1. Priority service contract with incremental demand function

The consumer submits an incremental demand function ϕ, where $\int_p^{\bar{p}} \phi(z)dz$ is the total demand he is willing pay at price p. The price range tops out at a fixed constant $\bar{p}$. The consumer's true demand funciton is θ. We define an index function $v(\theta, p, \phi)$ as

$$\int_{v(\theta,p,\phi)}^{\bar{p}} \theta(z)dz \equiv \int_p^{\bar{p}} \phi(z)dz, \tag{1.1}$$

which tracks the lower integrating limit when we calculate his utility from consuming electricity. The consumer's expected utility when he trufully reveals θ is

$$\begin{aligned} V(\theta,\theta) &= \mathbb{E}_p\left[\int_{v(\theta,p,\theta)}^{\bar{p}} \theta(z)zdz\right] - \mathbb{E}_p\left[p\int_p^{\bar{p}} \theta(z)dz\right] \\ &= \mathbb{E}_p\left[\int_p^{\bar{p}} \theta(z)(z-p)dz\right] \end{aligned} \tag{1.2}$$

Now we calculate the consumer's expected utility when he is reporting ϕ,

$$\begin{aligned} V(\theta,\phi) &= \mathbb{E}_p\left[\int_{v(\theta,p,\phi)}^{\bar{p}} \theta(z)zdz\right] - \mathbb{E}_p\left[p\int_p^{\bar{p}} \phi(z)dz\right] \\ &= \mathbb{E}_p\left[\int_{v(\theta,p,\phi)}^{\bar{p}} \theta(z)zdz - \int_p^{\bar{p}} p\phi(z)dz\right] \end{aligned} \tag{1.3}$$

We want to show that the θ is a solution to the problem $\max_\phi V(\theta,\phi)$. In the subsequent analysis, we fix p; this is without loss of generality because if we can prove a property for a degenerate distribution of p, for all $p \in [0, \bar{q}]$, then the property would also hold for any distribution of p in expectation. We use the Gateaux differential notation to show that θ is an extremal point for $V(\theta,\phi)$, and also that $V(\theta,\phi)$ is a convex functional in its second variable. For any $\phi \mapsto F(x,\phi)$, we define the differential

$$\delta_2 F(x,\phi;\epsilon) \equiv \lim_{\lambda\to 0} \frac{1}{\lambda}\Big(F(x,\phi+\lambda\epsilon) - F(x,\phi)\Big).$$

We call $\delta_2 F(x, \phi; \epsilon)$ the Gateaux differential[1] of F at ϕ in the direction of ϵ. The subscritp in $\delta_2 F$ denotes the differential is taken with respect to the second variable. As a preliminary exercise, let us differentiating both sizes of (1.1) with respect to ϕ,

$$-\theta(v(\theta,p,\phi))\delta_3 v(\theta,p,\phi;\epsilon) = \lim_{\lambda\to 0}\frac{1}{\lambda}\int_p^{\bar{p}} \lambda\epsilon(z)dz;$$

that is,

$$\delta_3 v(\theta,p,\phi;\epsilon) = -\frac{\int_p^{\bar{p}} \epsilon(z)dz}{\theta(v(\theta,p,\phi))}. \tag{1.4}$$

Proposition 1.1. *θ is an extremal point of the funtional $\phi \mapsto V(\theta,\phi)$.*

Proof. We are checking that $\delta_2 V(\theta,\phi;\epsilon)\big|_{\phi=\theta} = 0$, for any ϵ.

$$\begin{aligned}
\delta_2 V(\theta,\phi;\epsilon) &= -\theta(v(\theta,p,\phi))v(\theta,p,\phi)\delta_3 v(\theta,p,\phi;\epsilon) - p\int_p^{\bar{p}} \epsilon(z)dz \\
&= \Big(v(\theta,p,\phi) - p\Big)\int_p^{\bar{p}} \epsilon(z)dz. \qquad (1.5)
\end{aligned}$$

In the last equality, we substitute (1.4) into the equation. Evaluating (1.5) at $\phi = \theta$ and recognizing $v(\theta,p,\theta) = p$, we show that truful reporting satifies the necessary condition for the optimization problem. □

Proposition 1.2. *The functional $\phi \mapsto V(\theta,\phi)$ is concave.*

Proof. $V(\theta,\cdot)$ is concave if and only if[2] for any ϵ_1 and ϵ_2,

$$0 \le -\delta_2 V(\theta,\epsilon_1;\epsilon_1) - \delta_2 V(\theta,\epsilon_2;\epsilon_2) + \delta_2 V(\theta,\epsilon_1;\epsilon_2) + \delta_2 V(\theta,\epsilon_1;\epsilon_2) \tag{1.6}$$

Substituting (1.5) into (1.6), we get

$$\begin{aligned}
0 &\le \Big(v(\theta,p,\epsilon_1) - p\Big)\int_p^{\bar{p}} \Big(\epsilon_2(z) - \epsilon_1(z)\Big)dz + \Big(v(\theta,p,\epsilon_2) - p\Big)\int_p^{\bar{p}} \Big(\epsilon_1(z) - \epsilon_2(z)\Big)dz \\
&\le \Big(v(\theta,p,\epsilon_1) - v(\theta,p,\epsilon_2)\Big)\int_p^{\bar{p}} \Big(\epsilon_2(z) - \epsilon_1(z)\Big)dz
\end{aligned}$$

[1]See Chapter 7 in Luenberger (1969) for details.

[2]For example, see Ekeland and Temam (1999); Chapter 2, Prop 5.5.

The last line is in fact greater than or equal zero. To see that, note that

$$
\begin{aligned}
& \int_{p}^{\bar{p}} \Big(\epsilon_2(z) - \epsilon_1(z)\Big) dz \geq 0 \\
\implies \quad & \int_{p}^{\bar{p}} \epsilon_2(z) dz \geq \int_{p}^{\bar{p}} \epsilon_1(z) dz \\
\implies \quad & \int_{v(\theta,p,\epsilon_2}^{\bar{p}} \theta(z) dz \geq \int_{v(\theta,p,\epsilon_1}^{\bar{p}} \theta(z) dz \\
\implies \quad & v(\theta,p,\epsilon_1) \geq v(\theta,p,\epsilon_2).
\end{aligned}
$$

Also, $\int_{p}^{\bar{p}} \Big(\epsilon_2(z) - \epsilon_1(z)\Big) dz \leq 0 \implies v(\theta,p,\epsilon_1) \leq v(\theta,p,\epsilon_2)$. The concavity condition (1.6) holds. □

The two propositions show that θ is a global maximum of the mapping $\phi \mapsto V(\theta,\phi)$. The pricing scheme given that the consumer reports ϕ,

$$
\pi(\phi) = \mathbb{E}_p \left[p \int_{p}^{\bar{p}} \psi(z) dz \right],
$$

is incentive compatible.