Inner Minkowski Dimension of Products of Fractal Strings

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What is a Fractal String

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The fractal string we studied the most in particular is the **Cantor String**

Cantor String

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Counting function: $N_\Omega(x) = \sum_{\ell_j^{-1} \leq x} 1$

More Notation: $\mathcal{W}_\Omega(x) = \sum_{j : \ell_j^{-1} > x} \ell_j$

Now we can write: $V(\epsilon) = 2\epsilon \cdot N_\Omega(2\epsilon) + \mathcal{W}_\Omega(2\epsilon)$

Cantor String Volume

Inner Minkowski Dimension of a Fractal String

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Cantor String Dimension: $D_{cs} = \log_3 2$

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$$\ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots \geq 0, \quad p_1 \geq p_2 \geq p_3 \geq \cdots \geq 0$$

Cantor Lawn

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- $\mathcal{L}_i = \sum_{j \in \mathbb{N}} \ell_j$

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Useful Definitions:

- $\mathcal{L}_i = \sum_{j \in \mathbb{N}} \ell_j$
- $\mathcal{W}_i(x) = \sum_{j:\ell_j < x} \ell_j$
- $\sum_{j=1}^{N_{\Omega_i}(x)} \ell_j = \mathcal{L}_i - \mathcal{W}_i(x)$

Fractal Lawn Volume II

Partially Covered Volume Box

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- Partially covered volume
  - $V_{partial}(\epsilon) = 2\epsilon N_{\Omega_1}(2\epsilon) \sum_{j=1}^{N_{\Omega_2}(2\epsilon)} \ell_j + 2\epsilon N_{\Omega_2}(2\epsilon) \sum_{j=1}^{N_{\Omega_1}(2\epsilon)} (p_j - 2\epsilon)$

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- Fully covered volume
  - $V_{full}(\epsilon) = \mathcal{L}_1 \mathcal{W}_2(2\epsilon) + \mathcal{L}_2 \mathcal{W}_1(2\epsilon)$

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- Partially covered volume
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- Fully covered volume
  - $V_{full}(\epsilon) = \mathcal{L}_1 \mathcal{W}_2(2\epsilon) + \mathcal{L}_2 \mathcal{W}_1(2\epsilon)$
- Double counted volume
  - $V_{double}(\epsilon) = \mathcal{W}_1(2\epsilon)\mathcal{W}_2(2\epsilon)$

Fractal Lawn Volume III

$$V(\epsilon) = V_{partial}(\epsilon) + V_{full}(\epsilon) - V_{double}(\epsilon)$$

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$$V(\epsilon) = 2\epsilon \cdot N_{\Omega_1}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(2\epsilon)} p_j \right) + 2\epsilon \cdot N_{\Omega_2}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(2\epsilon)} (\ell_j - 2\epsilon) \right) + \mathcal{L}_1 \mathcal{W}_2(2\epsilon) + \mathcal{L}_2 \mathcal{W}_1(2\epsilon) - \mathcal{W}_1(2\epsilon)\mathcal{W}_2(2\epsilon)$$

$$= 2\epsilon \cdot N_{\Omega_1} \cdot (\mathcal{L}_2 - \mathcal{W}_2) + 2\epsilon \cdot N_{\Omega_2} \cdot (\mathcal{L}_1 - \mathcal{W}_1 - 2\epsilon \cdot N_{\Omega_1}) + \mathcal{L}_1 \mathcal{W}_2 + \mathcal{L}_2 \mathcal{W}_1 - \mathcal{W}_1 \mathcal{W}_2$$

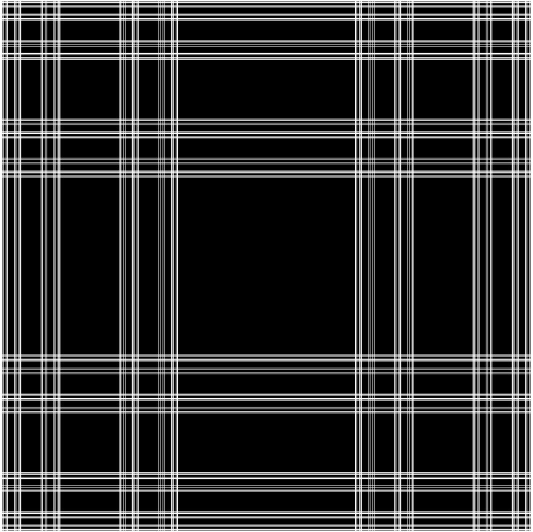
Fractal Lawn Volume IV

$$V(\epsilon) = \mathcal{L}_1 \cdot (2\epsilon \cdot N_{\Omega_2} + \mathcal{W}_2) + \mathcal{L}_2 \cdot (2\epsilon \cdot N_{\Omega_1} + \mathcal{W}_1) - (2\epsilon \cdot N_{\Omega_1} + \mathcal{W}_1) \cdot (2\epsilon \cdot N_{\Omega_2} + \mathcal{W}_2)$$

$$= \mathcal{L}_1 \cdot (2\epsilon \cdot N_{\Omega_2} + \mathcal{W}_2) + \mathcal{L}_2 \cdot (2\epsilon \cdot N_{\Omega_1} + \mathcal{W}_1) - (2\epsilon \cdot N_{\Omega_1} + \mathcal{W}_1) \cdot (2\epsilon \cdot N_{\Omega_2} + \mathcal{W}_2)$$

$$= \mathcal{L}_1 V_2 + \mathcal{L}_2 V_1 - V_1 V_2$$

Cantor Lawn Volume



Dimension of Fractal Lawn I

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$$V(\epsilon) = \mathcal{L}_1 V_2 + \mathcal{L}_2 V_1 - V_1 V_2$$

$$lim_{\epsilon \to 0^+} V(\epsilon) = O(\epsilon^{1-D_{\Omega_2}}) + O(\epsilon^{1-D_{\Omega_1}}) + O(\epsilon^{2-(D_{\Omega_1}+D_{\Omega_2})})$$

$$lim_{\epsilon \to 0^+} \frac{V(\epsilon)}{\epsilon^{2-\alpha}} = O(\epsilon^{1-D_{\Omega_2}} \epsilon^{2-\alpha}) + O(\epsilon^{1-D_{\Omega_1}} \epsilon^{2-\alpha}) + O(\epsilon^{2-(D_{\Omega_1}+D_{\Omega_2})} \epsilon^{2-\alpha})$$

Dimension of Fractal Lawn II

$$lim_{\epsilon \to 0^+} \frac{V(\epsilon)}{\epsilon^{2-\alpha}} = O(\epsilon^{1-\alpha(1+D_{\Omega_2})}) + O(\epsilon^{1-\alpha(1+D_{\Omega_1})}) + O(\epsilon^{\alpha-(D_{\Omega_1}+D_{\Omega_2})})$$

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$$D_{\Omega^2} = \max\{(1+D_{\Omega_1}),(1+D_{\Omega_2})\}$$

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$$\begin{aligned} V(\epsilon)_{\Omega^2} &= \mathcal{L}V(\epsilon)_\Omega + \mathcal{L}V(\epsilon)_\Omega - V(\epsilon)_\Omega \cdot V(\epsilon)_\Omega \\ &= 2\mathcal{L}V(\epsilon)_\Omega - (V(\epsilon)_\Omega)^2 \end{aligned}$$

Dimension of the Cantor Lawn II

Let $t = \{-log_3(2\epsilon)\}$ be defined as the fractional portion of $-log_3(2\epsilon)$

$$V_{\Omega^2}(\epsilon) = (2\epsilon)^{1-log_3 2}\left(2\left(\frac{1}{2}\right)^t + 2\left(\frac{3}{2}\right)^t\right) + (2\epsilon)^{2-log_3 2}\left(2\left(\frac{3}{2}\right)^t + 2\left(\frac{1}{2}\right)^t\right)$$
$$-(2\epsilon)^{2-log_3 4}\left(2\left(\frac{3}{4}\right)^t + \left(\frac{1}{4}\right)^t + \left(\frac{9}{4}\right)^t\right) - (4\epsilon + 4\epsilon^2)$$

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$$- (2\epsilon)^{2-log_3 4}\left(2\left(\frac{3}{4}\right)^t + \left(\frac{1}{4}\right)^t + \left(\frac{9}{4}\right)^t\right) - (4\epsilon + 4\epsilon^2)$$

$$\begin{cases} -log_3 2 + 1 - (2-\alpha) \geq 0 \\ -log_3 2 + 2 - (2-\alpha) \geq 0 \\ -log_3 4 + 2 - (2-\alpha) \geq 0 \end{cases}$$

Dimension of the Cantor Lawn III

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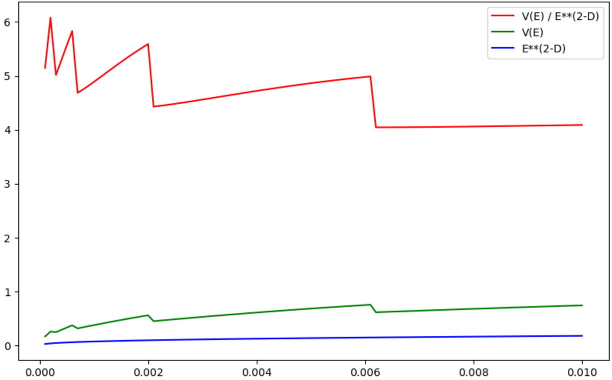
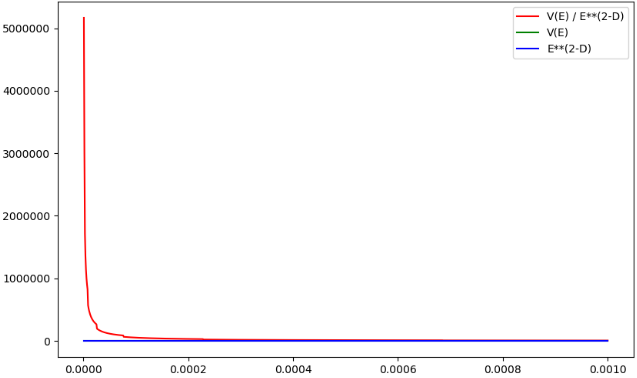
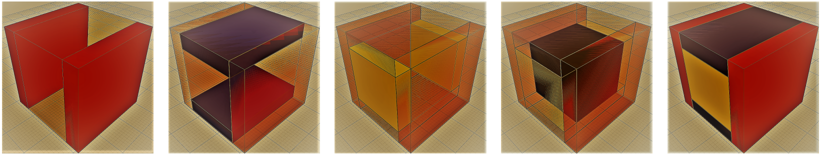
Dimension of the Cantor Lawn IV



Dimension of the Cantor Lawn V



3 Dimensional Fractal Objects



$$V(\epsilon) = 2\epsilon \cdot N_{\Omega_1}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(2\epsilon)} p_j \right) \cdot \left(\sum_{j=1}^{N_{\Omega_3}(2\epsilon)} q_j \right) +$$

$$2\epsilon \cdot N_{\Omega_2}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(2\epsilon)} \ell_j - 2\epsilon \right) \cdot \left(\sum_{j=1}^{N_{\Omega_3}(2\epsilon)} q_j \right) +$$

$$2\epsilon \cdot N_{\Omega_3}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(2\epsilon)} p_j - 2\epsilon \right) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(2\epsilon)} \ell_j - 2\epsilon \right) +$$

$$\mathcal{W}_1\mathcal{L}_2\mathcal{L}_3 + \mathcal{L}_1\mathcal{W}_2\mathcal{L}_3 + \mathcal{L}_1\mathcal{L}_2\mathcal{W}_3 - \mathcal{L}_1\mathcal{W}_2\mathcal{W}_3 - \mathcal{W}_1\mathcal{L}_2\mathcal{W}_3 -$$

$$\mathcal{W}_1\mathcal{W}_2\mathcal{L}_3 + \mathcal{W}_1\mathcal{W}_2\mathcal{W}_3$$

3 Dimensional Fractal Volume

$$V(\epsilon) = 2\epsilon \cdot N_{\Omega_1} \cdot (\mathcal{L}_2 - \mathcal{W}_3) \cdot (\mathcal{L}_3 - \mathcal{W}_3) + 2\epsilon \cdot N_{\Omega_2} \cdot (\mathcal{L}_1 - \mathcal{W}_1 - 2\epsilon \cdot N_{\Omega_1}) \cdot (\mathcal{L}_3 - \mathcal{W}_3) + 2\epsilon \cdot N_{\Omega_3} \cdot (\mathcal{L}_2 - \mathcal{W}_2 - 2\epsilon \cdot N_{\Omega_2}) \cdot (\mathcal{L}_1 - \mathcal{W}_1 - 2\epsilon \cdot N_{\Omega_1}) + \mathcal{W}_1 \mathcal{L}_2 \mathcal{L}_3 + \mathcal{L}_1 \mathcal{W}_2 \mathcal{L}_3 + \mathcal{L}_1 \mathcal{L}_2 \mathcal{W}_3 - \mathcal{L}_1 \mathcal{W}_2 \mathcal{W}_3 - \mathcal{W}_1 \mathcal{L}_2 \mathcal{W}_3 - \mathcal{W}_1 \mathcal{W}_2 \mathcal{L}_3 + \mathcal{W}_1 \mathcal{W}_2 \mathcal{W}_3$$

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$$V(\epsilon) = 2\epsilon \cdot N_{\Omega_1} \cdot (\mathcal{L}_2 - \mathcal{W}_3) \cdot (\mathcal{L}_3 - \mathcal{W}_3) + 2\epsilon \cdot N_{\Omega_2} \cdot (\mathcal{L}_1 - \mathcal{W}_1 - 2\epsilon \cdot N_{\Omega_1}) \cdot (\mathcal{L}_3 - \mathcal{W}_3) + 2\epsilon \cdot N_{\Omega_3} \cdot (\mathcal{L}_2 - \mathcal{W}_2 - 2\epsilon \cdot N_{\Omega_2}) \cdot (\mathcal{L}_1 - \mathcal{W}_1 - 2\epsilon \cdot N_{\Omega_1}) + \mathcal{W}_1 \mathcal{L}_2 \mathcal{L}_3 + \mathcal{L}_1 \mathcal{W}_2 \mathcal{L}_3 + \mathcal{L}_1 \mathcal{L}_2 \mathcal{W}_3 - \mathcal{L}_1 \mathcal{W}_2 \mathcal{W}_3 - \mathcal{W}_1 \mathcal{L}_2 \mathcal{W}_3 - \mathcal{W}_1 \mathcal{W}_2 \mathcal{L}_3 + \mathcal{W}_1 \mathcal{W}_2 \mathcal{W}_3$$

$$V(\epsilon) = \mathcal{L}_1 \mathcal{L}_2 V_3(\epsilon) + \mathcal{L}_1 V_2(\epsilon) \mathcal{L}_3 + V_1(\epsilon) \mathcal{L}_2 \mathcal{L}_3 - V_1(\epsilon) V_2(\epsilon) \mathcal{L}_3 - V_1(\epsilon) \mathcal{L}_2 V_3(\epsilon) - \mathcal{L}_1 V_2(\epsilon) V_3(\epsilon) + V_1(\epsilon) V_2(\epsilon) V_3(\epsilon)$$

4 Dimensional Fractal Objects

$$V(\epsilon) = 2\epsilon \cdot N_{\Omega_1}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(2\epsilon)} p_j\right) \cdot \left(\sum_{j=1}^{N_{\Omega_3}(2\epsilon)} q_j\right) \cdot \left(\sum_{j=1}^{N_{\Omega_4}(2\epsilon)} z_j\right) + 2\epsilon \cdot N_{\Omega_2}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(2\epsilon)} \ell_j - 2\epsilon\right) \cdot \left(\sum_{j=1}^{N_{\Omega_3}(2\epsilon)} q_j\right) \cdot \left(\sum_{j=1}^{N_{\Omega_4}(2\epsilon)} z_j\right) + 2\epsilon \cdot N_{\Omega_3}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(2\epsilon)} p_j - 2\epsilon\right) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(2\epsilon)} \ell_j - 2\epsilon\right) \cdot \left(\sum_{j=1}^{N_{\Omega_4}(2\epsilon)} z_j\right) + 2\epsilon \cdot N_{\Omega_4}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(2\epsilon)} p_j - 2\epsilon\right) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(2\epsilon)} \ell_j - 2\epsilon\right) \cdot \left(\sum_{j=1}^{N_{\Omega_3}(2\epsilon)} q_j - 2\epsilon\right) + \mathcal{W}_1\mathcal{L}_2\mathcal{L}_3\mathcal{L}_4 + \mathcal{L}_1\mathcal{W}_2\mathcal{L}_3\mathcal{L}_4 + \mathcal{L}_1\mathcal{L}_2\mathcal{W}_3\mathcal{L}_4 + \mathcal{L}_1\mathcal{L}_2\mathcal{L}_3\mathcal{W}_4 - \mathcal{L}_1\mathcal{L}_2\mathcal{W}_3\mathcal{W}_4 - \mathcal{L}_1\mathcal{W}_2\mathcal{L}_3\mathcal{W}_4 - \mathcal{L}_1\mathcal{W}_2\mathcal{W}_3\mathcal{L}_4 - \mathcal{W}_1\mathcal{L}_2\mathcal{L}_3\mathcal{W}_4 - \mathcal{W}_1\mathcal{L}_2\mathcal{W}_3\mathcal{L}_4 - \mathcal{W}_1\mathcal{W}_2\mathcal{L}_3\mathcal{L}_4 + \mathcal{L}_1\mathcal{W}_2\mathcal{W}_3\mathcal{W}_4 + \mathcal{W}_1\mathcal{L}_2\mathcal{W}_3\mathcal{W}_4 + \mathcal{W}_1\mathcal{W}_2\mathcal{L}_3\mathcal{W}_4 + \mathcal{W}_1\mathcal{W}_2\mathcal{W}_3\mathcal{L}_4 - \mathcal{W}_1\mathcal{W}_2\mathcal{W}_3\mathcal{W}_4$$

4 Dimensional Volume Formula

$$V^4(\epsilon) = \mathcal{L}_1\mathcal{L}_2\mathcal{L}_3 V_4(\epsilon) + \mathcal{L}_1\mathcal{L}_2 V_3(\epsilon)\mathcal{L}_4 + \mathcal{L}_1 V_2(\epsilon)\mathcal{L}_3\mathcal{L}_4 + V_1(\epsilon)\mathcal{L}_2\mathcal{L}_3\mathcal{L}_4 - V_1(\epsilon)V_2(\epsilon)\mathcal{L}_3\mathcal{L}_4 - V_1(\epsilon)\mathcal{L}_2 V_3(\epsilon)\mathcal{L}_4 - V_1(\epsilon)\mathcal{L}_2\mathcal{L}_3 V_4(\epsilon) - \mathcal{L}_1 V_2(\epsilon)V_3(\epsilon)\mathcal{L}_4 - \mathcal{L}_1 V_2(\epsilon)\mathcal{L}_3 V_4(\epsilon) - \mathcal{L}_1\mathcal{L}_2 V_3(\epsilon)V_4(\epsilon) + V_1(\epsilon)V_2(\epsilon)V_3(\epsilon)\mathcal{L}_4 + V_1(\epsilon)V_2(\epsilon)\mathcal{L}_3 V_4(\epsilon) + V_1(\epsilon)\mathcal{L}_2 V_3(\epsilon)V_4(\epsilon) + \mathcal{L}_1 V_2(\epsilon)V_3(\epsilon)V_4(\epsilon) - V_1(\epsilon)V_2(\epsilon)V_3(\epsilon)V_4(\epsilon)$$

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4 Dimensional Volume Formula

$$V^4(\epsilon) = \mathcal{L}_1\mathcal{L}_2\mathcal{L}_3 V_4(\epsilon) + \mathcal{L}_1\mathcal{L}_2 V_3(\epsilon)\mathcal{L}_4 + \mathcal{L}_1 V_2(\epsilon)\mathcal{L}_3\mathcal{L}_4 + V_1(\epsilon)\mathcal{L}_2\mathcal{L}_3\mathcal{L}_4 - V_1(\epsilon)V_2(\epsilon)\mathcal{L}_3\mathcal{L}_4 - V_1(\epsilon)\mathcal{L}_2 V_3(\epsilon)\mathcal{L}_4 - V_1(\epsilon)\mathcal{L}_2\mathcal{L}_3 V_4(\epsilon) - \mathcal{L}_1 V_2(\epsilon)V_3(\epsilon)\mathcal{L}_4 - \mathcal{L}_1 V_2(\epsilon)\mathcal{L}_3 V_4(\epsilon) - \mathcal{L}_1\mathcal{L}_2 V_3(\epsilon)V_4(\epsilon) + V_1(\epsilon)V_2(\epsilon)V_3(\epsilon)\mathcal{L}_4 + V_1(\epsilon)V_2(\epsilon)\mathcal{L}_3 V_4(\epsilon) + V_1(\epsilon)\mathcal{L}_2 V_3(\epsilon)V_4(\epsilon) + \mathcal{L}_1 V_2(\epsilon)V_3(\epsilon)V_4(\epsilon) - V_1(\epsilon)V_2(\epsilon)V_3(\epsilon)V_4(\epsilon)$$

$$V^3(\epsilon) = \mathcal{L}_1\mathcal{L}_2 V_3(\epsilon) + \mathcal{L}_1 V_2(\epsilon)\mathcal{L}_3 + V_1(\epsilon)\mathcal{L}_2\mathcal{L}_3 - V_1(\epsilon)V_2(\epsilon)\mathcal{L}_3 - V_1(\epsilon)\mathcal{L}_2 V_3(\epsilon) - \mathcal{L}_1 V_2(\epsilon)V_3(\epsilon) + V_1(\epsilon)V_2(\epsilon)V_3(\epsilon)$$

$$V^2(\epsilon) = \mathcal{L}_1 V_2(\epsilon) + V_1(\epsilon)\mathcal{L}_2 - V_1(\epsilon)V_2(\epsilon)$$

Algebraic Manipulations

$$
\begin{aligned}
V^2(\epsilon) &= \mathcal{L}_1 V_2(\epsilon) + V_1(\epsilon)\mathcal{L}_2 - V_1(\epsilon) V_2(\epsilon) \\
&= (-1) \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) + \mathcal{L}_1 \mathcal{L}_2
\end{aligned}
$$

Algebraic Manipulations

$$V^2(\epsilon) = \mathcal{L}_1 V_2(\epsilon) + V_1(\epsilon)\mathcal{L}_2 - V_1(\epsilon) V_2(\epsilon)$$

$$= (-1) \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) + \mathcal{L}_1 \mathcal{L}_2$$

$$V^3(\epsilon) = \mathcal{L}_1\mathcal{L}_2 V_3(\epsilon) + \mathcal{L}_1 V_2(\epsilon)\mathcal{L}_3 + V_1(\epsilon)\mathcal{L}_2\mathcal{L}_3 - V_1(\epsilon) V_2(\epsilon)\mathcal{L}_3 - V_1(\epsilon)\mathcal{L}_2 V_3(\epsilon) - \mathcal{L}_1 V_2(\epsilon) V_3(\epsilon) + V_1(\epsilon) V_2(\epsilon) V_3(\epsilon)$$

$$= (-1)^2 \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) \cdot (V_3(\epsilon) - \mathcal{L}_3) + \mathcal{L}_1\mathcal{L}_2\mathcal{L}_3$$

Algebraic Manipulations

$$V^4(\epsilon) = \mathcal{L}_1\mathcal{L}_2\mathcal{L}_3 V_4(\epsilon) + \mathcal{L}_1\mathcal{L}_2 V_3(\epsilon)\mathcal{L}_4 + \mathcal{L}_1 V_2(\epsilon)\mathcal{L}_3\mathcal{L}_4 + V_1(\epsilon)\mathcal{L}_2\mathcal{L}_3\mathcal{L}_4 - V_1(\epsilon) V_2(\epsilon)\mathcal{L}_3\mathcal{L}_4 - V_1(\epsilon)\mathcal{L}_2 V_3(\epsilon)\mathcal{L}_4 - V_1(\epsilon)\mathcal{L}_2\mathcal{L}_3 V_4(\epsilon) - \mathcal{L}_1 V_2(\epsilon) V_3(\epsilon)\mathcal{L}_4 - \mathcal{L}_1 V_2(\epsilon)\mathcal{L}_3 V_4(\epsilon) - \mathcal{L}_1\mathcal{L}_2 V_3(\epsilon) V_4(\epsilon) + V_1(\epsilon) V_2(\epsilon) V_3(\epsilon)\mathcal{L}_4 + V_1(\epsilon) V_2(\epsilon)\mathcal{L}_3 V_4(\epsilon) + V_1(\epsilon)\mathcal{L}_2 V_3(\epsilon) V_4(\epsilon) + \mathcal{L}_1 V_2(\epsilon) V_3(\epsilon) V_4(\epsilon) - V_1(\epsilon) V_2(\epsilon) V_3(\epsilon) V_4(\epsilon)$$

$$= (-1)^3\cdot(V_1(\epsilon)-\mathcal{L}_1)\cdot(V_2(\epsilon)-\mathcal{L}_2)\cdot(V_3(\epsilon)-\mathcal{L}_3)\cdot(V_4(\epsilon)-\mathcal{L}_4)+\mathcal{L}_1\mathcal{L}_2\mathcal{L}_3\mathcal{L}_4$$

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General Volume Formula

$$V^2(\epsilon) = (-1) \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) + \mathcal{L}_1\mathcal{L}_2$$

$$V^3(\epsilon) = (-1)^2 \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) \cdot (V_3(\epsilon) - \mathcal{L}_3) + \mathcal{L}_1\mathcal{L}_2\mathcal{L}_3$$

$$V^4(\epsilon) = (-1)^3 \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) \cdot (V_3(\epsilon) - \mathcal{L}_3) \cdot (V_4(\epsilon) - \mathcal{L}_4) + \mathcal{L}_1\mathcal{L}_2\mathcal{L}_3\mathcal{L}_4$$

$$V^n(\epsilon) = \left[(-1)^{n-1} \cdot \prod_{k=1}^{n} (V_k(\epsilon) - \mathcal{L}_k) \right] + \prod_{k=1}^{n} \mathcal{L}_k$$

Future Work

- Prove the general volume formula
- Find a general formula for dimension (based on general volume formula)

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