Inner Minkowski Dimension of Products of Fractal Strings

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What is a fractal string

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Introduce the Cantor string so that the audience can see an example right away. A movie would be nice!

Cantor String

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Lapidus, van Frankenhuijsen, [9, 10, 11]

$$D_{\mathcal{L}} = \inf\{\alpha \in \mathbb{R} \mid \sum_{j=1}^{\infty} \ell_j^{\alpha} < \infty\} \implies 0 < D_{\mathcal{L}} < 1$$

Cantor string volume

Fractal lawn

Definition 2

A **fractal lawn** $\mathcal{L}^2$ is the Cartesian product of two fractal strings $\Omega = \Omega_1 \times \Omega_2$.

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The Cartesian product of two fractal strings is a bounded open subset of $\mathbb{R}^2$ since a fractal string is a bounded open subset of $\mathbb{R}$.

$$\Omega_1 = \bigcup_{j=1}^{\infty} \ell_j, \, \Omega_2 = \bigcup_{i=1}^{\infty} l_i$$

$$\ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots \geq 0, \, l_1 \geq l_2 \geq l_3 \geq \cdots \geq 0$$

Cantor lawn

Dimension of fractal lawn

Inner-tubular neighborhood: $V(\epsilon) = \mathrm{vol}_2\{x \in \Omega \mid d(x, \partial\Omega) < \epsilon\}$

- $\partial\Omega = (\partial\Omega_1 \times \Omega_2) \cup (\Omega_1 \times \partial\Omega_2)$

Counting function: Counts the critical length for any $x > 0$

- $N_{\Omega_i}(x) = \sum_{j:\ell_j^{-1} \leq x} 1$

Useful Definitions:

- $\mathcal{W}_i(x) = \sum_{j:\ell_j^{-1} \leq x} \ell_j$
- $\mathcal{L}_i(x) = \sum_{j=1}^{N_{\Omega_i}(x)} \ell_j = \Omega_i - \mathcal{W}_i(x)$

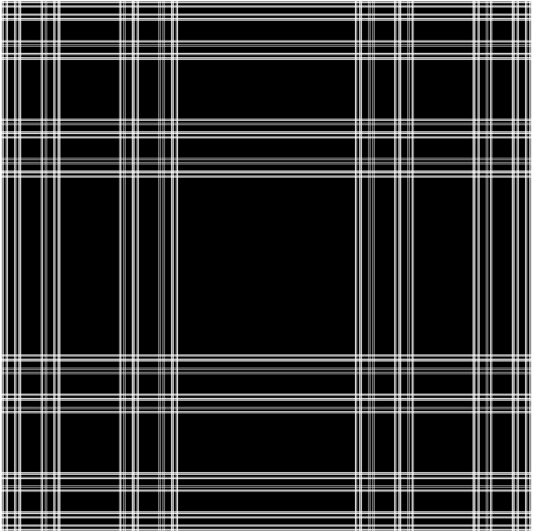
Dimension of fractal lawn

- Partially covered volume
  - $V_{partial}(\epsilon) = 2\epsilon N_{\Omega_1}(\frac{1}{2\epsilon})\mathcal{L}_1(\frac{1}{2\epsilon}) + 2\epsilon N_{\Omega_2}(\frac{1}{2\epsilon})\mathcal{L}_2(\frac{1}{2\epsilon})$
- Fully covered volume
  - $V_{full}(\epsilon) = \Omega_1 \mathcal{W}_2(\frac{1}{2\epsilon}) + \Omega_2 \mathcal{W}_1(\frac{1}{2\epsilon})$
- Double counted volume
  - $V_{double}(\epsilon) = \mathcal{W}_1(\frac{1}{2\epsilon})\mathcal{W}_2(\frac{1}{2\epsilon})$

Dimension of fractal lawn

$$V(\epsilon) = V_{partial}(\epsilon) + V_{full}(\epsilon) - V_{double}(\epsilon)$$
$$V(\epsilon) = 2\epsilon N_{\Omega_1}(\tfrac{1}{2\epsilon})\mathcal{L}_1(\tfrac{1}{2\epsilon}) + 2\epsilon N_{\Omega_2}(\tfrac{1}{2\epsilon})\mathcal{L}_2(\tfrac{1}{2\epsilon}) + \Omega_1\mathcal{W}_2(\tfrac{1}{2\epsilon}) + \Omega_2\mathcal{W}_1(\tfrac{1}{2\epsilon}) - \mathcal{W}_1(\tfrac{1}{2\epsilon})\mathcal{W}_2(\tfrac{1}{2\epsilon})$$

Cantor Lawn Volume GIF



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