Badass Title

Isaac Ashton, Joseph Ingenito, Peter Tonuzi

The College of New Jersey

What is a fractal string

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Introduce the Cantor string so that the audience can see an example right away. A movie would be nice!

Cantor String

Dimension of fractal string

Inner-tubular neighborhood: $V(\epsilon) = \mathrm{vol}_1\{x \in \Omega \mid d(x, \partial\Omega) < \epsilon\}$

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Counting function: $N_\Omega(x) = \sum_{\ell_j^{-1} \leq x}$

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Lapidus, van Frankenhuijsen, [9, 10, 11]

$$D_{\mathcal{L}} = \inf\{\alpha \in \mathbb{R} \mid \sum_{j=1}^{\infty} \ell_j^{\alpha} < \infty\} \implies 0 < D_{\mathcal{L}} < 1$$

Cantor string volume

Fractal lawn

Because a fractal string is a bounded open subset of $\mathbb{R}$, the Cartesian product of two fractal strings is a bounded open subset of $\mathbb{R}^2$.

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A **fractal lawn** $\mathcal{L}^2$ is a bounded open subset $\Omega = \{(x, y) : x \in \Omega_1$ and $y \in \Omega_2\}$ of $\mathbb{R}^2$.

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$$\Omega_1 = \bigcup_{j=1}^{\infty} \ell_j, \; \Omega_2 = \bigcup_{i=1}^{\infty} l_i$$

$$\ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots \geq 0, \; l_1 \geq l_2 \geq l_3 \geq \cdots \geq 0$$

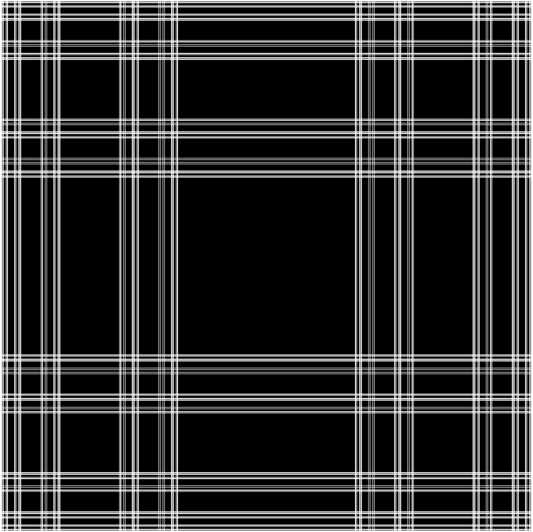
Cantor lawn

Dimension of fractal lawn

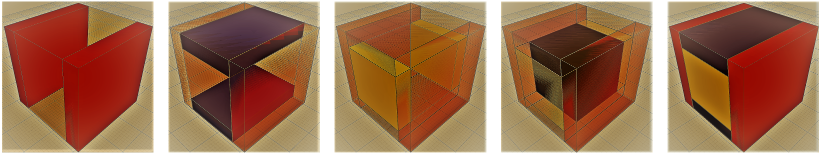
Inner-tubular neighborhood: $V(\epsilon) = \text{vol}_2\{x \in \Omega \mid d(x, \partial\Omega) < \epsilon\}$

Counting functions: $N_{\Omega_1}(x) = \sum_{\ell_j^{-1} \leq x}$, $N_{\Omega_2}(x) = \sum_{l_j^{-1} \leq x}$

Cantor Lawn Volume



3 Dimensional Fractal Objects



$$V(\epsilon) = 2\epsilon \cdot N_{\Omega_1}(\tfrac{1}{2\epsilon}) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(\frac{1}{2\epsilon})} p_j\right) \cdot \left(\sum_{j=1}^{N_{\Omega_3}(\frac{1}{2\epsilon})} q_j\right) +$$

$$2\epsilon \cdot N_{\Omega_2}(\frac{1}{2\epsilon}) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(\frac{1}{2\epsilon})} \ell_j - 2\epsilon\right) \cdot \left(\sum_{j=1}^{N_{\Omega_3}(\frac{1}{2\epsilon})} q_j\right) +$$

$$2\epsilon \cdot N_{\Omega_3}(\frac{1}{2\epsilon}) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(\frac{1}{2\epsilon})} p_j - 2\epsilon\right) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(\frac{1}{2\epsilon})} \ell_j - 2\epsilon\right)$$

$$+\mathcal{W}_1\mathcal{L}_2\mathcal{L}_3 + \mathcal{L}_1\mathcal{W}_2\mathcal{L}_3 + \mathcal{L}_1\mathcal{L}_2\mathcal{W}_3 - \mathcal{L}_1\mathcal{W}_2\mathcal{W}_3 - \mathcal{W}_1\mathcal{L}_2\mathcal{W}_3 -$$

$$\mathcal{W}_1\mathcal{W}_2\mathcal{L}_3 + \mathcal{W}_1\mathcal{W}_2\mathcal{W}_3$$

3 Dimensional Fractal Volume

$$V(\epsilon) = N_{\Omega_1} \cdot (\mathcal{L}_2 - \mathcal{W}_3) \cdot (\mathcal{L}_3 - \mathcal{W}_3) + N_{\Omega_2} \cdot (\mathcal{L}_1 - \mathcal{W}_1 - 2\epsilon \cdot N_1) \cdot (\mathcal{L}_3 - \mathcal{W}_3) + N_{\Omega_3} \cdot (\mathcal{L}_2 - \mathcal{W}_2 - 2\epsilon \cdot N_2) \cdot (\mathcal{L}_3 - \mathcal{W}_3 - 2\epsilon \cdot N_3) + \mathcal{W}_1\mathcal{L}_2\mathcal{L}_3 + \mathcal{L}_1\mathcal{W}_2\mathcal{L}_3 + \mathcal{L}_1\mathcal{L}_2\mathcal{W}_3 - \mathcal{L}_1\mathcal{W}_2\mathcal{W}_3 - \mathcal{W}_1\mathcal{L}_2\mathcal{W}_3 - \mathcal{W}_1\mathcal{W}_2\mathcal{L}_3 + \mathcal{W}_1\mathcal{W}_2\mathcal{W}_3$$

3 Dimensional Fractal Volume

$$V(\epsilon) = N_{\Omega_1} \cdot (\mathcal{L}_2 - \mathcal{W}_3) \cdot (\mathcal{L}_3 - \mathcal{W}_3) + N_{\Omega_2} \cdot (\mathcal{L}_1 - \mathcal{W}_1 - 2\epsilon \cdot N_1) \cdot (\mathcal{L}_3 - \mathcal{W}_3) + N_{\Omega_3} \cdot (\mathcal{L}_2 - \mathcal{W}_2 - 2\epsilon \cdot N_2) \cdot (\mathcal{L}_3 - \mathcal{W}_3 - 2\epsilon \cdot N_3) + \mathcal{W}_1\mathcal{L}_2\mathcal{L}_3 + \mathcal{L}_1\mathcal{W}_2\mathcal{L}_3 + \mathcal{L}_1\mathcal{L}_2\mathcal{W}_3 - \mathcal{L}_1\mathcal{W}_2\mathcal{W}_3 - \mathcal{W}_1\mathcal{L}_2\mathcal{W}_3 - \mathcal{W}_1\mathcal{W}_2\mathcal{L}_3 + \mathcal{W}_1\mathcal{W}_2\mathcal{W}_3$$

$$V(\epsilon) = \mathcal{L}_1\mathcal{L}_2 V_3(\epsilon) + \mathcal{L}_1 V_2(\epsilon)\mathcal{L}_3 + V_1(\epsilon)\mathcal{L}_2\mathcal{L}_3 - V_1(\epsilon)V_2(\epsilon)\mathcal{L}_3 - V_1(\epsilon)\mathcal{L}_2 V_3(\epsilon) - \mathcal{L}_1 V_2(\epsilon)V_3(\epsilon) + V_1(\epsilon)V_2(\epsilon)V_3(\epsilon)$$

4 Dimensional Fractal Objects

$$V(\epsilon) = 2\epsilon \cdot N_{\Omega_1}(\tfrac{1}{2\epsilon}) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(\frac{1}{2\epsilon})} p_j\right) \cdot \left(\sum_{j=1}^{N_{\Omega_3}(\frac{1}{2\epsilon})} q_j\right) \cdot \left(\sum_{j=1}^{N_{\Omega_4}(\frac{1}{2\epsilon})} z_j\right) + 2\epsilon \cdot N_{\Omega_2}(\tfrac{1}{2\epsilon}) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(\frac{1}{2\epsilon})} \ell_j - 2\epsilon\right) \cdot \left(\sum_{j=1}^{N_{\Omega_3}(\frac{1}{2\epsilon})} q_j\right) \cdot \left(\sum_{j=1}^{N_{\Omega_4}(\frac{1}{2\epsilon})} z_j\right) + 2\epsilon \cdot N_{\Omega_3}(\tfrac{1}{2\epsilon}) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(\frac{1}{2\epsilon})} p_j - 2\epsilon\right) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(\frac{1}{2\epsilon})} \ell_j - 2\epsilon\right) \cdot \left(\sum_{j=1}^{N_{\Omega_4}(\frac{1}{2\epsilon})} z_j\right) + 2\epsilon \cdot N_{\Omega_4}(\tfrac{1}{2\epsilon}) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(\frac{1}{2\epsilon})} p_j - 2\epsilon\right) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(\frac{1}{2\epsilon})} \ell_j - 2\epsilon\right) \cdot \left(\sum_{j=1}^{N_{\Omega_3}(\frac{1}{2\epsilon})} q_j - 2\epsilon\right) + \mathcal{W}_1\mathcal{L}_2\mathcal{L}_3\mathcal{L}_4 + \mathcal{L}_1\mathcal{W}_2\mathcal{L}_3\mathcal{L}_4 + \mathcal{L}_1\mathcal{L}_2\mathcal{W}_3\mathcal{L}_4 + \mathcal{L}_1\mathcal{L}_2\mathcal{L}_3\mathcal{W}_4 - \mathcal{L}_1\mathcal{L}_2\mathcal{W}_3\mathcal{W}_4 - \mathcal{L}_1\mathcal{W}_2\mathcal{L}_3\mathcal{W}_4 - \mathcal{L}_1\mathcal{W}_2\mathcal{W}_3\mathcal{L}_4 - \mathcal{W}_1\mathcal{L}_2\mathcal{L}_3\mathcal{W}_4 - \mathcal{W}_1\mathcal{L}_2\mathcal{W}_3\mathcal{L}_4 - \mathcal{W}_1\mathcal{W}_2\mathcal{L}_3\mathcal{L}_4 + \mathcal{L}_1\mathcal{W}_2\mathcal{W}_3\mathcal{W}_4 + \mathcal{W}_1\mathcal{L}_2\mathcal{W}_3\mathcal{W}_4 + \mathcal{W}_1\mathcal{W}_2\mathcal{L}_3\mathcal{W}_4 + \mathcal{W}_1\mathcal{W}_2\mathcal{W}_3\mathcal{L}_4 - \mathcal{W}_1\mathcal{W}_2\mathcal{W}_3\mathcal{W}_4$$

4 Dimensional Volume Formula

$$V^4(\epsilon) = \mathcal{L}_1\mathcal{L}_2\mathcal{L}_3 V_4(\epsilon) + \mathcal{L}_1\mathcal{L}_2 V_3(\epsilon)\mathcal{L}_4 + \mathcal{L}_1 V_2(\epsilon)\mathcal{L}_3\mathcal{L}_4 + V_1(\epsilon)\mathcal{L}_2\mathcal{L}_3\mathcal{L}_4 - V_1(\epsilon)V_2(\epsilon)\mathcal{L}_3\mathcal{L}_4 - V_1(\epsilon)\mathcal{L}_2 V_3(\epsilon)\mathcal{L}_4 - V_1(\epsilon)\mathcal{L}_2\mathcal{L}_3 V_4(\epsilon) - \mathcal{L}_1 V_2(\epsilon)V_3(\epsilon)\mathcal{L}_4 - \mathcal{L}_1 V_2(\epsilon)\mathcal{L}_3 V_4(\epsilon) - \mathcal{L}_1\mathcal{L}_2 V_3(\epsilon)V_4(\epsilon) + V_1(\epsilon)V_2(\epsilon)V_3(\epsilon)\mathcal{L}_4 + V_1(\epsilon)V_2(\epsilon)\mathcal{L}_3 V_4(\epsilon) + V_1(\epsilon)\mathcal{L}_2 V_3(\epsilon)V_4(\epsilon) + \mathcal{L}_1 V_2(\epsilon)V_3(\epsilon)V_4(\epsilon) - V_1(\epsilon)V_2(\epsilon)V_3(\epsilon)V_4(\epsilon)$$

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$$V^3(\epsilon) = \mathcal{L}_1\mathcal{L}_2 V_3(\epsilon) + \mathcal{L}_1 V_2(\epsilon)\mathcal{L}_3 + V_1(\epsilon)\mathcal{L}_2\mathcal{L}_3 - V_1(\epsilon)V_2(\epsilon)\mathcal{L}_3 - V_1(\epsilon)\mathcal{L}_2 V_3(\epsilon) - \mathcal{L}_1 V_2(\epsilon)V_3(\epsilon) + V_1(\epsilon)V_2(\epsilon)V_3(\epsilon)$$

$$V^2(\epsilon) = \mathcal{L}_1 V_2(\epsilon) + V_1(\epsilon)\mathcal{L}_2 - V_1(\epsilon)V_2(\epsilon)$$

Algebraic Manipulations

$$
\begin{aligned}
V^2(\epsilon) &= \mathcal{L}_1 V_2(\epsilon) + V_1(\epsilon)\mathcal{L}_2 - V_1(\epsilon) V_2(\epsilon) \\
&= (-1) \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) + \mathcal{L}_1 \mathcal{L}_2
\end{aligned}
$$

Algebraic Manipulations

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$$= (-1) \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) + \mathcal{L}_1\mathcal{L}_2$$

$$V^3(\epsilon) = \mathcal{L}_1\mathcal{L}_2 V_3(\epsilon) + \mathcal{L}_1 V_2(\epsilon)\mathcal{L}_3 + V_1(\epsilon)\mathcal{L}_2\mathcal{L}_3 - V_1(\epsilon) V_2(\epsilon)\mathcal{L}_3 - V_1(\epsilon)\mathcal{L}_2 V_3(\epsilon) - \mathcal{L}_1 V_2(\epsilon) V_3(\epsilon) + V_1(\epsilon) V_2(\epsilon) V_3(\epsilon)$$

$$= (-1)^2 \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) \cdot (V_3(\epsilon) - \mathcal{L}_3) + \mathcal{L}_1\mathcal{L}_2\mathcal{L}_3$$

Algebraic Manipulations

$$V^4(\epsilon) = \mathcal{L}_1\mathcal{L}_2\mathcal{L}_3V_4(\epsilon) + \mathcal{L}_1\mathcal{L}_2V_3(\epsilon)\mathcal{L}_4 + \mathcal{L}_1V_2(\epsilon)\mathcal{L}_3\mathcal{L}_4 + V_1(\epsilon)\mathcal{L}_2\mathcal{L}_3\mathcal{L}_4 - V_1(\epsilon)V_2(\epsilon)\mathcal{L}_3\mathcal{L}_4 - V_1(\epsilon)\mathcal{L}_2V_3(\epsilon)\mathcal{L}_4 - V_1(\epsilon)\mathcal{L}_2\mathcal{L}_3V_4(\epsilon) - \mathcal{L}_1V_2(\epsilon)V_3(\epsilon)\mathcal{L}_4 - \mathcal{L}_1V_2(\epsilon)\mathcal{L}_3V_4(\epsilon) - \mathcal{L}_1\mathcal{L}_2V_3(\epsilon)V_4(\epsilon) + V_1(\epsilon)V_2(\epsilon)V_3(\epsilon)\mathcal{L}_4 + V_1(\epsilon)V_2(\epsilon)\mathcal{L}_3V_4(\epsilon) + V_1(\epsilon)\mathcal{L}_2V_3(\epsilon)V_4(\epsilon) + \mathcal{L}_1V_2(\epsilon)V_3(\epsilon)V_4(\epsilon) - V_1(\epsilon)V_2(\epsilon)V_3(\epsilon)V_4(\epsilon)$$

$$= (-1)^3\cdot(V_1(\epsilon)-\mathcal{L}_1)\cdot(V_2(\epsilon)-\mathcal{L}_2)\cdot(V_3(\epsilon)-\mathcal{L}_3)\cdot(V_4(\epsilon)-\mathcal{L}_4)+\mathcal{L}_1\mathcal{L}_2\mathcal{L}_3\mathcal{L}_4$$

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General Volume Formula

$$V^2(\epsilon) = (-1) \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) + \mathcal{L}_1\mathcal{L}_2$$

$$V^3(\epsilon) = (-1)^2 \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) \cdot (V_3(\epsilon) - \mathcal{L}_3) + \mathcal{L}_1\mathcal{L}_2\mathcal{L}_3$$

$$V^4(\epsilon) = (-1)^3 \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) \cdot (V_3(\epsilon) - \mathcal{L}_3) \cdot (V_4(\epsilon) - \mathcal{L}_4) + \mathcal{L}_1\mathcal{L}_2\mathcal{L}_3\mathcal{L}_4$$

$$V^n(\epsilon) = \left[(-1)^{n-1} \cdot \prod_{k=1}^{n}(V_k(\epsilon) - \mathcal{L}_k)\right] + \prod_{k=1}^{n}\mathcal{L}_k$$

Future Work

- Prove the general volume formula
- Find a general formula for dimension (based on general volume formula)

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