# Optimal Dynamic Contract for Electricity Procurement*

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**Abstract**

Bilateral procurement contracts in the electricity industry use fixed unit prices to settle payments and deliveries. This type of purchasing arrangement creates perverse production incentives and ignores dynamic uncertainties that are inherent in electricity markets. We consider a continuous-time contracting environment, in which the electricity price is modeled as a mean reversion jump diffusion process. In the optimal contract, the principal provides incentive to the agent through a reward-and-punishment scheme that correlates with market fluctuation. The contracting parties keep track of two state variables (the electricity price and the agent's continuation value) to calculate payment and to determine optimal action that the agent follows in the optimal path. The analysis provides a guideline on how to write a procurement contract using the electricity price as a performance measure.

## 1. Introduction

### 1.1 Motivation

In most of the electricity markets that have undergone a deregulation process, load serving entities (LSE) meet their end-use customers' demand by procuring electricity either through a centralized wholesale market, or through bilateral contracts with electricity generators. A wholesale market is usually operated by an independent system operator (ISO), organizing an auction exchange to match supply and demand. A bilateral market lets suppliers and buyers to negotiate one-on-one contracts, and the ISO only acts as an intermediary to enforce operational limits. There was much debate over which of the two market approaches was better in the early days of the deregulation effort (see Joskow, 1999), and LSEs procure energy both ways in today's markets.

A design of electricity market could require all transactions to go through a centralized exchange, eliminating the needs for bilateral contracts. Most electricity exchanges use an auction system

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to set market clearing prices, and market experiences have shown that of aggregating all the buying and selling into an electricity pool are problematic. Prices can become extremely volatile due to inelastic demand and auction rules, and large generators can easily manipulate prices by withholding supplies in peaked hours (see Joskow, 2001; Borenstein, 2002). The long-run average wholesale price might not be sufficiently high to justify investments into new capacities, which are necessary to demand growth (see Joskow, 2006). A pure auction approach favors traditional generation technology and discourages the development of alternative energy sources (see Hausman, Hornby, and Smith, 2008).

Bilateral contract cannot be the only procuring channel, but the use of bilateral contracts can mitigate some market uncertainties that plague the centralized auction approach. Before deregulation, a vertically integrated industry did not have an auction market and was an example of pure bilateral market. Under that traditional framework, the regulated prices were stable. Costs and profits were transparent, and price manipulation did not exist. The bilateral contract environment offered stability and predictability. Public utilities were able to commit large up-front capital for long-term investments that were critical for bringing new capacities and renewable sources into the grid.

The prevailing model of bilateral contract is simple but flawed. In its current form, a contract specifies a fixed unit price, the amount of electricity the supplier is expected to deliver, and the duration of the agreement. This contract is problematic in two ways. First, it ignores incentives. For example, a supplier's bottom line is inversely proportional to the cost of generating fuel. In the event that fuel price drops, the supplier delivers the maximum amount he is contractually allowed. The timing of this surplus may not be desirable because the spot market price is likely to be low, and a LSE can buy from the market below the contract price. Second, it ignores stochastic dynamics of supply and demand. For example, there exist unpredictable supply fluctuations coming from renewable sources such as wind and solar. In peak hours when supply is short, more energy is needed; and in off-peak hours when baseload power plants and renewable sources oversupply, energy is so abundant that price could be negative. More generally, the primary shortcoming of a static, fixed price contract is that the supplier does not respond to market conditions. One of the main impetuses for deregulation is to introduce market responsiveness. Hence, we want to understand how bilateral contracts should be designed to give the suppliers incentives to respond to stochastic and dynamic market conditions.

In this paper, we investigate a continuous-time contracting problem. We derive an optimal bilateral contract. The contract uses the electricity price as a performance measure to determine a two-tier payment to the supplier. First, the supplier receives a payment that is the market value of the quantity of electricity delivered to the LSE, i.e. $\int_0^T P_t D_t dt$. Because the market price already aggregates a wealth of market information, such as operational uncertainties, fuel costs, weather, and demand fluctuations, the direct payment is sensitive to those market conditions. Second, the supplier receives an adjustment fee $C_T$ at the terminal time of the contract, which is calculated using a reward-penalty scheme to make sure that the supplier does not manipulate prices.

### 1.2 Related literature

There is a growing literature on the topic of market design for the electricity industry. The experiences of the industry on various market rules and institutions in the United States and in other parts of the world have been mixed. Most deregulated markets have introduced some versions of a centralized wholesale market by way of an auction system, but this system has been exposed to have many flaws. Borenstein (2002) uses the Californian energy crisis in 2000 as an example to highlight the many problems in the current market design. Most industry insiders and researchers agree that we are still experimenting with what market institutions work well and what does not (for example, see Joskow, 2008). There is the need to develop a systematic research agenda to investigate how to design the electricity market as a whole. Th current literature is scattered, and many connecting components of market are studied in isolation. For example, the day-ahead market (Giabardo, Zugno, Pinson, and Madsen, 2010), the forward market (Murphy and Smeers, 2010), the capacity market (Chao and Wilson, 2002), and bilateral contracts for wind power generators (Cai, Adlakha, and Chandy, 2011) are separately investigated, but there has not been any attempt to understand how they might affect one another. Further, those studies only focus on the supply side. On the retail side of the market, far less is known on how to coordinate consumption and create incentives for consumers to respond to prices.

Because one of the goals of market reform is to make the power grid more reliable and more accommodating to changing technologies and a growing demand, good market designs must consider both physical and economic constraints that are idiosyncratic to the generation, transmission, and consumption of electricity. Unlike other commodities such as fossil fuels, foods, or consumer goods, the cost of storage of electricity is extremely high. The needs for system reliability require the ISO to acquire reserve capacity well over the predicted demand to the point that in real time market, it is not uncommon to observe negative prices due to oversupply and the difficulty of disposing energy. Supply is increasingly unpredictable due to the introduction of larger renewable sources such as wind and solar, and demand can be inelastic. A system operator needs to implement automatic frequency, voltage, and reactive power control in a fast time scale to balance the grid. These characteristics are not independent, and they often interact in ways that require well-coordinated market and control designs to prevent both market failures and grid instabilities. Cho and Meyn (2010) show that even a small supplying friction in the form of ramping constraints can cause severe price instability, where the price fluctuates between zero and a choke-up price. The effects of transmission and generation constraints on the equilibrium price are investigated by Wang, Kowli, Negrete-Pincetic, Shafieepoorfard, and Meyn (2011) from a control theorist's perspective. Mathieu, Haring, Ledyard, and Andersson (2013) give an overview of the engineering and economic pros and cons of the different rules to implement a demand response program. The task of designing an electricity market is complex and multi-layered.

The formal model used in this paper is a continous-time version of a principal-agent problem with moral hazard and with the state process following a mean reversion jump diffusion. The seminal paper on the topic of contract theory in continuous-time is Holmstrom and Milgrom (1987), which focuses on the case of exponential utilities. There are different modeling variations with respect to time horizons, state processes, information structures, and payment methods. For instance, the influential paper by Sannikov (2008) analyzes a model in which the time horizon is infinite, the state process is an arithmetic Brownian motion, the agent controls the drift (moral hazard), and payment is continuous. Cvitanić, Wan, and Zhang (2009) study a finite horizon model with only a lump-sum payment. The paper by He (2009) modifies the state process to a geometric Brownian motion. Sung (2005) and Cvitanić, Wan, and Yang (2012) investigate models that have both moral hazard and adverse selection constraint. The recent book by Cvitanić and Zhang (2013) provides a nice summary of the literature and also develops a general theory to study these models. The model we develop here is different from the literature in three ways. First, the state process is a mean reversion process with jumps; second, the payment method is constrained so that the continuous payment is $\int P_s D_s ds$ and the principal can only directly control the end of period lump-sum payment $C_T$; third, the dynamic programming principle approach is used to describe a finite horizon model (in contrast to the general theory used in Cvitanić and Zhang (2013)).

While the literature of contract theory in continuous-time has become quite extensive, the application of these theories are still limited. Most of the applications have been devoted to corporate finance topics, for examples, executive compensation and career path (He, 2009; Zhu, 2013), capital structure (DeMarzo and Sannikov, 2006), and asset pricing (Biais, Mariotti, Plantin, and Rochet, 2007). One exception is the paper by Fong (2009), in which the theory is applied to evaluate the qualities. We are not aware of any other application outside the corporate finance literature. This paper applies a continuous-time model to study how to design procurement contracts in the electricity market.

The paper is organized as follows. Section 2 describes the contracting problem. Section 3 presents the solution of the problem when the principal can observe the manipulation process. Section 4 presents the analysis of the incentive compatibility condition and the derivation of an optimal contract. The contract is summarized in Theorem 4.5. Section 5 considers a few special cases. Section 6 discusses the optimal contract's properties and its limitations. Section 7 concludes.

## 2. Contracting environment

We use a princinpal-agent setup to model the procurement problem between an electricity buyer and a seller. The contracting period terminates at time $T$. The procuring principal wants to the electricity producing agent to supply a deterministic amount of energy $D = \{D_t : 0 \le t \le T\}$. The principal pays the agent an instantaneous direct payment $P_t D_t$ and also an adjustment fee $C_T$ at the terminal time, where $P = \{P_t : 0 \le t \le T\}$ is the price of the electricity. The direct payment is the market value of the electricity supplied, the adjustment fee can be viewed as a subsidy to the supplier. In many electricity markets, prices are posted and demand dispatches are issued at five-minutes interval by the ISO, but we model the price and quantity as continuous processes. In particular, the electricity is assumed to take the form[1]

$$P_t = P_0 + \int_0^t \rho(A_s - P_s)ds + \int_0^t \sigma dM_s + \sum_{i=1}^{N_t} L_i, \tag{2.1}$$

where $P_0 > 0$ and $\rho > 0$ are both constants. $M$ is a standard Brownian motion on the probability space $(\Omega, \mathbf{F}, \mathbb{Q})$. The jump size $L_i$ has a uniform distribution on $[-L, L]$, and $N$ is a Poisson process with intensity $\lambda$. We assume that the agent does not control the jump portion of the state process. We denote the compound Poisson process and its jumps

$$J_t \triangleq \sum_{i=1}^{N_t} L_i \text{ and } \Delta J_t \triangleq J_t - J_{t-},$$

where $J_{t-}$ is the value of $J_t$ right before the jump, i.e. $J_{t-} \triangleq \lim_{s\uparrow t} J_s$. The Brownian motion $M$ and the compound Poisson process are independent. We denote $\mathcal{F}^{M,J}$ as the augmented natural filtration generated by $(M, J)$, where $\mathcal{F}_t^{M,J} = \sigma\left(\{(M_s, J_s) : 0 \le s \le t\}\right)$. The agent manipulates the price process through $A$, where $A_t$ takes value in a compact subset $\mathbf{A} \subseteq \mathbb{R}$. We use $\mathcal{A}$ to denote the set of all predictable $A$.[2]

The price process is assumed to have two features: it takes on the form of mean-reverting jump diffusion, and the agent has partial control. Time series of electricity prices have shown to be remarkably consistent across a variety of auction markets, and mean reverting jump diffusion models capture most of the key dynamic characteristics of empirical prices(Cartea and Figueroa, 2005; Geman and Roncoroni, 2006). The price is subject to both frequent small ($M$) and occasionally large ($J$) variations, and the parameter $\rho$ is the rate at which the price adjust to its mean. This paper does not specify a microfoundation to show that price processes must be this form; rather, it assumes that there already exists a market mechanism from which prices are generated. As long as some version of an auction system is in placed, it is likely that price paths resembles a mean reverting jump diffusion. The perspective we takes here is similar to a well-known practice in option pricing theory in which the stock price is assumed to follow geometric Brownian motion. However, the assumption has its drawback because for example, if all the wholesale transactions are conducted by bilateral contracts (possibly using the format described here), then the evolution of price would likely be different from what were assumed. Given the regulatory environment, changes to the current market institutions

[1] Throughout, all Lévy processes are restricted so they have bounded jumps and finitely many jumps over the interval $[0, T]$. The associated random jump measure is finite, and the purely discontinuous part can be represented by a finite sum. Further, a stochastic integral w.r.t. a finite random measure is also a Lévy process whose characteristic random jump measure is also finite. Hence, we conveniently denote the pure jump portion of stochastic integral as a finite sum.

[2] A stochastic process $A$ is said to be *predictable* w.r.t the filtration $\mathcal{F}$ if for every $t \in [0, T]$, the mapping $(t, \omega) \to A_t(\omega)$ is measurable with respect to the sigma algebra generated by all adapted, continuous processes. Any adapted, left-continuous process is predictable.

are likely to be incremental. We should expect that electricity price continues to follow the trend of its recent past.

In this contracting environment, the agent is assumed to be able to control price without the principal's knowledge. There are a number of ways an electricity supplier can influence the price. It is often discussed that the combination of inelastic demand and a uniform price rule in most auction markets allow suppliers to easily push up prices by withholding small amount of generation, especially when the price is already high in peak hours (for examples, see Borenstein (2002); Joskow (2001). In the demand side, a large electricity customer can also shift its consumption to affect price without suffering the consequence if he has already locked into a fixed rate. Market participants can also submit virtual bids or trade financial contracts to influence prices. Because the variety of ways and number of players that could influence prices, a supplying agent could also manipulate prices by paying a proxy.

The setup is not limited to the procurement problem between an electricity supplier and an LSE, it is a model that generalizes a bilateral contracting problem that uses the electricity price as a performance measure. For example, an independent system operator needs to secure sufficient reserve capacities to ensure the grid's reliability. The operator would want to pay a supplier for the capacity commitment and delivered electricity based on the realized spot market prices. For another example, in a retail market, large commercial customers could be concerned that the distribution company would manipulate the prices to extract higher payments, especially when the distribution company could already own the generating capacities. In both examples, the operator and the commercial customers have to deal with a moral hazard problem. The analysis in this paper provides a qualitative guide to how the payment should be structured to provide the correct incentives so one one could benefit from manipulation the market.

The contracting parties commit to an agreement at time $t = 0$. The agent gets paid a stream payment $P_t D_t$ and a lump-sum $C_T$ at time $T$, and he obtains an utility $u(P_t D_t)$ and $U(C_T)$. Under those payments, the agent

$$\max_{\tilde{A}} \mathbb{E}\left[U(C_T) + \int_0^T \Big(u(P_s D_s) - h(\tilde{A}_s)\Big) ds\right], \tag{2.2}$$

where $h(\tilde{A}_s)$ is the cost of performing the action $\tilde{A}_s$. The choice of the sequence of action $\tilde{A}$ affects the price $P$ as in (2.1). We say that $(C_T, A)$ is *incentive compatible* if there exists a control proces $A$ such that it solves (2.2), and also, we say that $C_T$ *enforces* $A$. We refer the pair $(C_T, A)$ as the contract. Let $\mathcal{C}$ be the set of all incentive compatible contracts.

The principal seeks to minimize payment

$$\min_{C_T \in R} \mathbb{E}\left[C_T + \int_0^T P_s D_s ds\right].$$

The agent agrees to the contract $(C_T, A)$ if and only if he is better off with it than without it,

$$\mathbb{E}\left[U(C_T) + \int_0^T \Big(u(P_s D_s) - h(A_s)\Big) ds\right] \geq R, \tag{2.3}$$

where $R$ is agent's utility of his outside option. After accepting the contract, the agent does not deviate from recommended action $A$ if and only if it solves the optimization problem in (2.2). The optimal contract is the solution to the constrained optimization problem,

$$\min_{(C_T, A) \in \mathcal{C}} \mathbb{E}\left[C_T + \int_0^T P_s D_s ds\right] \text{ subject to (2.3).} \tag{2.4}$$

We make the following simplifying assumptions to avoid some technical difficulties in the next two sections.

**Assumptions:**

(**A1**) The principal is risk neutral.

(**A2**) The agent's utility and cost functions $(U, u, h)$ are twice continuously differentiable such that $U', u', h' > 0$ and $U'', u'', h'' < 0$.

Assumption (**A1**) states that the principal seeks to minimize the total payment made to agent. The principal might have access to the credit market, or he might have a well-diversified portfolio of assets that shield him from shocks. Assumption (**A2**) on $(U, u, h)$ is used to make sure that the agent's optimal response is well-defined and always unique. It allows us to write the agent's incentive compatibility constraint in terms of a first-order condition. This approach is commonly used in both static and dynamic model in contract theory.

## 3. Optimal contract without moral hazard

In this section, we assume that the electricity buyer observe the action $A$ of the generators. The principal takes away the market power of the agent by enforcing any desirable $A$ by imposing a heavy penalty in case of a deviation. The optimal contract under symmetric information is referred to as the first-best contract. The contract is obtained by solving

$$\min_{(C_T, A) \in (R, \mathcal{A})} \mathbb{E}\left[C_T + \int_0^T P_s D_s ds\right] \text{ subject to (2.3).}$$

The participation contraint (2.3) holds at equality. If not, let $(C_T, A)$ be optimal and pick $C_T^\epsilon = C_T - \epsilon$ so that $\mathbb{E}\left[U(C_T^\epsilon) + \int_0^T (u(P_s D_s) - h(A_s)) ds\right] > R$, whose existence is well-defined because $U(\cdot)$ is continuous. The principal would prefer $C_T^\epsilon$ to $C_T$. Incorporating the constraint into the principal's objective by way of Langrange multiplier, we get the relaxed problem

$$\min_{A,C_T} E\left[C_T - \lambda U(C_T) + \int_0^T \Big(P_s D_s - \lambda u(P_s D_s) + \lambda h(A_s)\Big) ds + \lambda R\right].$$

The first order condition with respect to $C_T$ is

$$\frac{1}{U'(C_T)} = \lambda. \tag{3.1}$$

Let

$$F(t,p) \triangleq \min_{A\in\mathcal{A}} \mathbb{E}\left[\int_t^T \Big(P_s^{t,p} D_s - \lambda u(P_s^{t,p} D_s) + \lambda h(A_s)\Big) ds\right] \tag{3.2}$$

where $P_s^{t,p} = p + \int_t^s \rho\big(A_v - P_v^{t,p}\big)ds + \int_t^s \sigma dM_v + \sum_{t<v\le s} \Delta J_v$. We assume that $F(\cdot)$ is continuous differentiable in $t$, and twice continuously differentiable in $p$. Following the standard dynamic programming argument,[3] we get a Hamiltonian-Jacobi-Bellman (HJB) equation,

$$\begin{aligned} -\partial_t F(t,p) &= \min_a pD_t - \lambda u(pD_t) + \lambda h(a) + \rho(a-p)\partial_p F(t,p) + \frac{1}{2}\sigma^2 \partial_{pp} F(t,p) \\ &\quad + \int_{-L}^{L} \Big(F(t, p+dp') - F(t,p)\Big)\frac{\lambda}{2L} dp', \end{aligned} \tag{3.3}$$

and the terminal condition $F(T,p) = 0$, for all $p$. The first order condition with respect to $a$ is,

$$\lambda h'(a) + \rho \partial_p F(t,p) = 0. \tag{3.4}$$

Let $C^{*,\lambda}$ and $a^{*,\lambda}(t,p)$ be the solution to both of the first order conditions, then the Lagrange multiplier $\lambda$ is found by

$$R - U(C^{*,\lambda}) = \mathbb{E}\left[\int_0^T \Big(u(P_s^{*,\lambda} D_s) - h(a^{*,\lambda}(t, P_s^{*,\lambda}))\Big) ds\right].$$

The price $P_t^{*,\lambda} = P_0 + \int_0^t \rho\big(a^*(t, P_s^{*,\lambda}) - P_s^{*,\lambda}\big)ds + \int_0^t \sigma dM_s + \sum_{0<s\le t} \Delta J_s$, which has a unique solution if $a^{*,\lambda}(\cdot)$ satisfies the Lipschitz and linear growth condition in the second variable.

We summarize the first-best contract in the following proposition.

**Proposition 3.1.** *Let $F$ be the solution to (3.3) and $\Big(C_T^{*,\lambda}, a^{*,\lambda}(t,p)\Big)$ be the solution to the first order conditions (3.1) and (3.4). The price process evolves according to the dynamics,*

$$P_t = P_0 + \int_0^t \rho\Big(a^*(s, P_{s-}) - P_s\Big) ds + \int_0^t \sigma dM_s + \sum_{0<s\le t} \Delta J_s.$$

*The optimal contract is $(C_T, A) = \big(C_T, \{a^*(t, P_{t-}, W_t) : 0 \le t \le T\}\big)$.*

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[3] See Chapter 3 in Pham (2009) and the proof of the generalized setup in the next section, Theorem 4.5.

*Remark* 3.2. The key feature of the first-best contract is that the lump-sum payment is a constant for any price realization. The payment is solely determined by the agent's participation constraint. In particular, the inverse of the marginal utility at the optimal is precisely the shadow price of the constraint. This is the Borch-rule for risk sharing, which states that the ratio of marginal utilities between the Principal and the agent is contant.

*Remark* 3.3. Since the principal can force the agent to perform any action, the incentive compatibility constraint does not play a role in the calculation of payment. The desirable action maximizes

$$\lambda h(a) + \rho(a-p)\partial_p F(t,p).$$

The optimal action is not deterministic but depends on $P_t$ at each moment in time. The principal does not only have to pay the agent $P_tD_t$, but also compensates for the cost of action $h(A_t)$ by way of the participation constraint. The two payments have opposing effects in that when the agent is not getting paid enough through direct payment, his participation constraint becomes more severe. If the utilities are linear, then the principal would simply require the lowest action, $a^* = \min\{\mathbf{A}\}$ (see Section 5.1). When the agent has risk aversion, the principal has to balance the instantaneous effect $\lambda h(a)$ and the long term effect $\rho a \partial_p F(t,p)$. In Lemma A.1, we show that $\frac{\partial a^{*,\lambda}}{\partial p} > 0$. The principal allows $a^*$ to deviate from $\min\{\mathbf{A}\}$, and at a higher price, the principal allows a slightly higher $a^*$ to balance the short term and long term effect. We will not explore the comparative statics of $a^*$ with respect to $t$ because the result depends on the dynamics of demand process $D$, which we do not make any restriction.

*Remark* 3.4. $F(t,p)$ is strictly increasing in $p$ and strictly convex in $p$ (see Lemma A.2). This is primarily because the agent's utility function is $u$ is strictly concave. The buyer has to pay more when the price is higher. The convexity is a reflection of the income effect in the agent's utility $u$. At a higher $p$, it is more expensive to compensate the agent because $u$ is concave.

## 4. Optimal contract

We solve the model proposed in Section 2 to derive the optimal contract. The contract specifies that the agent's terminal payment $C_T$ is calculated as $U(C_T) = W_T$, where $t \to W_t$ is the agent's continuation value. The dynamics of $W$ depends on an incentive device, encoded in $\xi(\cdot)$ and Proposition 4.2, that adjusts the payment up and down based on the realized market price $P$.

**Agent's problem**

Let the probability space be $(\Omega, \mathbf{F}, \mathbb{P})$. The price $P$ is assumed to be the solution of the SDE

$$P_t = P_0 + \int_0^t \rho P_s ds + \int_0^t \sigma dB_s + \sum_{0<s\leq t} \Delta J_s,$$

where $B$ is a standard Brownian motion (see Remark A.3 and A.4). The choice of price manipulation $A$ shifts the distribution of $P$ from $\mathbb{P}$ to $\mathbb{P}^A$ by the Girsanov transformation, where the probability measure $\mathbb{P}^A$ is

$$d\mathbb{P}^A \triangleq e^{\int_0^T \theta_s dB_s - \frac{1}{2}\int_0^T \theta_s^2 ds} d\mathbb{P} \tag{4.1}$$

with $\theta_t \triangleq \sigma^{-1}\rho A_t$. The shifted process

$$B_t^A \triangleq B_t - \int_0^t \sigma^{-1}\rho A_s ds = B_t - \int_0^t \theta_s ds \tag{4.2}$$

is a Brownian motion under the probability measure $\mathbb{P}^A$ if the stochastic exponential $e^{\int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t \theta_s^2 ds}$ is a martingale. This condition can be verified by the Novikov condition when $A_s$ takes value from a compact interval in $\mathbb{R}$. Furthermore, the distribution of the $J$ remains unchanged under $\mathbb{P}^A$. The probability shift from $\mathbb{P}$ to $\mathbb{P}^A$ only affects the drift of the continuous part of price process $P$.

**Lemma 4.1.** *Under the probability measure $\mathbb{P}^A$, $B_t^A$ and $J_t$ are independent, $B_t^A$ is a Gaussian with mean $0$ and variance $t$, and $J_t$ has jump intensity $\lambda$ and its jump sizes is uniformly distributed over $[-L, L]$.*

Under the weak formulation, $P$ is considered fixed regardless of the level of price manipulation $A$ performed by the agent, but the choice of $A$ controls the distribution of $P$. We note that $P$ satisfies the following SDE,

$$P_t = P_0 + \int_0^t \rho(A_s - P_s)ds + \int_0^t \sigma dB_s^A + \sum_{0<s\leq t} \Delta J_s.$$

Let $W_t^A \triangleq \mathbb{E}_t^A\left[U(C_T) + \int_t^T \Big(u(P_sD_s) - h(A_s)\Big)ds\right]$ to denote the agent's remaining utility, where the conditional expectation $\mathbb{E}_t^A[\,\cdot\,]$ under the weak formulation is taken with respect to $(\mathbb{P}^A, \mathcal{F}^{B^A,J})$, i.e. $\mathbb{E}_t^A[\,\cdot\,] = \mathbb{E}^A[\,\cdot\,|\mathcal{F}_t^{B^A,J}]$. The full discussion of the different filtrations is given in the Appendix (see Remark A.4 and A.7). We also say $W^A$ is the agent's continuation value. Given a contract $(C_T, A)$, $W_t^A$ can be written as

$$W_t^A = U(C_T) + \int_t^T \Big(u(P_sD_s) - h(A_s)\Big)ds - \int_t^T Z_s^c \sigma dB_s^A - \sum_{t<s\leq T} Z_s^d \Delta J_s, \tag{4.3}$$

for some predictable processes $(Z^c, Z^d)$. Switching the Brownian motion from $B^A$ to $B$, the representation of the agent's remaining utility is

$$\begin{aligned} W_t^A &= U(C_T) + \int_t^T \Big(u(P_sD_s) - h(A_s) + \rho Z_s A_s\Big)ds - \int_t^T Z_s^c \sigma dB_s - \sum_{t<s\leq T} Z_s^d \Delta J_s \\ &= W_0 - \int_0^t \Big(u(P_sD_s) - h(A_s) + \rho Z_s A_s\Big)ds + \int_0^t Z_s^c \sigma dB_s + \sum_{0<s\leq t} Z_s^d \Delta J_s. \end{aligned}$$

The agent controls $A$ to maximize his expected initial utility.

**Proposition 4.2.** *The following two conditions are equivalent.*

*1. $A$ is an optimal response to the payment $C_T$.*

*2. $(C_T, A, Z^c, Z^d)$ satisfies (4.3), and $A_t(\omega) \in \arg\max_a\{\rho Z^c_t(\omega)a - h(a)\}$, for almost every $(t,\omega) \in [0,T] \times \Omega$.*

The proposition describes the set of all incentive compatible contracts $\mathcal{C}$. For any contract $(C_T, A) \in \mathcal{C}$ with the associated $(Z^c, Z^d)$ according to (4.3), the quadruple $(C_T, A, Z^c, Z^d)$ must satisfy condition 2 of Proposition 4.2. Let

$$\xi(a) \triangleq \{z : a \in \arg\max_{a'} \ \rho z a' - h(a')\} \text{ and } \eta(z) \triangleq \arg\max_{a'} \ \rho z a' - h(a').$$

By Assumption (**A2**), $\xi(\cdot)$ and $\eta(\cdot)$ are singletons. If the principal offers $C_T$, the agent finds the solution $(W, Z^c, Z^d)$ to the BSDE driven by $(B, J)$, where $(Z^c, Z^d)$ are predictable, square-integrable processes,

$$\begin{cases} P_t = P_0 + \int_0^t \sigma B_s \\ W_t = U(C_T) + \int_t^T \Big(u(P_s D_s) - h(\eta(Z^c_s)) + \rho Z^c_s \eta(Z^c_s)\Big) ds - \int_t^T Z^c_s \sigma dB_s - \sum_{t<s\le T} Z^d_s \Delta J_s. \end{cases}$$

The BSDE has a unique solution if $h(\eta(\cdot))$ and $u(\cdot)$ are Lipschitz continuous (see Nualart and Schoutens (2001)). Let the agent's response be $A^*_s = \eta(Z_s)$. If $W_t$ is in fact the agent's remaining utility under the contract $(C_T, A^*)$, then $(C_T, A^*)$ is an optimal pair for the agent because $(C_T, A^*, Z, 0)$ satisfies condition 2 in Proposition 4.2.

**Lemma 4.3.** *Under the contract $(C_T, A^*)$, the expected remaining utility is*

$$W_t = U(C_T) + \int_t^T \Big(u(P_s D_s) - h(\eta(Z_s)) + \rho Z^c_s \eta(Z^c_s)\Big) ds - \int_t^T Z^c_s \sigma dB_s - \sum_{t<s\le T} Z^d_s \Delta J_s.$$

On the other hand, if the principal wants to enforce $A$ and guarantees that the agent receives an initial expected utility $W_0$, the principal pays the agent $C^*_T = I(W_T)$, where $I(\cdot)$ is the inverse function of $U(\cdot)$ and

$$W_T = W_0 - \int_0^T \Big(u(P_s D_s) - h(A_s) + \rho\xi(A_s)A_s\Big) ds + \int_0^T \xi(A_s)\sigma dB_s.$$

**Lemma 4.4.** *$(C^*_T, A)$ is incentive compatible.*

The volatility with respect to the jumps does not play a role because the agent cannot control neither the intensity nor the jump sizes, $Z^d$ does not affect incentives. We choose $Z^d = 0$, but any predictable $Z^d$ would have worked and hence, the payment $C^*_T$ can be designed in many different ways. In Lemma A.9, we show that under a mild assumption that the principal does not use $Z^d$ to provide insurance to offset the volatility of the agent's income, principal prefers the incentive compatible contract with

$Z^d = 0$ to any other incentive compatible contract.

**Principal's problem and the optimal contract**

We take the perspective that the principal controls $(A, W_0)$ and Lemma 4.4 determines the value of the enforcing adjustment fee $C_T$. The principal considers the problem

$$\min_{A, W_0 \geq R} \mathbb{E}\left[C_T^* + \int_0^T P_s D_s ds\right] = \min_{A, W_0 \geq R} \mathbb{E}\left[I(W_T) + \int_0^T P_s D_s ds\right],$$

with $W_t = W_0 - \int_0^t \big(u(P_s D_s) - h(A_s)\big) ds + \int_0^t \xi(A_s)\sigma dM_s$. We ignore the control $W_0$ for now, but we will later show that any incentive compatible contract with $W_0 > R$ is not optimal for the principal. We define the principal's value function

$$F(t,p,w) \triangleq \min_{A \in \mathcal{A}} \mathbb{E}\left[I(W_T^{t,p,w}) + \int_t^T P_s^{t,p} D_s ds\right], \tag{4.4}$$

where

$$\begin{aligned} P_s^{t,p} &\triangleq p + \int_t^s \rho(A_v - P_v) dv + \int_t^s \sigma dM_v + \sum_{t < v \leq s} \Delta J_v \quad \text{and} \\ W_s^{t,p,w} &\triangleq w - \int_t^s \Big[u(P_v^{t,p} D_v) - h(A_v)\Big] dv + \int_t^s \xi(A_v)\sigma dM_v. \end{aligned}$$

Let us assume that $F(\cdot)$ is continuously differentiable in $t$, and twice continuously differentiable in $p$ and $w$. The process $P_s^{t,p}$ has finitely many jumps, and the process $s \to W_s^{t,p,w}$ is continuous. The following equation is analogous to the HJB-PDE for a stochastic control problem driven by continuous Ito diffusion processes,[4]

$$\begin{aligned} -\partial_t F &= \min_a \rho(a-p)\partial_p F + \frac{1}{2}\sigma^2 \partial_{pp} F + (h(a) - u(pD_t))\partial_w F + \frac{1}{2}\xi(a)^2\sigma^2 \partial_{ww} F \\ &\quad + \xi(a)\sigma^2 \partial_{pw} F + \int_{-L}^{L} \Big(F(t, p+dp', w) - F(t,p,w)\Big)\frac{\lambda}{2L} dp' + pD_t, \end{aligned} \tag{4.5}$$

where $F$ denotes $F(t,p,w)$. The terminal condition is $F(T,p,w) = I(w)$, for all $p$ and $w$. We denote $a^*(t,p,w)$ as the minimizer of

$$\rho a \partial_p F + h(a)\partial_w F + \frac{1}{2}\xi(a)^2 \sigma^2 \partial_{ww} F + \xi(a)\sigma^2 \partial_{pw} F. \tag{4.6}$$

The search for the optimal $A$ is reduced to a family of minimization problems that are parametrized by $(t,p,w)$, and the optimal $A$ is a Markov policy that depends on two state variables $(p,w)$. We characterize the optimal contract in the following theorem.

[4] See Theorem 3.1 in Oksendal and Sulem (2007).

**Theorem 4.5.** *Let $a^*(t,p,w)$ be the minimizer in (4.6), and the agent is paid $C_T = I(W_T)$, where*

$$\begin{cases} P_t = P_0 + \int_0^t \rho\Big(a^*(s,P_{s-},W_s) - P_s\Big)ds + \int_0^t \sigma dM_s + \sum_{0<s\le t}\Delta J_s \\ W_t = R + \int_0^t \Big(h(a^*(s,P_{s-},W_s)) - u(P_sD_s)\Big)ds + \int_0^t \xi(a^*(s,P_{s-},W_s))\sigma dM_s. \end{cases} \tag{4.7}$$

*Then, the contract $(C_T, A) = \big(C_T, \{a^*(t,P_{t-},W_t) : 0 \le t \le T\}\big)$ is incentive compatible for the agent, and optimal for the principal among all incentive compatible contracts that deliver an initial expected value of at least $W_0$ to the agent.*

*Remark* 4.6. The recommended action $t \to A_t$ is left-continuous and in particular, it is predictable w.r.t. $\mathcal{F}^P$ (see Remark A.7). In other words, $A$ can be determined from observable information alone. Further, the calculation of $A_t$ only depends the current price and agent's continuation value. At time $t$, the principal doe snot need to consider the entire history of the price up to time $t$, $\{P_s, : 0 \le s \le t\}$. The agent's continuation value summarizes all the relevant incentive information from the past, and the pair $(P_t, W_t)$ has the Markov property.

*Remark* 4.7. The implementation of the optimal contract requires the contracting parties to keep track of two state variables: the price $P$ and the agent's continuation value $W$. The first-best contract only keeps track of the price, and when the contractual setting does not make a restriction on direct payment (see Section 5.1), the contract only keeps track of the agent's continuation value. The information in $W_t$ describes agent's incentive, and the information in $P_t$ is necessary for the principal to find the minimizing combination of paying the agent through the direct payment $P_tD_t$ and the adjustment fee $C_T$ (also see Remark 3.3).

*Remark* 4.8. The signs of the partials of the principal's value function tell us how the state variables affect the principal's utilities. Lemma A.14 shows that the agent's remaining utility has a straightforward effect to the principal. $\partial_w F > 0$ states that the principal would prefer to promise the agent as little as possible. $\partial_{ww} F > 0$ states that there is an income effect that makes compensation to the agent more and more costly to the principal as the promised payment gets large. The income effect comes from the concavity Assumption (**A2**). The direction of $\partial_p F$ is not clear-cut. On the one hand, an increase in $p$ increases the direct payment, but on the other, a higher payment now reduces the promised payment later. The two effects work in opposing directions, and because $I$ and $u$ is not linear, the sign of $\partial_p F(t,p,w)$ does not always point in one direction but depending on $(t,p,w)$.

*Remark* 4.9. The optimal effort minimizes the total cost of an action $a$,

$$\rho a \partial_p F + h(a)\partial_w F + \frac{1}{2}\xi(a)^2\sigma^2\partial_{ww}F + \xi(a)\sigma^2\partial_{pw}F.$$

An action does not incur a direct cost to the principal, but somehow, the principal must compensate the agent for the cost of action $h(a)$. The principal bears effect of $a$ through one of the two state variables. The first and the second term measure the direct impact through the price and the promised payment. The last two terms represent the additional cost associated with providing incentives. Unlike the result in the first-best contract (see Lemma A.1), the direction of both $\frac{\partial a^*}{\partial p}|_{t,p,w}$ and $\frac{\partial a^*}{\partial w}|_{t,p,w}$ are indeterminate.

*Remark* 4.10. When $\partial_p F > 0$ and $\partial_{pw} F > 0$, then (4.6) is minimized at $a^* = \min \mathbf{A}$. But in general, it could be optimal for the principal to allow te agent to deviate from $\min \mathbf{A}$. The rationale is similar to the first-best case (see Remark 3.3). At each moment in time, the principal balances the cost of paying the agent now and paying the later.

# 5. Special cases

## 5.1 Linear utilities

In this subsection, let the agent's utilities $U$ and $u$ be linear. The assumption of linear utilities could be justified if both the contractual parties are large institutions and have access to credit markets. The first-best contract $(C_T^*, A^*)$ is obtained by

$$\min_{(C_T, A)\in(\mathbb{R},\mathcal{A})} \mathbb{E}\left[C_T + \int_0^T P_s D_s ds\right] \text{ subject to } \mathbb{E}\left[C_T + \int_0^T \Big(P_s D_s - h(A_s)\Big) ds\right] \geq R.$$

The participation contraint holds at equality, and by substitution, the problem is reduced to

$$\min_A \mathbb{E}\left[R + \int_0^T h(A_s) ds\right].$$

Let $a^* = \arg\max_{a\in\mathbf{A}} h(a)$, or equivalently, $a^* = min\{\mathbf{A}\}$. The optimal action is $A_t^* = a^*$, for all $t$. Let $C_T^*$ the solution to $\mathbb{E}\left[C_T + \int_0^T \Big(P_s D_s - h(a^*)\Big) ds\right] = R$, where $P_t = P_0 + \int_0^s \rho(a^* - P_s) ds + \int_0^t \sigma dM_s + \sum_{0<s\leq t} \Delta J_s$.

The second best contract is obtained by

$$\min_{A, W_0 \geq R} \mathbb{E}\left[C_T^* + \int_0^T P_s D_s ds\right]$$

with $C_T^* = W_0 - \int_0^t \Big(P_s D_s - h(A_s)\Big) ds + \int_0^t \xi(A_s) \sigma dM_s$. By the same argument as in Lemma A.12, $W_0 = R$, and through substitutions, the problem is reduced to

$$\min_A \mathbb{E}\left[R + \int_0^T h(A_s) ds\right] + \mathbb{E}\left[\int_0^T \xi(A_s) \sigma dM_s\right].$$

The second expectation is zero, and it follows that the second-best contract is the same as the first-best contract when both the principal and the agent's utilities are linear. When the agent's utilities are linear, then the principal can enforce an action sequence in a straightforward way so that the provision of incentives has a zero expected cost to the principal.

### 5.2 Lump-sum payment

In this subsection, we assume that the buyer is not required to pay the seller the amount $\int_0^T P_t D_t dt$, and all settlement can be made in one lump-sum payment. The agent's problem is solved similarly, as in Proposition 4.2 and Lemma A.9. The optimal contract is obtained by

$$\min_{A, W_0 \geq R} \mathbb{E}\left[I(W_T)\right],$$

where $W_t = W_0 + \int_0^t h(A_s)ds + \int_0^t \xi(A_s)\sigma dM_s$. By the same argument as in Lemma A.12, $W_0 = R$. Letting $F(t,w) \triangleq \min_A \mathbb{E}_t\left[I(W_T^{t,w})\right]$, where

$$W_T^{t,w} = w + \int_t^T h(A_s)ds + \int_t^T \xi(A_s)\sigma dM_s.$$

The HJB equation is

$$\begin{cases} -\partial_t F(t,w) = \min_a \; h(a)\partial_w F(t,w) + \frac{1}{2}\xi(a)^2\sigma^2\partial_{ww}F(t,w), \\ F(T,w) = w, \text{ for all } w. \end{cases} \tag{5.1}$$

Let $a^*(t,w)$ be the minimizer in the right hand side of (5.1), and

$$W_t^* = R + \int_0^t h(a*(s, W_s^*)ds + \int_0^t \xi(a^*(s, W_s^*))\sigma dM_s.$$

The contract is $(I(W_T^*), A^* = \{A_t^* = a^*(t, W_t^*) : 0 \leq t \leq T\})$.

### 5.3 No jumps

We assume that the price does not exhibit jumps. The agent's remaining utility in (4.3) is written as

$$W_t^A = U(C_T) + \int_t^T \Big(u(P_s D_s) - h(A_s)\Big)ds - \int_t^T Z_s \sigma dM_s,$$

for some predictable processes $Z$. To enforce any action $A$, we set $C_T^* = I(W_T)$, where

$$W_T = W_0 - \int_0^T \Big(u(P_s D_s) - h(A_s)\Big)ds + \int_0^T \xi(A_s)\sigma dM_s.$$

By Lemma 4.4, $(C_T^*, A)$ is incentive compatible.

Let principal's value function be

$$F(t,p,w) \triangleq \min_{A} \mathbb{E}\left[J(W_T^{t,p,w}) + \int_t^T P_s^{t,p} D_s ds\right]. \tag{5.2}$$

The following HJB-PDE is a simplification of (4.5),

$$\begin{cases} -\partial_t F = \min_a \; pD_t + \rho(a-p)\partial_p F + \frac{1}{2}\sigma^2\partial_{pp}F + (h(a) - u(pD_t))\partial_w F + \frac{1}{2}\xi(a)^2\sigma^2\partial_{ww}F + \xi(a)\sigma^2\partial_{pw}F \\ F(T,p,w) = J(w) \quad \text{for all } p \text{ and } w. \end{cases} \tag{5.3}$$

Let $a^*(t,p,w)$ be the optimizer in (5.3). The state processes are

$$\begin{cases} P_t = P_0 + \int_0^t \rho\big(a^*(s,P_s,W_s) - P_t\big)ds + \int_0^t \sigma dM_s \\ W_t = R + \int_0^t \Big[h(a^*(s,P_s,W_s)) - u(P_s D_s)\Big]ds + \int_0^t \xi(a^*(s,P_s,W_s))\sigma dM_s. \end{cases} \tag{5.4}$$

The terminal payment is $C_T^* = J(W_T)$ and is implemented by the action $\{a^*(t,P_t,W_t) : 0 \le t \le T\}$.

## 6. Discussion

The proposed contract in Theorem 4.5 is optimal for a procurement problem that is described in Section 2. The abstract setting approximates an electricity market, but still, if we want to turn the analyses here into an implementable contract between a LSE and a generating company, there are two major challenges that must be overcome. First, the contract appears to be difficult to compute. The contract has to solve for $F(\cdot)$ in equation (4.5). This could be solved by numerical methods. Once this second order differential equation is computed, $a^*(\cdot)$ is known. Hence, the hard computation can be done before the contracting period, and during the implementation stage, $P_t$ is observed and knowing $W_t$ is a matter of accounting. The second shortcoming stems from the limited knowledge that contractual parties have on the utility function $(U,u,h)$. These utilities could the agent's private information, or worse, it could simplify be the case that this information cannot be obtained. While there are econometric tools available to estimate an economic agent's utility function, it is unlikely that in an electricity market, the contractual parties will share enough information to agree on the total structure of the contracting environment. Even if we justify that both the principal and the agent have linear utilities, it would be difficult to nail down the form of the cost function $h$. Still, there are qualitative lessons to be learned from the general features of the proposed contract.

One of the key features is that the contract keeps track of the continuation value $W_t$ as a state variable and the agent is paid an amount proportional to $W_T$. The evolution of $W$ is sensitive to market conditions that are partially influenced by the agent, and that is the reward-punishment scheme used to discourage agents from manipulating $P$. This lesson could be applied to any contractual design in the presence of market power. The payment scheme should be at least be such that a supplier gets an increasingly less subsidy payment if he already receives a large direct payment due to excessively high prices.

The optimal contract allows $A_t$ to deviate from $\min\{\mathbf{A}\}$, which is the action that would keep the price at its lowest level. It suggests that a contract design should not unnecessarily require a supplier to behave absolutely at all time. Instead of trying to limit manipulations of any form, it could be beneficial for a procuring party to give a little leeway to a supplier who possesses market power. The contract should have a feedback mechanism that dampens high prices, but if this goal is pursued without limit, then the buyer could be paying too much to eliminate a superficial phenomenon, which in effect, employing a suboptimal solution.

Because of the complexity of the contract, we might want to look for ways to simplify how payments are made to the agent. The model imposes a two-tier payment. One way is to do away with the adjustment fee $C_T$ and only allow the direct payment $\int_0^T P_s D_s ds$. But then the principal restricts her ability to provide incentives. The agent will manipulate the market to achieve the highest price possible. The other way is to offer the lump-sum fee $C_T$ only. Section 5.2 shows that the resulting contract is easier to calculate and implement. The contract has only one state variable, the agent's continuation value. This simplification might not be practical for two reasons. First, its payment is more sensitive to the form of one utility function $U$. Second, it is not as intuitive appealing as the contract that would pay the agent an amount that is equivalent to selling the electricity in the open market ($\int_0^T P_s D_s ds$).

It might be politically infeasible to justify the adjustment fee $C_T$ in addition to the direct payment. After all, the amount in $\int_0^T P_s D_s ds$ would already seem like a fair compensation for a supplier. We have discussed that the adjustment fee acts as a reward and penalty mechanism, and it can also be viewed as a form of subsidies. Many current regulatory markets already provide a variety of subsidies to electricity generators in an ad-hoc manner. These subsidies include a variety of incentives to generating companies for adding renewables sources, switching across fossil fuels, and meeting peak-hour energy demand (Kitson, Wooders, and Moerenhout, 2011). The subsidies are in the form of direct cash payments, tax credits, lower grid connection fees, priority access to infrastructures, etc. From this perspective, the adjustment payments $C_T$ can be seen as restructuring of the current subsidies in a way that accounts for strategic behaviors.

## 7. Conclusion

We identify that the electricity procurement contracts between LSEs and suppliers overlook two issues: incentive and dynamics. We address these problems by solving a dynamic principal-agent problem with moral hazard. The existing literature on electricity market has many papers that use static equilibrium and bid auction models. Not taking incentive constraints into consideration does not only lead to the potential problems of price manipulation and insufficient capacity, but also suppresses market responsiveness. The electricity market is unique in that supply and demand response themselves could be sold as resources (for example, in the ancillary services market), which could be used as control tools to reduce the need for a high reserve capacity and increase grid stability. Unlike canonical principal-agent problems where invariably, moral hazard is a costly constraint to the principal, the provision of incentives in this market could be beneficial to both the principal and the agent. This possibility is not explored here because our model does not describe how the power grid is controlled.

The view that the market is static is often justified by the argument that a lack of cheap storage technology (resp. inelastic demand) makes it impossible to substitue generation (resp. consumption) from one period with that from another. Each settlement interval (e.g. 5 minutes) are treated as an independent one-period model. This view is deeply problematic. First, market participants condition their actions on past history. For example, end-users exhibiting demand-response behavior are likely to respond to prices from previous time, and price dynamics depend crucially on how they respond to lagged prices (for example, see Bossaerts and Ledyard, 2013). Second, information accumulates over time. For example, an agent might only be willing to reveal information gradually, knowing that his counter-party would exploit the information indefinitely (see *ratchet effect* in Salanié, 1997). The design of the overall market has many dynamically connected components, e.g. frequency and voltage control, ramping constraints, demand response, supply stochasticities, etc. The electricity market is inherently dynamic and adjusted in a fast timescale. A continuous-time approach has obvious modeling advantages to answer the many open questions in the field.

# A. Appendix

In the appendix, we discuss some technical details and provide missing proofs.

**Lemma A.1.** *Let $a^*$ be the minimizer of $\lambda h(a) + \rho(a-p)\partial_p F(t,p)$. Then, $\frac{\partial a^{*,\lambda}}{\partial p} > 0$.*

*Proof.* By Assumption (**A2**) and the differentiability assumption on $F(t,p)$, the first order condition is sufficient to define $a^*$. Using the envelope theorem,

$$\lambda h''(a^*)\frac{\partial a^*}{\partial p} + \rho\partial_{pp}F(t,p) = 0.$$

By Lemma A.2, $\partial_{pp}F(t,p) > 0$. $h'' < 0$ by Assumption (**A2**). Then, $\frac{\partial a^{*,\lambda}}{\partial p} > 0$. □

**Lemma A.2.** *The principal's value function in (3.2) is strictly convex.*

*Proof.* Let $\alpha \in (0,1)$, and $p^\alpha \triangleq \alpha p + (1-\alpha)p'$. We let $A$ (resp. $A'$) be the minimizing control for the starting point $(t,p)$ (resp. $(t,p')$). Then,

$$\begin{aligned}
F(t,p^\alpha) &= \min_{\tilde{A}} \mathbb{E}\left[\int_t^T \Big(P_s^{t,p^\alpha,\tilde{A}}D_s - \lambda u(P_s^{t,p^\alpha,\tilde{A}}D_s) + \lambda h(\tilde{A}_s)\Big)ds\right] \\
&\leq \mathbb{E}\left[\int_t^T \Big(P_s^{t,p^\alpha,\bar{A}}D_s - \lambda u(P_s^{t,p^\alpha,\bar{A}}D_s) + \lambda h(\bar{A}_s)\Big)ds\right], \text{ for some } \bar{A} \\
&= \mathbb{E}\left[\int_t^T \Big((\alpha P_s^{t,p,A} + (1-\alpha)P_s^{t,p,A'})D_s - \lambda u((\alpha P_s^{t,p,A} + (1+\alpha)P_s^{t,p,A'})D_s) + \lambda h(\bar{A}_s)\Big)ds\right] \\
&< \mathbb{E}\left[\int_t^T \Big((\alpha P_s^{t,p,A} + (1-\alpha)P_s^{t,p',A'})D_s - \lambda(\alpha u(P_s^{t,p,A}D_s) + (1-\alpha)u(P_s^{t,p',A'}D_s)) + \lambda h(\bar{A}_s)\Big)ds\right] \\
&= \alpha F(t,p) + (1-\alpha)F(t,p)
\end{aligned}$$

The strict inequality is by the strict concavity of $u$. $P_s^{t,p,\tilde{A}}$ denotes the price process starts at $(t,p)$ and is controlled by $\tilde{A}$, and $\bar{A}$ is chosen so that $h(\bar{A}_t) = \alpha h(A_t) + (1-\alpha)h(A'_t)$. □

*Remark* A.3. We make note of a key technical detail: the agent's problem is solved by a weak formulation and the principal's problem is solved by a strong formulation. In general, the two formulations are not the same; for example, see discussions in Cvitanić, Wan, and Zhang (2009). In Lemma A.6, we show that the two formulations are consistent with each other.

*Remark* A.4. We denote $\mathcal{F} \triangleq \mathcal{F}^P$ the augmented filtration generated by $P$, and since $\sigma$ is a constant, the filtration generated by $(B,J)$ are equivalent to that generated by $P$, i.e. $\mathcal{F}^{B,J} = \mathcal{F}^P$. To see it, we note that for each $\omega$, we can exactly derive the value of $B(\omega)$ and $I(\omega)$ from $P(\omega)$, and vice versa. We denote $\mathcal{F}^{B^A,J}$ as the augmented filtration generated by $(B^A,J)$. $A$ is assumed to be $\mathcal{F}^P$-adapted, and so $\mathcal{F}^P \subseteq \mathcal{F}^{B^A,J}$.

*Proof of Lemma 4.1.* We follow the argument of Theorem 11.6.9 in Shreve (2004). We investigate the joint moment generating function of $B_t^A$ and $J_t$; in particular, we want to show that

$$\mathbb{E}^A\left[e^{u_1 B_t^A + u_2 J_t}\right] = exp\Big(\frac{1}{2}tu_1^2\Big)exp\Big(\lambda t(\mathbb{E}[e^{L_1 u_2}] - 1)\Big). \tag{A.1}$$

Notice that the first factor of the RHS is the moment generating function of a normal random variable of $(0,t)$ and the second factor is of a compound Poisson with intensity $\lambda$ and jump side $L_1$, and the separating of the moment generating function implies that the two random variables are independent. Denote

$$\begin{aligned}
\mathcal{E}_t &\triangleq exp\Big(\int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t \theta_s^2 ds\Big), \\
G_t &\triangleq exp\Big(\int_0^t \theta_s dB_s^A - \frac{1}{2}tu_1^2\Big), \\
H_t &\triangleq exp\Big(u_2 J_t - \lambda t(e^{\frac{1}{2}\nu^2 u_2^2} - 1)\Big).
\end{aligned}$$

$\mathcal{E}_tG_t$ is a $\mathcal{P}$-martingale because the differential of $\mathcal{E}_tG_t$

$$d[\mathcal{E}_tG_t] = \mathcal{E}_tG_t(u_1 + \theta_t)dB_t,$$

has no drift term. To see that $H_t$ is a martingale, we look at

$$\begin{aligned}
\mathbb{E}\left[H_t|\mathcal{F}_s\right] &= \mathbb{E}\left[exp\left\{(J_t - J_s + J_s)u_2 - \lambda t(\mathbb{E}(e^{L_1u_2}) - 1)\right\}|\mathcal{F}_s\right] \\
&= H_s\mathbb{E}\left[exp\left\{(J_t - J_s)u_2 - \lambda(t-s)(\mathbb{E}(e^{L_1u_2}) - 1)\right\}|\mathcal{F}_s\right] \\
&= H_s\mathbb{E}\left[exp\left\{(J_{t-t})u_2 - \lambda(t-s)(\mathbb{E}(e^{L_1u_2}) - 1)\right\}\right] = H_s.
\end{aligned}$$

We use the property of stationary independent increment of the process $J$ in the second and the third equality. We apply Ito's product rule,

$$\mathcal{E}_tG_tH_t - \mathcal{E}_0G_0H_0 = \int_0^t H_{s-}d[\mathcal{E}_sG_s] + \int_0^t \mathcal{E}_{s-}G_{s-}dH_s + cv(\mathcal{E}_tG_t, H_t), \tag{A.2}$$

where $cv$ denotes cross variation. The cross variation is zero because $H$ does not have any randomness attributed to $B$. The first and the second term of the RHS in (A.2) are $\mathbb{P}$-martingale because the integrands are predictable and the integrators are martingale. It follows that $\mathcal{E}_tG_tH_t$ is a $\mathbb{P}$-martingale, then we have

$$\mathbb{E}[\mathcal{E}_tG_tH_t] = \mathbb{E}[\mathcal{E}_0G_0H_0] = 1. \tag{A.3}$$

Notice that $\mathbb{E}^A\left[e^{u_1B_t^A + u_2J_t}\right] = \mathbb{E}\left[\mathcal{E}_Te^{u_1B_t^A+u_2J_t}\right] = \mathbb{E}\left[\mathcal{E}_te^{u_1B_t^A+u_2J_t}\right]$, and combining with (A.3), we get (A.1). □

**Proposition A.5.** *There exist unique, up to measure zero sets in $[0,T]\times\Omega$, predictable processes $(Z^c, Z^d)$ with respect to the filtration $\mathcal{F}^{B^A,J}$ such that*

$$W_t^A = U(C_T) + \int_t^T \Big(u(P_sD_s) - h(A_s)\Big)ds - \int_t^T Z_s^c\sigma dB_s^A - \sum_{t<s\leq T} Z_s^d\Delta J_s. \tag{A.4}$$

*Proof.* The term $\mathbb{E}_t^A\left[U(C_T) + \int_0^T \Big(u(P_sD_s) - h(A_s)\Big)ds\right]$ is a $\mathbb{P}^A$-martingale. Using the Martingale Representation Theorem for Lévy processes, there are unique predictable processes $Z^c$ w.r.t. $\mathcal{F}^{B^A}$ and $Z^d$ w.r.t. $\mathcal{F}^J$ such that[5]

$$W_t^A + \int_0^t \Big(u(P_sD_s) - h(A_s)\Big)ds = \mathbb{E}_0^A\left[U(C_T) + \int_0^T \Big(u(P_sD_s) - h(A_s)\Big)ds\right] + \int_0^t Z_s^c\sigma dB_s^A + \sum_{0<s\leq t} Z_s^d\Delta J_s.$$

Denoting $W_0 = \mathbb{E}_0^A\left[U(C_T) + \int_0^T \Big(u(P_sD_s) - h(A_s)\Big)ds\right]$ and subtracting from both size $\int_0^t (u(P_sD_s) - h(A_s))\,ds$

[5]See Theorem 5.3.5 in Applebaum (2004) and Theorem T8 in Chapter VIII in Brémaud (1981).

gives us,

$$W_t^A = W_0 - \int_0^t \Big(u(P_sD_s) - h(A_s)\Big)ds + \int_0^t Z_s^c \sigma dB_s^A + \sum_{0<s\leq t} Z_s^d \Delta J_s. \tag{A.5}$$

In particular,

$$W_T^A = U(C_T) + \int_0^T \Big(u(P_sD_s) - h(A_s)\Big)ds = W_0 + \int_0^T Z_s^c \sigma dB_s^A + \sum_{0<s\leq T} Z_s^d \Delta J_s. \tag{A.6}$$

Plugging (A.6) into (A.5), we get the desired result. □

Notice that $(Z^c, Z^d)$ are predicable with respect to $\mathcal{F}^{B^A,J}$, but in fact, we could have shown that we could have shown that $(Z^c, \mathbb{Z}^d)$ to be predictable w.r.t. $\mathcal{F}$ as well.[6] In the specific setting of this paper, the process $A$ is going to be a functional of the observables at each moment in time, and the two filtrations $\mathcal{F}^{B,J}$ and $\mathcal{F}^{B^A,J}$ are the same. We show this in Lemma A.6, and from here on, we no longer make the distinction between the two filtrations in the setting of the weak formulation. Switching the Brownian motion from $B^A$ to $B$, we also have

$$\begin{aligned} W_t^A &= U(C_T) + \int_t^T \Big(u(P_sD_s) - h(A_s) + \rho Z_s A_s\Big)ds - \int_t^T Z_s^c \sigma dB_s - \sum_{t<s\leq T} Z_s^d \Delta J_s \\ &= W_0 - \int_0^t \Big(u(P_sD_s) - h(A_s) + \rho Z_s A_s\Big)ds + \int_0^t Z_s^c \sigma dB_s + \sum_{0<s\leq t} Z_s^d \Delta J_s. \end{aligned}$$

*Proof of Proposition 4.2.* We recall that $W_t^A$ denotes the agent's remaining utility under the contract $(C_T, A)$, hence both the principal and the agent are indifferent between the promise of the $C_T$ at the terminal time and the promise of $I(W_t^A)$ at each moment in time because under both promises, the agent gets a settlement at time $T$, $I(W_T^A) = C_T$. It is important to note that the regardless of the actual action performed by the agent, the final payment is $I(W_T^A) = C_T$. If the agent follows the recommended action, his utility by time $t$ is

$$G_t^A(A) = W_t^A + \int_0^t \Big(u(P_sD_s) - h(A_s)\Big)ds,$$

and in particular, his initial expected utility is

$$\mathbb{E}^A[G_T^A(A)] = \mathbb{E}^A\left[W_T^A + \int_0^T \Big(u(P_sD_s) - h(A_s)\Big)ds\right] = W_0.$$

If he deviates to $\tilde{A}$, the principal continues to promise him $W_t^A$ and his utility by time $t$ is

$$G_t^A(\tilde{A}) = W_t^A + \int_0^t \Big(u(P_sD_s) - h(\tilde{A}_s)\Big)ds,$$

[6]See Lemma 10.4.6 in Cvitanić and Zhang (2013).

and in particular, his initial expected utility is

$$\begin{aligned}
\mathbb{E}^{\tilde{A}}[G_T^A(\tilde{A})] &= \mathbb{E}^{\tilde{A}}\left[W_T^A + \int_0^T \Big(u(P_sD_s) - h(\tilde{A}_s)\Big)ds\right] \\
&= \mathbb{E}^{\tilde{A}}\left[W_0 - \int_0^T \Big(u(P_sD_s) - h(A_s)\Big)ds + \int_0^T Z_s^c\sigma dB_s^A + \sum_{0<s\le T} Z_s^d\Delta J_s + \int_0^T \Big(u(P_sD_s) - h(\tilde{A}_s)\Big)ds\right] \\
&= \mathbb{E}^{\tilde{A}}\left[W_0 + \int_0^T \Big[\big(\rho Z_s\tilde{A}_s - h(\tilde{A}_s)\big) - \big(\rho Z_sA_s - h(A_s)\big)\Big]ds + \int_0^t Z_s^c\sigma dB_s^{\tilde{A}} + \sum_{0<s\le T} Z_s^d\Delta J_s\right].
\end{aligned}$$

Under the probability measure $\mathbb{P}^{\tilde{A}}$, the distribution of $J$ is still a compound Poisson with jump sizes of mean zero. The last two terms have zero expected mean.

If 1 holds but 2 does not on a set $S \subset [0,T]\times\Omega$ with respect to $\tilde{A}$, then let $A_t^*(\omega) = \tilde{A}_t(\omega)$ whenever $(t,\omega)\in S$ and $A_t^*(\omega) = A_t(\omega)$ otherwise. $\mathbb{E}^{A^*}[G_T^A(A^*)] > W_0 = \mathbb{E}^A[G_T^A(A)]$; that is, $A^*$ is a profitable deviation. If 2 holds, then $\mathbb{E}^{\tilde{A}}[G_T^A(\tilde{A})] \le W_0 = \mathbb{E}^A[G_T^A(A)]$. That is, $\tilde{A}$ is not a profitable deviation. □

*Proof of Lemma 4.3.* Let $P^{A^*}$ be the probability measure as defined by (4.1) with $\theta_t = \sigma^{-1}\rho A_t^*$. Using $A_t^* = \eta(Z_t^c)$,

$$W_t + \int_0^t \Big(u(P_sD_s) - h(A_s^*)\Big)ds = U(C_T) + \int_0^T \Big(u(P_sD_s) - h(A_s^*)\Big)ds - \int_t^T Z_s^c\sigma dB_s^{A^*} - \sum_{t<s\le T} Z_s^d\Delta J_s.$$

The process $t \to W_t + \int_0^t \Big(u(P_sD_s) - h(A_s^*)\Big)ds$ is a $P^{A^*}$-martingale. Then,

$$\begin{aligned}
W_t + \int_0^t \Big(u(P_sD_s) - h(A_s^*)\Big)ds &= \mathbb{E}_t^{A^*}\left[W_T + \int_0^T \Big(u(P_sD_s) - h(A_s^*)\Big)ds\right] \\
&= \mathbb{E}_t^{A^*}\left[U(C_T) + \int_0^T \Big(u(P_sD_s) - h(A_s^*)\Big)ds - \int_T^T Z_s^c\sigma dB_s^{A^*} - \sum_{t<s\le T} Z_s^d\Delta J_s\right].
\end{aligned}$$

That is, $W_t = E_t^{A^*}\Big[U(C_T) + \int_t^T \Big(u(P_sD_s) - h(A_s^*)\Big)ds\Big]$; in other words, $W_t$ represents the agent's expected remaining utility. □

*Proof of Lemma 4.4.* We note that the remaining expected utility under $(C_T^*, A)$ is

$$\begin{aligned}
W_t^A &= \mathbb{E}_t^A\left[U(C_T^*) + \int_t^T \Big(u(P_sD_s) - h(A_s)\Big)ds\right] = \mathbb{E}_t^A\left[W_T + \int_t^T \Big(u(P_sD_s) - h(A_s)\Big)ds\right] \\
&= \mathbb{E}_t^A\left[W_0 - \int_0^T \Big(u(P_sD_s) - h(A_s)\Big)ds + \int_0^T \xi(A_s)\sigma dB_s^A + \int_t^T \Big(u(P_sD_s) - h(A_s)\Big)ds\right] \\
&= W_0 - \int_0^t \Big(u(P_sD_s) - h(A_s)\Big)ds + \int_0^t \xi(A_s)\sigma dB_s^A.
\end{aligned}$$

Letting $Z^c_t = \xi(A_t)$ and $Z^d = 0$, then $(C^*_T, A, Z^c, Z^d)$ satisfies condition 2 in Proposition 4.2, hence $(C^*_T, A)$ is optimal for the agent. $\square$

The resolution of the agent's problem relies on a weak formulation while the contract in Theorem 4.5 is specified via a strong formulation. We need to first show that the two formulations are equivalent.

**Lemma A.6.** $\mathcal{F}^{M,J} = \mathcal{F}^{B,J}$.

*Proof.* Let the filtered probability space in the weak formulation be $(\Omega, \mathbf{F}, \mathcal{F}, \mathbb{P}) = (\Omega, \mathbf{F}, \mathcal{F}^{B,J}, \mathbb{Q}^B)$, where $\mathcal{F}^{B,J}$ is the filtration generated $(B, J)$. The continuous process $B$ is defined as

$$B_t \triangleq M_t + \int_0^t \sigma^{-1}\rho a^*(s, P_s, W_s)ds = M_t + \int_0^t \theta_s ds,$$

where $\theta_t = \sigma^{-1}\rho a^*(t, P_{t-}, W_t)$. The probability measure $\mathbb{Q}^B$ is defined by

$$d\mathbb{Q}^B = e^{-\int_0^t \theta_s dM_s - \int_0^t \theta_s^2 ds} d\mathbb{Q}.$$

Because $a^*(t, \cdot)$ is a functional of $(P_{t-}, W_t)$, the filtration generated by $(B, J)$ is smaller than the filtration generated by $(P, W, M)$, i.e. $\mathcal{F}^{B,J} \subseteq \mathcal{F}^{P,W,M}$. From (4.7), we can see that the filtration generated by $P, W$ is smaller than $\mathcal{F}^{M,J}$, i.e. $\mathcal{F}^{P,W} \subseteq \mathcal{F}^{M,J}$; then, $\mathcal{F}^{P,W,M} = \mathcal{F}^{M,J}$. We have shown that $\mathcal{F}^{B,J} \subseteq \mathcal{F}^{P,W,M} = \mathcal{F}^{M,J}$.

On the other hand, we rewrite (4.7) as

$$\begin{cases} P_t = P_0 + \int_0^t a^*(s, P_s, W_s)ds + \int_0^t \sigma(dB_s - \theta_s ds) + \sum_{0<s\leq t} \Delta J_s \\ W_t = R + \int_0^t \Big[h(a^*(s, P_s, W_s)) - u(P_s D_s)\Big]ds + \int_0^t \xi(a^*(s, P_s, W_s))\sigma(dB_s - \theta_s ds). \end{cases}$$

We can see that $\mathcal{F}^{P,W} \subseteq \mathcal{F}^{B,J}$, then by the same argument, $\mathcal{F}^{M,J} \subseteq \mathcal{F}^{P,W,B} = \mathcal{F}^{B,J}$. Finally, we have $\mathcal{F}^{M,J} = \mathcal{F}^{B,J}$. $\square$

*Remark* A.7. In the optimal contract, $A$ is predictable w.r.t. $\mathcal{F}^{P,W}$. In the proof of the Lemma A.6, we show that $\mathcal{F}^{P,W} \subseteq \mathcal{F}^{B,J}$. Since we already know that $\mathcal{F}^{B,J} = \mathcal{F}^P$ from Remark A.4. There is a subtlety on the timing at which the action $A_t$ is determined. At each time $t$, the agent is assumed to make the decision $A_t$ before the observation of $\Delta J_t$. In other words, it is requisite that $A$ is predictable w.r.t. observable information. Denote $\tilde{A} \triangleq \{a^*(t, P_t, W_t) : 0 \leq t \leq T\}$. While $A = \tilde{A}$ for almost all $(t, \omega) \in [0, T] \times \Omega$, $A$ is predictable with respect to $\mathcal{F}^P$ but $\tilde{A}$ is not.

Next, we prove the main result Theorem 4.5. We split up the proof into five lemmas. We fix the filtration $\mathcal{F}^{M,J}$ on the probability space $(\Omega, \mathbf{F}, \mathbb{Q})$.

**Lemma A.8.** *The contract $(C_T, A)$ is incentive compatible for the agent, and his initial expected utility is $R$.*

*Proof.* Applying Lemma 4.4, we see that the contract $(C_T, A)$ is optimal. We also verify that the agent's initial expected utility is

$$\mathbb{E}\left[U(C_T) + \int_0^T \Big(u(P_sD_s) - h(A_s)\Big)ds\right] = \mathbb{E}\left[W_T + \int_0^T \Big(u(P_sD_s) - h(A_s)\Big)ds\right] = R.$$

□

The next lemma shows that the principal does not need to pay attention to jumps in providing incentives. In particular, the principal can set $Z^d = 0$. We assume that $U(\cdot)$ is twice continuously differentiable, strictly increasing, and concave. Note that by Assumption (**A2**), $I' > 0$ and $I'' > 0$.

**Lemma A.9.** *Let the contract $(\bar{C}_T, \bar{A})$ be incentive compatible and delivers an initial expected utility $W_0$ to the agent. Then, the contract $(I(\tilde{W}_T), \bar{A})$ is also incentive compatible and delivers $W_0$, where*

$$\tilde{W}_t = W_0 + \int_0^t \Big[h(\bar{A}_s) - u(\bar{P}_sD_s)\Big]ds + \int_0^t \xi(\bar{A}_s)\sigma dM_s.$$

*If we assume that the principal cannot learn anything about the jump sizes from observing the history of $\tilde{W}_t$, then the principal does not strictly prefer $(\bar{C}_T, \bar{A})$ to $(I(\tilde{W}_T), \bar{A})$.*

*Proof.* By Lemma 4.4, the contract $(I(\tilde{W}_T), \bar{A})$ is incentive compatible. We need to show $\mathbb{E}[I(\tilde{W}_T)] \leq \mathbb{E}[I(\bar{W}_T)]$. $(\bar{C}_T, \bar{A})$ is incentive compatible, then by Proposition A.5 and 4.2, there is a predictable $Z^d$ such that

$$\bar{W}_T = W_0 + \int_0^T \Big[h(\bar{A}_s) - u(\bar{P}_sD_s)\Big]ds + \int_0^T \xi(\bar{A}_s)\sigma dM_s + \sum_{0<s\leq T} Z_s^d \Delta J_s.$$

Using the convexity of $I(\cdot)$ and linear expand around $\tilde{W}_T$,

$$\begin{aligned} I(\bar{W}_T) &= I(\tilde{W}_T + \sum_{0<s\leq T} Z_s^d \Delta J_s) \\ &\geq I(\tilde{W}_T) + I'(\tilde{W}_T) \sum_{0<s\leq T} Z_s^d \Delta J_s \end{aligned} \tag{A.7}$$

Let $\mathcal{F}^{\tilde{W}}$ be the filtration generated by $\tilde{W}$, and it is left continuous because $\tilde{W}$ is a continuous process. Let $\mathcal{F}^N$ be the filtration generated by the Poisson process; that is, $\mathcal{F}_t^N$ contains the information about when the jumps occur up to time $t$. It is clear that $\mathcal{F}^{\tilde{W}}$ and $\mathcal{F}^N$ are independent. We make the economic assumption that $\mathcal{F}^{\tilde{W}}$ does not contain information on jump sizes.[7] Then, $\mathcal{F}^{\tilde{W}}$ and $\mathcal{F}^J$ are

[7]In principle, some information of the sign and magnitutude (but not the timing) of the jumps ($J$) can be inferred from $\tilde{W}$. If this is the case, then the principal can reduce the expected payment by providing a form of insurance via $Z^d$. That is, $Z^d$ is set to correlate with $\tilde{W}$ in such a way to reduce the variance of the agent's utility. While this is mathematically feasible, we rule out this possibility to the analysis. Allowing for this possibility does not add more insights to the overall contract design.

independent. From the proof in Proposition A.5, $Z^d$ is predictable w.r.t. $\mathcal{F}^J$, so $\sum_{0<s\le t} Z_s^d \Delta J_s$ is adapted to $\mathcal{F}_t^J$. Hence,

$$\begin{aligned}
\mathbb{E}\Big[I'(\tilde{W}_t)\sum_{0<s\le t} Z_s^d\Delta J_s\Big] &= \mathbb{E}\Big[\mathbb{E}\big[I'(\tilde{W}_t)\sum_{0<s\le t} Z_s^d\Delta J_s|\mathcal{F}_t^{\tilde{W}}\big]\Big] \\
&= \mathbb{E}\Big[I'(\tilde{W}_t)\mathbb{E}\big[\sum_{0<s\le t} Z_s^d\Delta J_s|\mathcal{F}_t^{\tilde{W}}\big]\Big] = \mathbb{E}\Big[I'(\tilde{W}_t)\mathbb{E}\big[\sum_{0<s\le t} Z_s^d\Delta J_s\big]\Big] = 0
\end{aligned}$$

Taking expectation of both sides of (A.7), we get $\mathbb{E}\left[I(\bar{W}_T)\right] \ge \mathbb{E}\left[I(\tilde{W}_T)\right]$. $\square$

**Lemma A.10.** *If a contract $(\bar{C}_T, \bar{A})$ is incentive compatible that delivers an initial expected utility $R$ to the agent, then the principal's initial payment is at least $F(0, P_0, R)$, where $F(\cdot)$ is the solution to (4.5). That is,*

$$\mathbb{E}\left[\bar{C}_T + \int_0^T \bar{P}_s D_s ds\right] \ge F(0, P_0, R),$$

*where $\bar{P}_t = P_0 + \int_0^t \rho(\bar{A}_s - \bar{P}_s)ds + \int_0^t \sigma dM_s + \sum_{0<s\le t} \Delta J_s$.*

*Proof.* The contract $(\bar{C}_T, \bar{A})$ is incentive compatible, then by Proposition 4.2 and Lemma A.9, the agent's continuation utility must be

$$\bar{W}_t = R + \int_0^t \Big[h(\bar{A}_s) - u(\bar{P}_s D_s)\Big]ds + \int_0^t \xi(\bar{A}_s)\sigma dM_s.$$

Apply Ito's rule to $F$ from 0 to $t$,

$$\begin{aligned}
F(t, \bar{P}_t, \bar{W}_t) &= F(0, P_0, R) + \int_0^t \Big(\partial_t F + \rho\big(\bar{A}_s - \bar{P}_s\big)\partial_p F + \frac{1}{2}\sigma^2\partial_{pp}F + \big(h(\bar{A}_s) - u(\bar{P}_s D_s)\big)\partial_w F \\
&\quad + \frac{1}{2}\,\xi(\bar{A}_s)^2\sigma^2\partial_{ww}F + \xi(\bar{A}_s)\sigma^2\partial_{pw}F\Big)(s, \bar{P}_s, \bar{W}_s)ds \\
&\quad + \int_0^t \Big(\partial_p F + \xi(\bar{A}_s)\partial_w F\Big)(s, \bar{P}_s, \bar{W}_s)\sigma dM_s + \sum_{0<s\le t} F(s, \bar{P}_s, \bar{W}_s) - F(s, \bar{P}_{s-}, \bar{W}_s)
\end{aligned}$$

Taking the expectation of both sides, we get

$$\begin{aligned}
\mathbb{E}\left[F(t, \bar{P}_t, \bar{W}_t)\right] &= \mathbb{E}\Big[\int_0^t \Big(\partial_t F + \rho\big(\bar{A}_s - \bar{P}_s\big)\partial_p F + \frac{1}{2}\sigma^2\partial_{pp}F + \big(h(\bar{A}_s) - u(\bar{P}_s D_s)\big)\partial_w F \\
&\quad + \frac{1}{2}\xi(\bar{A}_s)^2\sigma^2\partial_{ww}F + \xi(\bar{A}_s)\sigma^2\partial_{pw}F\Big)(s, \bar{P}_s\bar{W}_s)ds\Big] \\
&\quad + \mathbb{E}\left[\sum_{0<s\le t} F(s, \bar{P}_{s-} + \Delta J_s, \bar{W}_s) - F(s, \bar{P}_{s-}, \bar{W}_s)\right] + F(0, P_0, R) \\
&= \mathbb{E}\Big[\int_0^t \Big(\partial_t F + \rho\big(\bar{A}_s - \bar{P}_s\big)\partial_p F + \frac{1}{2}\sigma^2\partial_{pp}F + \big(h(\bar{A}_s) - u(\bar{P}_s D_s)\big)\partial_w F \\
&\quad + \frac{1}{2}\xi(\bar{A}_s)^2\sigma^2\partial_{ww}F + \xi(\bar{A}_s)\sigma^2\partial_{pw}F\Big)(s, \bar{P}_s\bar{W}_s)ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \int_{-L}^{L} \Big(F(s, \bar{P}_{s-} + dp', \bar{W}_s) - F(s, \bar{P}_{s-}, \bar{W}_s)\frac{\lambda}{2L}\Big) dp' ds\Big] \\
&+ F(0, P_0, R) \qquad \text{(A.8)}
\end{aligned}
$$

The second equality holds because

$$
\begin{aligned}
& \mathbb{E}\Big[\sum_{0<s\le t} F(s, \bar{P}_{s-} + \Delta J_s, \bar{W}_s) - F(s, \bar{P}_{s-}, \bar{W}_s)\Big] \\
= \quad & \mathbb{E}\Big[\mathbb{E}\Big[\sum_{0<s\le t} F(s, \bar{P}_{s-} + \Delta J_s, \bar{W}_s) - F(s, \bar{P}_{s-}, \bar{W}_s)|\mathcal{F}_t^J\Big]\Big] \\
= \quad & \mathbb{E}\Big[\int_0^t \int_{-L}^{L} \Big(F(s, \bar{P}_{s-} + dp', \bar{W}_s) - F(s, \bar{P}_{s-}, \bar{W}_s)\frac{\lambda}{2L}\Big) dp' ds\Big]
\end{aligned}
$$

by noting that $(P_{s-}, W_s)$ and $J_s$ are independent, and $J$ has a jump intensity of $\lambda$ and jump size distributed uniformly over $[-L, L]$. On the other hand, $F$ is the solution to (4.5), then for any $(s, p, w, a)$,

$$
\begin{aligned}
&\Big(\partial_t F + \rho(a-p)\partial_p F + \frac{1}{2}\sigma^2 \partial_{pp} F + (h(a) - u(pD_s))\partial_w F + \frac{1}{2}\xi(a)^2\sigma^2\partial_{ww} F + \xi(a)\sigma^2 \partial_{pw} F\Big)(s, p, w) \\
&+ \int_{-L}^{L} \Big(F(s, p + dp', w) - F(s, p, w)\Big)\frac{\lambda}{2L} dp' + pD_s \ge 0 \qquad \text{(A.9)}
\end{aligned}
$$

Applying (A.9) to (A.8), we have

$$
\mathbb{E}\Big[F(t, \bar{P}_t, \bar{W}_t)\Big] \ge F(0, P_0, R) - \mathbb{E}\left[\int_0^t \bar{P}_s D_s ds\right].
$$

Letting $t = T$ and use the terminal condition of that $F(T, \bar{P}_T, \bar{W}_T) = I(\bar{W}_T) = \bar{C}_T$, we have the result

$$
\mathbb{E}\left[\bar{C}_T\right] \ge F(0, P_0, R) - \mathbb{E}\left[\int_0^T \bar{P}_s D_s ds\right].
$$

□

**Lemma A.11.** *Under the contract $(C_T, A)$, the principal's initial expected payment is $F(0, P_0, R)$.*

*Proof.* We follow the same steps as in the previous lemma, except replacing $(\bar{C}_T, \bar{A})$ with $(C_T, A)$ and also except that in (A.9), we have equality because $A_t$ is the minimizer in (4.6). We have

$$
\mathbb{E}\left[C_T + \int_0^T P_s D_s ds\right] = F(0, P_0, R).
$$

□

**Lemma A.12.** *Let $(\bar{C}_T, \bar{A})$ be an incentive compatible contract that deliver an initial expected value*

*$\bar{W}_0$ to the agent. If $\bar{W}_0 > R$, then*

$$\mathbb{E}\left[\bar{C}_T + \int_0^T \bar{P}_s D_s ds\right] > \mathbb{E}\left[C_T + \int_0^T P_s D_s ds\right].$$

*In other words, the principal never wants to offer an incentive compatible contract that gives the agent an initial value higher than $R$.*

*Proof.* The contract $(\bar{C}_T, \bar{A})$ is incentive compatible, then by Proposition 4.2 and Lemma A.9, the agent's continuation utility must be

$$\bar{W}_t = \bar{W}_0 + \int_0^t \left[h(\bar{A}_s) - u(\bar{P}_s D_s)\right] ds + \int_0^t \xi(\bar{A}_s)\sigma dM_s \text{ and } \bar{W}_T = U(\bar{C}_T).$$

From Lemma A.12, we know that $\mathbb{E}\left[C_T + \int_0^T P_s D_s ds\right] = F(0, P_0, R)$, Then,

$$\begin{aligned} F(0, P_0, R) &= \min_{\hat{A}} \mathbb{E}\left[I(\hat{W}_T^{0,P_0,R}) + \int_0^T \hat{P}_s D_s ds\right] \\ &\leq \mathbb{E}\left[I(\bar{W}_T^{0,P_0,R}) + \int_0^T \bar{P}_s D_s ds\right] \\ &< \mathbb{E}\left[I(\bar{W}_T^{0,P_0,R} + \bar{W}_0 - R) + \int_0^T \bar{P}_s D_s ds\right] = \mathbb{E}\left[I(\bar{W}_T) + \int_0^T \bar{P}_s D_s ds\right] \end{aligned}$$

where $\hat{W}_t^{0,P_0,R} = R + \int_0^t \left[h(\hat{A}_s) - u(\hat{P}_s D_s)\right] ds + \int_0^t \xi(\hat{A}_s)\sigma dM_s$. The strict inequality holds because $I' > 0$. $\square$

*Proof of Theorem 4.5.* By Lemma A.8, the contract is incentive compatible. By Lemma A.12, the principal should start the calculation of the agent's continuation utility process at $W_0 = R$. Lemma A.10 and A.11 say that among all incentive compatible contracts in which the agent gets an initial expected payoff $R$, $(C_T, A)$ is at least as good to the principal as any others. Hence, $(C_T, A)$ is optimal for the principal among all incentive compatible contracts. $\square$

**Lemma A.13.** *$\xi > 0$ and $\xi' > 0$.*

*Proof.* By the strict concavity of $h$, $\xi(a) = h'(a)/p$. Then, $\xi'(a) = h''(a)/p$. $\square$

**Lemma A.14.** *$\partial_w F > 0$, and $\partial_{ww} F > 0$.*

*Proof.* $F$ is assumed to continuously differentiable, and hence it is sufficient to show that $F$ is increasing in $w$. Let $\Delta w > 0$. Then an increase to $w$ would result in

$$
\begin{aligned}
F(t,p,w+\Delta w) &= \min_A \mathbb{E}\left[I(W_T^{t,p,w+\Delta w}) + \int_t^T P_s^{t,p} D_s ds\right] \\
&= \min_A \mathbb{E}\left[I(W_T^{t,p,w} + \Delta w) + \int_t^T P_s^{t,p} D_s ds\right] \\
&> \min_A \mathbb{E}\left[I(W_T^{t,p,w}) + \int_t^T P_s^{t,p} D_s ds\right].
\end{aligned}
$$

To show convexity, let $\alpha \in (0,1)$, and $w^\alpha \triangleq \alpha w + (1-\alpha)w'$. Let $A^1$ (resp. $A^2$) be the optimal control when the starting condition is $(t,p,w)$ (resp. $(t,p,w')$), and $\tilde{A} = \alpha A^1 + (1+\alpha)A^2$ is admissible (i.e. $\tilde{A} \in \mathcal{A}$) because $\mathbf{A}$ is a convex set and hence, $\mathcal{A}$ is also convex. Then,

$$
\begin{aligned}
F(t,p,w^\alpha) &= \min_A \mathbb{E}\left[I(W_T^{t,p,w^\alpha}) + \int_t^T P_s^{t,p} D_s ds\right] \\
&\leq \mathbb{E}\left[I(\alpha W_T^{t,p,w,\tilde{A}} + (1-\alpha)W_T^{t,p,w',\tilde{A}}) + \int_t^T P_s^{t,p,\tilde{A}} D_s ds\right] \\
&< \mathbb{E}\left[\alpha I(W_T^{t,p,w,\tilde{A}}) + (1-\alpha)I(W_T^{t,p,w',\tilde{A}}) + \int_t^T P_s^{t,p,\tilde{A}} D_s ds\right] \\
&= \alpha \mathbb{E}\left[I(W_T^{t,p,w,A^1}) + \int_t^T P_s^{t,p,A^1} D_s ds\right] + (1-\alpha)\mathbb{E}\left[I(W_T^{t,p,w',A^2}) + \int_t^T P_s^{t,p,A^2} D_s ds\right] \\
&= \alpha F(t,p,w) + (1-\alpha)F(t,p,w')
\end{aligned}
$$

The first inequality is due to optimality, and the strict inequality is due to $I'' > 0$. The subscripts in $W$ and $P$ denote how the state processes are controlled. We conclude that $F$ is strictly convex in $w$. □

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