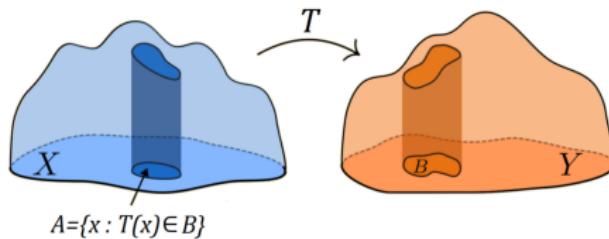


Introduction to Optimal Transport¹²³

“Moving Sandcastles in the Air”

J. Setpal

March 26, 2025



¹Peyré, Cuturi. [Arxiv 2020]

²Arjovsky, et. al. [Arxiv 2017]

³Heitz, et. al. [CVPR 2021]

Outline

- ① Motivation
- ② Monge Problem, Kantorovich Relaxation
- ③ Kantorovich Problem's Dual Formulation
- ④ Optimal Transport Induces a Distance
- ⑤ Wasserstein GANs

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Why Should We Care? (1/3)

Monge likes playing with sandcastles.

He wonders, “What is the most efficient way to move this marvellous sandcastle from the beach to my house?”

And **Optimal Transport** was born.

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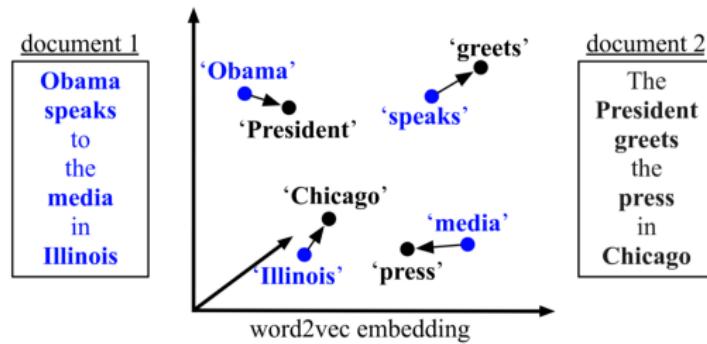
Why should you care:

1. You like playing with sandcastles.
2. You're interested in any of the following research foci:
 - a. **Neural Style Transfer:**



Why Should We Care? (2/3)

2. b. Sentence Similarity (Word Mover's Distance):

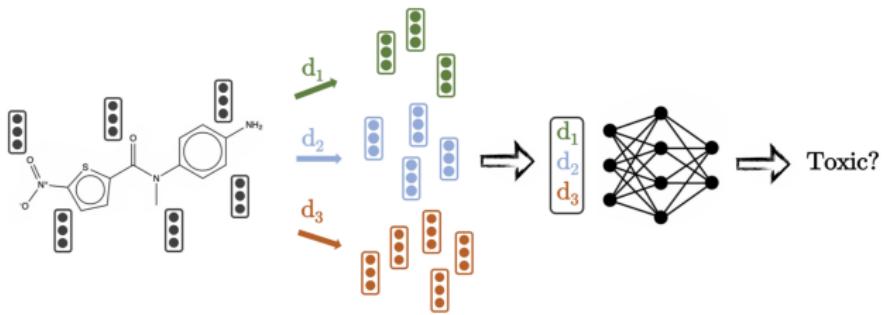


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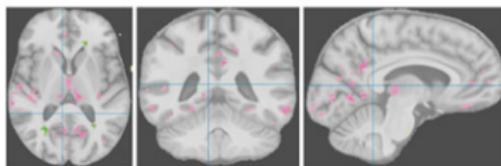
c. Graph Neural Networks (Better Representation Learning):



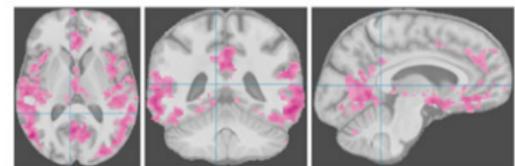
Why Should We Care? (3/3)

2. d. **Medical Imaging** (Gray Matter Tissue loss for Dementia):

TBM



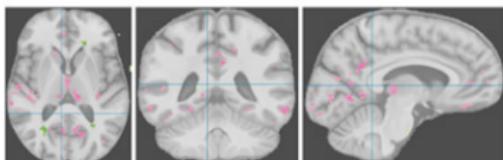
TBM with OTF



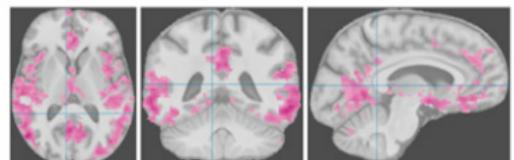
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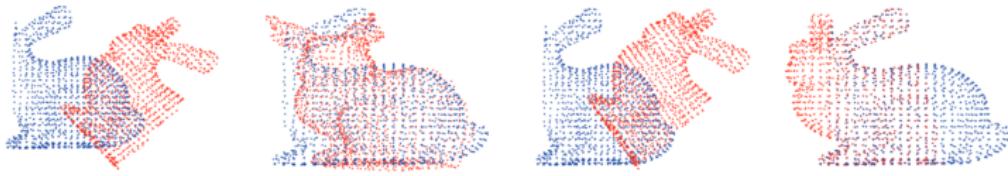


TBM with OTF



A horizontal color bar with a gradient from dark purple on the left to dark green on the right. The values -0.65 and 0.65 are labeled at the ends of the bar.

e. Robust Point-Cloud Matching:



Initialization

CPD

Trimmed ICP

Proposed

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Geometry Induced by OT on the Probability Simplex

We start with the probability simplex:

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Aside

OT literature deals with both discrete and continuous measures using the same framework. We'll focus mostly on the discrete setting.

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Monge asks us to transfer measure α to a new measure β while also minimizing the total cost of transportation.

$$\alpha(x) = \sum_{i=1}^n \mathbf{a}_i \chi_{x_i}(x), \quad \beta(y) = \sum_{i=1}^m \mathbf{b}_i \chi_{y_i}(y) \quad (3)$$

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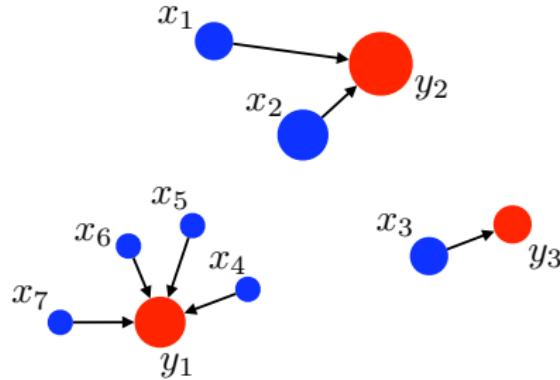
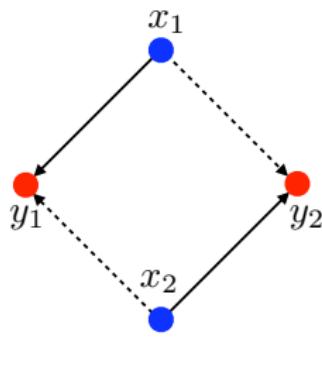
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If $n = m$, $T \in \text{Perm}(n)$.

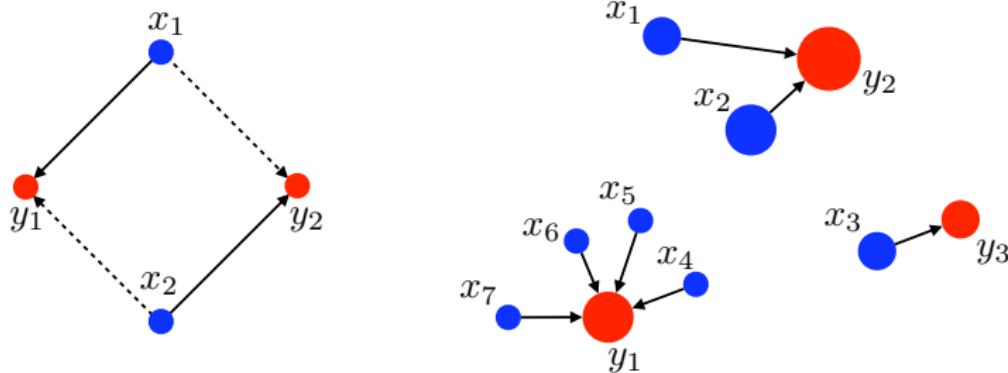
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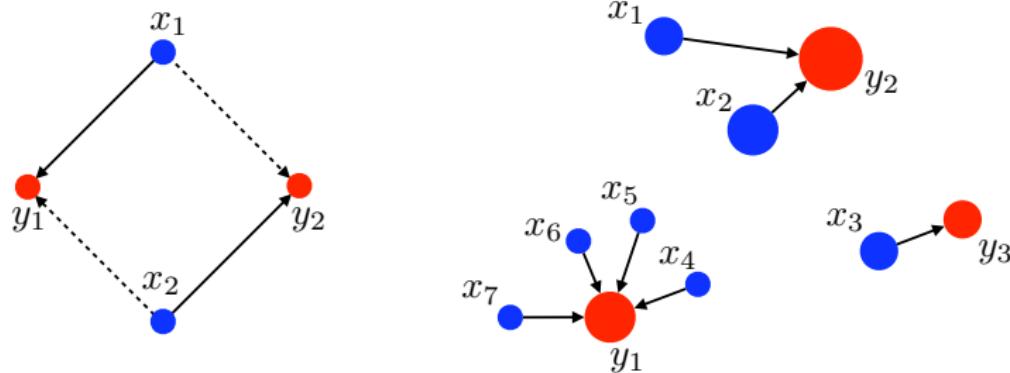


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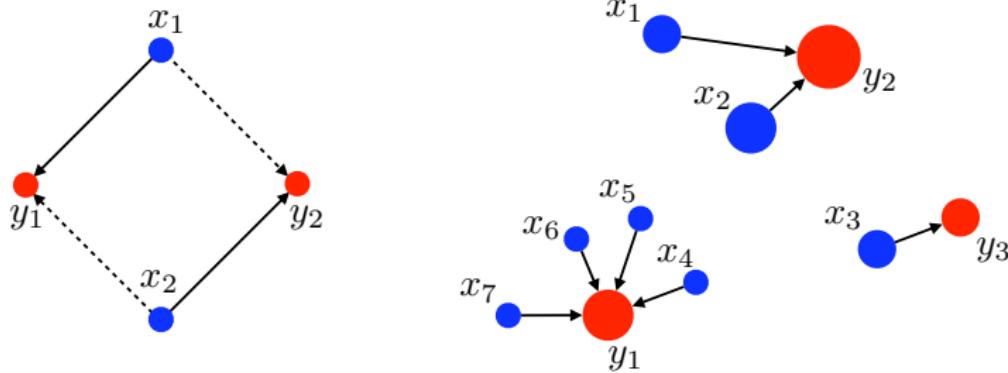


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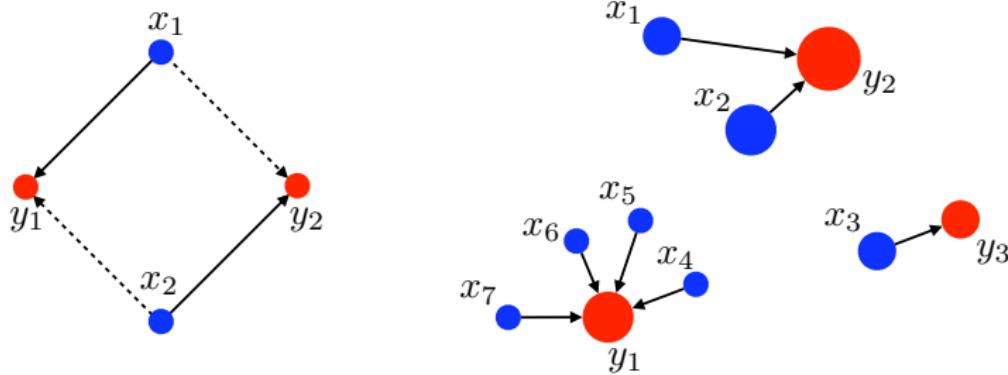


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4. Complexity scales sharply and optimization landscape is non-convex.

Push-Forward & Pull-Back Operators

For every valid transport map, we know that the following is satisfied:

$$\forall j \in \{1, \dots, m\}, \quad \mathbf{b}_j = \sum_{i: T(i)=y_j} \mathbf{a}_i \quad (5)$$

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Push-Forward and Pull-Back operators are related as follows:

$$\forall (\alpha, g) \in \mathcal{M}(X) \times \mathcal{C}(Y), \quad \int_Y g d(T_\sharp \alpha) = \int_X T^\sharp g d\alpha \quad (8)$$

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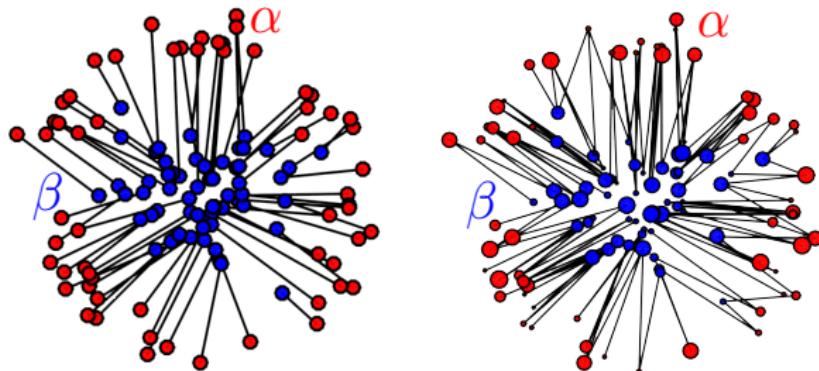
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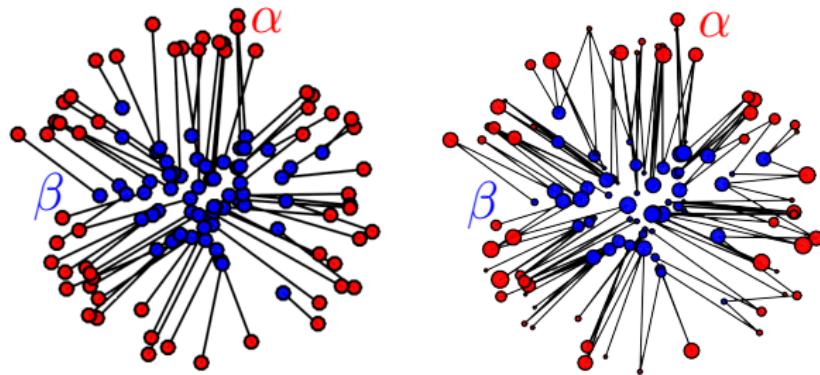
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BIG Observation: This is a linear program.

Kantorovich Relaxation (2/2)



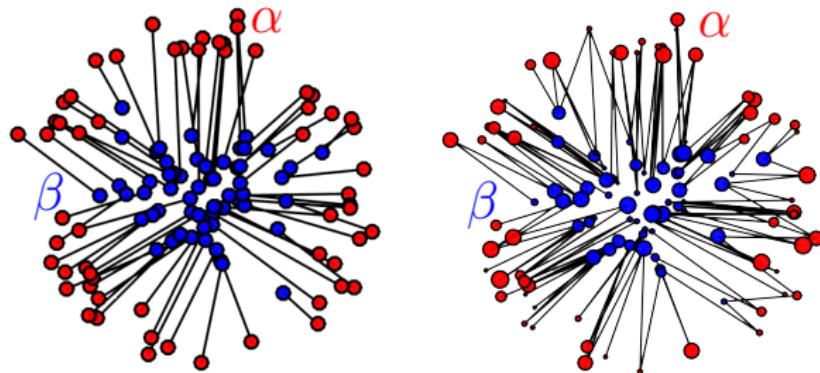
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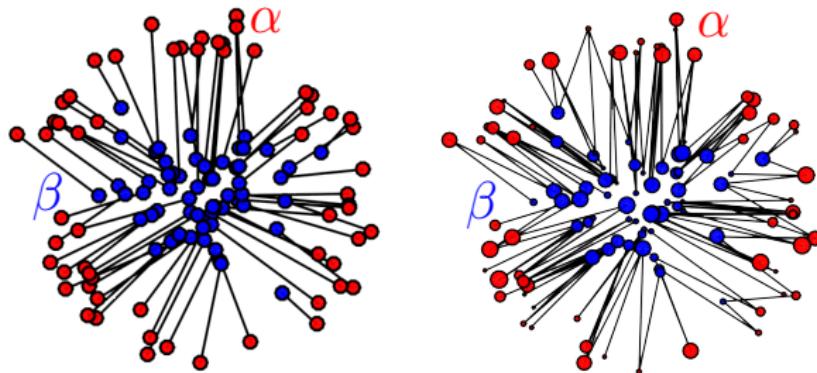


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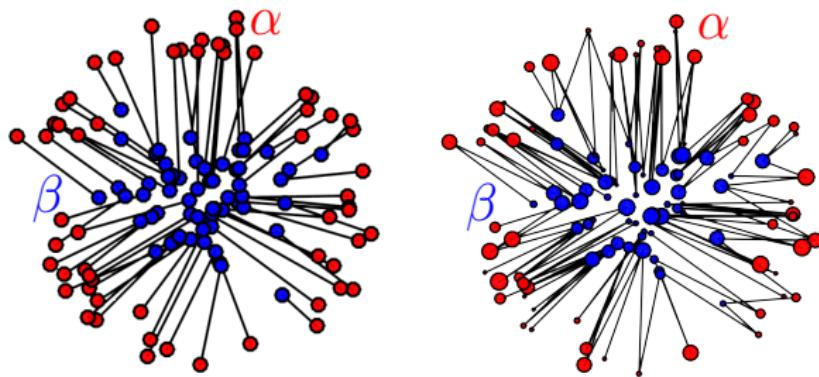
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2. Each coupling \mathbf{P} is symmetric: $\mathbf{P} \in \mathcal{U}(\mathbf{a}, \mathbf{b}) \iff \mathbf{P}^T \in \mathcal{U}(\mathbf{a}, \mathbf{b})$.

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5. If we know an optimal solution exists, we can choose to solve the easier problem and get the same answer.

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From there, we have the following dual problem:

$$L_{\mathbf{C}}(\mathbf{a}, \mathbf{b}) = \max_{\mathbf{f}, \mathbf{g} \in \mathcal{R}(\mathbf{C})} \langle \mathbf{f}, \mathbf{a} \rangle + \langle \mathbf{g}, \mathbf{b} \rangle \quad (13)$$

The dual variables, here \mathbf{f}, \mathbf{g} are called Kantorovich Potentials.

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To check the optimality of the vendor's prices, the operator can use $\mathbf{C}_{i,j}$:

$$\forall (i,j), \quad \mathbf{f}_i + \mathbf{g}_j \stackrel{?}{\leq} \mathbf{C}_{i,j} \quad (16)$$

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Let $n = m$, $p \geq 1$, $\mathbf{C} = \mathbf{D}^p = (\mathbf{D}_{i,j}^p)_{i,j} \in \mathbb{R}^{n \times n}$. We can verify:

1. $\mathbf{D} \in \mathbb{R}_+^{n \times n}$ is symmetric.

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Using this, we define the **Wasserstein Distance**:

$$W_p(\mathbf{a}, \mathbf{b}) := L_{\mathbf{D}^p}(\mathbf{a}, \mathbf{b})^{1/p} \quad (17)$$

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$$W_2(T_{\tau\sharp}\alpha, T_{\tau'\sharp}\beta)^2 = W_2(\tilde{\alpha}, \tilde{\beta})^2 - \|\mathbf{m}_\alpha - \mathbf{m}_\beta\|^2 \quad (18)$$

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This distinction implies a two-fold comparison: the shapes of measures α and β , and the distance between their means.

Sliced Wasserstein Distance (1/4)

One special case of Optimal Transport is the 1-D case; $\mathcal{X} = \mathbb{R}$. Assuming uniform weights⁴ and $c(x, y) = \|x - y\|_p^p$, we have:

$$\alpha = \frac{1}{n} \sum_{i=1}^n \chi_{x_i}, \quad \beta = \frac{1}{n} \sum_{i=1}^n \chi_{y_i} \quad (19)$$

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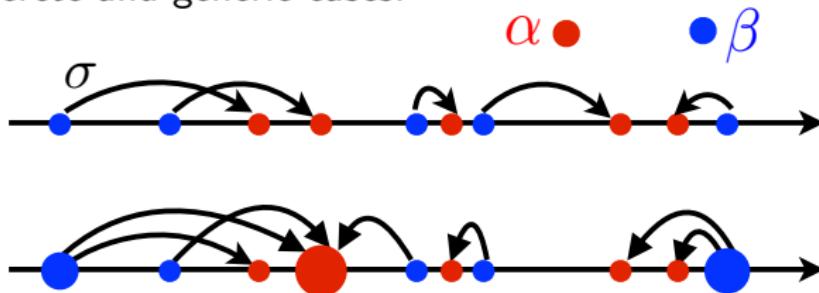
$$W_p(\alpha, \beta)^p = \frac{1}{n} \sum_{i=1}^n |x_i - y_i|^p \quad (21)$$

This **reduces OT to a sorting problem**, and can be solved in $\mathcal{O}(n \log n)$.

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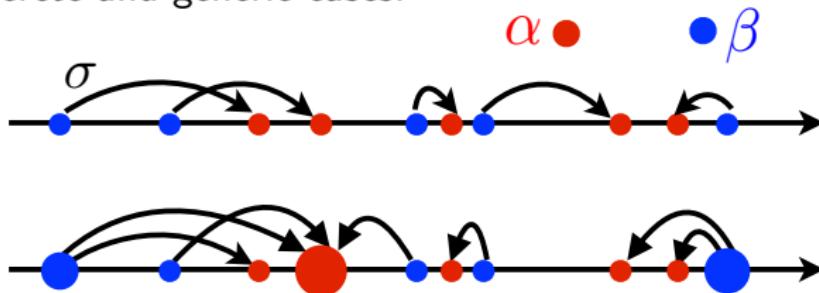
Sliced Wasserstein Distance (2/4)

Visual for discrete and generic cases:



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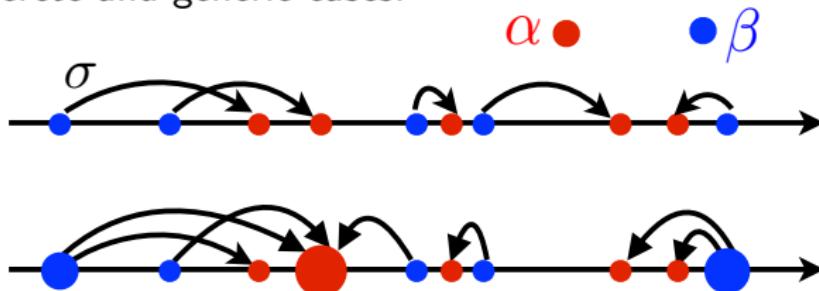
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This is nice, but somewhat limited. Can we extend this notion to \mathbb{R}^n ?

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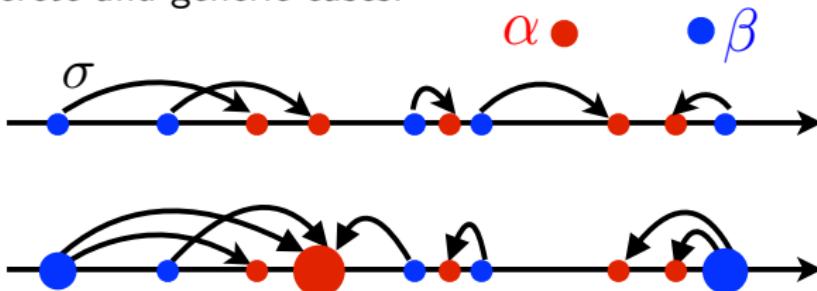
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1. Project n features onto d random directions.

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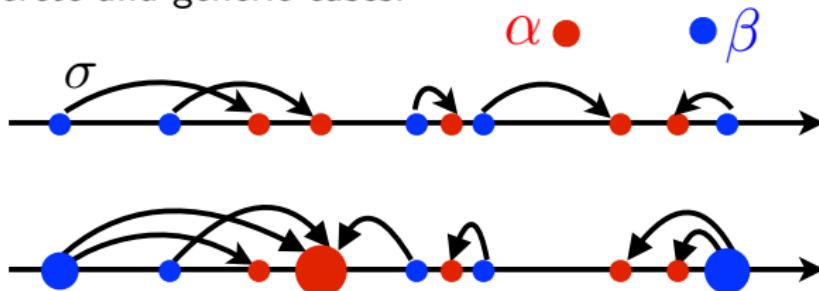
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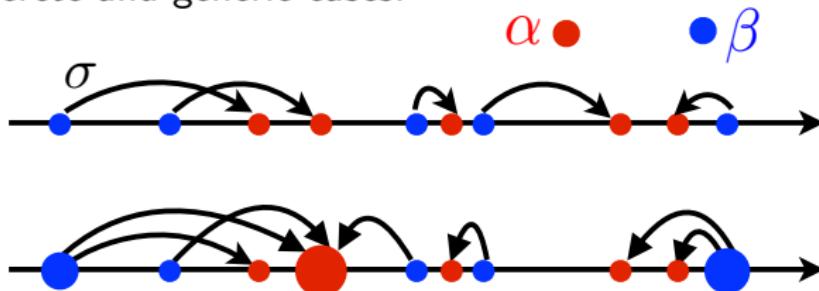
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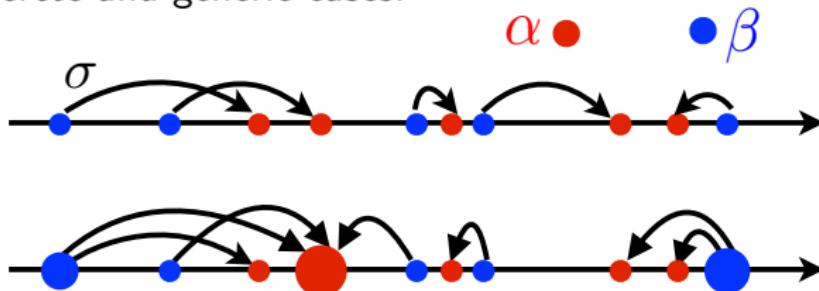
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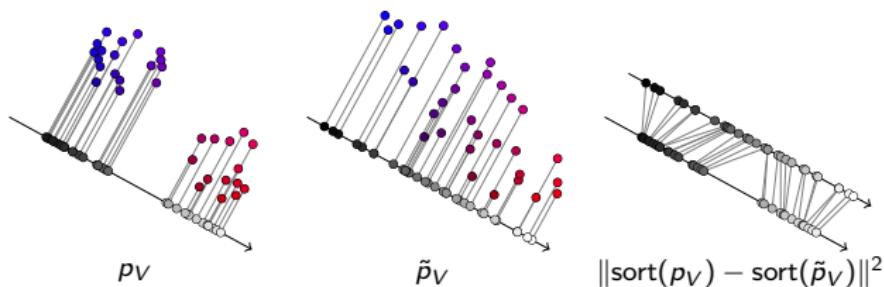
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Caveat: This is no longer the p -Wasserstein Distance.

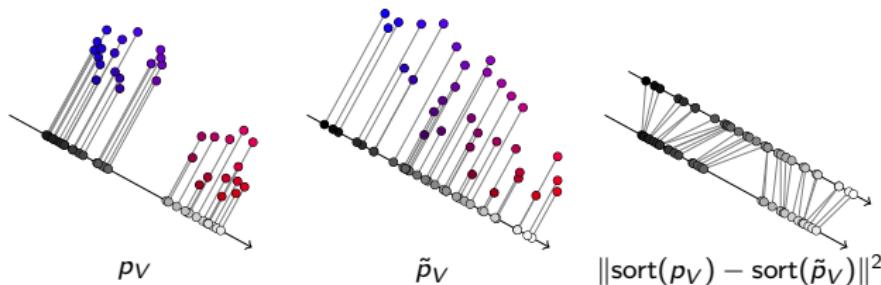
Sliced Wasserstein Distance ($^{3/4}$)

Here's what that looks visually, for a single direction:

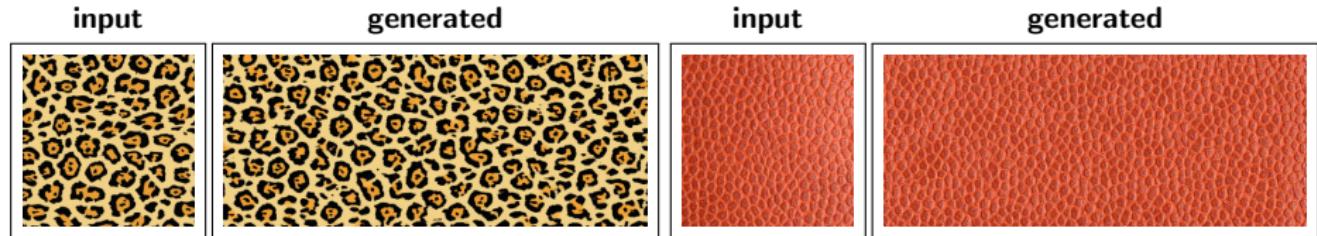


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Crucially, Sliced Wasserstein Distance is **differentiable**, which enables us to use optimize transport cost using neural nets. E.g. texture matching:



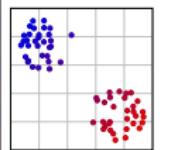
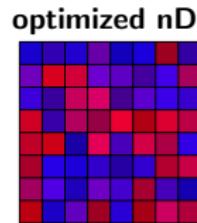
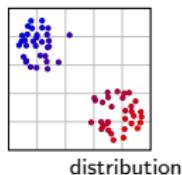
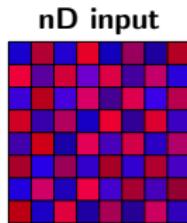
Sliced Wasserstein Distance (4/4)

Spatial Priors: Projections act on point clouds, which rids spatial information in learning the input distribution.

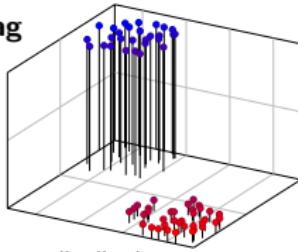
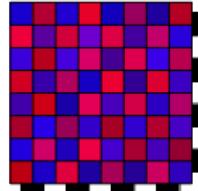
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A trick to recover spatial structure is to cluster-sort by spatial dimension:

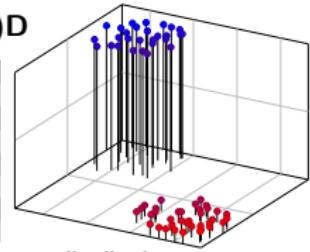
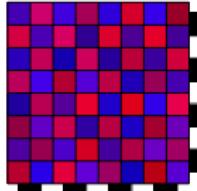


nD input + 1D tag



features + tag

optimized (n+1)D



distribution

Outline

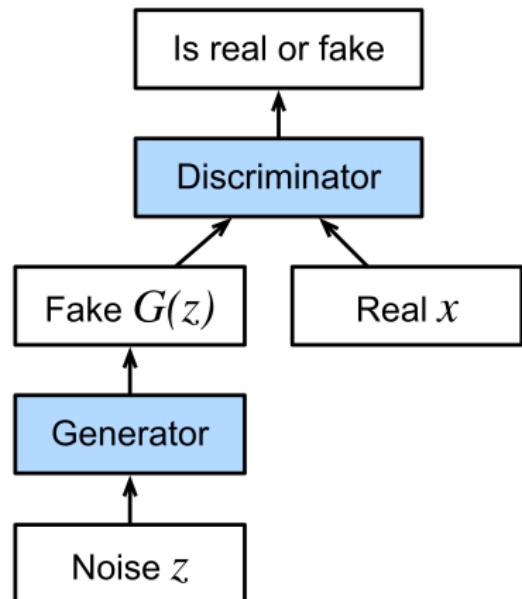
- ① Motivation
- ② Monge Problem, Kantorovich Relaxation
- ③ Kantorovich Problem's Dual Formulation
- ④ Optimal Transport Induces a Distance
- ⑤ Wasserstein GANs

Wasserstein GAN Setup

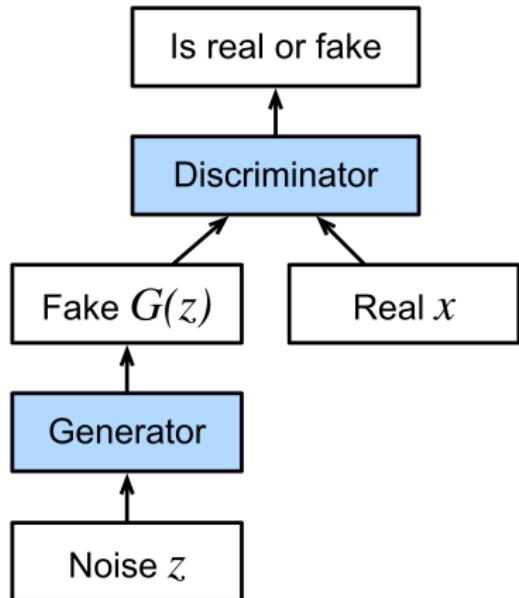
GANs have the following setup:

Discriminator $f_\xi : \mathbb{R}^{C \times D_1 \times D_2} \rightarrow [0, 1]$

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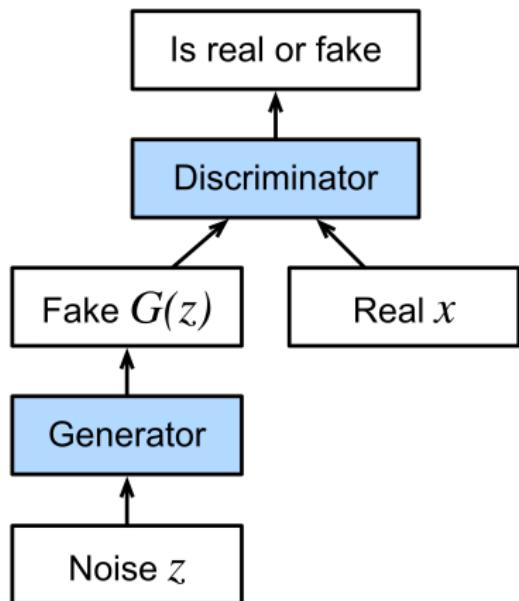
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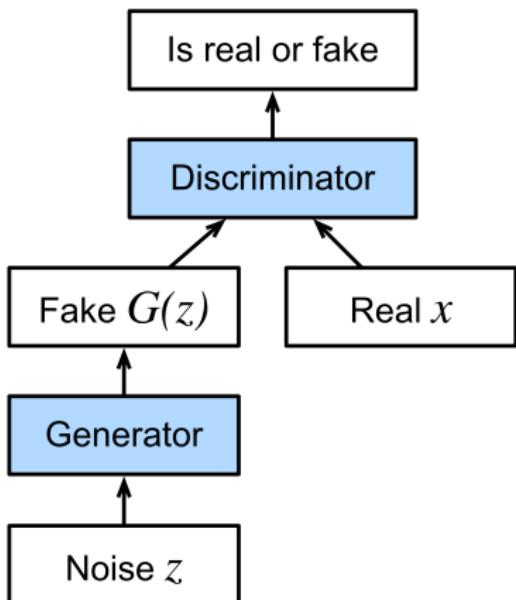
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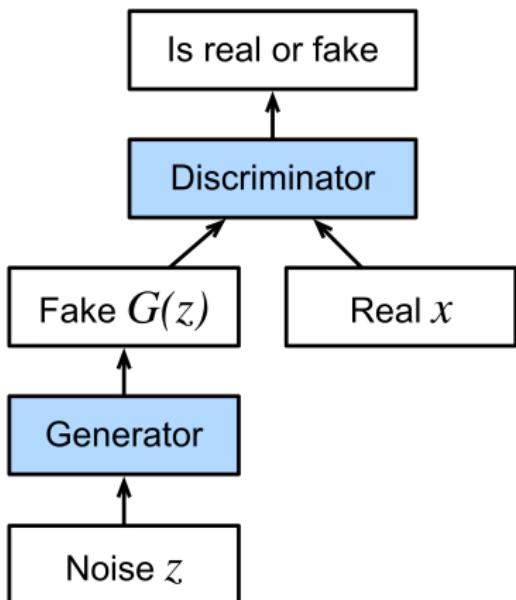
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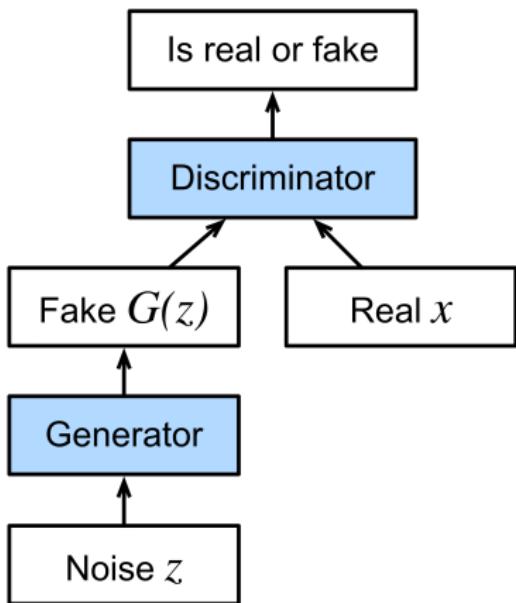
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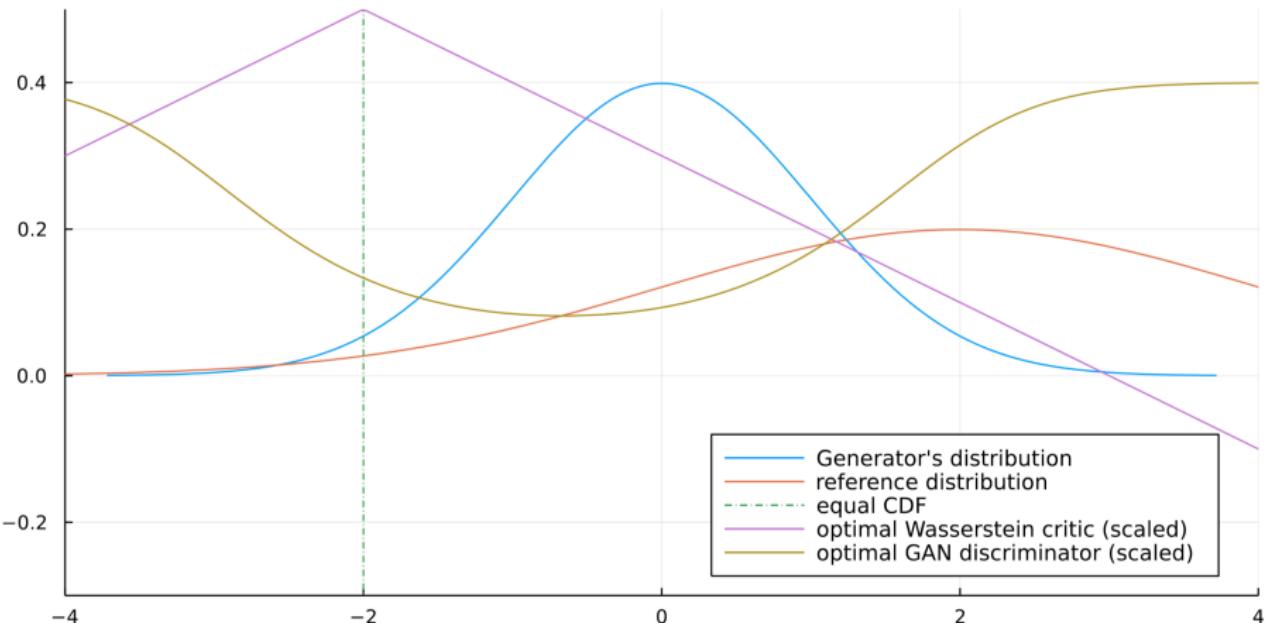
New **Discriminator** $f_\xi : \mathbb{R}^{C \times D_1 \times D_2} \rightarrow \mathbb{R}$
which models Wasserstein Distance.

Training WGANs

Algorithm 1 WGAN training algorithm. $\eta = 10^{-5}$, $c = 0.01$, $n_{\text{critic}} = 5$, $n_{\text{iter}} = 500$.

```
1: for  $t = 0, \dots, n_{\text{iter}}$  do
2:   for  $t = 0, \dots, n_{\text{critic}}$  do
3:     Sample  $\{x_i\}_{i=1}^B \sim \mathcal{D}^B$  a batch from the real data.
4:     Sample  $\{z_i\}_{i=1}^B \sim \mathcal{P}^B$  a batch of prior samples.
5:      $g_\xi \leftarrow \nabla_\xi \left[ \frac{1}{B} \sum_{i=1}^B f_\xi(x_i) - \frac{1}{B} \sum_{i=1}^B f_\xi(G_\theta(z_i)) \right]$ 
6:      $\xi \leftarrow \xi + \eta \cdot \text{Adam}(g_\xi)$ 
7:      $\xi \leftarrow \text{clip}(\xi, -c, c)$ 
8:   end for
9:   Sample  $\{z_i\}_{i=1}^B \sim \mathcal{P}(z)$  a batch of prior samples.
10:   $g_\theta \leftarrow -\nabla_\theta \frac{1}{B} \sum_{i=1}^B f_\xi(G_\theta(z_i))$ 
11:   $\theta \leftarrow \theta - \eta \cdot \text{Adam}(g_\theta)$ 
12: end for
```

Critic Improvements from Wasserstein GANs



Code Example – Training WGANs

If you can view this screen, I am making a mistake.

Thank you!

Have an awesome rest of your day!

Slides: <https://jinen.setpal.net/slides/ot.pdf>