Badass Title

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The fractal string we studied the most in particular is the Cantor String



Cantor String



Inner-Tubular Neighborhood

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Counting function:
$$N_{\Omega}(x) = \sum_{\ell_j^{-1} \le x} 1$$

More Notation:
$$\mathcal{W}_{\Omega}(x) = \sum_{j:\ell_j^{-1} > x} \ell_j$$

Now we can write: $V(\epsilon) = 2\epsilon \cdot N_{\Omega}\left(\frac{1}{2\epsilon}\right) + \mathcal{W}_{\Omega}\left(\frac{1}{2\epsilon}\right)$



Cantor String Volume



Inner Minkowski Dimension of a Fractal String

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Dimension of Ω:
$$D_{\Omega} = \inf \left\{ \alpha \geq 0 \mid V(\epsilon) = O\left(\epsilon^{1-\alpha}\right) \text{ as } \epsilon \to 0^+ \right\}$$



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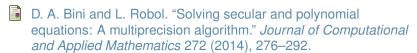
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Cantor String Dimension: $D_{cs} = \log_3 2$



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