

Inner Minkowski Dimension of Products of Fractal Strings

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What is a Fractal String

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The fractal string we studied the most in particular is the **Cantor String**



Cantor String

Inner-Tubular Neighborhood

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Counting function: $N_\Omega(x) = \sum_{\ell_j^{-1} \leq x} 1$

More Notation: $\mathcal{W}_\Omega(x) = \sum_{j: \ell_j^{-1} > x} \ell_j$

Now we can write: $V(\epsilon) = 2\epsilon \cdot N_\Omega(2\epsilon) + \mathcal{W}_\Omega(2\epsilon)$



Cantor String Volume



Inner Minkowski Dimension of a Fractal String

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Dimension of Ω : $D_\Omega = \inf \{ \alpha \geq 0 \mid V(\epsilon) = O(\epsilon^{1-\alpha}) \text{ as } \epsilon \rightarrow 0^+ \}$

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Cantor String Dimension: $D_{cs} = \log_3 2$

Fractal Lawn

Definition 4

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$$\ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots \geq 0, \quad p_1 \geq p_2 \geq p_3 \geq \cdots \geq 0$$



Cantor Lawn



Fractal Lawn Volume I

Inner-tubular neighborhood: $V(\epsilon) = \text{vol}_2\{x \in \Omega^2 \mid d(x, \partial\Omega^2) < \epsilon\}$



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- $\mathcal{L}_i = \sum_{j \in \mathbb{N}} \ell_j$

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- $\mathcal{L}_i = \sum_{j \in \mathbb{N}} \ell_j$

- $\mathcal{W}_i(x) = \sum_{j: \ell_j < x} \ell_j$

- $\sum_{j=1}^{N_{\Omega_i}(x)} \ell_j = \mathcal{L}_i - \mathcal{W}_i(x)$



Fractal Lawn Volume II

Partially Covered Volume Box



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Partially Covered Volume Box



■ Partially covered volume

$$V_{partial}(\epsilon) = 2\epsilon N_{\Omega_1}(2\epsilon) \sum_{j=1}^{N_{\Omega_2}(2\epsilon)} \ell_j + 2\epsilon N_{\Omega_2}(2\epsilon) \sum_{j=1}^{N_{\Omega_1}(2\epsilon)} (p_j - 2\epsilon)$$

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- Fully covered volume

- $$V_{full}(\epsilon) = \mathcal{L}_1 \mathcal{W}_2(2\epsilon) + \mathcal{L}_2 \mathcal{W}_1(2\epsilon)$$

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■ Partially covered volume

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■ Fully covered volume

$$V_{full}(\epsilon) = \mathcal{L}_1 \mathcal{W}_2(2\epsilon) + \mathcal{L}_2 \mathcal{W}_1(2\epsilon)$$

■ Double counted volume

$$V_{double}(\epsilon) = \mathcal{W}_1(2\epsilon) \mathcal{W}_2(2\epsilon)$$

Fractal Lawn Volume III

$$V(\epsilon) = V_{partial}(\epsilon) + V_{full}(\epsilon) - V_{double}(\epsilon)$$



Fractal Lawn Volume III

$$V(\epsilon) = V_{partial}(\epsilon) + V_{full}(\epsilon) - V_{double}(\epsilon)$$

$$V(\epsilon) = 2\epsilon \cdot N_{\Omega_1}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(2\epsilon)} p_j \right) + 2\epsilon \cdot N_{\Omega_2}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(2\epsilon)} (\ell_j - 2\epsilon) \right) +$$

$$\mathcal{L}_1 \mathcal{W}_2(2\epsilon) + \mathcal{L}_2 \mathcal{W}_1(2\epsilon) - \mathcal{W}_1(2\epsilon) \mathcal{W}_2(2\epsilon)$$

$$= 2\epsilon \cdot N_{\Omega_1} \cdot (\mathcal{L}_2 - \mathcal{W}_2) + 2\epsilon \cdot N_{\Omega_2} \cdot (\mathcal{L}_1 - \mathcal{W}_1 - 2\epsilon \cdot N_{\Omega_1}) + \\ \mathcal{L}_1 \mathcal{W}_2 + \mathcal{L}_2 \mathcal{W}_1 - \mathcal{W}_1 \mathcal{W}_2$$

Fractal Lawn Volume IV

$$\begin{aligned} V(\epsilon) &= \mathcal{L}_1 \cdot (2\epsilon \cdot N_{\Omega_2} + \mathcal{W}_2) + \mathcal{L}_2 \cdot (2\epsilon \cdot N_{\Omega_1} + \mathcal{W}_1) - \\ &\quad (2\epsilon \cdot N_{\Omega_1} + \mathcal{W}_1) \cdot (2\epsilon \cdot N_{\Omega_2} + \mathcal{W}_2) \\ &= \mathcal{L}_1 \cdot (2\epsilon \cdot N_{\Omega_2} + \mathcal{W}_2) + \mathcal{L}_2 \cdot (2\epsilon \cdot N_{\Omega_1} + \mathcal{W}_1) - \\ &\quad (2\epsilon \cdot N_{\Omega_1} + \mathcal{W}_1) \cdot (2\epsilon \cdot N_{\Omega_2} + \mathcal{W}_2) \\ &= \mathcal{L}_1 V_2 + \mathcal{L}_2 V_1 - V_1 V_2 \end{aligned}$$

Cantor Lawn Volume



Dimension of Fractal Lawn I

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$$V(\epsilon) = \mathcal{L}_1 V_2 + \mathcal{L}_2 V_1 - V_1 V_2$$

$$\lim_{\epsilon \rightarrow 0^+} V(\epsilon) = O(\epsilon^{1-D_{\Omega_2}}) + O(\epsilon^{1-D_{\Omega_1}}) + O(\epsilon^{2-(D_{\Omega_1}+D_{\Omega_2})})$$

$$\lim_{\epsilon \rightarrow 0^+} \frac{V(\epsilon)}{\epsilon^{2-\alpha}} = O(\epsilon^{1-D_{\Omega_2}} \epsilon^{2-\alpha}) + O(\epsilon^{1-D_{\Omega_1}} \epsilon^{2-\alpha}) + O(\epsilon^{2-(D_{\Omega_1}+D_{\Omega_2})} \epsilon^{2-\alpha})$$



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$$\lim_{\epsilon \rightarrow 0^+} \frac{V(\epsilon)}{\epsilon^{2-\alpha}} = O(\epsilon^{1-\alpha(1+D_{\Omega_2})}) + O(\epsilon^{1-\alpha(1+D_{\Omega_1})}) + O(\epsilon^{\alpha-(D_{\Omega_1}+D_{\Omega_2})})$$

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$$\begin{cases} \alpha - (1 + D_{\Omega_1}) \geq 0 \\ \alpha - (1 + D_{\Omega_2}) \geq 0 \\ \alpha - (D_{\Omega_1} + D_{\Omega_2}) \geq 0 \end{cases}$$

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$$\begin{cases} \alpha \geq 1 + D_{\Omega_1} \\ \alpha \geq 1 + D_{\Omega_2} \end{cases}$$

$$D_{\Omega^2} = \max\{(1 + D_{\Omega_1}), (1 + D_{\Omega_2})\}$$



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$$\begin{aligned}V(\epsilon)_{\Omega^2} &= \mathcal{L} V(\epsilon)_\Omega + \mathcal{L} V(\epsilon)_\Omega - V(\epsilon)_\Omega \cdot V(\epsilon)_\Omega \\&= 2\mathcal{L} V(\epsilon)_\Omega - (V(\epsilon)_\Omega)^2\end{aligned}$$



Dimension of the Cantor Lawn II

Let $t = \{-\log_3(2\epsilon)\}$ be defined as the fractional portion of $-\log_3(2\epsilon)$

$$\begin{aligned}V_{\Omega^2}(\epsilon) &= (2\epsilon)^{1-\log_3 2} \left(2 \left(\frac{1}{2}\right)^t + 2 \left(\frac{3}{2}\right)^t \right) + (2\epsilon)^{2-\log_3 2} \left(2 \left(\frac{3}{2}\right)^t + 2 \left(\frac{1}{2}\right)^t \right) \\&= -(2\epsilon)^{2-\log_3 4} \left(2 \left(\frac{3}{4}\right)^t + \left(\frac{1}{4}\right)^t + \left(\frac{9}{4}\right)^t \right) - (4\epsilon + 4\epsilon^2)\end{aligned}$$



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$$\begin{cases} -\log_3 2 + 1 - (2 - \alpha) \geq 0 \\ -\log_3 2 + 2 - (2 - \alpha) \geq 0 \\ -\log_3 4 + 2 - (2 - \alpha) \geq 0 \end{cases}$$



Dimension of the Cantor Lawn III

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Dimension of the Cantor Lawn III

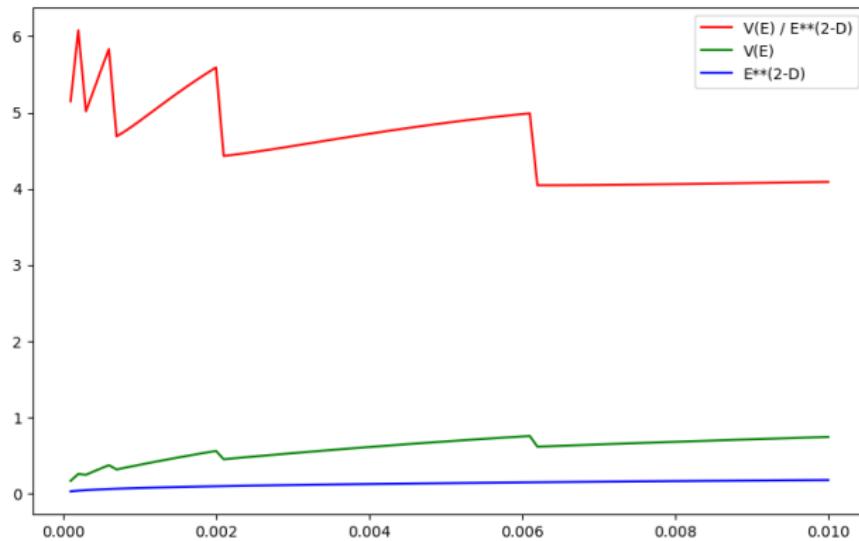
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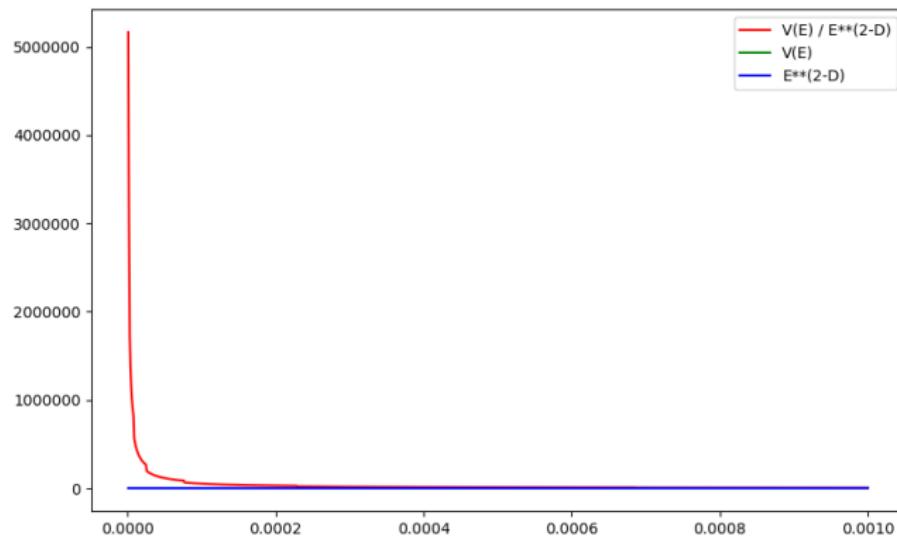
$$D_{\Omega^2} = 1 + \log_3 2$$



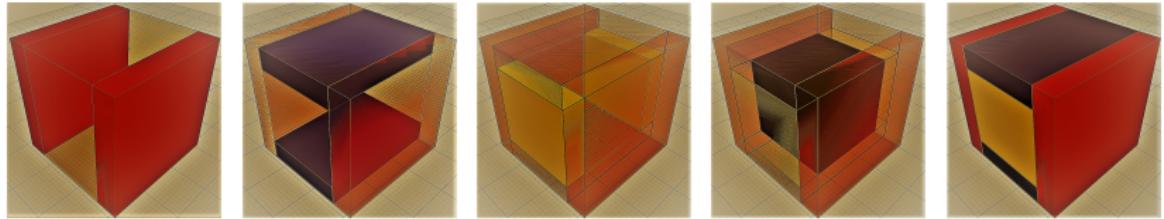
Dimension of the Cantor Lawn IV



Dimension of the Cantor Lawn V



3 Dimensional Fractal Objects



$$\begin{aligned} V(\epsilon) = & 2\epsilon \cdot N_{\Omega_1}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(2\epsilon)} p_j \right) \cdot \left(\sum_{j=1}^{N_{\Omega_3}(2\epsilon)} q_j \right) + \\ & 2\epsilon \cdot N_{\Omega_2}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(2\epsilon)} \ell_j - 2\epsilon \right) \cdot \left(\sum_{j=1}^{N_{\Omega_3}(2\epsilon)} q_j \right) + \\ & 2\epsilon \cdot N_{\Omega_3}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(2\epsilon)} p_j - 2\epsilon \right) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(2\epsilon)} \ell_j - 2\epsilon \right) + \\ & \mathcal{W}_1 \mathcal{L}_2 \mathcal{L}_3 + \mathcal{L}_1 \mathcal{W}_2 \mathcal{L}_3 + \mathcal{L}_1 \mathcal{L}_2 \mathcal{W}_3 - \mathcal{L}_1 \mathcal{W}_2 \mathcal{W}_3 - \mathcal{W}_1 \mathcal{L}_2 \mathcal{W}_3 - \\ & \mathcal{W}_1 \mathcal{W}_2 \mathcal{L}_3 + \mathcal{W}_1 \mathcal{W}_2 \mathcal{W}_3 \end{aligned}$$

3 Dimensional Fractal Volume

$$V(\epsilon) = 2\epsilon \cdot N_{\Omega_1} \cdot (\mathcal{L}_2 - \mathcal{W}_3) \cdot (\mathcal{L}_3 - \mathcal{W}_3) + 2\epsilon \cdot N_{\Omega_2} \cdot (\mathcal{L}_1 - \mathcal{W}_1 - 2\epsilon \cdot N_{\Omega_1}) \cdot (\mathcal{L}_3 - \mathcal{W}_3) + 2\epsilon \cdot N_{\Omega_3} \cdot (\mathcal{L}_2 - \mathcal{W}_2 - 2\epsilon \cdot N_{\Omega_2}) \cdot (\mathcal{L}_1 - \mathcal{W}_1 - 2\epsilon \cdot N_{\Omega_1}) + \mathcal{W}_1 \mathcal{L}_2 \mathcal{L}_3 + \mathcal{L}_1 \mathcal{W}_2 \mathcal{L}_3 + \mathcal{L}_1 \mathcal{L}_2 \mathcal{W}_3 - \mathcal{L}_1 \mathcal{W}_2 \mathcal{W}_3 - \mathcal{W}_1 \mathcal{L}_2 \mathcal{W}_3 - \mathcal{W}_1 \mathcal{W}_2 \mathcal{L}_3 + \mathcal{W}_1 \mathcal{W}_2 \mathcal{W}_3$$

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$$V(\epsilon) = \mathcal{L}_1 \mathcal{L}_2 V_3(\epsilon) + \mathcal{L}_1 V_2(\epsilon) \mathcal{L}_3 + V_1(\epsilon) \mathcal{L}_2 \mathcal{L}_3 - V_1(\epsilon) V_2(\epsilon) \mathcal{L}_3 - V_1(\epsilon) \mathcal{L}_2 V_3(\epsilon) - \mathcal{L}_1 V_2(\epsilon) V_3(\epsilon) + V_1(\epsilon) V_2(\epsilon) V_3(\epsilon)$$

4 Dimensional Fractal Objects

$$\begin{aligned} V(\epsilon) = & 2\epsilon \cdot N_{\Omega_1}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_2}(2\epsilon)} p_j \right) \cdot \left(\sum_{j=1}^{N_{\Omega_3}(2\epsilon)} q_j \right) \cdot \left(\sum_{j=1}^{N_{\Omega_4}(2\epsilon)} z_j \right) + 2\epsilon \cdot \\ & N_{\Omega_2}(2\epsilon) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(2\epsilon)} \ell_j - 2\epsilon \right) \cdot \left(\sum_{j=1}^{N_{\Omega_3}(2\epsilon)} q_j \right) \cdot \left(\sum_{j=1}^{N_{\Omega_4}(2\epsilon)} z_j \right) + 2\epsilon \cdot N_{\Omega_3}(2\epsilon) \cdot \\ & \left(\sum_{j=1}^{N_{\Omega_2}(2\epsilon)} p_j - 2\epsilon \right) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(2\epsilon)} \ell_j - 2\epsilon \right) \cdot \left(\sum_{j=1}^{N_{\Omega_4}(2\epsilon)} z_j \right) + 2\epsilon \cdot N_{\Omega_4}(2\epsilon) \cdot \\ & \left(\sum_{j=1}^{N_{\Omega_2}(2\epsilon)} p_j - 2\epsilon \right) \cdot \left(\sum_{j=1}^{N_{\Omega_1}(2\epsilon)} \ell_j - 2\epsilon \right) \cdot \left(\sum_{j=1}^{N_{\Omega_3}(2\epsilon)} q_j - 2\epsilon \right) + W_1 L_2 L_3 L_4 + \\ & L_1 W_2 L_3 L_4 + L_1 L_2 W_3 L_4 + L_1 L_2 L_3 W_4 - L_1 L_2 W_3 W_4 - L_1 W_2 L_3 W_4 - \\ & L_1 W_2 W_3 L_4 - W_1 L_2 L_3 W_4 - W_1 L_2 W_3 L_4 - W_1 W_2 L_3 L_4 + L_1 W_2 W_3 W_4 + \\ & W_1 L_2 W_3 W_4 + W_1 W_2 L_3 W_4 + W_1 W_2 W_3 L_4 - W_1 W_2 W_3 W_4 \end{aligned}$$

4 Dimensional Volume Formula

$$V^4(\epsilon) = \mathcal{L}_1 \mathcal{L}_2 \mathcal{L}_3 V_4(\epsilon) + \mathcal{L}_1 \mathcal{L}_2 V_3(\epsilon) \mathcal{L}_4 + \mathcal{L}_1 V_2(\epsilon) \mathcal{L}_3 \mathcal{L}_4 + V_1(\epsilon) \mathcal{L}_2 \mathcal{L}_3 \mathcal{L}_4 - V_1(\epsilon) V_2(\epsilon) \mathcal{L}_3 \mathcal{L}_4 - V_1(\epsilon) \mathcal{L}_2 V_3(\epsilon) \mathcal{L}_4 - V_1(\epsilon) \mathcal{L}_2 \mathcal{L}_3 V_4(\epsilon) - \mathcal{L}_1 V_2(\epsilon) V_3(\epsilon) \mathcal{L}_4 - \mathcal{L}_1 V_2(\epsilon) \mathcal{L}_3 V_4(\epsilon) - \mathcal{L}_1 \mathcal{L}_2 V_3(\epsilon) V_4(\epsilon) + V_1(\epsilon) V_2(\epsilon) V_3(\epsilon) \mathcal{L}_4 + V_1(\epsilon) V_2(\epsilon) \mathcal{L}_3 V_4(\epsilon) + V_1(\epsilon) \mathcal{L}_2 V_3(\epsilon) V_4(\epsilon) + \mathcal{L}_1 V_2(\epsilon) V_3(\epsilon) V_4(\epsilon) - V_1(\epsilon) V_2(\epsilon) V_3(\epsilon) V_4(\epsilon)$$



4 Dimensional Volume Formula

$$V^4(\epsilon) = \mathcal{L}_1 \mathcal{L}_2 \mathcal{L}_3 V_4(\epsilon) + \mathcal{L}_1 \mathcal{L}_2 V_3(\epsilon) \mathcal{L}_4 + \mathcal{L}_1 V_2(\epsilon) \mathcal{L}_3 \mathcal{L}_4 + V_1(\epsilon) \mathcal{L}_2 \mathcal{L}_3 \mathcal{L}_4 - V_1(\epsilon) V_2(\epsilon) \mathcal{L}_3 \mathcal{L}_4 - V_1(\epsilon) \mathcal{L}_2 V_3(\epsilon) \mathcal{L}_4 - V_1(\epsilon) \mathcal{L}_2 \mathcal{L}_3 V_4(\epsilon) - \mathcal{L}_1 V_2(\epsilon) V_3(\epsilon) \mathcal{L}_4 - \mathcal{L}_1 \mathcal{L}_2 V_3(\epsilon) V_4(\epsilon) + V_1(\epsilon) V_2(\epsilon) V_3(\epsilon) \mathcal{L}_4 + V_1(\epsilon) V_2(\epsilon) \mathcal{L}_3 V_4(\epsilon) + V_1(\epsilon) \mathcal{L}_2 V_3(\epsilon) V_4(\epsilon) + \mathcal{L}_1 V_2(\epsilon) V_3(\epsilon) V_4(\epsilon) - V_1(\epsilon) V_2(\epsilon) V_3(\epsilon) V_4(\epsilon)$$

$$V^3(\epsilon) = \mathcal{L}_1 \mathcal{L}_2 V_3(\epsilon) + \mathcal{L}_1 V_2(\epsilon) \mathcal{L}_3 + V_1(\epsilon) \mathcal{L}_2 \mathcal{L}_3 - V_1(\epsilon) V_2(\epsilon) \mathcal{L}_3 - V_1(\epsilon) \mathcal{L}_2 V_3(\epsilon) - \mathcal{L}_1 V_2(\epsilon) V_3(\epsilon) + V_1(\epsilon) V_2(\epsilon) V_3(\epsilon)$$

$$V^2(\epsilon) = \mathcal{L}_1 V_2(\epsilon) + V_1(\epsilon) \mathcal{L}_2 - V_1(\epsilon) V_2(\epsilon)$$

Algebraic Manipulations

$$\begin{aligned}V^2(\epsilon) &= \mathcal{L}_1 V_2(\epsilon) + V_1(\epsilon) \mathcal{L}_2 - V_1(\epsilon) V_2(\epsilon) \\&= (-1) \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) + \mathcal{L}_1 \mathcal{L}_2\end{aligned}$$

Algebraic Manipulations

$$\begin{aligned}V^2(\epsilon) &= \mathcal{L}_1 V_2(\epsilon) + V_1(\epsilon) \mathcal{L}_2 - V_1(\epsilon) V_2(\epsilon) \\&= (-1) \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) + \mathcal{L}_1 \mathcal{L}_2\end{aligned}$$

$$\begin{aligned}V^3(\epsilon) &= \mathcal{L}_1 \mathcal{L}_2 V_3(\epsilon) + \mathcal{L}_1 V_2(\epsilon) \mathcal{L}_3 + V_1(\epsilon) \mathcal{L}_2 \mathcal{L}_3 - V_1(\epsilon) V_2(\epsilon) \mathcal{L}_3 - V_1(\epsilon) \mathcal{L}_2 V_3(\epsilon) - \\&\quad \mathcal{L}_1 V_2(\epsilon) V_3(\epsilon) + V_1(\epsilon) V_2(\epsilon) V_3(\epsilon) \\&= (-1)^2 \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) \cdot (V_3(\epsilon) - \mathcal{L}_3) + \mathcal{L}_1 \mathcal{L}_2 \mathcal{L}_3\end{aligned}$$

Algebraic Manipulations

$$\begin{aligned} V^4(\epsilon) &= \mathcal{L}_1 \mathcal{L}_2 \mathcal{L}_3 V_4(\epsilon) + \mathcal{L}_1 \mathcal{L}_2 V_3(\epsilon) \mathcal{L}_4 + \mathcal{L}_1 V_2(\epsilon) \mathcal{L}_3 \mathcal{L}_4 + V_1(\epsilon) \mathcal{L}_2 \mathcal{L}_3 \mathcal{L}_4 - \\ &\quad V_1(\epsilon) V_2(\epsilon) \mathcal{L}_3 \mathcal{L}_4 - V_1(\epsilon) \mathcal{L}_2 V_3(\epsilon) \mathcal{L}_4 - V_1(\epsilon) \mathcal{L}_2 \mathcal{L}_3 V_4(\epsilon) - \mathcal{L}_1 V_2(\epsilon) V_3(\epsilon) \mathcal{L}_4 - \\ &\quad \mathcal{L}_1 V_2(\epsilon) \mathcal{L}_3 V_4(\epsilon) - \mathcal{L}_1 \mathcal{L}_2 V_3(\epsilon) V_4(\epsilon) + V_1(\epsilon) V_2(\epsilon) V_3(\epsilon) \mathcal{L}_4 + V_1(\epsilon) V_2(\epsilon) \mathcal{L}_3 V_4(\epsilon) + \\ &\quad V_1(\epsilon) \mathcal{L}_2 V_3(\epsilon) V_4(\epsilon) + \mathcal{L}_1 V_2(\epsilon) V_3(\epsilon) V_4(\epsilon) - V_1(\epsilon) V_2(\epsilon) V_3(\epsilon) V_4(\epsilon) \\ &= (-1)^3 \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) \cdot (V_3(\epsilon) - \mathcal{L}_3) \cdot (V_4(\epsilon) - \mathcal{L}_4) + \mathcal{L}_1 \mathcal{L}_2 \mathcal{L}_3 \mathcal{L}_4 \end{aligned}$$

General Volume Formula

$$V^2(\epsilon) = (-1) \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) + \mathcal{L}_1 \mathcal{L}_2$$

$$V^3(\epsilon) = (-1)^2 \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) \cdot (V_3(\epsilon) - \mathcal{L}_3) + \mathcal{L}_1 \mathcal{L}_2 \mathcal{L}_3$$

$$V^4(\epsilon) = (-1)^3 \cdot (V_1(\epsilon) - \mathcal{L}_1) \cdot (V_2(\epsilon) - \mathcal{L}_2) \cdot (V_3(\epsilon) - \mathcal{L}_3) \cdot (V_4(\epsilon) - \mathcal{L}_4) + \mathcal{L}_1 \mathcal{L}_2 \mathcal{L}_3 \mathcal{L}_4$$

$$V^n(\epsilon) = \left[(-1)^{n-1} \cdot \prod_{k=1}^n (V_k(\epsilon) - \mathcal{L}_k) \right] + \prod_{k=1}^n \mathcal{L}_k$$

Future Work

- Prove the general volume formula
- Find a general formula for dimension (based on general volume formula)

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