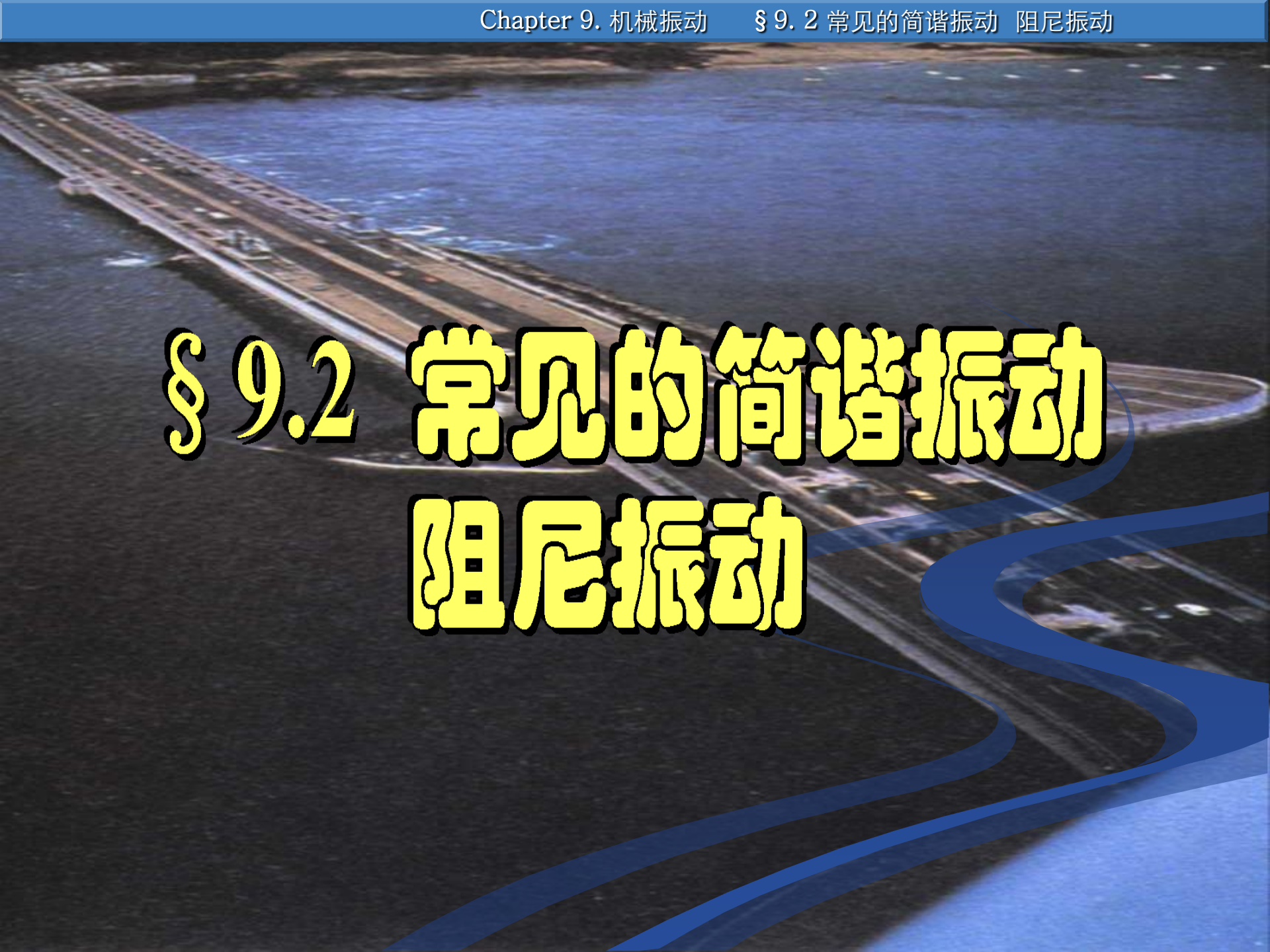


§ 9.2 常见的简谐振动 阻尼振动

An aerial photograph of a long bridge spanning a body of water. The bridge has multiple lanes and is supported by several piers. The water is dark blue, and the sky is a lighter blue. At the bottom of the image, there are decorative, stylized blue wavy lines that resemble a river or a path.

一、简谐振动的判断

满足下列条件之一的振动即为简谐振动：

$$\blacktriangle x = A \cos(\omega t + \varphi)$$

$$\blacktriangle F = -kx$$

$$\blacktriangle \ddot{x} + \omega^2 x = 0$$

F 为合外力， x 为离开平衡位置的位移， k 为常数。

二、常见的简谐振动

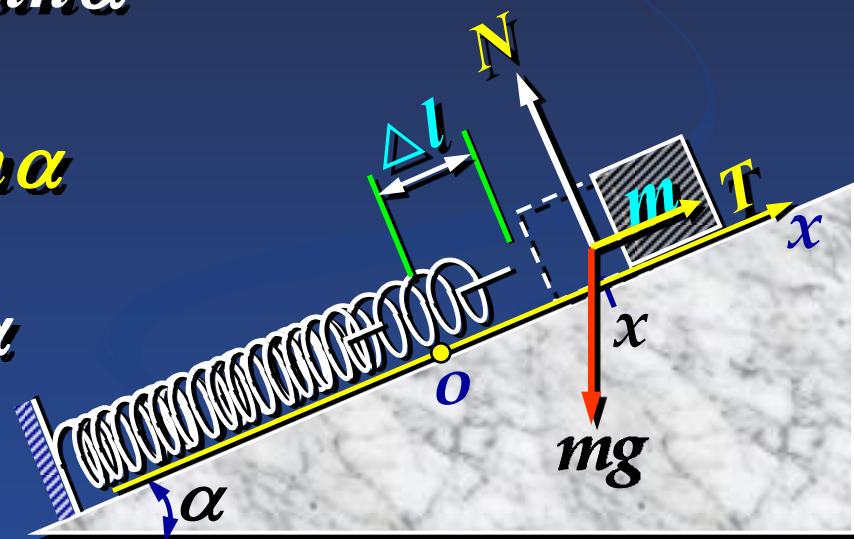
1. 光滑斜面上的弹簧振子:

平衡位置 o 处: $k \cdot \Delta l = mg \cdot \sin \alpha$

在 x 处: 合力 $F = T - mg \cdot \sin \alpha$

$$T = -k(x - \Delta l) = -kx + mg \cdot \sin \alpha$$

$$\therefore F = -kx$$



$$\blacktriangle x = A \cos(\omega t + \varphi) \quad \blacktriangle F = -kx \quad \blacktriangle \ddot{x} + \omega^2 x = 0$$

二、常见的简谐振动

1. 光滑斜面上的弹簧振子:

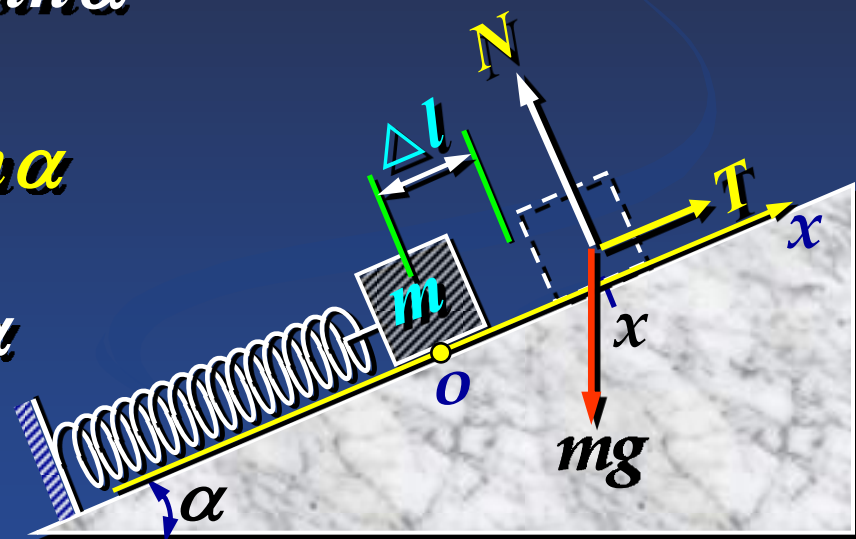
平衡位置 o 处: $k \cdot \Delta l = mg \cdot \sin \alpha$

在 x 处: 合力 $F = T - mg \cdot \sin \alpha$

$$T = -k(x - \Delta l) = -kx + mg \cdot \sin \alpha$$

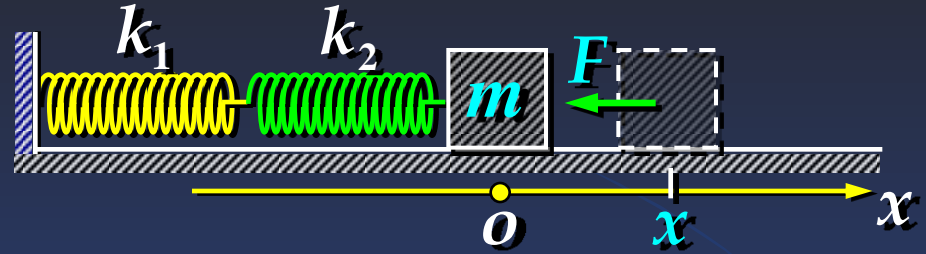
$$\therefore F = -kx$$

即作简谐运动: $\omega = \sqrt{\frac{k}{m}}$, $T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}}$



2. 光滑平面上的复合弹簧:

$$\begin{cases} x_1 + x_2 = x \\ k_1 x_1 = k_2 x_2 \end{cases}$$



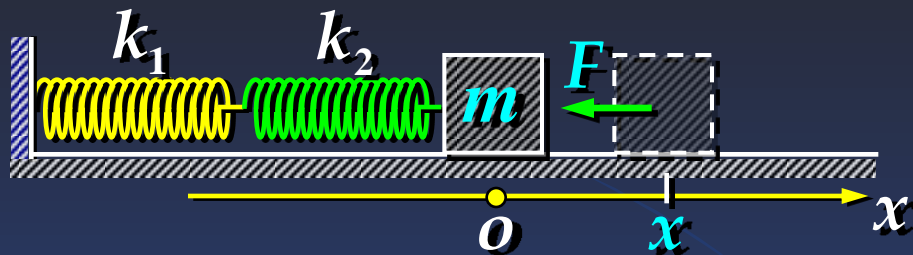
$$F = -k_2 x_2 = -\frac{k_1 k_2}{k_1 + k_2} x = -kx \quad \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{k_1 k_2}{m(k_1 + k_2)}}$$

$$\therefore F = -kx$$

即作简谐运动: $\omega = \sqrt{\frac{k}{m}}, T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}}$

2. 光滑平面上的复合弹簧:

$$\begin{cases} x_1 + x_2 = x \\ k_1 x_1 = k_2 x_2 \end{cases}$$



$$F = -k_2 x_2 = -\frac{k_1 k_2}{k_1 + k_2} x = -kx \quad \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{k_1 k_2}{m(k_1 + k_2)}}$$

试证明下图中系统角频率为:

$$\omega = \sqrt{\frac{k_1 + k_2}{m}} \quad ?$$



3. 复摆与单摆:

设复摆作小角度摆动。

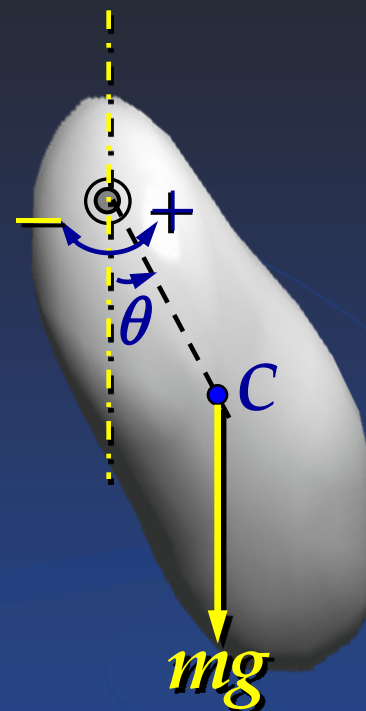
$$M = J\alpha = J \frac{d^2\theta}{dt^2} = J\ddot{\theta} \quad (M \text{ 为重力矩})$$

$$M = -mgl \cdot \sin\theta \approx -mgl\theta$$

$$\longrightarrow \ddot{\theta} + \frac{mgl}{J}\theta = 0 \quad \ddot{\theta} + \omega^2\theta = 0$$

$$\theta = \theta_0 \cos(\omega t + \varphi)$$

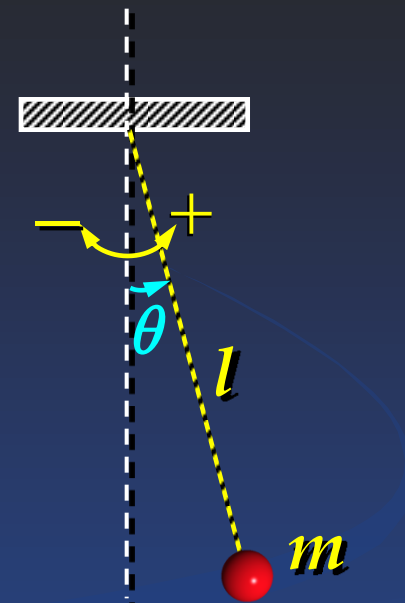
$$\omega = \sqrt{\frac{mgl}{J}}, T = 2\pi \sqrt{\frac{J}{mgl}}$$



对作小角度摆动的单摆: $J = ml^2$

$$\omega = \sqrt{\frac{g}{l}}, \quad T = 2\pi \sqrt{\frac{l}{g}}$$

$$\theta = \theta_{\max} \cos(\omega t + \varphi)$$



$$\longrightarrow \ddot{\theta} + \frac{mgl}{J} \theta = 0 \quad \ddot{\theta} + \omega^2 \theta = 0$$

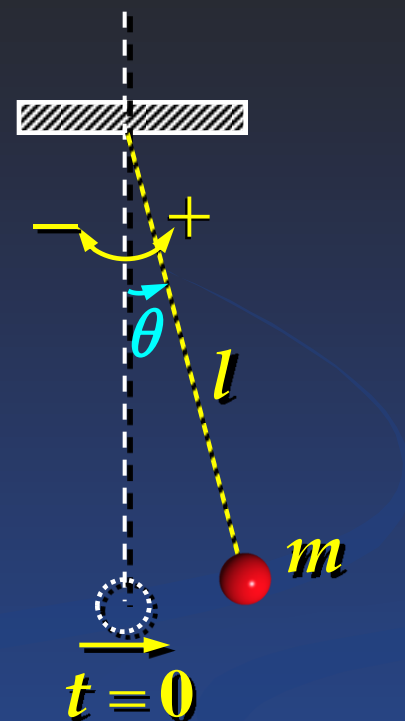
$$\theta = \theta_0 \cos(\omega t + \varphi)$$

$$\omega = \sqrt{\frac{mgl}{J}}, \quad T = 2\pi \sqrt{\frac{J}{mgl}}$$

对作小角度摆动的单摆: $J = ml^2$

$$\omega = \sqrt{\frac{g}{l}}, \quad T = 2\pi \sqrt{\frac{l}{g}}$$

$$\theta = \theta_{\max} \cos(\omega t + \varphi)$$



例 若单摆最大摆角为 5° ，起始状态

如图，则由旋转矢量图可知：

$$\varphi = -\frac{\pi}{2}, \quad \theta = \frac{\pi}{36} \cos\left(\sqrt{\frac{g}{l}} t - \frac{\pi}{2}\right)$$

$$\theta_{\max} = \sqrt{\theta_0^2 + \left(\frac{\omega_0}{\omega}\right)^2}$$

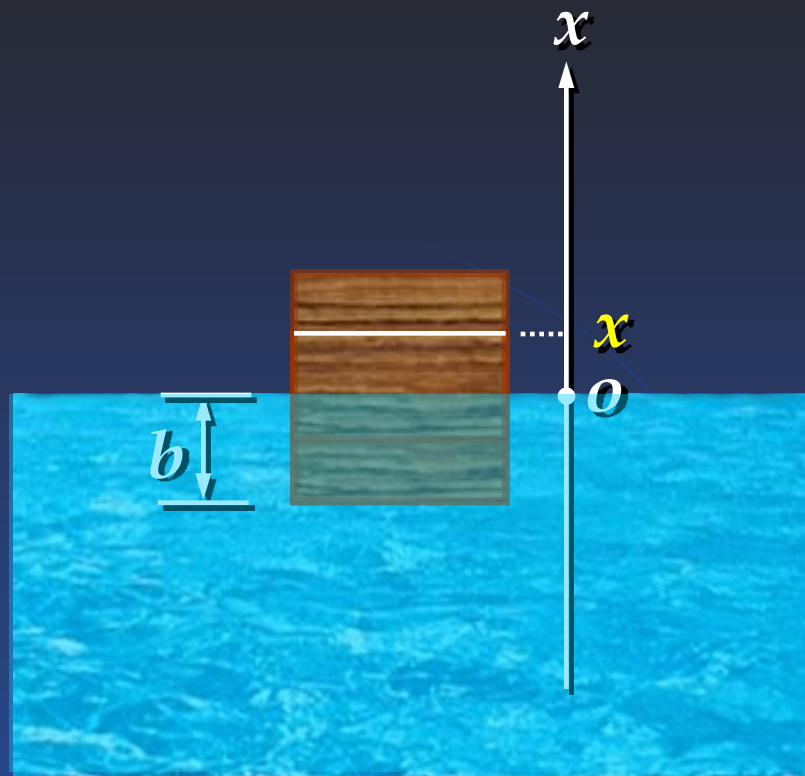
4. 水中浮块:

忽略水的阻力, 设:

木块: m, ρ, S ; 水: ρ'

平衡时: $mg = Sb\rho'g$

x 处所受合力:



$$\varphi = -\frac{\pi}{2}, \quad \theta = \frac{\pi}{36} \cos\left(\sqrt{\frac{g}{l}} t - \frac{\pi}{2}\right)$$

$$\theta_{\max} = \sqrt{\theta_0^2 + \left(\frac{\omega_0}{\omega}\right)^2}$$

4. 水中浮块:

忽略水的阻力, 设:

木块: m, ρ, S ; 水: ρ'

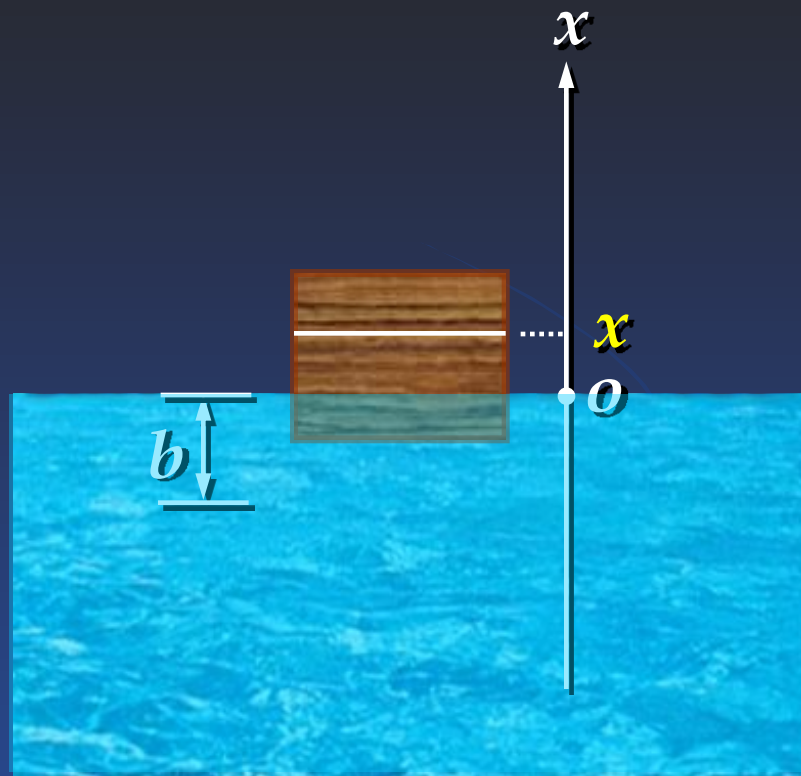
平衡时: $mg = Sb\rho'g$

x 处所受合力:

$$F = F_{\text{浮}} - mg$$

$$= S(b-x)\rho'g - mg$$

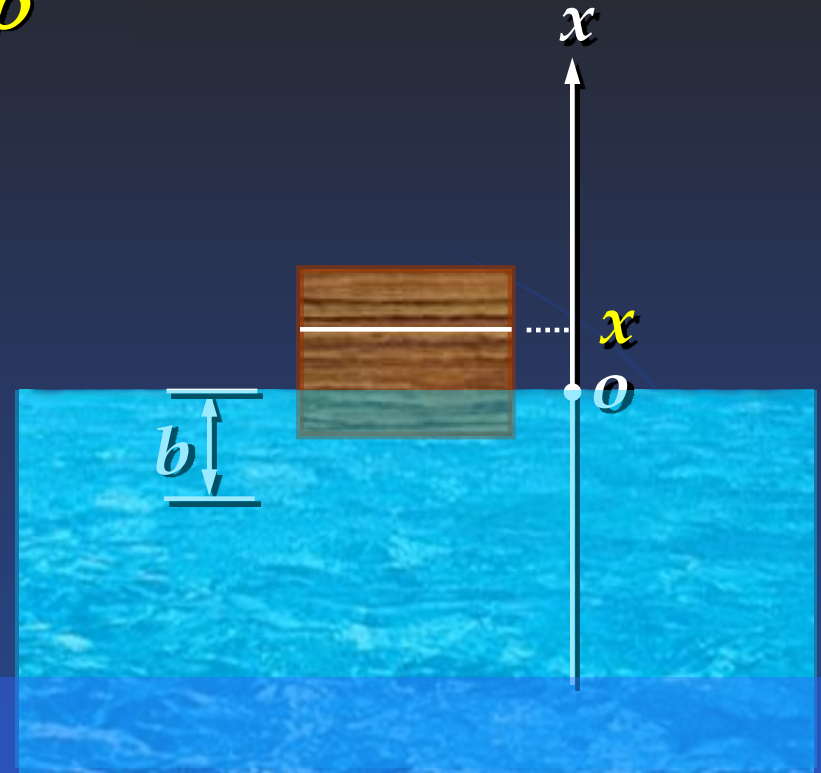
$$= -S\rho'g \cdot x = -kx \quad (k = S\rho'g) \quad \text{作简谐振动!}$$



$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{S\rho'g}{m}} \quad m = Sb\rho'$$

$$\omega = \sqrt{\frac{g}{b}}, \quad T = 2\pi \sqrt{\frac{b}{g}}$$

$$x = A \cos\left(\sqrt{\frac{g}{b}} t + \varphi\right)$$



$$F = F_{\text{浮}} - mg$$

$$= S(b-x)\rho'g - mg$$

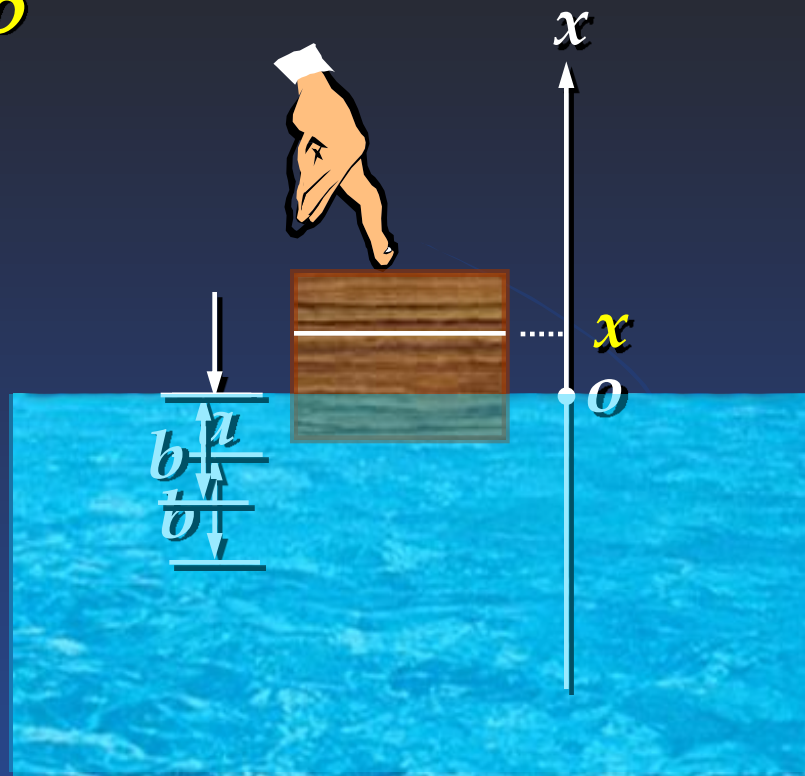
$$= -S\rho'g \cdot x = -kx \quad (k = S\rho'g) \quad \text{作简谐振动!}$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{S\rho'g}{m}} \quad m = Sb\rho'$$

$$\omega = \sqrt{\frac{g}{b}}, \quad T = 2\pi \sqrt{\frac{b}{g}}$$

$$x = A \cos\left(\sqrt{\frac{g}{b}} t + \varphi\right)$$

若 $t=0$ 时按住木块恰使其完全浸在水中，在由静止释放，则：



$$A = a, \quad \varphi = \pi \quad x = a \cdot \cos\left(\sqrt{\frac{g}{b}} t + \pi\right)$$

A diagram showing the initial condition for the oscillation. A horizontal axis x is shown with the origin O . A yellow arrow labeled \vec{A} points to the left from O , representing the amplitude A . A green arrow labeled $t=0$ points downwards. A blue arc indicates a phase shift of π from the positive x axis.

三、阻尼振动 *

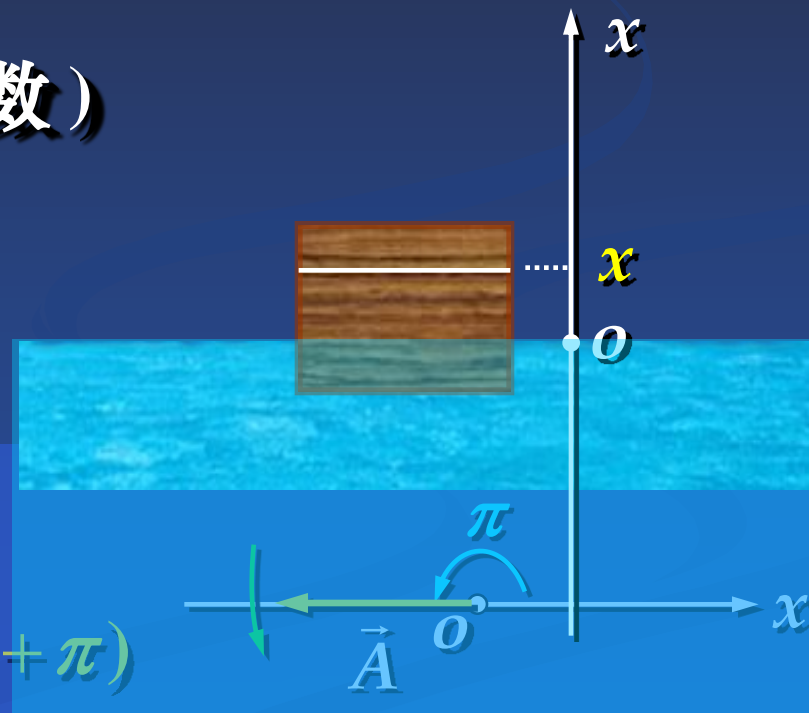
阻尼振动：振幅随时间而减小的振动。

在粘性的介质中，物体速度较小时，阻力有下列关系：

$$F_r = -Cv \quad (C \text{ 为阻力常数})$$

$$F = -kx + F_r = -kx - Cv = ma$$

$$A = a, \quad \varphi = \pi \quad x = a \cdot \cos\left(\sqrt{\frac{g}{b}} t + \pi\right)$$



三、阻尼振动 *

阻尼振动：振幅随时间而减小的振动。

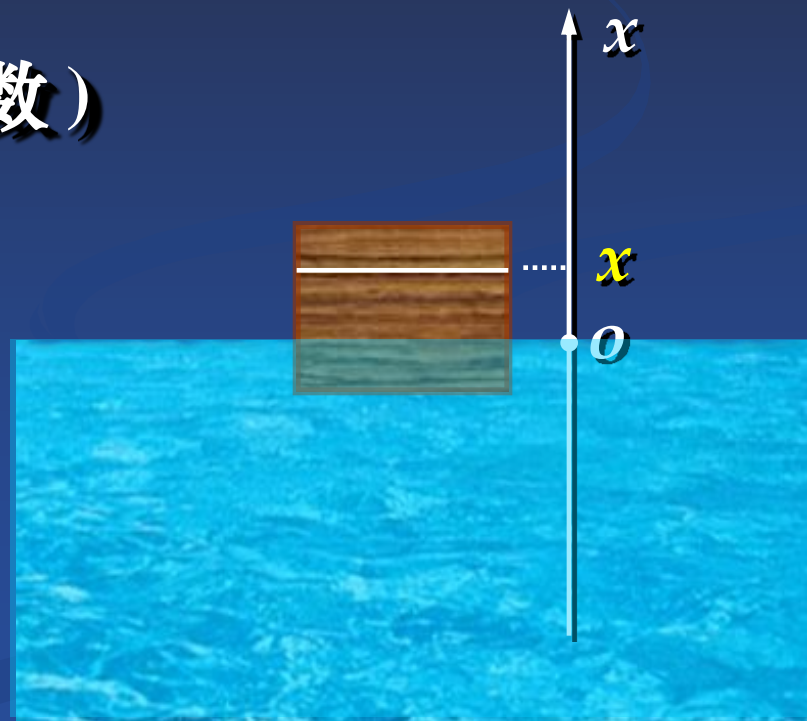
在粘性的介质中，物体速度较小时，阻力有下列关系：

$$F_r = -Cv \quad (C \text{ 为阻力常数})$$

$$F = -kx + F_r = -kx - Cv = ma$$

$$m\ddot{x} + C\dot{x} + kx = 0$$

$$\ddot{x} + 2\delta\dot{x} + \omega_0^2 x = 0$$



其中 $\delta = \frac{C}{2m}$ (阻尼系数)

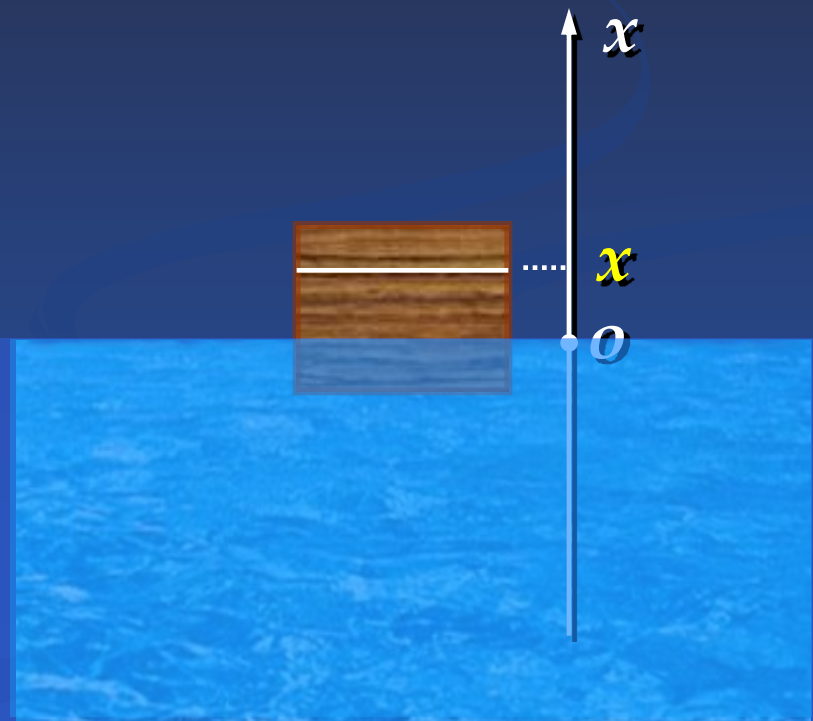
$\omega_0 = \sqrt{\frac{k}{m}}$ (系统固有角频率)

$x = Ae^{-\delta t} \cos(\omega t + \varphi)$

$\omega = \sqrt{\omega_0^2 - \delta^2}$

$m\ddot{x} + C\dot{x} + kx = 0$

$\ddot{x} + 2\delta\dot{x} + \omega_0^2 x = 0$



其中 $\delta = \frac{C}{2m}$ (阻尼系数)

$$\omega_0 = \sqrt{\frac{k}{m}} \text{ (系统固有角频率)}$$

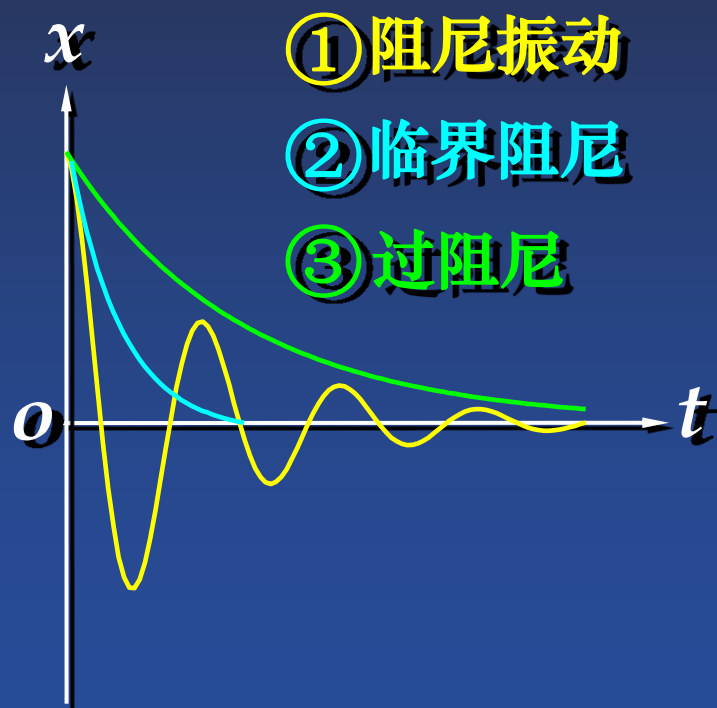
$$x = Ae^{-\delta t} \cos(\omega t + \varphi)$$

$$\omega = \sqrt{\omega_0^2 - \delta^2}$$

① $\delta < \omega_0$: 阻尼振动

② $\delta = \omega_0$: 临界阻尼

③ $\delta > \omega_0$: 过阻尼



归纳:

1. 简谐振动满足:

▲ $x = A \cos(\omega t + \varphi)$

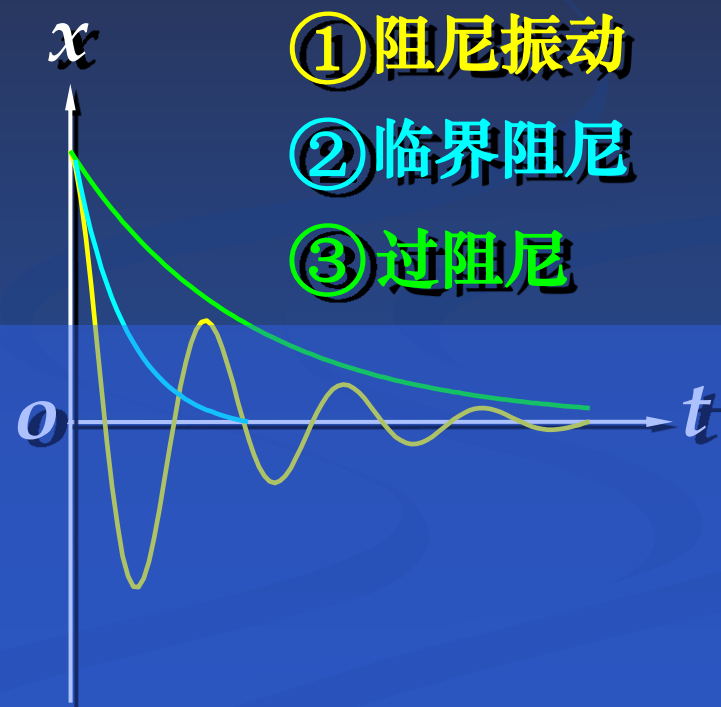
▲ $F = -kx$

▲ $\ddot{x} + \omega^2 x = 0$

① $\delta < \omega_0$: 阻尼振动

② $\delta = \omega_0$: 临界阻尼

③ $\delta > \omega_0$: 过阻尼



归纳:

1. 简谐振动满足:

▲ $x = A \cos(\omega t + \varphi)$

▲ $F = -kx$

▲ $\ddot{x} + \omega^2 x = 0$

***2. 阻尼振动的三种状态:**

阻尼、临界阻尼、过阻尼

(请看录像)