Discounting, values, decisions

a short review

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Introduction

Develop a methodology to compare utility processes with respect to discounting functions.

[+] Results: more patience $\Rightarrow$

i. stop later
ii. higher payoff
iii. larger continuation domain

[−] Results: contrary to the the [+] results.

We continue our investigation of comparative statics

- There are objective functions $f, g : X \to \mathbb{R}$, where $X \in R$.

- We want the optimizers to be well-behaved:

$$\arg\max_{x \in X} g(x) \geq \arg\max_{x \in X} f(x).$$

- Answers: SCP and IDO.

Terminology

- Let $T = [\,0, \bar{t}\,]$.
- Let the probability space with a filtration be $(\Omega, \mathbb{P}, \{\mathcal{F}_t\}_{t \in T})$.

- $\alpha(t)$ denotes a discount rate, e.g. $\alpha(t) = e^{-rt}$.
- τ denotes a stopping time w.r.t. the filtration.
- $u(t)$ denotes a payoff stream.
- $G(t)$ is the termination payoff if the process is stopped at time t.

A lattice order ($\succ$) on the set of discounting functions

Considering the problem: $\max_{\tau \in \mathcal{T}} U(\tau; \alpha)$

The general formulation:

$$\beta \succ \alpha \Leftrightarrow \frac{\beta(s)}{\alpha(s)} \text{ is increasing in } s$$

A simple example:

Let $\alpha(s) = e^{-rs}$ and $\beta(s) = e^{-r's}$, then

$$\beta \succ \alpha \Leftrightarrow r \geq r'.$$

That is, the agent with the β discounting function is more patient.

Setup: the optimal stopping problem

A general formulation:

$$\begin{aligned} V(\alpha) &= \sup_{\tau \in \mathcal{T}} U(\tau; \alpha) \\ &= \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[\int_0^{\tau} \alpha(s)u(s)ds + \alpha(\tau)G(\tau)\right] \end{aligned}$$

A simple example:

$$V(r) = \sup_{\tau \in \mathcal{T}} \left[\int_0^{\tau} e^{-rt}u(t)dt\right],$$

where $u(t)$ is a deterministic function of t.

Main idea: comparing optimal stopping rules and value functions

Theorem (1)

Let G_t be a non-negative process. If $\beta \succ \alpha$, then

$$\tau(\beta) > \tau(\alpha) \text{ and } \frac{V(\beta)}{\beta(0)} \geq \frac{V(\alpha)}{\alpha(0)}.$$

Proposition (1)

Let α and β be continuous. If $\beta \not\succ \alpha$, the there exists a deterministic cash-flow stream $(\pi_t)_{t \in T}$ such that, with zero termination value,

$$\tau(\alpha) > \tau(\beta) \text{ and } \frac{V(\alpha)}{\alpha(0)} \geq \frac{V(\beta)}{\beta(0)}.$$

Example: The IOD perspective – 1

Let the payoff stream to be deterministic. The discounted payoff payoff function is

$$U(\tau; r) = \int_0^\tau e^{-rt} u(t) dt.$$

Claim: If $r'' > r' > 0$, then

$$\arg\max_{\tau > 0} U(\tau; r') \geq \arg\max_{\tau > 0} U(\tau; r'').$$

Statement of Theorem (1):

Lower discount rate

$\Rightarrow$ later stopping (and higher utility)

Example: The IOD perspective – 2

Recall two facts: First, let J be an interval in X, where $X \subset \mathbb{R}$, then

$$g \succ_I f \Rightarrow \arg\max_{x \in J} g(x) \geq \arg\max_{x \in J} f(x).$$

Second, a sufficient condition for $g \succ_I f$ is that $\exists$ a positive function $d(\cdot)$ such that

$$g'(x) \geq d(x) f'(x).$$

Proof:

$$\begin{aligned} U_r(t;r)|_{r=r'} &= e^{-r't}u(t) = e^{(r''-r')t}e^{-r''t}u(t) \\ &\geq e^{(r''-r')t}U_r(t;r)|_{r=r''} \end{aligned}$$

$\Rightarrow U(t;r') \succ_I U(t;r'')$

Theorem (1) and Proposition (1)

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Proposition (1)

Let α and β be continuous. If $\beta \not\succ \alpha$, the there exists a deterministic payoff stream $(\pi_t)_{t \in T}$ and zero termination value such that,

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Proof of the theorem – 1

Lemma (1)

Let τ is optimal w.r.t. the discounting function $\alpha(t)$. Then,

$$\mathbb{P}\{\omega : t \leq \tau(\omega) \text{ and } \mathbb{E}\left[\int_t^{\tau(\omega)} \alpha(s)u(s)ds|\mathcal{F}_t\right] < 0\} = 0.$$

Lemma (2)

Let $v(s) \stackrel{\text{def}}{=} \mathbb{E}\left[u(s)\chi_{\{s\leq\tau\}}\right]$. Then for all $t \in [0, \bar{t})$,

$$\int_t^{\bar{t}} \alpha(s)v(s)ds \geq 0.$$

Proof of the theorem – 2

Lemma (3)

Let γ be increasing and $\int_x^{x''} h(s) > 0$ for all $x \in [x', x'']$. Then,

$$\int_{x'}^{x''} \gamma(s)h(s)ds \geq \gamma(x') \int_{x'}^{x''} h(s)ds.$$

Proof: (The part about the value function)

$$\begin{aligned} V(\beta) &\geq \int_0^{\bar{t}} \beta(s)v(s)ds = \int_0^{\bar{t}} \frac{\beta(s)}{\alpha(s)}\alpha(s)v(s)ds \\ &\geq \frac{\beta(0)}{\alpha(0)} \int_0^{\bar{t}} \alpha(s)v(s)ds = \frac{\beta(0)}{\alpha(0)}V(\alpha) \geq 0. \end{aligned}$$

Proof of the theorem – 3

Proof: (The part about the optimal stopping time)

Suppose not; then there is a set of non-zero measure such that

$$B = \{\omega : \tau(\beta) < \tau(\alpha)\}.$$

Consider the stopping rule $\tau^* = \max\{\tau(\alpha), \tau(\beta)\}$, then

$$U(\tau^*; \beta) = \mathbb{E}[\chi_{\{\Omega \setminus B\}} \int_0^{\tau(\beta)} ds + \chi_{\{B\}} \int_0^{\tau(\beta)} ds + \underbrace{\chi_{\{B\}} \int_{\tau(\beta)}^{\tau(\alpha)} \beta(s) u(s) ds]}_{>0}$$

which is greater than $V(\tau(\beta); \beta)$.

The problem of optimal stopping and control

Agents simultaneously decide when to stop (τ) and how to control (λ) her payoff streams.

$$\begin{aligned} V(\alpha) &= \sup_{\tau,\,\lambda} U(\tau, \lambda; \alpha) \\ &= \sup_{\tau,\,\lambda} \mathbb{E}\left[\int_0^{\tau} \alpha(s) u(x(\lambda), \lambda(s), s) ds + \alpha(\tau) G(\tau)\right], \end{aligned}$$

where $\{x(\lambda(t)\}_{t \in T}, \{\lambda(t)\}$, and τ are adapted to the filtration $\{\mathcal{F}_t\}$.

Existence of a solution

The existence of solution to the problem $V(\alpha) = \sup_{\tau \in \mathcal{T}} U(\tau; \alpha)$ is relatively easy: Continuity condition on $G(t)$ and $\{\mathcal{F}_t\}_t$ is sufficient.

The existence of optimal control and optimal stopping to the problem $V(\alpha) = \sup_{\tau,\, \lambda} U(\tau, \lambda; \alpha)$ is less obvious. The stopping part can be parsed out by the DDP principle, but finding the optimal control is not easy. One approach is to use Hamilton-Jacobi-Bellman PDE. For the finite horizon problem: A function satisfies

$$-V_t - \sup_{\lambda} \left[\mu V_x + 1/2\sigma^2 V_{xx} + u(t)\right] = 0.$$

is only a candidate to be the valuation function.

Continuation Domain – 1 [+]

The continuation domain is the set of states that the decision maker should wait, that is,

$$C(\alpha, t) = \{x : V(\alpha, t, x) > G(t, x)\},$$

where

$$V(\alpha, t, x) = \sup_{\tau \geq t} \mathbb{E}\left[\frac{1}{\alpha(t)}\left[\int_t^\tau \alpha(s)u(s)ds + \alpha(\tau)G(\tau)\right]\right]$$

Continuation domain – 2 [+]

Theorem (3)

If $\beta \succ \alpha$*, then*

$$C(\alpha, t) \subset C(\beta, t).$$

Proof:

$$\begin{aligned} x \in C(\alpha, t) &\Rightarrow V(\beta, t, x) \geq V(\alpha, t, x) > G(t, x) \\ &\Rightarrow x \in C(\beta, t) \end{aligned}$$

Timing of decisions– 1 [–]

With both stopping and control decisions to be made, a more patient player do not always stop later.

- controls: $\lambda \in \{1, 2\}$
- For $x \in [1, 10), u(x, \lambda) = M$
- For $x \in [10, \infty)$, $u(x, \lambda) = -M$
- For $x \in [0, 1), u(x, 1) = 1$ and $u(x, 2) = -0.01$.
- $\frac{dx}{dt} = \lambda(t) > 0$; hence $x(t)$ is strictly increasing.

Then,

1. An impatient player ($\alpha = \epsilon > 0$) chooses control 1 and stops at $t = 10$.
2. A moderate patient player chooses control 2 to get to the M payoff sooner, and also stops at $t = 10$.

Timing of decisions– 2.a [+]

The problem:

$$\max_{\tau,\lambda} U(\tau,\lambda;r) = \max_{\tau,\lambda} E \int_0^\tau e^{-rs} u(x_t,\lambda_t,t)dt$$

Assume that

- $x(t)$ is an Ito process with drift, i.e. $dx(t) = \mu dt + \sigma dB(t)$.
- $m^*(t;r) = \arg\max_m \{u(x,\lambda,t) : \mu(x,\lambda,t) = m\}$
- u is increasing in x for all λ and t.

Timing of decisions – 2.b [+]

Theorem (4)

$V(r), m^*(t;r), x^*(t;r)$, and $\tau(r)$ are all decreasing in r.

In words, the value function, the optimal drift, the state path, and the optimal stopping time are all increasing with patience. This is analogous to Theorem (1).

Time consistency – 1

An agent has a time in-homogenous discounting function.
At time s, the agent has a discounting function $\alpha(s, \cdot)$.

Let's define a slightly different partial order, and let
$\beta \succ \alpha$ if at each time s, the ratio

$$\frac{\beta(s,t)}{\alpha(s,t)}$$

is increasing in t, for all $t > s$.

Time consistency – 2 : naive agents [+]

Theorem (6)

If $\beta \succ \alpha$, *then*

$$\tau(\beta) > \tau(\alpha) \ \textit{and} \ \frac{V(\beta(s,\cdot)}{\beta(s,s)} \geq \frac{V(\alpha(s,\cdot)}{\alpha(s,s)}.$$

Almost the same as Theorem (1). In fact, it is implied by Theorem (1).

Time consistency – 3 : sophisticated agents [–]

A agent with more patience stops earlier.

Let the cash flow be:

$$1, -M, \underbrace{M/n, M/n, \cdots, M/n}_{n+1}, 0, 0, 0, \cdots$$

1. Agent A has $\alpha(s,t) = (1/2)^t$. She stops at period $t = 1$.

2. Agent B's $\alpha(0,t) = (1/2)^t$ and $\alpha(s,t) = 1$, for all $s \geq 1$. She stops at period $t = 0$.

A few ideas

- Apply similar comparative statics results for a different class of optimization problems
- Extend the analysis here to a repeated game setting, e.g. different players have different but comparable discounting functions.
- For $dx(t) = \mu dt + \sigma dB(t)$, put a ordering structure $(\succ)$ on σ.
- A large literature in finance that use a stochastic control approach to solve the problems of portfolio optimization and asset pricing. Most solutions are numerical, and comparative results would be very useful.