Inner Minkowski Dimension of Products of Fractal Strings

Isaac Ashton, Joseph Ingenito, Peter Tonuzi

The College of New Jersey

What is a fractal string

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Introduce the Cantor string so that the audience can see an example right away. A movie would be nice!

Cantor String

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Lapidus, van Frankenhuijsen, [9, 10, 11]

$$D_{\mathcal{L}} = \inf\{\alpha \in \mathbb{R} \mid \sum_{j=1}^{\infty} \ell_j^\alpha < \infty\} \implies 0 < D_{\mathcal{L}} < 1$$

Cantor string volume

Fractal Lawn

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- $\sum_{j=1}^{N_{\Omega_i}(x)} \ell_j = \mathcal{L}_i - \mathcal{W}_i(x)$

Fractal Lawn Volume II

Partially Covered Volume Box

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- Partially covered volume
  - $V_{partial}(\epsilon) = 2\epsilon N_{\Omega_1}(2\epsilon) \sum_{j=1}^{N_{\Omega_2}(2\epsilon)} \ell_j + 2\epsilon N_{\Omega_2}(2\epsilon) \sum_{j=1}^{N_{\Omega_1}(2\epsilon)} (p_j - 2\epsilon)$

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- Fully covered volume
  - $V_{full}(\epsilon) = \mathcal{L}_1 \mathcal{W}_2(2\epsilon) + \mathcal{L}_2 \mathcal{W}_1(2\epsilon)$

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- Fully covered volume
  - $V_{full}(\epsilon) = \mathcal{L}_1 \mathcal{W}_2(2\epsilon) + \mathcal{L}_2 \mathcal{W}_1(2\epsilon)$
- Double counted volume
  - $V_{double}(\epsilon) = \mathcal{W}_1(2\epsilon)\mathcal{W}_2(2\epsilon)$

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$$V(\epsilon) = 2\epsilon \cdot N_{\Omega_1}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(2\epsilon)} p_j \right) + 2\epsilon \cdot N_{\Omega_2}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(2\epsilon)} (\ell_j - 2\epsilon) \right) +$$
$$\mathcal{L}_1 \mathcal{W}_2(2\epsilon) + \mathcal{L}_2 \mathcal{W}_1(2\epsilon) - \mathcal{W}_1(2\epsilon)\mathcal{W}_2(2\epsilon)$$

$$= 2\epsilon \cdot N_{\Omega_1} \cdot (\mathcal{L}_2 - \mathcal{W}_2) + 2\epsilon \cdot N_{\Omega_2} \cdot (\mathcal{L}_1 - \mathcal{W}_1 - 2\epsilon \cdot N_{\Omega_1}) +$$
$$\mathcal{L}_1 \mathcal{W}_2 + \mathcal{L}_2 \mathcal{W}_1 - \mathcal{W}_1 \mathcal{W}_2$$

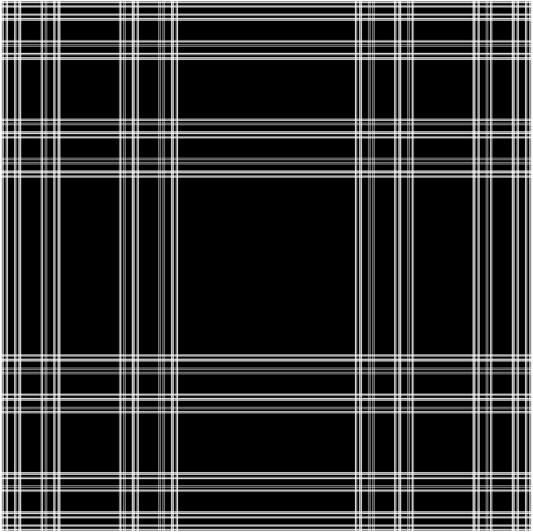
Fractal Lawn Volume IV

$$V(\epsilon) = \mathcal{L}_1 \cdot (2\epsilon \cdot N_{\Omega_2} + \mathcal{W}_2) + \mathcal{L}_2 \cdot (2\epsilon \cdot N_{\Omega_1} + \mathcal{W}_1) - (2\epsilon \cdot N_{\Omega_1} + \mathcal{W}_1) \cdot (2\epsilon \cdot N_{\Omega_2} + \mathcal{W}_2)$$

$$= \mathcal{L}_1 \cdot (2\epsilon \cdot N_{\Omega_2} + \mathcal{W}_2) + \mathcal{L}_2 \cdot (2\epsilon \cdot N_{\Omega_1} + \mathcal{W}_1) - (2\epsilon \cdot N_{\Omega_1} + \mathcal{W}_1) \cdot (2\epsilon \cdot N_{\Omega_2} + \mathcal{W}_2)$$

$$= \mathcal{L}_1 V_2 + \mathcal{L}_2 V_1 - V_1 V_2$$

Cantor Lawn Volume



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$$V(\epsilon) = \mathcal{L}_1 V_2 + \mathcal{L}_2 V_1 - V_1 V_2$$

$$lim_{\epsilon \to 0^+} V(\epsilon) = O(\epsilon^{1-D_{\Omega_2}}) + O(\epsilon^{1-D_{\Omega_1}}) + O(\epsilon^{2-(D_{\Omega_1}+D_{\Omega_2})})$$

$$lim_{\epsilon \to 0^+} \frac{V(\epsilon)}{\epsilon^{2-\alpha}} = O(\epsilon^{1-D_{\Omega_2}} \epsilon^{2-\alpha}) + O(\epsilon^{1-D_{\Omega_1}} \epsilon^{2-\alpha}) + O(\epsilon^{2-(D_{\Omega_1}+D_{\Omega_2})} \epsilon^{2-\alpha})$$

Dimension of Fractal Lawn II

$$lim_{\epsilon \to 0^+} \frac{V(\epsilon)}{\epsilon^{2-\alpha}} = O(\epsilon^{1-\alpha(1+D_{\Omega_2})}) + O(\epsilon^{1-\alpha(1+D_{\Omega_1})}) + O(\epsilon^{\alpha-(D_{\Omega_1}+D_{\Omega_2})})$$

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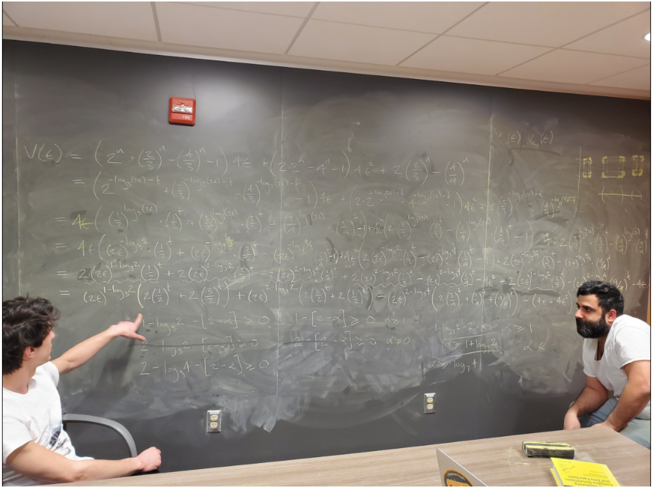
$$\begin{aligned} V(\epsilon)_{\Omega^2} &= \mathcal{L} V(\epsilon)_\Omega + \mathcal{L} V(\epsilon)_\Omega - V(\epsilon)_\Omega \cdot V(\epsilon)_\Omega \\ &= 2\mathcal{L} V(\epsilon)_\Omega - (V(\epsilon)_\Omega)^2 \end{aligned}$$

Dimension of the Cantor Lawn II



Dimension of the Cantor Lawn III

Let $t = \{-log_3(2\epsilon)\}$ be defined as the fractional portion of $-log_3(2\epsilon)$

$$V_{\Omega^2}(\epsilon) = (2\epsilon)^{1-log_3 2}\left(2\left(\frac{1}{2}\right)^t + 2\left(\frac{3}{2}\right)^t\right) + (2\epsilon)^{2-log_3 2}\left(2\left(\frac{3}{2}\right)^t + 2\left(\frac{1}{2}\right)^t\right)$$

$$= -(2\epsilon)^{2-log_3 4}\left(2\left(\frac{3}{4}\right)^t + \left(\frac{1}{4}\right)^t + \left(\frac{9}{4}\right)^t\right) - (4\epsilon + 4\epsilon^2)$$

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$$\begin{cases} -log_3 2 + 1 - (2-\alpha) \geq 0 \\ -log_3 2 + 2 - (2-\alpha) \geq 0 \\ -log_3 4 + 2 - (2-\alpha) \geq 0 \end{cases}$$

Dimension of the Cantor Lawn IV

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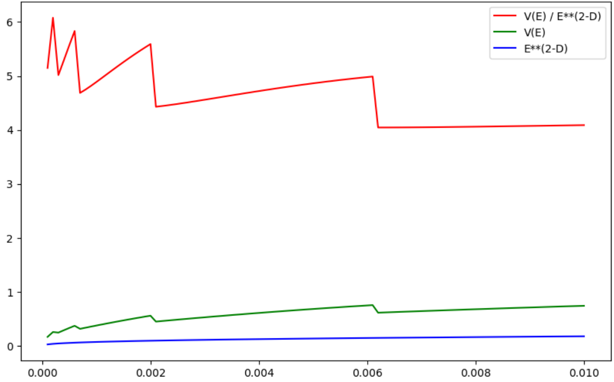
Dimension of the Cantor Lawn V



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