A.1.4. Proof of Theorem 4.1

Theorem A.3. *(Stability guarantee) For a candidate controller* $\mathbf{u}$ *and the stable controller space* $\mathcal{U}(V) = \{\boldsymbol{u} : \mathcal{L}_{\boldsymbol{f_u}} V + V \leq 0\}$*, we define the projection operator as,*

$$\pi(\boldsymbol{u}, \mathcal{U}(V)) \triangleq \boldsymbol{u} - \frac{\max(0, \mathcal{L}_{\boldsymbol{f_u}} V - V)}{\|\nabla V\|^2} \cdot \nabla V.$$

If the controller has affine actuator, then we have $\pi(\boldsymbol{u}, \mathcal{U}(V)) \in \mathcal{U}(V)$*, the projected controller is Lipschitz continuous over the state space* $\mathcal{D}$ *if and only if* $\mathcal{D}$ *is bounded. Furthermore, under the triggering mechanism*

$$\begin{aligned} &\nabla V(\boldsymbol{x}) \cdot [\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x} + \boldsymbol{e})) - \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}))] - \sigma V(\boldsymbol{x}) = 0, \\ &\sigma \in (0, 1), \boldsymbol{e} = \boldsymbol{x}(t_k) - \boldsymbol{x}(t), t \in [t_k, t_{k+1}) \end{aligned}$$

the controlled system under $\pi(\boldsymbol{u}, \mathcal{U})$ *is assured exponential stable with decay rate* $1 - \sigma$*, and the inter-event time has positive lower bound.*

Proof. To begin with, we check the inequality constraint in $\mathcal{U}(V)$ is satisfied by the projection element, that is

$$\mathcal{L}_{\boldsymbol{f_u}} V\big|_{\boldsymbol{u}=\pi(\boldsymbol{u}, \mathcal{U}(V))} \leq -V.$$

Since the controller has affine actuator, from the definition of the Lie derivative operator, we have

$$\begin{aligned} \mathcal{L}_{\boldsymbol{f_u}} V\big|_{\boldsymbol{u}=\pi(\boldsymbol{u}, \mathcal{U}(V))} &= \nabla V \cdot (\boldsymbol{f} + \boldsymbol{u} - \frac{\max(0, \mathcal{L}_{\boldsymbol{u}} V + V)}{\|\nabla V\|^2} \cdot \nabla V) \\ &= \nabla V \cdot (\boldsymbol{f} + \boldsymbol{u}) - \nabla V \cdot \frac{\max(0, \mathcal{L}_{\boldsymbol{u}} V + V)}{\|\nabla V\|^2} \cdot \nabla V \\ &= \mathcal{L}_{\boldsymbol{u}} V - \max(0, \mathcal{L}_{\boldsymbol{u}} V + V) \leq -V. \end{aligned}$$

Next, we show the equivalent condition of the Lipschitz continuity of projection element. Notice that $\boldsymbol{u} \in \mathrm{Lip}(\mathcal{D})$, then we have

$$\hat{\pi}(\boldsymbol{u}, \mathcal{U}(V)) \in \mathrm{Lip}(\mathcal{D}) \iff \frac{\max(0, \mathcal{L}_{\boldsymbol{u}} V + V)}{\|\nabla V\|^2} \cdot \nabla V \in \mathrm{Lip}(\mathcal{D}).$$

Since $\|\dfrac{\nabla V}{\|\nabla V\|}\|$ is a continuous unit vector, and naturally is Lipschitz continuous, we only need to consider the remaining term $\dfrac{\max(0, \mathcal{L}_{\boldsymbol{u}} V + V)}{\|\nabla V\|}$. According to the definition, all the functions occured in this term are continuous, so we only need to bound this term to obtain the global Lipschitz continuity, that is

$$\frac{\max(0, \mathcal{L}_{\boldsymbol{u}} V + V)}{\|\nabla V\|} \in \mathrm{Lip}(\mathcal{D}) \iff \sup_{\boldsymbol{x} \in \mathcal{D}} \frac{\max(0, \mathcal{L}_{\boldsymbol{u}} V + V)}{\|\nabla V\|} < +\infty.$$

When $\mathcal{L}_{\boldsymbol{u}} V \leq -V$, obviously we have $\max(0, \mathcal{L}_{\boldsymbol{u}} V + V) = 0 < +\infty$, otherwise, since $V \geq \varepsilon \|\boldsymbol{x}\|^p$, we have

$$\mathcal{L}_{\boldsymbol{u}} V + V \geq \mathcal{L}_{\boldsymbol{u}} V + \varepsilon \|\boldsymbol{x}\|^p \approx \mathcal{O}(\|\boldsymbol{x}\|^p) \to \infty (\|\boldsymbol{x}\| \to \infty).$$

Thus, we have

$$\sup_{\boldsymbol{x} \in \mathcal{D}} \frac{\max(0, \mathcal{L}_{\boldsymbol{u}} V + V)}{\|\nabla V\|} < +\infty \iff \sup_{\boldsymbol{x} \in \mathcal{D}} \|\boldsymbol{x}\| < +\infty.$$

The positive lower bound of the inter-event time comes from the Theorem 3.2. We complete the proof.