Ambiguity and Second Order Belief

An axiomatic approach to second order subjective expected utility representation

October 8, 2011

A very basic review of decision theory models

- The preference relation $\succeq$ is defined on X
- Utility representation $V(\cdot)$ of the preference relation on X: x and y are elements in the choice set, then V is a representation if

$$x \succeq y \Leftrightarrow V(x) \geq V(y)$$

One way to think about the progression of utility representation models is by looking at the choice sets

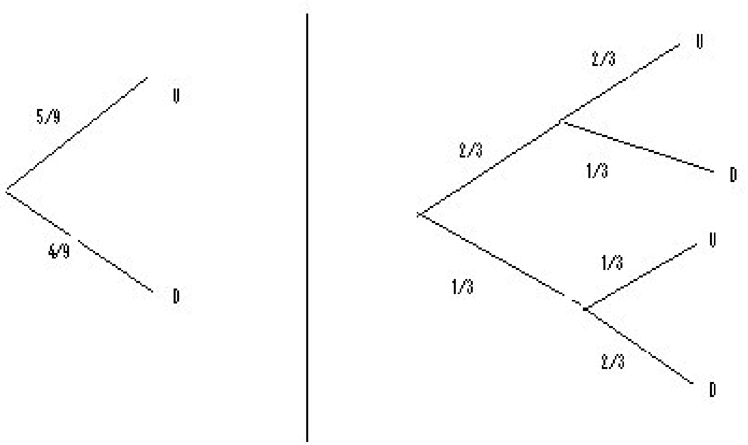
- Debreu: $\succeq$ is defined on the set of prizes(consequences) Z
- Expected utility: $\succeq$ is defined on the set of probability measures $\Delta(Z)$

- Ascombe-Aumann (AA): $\succeq$ is defined on the set of acts $\mathcal{H}$
- What about $\succeq$ defined on $\Delta(\mathcal{H})$?

Another way is by looking at different sources of uncertainties

- Objective uncertainty is tackled by the expected utility theorem; simple one-stage lotteries
- Subjective belief of the states of the world (S) is tackled by the AA representation
- What about two stages of objective uncertainties, as in compound lotteries? How does a Decision Maker (DM) deals with two-stage lotteries?
- What about second order belief (DM's belief over the set of *probability distribution* of S?)

The two different ways to think about a bet on Up



Setup and notations

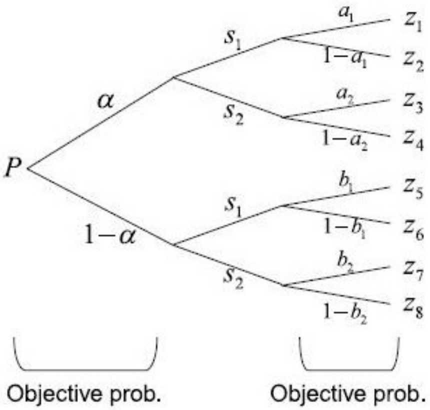
1. Z: the set of prizes/consequences
2. $\Delta(Z)$: the set of probability measures on Z
3. S: the set of states (indexed by $\{1, \cdots, n\}$)
4. $\Delta(S)$: the set of probability measures on S
5. $\mathcal{H}$: the set of all acts, mappings from S to $\Delta(Z)$
6. $\Delta(\mathcal{H})$: the set of probability measures on $\mathcal{H}$

Notes: denote elements of $\Delta(Z)$ by p, q, r; $\mathcal{H}$ by f, g, h; and $\Delta(\mathcal{H})$ by P, Q, R.

Setup and notations

- Note that $\Delta(Z)$ are one-stage lotteries
- Note that $\Delta(\Delta(Z))$ are two-stage lotteries
- Note that $\Delta(Z) \subset \mathcal{H} \subset \Delta(\mathcal{H})$
- Then, $\Delta(\Delta(Z)) \subset \Delta(\mathcal{H})$
- $\succeq$ is defined on $\Delta(\mathcal{H})$
- Thus, $\succeq$ also works for elements of $\Delta(Z)$ and $\Delta(\Delta(Z))$

Look at a typical element of $\Delta(\mathcal{H})$



AA model deals with first order subjective belief on the space of acts $\mathcal{H}$

Let $\succeq$ be defined on defined on $\mathcal{H}$, then for any $f, g \in \mathcal{H}$,

$$f \succeq g \Leftrightarrow \sum_{s \in S} U(f_s) \geq \sum_{s \in S} U(g_s)$$

$$\Leftrightarrow$$

$$\sum_{s \in S} \mu(s) \sum_{z \in Z} u(z) * f_s(z) \geq \sum_{s \in S} \mu(s) \sum_{z \in Z} u(z) * g_s(z)$$

$$\Leftrightarrow$$

$$\mathbb{E}_\mu u(f) \geq \mathbb{E}_\mu u(g)$$

Notes:

1. u is the vNM utility
2. μ is the first order subjective belief
3. Denote $u(f) = \mathbb{E}_f u$

We can expand the AA model to the choice set $\Delta(\mathcal{H})$

Let $\succeq$ be defined on $\Delta(\mathcal{H})$; with the axioms defined on the new space, we can get

$$V(P) = \sum_{f \in \mathcal{H}} P(f)U(f)$$

$$\text{and } U(f) = \sum_{s \in S} \mu(s)u(f_s) = E_\mu u \circ f$$

Notes:

1. u is risk attitude; μ is risk belief
2. $u \circ f$ is the utility of a simple lottery

We want a model allowing the DM to have second order subjective belief to account for ambiguity behavior

Let $\succeq$ be defined on $\Delta(\mathcal{H})$, and we want the utility representation be

$$V(P) = \sum_{f \in \mathcal{H}} P(f)U(f)$$

$$\text{and } U(f) = \sum_{\mu \in \Delta(S)} m(\mu)\phi(\sum_{s \in S} \mu(s)u(f_s)) = \mathbb{E}_m \phi(\mathbb{E}_\mu u \circ f))$$

Notes:

1. u represents the risk attitude; μ represents the risk spread/belief
2. ϕ is ambiguity attitude; m is ambiguity belief

For the remaining time, we will

1. Discuss the axioms needed for the SEU and SOSEU representation defined on $\Delta(\mathcal{H})$
2. State the two main theorems and sketch a proof of the SOSEU theorem
3. Interprete the notion ambiguity and the connection between SOSEU and SEU
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Axiom 1 and 2: Order and Continuity

A1. Order: $\succeq$ is complete and transitive

A2. Continuity: $\succeq$ is continuous; i.e. the graph $\{(P, Q) \in (\Delta(\mathcal{H}))^2 : P \succeq Q\}$ is closed in the product topology, and the topological space of $\mathcal{H}$ is defined by yet another product topology ($\mathcal{H} = (\Delta(Z))^S$).

Axiom 3 and 4: First-stage independence and second-stage independence

A3. First-stage Independence: for any P, Q, R and $a \in (0, 1)$,

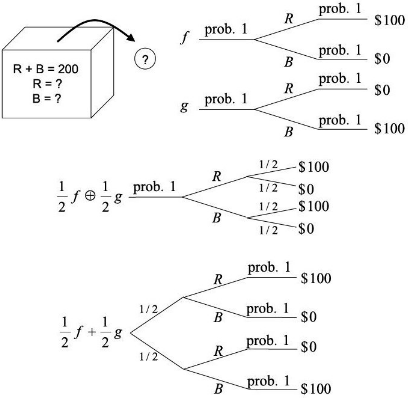
$$P \succeq Q \Leftrightarrow aP + (1-a)R \succeq aQ + (1-a)R$$

A4. Second-stage Independence: for any $p, q, r \in \Delta(Z)$ and a

$$p \succeq q \Leftrightarrow ap \oplus (1-a)r \succeq aq \oplus (1-a)r$$

Axiom 3 and 4: an example

Think of f, g as degenerate acts in $\Delta(\mathcal{H})$; also think of f, g as degenerate first-stage lotteries in $\Delta(Z)$ for every state.
First-stage mixing is a mixing of acts; second-stage mixing is a mixing of the outputs of acts.



Axiom 5 and 6: AA dominance and Dominance

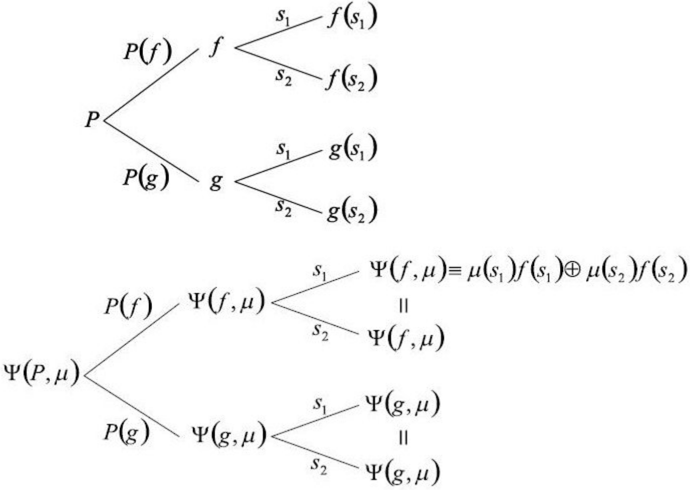
A5. AA dominance: Let $f, g \in \mathcal{H}$ and $s \in S$. If $f(s') = g(s')$ for all $s' \neq s$ and $f(s) \succeq g(s)$, then $f \succeq g$

A6. Dominance: Let $P, Q \in \Delta(\mathcal{H})$. If $\Psi(P, \mu) \succeq \Psi(Q, \mu)$ for all $\mu \in \Delta(S)$, then $P \succeq Q$.

Notation:

- $\Psi(f, \mu) \equiv \mu(s_1) f_{s_1} \oplus \cdots \oplus \mu(s_n) f_{s_n} \in \Delta(Z)$
- $\Psi(P, \mu)(B) = Pr\left(\{f \in \mathcal{H} : \Psi(f, \mu) \in B\}\right)$, for all $B \in \mathcal{B}$, where $\mathcal{B}$ is a sigma-algebra on $\mathcal{H}$.
- for a probability belief μ of the states of world, $\Psi(P, \mu) \in \Delta(\Delta(Z))$ is simply a compound lottery

Reduction of an element P of $\Delta(\mathcal{H})$ to a two-stage lottery



Axiom 7 and 8: Reversal of Order (RofO) and Reduction of Compound Lotteries (ROCL)

A7. RofO: For any $f, g \in \mathcal{H}$,

$$af \oplus (1-a)g \sim af + (1-a)g$$

A8. ROCL: For any $p, q \in \Delta(Z)$,

$$ap \oplus (1-a)q \sim ap + (1-a)q$$

Each of the pairs of axioms (3,4; 5,6; 7,8) are closely related.

- But we do not discuss them right now.
- We will first look at the major results and then we come back to discuss the axioms.

Subjective expected utility (SEU) representation

Theorem (SEU)

Preference $\succeq$ on $\Delta(\mathcal{H})$ satisfies order, continuity, second-stage independence, first-stage independence, reversal of order, and AA dominance if and only if it has an SEU representation.

$$V(P) = \sum_{f \in \mathcal{H}} P(f)U(f)$$

$$U(f) = \sum_{s \in S} \mu(s)u(f_s) = E_\mu u \circ f$$

Moreover, the belief μ is unique and the vNM utility u is unique up to positive affine transformations.

Second order subjective expected utility (SOSEU) representation

Theorem (SOSEU)

Preference $\succeq$ on $\Delta(\mathcal{H})$ satisfies order, continuity, second-stage independence, first-stage independence, and dominance if and only if it has an SOSEU representation.

$$V(P) = \sum_{f \in \mathcal{H}} P(f)U(f)$$

$$U(f) = \mathbb{E}_m \phi(\mathbb{E}_\mu u \circ f)$$

Moreover, u and $\phi \circ u$ are unique up to positive affine transformations; m may not be unique.

The proof the SOSEU theorem is essentially an application of generalized version of the Farka's lemma

Proof:
We will only prove the sufficiency condition.

1. First-stage independence reduces the representation of $V(P) = \sum_{f \in \mathcal{H}} P(f)U(f)$
2. Use the Ψ notation to restrict the U's domain to simply one-stage lottery $\Delta(Z)$; that is

$$U(f) = \sum_{\mu \in \Delta(S)} m_\mu U(\Psi(f, \mu)) \qquad (1)$$

3. We need to find a specific probability measure m on $\Delta(S)$ such that the system of equations in (1) have a solution.

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Proof of the SOSEU theorem continues

4. Farka's lemma states that there is a non-negative m satisfying (1) if and only if the following condition holds for all measures t on $\mathcal{H}$ and $v \in \Delta(S)$.

Proof of the SOSEU theorem continues

5. To show (2), we use the dominance condition, essentially decomposing the measure t_f into a linear combination of two acts $P, Q \in \Delta(\mathcal{H})$ and their respective weights a and b and showing that

$$P \succeq \frac{b}{a}Q + (1 - \frac{b}{a}\bar{R}$$

where $\bar{R}$ is normaled to to have $V(\bar{R}) = 0$.

6. Hence, $V(aP - bQ) \geq 0$, proving the condition.
7. Lastly, use second-stage independence to show that there exists a ϕ s.t. $U(\cdot) = \phi(u(\cdot))$.

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Compare the two theorems side by size

Theorem (SEU)

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AA dominance and dominance are equivalent under certain conditions

1. Order, continutin, and reversal of order and AA dominance imply dominance
2. Dominance and second-stage independence imply AA dominance

- Corollary of the two theorems: SEU $\Leftrightarrow$ SOSEU + reversal of order

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An experiment supports the behavioral equivalence of ROCL and ambiguity neutrality

Participants are paid to guess the color of the balls inside a box. There are three boxes.

1st Box has 5 red balls and 5 blue balls

2nd Box has an unknown distribution

3rd Box is a two-stage lottery; a number is drawn from a uniform 0-10 to determine the number of red, and then the ball is picked from the box

Neutrality $\Rightarrow$ box 1 $\sim$ box 2; ROCl $\Rightarrow$ box 1 $\sim$ box 3

Result: Almost every subjet who is indifferent between 1 and 2 is also indifferent between 1 and 3.

Conclusion

- Extend the AA model to the choice set $\Delta(\mathcal{H})$
- Establish an axiomatic construction of SEU and SOSEU
- SOSEU is compatiable with the type of ambiguity-aversion behaviors exhibited in the Ellsberg paradox
- The model captures the intuitive notion that a decision maker may use a belief over the set of probabilities of the set of states to inform her decision process