

Badass Title

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What is a Fractal String

Definition 1

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The fractal string we studied the most in particular is the **Cantor String**



Cantor String

Inner-Tubular Neighborhood

Definition 2

Inner-tubular neighborhood: $V(\epsilon) = \text{vol}_1\{x \in \Omega \mid d(x, \partial\Omega) < \epsilon\}$



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Counting function: $N_\Omega(x) = \sum_{\ell_j^{-1} \leq x} 1$

More Notation: $\mathcal{W}_\Omega(x) = \sum_{j: \ell_j^{-1} > x} \ell_j$

Now we can write: $V(\epsilon) = 2\epsilon \cdot N_\Omega\left(\frac{1}{2\epsilon}\right) + \mathcal{W}_\Omega\left(\frac{1}{2\epsilon}\right)$



Cantor String Volume



Inner Minkowski Dimension of a Fractal String

Definition 3

Dimension of Ω : $D_\Omega = \inf \{ \alpha \geq 0 \mid V(\epsilon) = O(\epsilon^{1-\alpha}) \text{ as } \epsilon \rightarrow 0^+ \}$

Inner Minkowski Dimension of a Fractal String

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In other terms: $D_\Omega = \text{the smallest } \alpha \text{ st } \lim_{\epsilon \rightarrow 0^+} \frac{V(\epsilon)}{\epsilon^{1-\alpha}} = c, \ c \in \mathbb{R}$

Inner Minkowski Dimension of a Fractal String

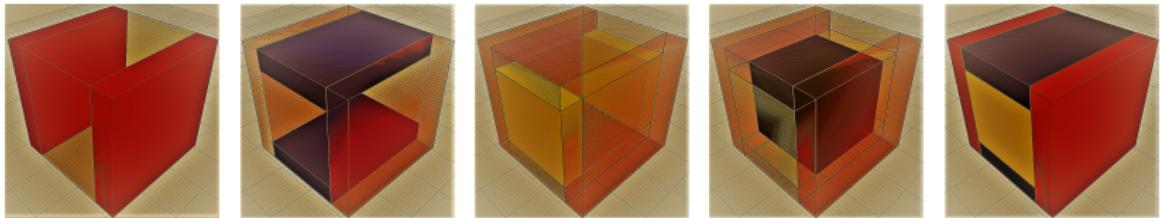
Definition 3

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Cantor String Dimension: $D_{cs} = \log_3 2$

3 Dimensional Fractal Objects



$$V(\epsilon) = 2\epsilon \cdot N_{\Omega_1}(\frac{1}{2\epsilon}) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(\frac{1}{2\epsilon})} p_j \right) \cdot \left(\sum_{j=1}^{N_{\Omega_3}(\frac{1}{2\epsilon})} q_j \right) + \\ 2\epsilon \cdot N_{\Omega_2}(\frac{1}{2\epsilon}) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(\frac{1}{2\epsilon})} \ell_j - 2\epsilon \right) \cdot \left(\sum_{j=1}^{N_{\Omega_3}(\frac{1}{2\epsilon})} q_j \right) +$$

$$2\epsilon \cdot N_{\Omega_3}(\frac{1}{2\epsilon}) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(\frac{1}{2\epsilon})} p_j - 2\epsilon \right) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(\frac{1}{2\epsilon})} \ell_j - 2\epsilon \right)$$

$$+ \mathcal{W}_1 \mathcal{L}_2 \mathcal{L}_3 + \mathcal{L}_1 \mathcal{W}_2 \mathcal{L}_3 + \mathcal{L}_1 \mathcal{L}_2 \mathcal{W}_3 - \mathcal{L}_1 \mathcal{W}_2 \mathcal{W}_3 - \mathcal{W}_1 \mathcal{L}_2 \mathcal{W}_3 -$$

$$\mathcal{W}_1 \mathcal{W}_2 \mathcal{L}_3 + \mathcal{W}_1 \mathcal{W}_2 \mathcal{W}_3$$

3 Dimensional Fractal Volume

$$V(\epsilon) = N_{\Omega_1} \cdot (\mathcal{L}_2 - \mathcal{W}_3) \cdot (\mathcal{L}_3 - \mathcal{W}_3) + N_{\Omega_2} \cdot (\mathcal{L}_1 - \mathcal{W}_1 - 2\epsilon \cdot N_1) \cdot (\mathcal{L}_3 - \mathcal{W}_3) + N_{\Omega_3} \cdot (\mathcal{L}_2 - \mathcal{W}_2 - 2\epsilon \cdot N_2) \cdot (\mathcal{L}_3 - \mathcal{W}_3 - 2\epsilon \cdot N_3) + \mathcal{W}_1 \mathcal{L}_2 \mathcal{L}_3 + \mathcal{L}_1 \mathcal{W}_2 \mathcal{L}_3 + \mathcal{L}_1 \mathcal{L}_2 \mathcal{W}_3 - \mathcal{L}_1 \mathcal{W}_2 \mathcal{W}_3 - \mathcal{W}_1 \mathcal{L}_2 \mathcal{W}_3 - \mathcal{W}_1 \mathcal{W}_2 \mathcal{L}_3 + \mathcal{W}_1 \mathcal{W}_2 \mathcal{W}_3$$

3 Dimensional Fractal Volume

$$V(\epsilon) = N_{\Omega_1} \cdot (\mathcal{L}_2 - \mathcal{W}_3) \cdot (\mathcal{L}_3 - \mathcal{W}_3) + N_{\Omega_2} \cdot (\mathcal{L}_1 - \mathcal{W}_1 - 2\epsilon \cdot N_1) \cdot (\mathcal{L}_3 - \mathcal{W}_3) + N_{\Omega_3} \cdot (\mathcal{L}_2 - \mathcal{W}_2 - 2\epsilon \cdot N_2) \cdot (\mathcal{L}_3 - \mathcal{W}_3 - 2\epsilon \cdot N_3) + \mathcal{W}_1 \mathcal{L}_2 \mathcal{L}_3 + \mathcal{L}_1 \mathcal{W}_2 \mathcal{L}_3 + \mathcal{L}_1 \mathcal{L}_2 \mathcal{W}_3 - \mathcal{L}_1 \mathcal{W}_2 \mathcal{W}_3 - \mathcal{W}_1 \mathcal{L}_2 \mathcal{W}_3 - \mathcal{W}_1 \mathcal{W}_2 \mathcal{L}_3 + \mathcal{W}_1 \mathcal{W}_2 \mathcal{W}_3$$

$$V(\epsilon) = \mathcal{L}_1 \mathcal{L}_2 V_3(\epsilon) + \mathcal{L}_1 V_2(\epsilon) \mathcal{L}_3 + V_1(\epsilon) \mathcal{L}_2 \mathcal{L}_3 - V_1(\epsilon) V_2(\epsilon) \mathcal{L}_3 - V_1(\epsilon) \mathcal{L}_2 V_3(\epsilon) - \mathcal{L}_1 V_2(\epsilon) V_3(\epsilon) + V_1(\epsilon) V_2(\epsilon) V_3(\epsilon)$$

4 Dimensional Fractal Objects

$$\begin{aligned} V(\epsilon) = & 2\epsilon \cdot N_{\Omega_1}(\frac{1}{2\epsilon}) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(\frac{1}{2\epsilon})} p_j \right) \cdot \left(\sum_{j=1}^{N_{\Omega_3}(\frac{1}{2\epsilon})} q_j \right) \cdot \left(\sum_{j=1}^{N_{\Omega_4}(\frac{1}{2\epsilon})} z_j \right) + \\ & 2\epsilon \cdot N_{\Omega_2}(\frac{1}{2\epsilon}) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(\frac{1}{2\epsilon})} \ell_j - 2\epsilon \right) \cdot \left(\sum_{j=1}^{N_{\Omega_3}(\frac{1}{2\epsilon})} q_j \right) \cdot \left(\sum_{j=1}^{N_{\Omega_4}(\frac{1}{2\epsilon})} z_j \right) + 2\epsilon \cdot \\ & N_{\Omega_3}(\frac{1}{2\epsilon}) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(\frac{1}{2\epsilon})} p_j - 2\epsilon \right) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(\frac{1}{2\epsilon})} \ell_j - 2\epsilon \right) \cdot \left(\sum_{j=1}^{N_{\Omega_4}(\frac{1}{2\epsilon})} z_j \right) + 2\epsilon \cdot \\ & N_{\Omega_4}(\frac{1}{2\epsilon}) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(\frac{1}{2\epsilon})} p_j - 2\epsilon \right) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(\frac{1}{2\epsilon})} \ell_j - 2\epsilon \right) \cdot \left(\sum_{j=1}^{N_{\Omega_3}(\frac{1}{2\epsilon})} q_j - 2\epsilon \right) + \\ & W_1 L_2 L_3 L_4 + L_1 W_2 L_3 L_4 + L_1 L_2 W_3 L_4 + L_1 L_2 L_3 W_4 - L_1 L_2 W_3 W_4 - \\ & L_1 W_2 L_3 W_4 - L_1 W_2 W_3 L_4 - W_1 L_2 L_3 W_4 - W_1 L_2 W_3 L_4 - W_1 W_2 L_3 L_4 + \\ & L_1 W_2 W_3 W_4 + W_1 L_2 W_3 W_4 + W_1 W_2 L_3 W_4 + W_1 W_2 W_3 L_4 - W_1 W_2 W_3 W_4 \end{aligned}$$

4 Dimensional Volume Formula

$$V^4(\epsilon) = \mathcal{L}_1 \mathcal{L}_2 \mathcal{L}_3 V_4(\epsilon) + \mathcal{L}_1 \mathcal{L}_2 V_3(\epsilon) \mathcal{L}_4 + \mathcal{L}_1 V_2(\epsilon) \mathcal{L}_3 \mathcal{L}_4 + V_1(\epsilon) \mathcal{L}_2 \mathcal{L}_3 \mathcal{L}_4 - V_1(\epsilon) V_2(\epsilon) \mathcal{L}_3 \mathcal{L}_4 - V_1(\epsilon) \mathcal{L}_2 V_3(\epsilon) \mathcal{L}_4 - V_1(\epsilon) \mathcal{L}_2 \mathcal{L}_3 V_4(\epsilon) - \mathcal{L}_1 V_2(\epsilon) V_3(\epsilon) \mathcal{L}_4 - \mathcal{L}_1 V_2(\epsilon) \mathcal{L}_3 V_4(\epsilon) - \mathcal{L}_1 \mathcal{L}_2 V_3(\epsilon) V_4(\epsilon) + V_1(\epsilon) V_2(\epsilon) V_3(\epsilon) \mathcal{L}_4 + V_1(\epsilon) V_2(\epsilon) \mathcal{L}_3 V_4(\epsilon) + V_1(\epsilon) \mathcal{L}_2 V_3(\epsilon) V_4(\epsilon) + \mathcal{L}_1 V_2(\epsilon) V_3(\epsilon) V_4(\epsilon) - V_1(\epsilon) V_2(\epsilon) V_3(\epsilon) V_4(\epsilon)$$



4 Dimensional Volume Formula

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$$V^3(\epsilon) = \mathcal{L}_1 \mathcal{L}_2 V_3(\epsilon) + \mathcal{L}_1 V_2(\epsilon) \mathcal{L}_3 + V_1(\epsilon) \mathcal{L}_2 \mathcal{L}_3 - V_1(\epsilon) V_2(\epsilon) \mathcal{L}_3 - V_1(\epsilon) \mathcal{L}_2 V_3(\epsilon) - \mathcal{L}_1 V_2(\epsilon) V_3(\epsilon) + V_1(\epsilon) V_2(\epsilon) V_3(\epsilon)$$

$$V^2(\epsilon) = \mathcal{L}_1 V_2(\epsilon) + V_1(\epsilon) \mathcal{L}_2 - V_1(\epsilon) V_2(\epsilon)$$

Algebraic Manipulations

$$\begin{aligned}V^2(\epsilon) &= \mathcal{L}_1 V_2(\epsilon) + V_1(\epsilon) \mathcal{L}_2 - V_1(\epsilon) V_2(\epsilon) \\&= (-1) \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) + \mathcal{L}_1 \mathcal{L}_2\end{aligned}$$

Algebraic Manipulations

$$\begin{aligned}V^2(\epsilon) &= \mathcal{L}_1 V_2(\epsilon) + V_1(\epsilon) \mathcal{L}_2 - V_1(\epsilon) V_2(\epsilon) \\&= (-1) \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) + \mathcal{L}_1 \mathcal{L}_2\end{aligned}$$

$$\begin{aligned}V^3(\epsilon) &= \mathcal{L}_1 \mathcal{L}_2 V_3(\epsilon) + \mathcal{L}_1 V_2(\epsilon) \mathcal{L}_3 + V_1(\epsilon) \mathcal{L}_2 \mathcal{L}_3 - V_1(\epsilon) V_2(\epsilon) \mathcal{L}_3 - V_1(\epsilon) \mathcal{L}_2 V_3(\epsilon) - \\&\quad \mathcal{L}_1 V_2(\epsilon) V_3(\epsilon) + V_1(\epsilon) V_2(\epsilon) V_3(\epsilon) \\&= (-1)^2 \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) \cdot (V_3(\epsilon) - \mathcal{L}_3) + \mathcal{L}_1 \mathcal{L}_2 \mathcal{L}_3\end{aligned}$$

Algebraic Manipulations

$$\begin{aligned}V^4(\epsilon) &= \mathcal{L}_1 \mathcal{L}_2 \mathcal{L}_3 V_4(\epsilon) + \mathcal{L}_1 \mathcal{L}_2 V_3(\epsilon) \mathcal{L}_4 + \mathcal{L}_1 V_2(\epsilon) \mathcal{L}_3 \mathcal{L}_4 + V_1(\epsilon) \mathcal{L}_2 \mathcal{L}_3 \mathcal{L}_4 - \\& V_1(\epsilon) V_2(\epsilon) \mathcal{L}_3 \mathcal{L}_4 - V_1(\epsilon) \mathcal{L}_2 V_3(\epsilon) \mathcal{L}_4 - V_1(\epsilon) \mathcal{L}_2 \mathcal{L}_3 V_4(\epsilon) - \mathcal{L}_1 V_2(\epsilon) V_3(\epsilon) \mathcal{L}_4 - \\& \mathcal{L}_1 V_2(\epsilon) \mathcal{L}_3 V_4(\epsilon) - \mathcal{L}_1 \mathcal{L}_2 V_3(\epsilon) V_4(\epsilon) + V_1(\epsilon) V_2(\epsilon) V_3(\epsilon) \mathcal{L}_4 + V_1(\epsilon) V_2(\epsilon) \mathcal{L}_3 V_4(\epsilon) + \\& V_1(\epsilon) \mathcal{L}_2 V_3(\epsilon) V_4(\epsilon) + \mathcal{L}_1 V_2(\epsilon) V_3(\epsilon) V_4(\epsilon) - V_1(\epsilon) V_2(\epsilon) V_3(\epsilon) V_4(\epsilon) \\& = (-1)^3 \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) \cdot (V_3(\epsilon) - \mathcal{L}_3) \cdot (V_4(\epsilon) - \mathcal{L}_4) + \mathcal{L}_1 \mathcal{L}_2 \mathcal{L}_3 \mathcal{L}_4\end{aligned}$$

General Volume Formula

$$V^2(\epsilon) = (-1) \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) + \mathcal{L}_1 \mathcal{L}_2$$

$$V^3(\epsilon) = (-1)^2 \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) \cdot (V_3(\epsilon) - \mathcal{L}_3) + \mathcal{L}_1 \mathcal{L}_2 \mathcal{L}_3$$

$$V^4(\epsilon) = (-1)^3 \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) \cdot (V_3(\epsilon) - \mathcal{L}_3) \cdot (V_4(\epsilon) - \mathcal{L}_4) + \mathcal{L}_1 \mathcal{L}_2 \mathcal{L}_3 \mathcal{L}_4$$

$$V^n(\epsilon) = \left[(-1)^{n-1} \cdot \prod_{k=1}^n (V_k(\epsilon) - \mathcal{L}_k) \right] + \prod_{k=1}^n \mathcal{L}_k$$

Future Work

- Prove the general volume formula
- Find a general formula for dimension (based on general volume formula)

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