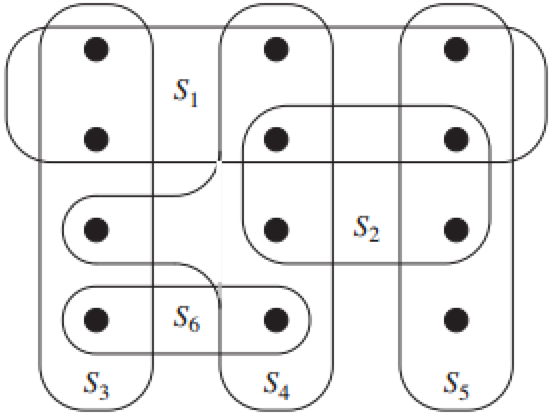
- If the original graph G does not have a Hamiltonian cycle
  - Then any Hamiltonian cycle in G' must use some (at least one) edges that are not in E, i.e., some newly added edges.
    - In the best case, we use only one newly added edge, we have (ρ|V|+1)+(|V|-1) = ρ|V| + |V| > ρ|V|
    - This example: 1+1+1+9=12 > ρ|V| = 8

Hamiltonian cycle problem to General TSP

- So this means that, if the ρ-approximation algorithm A returns
  - A cycle whose cost is at most $\rho|V|$, G has a Hamiltonian cycle.
  - A cycle whose cost is more than $\rho|V|$, G has no Hamiltonian cycle.
- Therefore, we can use A to solve the Hamiltonian-cycle in polynomial time because A is a polynomial approximation algorithm.
- Since the Hamiltonian-cycle problem is NP-complete, there does not exist a polynomial time algorithm unless P=NP.
- This is a contradiction unless P=NP.
  - This is a contradiction if $P \neq NP$.

Set-covering problem

- The set-covering problem
- Given a finite set X and a family F of subsets of X. The problem is to find a minimum-size subset C⊆ F whose members cover all of X.
  - Black dots are the elements in X.
  - $F=\{S_1, S_2, S_3, S_4, S_5, S_6\}$ each Si contains some elements in X (black dots).
  - $C^*=\{S_3, S_4, S_5\}$



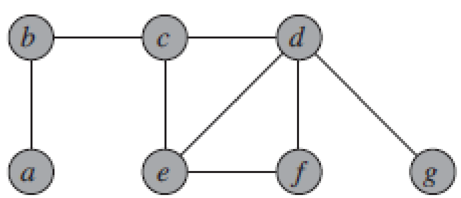
- A greedy approximation algorithm
  - At each stage, picking up the set S that covers the greatest number of remaining uncovered elements.
  - $C=\{S_1, S_4, S_5, S_3\}$
  - $(\ln|X|+1)$-approximation algorithm
    - Approximation ratio is not a constant anymore.

Set-covering vs. vertex cover

- Exam 2016

2 In Lecture 11, we have seen a 2-approximation algorithm (denoted as ALG1) for solving the **vertex cover** problem. We also briefly talked about a $(\ln |X| + 1)$-approximation algorithm (denoted as ALG2) for solving the **set cover** problem.

- (*10 points*) Actually, the **set cover** problem can be regarded as a generalization of the vertex cover problem. Show how can you transform a vertex cover problem into a set cover problem.



- X represents all edges.
- Each vertex is a subset of X, which contains the edges that are incident to the vertex.

ILO of Lecture 11

- Approximation algorithms
  - to understand the concepts of **approximation ratio**, **approximation algorithm**;
  - to understand the examples of approximation algorithms for the problems of vertex-cover and traveling-salesman.