Principal-Agent Problems in Continuous Time

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Outline

- Contract theory in continuous-time models
- Sannikov's model with infinite time horizon
- The optimal contract depends on the agent's continuation value and has a lower and upper boundary
- A procurement model in finite time horizon
- The optimal contract depends on the continuation value and the state process

Related literature

- Books:
  - Discrete-time contract theory: Salanie (1997), Laffont and Martinmort (2001), Bolton and Dewatripont (2004)
  - Continuous-time: Cvitanic and Zhang (2013)
- Holmstrong and Milgrom (1987), Schatterler and Sung (1993), Williams (2003), DeMarzo and Sannikov (2007), Sannikov (2008), He (2008), Cvitanic and Wang and Zhang (2009), He (2008), Zhu (2012), Cvitanic and Wan and Yang (2012)

Modeling variations

- Time horizons: finite or infinite
- State processes: Brownian motion plus drift, geometric Brownian motion, or mean-reverting
- Payments: instantaneous payments C_t, or lump-sum payment C_T at the terminal time
- Information: moral hazard, adverse selection, and/or learning
- Modeling approaches: weak or strong formulation

The contracting environment of Sannikov's model

- $t \in [0, \infty)$
- The output (state) process:

$$dX_t = A_t dt + \sigma dB_t$$

- Effort: A_t
- Hidden action (moral hazard)
- Per-period payment: C_t
- Random fluctuations: B_t

The contracting problem

- r is the common discounting factor
- The principal offers C to

$$\begin{aligned} & \max_C \mathbb{E}\left[r \int_0^\infty e^{-rt}(dX_t - C_t dt)\right] \\ = \quad & \max_C \mathbb{E}\left[r \int_0^\infty e^{-rt}(A_t - C_t)dt\right] \end{aligned}$$

- The agent best-responds with A to

$$\max_A \mathbb{E}\left[r \int_0^\infty e^{-rt}\Big(u(C_t) - h(A_t)\Big)dt\right]$$

Principal's constrained problem

$$\max_{A,C} \mathbb{E}\left[r\int_0^\infty e^{-rt}(A_t - C_t)dt\right]$$

subject to:

(P): $\mathbb{E}\left[r\int_0^\infty e^{-rt}\Big(u(C_t) - h(A_t)\Big)dt\right] \geq R$

(IC): A is agent's optimal response to C.

Representations of agent's continuation value

- Define $W_t \equiv \mathbb{E}_t^A \left[r \int_t^\infty e^{-rs} \big(u(C_s) - h(A_s) \big) ds \right]$
- By the Martingale Representation Theorem, there is a unique Y such that

$$W_t = W_0 + r \int_0^t \big(W_s - u(C_s) + h(A_s) \big) ds + r \int_0^t Y_s dZ_s^A$$

- The agent is indifferent between getting a promise of
  1. $\{C_s : t \leq s \leq \infty\}$, or
  2. W_t

Incentive compatibility

Proposition (Incentives)

The following two conditions are equivalent.

1. *A is an optimal response to the payment C.*
2. $A_t(\omega) \in \arg\max_a \{Y_t(\omega)a - h(a)\}$, *for almost every (t, ω).*

It is a continuous-time version of the one-shot deviation principle in that A is optimal if and only if A_t is optimal at each moment t.

The main idea behind the IC proposition

Let A^* be optimal, and assume that the agent follows A^* up to time t and switch to an alternative A after time t.

- $V_t = r\int_0^t e^{-rs}\Big(u(C_s) - h(A_s^*)\Big)ds + e^{-rt}W_t$
- Drift of V_t:

$$re^{-rt}\Big([Y_tA_t^* - h(A_t^*)] - [Y_tA_t - h(A_t)]\Big)dt$$

Designing the optimal pair (A, C)

Proposition (Transversality condition)
If $\mathbb{E}_t^A[e^{-rs}\tilde{W}_{t+s}] \to 0$ *as* $s \to \infty$, *then* $\tilde{W}_t = W_t$.

- $\tilde{W}_t = W_0 + r\int_0^t \big(\tilde{W}_s - u(C_s) + h(A_s)\big)ds + r\int_0^t \beta_s dZ_s^A$.
- Principal must pay the agent eventually.
- Let $\gamma(a) = \{y : a \in \arg\max_{a'} ya' - h(a')\}$. To enforce A, the principal promises

$$W_t = W_0 + r\int_0^t \big(W_s - u(C_s) + h(A_s)\big)ds + r\int_0^t \gamma(A_s)dZ_s^A.$$

Converting the principal's problem into a stochastic control problem

Let $F_0(w) = u^{-1}(w)$. The principal's problem

$$\max_{A,C,\tau} \mathbb{E}\left[r\int_0^\tau e^{-rs}(A_s - C_s)ds + e^{-r\tau}F_0(W_\tau)\right],$$

where

$$W_t = W_0 + r\int_0^t \big(W_s - u(C_s) + h(A_s)\big)ds + r\int_0^t \gamma(A_s)dB_s.$$

The dynamic programming principal and the HJB equation

Let

$$F(w) = \max_{A,C,\tau} \mathbb{E}_t \left[r \int_t^\tau e^{-rs}(A_s - C_s)ds + e^{-r\tau} F_0(W_\tau^{t,w}) \right].$$

Following the standard arguments of DPP, $F(\cdot)$ is a solution of the following ODE:

$$\begin{cases} rF = \max_{a,c} r(a-c) + r\Big(w - u(c) + h(a)\Big)F' + \frac{1}{2}r^2\gamma(a)^2\sigma^2 F'' \\ F(0) = 0 \\ F(W_{gp}) = F_0(W_{gp}), \text{ and } F'(W_{gp}) = F_0'(W_{gp}) \end{cases}$$

Let $a^*(w)$ and $c^*(w)$ be the maximizers.

Description of the optimal contract

Theorem (Optimal contract)

Let

$$W_t = W_0 + r \int_0^t \Big(W_s - u(c^*(W_s)) + h(a^*(W_s)) \Big) ds + r \int_0^t \gamma(a^*(W_s)) dB_s.$$

The stopping rule τ is the first time W_t hits lower boundary $W_t = 0$ or the upper boundary $W_t = W_{gp}$. The payment and requested efforts before τ is

$$A_t = a^*(W_t) \text{ and } C_t = c^*(W_t)$$

and after τ, $A_t = 0$ and $C_t = F_0(W_\tau)$.

Features of the contract

- W_t summarizes the past history
- Lower boundary serves as a punishment scheme for incentives
- Upper boundary is due to income effect; too costly to compensate for the agent for his effort when W_t is too high
- Probational period for low W_t.

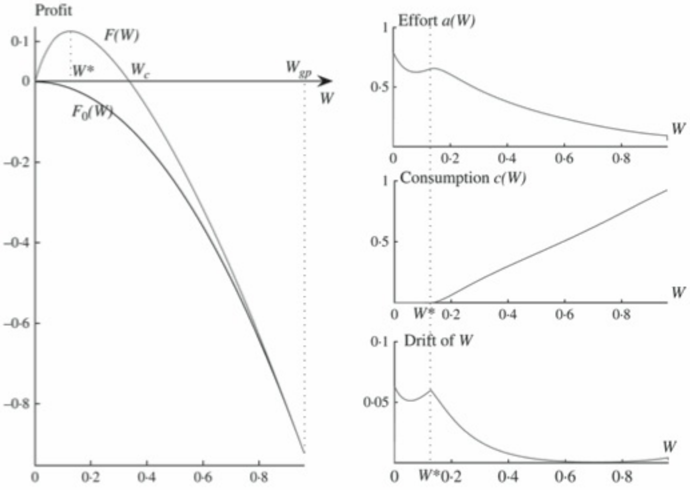

FIGURE 1

Function F for $u(c) = \sqrt{c}$, $h(a) = 0{\cdot}5a^2 + 0.4a$, $r = 0{\cdot}1$ and $\sigma = 1$. Point W^* is the maximum of F

Implementation

1. Find W^* that is the best starting point for the principal. It may be higher than the agent's reservation value R
2. At time t, calculate W_t
3. Once W_t is known, so is $A_t = a^*(W_t)$ and $C_t = c^*(W_t)$
4. Retire the agent once W_t hits the boundary

The contracting environment of a procurement problem

- $t \in [0, T]$
- The price (state) process is mean reverting:

$$P_t = P_0 + \int_0^t \lambda(A_s - P_s)ds + \int_0^t \sigma dM_s.$$

- Effort: A_t
- Hidden action (moral hazard); the principal cannot contract on actions
- Lump-sum payment at terminal time: C_T
- Random fluctuations: M_t

An example: a bilateral contract to supply ancillary services

- A supplier can manipulate the market price without detection, for example
  1. manipulate supplies
  2. through virtual bids
  3. by proxy

- The supplier manipulate the price to maximize payment
- The utility company seeks to pay as little as possible
- We want to know how to mollify the agent's incentives to manipulate prices through an adjustment C_T

The contracting problem

- The agent's problem:

$$\max_{A} \mathbb{E}\left[U(C_T) + \int_0^T \Big(u(P_s D_s) - h(A_s)\Big) ds\right]$$

- The principal's problem:

$$\min_{C_T} \mathbb{E}\left[C_T + \int_0^T P_s D_s ds\right]$$

The same problem with the ABM state proces

$$\min_{A,C_T} \mathbb{E}\left[C_T + \int_0^T P_s D_s ds\right]$$

subject to:

1. Incentive compatibility constraint in that A is an optimal response to C_T
2. The participation contraint

$$\mathbb{E}\left[U(C_T) + \int_0^T \big(u(P_s D_s) - h(A_s)\big) ds\right] \geq R$$

3. The state process is Arithmetic Brownian (not mean-reverting)

$$P_t = P_0 + \int_0^t A_s ds + \int_0^t \sigma dM_s$$

First-best (1)

1. Let $F(\cdot)$ be the solution of the PDE

$$\begin{cases} -\partial_t F(t,p) = \min_a \; pD_t - \lambda u(pD_t) + \lambda h(a) + a\partial_p F(t,p) \\ \qquad\qquad\qquad\qquad\qquad + \frac{1}{2}\sigma^2 \partial_{pp} F(t,p) \\ F(T,p) = 0, \quad \text{for all } p, \end{cases}$$

and $a^{*,\lambda}(t,p)$ be the solution of

$$\lambda h'(a) + \partial_p F(t,p) = 0.$$

2. Let $C^{*,\lambda}$ be the solution of

$$1 = \lambda U'(C_T).$$

First-best (2)

3. And λ is found by the participation constraint

$$R - U(C^{*,\lambda}) = \mathbb{E}\left[\int_0^T \Big(u(P_s^{*,\lambda} D_s) - h(a^{*,\lambda}(t, P_s^{*,\lambda}))\Big) ds\right],$$

where $P_t^{*,\lambda} = P_0 + \int_0^t a^*(t, P_s^{*,\lambda}) ds + \int_0^t \sigma dM_s$.

Proposition (First-best contract)

The first best contract is $(C^{*,\lambda}, A^*)$, *where* $A^* = \{a^{*,\lambda}(t, P_t^*) : 0 \leq t \leq T\}$.

First-best (3): Steps to obtain the contract (numerically)

1. Solve the PDE and obtain $a^{*,\lambda}(\cdot)$
2. Obtain $C^{*,\lambda}$ by inverting $U(\cdot)$
3. Find λ by setting participation constraint at equality
4. Calculate $(C^{*,\lambda}, A^{*})$.

Second-best: agent's problem

The agent's continuation value is

$$W_t = \mathbb{E}_t^A \left[U(C_T) + \int_t^T (u(P_s D_s) - h(A_s)) ds \right],$$

which has a representation:

$$W_t^A = U(C_T) + \int_t^T (u(P_s D_s) - h(A_s) + Z_s A_s) ds - \int_t^T Z_s \sigma dB_s.$$

Proposition (Incentives)

The following two conditions are equivalent.
1. A is an optimal response to the payment C_T.
2. $A_t(\omega) \in \arg\max_a \{Z_t(\omega) a - h(a)\}$, for almost every (t, ω).

The resolution of the problem of agency

Let $\gamma(a)$ (resp. $\eta(z)$) be the solution of $z - h'(a) = 0$ in terms of z (resp. a).

1. Given C_T, the response $A_t = \eta(Z_t)$ is optimal, where (W, Z) is the unique solution of the BSDE

$$W_t = U(C_T) + \int_t^T (u(P_s D_s) - h(\eta(Z_s)) + Z_s \eta(Z_s)) ds - \int_t^T Z_s \sigma dB_s.$$

2. Given (A, W_0), the enforcing $C_T = J(W_T^A)$ where J is the inverse of $U(\cdot)$, and

$$W_t^A = W_0 - \int_0^t \Big(u(P_s D_s) - h(A_s) \Big) ds + \int_0^t \gamma(A_s) \sigma dB_s^A.$$

Second-best: principal's problem

Let F be the solution to the PDE

$$\begin{cases} -\partial_t F = \min_a \; pD_t + a\partial_p F + \frac{1}{2}\sigma^2 \partial_{pp} F + (h(a) - u(pD_t))\partial_w F \\ \qquad\qquad\qquad\qquad + \frac{1}{2}\gamma(a)^2\sigma^2\partial_{ww} F + \gamma(a)\sigma^2\partial_{pw} F \\ F(T,p,w) = J(w), \quad \text{for all } p \text{ and } w, \end{cases} \tag{1}$$

and $a^*(t,p,w)$ be the optimizer of

$$\min_a \; a\partial_p F + h(a)\partial_w F + \frac{1}{2}\gamma(a)^2\sigma^2\partial_{ww} F + \gamma(a)\sigma^2\partial_{pw} F.$$

Second-best: the contract

Theorem (Optimal Contract)

Let $a^*(t,p,w)$ be the minimizer in (1), and the agent is paid $C_T = J(\tilde{W}_T)$, where

$$\begin{cases} P_t = P_0 + \int_0^t a^*(s, P_s, \tilde{W}_s)ds + \int_0^t \sigma dM_s \\ \tilde{W}_t = R + \int_0^t \Big[h(a^*(s, P_s, \tilde{W}_s)) - u(P_s D_s)\Big] ds \\ \qquad\qquad\qquad + \int_0^t \gamma(a^*(s, P_s, \tilde{W}_s))\sigma dM_s. \end{cases}$$

Then, the the contract $(A, C_T) = \big(\{a^*(t, P_t, \tilde{W}_t) : 0 \le t \le T\}, C_T\big)$ is incentive compatible for the agent, and optimal for the principal among all incentive compatible contracts that deliver an initial expected value of at least W_0 to the agent.

Features of the contract

- The pair (P_t, W_t) summarizes the past history; P_t is needed because the restriction on payments
- There are no boundaries on which the agent is retired. This is due to finite time horizon
- The calculation of the contract relies on solving a PDE, as opposed to an ODE in infinite time horizon
- It is never optimal for the principal to gives the agent an initial value more than R

An implementation

1. The agent is asked to perform A^*
2. By time t, the agent is paid $\int_0^t P_s D_s ds$
3. By time t, the agent is also paid

$$R - \int_0^t \Big(u(P_s D_s) - h(A_s^*)\Big) ds + \int_0^t \gamma(A_s^*) \sigma dM_s$$

Note that P is observable and M is the standard Brownian motion, hence the payments can be calculated by time t.

Proof of the optimal contract theorem

Verify four lemmas:

1. (C_T, A) is optimal for the agent.
2. Any IC contract delivering R to the agent costs the principal at least $F(0, P_0, R)$.
3. (C_T, A) costs the principal $F(0, P_0, R)$.
4. Any IC contract delivering more than R is not optimal for the principal.

First-best vs. second-best

- If $U(\cdot)$ and $u(\cdot)$ is linear, then first-best and second-best are identical

- Using a different approach, the condition for the second-best payment can be reduced to

$$1 = \Gamma_T U'(C_T),$$

comparing to the first best

$$1 = \lambda U'(C_T),$$

where $\Gamma_t \equiv \lambda + \int_0^t \sigma^{-1} A_s^* dM_s$

Thoughts

- Theories on continuous-time models are relatively developed
- Not too many completely solved models
- General frameworks have been developed but there are still limitations (Cvitanic and Zhang, 2013)
- Limited work on applying the theories
- Computational complexities because general theories relie on solving PDEs and BSDEs (also FBSDEs)