Badass Title

Isaac Ashton, Joseph Ingenito, Peter Tonuzi

The College of New Jersey

What is a Fractal String

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The fractal string we studied the most in particular is the **Cantor String**

Cantor String

Inner-Tubular Neighborhood

Definition 2

Inner-tubular neighborhood: $V(\epsilon) = \mathrm{vol}_1\{x \in \Omega \mid d(x, \partial\Omega) < \epsilon\}$

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Counting function: $N_\Omega(x) = \sum_{\ell_j^{-1} \leq x} 1$

More Notation: $\mathcal{W}_\Omega(x) = \sum_{j:\ell_j^{-1} > x} \ell_j$

Now we can write: $V(\epsilon) = 2\epsilon \cdot N_\Omega\left(\frac{1}{2\epsilon}\right) + \mathcal{W}_\Omega\left(\frac{1}{2\epsilon}\right)$

Cantor String Volume

Inner Minkowski Dimension of a Fractal String

Definition 3

Dimension of Ω: $D_\Omega = \inf\left\{\alpha \geq 0 \mid V(\epsilon) = O\left(\epsilon^{1-\alpha}\right) \text{ as } \epsilon \to 0^+\right\}$

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Cantor String Dimension: $D_{cs} = \log_3 2$

Fractal lawn

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A **fractal lawn** $\mathcal{L}^2$ is a bounded open subset $\Omega = \{(x, y) : x \in \Omega_1$ and $y \in \Omega_2\}$ of $\mathbb{R}^2$.

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$$\Omega_1 = \bigcup_{j=1}^{\infty} \ell_j,\ \Omega_2 = \bigcup_{i=1}^{\infty} l_i$$

$$\ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots \geq 0,\ l_1 \geq l_2 \geq l_3 \geq \cdots \geq 0$$

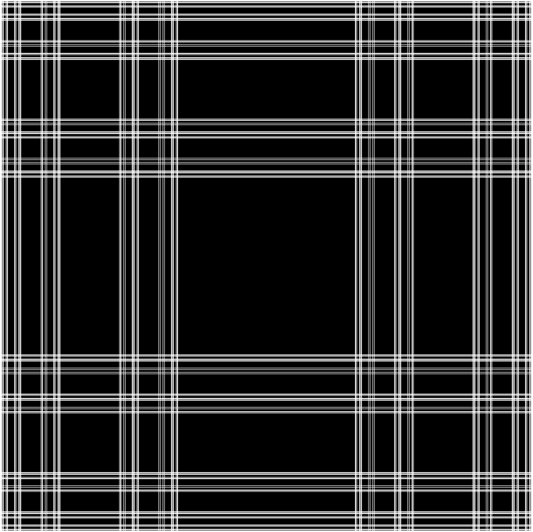
Cantor lawn

Dimension of fractal lawn

Inner-tubular neighborhood: $V(\epsilon) = \mathrm{vol}_2\{x \in \Omega \mid d(x, \partial\Omega) < \epsilon\}$

Counting functions: $N_{\Omega_1}(x) = \sum_{\ell_j^{-1} \leq x}$, $N_{\Omega_2}(x) = \sum_{l_j^{-1} \leq x}$

Cantor Lawn Volume GIF



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