Inner Minkowski Dimension of Products of Fractal Strings

Joseph Ingenito, Peter Tonuzi

The College of New Jersey

December 2020

What is a Fractal String

Definition

A **fractal string** Ω is a bounded open subset of the real line.

What is a Fractal String

Definition

A **fractal string** Ω is a bounded open subset of the real line.

$$\Omega = \bigcup_{j=1}^{\infty} \ell_j, \qquad \mathcal{L} = \mathrm{Vol}_1(\Omega) = \sum_{j=1}^{\infty} \ell_j \implies \lim_{j\to\infty} \ell_j = 0.$$

What is a Fractal String

Definition

A **fractal string** Ω is a bounded open subset of the real line.

$$\Omega = \bigcup_{j=1}^{\infty} \ell_j, \qquad \mathcal{L} = \mathrm{Vol}_1(\Omega) = \sum_{j=1}^{\infty} \ell_j \implies \lim_{j\to\infty} \ell_j = 0.$$

$$\ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots \geq 0$$

Inner-Tubular Neighborhood

Definition

Inner-tubular neighborhood:

$V(\epsilon) = \mathrm{vol}_1\{x \in \Omega \mid d(x, \partial\Omega) < \epsilon\}$

Inner-Tubular Neighborhood

Definition

Inner-tubular neighborhood:
$V(\epsilon) = \mathrm{vol}_1\{x \in \Omega \mid d(x, \partial\Omega) < \epsilon\}$

Counting function: $$N_\Omega(x) = \sum_{\ell_j \geq x} 1$$

More Notation: $$\mathcal{W}_\Omega(x) = \sum_{j : \ell_j < x} \ell_j$$

$$V(\epsilon) = 2\epsilon \cdot N_\Omega(2\epsilon) + \mathcal{W}_\Omega(2\epsilon)$$

Inner-Minkowski Dimension of a Fractal String

Definition

Inner-Minkowski Dimension of Ω:

$D_\Omega = \inf\left\{\alpha \geq 0 \mid V(\epsilon) = O\left(\epsilon^{1-\alpha}\right) \text{ as } \epsilon \to 0^+\right\}$

Inner-Minkowski Dimension of a Fractal String

Definition

Inner-Minkowski Dimension of Ω:
$D_\Omega = \inf\left\{\alpha \geq 0 \mid V(\epsilon) = O\left(\epsilon^{1-\alpha}\right) \text{ as } \epsilon \to 0^+\right\}$

The fractal string Ω is said to be Minkowski Measureable with Minkowski Content:

$$\mathcal{M} = \mathcal{M}(D;\Omega) = \lim_{\epsilon \to 0^+} V(\epsilon)\epsilon^{1-D_\Omega}$$

Fractal Lawn

Definition

A **fractal lawn** $\Omega \subset \mathbb{R}^2$ is the Cartesian product of two fractal strings $\Omega = \Omega_1 \times \Omega_2$.

Fractal Lawn

Definition

A **fractal lawn** $\Omega \subset \mathbb{R}^2$ is the Cartesian product of two fractal strings $\Omega = \Omega_1 \times \Omega_2$.

$$\mathcal{L}_1 = \sum_{j \in \mathbb{N}} \ell_j < \infty \implies \lim_{j \to \infty} \ell_j = 0$$

$$\mathcal{L}_2 = \sum_{j \in \mathbb{N}} p_j < \infty \implies \lim_{j \to \infty} p_j = 0$$

Fractal Lawn

Definition

A **fractal lawn** $\Omega \subset \mathbb{R}^2$ is the Cartesian product of two fractal strings $\Omega = \Omega_1 \times \Omega_2$.

$$\mathcal{L}_1 = \sum_{j \in \mathbb{N}} \ell_j < \infty \implies \lim_{j \to \infty} \ell_j = 0$$

$$\mathcal{L}_2 = \sum_{j \in \mathbb{N}} p_j < \infty \implies \lim_{j \to \infty} p_j = 0$$

$$\ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots \geq 0$$

$$p_1 \geq p_2 \geq p_3 \geq \cdots \geq 0$$

Fractal Lawn Volume I

Inner-tubular neighborhood:
$V(\epsilon) = \mathrm{vol}_2\{x \in \Omega \mid d(x, \partial\Omega) < \epsilon\}$

Fractal Lawn Volume I

Inner-tubular neighborhood:
$V(\epsilon) = \mathrm{vol}_2\{x \in \Omega \mid d(x, \partial\Omega) < \epsilon\}$

Useful Definition:

- $$\sum_{j=1}^{N_{\Omega_i}(x)} \ell_j = \mathcal{L}_i - \mathcal{W}_i(x)$$

Fractal Lawn Volume II

Partially Covered Volume Box

Fractal Lawn Volume II

Partially Covered Volume Box

- Partially covered volume
  - $V_{partial}(\epsilon) = 2\epsilon N_{\Omega_1}(2\epsilon) \sum_{j=1}^{N_{\Omega_2}(2\epsilon)} p_j + 2\epsilon N_{\Omega_2}(2\epsilon) \sum_{j=1}^{N_{\Omega_1}(2\epsilon)} (\ell_j - 2\epsilon)$

Fractal Lawn Volume II

Partially Covered Volume Box

- Partially covered volume
  - $V_{partial}(\epsilon) = 2\epsilon N_{\Omega_1}(2\epsilon) \sum_{j=1}^{N_{\Omega_2}(2\epsilon)} p_j + 2\epsilon N_{\Omega_2}(2\epsilon) \sum_{j=1}^{N_{\Omega_1}(2\epsilon)} (\ell_j - 2\epsilon)$
- Fully covered volume
  - $V_{full}(\epsilon) = \mathcal{L}_1 \mathcal{W}_2(2\epsilon) + \mathcal{L}_2 \mathcal{W}_1(2\epsilon)$

Fractal Lawn Volume II

Partially Covered Volume Box

- Partially covered volume
  - $V_{partial}(\epsilon) = 2\epsilon N_{\Omega_1}(2\epsilon) \sum_{j=1}^{N_{\Omega_2}(2\epsilon)} p_j + 2\epsilon N_{\Omega_2}(2\epsilon) \sum_{j=1}^{N_{\Omega_1}(2\epsilon)} (\ell_j - 2\epsilon)$
- Fully covered volume
  - $V_{full}(\epsilon) = \mathcal{L}_1 \mathcal{W}_2(2\epsilon) + \mathcal{L}_2 \mathcal{W}_1(2\epsilon)$
- Double counted volume
  - $V_{double}(\epsilon) = \mathcal{W}_1(2\epsilon)\mathcal{W}_2(2\epsilon)$

Fractal Lawn Volume III

$$V(\epsilon) = V_{partial}(\epsilon) + V_{full}(\epsilon) - V_{double}(\epsilon)$$

Fractal Lawn Volume III

$$V(\epsilon) = V_{partial}(\epsilon) + V_{full}(\epsilon) - V_{double}(\epsilon)$$

$$V(\epsilon) = 2\epsilon \cdot N_{\Omega_1}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(2\epsilon)} p_j \right) + 2\epsilon \cdot N_{\Omega_2}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(2\epsilon)} (\ell_j - 2\epsilon) \right) +$$
$$\mathcal{L}_1 \mathcal{W}_2(2\epsilon) + \mathcal{L}_2 \mathcal{W}_1(2\epsilon) - \mathcal{W}_1(2\epsilon)\mathcal{W}_2(2\epsilon)$$

$$= 2\epsilon \cdot N_{\Omega_1} \cdot (\mathcal{L}_2 - \mathcal{W}_2) + 2\epsilon \cdot N_{\Omega_2} \cdot (\mathcal{L}_1 - \mathcal{W}_1 - 2\epsilon \cdot N_{\Omega_1}) +$$
$$\mathcal{L}_1 \mathcal{W}_2 + \mathcal{L}_2 \mathcal{W}_1 - \mathcal{W}_1 \mathcal{W}_2$$

Fractal Lawn Volume IV

$$V(\epsilon) = \mathcal{L}_1 \cdot (2\epsilon \cdot N_{\Omega_2} + \mathcal{W}_2) + \mathcal{L}_2 \cdot (2\epsilon \cdot N_{\Omega_1} + \mathcal{W}_1) - (2\epsilon \cdot N_{\Omega_1} + \mathcal{W}_1) \cdot (2\epsilon \cdot N_{\Omega_2} + \mathcal{W}_2)$$

$$= \mathcal{L}_1 \cdot (2\epsilon \cdot N_{\Omega_2} + \mathcal{W}_2) + \mathcal{L}_2 \cdot (2\epsilon \cdot N_{\Omega_1} + \mathcal{W}_1) - (2\epsilon \cdot N_{\Omega_1} + \mathcal{W}_1) \cdot (2\epsilon \cdot N_{\Omega_2} + \mathcal{W}_2)$$

$$= \mathcal{L}_1 V_2 + \mathcal{L}_2 V_1 - V_1 V_2$$

Inner-Minkowski Dimension of Fractal Lawn I

Definition

The **Inner-Minkowski Dimension**
$D_\Omega = \inf\{\alpha \geq 0 \mid V(\epsilon) = O(\epsilon^{2-\alpha}) \text{ as } \epsilon \to 0^+\}$

Inner-Minkowski Dimension of Fractal Lawn I

Definition

The **Inner-Minkowski Dimension**
$D_\Omega = \inf\{\alpha \geq 0 \mid V(\epsilon) = O(\epsilon^{2-\alpha}) \text{ as } \epsilon \to 0^+\}$

- Since $\Omega \subset \mathbb{R}^2$, V is required to remain bounded by $\epsilon^{2-\alpha}$ as $\epsilon \to 0^+$ rather than the 1D case

Inner-Minkowski Dimension of Fractal Lawn I

Definition

The **Inner-Minkowski Dimension**
$D_\Omega = \inf\{\alpha \geq 0 \mid V(\epsilon) = O(\epsilon^{2-\alpha}) \text{ as } \epsilon \to 0^+\}$

- Since $\Omega \subset \mathbb{R}^2$, V is required to remain bounded by $\epsilon^{2-\alpha}$ as $\epsilon \to 0^+$ rather than the 1D case

$$V(\epsilon) = \mathcal{L}_1 V_2 + \mathcal{L}_2 V_1 - V_1 V_2$$

Inner-Minkowski Dimension of Fractal Lawn I

Definition

The **Inner-Minkowski Dimension**
$D_\Omega = \inf\{\alpha \geq 0 \mid V(\epsilon) = O(\epsilon^{2-\alpha}) \text{ as } \epsilon \to 0^+\}$

- Since $\Omega \subset \mathbb{R}^2$, V is required to remain bounded by $\epsilon^{2-\alpha}$ as $\epsilon \to 0^+$ rather than the 1D case

$$V(\epsilon) = \mathcal{L}_1 V_2 + \mathcal{L}_2 V_1 - V_1 V_2$$

$$\lim_{\epsilon \to 0^+} V(\epsilon) = O(\epsilon^{1-D_{\Omega_2}}) + O(\epsilon^{1-D_{\Omega_1}}) + O(\epsilon^{2-(D_{\Omega_1}+D_{\Omega_2})})$$

Heuristic Intuition

$$V(\epsilon) = c_1\epsilon^{1-D_{\Omega_2}} + c_2\epsilon^{1-D_{\Omega_1}} + c_3\epsilon^{2-(D_{\Omega_1}+D_{\Omega_2})}, c_1, c_2, c_3 \in \mathbb{R}$$

Heuristic Intuition

$$V(\epsilon) = c_1\epsilon^{1-D_{\Omega_2}} + c_2\epsilon^{1-D_{\Omega_1}} + c_3\epsilon^{2-(D_{\Omega_1}+D_{\Omega_2})}, c_1, c_2, c_3 \in \mathbb{R}$$

$$\frac{V(\epsilon)}{\epsilon^{2-\alpha}} = c_1\epsilon^{\alpha-(1+D_{\Omega_2})} + c_2\epsilon^{\alpha-(1+D_{\Omega_1})} + c_3\epsilon^{\alpha-(D_{\Omega_1}+D_{\Omega_2})}$$

Heuristic Intuition

$$V(\epsilon) = c_1\epsilon^{1-D_{\Omega_2}} + c_2\epsilon^{1-D_{\Omega_1}} + c_3\epsilon^{2-(D_{\Omega_1}+D_{\Omega_2})}, c_1, c_2, c_3 \in \mathbb{R}$$

$$\frac{V(\epsilon)}{\epsilon^{2-\alpha}} = c_1\epsilon^{\alpha-(1+D_{\Omega_2})} + c_2\epsilon^{\alpha-(1+D_{\Omega_1})} + c_3\epsilon^{\alpha-(D_{\Omega_1}+D_{\Omega_2})}$$

$$\begin{cases} \alpha - (1 + D_{\Omega_1}) \geq 0 \\ \alpha - (1 + D_{\Omega_2}) \geq 0 \\ \alpha - (D_{\Omega_1} + D_{\Omega_2}) \geq 0 \end{cases}$$

Heuristic Intuition

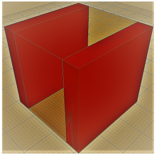
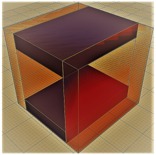
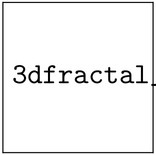
$$V(\epsilon) = c_1\epsilon^{1-D_{\Omega_2}} + c_2\epsilon^{1-D_{\Omega_1}} + c_3\epsilon^{2-(D_{\Omega_1}+D_{\Omega_2})}, c_1, c_2, c_3 \in \mathbb{R}$$

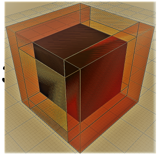
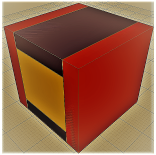
$$\frac{V(\epsilon)}{\epsilon^{2-\alpha}} = c_1\epsilon^{\alpha-(1+D_{\Omega_2})} + c_2\epsilon^{\alpha-(1+D_{\Omega_1})} + c_3\epsilon^{\alpha-(D_{\Omega_1}+D_{\Omega_2})}$$

$$\begin{cases} \alpha - (1 + D_{\Omega_1}) \geq 0 \\ \alpha - (1 + D_{\Omega_2}) \geq 0 \\ \alpha - (D_{\Omega_1} + D_{\Omega_2}) \geq 0 \end{cases}$$

$$D_\Omega = \max\{(1 + D_{\Omega_1}), (1 + D_{\Omega_2})\}$$

3 Dimensional Fractal Objects

$$\begin{aligned}
V(\epsilon) = {} & 2\epsilon \cdot N_{\Omega_1}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(2\epsilon)} p_j \right) \cdot \left(\sum_{j=1}^{N_{\Omega_3}(2\epsilon)} q_j \right) + \\
& 2\epsilon \cdot N_{\Omega_2}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(2\epsilon)} \ell_j - 2\epsilon \right) \cdot \left(\sum_{j=1}^{N_{\Omega_3}(2\epsilon)} q_j \right) + \\
& 2\epsilon \cdot N_{\Omega_3}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(2\epsilon)} p_j - 2\epsilon \right) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(2\epsilon)} \ell_j - 2\epsilon \right) + \\
& \mathcal{W}_1\mathcal{L}_2\mathcal{L}_3 + \mathcal{L}_1\mathcal{W}_2\mathcal{L}_3 + \mathcal{L}_1\mathcal{L}_2\mathcal{W}_3 - \mathcal{L}_1\mathcal{W}_2\mathcal{W}_3 - \mathcal{W}_1\mathcal{L}_2\mathcal{W}_3 - \\
& \mathcal{W}_1\mathcal{W}_2\mathcal{L}_3 + \mathcal{W}_1\mathcal{W}_2\mathcal{W}_3
\end{aligned}$$

3 Dimensional Fractal Volume

$$V(\epsilon) = 2\epsilon \cdot N_{\Omega_1} \cdot (\mathcal{L}_2 - \mathcal{W}_3) \cdot (\mathcal{L}_3 - \mathcal{W}_3) + 2\epsilon \cdot N_{\Omega_2} \cdot (\mathcal{L}_1 - \mathcal{W}_1 - 2\epsilon \cdot N_{\Omega_1}) \cdot (\mathcal{L}_3 - \mathcal{W}_3) + 2\epsilon \cdot N_{\Omega_3} \cdot (\mathcal{L}_2 - \mathcal{W}_2 - 2\epsilon \cdot N_{\Omega_2}) \cdot (\mathcal{L}_1 - \mathcal{W}_1 - 2\epsilon \cdot N_{\Omega_1}) + \mathcal{W}_1\mathcal{L}_2\mathcal{L}_3 + \mathcal{L}_1\mathcal{W}_2\mathcal{L}_3 + \mathcal{L}_1\mathcal{L}_2\mathcal{W}_3 - \mathcal{L}_1\mathcal{W}_2\mathcal{W}_3 - \mathcal{W}_1\mathcal{L}_2\mathcal{W}_3 - \mathcal{W}_1\mathcal{W}_2\mathcal{L}_3 + \mathcal{W}_1\mathcal{W}_2\mathcal{W}_3$$

3 Dimensional Fractal Volume

$$V(\epsilon) = 2\epsilon\cdot N_{\Omega_1}\cdot(\mathcal{L}_2-\mathcal{W}_3)\cdot(\mathcal{L}_3-\mathcal{W}_3)+2\epsilon\cdot N_{\Omega_2}\cdot(\mathcal{L}_1-\mathcal{W}_1-2\epsilon\cdot N_{\Omega_1})\cdot(\mathcal{L}_3-\mathcal{W}_3)+2\epsilon\cdot N_{\Omega_3}\cdot(\mathcal{L}_2-\mathcal{W}_2-2\epsilon\cdot N_{\Omega_2})\cdot(\mathcal{L}_1-\mathcal{W}_1-2\epsilon\cdot N_{\Omega_1})+\mathcal{W}_1\mathcal{L}_2\mathcal{L}_3+\mathcal{L}_1\mathcal{W}_2\mathcal{L}_3+\mathcal{L}_1\mathcal{L}_2\mathcal{W}_3-\mathcal{L}_1\mathcal{W}_2\mathcal{W}_3-\mathcal{W}_1\mathcal{L}_2\mathcal{W}_3-\mathcal{W}_1\mathcal{W}_2\mathcal{L}_3+\mathcal{W}_1\mathcal{W}_2\mathcal{W}_3$$

$$V(\epsilon) = \mathcal{L}_1\mathcal{L}_2V_3(\epsilon)+\mathcal{L}_1V_2(\epsilon)\mathcal{L}_3+V_1(\epsilon)\mathcal{L}_2\mathcal{L}_3-V_1(\epsilon)V_2(\epsilon)\mathcal{L}_3-V_1(\epsilon)\mathcal{L}_2V_3(\epsilon)-\mathcal{L}_1V_2(\epsilon)V_3(\epsilon)+V_1(\epsilon)V_2(\epsilon)V_3(\epsilon)$$

4 Dimensional Fractal Objects

$$V(\epsilon) = 2\epsilon \cdot N_{\Omega_1}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(2\epsilon)} p_j\right) \cdot \left(\sum_{j=1}^{N_{\Omega_3}(2\epsilon)} q_j\right) \cdot \left(\sum_{j=1}^{N_{\Omega_4}(2\epsilon)} z_j\right) + 2\epsilon \cdot N_{\Omega_2}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(2\epsilon)} \ell_j - 2\epsilon\right) \cdot \left(\sum_{j=1}^{N_{\Omega_3}(2\epsilon)} q_j\right) \cdot \left(\sum_{j=1}^{N_{\Omega_4}(2\epsilon)} z_j\right) + 2\epsilon \cdot N_{\Omega_3}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(2\epsilon)} p_j - 2\epsilon\right) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(2\epsilon)} \ell_j - 2\epsilon\right) \cdot \left(\sum_{j=1}^{N_{\Omega_4}(2\epsilon)} z_j\right) + 2\epsilon \cdot N_{\Omega_4}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(2\epsilon)} p_j - 2\epsilon\right) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(2\epsilon)} \ell_j - 2\epsilon\right) \cdot \left(\sum_{j=1}^{N_{\Omega_3}(2\epsilon)} q_j - 2\epsilon\right) + \mathcal{W}_1\mathcal{L}_2\mathcal{L}_3\mathcal{L}_4 + \mathcal{L}_1\mathcal{W}_2\mathcal{L}_3\mathcal{L}_4 + \mathcal{L}_1\mathcal{L}_2\mathcal{W}_3\mathcal{L}_4 + \mathcal{L}_1\mathcal{L}_2\mathcal{L}_3\mathcal{W}_4 - \mathcal{L}_1\mathcal{L}_2\mathcal{W}_3\mathcal{W}_4 - \mathcal{L}_1\mathcal{W}_2\mathcal{L}_3\mathcal{W}_4 - \mathcal{L}_1\mathcal{W}_2\mathcal{W}_3\mathcal{L}_4 - \mathcal{W}_1\mathcal{L}_2\mathcal{L}_3\mathcal{W}_4 - \mathcal{W}_1\mathcal{L}_2\mathcal{W}_3\mathcal{L}_4 - \mathcal{W}_1\mathcal{W}_2\mathcal{L}_3\mathcal{L}_4 + \mathcal{L}_1\mathcal{W}_2\mathcal{W}_3\mathcal{W}_4 + \mathcal{W}_1\mathcal{L}_2\mathcal{W}_3\mathcal{W}_4 + \mathcal{W}_1\mathcal{W}_2\mathcal{L}_3\mathcal{W}_4 + \mathcal{W}_1\mathcal{W}_2\mathcal{W}_3\mathcal{L}_4 - \mathcal{W}_1\mathcal{W}_2\mathcal{W}_3\mathcal{W}_4$$

4 Dimensional Volume Formula

$$V^4(\epsilon) = \mathcal{L}_1\mathcal{L}_2\mathcal{L}_3 V_4(\epsilon) + \mathcal{L}_1\mathcal{L}_2 V_3(\epsilon)\mathcal{L}_4 + \mathcal{L}_1 V_2(\epsilon)\mathcal{L}_3\mathcal{L}_4 + V_1(\epsilon)\mathcal{L}_2\mathcal{L}_3\mathcal{L}_4 - V_1(\epsilon)V_2(\epsilon)\mathcal{L}_3\mathcal{L}_4 - V_1(\epsilon)\mathcal{L}_2 V_3(\epsilon)\mathcal{L}_4 - V_1(\epsilon)\mathcal{L}_2\mathcal{L}_3 V_4(\epsilon) - \mathcal{L}_1 V_2(\epsilon)V_3(\epsilon)\mathcal{L}_4 - \mathcal{L}_1 V_2(\epsilon)\mathcal{L}_3 V_4(\epsilon) - \mathcal{L}_1\mathcal{L}_2 V_3(\epsilon)V_4(\epsilon) + V_1(\epsilon)V_2(\epsilon)V_3(\epsilon)\mathcal{L}_4 + V_1(\epsilon)V_2(\epsilon)\mathcal{L}_3 V_4(\epsilon) + V_1(\epsilon)\mathcal{L}_2 V_3(\epsilon)V_4(\epsilon) + \mathcal{L}_1 V_2(\epsilon)V_3(\epsilon)V_4(\epsilon) - V_1(\epsilon)V_2(\epsilon)V_3(\epsilon)V_4(\epsilon)$$

4 Dimensional Volume Formula

$$V^4(\epsilon) = \mathcal{L}_1\mathcal{L}_2\mathcal{L}_3V_4(\epsilon) + \mathcal{L}_1\mathcal{L}_2V_3(\epsilon)\mathcal{L}_4 + \mathcal{L}_1V_2(\epsilon)\mathcal{L}_3\mathcal{L}_4 + V_1(\epsilon)\mathcal{L}_2\mathcal{L}_3\mathcal{L}_4 - V_1(\epsilon)V_2(\epsilon)\mathcal{L}_3\mathcal{L}_4 - V_1(\epsilon)\mathcal{L}_2V_3(\epsilon)\mathcal{L}_4 - V_1(\epsilon)\mathcal{L}_2\mathcal{L}_3V_4(\epsilon) - \mathcal{L}_1V_2(\epsilon)V_3(\epsilon)\mathcal{L}_4 - \mathcal{L}_1V_2(\epsilon)\mathcal{L}_3V_4(\epsilon) - \mathcal{L}_1\mathcal{L}_2V_3(\epsilon)V_4(\epsilon) + V_1(\epsilon)V_2(\epsilon)V_3(\epsilon)\mathcal{L}_4 + V_1(\epsilon)V_2(\epsilon)\mathcal{L}_3V_4(\epsilon) + V_1(\epsilon)\mathcal{L}_2V_3(\epsilon)V_4(\epsilon) + \mathcal{L}_1V_2(\epsilon)V_3(\epsilon)V_4(\epsilon) - V_1(\epsilon)V_2(\epsilon)V_3(\epsilon)V_4(\epsilon)$$

$$V^3(\epsilon) = \mathcal{L}_1\mathcal{L}_2V_3(\epsilon) + \mathcal{L}_1V_2(\epsilon)\mathcal{L}_3 + V_1(\epsilon)\mathcal{L}_2\mathcal{L}_3 - V_1(\epsilon)V_2(\epsilon)\mathcal{L}_3 - V_1(\epsilon)\mathcal{L}_2V_3(\epsilon) - \mathcal{L}_1V_2(\epsilon)V_3(\epsilon) + V_1(\epsilon)V_2(\epsilon)V_3(\epsilon)$$

$$V^2(\epsilon) = \mathcal{L}_1V_2(\epsilon) + V_1(\epsilon)\mathcal{L}_2 - V_1(\epsilon)V_2(\epsilon)$$

Algebraic Manipulations

$$
\begin{aligned}
V^2(\epsilon) &= \mathcal{L}_1 V_2(\epsilon) + V_1(\epsilon)\mathcal{L}_2 - V_1(\epsilon)V_2(\epsilon) \\
&= (-1) \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) + \mathcal{L}_1\mathcal{L}_2
\end{aligned}
$$

Algebraic Manipulations

$$V^2(\epsilon) = \mathcal{L}_1 V_2(\epsilon) + V_1(\epsilon)\mathcal{L}_2 - V_1(\epsilon)V_2(\epsilon)$$

$$= (-1) \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) + \mathcal{L}_1\mathcal{L}_2$$

$$V^3(\epsilon) = \mathcal{L}_1\mathcal{L}_2 V_3(\epsilon) + \mathcal{L}_1 V_2(\epsilon)\mathcal{L}_3 + V_1(\epsilon)\mathcal{L}_2\mathcal{L}_3 - V_1(\epsilon)V_2(\epsilon)\mathcal{L}_3 - V_1(\epsilon)\mathcal{L}_2 V_3(\epsilon) - \mathcal{L}_1 V_2(\epsilon)V_3(\epsilon) + V_1(\epsilon)V_2(\epsilon)V_3(\epsilon)$$

$$= (-1)^2 \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) \cdot (V_3(\epsilon) - \mathcal{L}_3) + \mathcal{L}_1\mathcal{L}_2\mathcal{L}_3$$

Algebraic Manipulations

$$V^4(\epsilon) = \mathcal{L}_1\mathcal{L}_2\mathcal{L}_3V_4(\epsilon) + \mathcal{L}_1\mathcal{L}_2V_3(\epsilon)\mathcal{L}_4 + \mathcal{L}_1V_2(\epsilon)\mathcal{L}_3\mathcal{L}_4 + V_1(\epsilon)\mathcal{L}_2\mathcal{L}_3\mathcal{L}_4 - V_1(\epsilon)V_2(\epsilon)\mathcal{L}_3\mathcal{L}_4 - V_1(\epsilon)\mathcal{L}_2V_3(\epsilon)\mathcal{L}_4 - V_1(\epsilon)\mathcal{L}_2\mathcal{L}_3V_4(\epsilon) - \mathcal{L}_1V_2(\epsilon)V_3(\epsilon)\mathcal{L}_4 - \mathcal{L}_1V_2(\epsilon)\mathcal{L}_3V_4(\epsilon) - \mathcal{L}_1\mathcal{L}_2V_3(\epsilon)V_4(\epsilon) + V_1(\epsilon)V_2(\epsilon)V_3(\epsilon)\mathcal{L}_4 + V_1(\epsilon)V_2(\epsilon)\mathcal{L}_3V_4(\epsilon) + V_1(\epsilon)\mathcal{L}_2V_3(\epsilon)V_4(\epsilon) + \mathcal{L}_1V_2(\epsilon)V_3(\epsilon)V_4(\epsilon) - V_1(\epsilon)V_2(\epsilon)V_3(\epsilon)V_4(\epsilon)$$

$$= (-1)^3\cdot(V_1(\epsilon)-\mathcal{L}_1)\cdot(V_2(\epsilon)-\mathcal{L}_2)\cdot(V_3(\epsilon)-\mathcal{L}_3)\cdot(V_4(\epsilon)-\mathcal{L}_4)+\mathcal{L}_1\mathcal{L}_2\mathcal{L}_3\mathcal{L}_4$$

General Volume Formula

$$V^2(\epsilon) = (-1)\cdot(V_1(\epsilon)-\mathcal{L}_1)\cdot(V_2(\epsilon)-\mathcal{L}_2)+\mathcal{L}_1\mathcal{L}_2$$

$$V^3(\epsilon) = (-1)^2\cdot(V_1(\epsilon)-\mathcal{L}_1)\cdot(V_2(\epsilon)-\mathcal{L}_2)\cdot(V_3(\epsilon)-\mathcal{L}_3)+\mathcal{L}_1\mathcal{L}_2\mathcal{L}_3$$

$$V^4(\epsilon) = (-1)^3\cdot(V_1(\epsilon)-\mathcal{L}_1)\cdot(V_2(\epsilon)-\mathcal{L}_2)\cdot(V_3(\epsilon)-\mathcal{L}_3)\cdot(V_4(\epsilon)-\mathcal{L}_4)+\mathcal{L}_1\mathcal{L}_2\mathcal{L}$$

$$V^n(\epsilon) = \left[(-1)^{n-1}\cdot\prod_{k=1}^{n}(V_k(\epsilon)-\mathcal{L}_k)\right]+\prod_{k=1}^{n}\mathcal{L}_k$$