대학원 신입생 세미나 과제1: Latex 문서 만들기

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Borel-Cantelli lemma

Let $A_1, A_2, ...$ be a sequence of events in a probability space. If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then the probability that infinitely many of these events occur is 0. That is:

$$P(\limsup_{n\to\infty} A_n) = P(A_n \ i.o.) = 0$$

Here, $\limsup\limits_{n\to\infty} A_n$ denotes the set of outcomes that occur infinitely often (i.o.) within the infinite sequence of events. Explicitly, $\limsup\limits_{n\to\infty} A_n = \bigcap\limits_{n=1}^{\infty} \bigcup\limits_{m=n}^{\infty} A_m$. This states that if the sum of the probabilities is less than infinity for an infinite sequence of events, then the set of outcomes that are "repeated" infinitely many times must occur with probability 0.

Proof) Let $A'_n = \bigcup\limits_{m=n}^{\infty} A_m$. Since $\{A'_n\}$ is a non-increasing sequence of events, by continuity and subadditivity of P,

$$\begin{aligned}
\mathrm{P}(\limsup_{n\to\infty} A_n) &= \mathrm{P}(\bigcap_{n=1}^{\infty} A'_n) \\
&= \lim_{n\to\infty} \mathrm{P}(A'_n) \\
&\le \lim_{n\to\infty} \sum_{m=n}^{\infty} \mathrm{P}(A_m)
\end{aligned}$$

Since $\sum_{n=1}^{\infty} P(A_n) < \infty$ by assumption, the tail-sum $\sum_{m=n}^{\infty} \mathrm{P}(A_m)$ tends to 0 as $n \to \infty$.

Reference

Rick Durrett (2019) *Probability: Theory and Examples*, Cambridge University Press, 5th ed., p58