An Intertemporal General Equilibrium Model of Asset Prices

A short review

by Cox, Ingersoll, and Ross

November 20, 2012

Introduction

- Merton (1973) analyzes a continuous time, consumption-based asset pricing model. It makes parametric assumptions on the price processes, and there is no production in the economy.

- Lucas (1978) analyzes a discreet time, consumption-based model. It allows production, does not make parametric assumptions on how to price assets, and is a general equilibrium model.

- There is a companion paper by the same authors that discusses a specialization of the model in this paper.

The setup

The economy has

1. production processes; n of them
2. contingent claims; a lot of them, and the exact number is not important

Agents makes the following decisions

1. how to consume; $\{c_t : t > 0\} = c$
2. how to invest in assets; $\{a_t : t > 0\} = a$
3. how to invest in claims; $\{b_t : t > 0\} = b$

The economy equilibrates and determines

1. interest rate; r
2. expected rate of return of claims; β

Parametric assumptions of the economy

- The state variables Y has the dynamics

$$dY(t) = \mu(Y,t)dt + S(Y,t)dw(t)$$

- The production technology

$$I_\eta^{-1} d\eta(t) = \alpha(Y,t)dt + G(Y,t)dw(t)$$

- The contingent claims

$$dF^i(t) = [F^i(t)\beta_i(t) - \delta_i(t)] + F^i(t)h_i dw(t)$$

But note that β is an endogenous process.

The representative agent optimizes the investment and consumption

There is a state and time dependent utility function

$$U(c_t, Y_t, t),$$

and the agent makes the following decision at each t

1. a_t; investment in productive assets
2. b_t; investment in auxiliary assets
3. c_t; consumption

We refer to the triple (a, b, c) the control process and label it $v = \{v_t : t \geq 0\}$.

The wealth process balances the budget

Fix the control (a, b, c), the wealth process take the following dynamics. For simplicity, let $n = k = 1$.

$$
\begin{aligned}
dW_t = \ & [a_t W_t(\alpha - r) + b_t W_t(\beta_t - r_t) + r_t W_t - c_t]\, dt \\
& + a_t W_t[g_1 dw_1(t) + g_2 dw_1(t)] \\
& + b_t W_t[(h_1 dw_1(t) + h_2 dw_1(t)],
\end{aligned}
$$

where the index 1 refers to the production asset and index 2 refers to the contingent-claim asset.

The wealth process restricts the consumption and investment decision so that the budget constraint is not violated.

The agent's optimization problem

The agent solves the following stochastic control problem

$$\sup_{v} \mathbb{E}\left[\int_0^\infty U(c_s, Y_s, s)ds\right]$$
$$\text{subject to: } W_0 = w_0 \text{ and } W_t \geq 0.$$

Use the hat to denote optimal policy, i.e. $\hat{v}$ denotes an

optimal control.

Solving agent's problem (1): Dynamic programming principal (DPP)

Define the value function J as

$$J(t, w, y) = \sup_v \mathbb{E}\left[\int_t^\infty U(c_s, Y_s, s)ds\right]$$
$$\text{subject to: } W_t = w \text{ and } Y_t = y$$

Then locally, the optimality condition becomes

$$J(t, w, y) = \sup_v \mathbb{E}\left[\int_t^\tau U(c_s, Y_s^{w,y}, s)ds + J(\tau, W_\tau^{w,y}, Y_\tau^{w,y})\right]$$

Solving agent's problem (2): HJB-PDE

With DPP, Ito's formula, and the assumption that J is smooth enough, we get the following non-linear PDE

$$\sup_v [\mathcal{L}^v J + U(v, y, t)] = -J_t,$$

with some boundary conditions.

Solution of the PDE vs. solution to the original stochastic control problem

lemma (1)

If J is a solution to the PDE and also J is C^2, then

1. *J is a value function.*
2. *the $\hat{v}$ s.t. $\mathcal{L}^{\hat{v}}J + U(\hat{v}, y, t) = -J_t$ is an optimal policy.*

Note that the Bellman equation by itself is neither sufficient nor necessary.

To find the solution of the stochastic control problem:

1. Solve the PDE
2. Verification step

Solving agent's problem (3): Resolution

Make the explicit assumption that the value solution $t, w, y \mapsto J(t, w, y)$ exists and is unique.

lemma (2)

$J(t, w, y)$ is increasing and strictly concave in w.

Towards an equilibrium

The equilibrium conditions are:

1. $\sum_i = a_i = 1$
2. $b_i = 0$

Then, finding the solution to the agent's problem and imposing the equilibrium conditions pin down the following processes:

1. r; the interest rate
2. β; the expected return of claims
3. a investment strategy
4. c consumption

Key result (1): Interest rate

Theorem (1)

$$r(t, w, y) =$$

$$\textit{1. } a^t\alpha - \frac{-J_{ww}}{J_w}\frac{var(w)}{w} - \sum_{i=1}^{k}\frac{-J_{wy_i}}{J_w}\frac{cov(w, y_i)}{w}$$

$$\textit{2. } -\frac{J_{wt} + \mathcal{L}J_w}{J_w} = -\frac{D[J_w]}{J_w}$$

$$\textit{3. } a^t\alpha + \left[\frac{cov(w, J_w)}{wJ_w}\right],$$

where the evaluation is $w = W_t$ *and* $y = Y_t$.

Key result (2): Determining β

Theorem (2)

$$(\beta_i - r)F^i =$$

$$\textit{1. } [\phi_W \phi_{Y_i} \cdots \phi_{Y_k}][F^i_W F^i_{Y_1} \cdots F^i_{Y_k}]^T$$

$$\textit{2. } -\frac{cov(F^i, J_w)}{F^i J_w}$$

$\phi(\cdot)$ is specified in equation (20).

Key result (3): Pricing contingent claims

Theorem (3)

The pricing formula $F(t, w, y)$ *satisfies the following PDE*

$$\mathcal{L}F(t, w, y) + \delta_t = r_t F(t, w, y),$$

where r *is determined by Theorem (1) and* $\mathcal{L}$ *is the differential generator.*

Note that this PDE and along with boundary conditions, which depends on each particular contingent claim, would yield the deterministic pricing formula $F(t, w, y)$.

The equilibrium pricing formula is analogous to the Black-Scholes pricing formula)

In the BS setup, a contingent claim that pays $h(X_T)$ at the termination time T is priced at

$$F(t,s) = \mathbb{E}^{t,x}\left[e^{\int_t^T -r ds} h(X_T)\right].$$

Then, a Feynman-Kac type argument would require that $F(t,s)$ would also satisfy the PDE

$$\mathcal{L}F(t,s) = rF(t,s),$$

with terminal condition

$$f(T,s) = h(s), \text{ for all } s.$$

Lemma 3 makes this connection precise

lemma (3)

The solution to the pricing PDE with the boundary condition given in (34) *can be calculated by the following expectation formula,*

$$
\begin{aligned}
F(W,Y,t,T) = \mathbb{E}\{\ & \Theta(W(T),Y(T))[exp\{-\int_t^T \beta_u du\}]1_{\tau \geq T} \\
& + \Phi(W(\tau),Y(\tau),\tau)[exp\{-\int_t^\tau \beta_u du\}]1_{\tau < T} \\
& + \int_t^{\tau\wedge T} \delta_s [exp\{-\int_t^s \beta_u du\}] ds\ \}.
\end{aligned}
$$

Pricing in terms of marginal-utility-weighted expected value

Theorem (4)

$$
\begin{aligned}
F(W, Y, t, T) = \mathbb{E}\{\ & \Theta(W(T), Y(T)) \frac{J_W(W(T), Y(T), T)}{J_W(W(t), Y(t), t)} 1_{\tau \geq T} \\
& + \Phi(W(\tau), Y(\tau), \tau) \frac{J_W(W(\tau), Y(\tau), \tau)}{J_W(W(t), Y(t), t)} 1_{\tau < T} \\
& + \int_t^{\tau \wedge T} \delta_s \frac{J_W(W(s), Y(s), s)}{J_W(W(t), Y(t), t)} ds\ \}.
\end{aligned}
$$

Roughly, it states that the correct "pricing kernel" is the inter-temporal marginal rate of substitution: $\frac{J_W(W(s),Y(s),s)}{J_W(W(t),Y(t),t)}$.

Comparing to the Lucas model of a single asset

The lucas model prices the asset with payment stream x_t at

$$p_t = \mathbb{E}^t \left[\sum_{j=t}^{\infty} \beta^j \frac{u'(x_{j+1})}{u'(x_t)} x_{j+1} \right].$$

Now consider a similar contingent claim in the continuous-time economy in which the asset pays the stream δ_t without a stopping rule. Then this asset is priced at

$$p_t = \mathbb{E}^t \left[\int_t^{\infty} \frac{J_W(W(s), Y(s), s)}{J_W(W(t), Y(t), t)} \delta_s ds \right].$$

The lack of a discounting factor here is due to the assumption on $U(t, \cdot)$.

Let's see an application/specialization

1. Restrict U to be log utility, state-independent, and time-homogeneous after discounting in that

$$U(c,t) = e^{-\int_0^t r_s ds} log(c).$$

2. Let there be only one state variable with the following dynamics

$$dY(t) = (\xi Y(t) + \zeta)dt + \nu\sqrt{Y(t)}dw(t).$$

Solving this model yields the Cox-Ingersoll-Ross interest rate dynamics

1. $\hat{a} = (GG')^{-1}\alpha + \frac{e - e'(GG')^{-1}\alpha}{e'(GG')^{-1}e}(GG')^{-1}e$
2. r satisfies the SDE

$$dr_t = \kappa_t(\theta_t - r_t)dt + \sigma\sqrt{r_t}dw_t,$$

   where κ and θ are defined by some SDEs (see (15) of the companionl paper.)

The key result here is that the interest rate process is endogenously determined by an equilibrium model.

The Cox-Ingersoll-Ross interest rate model (1)

The process satisfying

$$dr_t = \kappa_t(\theta_t - r_t)dt + \sigma\sqrt{r_t}dw_t,$$

is known as a mean-reverting process, a.k.a. Ornstein-Uhlenbeck process.

Given this process, the price of a zero couple bond with duration T, $f(t,r)$, can be valued by solving the following PDE,

$$f_t(t,r) + \kappa_t(\theta_t - r_t)f_r(t,r) + \frac{1}{2}\sigma^2 r f_{rr}(t,r) = rf(t,r).$$

The Cox-Ingersoll-Ross interest rate model (2)

The solution turns out to be tractable and has a closed form

$$f(t, r) = exp\{-rC - A\},$$

for some constant deterministic functions $C(t, T)$ and $A(t, T)$. And this corresponds to an affine yield

$$Y(t, T) = \frac{1}{T - t}(rC + A),$$

which is a comforting result for some folks.