Some notes on homework 3

Question 1

It should be noted all auctions that have the same boundary condition (i.e. a losing bidder pays nothing), all bidders are risk neutral, are with independent private valuations, and that bidders having the highest valuations win the auction would satisfy the revenue equivalence property. (For example, see Milgrom and Weber (1982), A theory of auctions and competitive bidding.) Then it is without loss of generality to restrict attention to second price auction.

The task is to find the optimal reserve price. It turns out that the condition for the optimal r^* is

$$r^* = \frac{1 - H(r)}{H'(r)}. \tag{1}$$

Without providing too much detail, let us see a rough sketch for the one item auction. Let $v \mapsto a(v, r)$ denote the expected payment from a bidder with valuation v when the reserve price is r. We consider the bidder's problem first. The bidder is truthful, then

$$v = \arg\max_x H(x)^{n-1} v - a(x, r).$$

Using first order condition (and assume enough so that this is also sufficient) we can pin down the bidding function as

$$a(v, r) = vH(v)^{n-1} - \int_r^v H(x)^{n-1} dx.$$

Now using the agent's solution, the seller maximize his expected revenue,

$$\max_r \int_r^{\theta_1} a(v, r) dH(v).$$

Expanding and taking a first order condition, we can get (1). Note that a VCG mechanism always maximizes social welfare and a secod price auction with $r = 0$ is a VCG mechanim.

Question 2

Providing a counter-example would be the easiest thing to try when one is ask to disprove a general statement. Let $N = 2$. Assume that second agent will always lie that $\theta_2 = 1$. Then, the mechanism would decide that $x^*(\hat{\theta}_1) = \frac{2}{K}(1 + \hat{\theta}_1)$, where $\hat{\theta}_1$ is agent 1's report. Then, agent 1 best responds by

$$\max_{\hat{\theta}_1} \; \theta_1 \ln x^*(\hat{\theta}_1) - \frac{K}{2} x^*(\hat{\theta}_1) - t^1((\hat{\theta}_1, 1)).$$

Take the first order condition, and it should be clear that $\hat{\theta}_1^* \neq \theta_1$. If that doesn't work, then take the general case that $x^*(\hat{\theta}_1, \theta_2)$, and dominant strategy incentive compatible would require that $\hat{\theta}_1^* = \theta_1$, for all θ_2.

Question 3

Notice the mechnism is efficient, balanced budget, and dominant strategy IC for all but 1 players. As long as $h^i(\theta_{-i})$ does not depend on i's own report θ_i, then $t^i_{vcg} + h^i(\theta_{-i})$ is dominant strategy incentive compatible. It is sufficient to show that for the special agent N, he is Bayes IC for a specific $\{h^i(\cdot)\}_{i \neq N}$.

One idea to balance the budge is to have N to pay for all of i's externality and the rest of the i to pay for N's externality. Let $N = 2$. Then let

$$T^2(\theta_2) = \int v^1(x^*(\theta_1, \theta_2)) dH(\theta_1),$$

where $v^1(x^*(\theta_1, \theta_2)) = \theta_1 x^*(\theta_1, \theta_2) - \frac{K}{2} x^*(\theta_1, \theta_2)$. Now let $h^2(\theta_1) = t^1_{vcg}(\theta_1)$ and

$h^1(\theta_2) = T^2(\theta_2)$. In summary, the agent pays

$$\begin{cases} t^1 = -t^1_{vcg}(\theta) + h^1(\theta_2) \\ t^2 = -T^2(\theta_2) + h^2(\theta_1). \end{cases}$$

The payments are balanced, and for agent 2, the choice of $T^2(\theta_2)$ induce Bayes incentive compatibility (but h^2 does not affect incetives).

For the general case N, let

$$T^N(\theta_N) = \int \sum_{i \neq N} v^i(x^*(\theta_N, \theta_{-N}))dH(\theta_1) \cdots dH(\theta_{N-1}).$$

For $i \neq N$, let

$$h^i(\theta_{-i}) = T^N(\theta_N)/(N-1)$$

And the final payments are

$$\begin{cases} t^i = -t^1_{vcg}(\theta) + h^i(\theta_{-i}) \\ t^N = -T^N(\theta_N) + \sum_{i \neq N} t^i_{vcg}(\theta). \end{cases}$$

I am fairly sure this payment works, but I am not guaranteeing it. If anyone finds a mistake, let me know.

Question 4

(b) Mis-representation. For a concrete analysis, consider the utility function $u^i = \theta^i x + w^i - t^i$ and $C(x) \geq 0$. It would be clear that if $\theta_i = 0$, then i would want to report $\hat{\theta}_i$ so that $x(\theta_{-i}, \hat{\theta}_i) = 0$, assuming that $C(0) = 0$.

There might also be a problem of existence. $N = 2$. Let the cost function be quadratic and the utility be linear ($\theta_i x$), then for agent 1

$$u_1 = \theta_1(\hat{\theta}_1 + \hat{\theta}_2) - (\hat{\theta}_1 + \hat{\theta}_2)^2/2,$$

where $(\hat{\theta}_1, \hat{\theta}_2)$ are the reported messages. Best responding, $\hat{\theta}_1^* = \theta_1 - \hat{\theta}_2$. But the system of equation

$$\begin{cases} \hat{\theta}_1 = \theta_1 - \hat{\theta}_2 \\ \hat{\theta}_2 = \theta_2 - \hat{\theta}_1 \end{cases}$$

does not have a solution for $\theta_1 \neq \theta_2$.