Conjecture: Volume Formula for an Arbitrary Product of Fractal Strings

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Two View Points of a Fractal String

Definition

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Alternatively, a fractal string $\mathcal{L} = (\ell_j)_{j \in \mathbb{N}}$ is viewed as a nonincreasing sequence of lengths such that $(\ell_j) \to 0$ (possibly after reshuffling).

Cantor String Ex (n=1)

Cantor String Ex (n=2)

Cantor String Ex (n=3)

Cantor String Ex (n=4)

Cantor String Ex

Disjoint union of open intervals:

$$\Omega_{CS} = \left(\frac{1}{3}, \frac{2}{3}\right) \bigcup \left(\frac{1}{9}, \frac{2}{9}\right) \bigcup \left(\frac{7}{9}, \frac{8}{9}\right) \bigcup \left(\frac{1}{27}, \frac{2}{27}\right) \bigcup \left(\frac{7}{27}, \frac{8}{27}\right) \bigcup \left(\frac{19}{27}, \frac{20}{27}\right) \bigcup \left(\frac{25}{27}, \frac{26}{27}\right) \bigcup \cdots$$

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Associated sequence of lengths $\mathcal{L}_{CS}$ given by:

$$1/3, 1/9, 1/9, 1/27, 1/27, 1/27, 1/27, \cdots ;$$

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Associated sequence of lengths $\mathcal{L}_{CS}$ given by:

$$1/3, 1/9, 1/9, 1/27, 1/27, 1/27, 1/27, \cdots ;$$

Equivalently, $\mathcal{L}_{CS}$ consists of lengths $1/3^n$ counted with multiplicity 2^{n-1} for $n = 1, 2, 3, \cdots$.

Inner Tubular Volume

Definition

For $\varepsilon > 0$, let $V_{\mathcal{L}}(\varepsilon)$ be the *volume of the inner-tubular neighborhood* of $\mathcal{L}$, (or Ω),

$$V_{\mathcal{L}}(\varepsilon) = |\{x \in \Omega : d(x, \partial\Omega) < \varepsilon\}|,$$

where $d(x, \partial\Omega)$ denotes the distance in $\mathbb{R}$ from x to $\partial\Omega$.

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Definition

$\forall x \in \Omega$, let $N_{\mathcal{L}}(x)$ be the *geometric counting function* of $\mathcal{L}$,

$$N_{\mathcal{L}}(x) = \sum_{j:\, \ell_j \geq x} 1.$$

Inner Tubular Volume II

In hope of a simpler volume formula, let

$$W_{\mathcal{L}}(x) = \sum_{j\colon \ell_j < x} \ell_j$$

represent the lengths that are "too small" for ε.

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Then the Inner-tubular volume can be rewritten as:

$$V_{\mathcal{L}}(\varepsilon) = 2\varepsilon \cdot N_{\mathcal{L}}(2\varepsilon) + W_{\mathcal{L}}(2\varepsilon).$$

Inner Tubular Volume II

Figure: Image of the volume of the inner tubular neighborhood of the Cantor String at iteration $n = 4$ with $\varepsilon = 1/80$.

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In this case, the first fully covered length occurs at $n = 4$ for the given ε.

Inner-Minkowski Dimension

Definition

Let $f : \mathbb{R} \to \mathbb{R}$ be *estimated* by some strictly positive $g : \mathbb{R} \to \mathbb{R}$, written $f(x) = \mathcal{O}(g(x))$ as $x \to \infty$, iff $\exists M \in \mathbb{R}^+$ and $\exists x_0 \in \mathbb{R}$ such that

$$|f(x)| \leq Mg(x) \quad \forall x \geq x_0.$$

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Definition

Let $\mathcal{D}_{\mathcal{L}}$ be the Inner-Minkowski dimension of $\mathcal{L}$, given by

$$\mathcal{D}_{\mathcal{L}} = \inf\{\alpha \geq 0 : V_{\mathcal{L}}(\varepsilon) = \mathcal{O}(\varepsilon^{1-\alpha}) \text{ as } \varepsilon \to 0^+\}.$$

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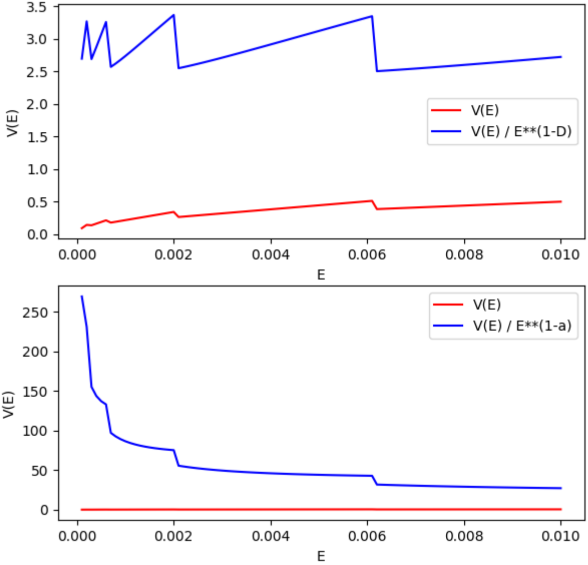
Equivalently, the following limit must be finite for any $\alpha \geq \mathcal{D}_{\mathcal{L}}$,

$$\lim_{\varepsilon \to 0^+} V_{\mathcal{L}}(\varepsilon)\varepsilon^{-(1-\alpha)}.$$

Cantor String Inner-Minkowski Dimension



Fractal Lawns

Definition

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two fractal strings with bounded open subsets Ω_1 and Ω_2, respectively. A **fractal lawn** is the bounded open set

$$\Omega := \Omega_1 \times \Omega_2.$$

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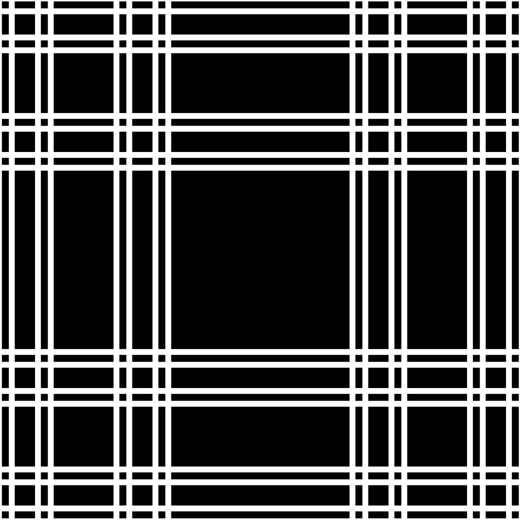
For the sake of this talk, take $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ as nothing more than a notation, but there is no meaningful extension of the sequence of lengths to the sequence of areas.

Cantor Lawn (n=4)



Inner-Tubular Volume (Fractal Lawn)

Partially Covered Volume Box

Inner-Tubular Volume (Fractal Lawn)

Partially Covered Volume Box

$$2\varepsilon N_{\mathcal{L}_1}(2\varepsilon) \sum_{j=1}^{N_{\mathcal{L}_2}(2\varepsilon)} \ell_{2_j} + 2\varepsilon N_{\mathcal{L}_2}(2\varepsilon) \sum_{j=1}^{N_{\mathcal{L}_1}(2\varepsilon)} (\ell_{1_j} - 2\varepsilon)$$
$$= 2\varepsilon N_{\mathcal{L}_1}(\mathcal{L}_2 - W_{\mathcal{L}_2}) + 2\varepsilon N_{\mathcal{L}_2}(\mathcal{L}_1 - W_{\mathcal{L}_1} - 2\varepsilon N_{\mathcal{L}_1})$$

Inner-Tubular Volume (Fractal Lawn)

The fully covered boxes are then counted by:

$$|\mathcal{L}_1|W_2 + |\mathcal{L}_2|W_1 - W_1 W_2.$$

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$$\begin{aligned}
V_{\mathcal{L}} &= 2\varepsilon N_1(|\mathcal{L}_2| - W_2) + 2\varepsilon N_2(|\mathcal{L}_1| - W_1 - 2\varepsilon N_1) \\
&\quad + |\mathcal{L}_1|W_2 + |\mathcal{L}_2|W_1 - W_1W_2 \\
&= \mathcal{L}_1(2\varepsilon N_2 + W_2) + \mathcal{L}_2(2\varepsilon N_1 + W_1) - (2\varepsilon N_2 + W_2)(2\varepsilon N_1 + W_1) \\
&= \boxed{|\mathcal{L}_1|V_{\mathcal{L}_2} + |\mathcal{L}_2|V_{\mathcal{L}_1} - V_{\mathcal{L}_1}V_{\mathcal{L}_2}}
\end{aligned}$$

Summary

Proposition

Let $n \in \mathbb{N}$, and $\mathcal{L} = \mathcal{L}_1 \times \cdots \times \mathcal{L}_n$ be a product of n fractal strings, then

$$V_{\mathcal{L}}(\varepsilon) = \left[(-1)^{n-1} \cdot \prod_{k=1}^{n} \left(V_{\mathcal{L}_k}(\varepsilon) - |\mathcal{L}_k| \right) \right] + \prod_{k=1}^{n} |\mathcal{L}_k|.$$