Badass Title

Isaac Ashton, Joseph Ingenito, Peter Tonuzi

The College of New Jersey

What is a Fractal String

Definition 1

A **fractal string** Ω is a bounded open subset of the real line.

What is a Fractal String

Definition 1

A **fractal string** Ω is a bounded open subset of the real line.

$$\Omega = \bigcup_{j=1}^{\infty} \ell_j, \qquad \mathcal{L} = \mathrm{Vol}_1(\Omega) = \sum_{j=1}^{\infty} \ell_j \implies \lim_{j \to \infty} \ell_j = 0.$$

What is a Fractal String

Definition 1

A **fractal string** Ω is a bounded open subset of the real line.

$$\Omega = \bigcup_{j=1}^{\infty} \ell_j, \qquad \mathcal{L} = \mathrm{Vol}_1(\Omega) = \sum_{j=1}^{\infty} \ell_j \implies \lim_{j \to \infty} \ell_j = 0.$$

$$\ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots \geq 0$$

What is a Fractal String

Definition 1

A **fractal string** Ω is a bounded open subset of the real line.

$$\Omega = \bigcup_{j=1}^{\infty} \ell_j, \qquad \mathcal{L} = \mathrm{Vol}_1(\Omega) = \sum_{j=1}^{\infty} \ell_j \implies \lim_{j\to\infty} \ell_j = 0.$$

$$\ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots \geq 0$$

The fractal string we studied the most in particular is the **Cantor String**

Cantor String

Inner-Tubular Neighborhood

Definition 2

Inner-tubular neighborhood: $V(\epsilon) = \mathrm{vol}_1\{x \in \Omega \mid d(x, \partial\Omega) < \epsilon\}$

Inner-Tubular Neighborhood

Definition 2

Inner-tubular neighborhood: $V(\epsilon) = \mathrm{vol}_1\{x \in \Omega \mid d(x, \partial\Omega) < \epsilon\}$

Counting function: $N_\Omega(x) = \sum_{\ell_j^{-1} \leq x} 1$

More Notation: $\mathcal{W}_\Omega(x) = \sum_{j:\ell_j^{-1} > x} \ell_j$

Now we can write: $V(\epsilon) = 2\epsilon \cdot N_\Omega(2\epsilon) + \mathcal{W}_\Omega(2\epsilon)$

Cantor String Volume

Inner Minkowski Dimension of a Fractal String

Definition 3

Dimension of Ω: $D_\Omega = \inf\left\{\alpha \geq 0 \mid V(\epsilon) = O\left(\epsilon^{1-\alpha}\right) \text{ as } \epsilon \to 0^+\right\}$

Inner Minkowski Dimension of a Fractal String

Definition 3

Dimension of Ω: $D_\Omega = \inf\left\{\alpha \geq 0 \mid V(\epsilon) = O\left(\epsilon^{1-\alpha}\right) \text{ as } \epsilon \to 0^+\right\}$

In other terms: $D_\Omega =$ the smallest α st $\displaystyle\lim_{\epsilon\to 0^+} \frac{V(\epsilon)}{\epsilon^{1-\alpha}} = c,\ c \in \mathbb{R}$

Inner Minkowski Dimension of a Fractal String

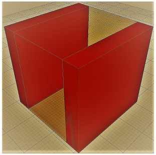
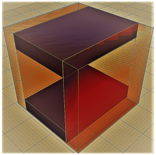
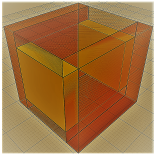
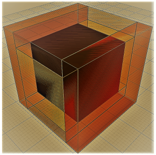
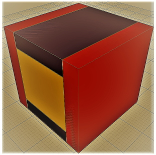
Definition 3

Dimension of Ω: $D_\Omega = \inf\left\{\alpha \geq 0 \mid V(\epsilon) = O\left(\epsilon^{1-\alpha}\right) \text{ as } \epsilon \to 0^+\right\}$

In other terms: $D_\Omega =$ the smallest α st $\lim\limits_{\epsilon \to 0^+} \dfrac{V(\epsilon)}{\epsilon^{1-\alpha}} = c,\ c \in \mathbb{R}$

Cantor String Dimension: $D_{cs} = \log_3 2$

3 Dimensional Fractal Objects

$$
\begin{aligned}
V(\epsilon) = {} & 2\epsilon \cdot N_{\Omega_1}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(2\epsilon)} p_j\right) \cdot \left(\sum_{j=1}^{N_{\Omega_3}(2\epsilon)} q_j\right) + \\
& 2\epsilon \cdot N_{\Omega_2}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(2\epsilon)} \ell_j - 2\epsilon\right) \cdot \left(\sum_{j=1}^{N_{\Omega_3}(2\epsilon)} q_j\right) + \\
& 2\epsilon \cdot N_{\Omega_3}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(2\epsilon)} p_j - 2\epsilon\right) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(2\epsilon)} \ell_j - 2\epsilon\right) + \\
& \mathcal{W}_1\mathcal{L}_2\mathcal{L}_3 + \mathcal{L}_1\mathcal{W}_2\mathcal{L}_3 + \mathcal{L}_1\mathcal{L}_2\mathcal{W}_3 - \mathcal{L}_1\mathcal{W}_2\mathcal{W}_3 - \mathcal{W}_1\mathcal{L}_2\mathcal{W}_3 - \\
& \mathcal{W}_1\mathcal{W}_2\mathcal{L}_3 + \mathcal{W}_1\mathcal{W}_2\mathcal{W}_3
\end{aligned}
$$

3 Dimensional Fractal Volume

$$V(\epsilon) = 2\epsilon \cdot N_{\Omega_1} \cdot (\mathcal{L}_2 - \mathcal{W}_3) \cdot (\mathcal{L}_3 - \mathcal{W}_3) + 2\epsilon \cdot N_{\Omega_2} \cdot (\mathcal{L}_1 - \mathcal{W}_1 - 2\epsilon \cdot N_{\Omega_1}) \cdot (\mathcal{L}_3 - \mathcal{W}_3) + 2\epsilon \cdot N_{\Omega_3} \cdot (\mathcal{L}_2 - \mathcal{W}_2 - 2\epsilon \cdot N_{\Omega_2}) \cdot (\mathcal{L}_1 - \mathcal{W}_1 - 2\epsilon \cdot N_{\Omega_1}) + \mathcal{W}_1\mathcal{L}_2\mathcal{L}_3 + \mathcal{L}_1\mathcal{W}_2\mathcal{L}_3 + \mathcal{L}_1\mathcal{L}_2\mathcal{W}_3 - \mathcal{L}_1\mathcal{W}_2\mathcal{W}_3 - \mathcal{W}_1\mathcal{L}_2\mathcal{W}_3 - \mathcal{W}_1\mathcal{W}_2\mathcal{L}_3 + \mathcal{W}_1\mathcal{W}_2\mathcal{W}_3$$

TCNJ THE COLLEGE OF NEW JERSEY

3 Dimensional Fractal Volume

$$V(\epsilon) = 2\epsilon \cdot N_{\Omega_1} \cdot (\mathcal{L}_2 - \mathcal{W}_3) \cdot (\mathcal{L}_3 - \mathcal{W}_3) + 2\epsilon \cdot N_{\Omega_2} \cdot (\mathcal{L}_1 - \mathcal{W}_1 - 2\epsilon \cdot N_{\Omega_1}) \cdot (\mathcal{L}_3 - \mathcal{W}_3) + 2\epsilon \cdot N_{\Omega_3} \cdot (\mathcal{L}_2 - \mathcal{W}_2 - 2\epsilon \cdot N_{\Omega_2}) \cdot (\mathcal{L}_1 - \mathcal{W}_1 - 2\epsilon \cdot N_{\Omega_1}) + \mathcal{W}_1 \mathcal{L}_2 \mathcal{L}_3 + \mathcal{L}_1 \mathcal{W}_2 \mathcal{L}_3 + \mathcal{L}_1 \mathcal{L}_2 \mathcal{W}_3 - \mathcal{L}_1 \mathcal{W}_2 \mathcal{W}_3 - \mathcal{W}_1 \mathcal{L}_2 \mathcal{W}_3 - \mathcal{W}_1 \mathcal{W}_2 \mathcal{L}_3 + \mathcal{W}_1 \mathcal{W}_2 \mathcal{W}_3$$

$$V(\epsilon) = \mathcal{L}_1 \mathcal{L}_2 V_3(\epsilon) + \mathcal{L}_1 V_2(\epsilon) \mathcal{L}_3 + V_1(\epsilon) \mathcal{L}_2 \mathcal{L}_3 - V_1(\epsilon) V_2(\epsilon) \mathcal{L}_3 - V_1(\epsilon) \mathcal{L}_2 V_3(\epsilon) - \mathcal{L}_1 V_2(\epsilon) V_3(\epsilon) + V_1(\epsilon) V_2(\epsilon) V_3(\epsilon)$$

4 Dimensional Fractal Objects

$$V(\epsilon) = 2\epsilon \cdot N_{\Omega_1}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(2\epsilon)} p_j\right) \cdot \left(\sum_{j=1}^{N_{\Omega_3}(2\epsilon)} q_j\right) \cdot \left(\sum_{j=1}^{N_{\Omega_4}(2\epsilon)} z_j\right) + 2\epsilon \cdot N_{\Omega_2}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(2\epsilon)} \ell_j - 2\epsilon\right) \cdot \left(\sum_{j=1}^{N_{\Omega_3}(2\epsilon)} q_j\right) \cdot \left(\sum_{j=1}^{N_{\Omega_4}(2\epsilon)} z_j\right) + 2\epsilon \cdot N_{\Omega_3}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(2\epsilon)} p_j - 2\epsilon\right) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(2\epsilon)} \ell_j - 2\epsilon\right) \cdot \left(\sum_{j=1}^{N_{\Omega_4}(2\epsilon)} z_j\right) + 2\epsilon \cdot N_{\Omega_4}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(2\epsilon)} p_j - 2\epsilon\right) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(2\epsilon)} \ell_j - 2\epsilon\right) \cdot \left(\sum_{j=1}^{N_{\Omega_3}(2\epsilon)} q_j - 2\epsilon\right) + \mathcal{W}_1\mathcal{L}_2\mathcal{L}_3\mathcal{L}_4 + \mathcal{L}_1\mathcal{W}_2\mathcal{L}_3\mathcal{L}_4 + \mathcal{L}_1\mathcal{L}_2\mathcal{W}_3\mathcal{L}_4 + \mathcal{L}_1\mathcal{L}_2\mathcal{L}_3\mathcal{W}_4 - \mathcal{L}_1\mathcal{L}_2\mathcal{W}_3\mathcal{W}_4 - \mathcal{L}_1\mathcal{W}_2\mathcal{L}_3\mathcal{W}_4 - \mathcal{L}_1\mathcal{W}_2\mathcal{W}_3\mathcal{L}_4 - \mathcal{W}_1\mathcal{L}_2\mathcal{L}_3\mathcal{W}_4 - \mathcal{W}_1\mathcal{L}_2\mathcal{W}_3\mathcal{L}_4 - \mathcal{W}_1\mathcal{W}_2\mathcal{L}_3\mathcal{L}_4 + \mathcal{L}_1\mathcal{W}_2\mathcal{W}_3\mathcal{W}_4 + \mathcal{W}_1\mathcal{L}_2\mathcal{W}_3\mathcal{W}_4 + \mathcal{W}_1\mathcal{W}_2\mathcal{L}_3\mathcal{W}_4 + \mathcal{W}_1\mathcal{W}_2\mathcal{W}_3\mathcal{L}_4 - \mathcal{W}_1\mathcal{W}_2\mathcal{W}_3\mathcal{W}_4$$

4 Dimensional Volume Formula

$$V^4(\epsilon) = \mathcal{L}_1\mathcal{L}_2\mathcal{L}_3 V_4(\epsilon) + \mathcal{L}_1\mathcal{L}_2 V_3(\epsilon)\mathcal{L}_4 + \mathcal{L}_1 V_2(\epsilon)\mathcal{L}_3\mathcal{L}_4 + V_1(\epsilon)\mathcal{L}_2\mathcal{L}_3\mathcal{L}_4 - V_1(\epsilon)V_2(\epsilon)\mathcal{L}_3\mathcal{L}_4 - V_1(\epsilon)\mathcal{L}_2 V_3(\epsilon)\mathcal{L}_4 - V_1(\epsilon)\mathcal{L}_2\mathcal{L}_3 V_4(\epsilon) - \mathcal{L}_1 V_2(\epsilon)V_3(\epsilon)\mathcal{L}_4 - \mathcal{L}_1 V_2(\epsilon)\mathcal{L}_3 V_4(\epsilon) - \mathcal{L}_1\mathcal{L}_2 V_3(\epsilon)V_4(\epsilon) + V_1(\epsilon)V_2(\epsilon)V_3(\epsilon)\mathcal{L}_4 + V_1(\epsilon)V_2(\epsilon)\mathcal{L}_3 V_4(\epsilon) + V_1(\epsilon)\mathcal{L}_2 V_3(\epsilon)V_4(\epsilon) + \mathcal{L}_1 V_2(\epsilon)V_3(\epsilon)V_4(\epsilon) - V_1(\epsilon)V_2(\epsilon)V_3(\epsilon)V_4(\epsilon)$$

4 Dimensional Volume Formula

$$V^4(\epsilon) = \mathcal{L}_1\mathcal{L}_2\mathcal{L}_3 V_4(\epsilon) + \mathcal{L}_1\mathcal{L}_2 V_3(\epsilon)\mathcal{L}_4 + \mathcal{L}_1 V_2(\epsilon)\mathcal{L}_3\mathcal{L}_4 + V_1(\epsilon)\mathcal{L}_2\mathcal{L}_3\mathcal{L}_4 - V_1(\epsilon)V_2(\epsilon)\mathcal{L}_3\mathcal{L}_4 - V_1(\epsilon)\mathcal{L}_2 V_3(\epsilon)\mathcal{L}_4 - V_1(\epsilon)\mathcal{L}_2\mathcal{L}_3 V_4(\epsilon) - \mathcal{L}_1 V_2(\epsilon)V_3(\epsilon)\mathcal{L}_4 - \mathcal{L}_1 V_2(\epsilon)\mathcal{L}_3 V_4(\epsilon) - \mathcal{L}_1\mathcal{L}_2 V_3(\epsilon)V_4(\epsilon) + V_1(\epsilon)V_2(\epsilon)V_3(\epsilon)\mathcal{L}_4 + V_1(\epsilon)V_2(\epsilon)\mathcal{L}_3 V_4(\epsilon) + V_1(\epsilon)\mathcal{L}_2 V_3(\epsilon)V_4(\epsilon) + \mathcal{L}_1 V_2(\epsilon)V_3(\epsilon)V_4(\epsilon) - V_1(\epsilon)V_2(\epsilon)V_3(\epsilon)V_4(\epsilon)$$

$$V^3(\epsilon) = \mathcal{L}_1\mathcal{L}_2 V_3(\epsilon) + \mathcal{L}_1 V_2(\epsilon)\mathcal{L}_3 + V_1(\epsilon)\mathcal{L}_2\mathcal{L}_3 - V_1(\epsilon)V_2(\epsilon)\mathcal{L}_3 - V_1(\epsilon)\mathcal{L}_2 V_3(\epsilon) - \mathcal{L}_1 V_2(\epsilon)V_3(\epsilon) + V_1(\epsilon)V_2(\epsilon)V_3(\epsilon)$$

$$V^2(\epsilon) = \mathcal{L}_1 V_2(\epsilon) + V_1(\epsilon)\mathcal{L}_2 - V_1(\epsilon)V_2(\epsilon)$$

Algebraic Manipulations

$$
\begin{aligned}
V^2(\epsilon) &= \mathcal{L}_1 V_2(\epsilon) + V_1(\epsilon)\mathcal{L}_2 - V_1(\epsilon) V_2(\epsilon) \\
&= (-1) \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) + \mathcal{L}_1 \mathcal{L}_2
\end{aligned}
$$

Algebraic Manipulations

$$V^2(\epsilon) = \mathcal{L}_1 V_2(\epsilon) + V_1(\epsilon)\mathcal{L}_2 - V_1(\epsilon) V_2(\epsilon)$$

$$= (-1) \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) + \mathcal{L}_1 \mathcal{L}_2$$

$$V^3(\epsilon) = \mathcal{L}_1 \mathcal{L}_2 V_3(\epsilon) + \mathcal{L}_1 V_2(\epsilon)\mathcal{L}_3 + V_1(\epsilon)\mathcal{L}_2\mathcal{L}_3 - V_1(\epsilon) V_2(\epsilon)\mathcal{L}_3 - V_1(\epsilon)\mathcal{L}_2 V_3(\epsilon) - \mathcal{L}_1 V_2(\epsilon) V_3(\epsilon) + V_1(\epsilon) V_2(\epsilon) V_3(\epsilon)$$

$$= (-1)^2 \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) \cdot (V_3(\epsilon) - \mathcal{L}_3) + \mathcal{L}_1 \mathcal{L}_2 \mathcal{L}_3$$

Algebraic Manipulations

$$V^4(\epsilon) = \mathcal{L}_1\mathcal{L}_2\mathcal{L}_3 V_4(\epsilon) + \mathcal{L}_1\mathcal{L}_2 V_3(\epsilon)\mathcal{L}_4 + \mathcal{L}_1 V_2(\epsilon)\mathcal{L}_3\mathcal{L}_4 + V_1(\epsilon)\mathcal{L}_2\mathcal{L}_3\mathcal{L}_4 - V_1(\epsilon)V_2(\epsilon)\mathcal{L}_3\mathcal{L}_4 - V_1(\epsilon)\mathcal{L}_2 V_3(\epsilon)\mathcal{L}_4 - V_1(\epsilon)\mathcal{L}_2\mathcal{L}_3 V_4(\epsilon) - \mathcal{L}_1 V_2(\epsilon)V_3(\epsilon)\mathcal{L}_4 - \mathcal{L}_1 V_2(\epsilon)\mathcal{L}_3 V_4(\epsilon) - \mathcal{L}_1\mathcal{L}_2 V_3(\epsilon)V_4(\epsilon) + V_1(\epsilon)V_2(\epsilon)V_3(\epsilon)\mathcal{L}_4 + V_1(\epsilon)V_2(\epsilon)\mathcal{L}_3 V_4(\epsilon) + V_1(\epsilon)\mathcal{L}_2 V_3(\epsilon)V_4(\epsilon) + \mathcal{L}_1 V_2(\epsilon)V_3(\epsilon)V_4(\epsilon) - V_1(\epsilon)V_2(\epsilon)V_3(\epsilon)V_4(\epsilon)$$

$$= (-1)^3\cdot(V_1(\epsilon)-\mathcal{L}_1)\cdot(V_2(\epsilon)-\mathcal{L}_2)\cdot(V_3(\epsilon)-\mathcal{L}_3)\cdot(V_4(\epsilon)-\mathcal{L}_4)+\mathcal{L}_1\mathcal{L}_2\mathcal{L}_3\mathcal{L}_4$$

General Volume Formula

$$V^2(\epsilon) = (-1) \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) + \mathcal{L}_1\mathcal{L}_2$$

$$V^3(\epsilon) = (-1)^2 \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) \cdot (V_3(\epsilon) - \mathcal{L}_3) + \mathcal{L}_1\mathcal{L}_2\mathcal{L}_3$$

$$V^4(\epsilon) = (-1)^3 \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) \cdot (V_3(\epsilon) - \mathcal{L}_3) \cdot (V_4(\epsilon) - \mathcal{L}_4) + \mathcal{L}_1\mathcal{L}_2\mathcal{L}_3\mathcal{L}_4$$

$$V^n(\epsilon) = \left[(-1)^{n-1} \cdot \prod_{k=1}^{n} (V_k(\epsilon) - \mathcal{L}_k)\right] + \prod_{k=1}^{n} \mathcal{L}_k$$

Future Work

- Prove the general volume formula
- Find a general formula for dimension (based on general volume formula)

References I

D. A. Bini and L. Robol. “Solving secular and polynomial equations: A multiprecision algorithm.” *Journal of Computational and Applied Mathematics* 272 (2014), 276–292.

M. R. Bremner. *Lattice Basis Reduction: An Introduction to the LLL Algorithm and its Applications*, Taylor & Francis, Boca Raton, 2011.

M. L. Lapidus. “Fractal drum, inverse spectral problems for elliptic operators and a partial resolution of the Weyl-Berry conjecture.” *Transactions of the American Mathematical Society.* (2) 325 (1991), 465–529.

References II

- M. L. Lapidus. “Vibrations of Fractal Drums, the Riemann Hypothesis, Waves in Fractal Media and the Weyl-Berry Conjecture.” In *Ordinary and Partial Differential Equations*, edited by B. D. Sleeman and R. J. Jarvis, Vol. IV, Proc. Twelfth Intern. Conf. (Dundee, Scotland, UK, June 1992), Pitman Research Notes in Math. Series 289, pp. 126–209, London: Longman Scientific and Technical, 1993.
- M. L. Lapidus. *In Search of the Riemann Zeros: Strings, fractal membranes and noncommutative spacetimes*, American Mathematical Society, Providence, Rhode Island, 2008.
- M. L. Lapidus. “An overview of the complex fractal dimensions: From fractal strings to fractal drums, and back,” *Contemporary Mathematics*, Vol. 731, Amer. Math. Soc., Providence, R.I., 2019, 143–269

References III

- M. L. Lapidus and C. Pomerance. “The Riemann Zeta-Function and the One-Dimensional Weyl-Berry Conjecture for Fractal Drums,” *Proc. London Math. Soc.* (3) 66 (1993), 41–69.
- M. L. Lapidus and H. Maier. “The Riemann Hypothesis and Inverse Spectral Problems for Fractal Strings.” *J. London Math. Soc.* (2) 52 (1995), 15–34.
- M. L. Lapidus and M. van Frankenhuijsen. *Fractal Geometry and Number Theory* (*Complex dimensions of fractal strings and zeros of zeta functions*), Birkhäuser, Boston, 2000.
- M. L. Lapidus and M. van Frankenhuijsen. “Complex Dimensions of Self-Similar Fractal Strings and Diophantine Approximation,” *J. Experimental Math.* (1) 12 (2003), 41–69.

References IV

- M. L. Lapidus and M. van Frankenhuijsen. *Fractal Geometry, Complex Dimensions and Zeta Functions: Geometry and Spectra of Fractal Strings*, second edition (of the 2006 edition), Springer Monographs in Mathematics, Springer, New York, 2012.
- M. L. Lapidus and G. Radunović and D. Žubrinić. *Fractal Zeta Functions and Fractal Drums: Higher-Dimensional Theory of Complex Dimensions*, Springer Monographs in Mathematics, Springer, New York, 2017.
- A. K. Lenstra and H. W. Lenstra and L. Lovász. "Factoring polynomials with rational coefficients," *Mathematische Annalen* (4) 261 (1982), 515–534.
- W. M. Schmidt. *Diophantine Approximation*, Springer, New York, 1980.

References V

J.-P. Serre. *A Course in Arithmetic*, English translation, Springer-Verlag, Berlin, 1973.

E. K. Voskanian. "On the Quasiperiodic Structure of the Complex Dimensions of Self-Similar Fractal Strings," Ph. D. Dissertation, University of California, Riverside, 2019.