Inner Minkowski Dimension of Products of Fractal Strings

Isaac Ashton, Joseph Ingenito, Peter Tonuzi

The College of New Jersey

What is a fractal string

Definition 1

A **fractal string** is a bounded open subset Ω of the real line.

What is a fractal string

Definition 1

A **fractal string** is a bounded open subset Ω of the real line.

$$\Omega = \bigcup_{j=1}^{\infty} \ell_j, \qquad \mathrm{Vol}_1(\Omega) = \sum_{j=1}^{\infty} \ell_j \implies \lim_{j\to\infty} \ell_j = 0.$$

What is a fractal string

Definition 1

A **fractal string** is a bounded open subset Ω of the real line.

$$\Omega = \bigcup_{j=1}^{\infty} \ell_j, \qquad \mathrm{Vol}_1(\Omega) = \sum_{j=1}^{\infty} \ell_j \implies \lim_{j\to\infty} \ell_j = 0.$$

$$\ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots \geq 0$$

What is a fractal string

Definition 1

A **fractal string** is a bounded open subset Ω of the real line.

$$\Omega = \bigcup_{j=1}^{\infty} \ell_j, \qquad \mathrm{Vol}_1(\Omega) = \sum_{j=1}^{\infty} \ell_j \implies \lim_{j\to\infty} \ell_j = 0.$$

$$\ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots \geq 0$$

Introduce the Cantor string so that the audience can see an example right away. A movie would be nice!

Cantor String

Dimension of fractal string

Inner-tubular neighborhood: $V(\epsilon) = \text{vol}_1\{x \in \Omega \mid d(x, \partial\Omega) < \epsilon\}$

Dimension of fractal string

Inner-tubular neighborhood: $V(\epsilon) = \mathrm{vol}_1\{x \in \Omega \mid d(x, \partial\Omega) < \epsilon\}$

Use a movie of the Cantor String to illustrate the volume formula!

Dimension of fractal string

Inner-tubular neighborhood: $V(\epsilon) = \mathrm{vol}_1\{x \in \Omega \mid d(x, \partial\Omega) < \epsilon\}$

Use a movie of the Cantor String to illustrate the volume formula!

Counting function: $N_\Omega(x) = \sum_{\ell_j^{-1} \leq x}$

$$D_{\mathcal{L}} = \inf\{\alpha \geq 0 \mid V(\epsilon) = O(\epsilon^{1-\alpha}) \text{ as } \epsilon \to 0^+\}$$

Dimension of fractal string

Inner-tubular neighborhood: $V(\epsilon) = \mathrm{vol}_1\{x \in \Omega \mid d(x, \partial\Omega) < \epsilon\}$

Use a movie of the Cantor String to illustrate the volume formula!

Counting function: $N_\Omega(x) = \sum_{\ell_j^{-1} \leq x}$

$D_\mathcal{L} = \inf\{\alpha \geq 0 \mid V(\epsilon) = O(\epsilon^{1-\alpha}) \text{ as } \epsilon \to 0^+\}$

Lapidus, van Frankenhuijsen, [9, 10, 11]

$$D_\mathcal{L} = \inf\{\alpha \in \mathbb{R} \mid \sum_{j=1}^{\infty} \ell_j^\alpha < \infty\} \implies 0 < D_\mathcal{L} < 1$$

Cantor string volume

Fractal Lawn

Definition 2

A **fractal lawn** Ω is the Cartesian product of two fractal strings $\Omega = \Omega \times \Omega_2$.

Fractal Lawn

Definition 2

A **fractal lawn** Ω is the Cartesian product of two fractal strings $\Omega = \Omega \times \Omega_2$.

The Cartesian product of two fractal strings is a bounded open subset of $\mathbb{R}^2$ since a fractal string is a bounded open subset of $\mathbb{R}$.

Fractal Lawn

Definition 2

A **fractal lawn** Ω is the Cartesian product of two fractal strings $\Omega = \Omega \times \Omega_2$.

The Cartesian product of two fractal strings is a bounded open subset of $\mathbb{R}^2$ since a fractal string is a bounded open subset of $\mathbb{R}$.

$$\Omega_1 = \bigcup_{j=1}^{\infty} \ell_j, \quad \Omega_2 = \bigcup_{i=1}^{\infty} p_i$$

Fractal Lawn

Definition 2

A **fractal lawn** Ω is the Cartesian product of two fractal strings $\Omega = \Omega \times \Omega_2$.

The Cartesian product of two fractal strings is a bounded open subset of $\mathbb{R}^2$ since a fractal string is a bounded open subset of $\mathbb{R}$.

$$\Omega_1 = \bigcup_{j=1}^{\infty} \ell_j, \quad \Omega_2 = \bigcup_{i=1}^{\infty} p_i$$

$$\ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots \geq 0, \quad p_1 \geq p_2 \geq p_3 \geq \cdots \geq 0$$

Cantor Lawn

Fractal Lawn Volume I

Inner-tubular neighborhood: $V(\epsilon) = \text{vol}_2\{x \in \Omega \mid d(x, \partial\Omega) < \epsilon\}$

Fractal Lawn Volume I

Inner-tubular neighborhood: $V(\epsilon) = \mathrm{vol}_2\{x \in \Omega \mid d(x, \partial\Omega) < \epsilon\}$

- $\partial\Omega = (\partial\Omega_1 \times \Omega_2) \cup (\Omega_1 \times \partial\Omega_2)$

Fractal Lawn Volume I

Inner-tubular neighborhood: $V(\epsilon) = \text{vol}_2\{x \in \Omega \mid d(x, \partial\Omega) < \epsilon\}$

- $\partial\Omega = (\partial\Omega_1 \times \Omega_2) \cup (\Omega_1 \times \partial\Omega_2)$

Counting function: Counts up to the critical length for any $x > 0$

Fractal Lawn Volume I

Inner-tubular neighborhood: $V(\epsilon) = \text{vol}_2\{x \in \Omega \mid d(x, \partial\Omega) < \epsilon\}$

- $\partial\Omega = (\partial\Omega_1 \times \Omega_2) \cup (\Omega_1 \times \partial\Omega_2)$

Counting function: Counts up to the critical length for any $x > 0$

- $N_{\Omega_i}(x) = \sum_{j:\ell_j \geq x} 1$

Fractal Lawn Volume I

Inner-tubular neighborhood: $V(\epsilon) = \text{vol}_2\{x \in \Omega \mid d(x, \partial\Omega) < \epsilon\}$

- $\partial\Omega = (\partial\Omega_1 \times \Omega_2) \cup (\Omega_1 \times \partial\Omega_2)$

Counting function: Counts up to the critical length for any $x > 0$

- $N_{\Omega_i}(x) = \sum_{j:\ell_j \geq x} 1$

Useful Definitions:

Fractal Lawn Volume I

Inner-tubular neighborhood: $V(\epsilon) = \mathrm{vol}_2\{x \in \Omega \mid d(x, \partial\Omega) < \epsilon\}$

- $\partial\Omega = (\partial\Omega_1 \times \Omega_2) \cup (\Omega_1 \times \partial\Omega_2)$

Counting function: Counts up to the critical length for any $x > 0$

- $N_{\Omega_i}(x) = \sum_{j:\ell_j \geq x} 1$

Useful Definitions:

- $\mathcal{L}_i = \sum_{j \in \mathbb{N}} \ell_j$

Fractal Lawn Volume I

Inner-tubular neighborhood: $V(\epsilon) = \text{vol}_2\{x \in \Omega \mid d(x, \partial\Omega) < \epsilon\}$

- $\partial\Omega = (\partial\Omega_1 \times \Omega_2) \cup (\Omega_1 \times \partial\Omega_2)$

Counting function: Counts up to the critical length for any $x > 0$

- $N_{\Omega_i}(x) = \sum_{j:\ell_j \geq x} 1$

Useful Definitions:

- $\mathcal{L}_i = \sum_{j \in \mathbb{N}} \ell_j$
- $\mathcal{W}_i(x) = \sum_{j:\ell_j < x} \ell_j$

Fractal Lawn Volume I

Inner-tubular neighborhood: $V(\epsilon) = \text{vol}_2\{x \in \Omega \mid d(x, \partial\Omega) < \epsilon\}$

- $\partial\Omega = (\partial\Omega_1 \times \Omega_2) \cup (\Omega_1 \times \partial\Omega_2)$

Counting function: Counts up to the critical length for any $x > 0$

- $N_{\Omega_i}(x) = \sum_{j:\ell_j \geq x} 1$

Useful Definitions:

- $\mathcal{L}_i = \sum_{j \in \mathbb{N}} \ell_j$
- $\mathcal{W}_i(x) = \sum_{j:\ell_j < x} \ell_j$
- $\sum_{j=1}^{N_{\Omega_i}(x)} \ell_j = \mathcal{L}_i - \mathcal{W}_i(x)$

Fractal Lawn Volume II

General Volume Box

Fractal Lawn Volume II

General Volume Box

- Partially covered volume
  - $V_{partial}(\epsilon) = 2\epsilon N_{\Omega_1}(2\epsilon) \sum_{j=1}^{N_{\Omega_2}(2\epsilon)} \ell_j + 2\epsilon N_{\Omega_2}(2\epsilon) \sum_{j=1}^{N_{\Omega_1}(2\epsilon)} (p_j - 2\epsilon)$

Fractal Lawn Volume II

General Volume Box

- Partially covered volume
  - $V_{partial}(\epsilon) = 2\epsilon N_{\Omega_1}(2\epsilon) \sum_{j=1}^{N_{\Omega_2}(2\epsilon)} \ell_j + 2\epsilon N_{\Omega_2}(2\epsilon) \sum_{j=1}^{N_{\Omega_1}(2\epsilon)} (p_j - 2\epsilon)$
- Fully covered volume
  - $V_{full}(\epsilon) = \mathcal{L}_1\mathcal{W}_2(2\epsilon) + \mathcal{L}_2\mathcal{W}_1(2\epsilon)$

Fractal Lawn Volume II

General Volume Box

- Partially covered volume
  - $V_{partial}(\epsilon) = 2\epsilon N_{\Omega_1}(2\epsilon) \sum_{j=1}^{N_{\Omega_2}(2\epsilon)} \ell_j + 2\epsilon N_{\Omega_2}(2\epsilon) \sum_{j=1}^{N_{\Omega_1}(2\epsilon)} (p_j - 2\epsilon)$
- Fully covered volume
  - $V_{full}(\epsilon) = \mathcal{L}_1 \mathcal{W}_2(2\epsilon) + \mathcal{L}_2 \mathcal{W}_1(2\epsilon)$
- Double counted volume
  - $V_{double}(\epsilon) = \mathcal{W}_1(2\epsilon)\mathcal{W}_2(2\epsilon)$

Fractal Lawn Volume III

- $V(\epsilon) = V_{partial}(\epsilon) + V_{full}(\epsilon) - V_{double}(\epsilon)$

Fractal Lawn Volume III

- $V(\epsilon) = V_{partial}(\epsilon) + V_{full}(\epsilon) - V_{double}(\epsilon)$
- $$V(\epsilon) = 2\epsilon \cdot N_{\Omega_1}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(2\epsilon)} p_j \right) + 2\epsilon \cdot N_{\Omega_2}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(2\epsilon)} (\ell_j - 2\epsilon) \right) + \mathcal{L}_1 \mathcal{W}_2(2\epsilon) + \mathcal{L}_2 \mathcal{W}_1(2\epsilon) - \mathcal{W}_1(2\epsilon) \mathcal{W}_2(2\epsilon)$$

Fractal Lawn Volume III

- $V(\epsilon) = V_{partial}(\epsilon) + V_{full}(\epsilon) - V_{double}(\epsilon)$
- $$V(\epsilon) = 2\epsilon \cdot N_{\Omega_1}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(2\epsilon)} p_j \right) + 2\epsilon \cdot N_{\Omega_2}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(2\epsilon)} (\ell_j - 2\epsilon) \right) + \mathcal{L}_1 \mathcal{W}_2(2\epsilon) + \mathcal{L}_2 \mathcal{W}_1(2\epsilon) - \mathcal{W}_1(2\epsilon)\mathcal{W}_2(2\epsilon)$$
- $$V(\epsilon) = 2\epsilon \cdot N_{\Omega_1} \cdot (\mathcal{L}_2 - \mathcal{W}_2) + 2\epsilon \cdot N_{\Omega_2} \cdot (\mathcal{L}_1 - \mathcal{W}_1 - 2\epsilon \cdot N_{\Omega_1}) + \mathcal{L}_1 \mathcal{W}_2 + \mathcal{L}_2 \mathcal{W}_1 - \mathcal{W}_1 \mathcal{W}_2$$

Fractal Lawn Volume III

- $V(\epsilon) = V_{partial}(\epsilon) + V_{full}(\epsilon) - V_{double}(\epsilon)$
- $$V(\epsilon) = 2\epsilon \cdot N_{\Omega_1}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(2\epsilon)} p_j \right) + 2\epsilon \cdot N_{\Omega_2}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(2\epsilon)} (\ell_j - 2\epsilon) \right) + \mathcal{L}_1 \mathcal{W}_2(2\epsilon) + \mathcal{L}_2 \mathcal{W}_1(2\epsilon) - \mathcal{W}_1(2\epsilon)\mathcal{W}_2(2\epsilon)$$
- $$V(\epsilon) = 2\epsilon \cdot N_{\Omega_1} \cdot (\mathcal{L}_2 - \mathcal{W}_2) + 2\epsilon \cdot N_{\Omega_2} \cdot (\mathcal{L}_1 - \mathcal{W}_1 - 2\epsilon \cdot N_{\Omega_1}) + \mathcal{L}_1 \mathcal{W}_2 + \mathcal{L}_2 \mathcal{W}_1 - \mathcal{W}_1 \mathcal{W}_2$$
- $$V(\epsilon) = \mathcal{L}_1 \cdot (2\epsilon \cdot N_{\Omega_2} + \mathcal{W}_2) + \mathcal{L}_2 \cdot (2\epsilon \cdot N_{\Omega_1} + \mathcal{W}_1) - (2\epsilon \cdot N_{\Omega_1} + \mathcal{W}_1) \cdot (2\epsilon \cdot N_{\Omega_2} + \mathcal{W}_2)$$

Fractal Lawn Volume III

$$V(\epsilon) = V_{partial}(\epsilon) + V_{full}(\epsilon) - V_{double}(\epsilon)$$

Fractal Lawn Volume III

$$V(\epsilon) = V_{partial}(\epsilon) + V_{full}(\epsilon) - V_{double}(\epsilon)$$

$$V(\epsilon) = 2\epsilon \cdot N_{\Omega_1}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(2\epsilon)} p_j \right) + 2\epsilon \cdot N_{\Omega_2}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(2\epsilon)} (\ell_j - 2\epsilon) \right) +$$
$$\mathcal{L}_1 \mathcal{W}_2(2\epsilon) + \mathcal{L}_2 \mathcal{W}_1(2\epsilon) - \mathcal{W}_1(2\epsilon)\mathcal{W}_2(2\epsilon)$$

$$= 2\epsilon \cdot N_{\Omega_1} \cdot (\mathcal{L}_2 - \mathcal{W}_2) + 2\epsilon \cdot N_{\Omega_2} \cdot (\mathcal{L}_1 - \mathcal{W}_1 - 2\epsilon \cdot N_{\Omega_1}) +$$
$$\mathcal{L}_1 \mathcal{W}_2 + \mathcal{L}_2 \mathcal{W}_1 - \mathcal{W}_1 \mathcal{W}_2$$

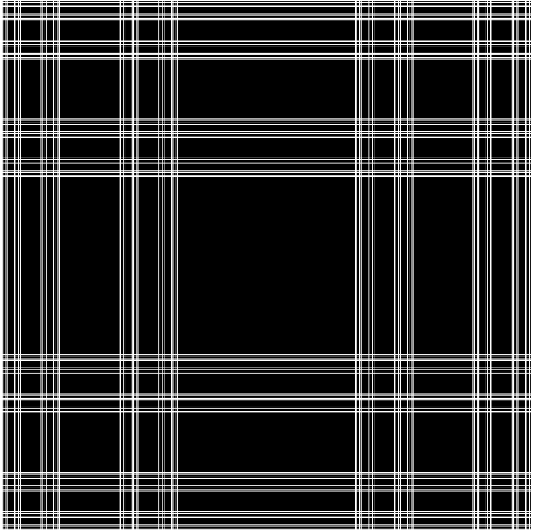
$$= \mathcal{L}_1 \cdot (2\epsilon \cdot N_{\Omega_2} + \mathcal{W}_2) + \mathcal{L}_2 \cdot (2\epsilon \cdot N_{\Omega_1} + \mathcal{W}_1) -$$
$$(2\epsilon \cdot N_{\Omega_1} + \mathcal{W}_1) \cdot (2\epsilon \cdot N_{\Omega_2} + \mathcal{W}_2)$$

Fractal Lawn Volume IV

$$V(\epsilon) = \mathcal{L}_1 \cdot (2\epsilon \cdot N_{\Omega_2} + \mathcal{W}_2) + \mathcal{L}_2 \cdot (2\epsilon \cdot N_{\Omega_1} + \mathcal{W}_1) - (2\epsilon \cdot N_{\Omega_1} + \mathcal{W}_1) \cdot (2\epsilon \cdot N_{\Omega_2} + \mathcal{W}_2)$$

$$= \mathcal{L}_1 V_2 + \mathcal{L}_2 V_1 - V_1 V_2$$

Cantor Lawn Volume



Dimension of Fractal Lawn

Definition 3

The **Dimension** $D_\Omega = \inf\{\alpha \geq 0 \mid V(\epsilon) = O(\epsilon^{2-\alpha}) \text{ as } \epsilon \to 0^+\}$

Dimension of Fractal Lawn

Definition 3

The **Dimension** $D_\Omega = \inf\{\alpha \geq 0 \mid V(\epsilon) = O(\epsilon^{2-\alpha}) \text{ as } \epsilon \to 0^+\}$

- Since $\Omega \subset \mathbb{R}^2$, V is required to remain bounded by $\epsilon^{2-\alpha}$ rather than the 1D case

Dimension of Fractal Lawn

Definition 3

The **Dimension** $D_\Omega = \inf\{\alpha \geq 0 \mid V(\epsilon) = O(\epsilon^{2-\alpha}) \text{ as } \epsilon \to 0^+\}$

- Since $\Omega \subset \mathbb{R}^2$, V is required to remain bounded by $\epsilon^{2-\alpha}$ rather than the 1D case

$$\begin{aligned} V(\epsilon) &= \mathcal{L}_1 V_2 + \mathcal{L}_2 V_1 - V_1 V_2 \\ &= O(\epsilon^{1-D_{\Omega_2}}) + O(\epsilon^{1-D_{\Omega_1}}) + O(\epsilon^{2-(D_{\Omega_1}+D_{\Omega_2})}) \end{aligned}$$

References I

D. A. Bini and L. Robol. "Solving secular and polynomial equations: A multiprecision algorithm." *Journal of Computational and Applied Mathematics* 272 (2014), 276–292.

M. R. Bremner. *Lattice Basis Reduction: An Introduction to the LLL Algorithm and its Applications*, Taylor & Francis, Boca Raton, 2011.

M. L. Lapidus. "Fractal drum, inverse spectral problems for elliptic operators and a partial resolution of the Weyl-Berry conjecture." *Transactions of the American Mathematical Society.* (2) 325 (1991), 465–529.

References II

- M. L. Lapidus. "Vibrations of Fractal Drums, the Riemann Hypothesis, Waves in Fractal Media and the Weyl-Berry Conjecture." In *Ordinary and Partial Differential Equations*, edited by B. D. Sleeman and R. J. Jarvis, Vol. IV, Proc. Twelfth Intern. Conf. (Dundee, Scotland, UK, June 1992), Pitman Research Notes in Math. Series 289, pp. 126–209, London: Longman Scientific and Technical, 1993.
- M. L. Lapidus. *In Search of the Riemann Zeros: Strings, fractal membranes and noncommutative spacetimes*, American Mathematical Society, Providence, Rhode Island, 2008.
- M. L. Lapidus. "An overview of the complex fractal dimensions: From fractal strings to fractal drums, and back," *Contemporary Mathematics*, Vol. 731, Amer. Math. Soc., Providence, R.I., 2019, 143–269

TCNJ THE COLLEGE OF NEW JERSEY

References III

- M. L. Lapidus and C. Pomerance. “The Riemann Zeta-Function and the One-Dimensional Weyl-Berry Conjecture for Fractal Drums,” *Proc. London Math. Soc.* (3) 66 (1993), 41–69.
- M. L. Lapidus and H. Maier. “The Riemann Hypothesis and Inverse Spectral Problems for Fractal Strings.” *J. London Math. Soc.* (2) 52 (1995), 15–34.
- M. L. Lapidus and M. van Frankenhuijsen. *Fractal Geometry and Number Theory* (*Complex dimensions of fractal strings and zeros of zeta functions*), Birkhäuser, Boston, 2000.
- M. L. Lapidus and M. van Frankenhuijsen. “Complex Dimensions of Self-Similar Fractal Strings and Diophantine Approximation,” *J. Experimental Math.* (1) 12 (2003), 41–69.

References IV

M. L. Lapidus and M. van Frankenhuijsen. *Fractal Geometry, Complex Dimensions and Zeta Functions: Geometry and Spectra of Fractal Strings*, second edition (of the 2006 edition), Springer Monographs in Mathematics, Springer, New York, 2012.

M. L. Lapidus and G. Radunović and D. Žubrinić. *Fractal Zeta Functions and Fractal Drums: Higher-Dimensional Theory of Complex Dimensions*, Springer Monographs in Mathematics, Springer, New York, 2017.

A. K. Lenstra and H. W. Lenstra and L. Lovász. "Factoring polynomials with rational coefficients," *Mathematische Annalen* (4) 261 (1982), 515–534.

W. M. Schmidt. *Diophantine Approximation*, Springer, New York, 1980.

References V

- J.-P. Serre. *A Course in Arithmetic*, English translation, Springer-Verlag, Berlin, 1973.
- E. K. Voskanian. “On the Quasiperiodic Structure of the Complex Dimensions of Self-Similar Fractal Strings,” Ph. D. Dissertation, University of California, Riverside, 2019.