Support Vector Machines

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Supervised Learning Models

Definition

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Types of Models:

- Neural Networks
- Naive Bayes Classifiers
- K-Nearest Neighbors
- Regression
- Support Vector Machines (SVM)

Neural Network Comparison

Advantages:

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Disadvantages:

- Longer training time.
- Non-guaranteed convergence due to local minima.
- Fixed size (now a disadvantage), since in the real world the actual problem could be more complex than anticipated.

SVM Goals

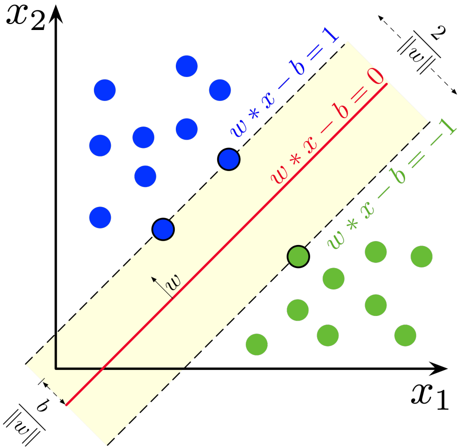

Figure: Goal of the SVM

Widest Street Approach

Goal: Maximize the width of the street $\frac{2}{||W||}$, or equivalently, minimize the ℓ^2-norm $||W||$.

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Mathematical Conveniences:

- Minimizing $||W||$ is equivalent to minimizing $\frac{1}{2}||W||^2$.
- Introduce a new variable $y_i = \begin{cases} 1, \text{if } W \cdot X_i + b \geq 1 \\ -1, \text{if } W \cdot X_i + b \leq -1. \end{cases}$

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Combine constraints into one condition:

$$y_i(W \cdot X_i + b) \geq 1 \iff y_i(W \cdot X_i + b) - 1 \geq 0.$$

Lagrange Mulitpliers

Goal from Lagrange: Maximize over all α_i,

$$\mathcal{L}(X) = \frac{1}{2}||W||^2 - \sum_{i=1}^{N} \alpha_i (y_i(W \cdot X_i + b) - 1)$$

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Take partial derivatives:

$$\frac{\partial}{\partial W}\mathcal{L} = W - \sum_i \alpha_i y_i X_i = 0 \iff W = \sum_i \alpha_i y_i X_i$$

$$\frac{\partial}{\partial b}\mathcal{L} = -\sum_i \alpha_i y_i = 0 \iff \sum_i \alpha_i y_i = 0.$$

Lagrange Continued

Represent the optimization problem in terms of the dot product of input vectors:

$$\begin{aligned}
\max \mathcal{L} &= \frac{1}{2}||W||^2 - \sum_i \alpha_i (y_i(W \cdot X_i + b) - 1) \\
&= \frac{1}{2} W^T W - W^T \sum_i \alpha_i y_i X_i - b \sum_i \alpha_i y_i + \sum_i \alpha_i \\
&= \sum_i \alpha_i - \frac{1}{2} \sum_i \sum_j \alpha_i \alpha_j y_i y_j X_i \cdot X_j
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\end{aligned}$$

Decision Rule: Given a vector U, we classify U according to the rule,

$$\text{sign}(W \cdot U + b) = \text{sign}(\sum_i \alpha_i y_i X_i \cdot U + b)$$

Support Vectors

Still need to optimize

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Result: most $\alpha_i = 0$ except for a small amount, referred to as the "support vectors". The decision rule simplifies:

$$\text{sign}\left(\sum_{i \in SV} \alpha_i y_i X_i \cdot U + b\right).$$

Non-Linearly Seperable Case

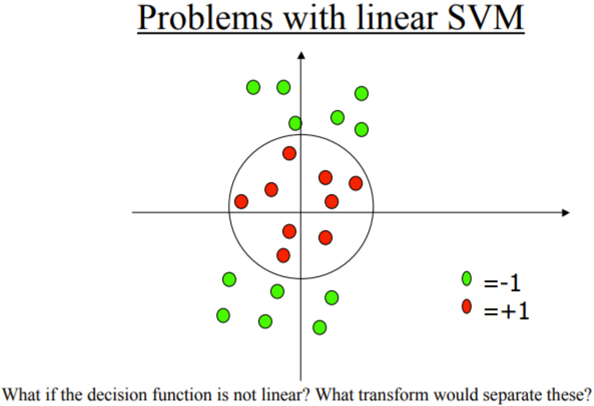

Figure: Non-linearly separable data.

Project to Higher Dimensions

Luckily we only need to define the dot product in the higher dimensional space!

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Suppose φ is some projection, then we just need a function K such that $K(X, Y) = \varphi(X) \cdot \varphi(Y)$, called the Kernel.

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We really only care about the Kernel since

$$\text{sign}\left(\sum_{i \in SV} \alpha_i y_i K(X_i, U) + b\right),$$

is the new decision rule.

Popular Kernels

Linear:

$$K(X, Y) = X \cdot Y + 1$$

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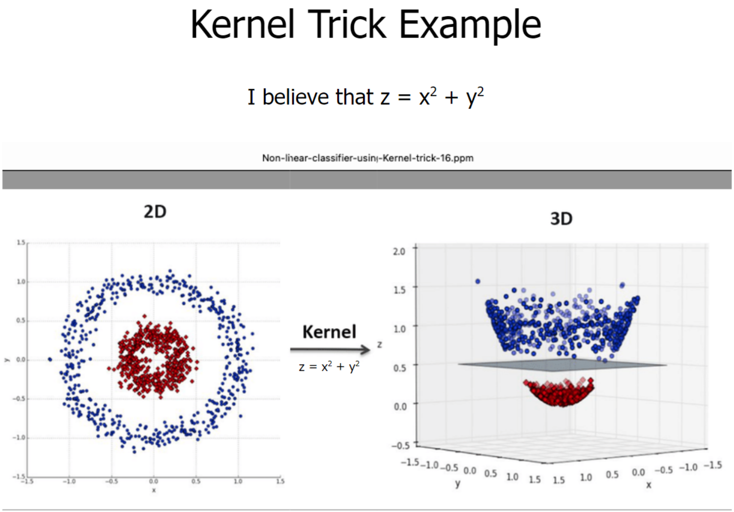
Radial Basis Function (Standard in Practice):

$$K(X, Y) = \exp\left(-\frac{||X - Y||^2}{2\sigma^2}\right) = \exp(-\gamma||X - Y||^2)$$

Kernel Trick Visualized



Jos Luis Rojo-lvarez; Manel Martnez-Ramn; Jordi Muoz-Mar; Gustau Camps-Valls, "Support Vector Machine and Kernel Classification Algorithms," in Digital Signal Processing with Kernel Methods , , IEEE, 2018, pp.433-502.