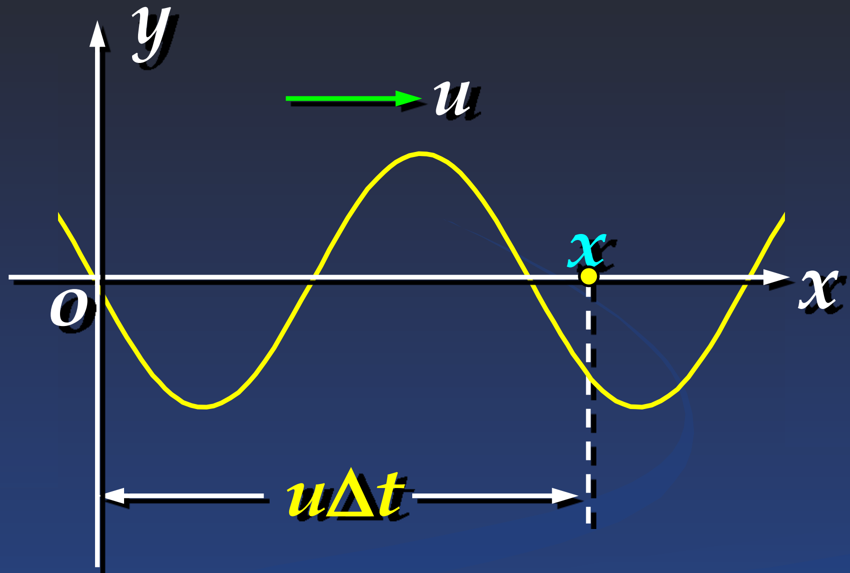
§ 10.2 平面简谐波的波函数

一、波函数

x 处质元 t 时刻的位移:

$$y = y(x,\ t) = ?$$

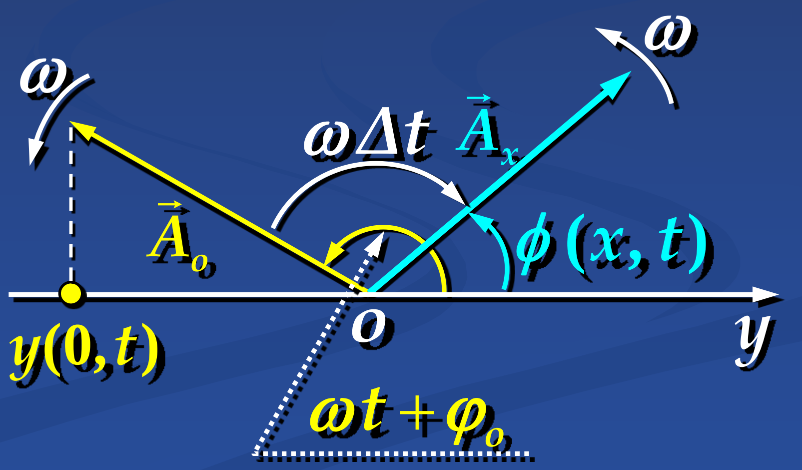


设：原点处质元的振动方程

$$y(0,\ t) = A\cos(\omega t + \varphi_o)$$

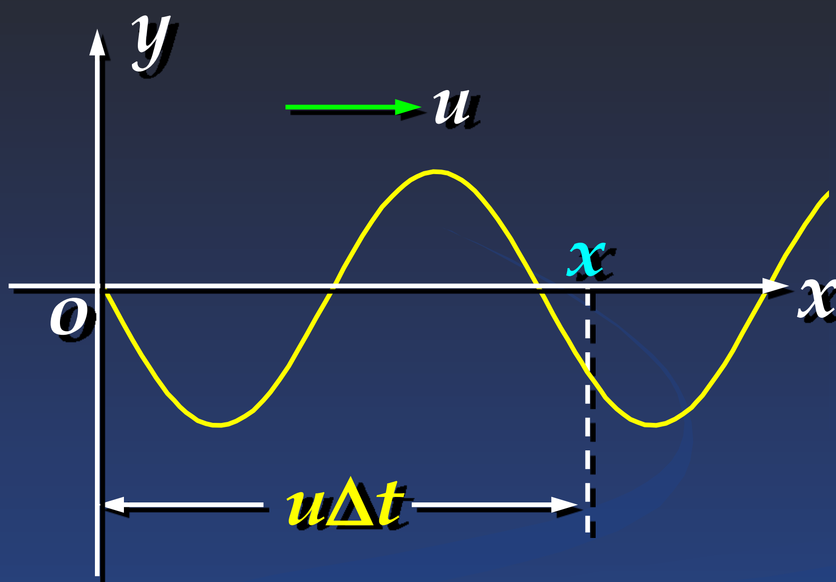
x 处质元的振动落后于 o 点

位相： $\omega\Delta t = \omega\dfrac{x}{u}$



$$\phi(x,\ t)=(\omega t+\varphi_o)-\omega\Delta t$$

$$=\omega(t-\frac{x}{u})+\varphi_o$$



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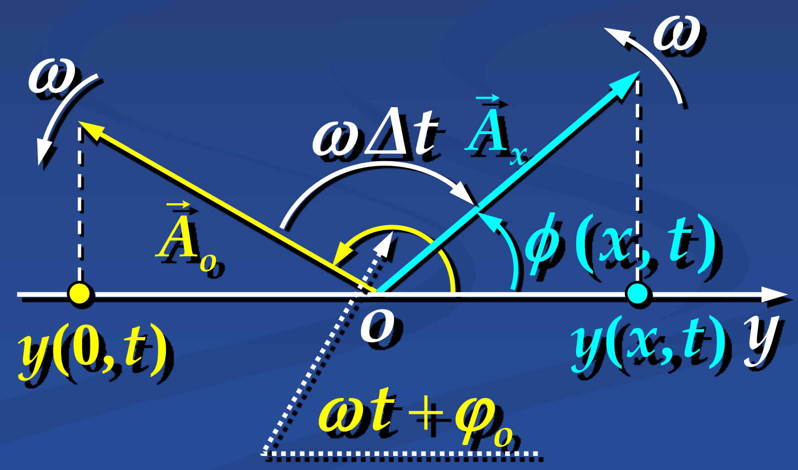
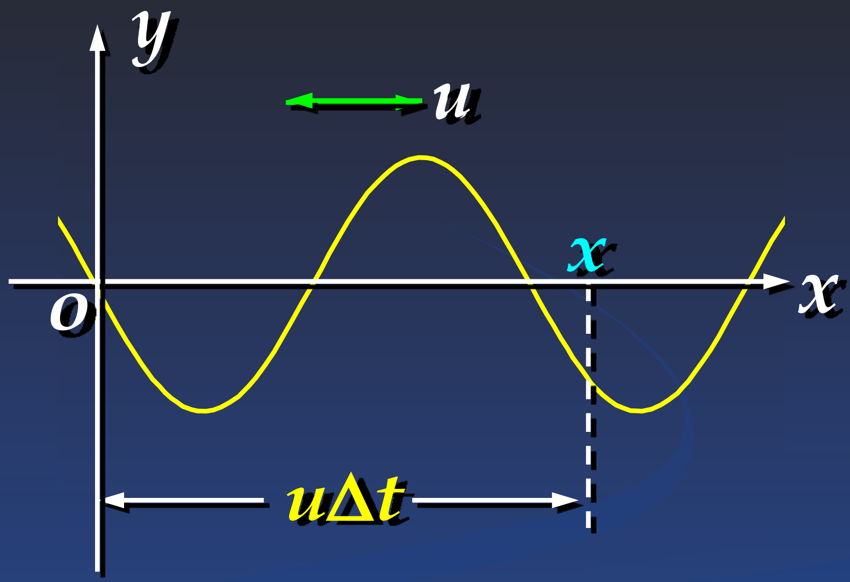


$$\phi(x,\ t) = (\omega t + \varphi_o) - \omega \Delta t$$

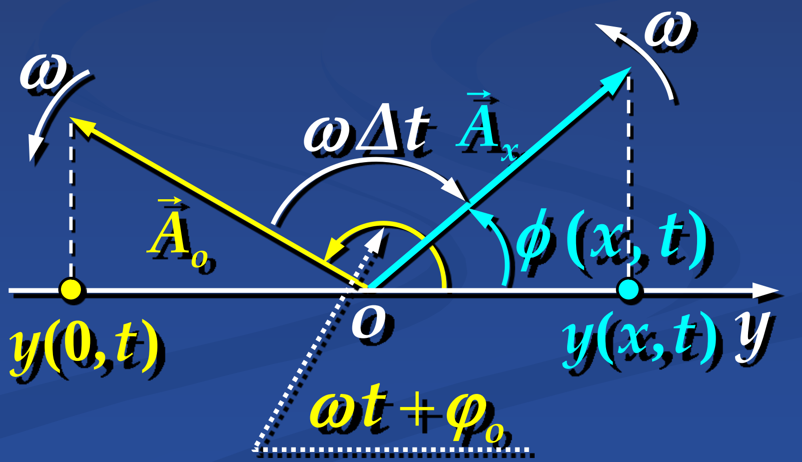
$$= \omega(t - \frac{x}{u}) + \varphi_o$$



x 处质元的振动方程：

$$y(x,\ t) = A\cos\phi(x,\ t)$$

$$\boxed{y = A\cos\left[\omega\left(t \mp \frac{x}{u}\right) + \varphi_o\right]}$$

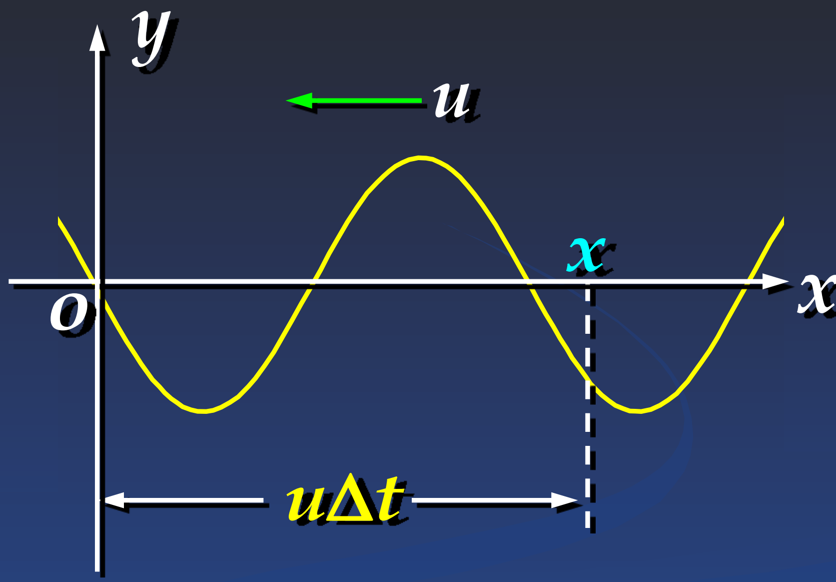


波函数亦称 波动方程 。

波动方程 的几种标准形式:

$$y = A\cos\left[\omega\left(t \mp \frac{x}{u}\right) + \varphi_o\right]$$

$$y = A\cos\left[\frac{2\pi}{T}\left(t \mp \frac{x}{u}\right) + \varphi_o\right]$$



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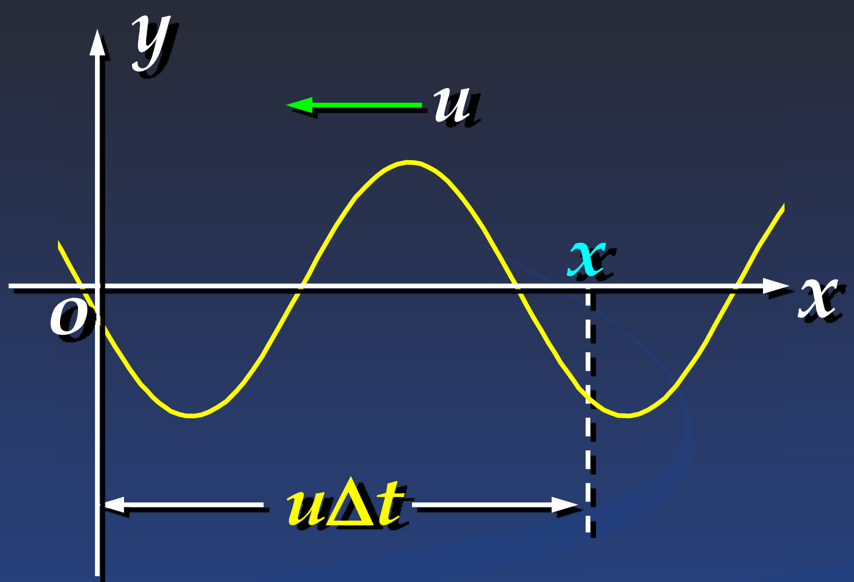
$$y = A\cos\left[2\pi\left(\frac{t}{T} \mp \frac{x}{\lambda}\right) + \varphi_o\right]$$

$$y = A\cos\left[\frac{2\pi}{\lambda}\left(ut \mp x\right) + \varphi_o\right]$$



右行波：取 — 号;

左行波：取 + 号。

即 ：x 处质元的振动方程，或在 t 时刻的位移!

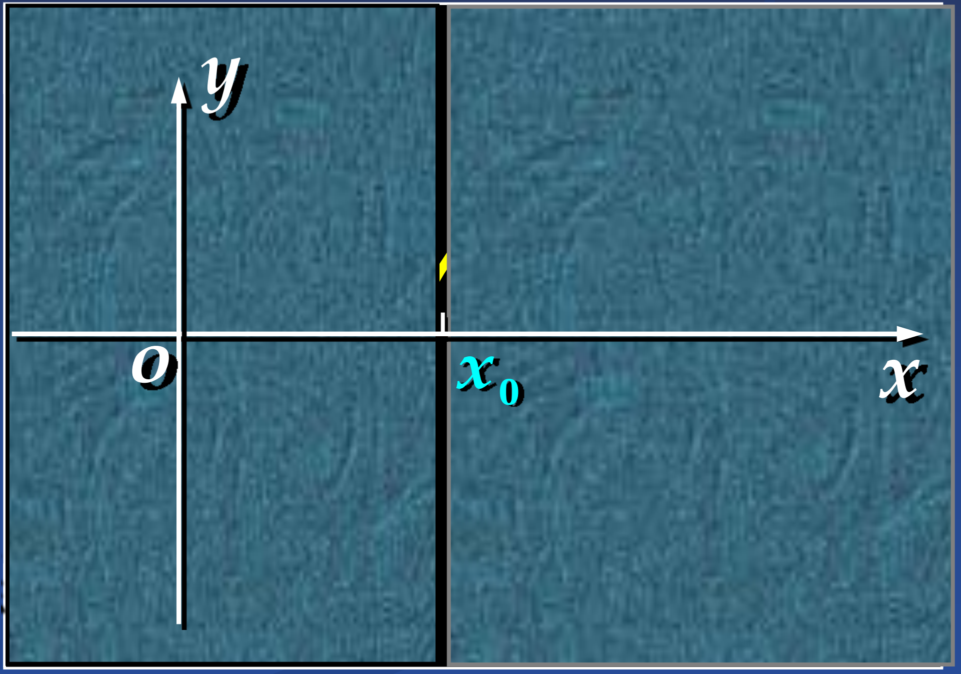
二、波动方程的物理含义

$$y = A\cos\left[\omega\left(t-\frac{x}{u}\right)+\varphi_o\right]$$

1. x 一定： $x = x_0$ ，$y(x_0,t) = A\cos\left[\omega\left(t-\frac{x_0}{u}\right)+\varphi_o\right]$

$$y = A\cos\left[2\pi\left(\frac{t}{T}\mp\frac{x}{\lambda}\right)+\varphi_o\right]$$

$$y = A\cos\left[\frac{2\pi}{\lambda}(ut\mp x)+\varphi_o\right]$$



即：x 处质元的振动方程

二、波动方程的物理含义

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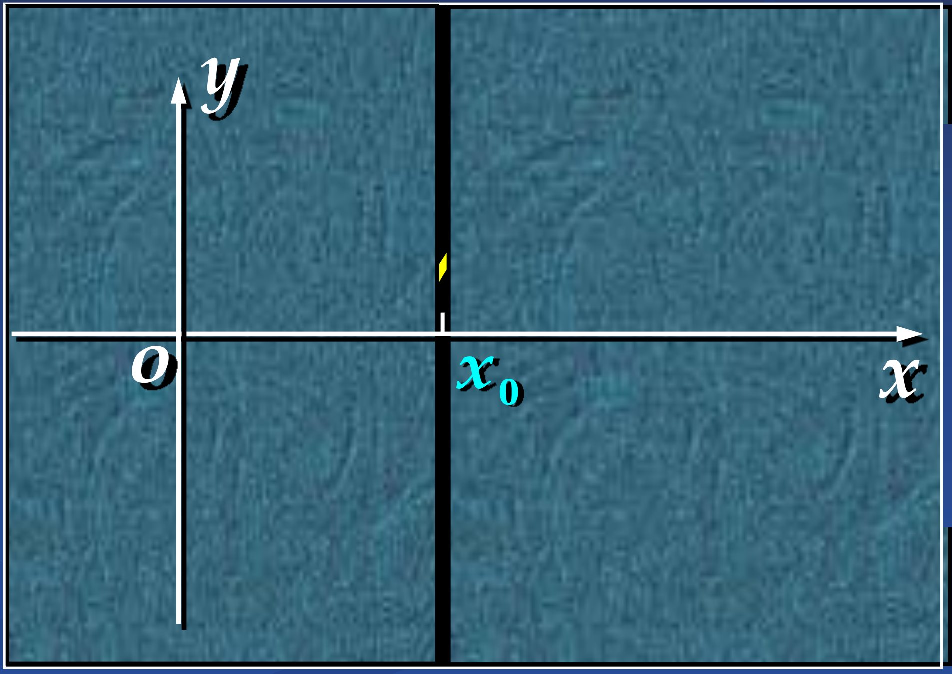
1. x 一定： $x = x_0$ ，　$y(x_0,t) = A\cos\left[\omega\left(t-\frac{x_0}{u}\right)+\varphi_o\right]$

$$y(x_0,t) = A\cos(\omega t+\varphi_x)$$

$$\varphi_x = -\frac{\omega x_0}{u}+\varphi_o$$

$$v = \frac{dy}{dt} = -\omega A\sin(\omega t+\varphi_x)$$

$$a = \frac{dv}{dt} = -\omega^2 A\cos(\omega t+\varphi_x)$$



$$y = A\cos\left[\omega\left(t - \frac{x}{u}\right) + \varphi_o\right]$$

2. t **一定：** $t = t_0$ ， $y(x, t_0) = A\cos\left[\omega\left(t_0 - \frac{x}{u}\right) + \varphi_o\right]$

$$y(x_0, t) = A\cos(\omega t + \varphi_x)$$

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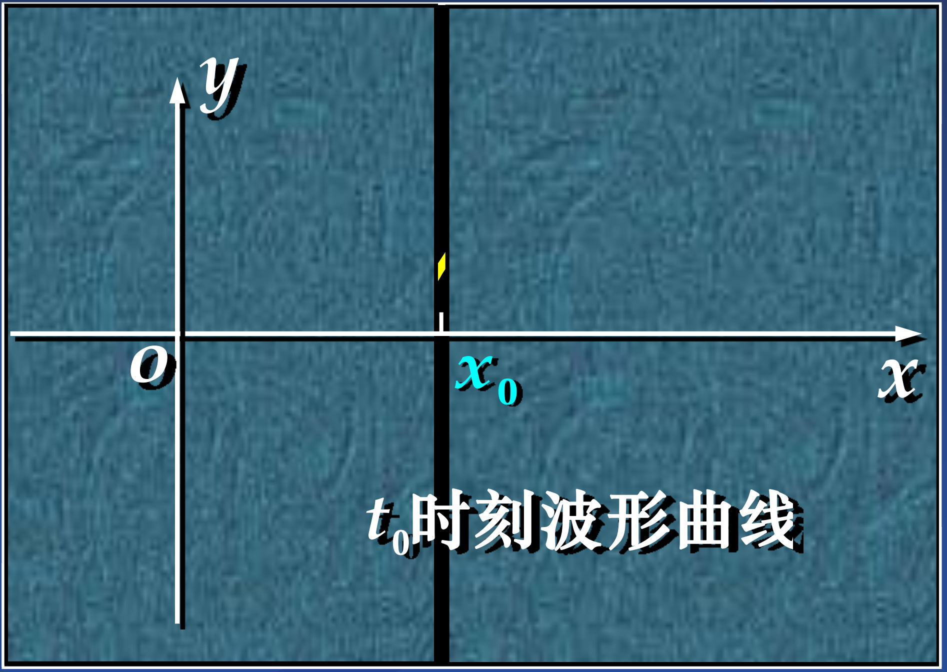


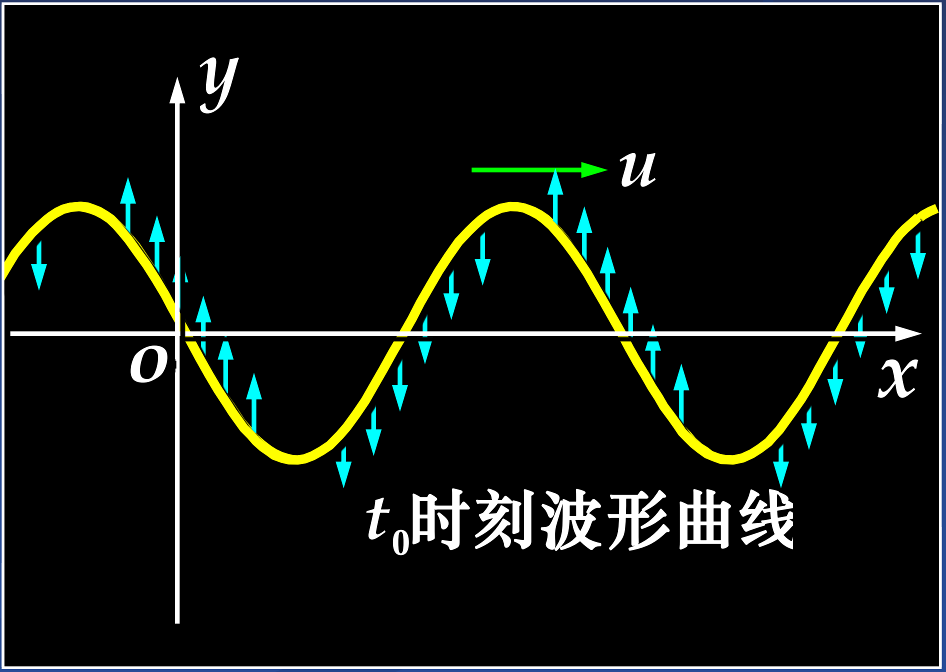
$$y = A\cos\left[\omega\left(t-\frac{x}{u}\right)+\varphi_o\right]$$

2. t 一定： $t = t_0$ ， $y(x,t_0) = A\cos\left[\omega\left(t_0-\frac{x}{u}\right)+\varphi_o\right]$

$$y(x,t_0) = A\cos\left(\frac{2\pi}{\lambda}x+\varphi^*\right)$$

判断：

右图中各点的速度方向或运动趋势。

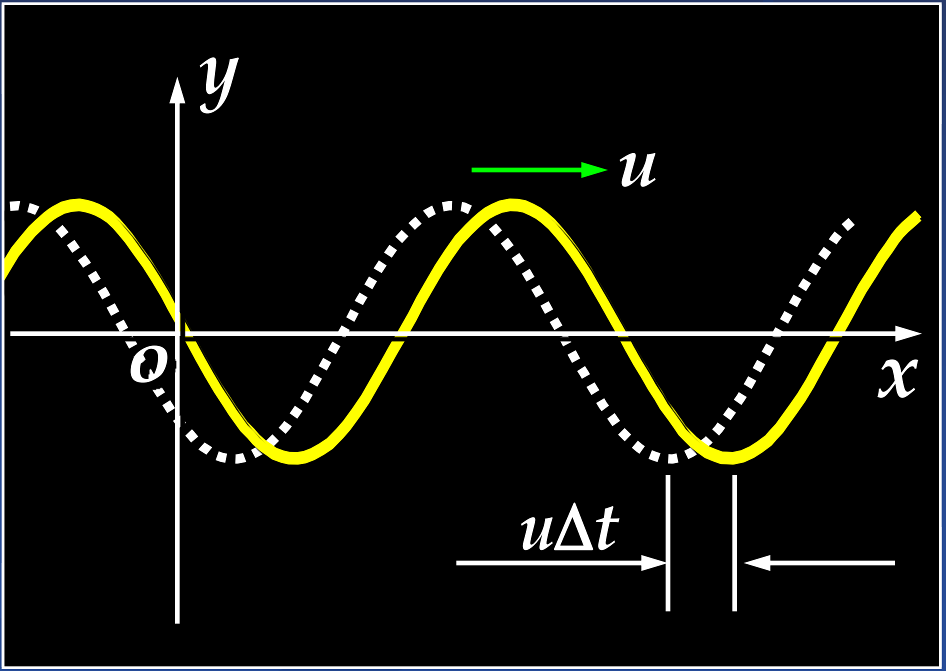


$$y = A\cos\left[\omega\left(t - \frac{x}{u}\right) + \varphi_o\right]$$

3. x、t 都不定： $y = A\cos\left[\omega\left(t - \frac{x}{u}\right) + \varphi_o\right]$

$$\omega\left(t - \frac{x}{u}\right) + \varphi_o = \omega\left(t + \Delta t - \frac{x + \Delta x}{u}\right) + \varphi_o$$

$$\Delta x = u\Delta t$$

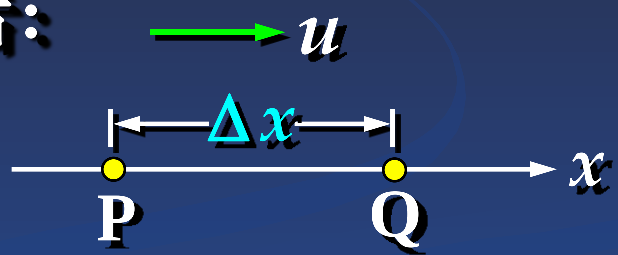


- 波速即为相位传播速度（**相速**）。
- 行波或前进波。

三、波函数的求解

简谐波波函数：$y = A\cos\left[\omega\left(t \mp \frac{x}{u}\right) + \varphi_o\right]$

- 波函数与坐标系的建立有关。
- 沿波的传播方向各点相位依次落后：



$$\phi_P(t) - \phi_Q(t) = \omega\Delta t = \omega\frac{\Delta x}{u} = \frac{2\pi}{\lambda}\Delta x$$

- 求波函数 ← 求 x 处质点的振动方程 ← 求 $\phi(x, t) = ?$
- x_0 处质元在 t_0 时刻的 v、a：

$$v(x_0,\ t_0) = \left.\frac{\partial y}{\partial t}\right|_{x_0, t_0} \qquad a(x_0,\ t_0) = \left.\frac{\partial^2 y}{\partial t^2}\right|_{x_0, t_0}$$

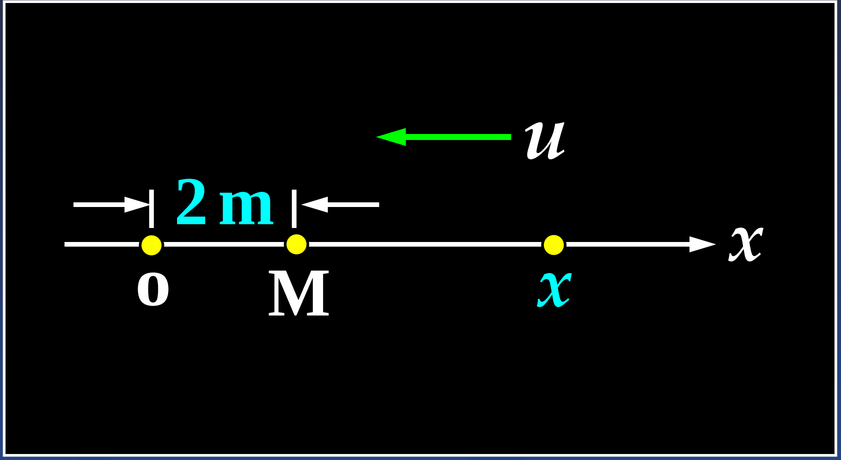
例 如图，波沿 –x方向传播，oM=2m，M点的振动方程:

$y = 10\cos(\pi t + \frac{\pi}{2})$ ，u=2m/s，求波动方程(即**波函数**)。

解 x 处相位超前 M 点:

$$\phi(x,t) - \phi(x_M, t) = \omega \frac{x - x_M}{u}$$

$$\phi(x_M, t) = \pi t + \frac{\pi}{2}$$



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例 如图，波沿 $-x$ 方向传播，oM=2m，M点的振动方程:

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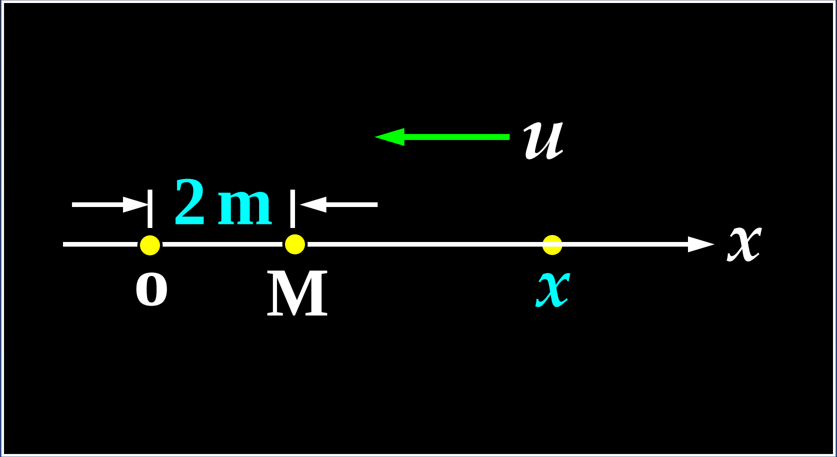


$$\phi(x,t)-\phi(x_M,t)=\omega\frac{x-x_M}{u}$$

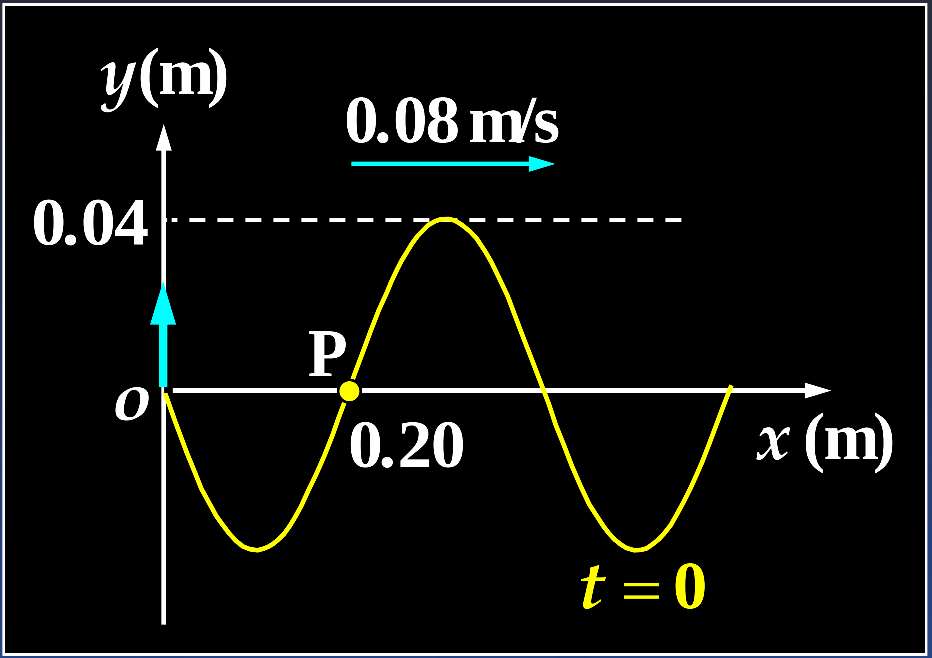
$$\phi(x_M,t)=\pi t+\frac{\pi}{2}$$

$$\phi(x,t)=(\pi t+\frac{\pi}{2})+\pi\frac{x-2}{2}=2\pi(\frac{t}{2}+\frac{x}{4})-\frac{\pi}{2}$$

$$y=10\cos[2\pi(\frac{t}{2}+\frac{x}{4})-\frac{\pi}{2}]$$

(the end)

例 图示为 $t=0$ 时的波形图，求波动方程及此时P点 v 。



解 $\lambda = 0.40\,\text{m}$

$$\omega = \frac{2\pi}{\lambda}u = \frac{2\pi}{5}$$

由旋转矢量图可知:

$$\phi(x,t) = (\pi t + \frac{\pi}{2}) + \pi\frac{x-2}{2} = 2\pi(\frac{t}{2}+\frac{x}{4}) - \frac{\pi}{2}$$

$$y = 10\cos[2\pi(\frac{t}{2}+\frac{x}{4}) - \frac{\pi}{2}]$$

(the end)

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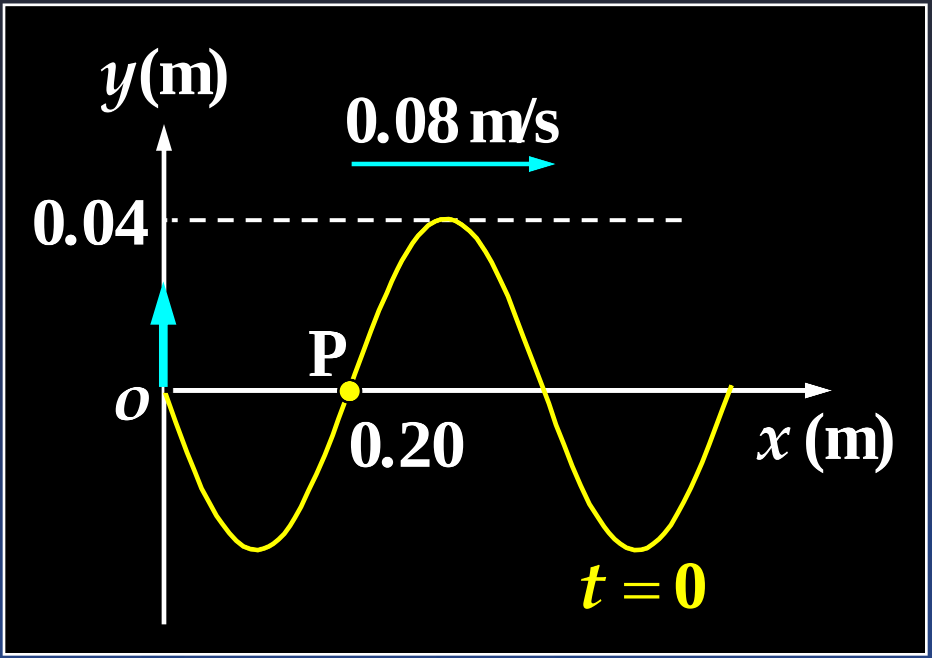
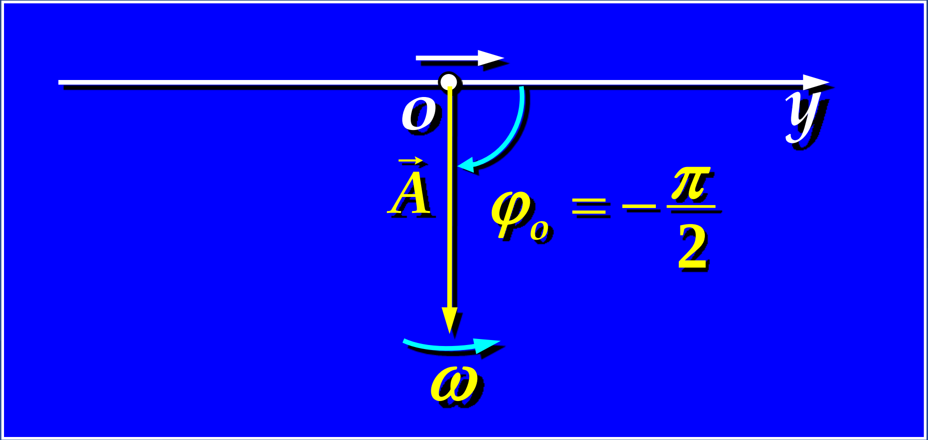


由旋转矢量图可知:

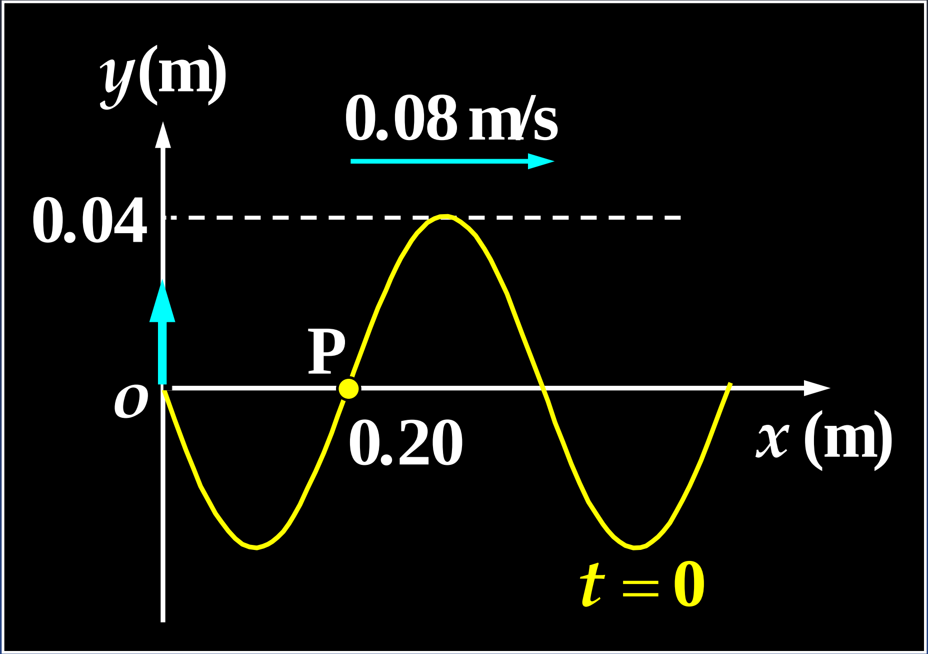
$$\varphi_o = -\frac{\pi}{2}$$



波函数的标准形式:

$$y = A\cos\left[\omega\left(t - \frac{x}{u}\right) + \varphi_o\right]$$

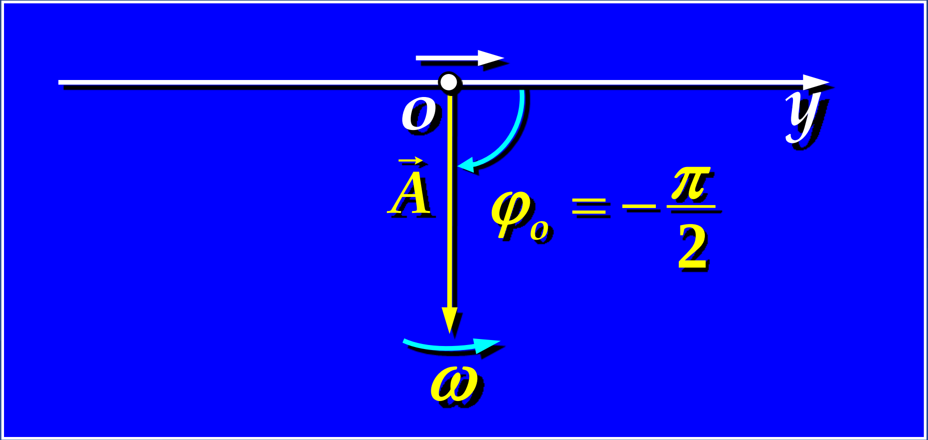
$$y = 0.04\cos\left[\frac{2\pi}{5}\left(t - \frac{x}{0.08}\right) - \frac{\pi}{2}\right]$$



$t = 0$ 时 P 点的速度：

$$v_P = \left.\frac{\partial y}{\partial t}\right|_{\substack{x=0.20\\ t=0}}$$

$$\varphi_o = -\frac{\pi}{2}$$



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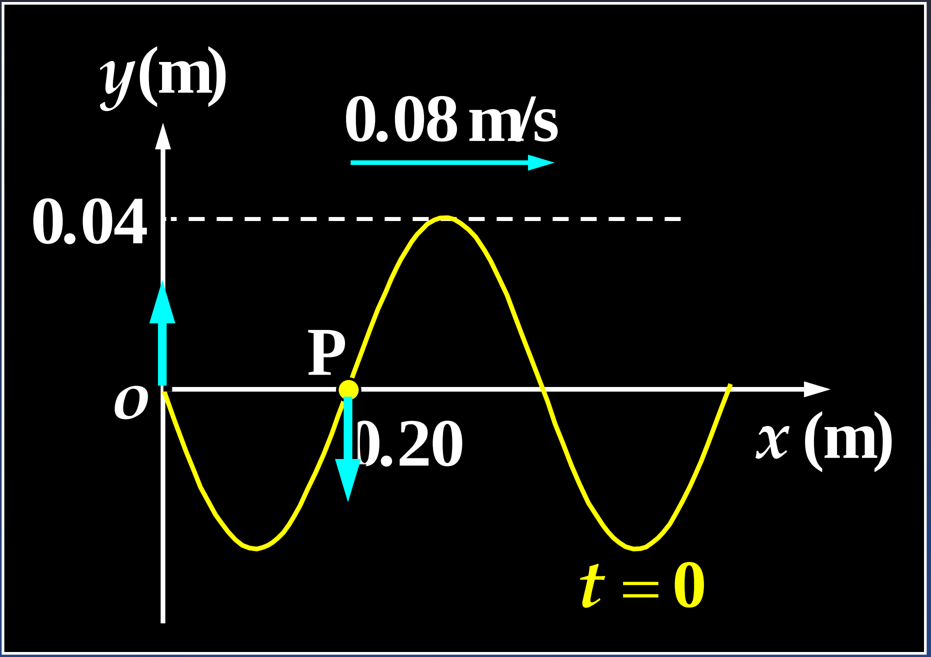
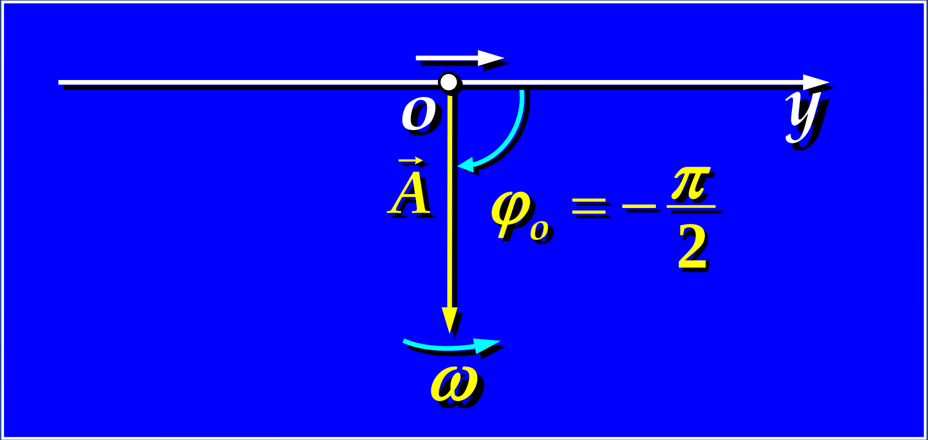


$t = 0$ 时 P 点的速度:

$$v_P = \left.\frac{\partial y}{\partial t}\right|_{\substack{x=0.20\\ t=0}}$$

$$= -0.04 \times \frac{2\pi}{5} \sin\left(-\frac{3\pi}{2}\right)$$

$$= -0.05\,(\mathrm{m/s})$$



沿 $-y$ 方向。

(the end)

课堂练习 图示为 $t = 1\text{s}$ 时的波形曲线，求波动方程。

提示 关键：求解原点o处质元初位相 φ_o！

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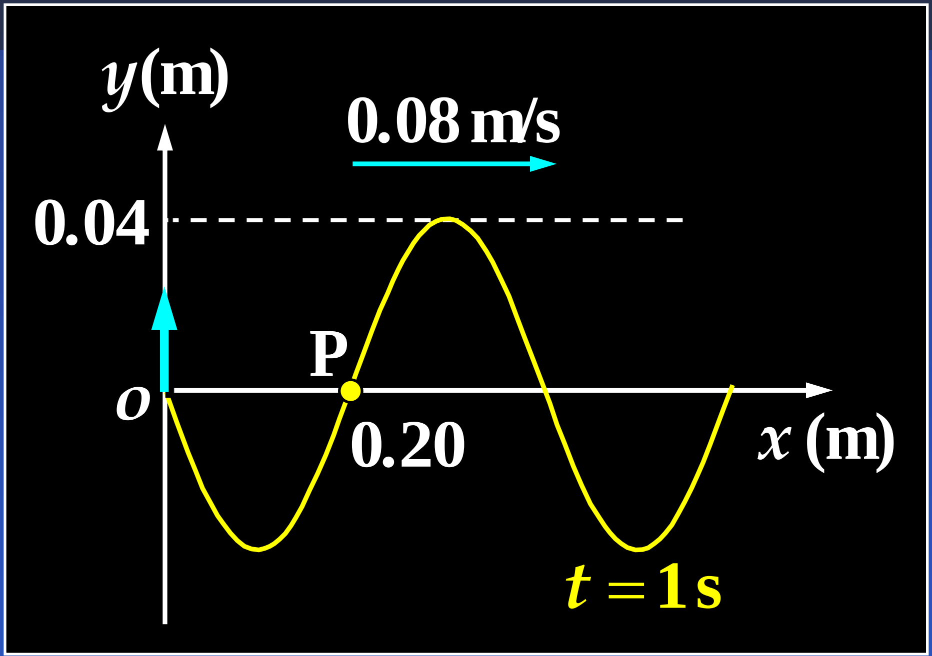

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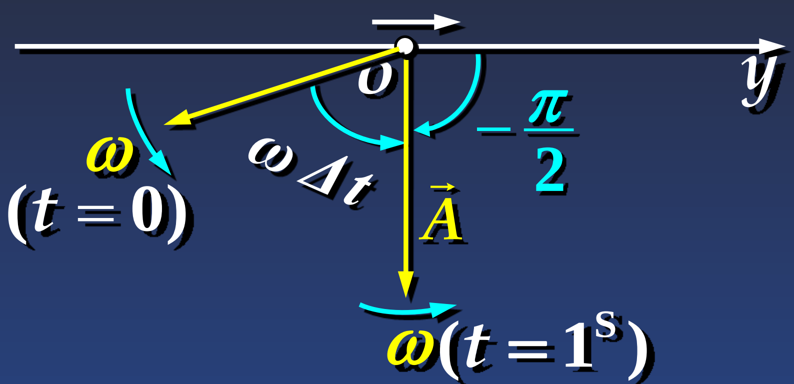

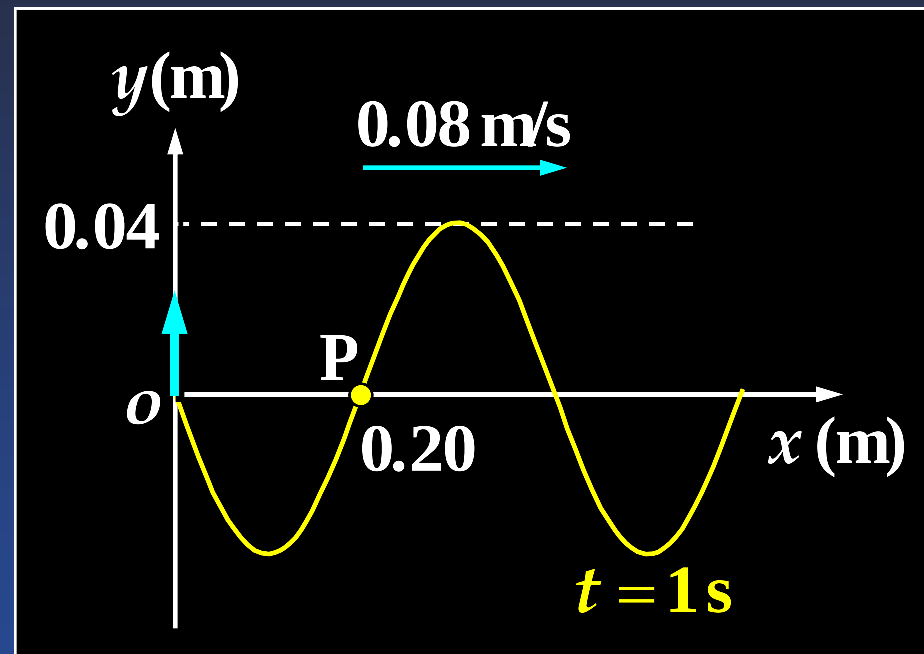

$$\omega\Delta t = \frac{2\pi}{5}(1-0) = \frac{2\pi}{5}$$

$$\therefore \varphi_o = -\frac{\pi}{2} - \frac{2\pi}{5} = -\frac{9\pi}{10}$$

答案：$y = 0.04\cos\left[\frac{2\pi}{5}\left(t - \frac{x}{0.08}\right) - \frac{9\pi}{10}\right]$

归纳:

1. 简谐波的波函数(波动方程):

$y = A\cos\left[\omega\left(t \mp \dfrac{x}{u}\right) + \varphi_o\right]$ 及其他几种标准形式。

$$\omega\Delta t = \frac{2\pi}{5}(1-0) = \frac{2\pi}{5}$$

$$\therefore \varphi_o = -\frac{\pi}{2} - \frac{2\pi}{5} = -\frac{9\pi}{10}$$

答案：$y = 0.04\cos\left[\frac{2\pi}{5}\left(t - \frac{x}{0.08}\right) - \frac{9\pi}{10}\right]$

归纳:

1. 简谐波的波函数(波动方程):

$$y = A\cos\left[\omega\left(t \mp \frac{x}{u}\right) + \varphi_o\right]$$及其他几种标准形式。

2. 波动方程的微分形式:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{u^2}\frac{\partial^2 y}{\partial t^2} \qquad \nabla^2 \psi = \frac{1}{u^2}\frac{\partial^2 \psi}{\partial t^2}$$

3. 波方程的求解。

(The end)