Suggested answers for problem 1 & 2

Problem 1

Question 1:

(a)

$$q^i(\theta|w^i) = \frac{f^i(w^i|\theta)f(\theta)}{\sum_\theta f^i(w^i|\theta)f(\theta)}$$

(b)

$$\pi(w^1, \cdots, w^i, \cdots, w^N, \theta|w^i) = \frac{q(\theta|w^i)\prod_{j\neq i} f^j(w^j|\theta)}{\sum_{\sigma\,:\,w^i\in\sigma} q(\theta|w^i)\prod_{j\neq i} f^j(w^j|\theta)}$$

(c) For any σ s.t. $w \notin \sigma$, $\pi^{FI}(\sigma|w) = 0$. So it is only interesting to look at $\pi^{FI}(\theta|w)$, which is

$$\pi^{FI}(\theta|w) = \frac{g(w,\theta)}{\sum_\theta f(\theta)\prod_i f(w^i|\theta)}$$

Question 2:

(a) The equilibrium is a consumption plan and a price for each possible contingency. Let $p_\omega \in \mathbb{R}^K$ to denote the price vector when the signals are ω. Fix an ω, then in equilibrium, for all i,

1. Individual rationality:

$$x^i_\omega \in \arg\max_{y\in\mathbb{R}^K} U^i(y) \text{ subject to: } p_\omega \cdot (e^i - y) \geq 0.$$

2. Market clears, i.e. $\sum_i x^i_\omega \leq \sum_i e^i$.

For different realizations of the signal vector ω, the consumption is different; however, the actual value of θ does not affect the consumption vector. This can be viewed as consumption decision and adjustment must be made before θ is observed. If consumption adjustment is possible after knowing θ even the decision of those adjustments must be made prior, then the state-contingent consumption plan would be of the form $x^i_{\omega,\theta}$.

(b) Let i's observation be $(\omega^i, \bar{p})$ and he also knows the equilibrium contingent pricing plan $\omega \to p_\omega$. First, i rules out two types of σ. If $\omega^i \notin \sigma$, $\pi^{RE}(\sigma|\omega^i, \bar{p}) = 0$; if $\bar{p} \neq p_\omega$, $\pi^{RE}(\omega, \theta|\omega^i, \bar{p}) = 0$; otherwise,

$$\begin{aligned} \pi^{RE}(\sigma|\omega^i, \bar{p}) &= \frac{Pr(\sigma, \omega^i, \bar{p})}{Pr(\omega^i, \bar{p})} = \frac{Pr(\sigma, \bar{p})}{Pr(\{\sigma' : \omega^i \in \sigma', p_{\omega'} = \bar{p}\})} \\ &= \frac{g(\sigma)}{\sum_{\sigma' : \omega^i \in \sigma'} \mathbb{1}\{\bar{p} = p_{\sigma'}\} g(\sigma')}, \end{aligned}$$

where $p_\sigma = p_{\omega,\theta} = p_\omega$.

(c) The only information from the equilibrium p on the state θ is through ω. Then, we can rephrase the question: under what condition is the information in (ω^i, p) fully revealing of ω? One sufficient condition is that p is unique for each ω; that is, there is a 1-1 correspondence between ω and equilibrium p. It is equivalent to know p or to know ω.

Question 3:
We need to price $\Omega \times \Theta$ state-contingent goods; $p \in \mathbb{R}^{\Omega \times \Theta}$. Let the signal be ω. In equilibrium,

1. Individual rationality:

$$(x^i_{1,\omega}, x^i_{2,\omega}) \in \underset{y_1 \in \mathbb{R}^\Theta, y_2 \in \mathbb{R}^\Theta}{\arg\max} \sum_\theta u^i(y_{1,\theta} + y_{2,\theta}) q^i(\theta|\omega^i),$$

   subject to

$$p_\omega \cdot (e - y_1) + 1 \cdot y_2 \geq 0.$$

2. Market clears, i.e. $\sum_i x^i_{1,\omega,\theta} \leq \sum_i e$ and $\sum_i x^i_{2,\omega,\theta} \leq \sum_i 1$, for all θ.

Note that $x^i_\omega \in \mathbb{R}^{2\times\Theta}_+$ is i's consumption plan in state ω. He consumes $x^i_{\omega,\theta}$ at (ω, θ). The information for his decision-making is ω^i, and a decision is made before the realization of θ.

(b) If all the agents have the same preferences and have the same information, the prices of each arrow-debreu asset is its posterior probability. The condition that all agents have the same information can be expressed as there is a 1-1 correspondence between ω and equilibrium prices p.

Problem 2

Question 4:

Let's assume that the sequence of action is such that the monopolist first offers the contract, and then workers decide to take the offer or not.

Unless restricted by the monopolist, if a worker is willing to work at all, he would want to work full time. The firm cannot offer fully separating contract because if $w(\theta) \neq w(\theta')$ for all $\theta \neq \theta'$, then all $\theta' < \theta_1$ would report θ_1 and get the highest wage.

The monopolist's problem:

$$\max_{w(\cdot)} \int_{\theta_0}^{\theta_1} (\theta - w(\theta)) I(\theta) dF(\theta),$$

such that

$$I(\theta) \in \arg\max_{\theta', I} \{I w(\theta') + (1 - I) r(\theta)\}.$$

This is a complicated optimization problem because the choice space is a infinite dimensional (choosing over functions). Note that the agent chooses two things: what θ' to report and what fraction of I to work in response to wage schedule $w(\cdot)$. It is clear that the agent always lie and report θ_1 regardless of type to earn $\bar{w} = \max_{\theta'} w(\theta')$. Further, $I(\theta) = 1$ if and only if $\bar{w} \geq r(\theta)$. Letting $\bar{w} = r(\bar{\theta})$, assuming $r(\cdot)$ is monotonic, and substituting, the monopolist's problem becomes,

$$\max_{\bar{w}} \int_{\bar{\theta}}^{\theta_1} (\theta - \bar{w}) dF(\theta) = \max_{\bar{w}} \int_{r^{-1}(\bar{w})}^{\theta_1} (\theta - \bar{w}) dF(\theta).$$

FOC w.r.t. $\bar{w}$,

$$0 = -\frac{1}{r'(r^{-1}(\bar{w}))}(r^{-1}(\bar{w}) - \bar{w})f(r^{-1}(\bar{w})) - \left[F(\theta_1) - F(r^{-1}(\bar{w}))\right]. \qquad (1)$$

Assuming (1) has a unique solution $\bar{w}^*$, then the principal offers a uniform price $\bar{w}^*$ and all workers with $\bar{w}^* \geq r(\theta)$ set $I(\theta) = 1$ and otherwise, do not participate.

When there is competition, w^c is such that

$$\int_{r^{-1}(w^c)}^{\theta_1} (\theta - w^c)dF(\theta) = 0.$$

With some mild assumptions on (r, F), we should be able to get $w^c > \bar{w}$. There are more workers in the workforce in the competitive equilibrium.

Question 5:

Let $A \equiv \{\theta : \theta \geq r(\theta)\}$ and $B \equiv \{\theta : \theta < r(\theta)\}$. The agent's problem remains unchage, hence the social planner solves the problem.

$$\max_{w} \int_A \theta dF(\theta) + \int_B r(\theta)dF(\theta). \qquad (2)$$

If $r(\cdot)$ single-crosses the 45 degree line, then w^* should be set at the crossing point, i.e. $r(w^*) = w^*$. Further, if $r(\cdot)$ is concave, then $A = \{\theta : \theta \geq w^*\} = \{\theta : w^* \geq r(\theta)\}$; also , the firm's profit is always strictly greater than 0 because those who work, their contribution $\theta > w^*$. Note that the social planner does not need to observe θ to implement the first-best outcome as long as he knows the property of r so that from the perspective of social welfare, he can separate those who are better off working from those who are better off receiving the outside option.