Identification and Estimation of Dynamic Games when Players' Belief Are Not in Equilibrium

A Short Review of
Aguirregabiria and Magesan (2010)

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Dynamics of the game

- Two players, $\{i, j\}$
- T periods
- X_t is the vector of state variables
- Y_{it} is player i's choice at time t
- $\epsilon_{it}(Y_{it})$ is i's private information; it is distributed by G
- $\pi_i(Y_{it}, Y_{jt}, X_t)$ is a real value function of i's own action, opponent's action, and state variables; it is the deterministic part of i's payoff
- payoff function

$$\Pi_{it}(Y_{it}, Y_{jt}, X_t) = \pi_i(Y_{it}, Y_{jt}, X_t) + \epsilon_i(Y_{it})$$

State variables and transition functions

- $X_t = (W_t, S_{1t}, S_{2t})$
- W_t are some exogenous, player independent market characteristics
- S_{it} is player i's specific characteristics
- $f^W(W_{t+1}|W_t)$ is the transition function of W
- $f^S(S_{t+1}|Y_{it}, S_{it})$ is the transition function of S; it does not depend on j's action or state

Strategies, choice probabilities, and beliefs

- $\sigma_{it}(X_t, \epsilon_{it})$ is i's strategy at time t
- $b_{jt}^{t_o}(X_t, \epsilon_{jt})$ is player i's belief at period t_o about the strategy of player j at time t
- $P_{it}(X_t) = Pr(\sigma_{it}(X_t, \epsilon_{it}) = 1|X_t)$ is i's choice probability
- $B_{jt}(X_t) = Pr(b_{jt}(X_t, \epsilon_{it}) = 1|X_t)$ is i's belief of player j's behavior at time t

Model assumptions

MOD1: Players' strategies depend on state variables

MOD2: Players maximize expected payoffs

MOD3: A player's belief of his own action is consistent with his expectation of his actual actions.

'equil': Players' beliefs about other players' actions are unbiased expectations of the actual actions of other players. That is,

$$B_{jt}(X_t) = P_{jt}(X_t)$$

MOD4: If $T < \infty$, $B_{jt}^{(t_o)} = B_{jt}$; if $T < \infty$, $B_{jt}^{(t_o)} = B_j$

i's belief of j's behavior is a TxT matrix

Table 1
Sequence of Beliefs $B_{jt}^{(t_0)}$

Period when beliefs are formed (t_0)	Period of the opponents' behavior (t) $t=1$	$t=2$	$t=3$	...	$t=T-1$	$t=T$
$t_0=1$	$B_{j1}^{(1)}$	$B_{j2}^{(1)}$	$B_{j3}^{(1)}$	...	$B_{j,T-1}^{(1)}$	$B_{jT}^{(1)}$
$t_0=2$	-	$B_{j2}^{(2)}$	$B_{j3}^{(2)}$	...	$B_{j,T-1}^{(2)}$	$B_{jT}^{(2)}$
$t_0=3$	-	-	$B_{j3}^{(3)}$	...	$B_{j,T-1}^{(3)}$	$B_{jT}^{(3)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$t_0=T-1$	-	-	-	...	$B_{j,T-1}^{(T-1)}$	$B_{jT}^{(T-1)}$
$t_0=T$	-	-	-	...	-	$B_{jT}^{(T)}$

Best response (1)

- Given a belief B_j, player i best responds by maximizing her expected utility payoff
- The key optimization criterion is the Bellman equation

$$V_{it}^B(X_t, \epsilon_{it}) = \underset{Y_{it}}{max} \left(Y_{it}\pi_i^B(X_t - \epsilon_{it}) + \beta \int V_i^B(X_{t+1}, \epsilon_{it+1}) f^B dG \right)$$

where:

$$\pi_i^B(X_t) = B_{jt}(X_t)\pi_i(1, X_t) + (1 - B_{jt}(X_t))\pi_i(0, X_t)$$

$$\begin{aligned} f_i^B(X_{t+1})|Y_{it}, X_t) = & f_i(X_{it+1}|Y_{it}, X_{it}) * \\ & [B_{jt}(X_t) f_j(X_{jt+1}|1, X_{jt}) + \\ & (1 - B_{jt}(X_t)) f_j(X_{jt+1}|0, X_{jt})] \end{aligned}$$

Best response (2)

- The best response function can be represented by the threshold function

  $$\{Y_{it} = 1\} \Leftrightarrow \{\epsilon_{it}(0) - \epsilon_{it}(1) \leq v_{it}^{B}(1, X_t) - v_{it}^{B}(0, X_t)\}$$

  where:

  $$v_{it}^{B} = \pi_{it}^{B}(Y_{it}, X_t) + \beta \int_{X', \epsilon'} V_i(X', \epsilon') f_i^{B}(X' | Y_{it}, X_t) dG(\epsilon')$$

- Denote Λ as the best response function using the explicit distribution function (G) of ϵ, i.e.

  $$Pr(Y_{it} = 1 | X_t) = \Lambda(v_{it}^{B}(1, X_t) - v_{it}^{B}(0, X_t))$$

Data

- There are M markets. The econometrician observes

$$\{Y_{imt}, Y_{jmt}, X_{mt}\}_{t=1}^{T}$$

  for every market m.
- We are going to suppress m for our discussion of how to estimate the model.

Identification assumptions

ID1: $X_{mt} = X_t$, $B_{jmt}(X) = B_{jt}(X)$

ID2: Normalization of the payoff function $\pi(\cdot)$

ID3: There are two values of player i's opponent's state, S_j^L and S_j^H, at which player i's beliefs are in equilibrium; that is,

$$B_{jt}(W_t, S_i, S_j^L) = P_{jt}(W_t, S_i, S_j^L)$$

$$B_{jt}(W_t, S_i, S_j^H) = P_{jt}(W_t, S_i, S_j^H)$$

Estimation with the assumption 'equil'

Suppose $T = \infty$,

1. Observe the data (Y_{it}, Y_{jt}, X_t); do not observe ϵ_{it}
2. Assume that G (hence Λ) and β are known
3. Estimate $(\widehat{f_t^S}, \widehat{f_t^S}, \widehat{P_{it}}, \widehat{P_{jt}})$ non-parametrically
4. Inverts Λ to obtain $\tilde{v}_{it}$
5. Solve the Bellman equation to obtain $\widetilde{V}$ and $\tilde{\pi}$

$$V_{it}^B(X_t, \epsilon_{it}) = \max_{Y_{it}} \{ v_{it}^B(Y_{it}, X_t) + \epsilon_{it}(Y_{it}) \}$$

$$v_{it}^B(Y_{it}, X_t) = \pi_{it}^B(Y_{it}, X_t) + \beta \int V_{it+1}^B(X_{t+1}, \epsilon_{it+1}) f_{it}^B dG$$

note:

$$B_{it}(X_t) = \Lambda(v_{it}^B(1, X_t) - v_{it}^B(0, X_t))$$

$$B = P$$

Estimation using backward induction

Same as the last slide, but suppose $T < \infty$

- Define player i's continuation payoff at time $t-1$

$$d_{it-1} = \beta \sum_{X'} \bar{V}_{it}^{B}(X') f_{t-1}(X'|Y_i, Yj, X)$$

- Let $\tilde{d}_{iT} = 0$
- Solve for $\widetilde{\pi}_{iT-1}$ and $\tilde{\bar{\tilde{V}}}_{iT-1}^{B}$
- Calculate $\tilde{d}_{iT-1}$
- Repeat

Identification assumptions without the assumption 'equil'

- Instead of the assumption 'equil', we assume MOD4 and ID3, which states that there are two values of opponent's state variable, S_j^L and S_j^H, at which player i's beliefs are in equilibrium.
- Proposition 2 states that these are sufficient conditions to non-parametrically estimate player i's belief function and payoff function.

Estimation without the assumption 'equil'

1. Let $\tilde{d}_{iT} = 0$
2. Calculate $\hat{B}_{jt}$ by formula (30) in the paper
3. Calculate $\hat{\tilde{V}}_{iT}^{B}$ by formula (31)
4. Calculate $\tilde{d}_{iT-1}$ of the previous period
5. Repeat

Testing unbiased beliefs (1)

Under the assumptions MOD1, MOD2, MOD3, MOD4, ID1, and ID2, we can test the null of unbiased belief, i.e. player i's belief of j's behavior is consistent with the j's actual behavior at time t, $B_{jt}(X_t) = P_{jt}(X_t)$.

- Define
$$q_{it}(X) = \Lambda^{-1}(P_{it}(X))$$
- Pick X^a, X^b, X^c, X^d s.t. each value has the same value in the component of (S_i, W), but different values of S_j.

Testing unbiased beliefs (2)

- Define

$$\delta = \left\{ \frac{q_{it}(X_a) - q_{it}(X_b)}{q_{it}(X_c) - q_{it}(X_d)} - \frac{P_{jt}(X_a) - P_{jt}(X_b)}{P_{jt}(X_c) - P_{jt}(X_d)} \right\}$$

- Further define

$$D = \sum_{h=1}^{H} \left(\frac{\bar{\delta}_i^h}{se(\bar{\delta})} \right)^2$$

where $\bar{\delta}$ is the sample mean, then D is asymptotically distributed as Chi-square with H degrees of freedom. H is the number of all possible combinations of four different values of S_j with $S_j^a \neq S_j^b$ and $S_j^d \neq S_j^d$.

Empirically testing the null of unbiased belief

Table 8
Estimated Bias in BK Beliefs
Difference Between B_{MD} and P_{MD}

	Stores of BK	
	0	**1**
Stores of MD		
1	**-0.17 (0.04)**	-0.10 (0.06)
2	-0.08 (0.07)	-0.06 (0.10)

Empirically testing the null of unbiased belief

Estimated Bias in MD Beliefs
Difference Between B_{BK} and P_{BK}

	Stores of MD	
	0	**1**
Stores of BK		
1	-0.03 (0.05)	0.02 (0.04)
2	0.03 (0.10)	0.04 (0.12)