Hysteresis in the Kuramoto Model With Inertia

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The College of New Jersey

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Synchronous Fireflies

Figure: Synchronous congregation of fireflies located in Congaree National Park, South Carolina. https://www.nps.gov/cong/fireflies.htm

Adjacency Matrix

We can study different types of networks by thinking of fireflies as a set of nodes in a graph, where an edge represents the influence of each firefly on another.

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Definition

The *adjacency matrix* representation of a graph is a matrix G such that

$$G_{ij} = \begin{cases} 1 & i \text{ is adjacent to } j \\ 0 & \text{otherwise.} \end{cases}$$

Example

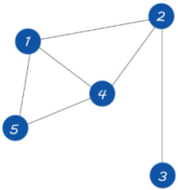

Figure: Simple graph with 5 nodes.

Example

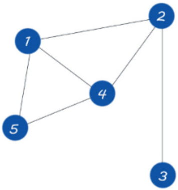

Figure: Simple graph with 5 nodes.

If G represents the graph shown, then

$$G = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Small-World Graphs

For the following definition, it is helpful to visualize nodes arranged around the unit circle.

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Definition

A Small-World Graph $G_{N,r,p}$ with parameters $p \in (0, 0.5), r \in (0, 0.5)$, is a graph in which each node is connected to $k = \lfloor rN \rfloor$ neighbors on each side with probability of forming an edge determined by

$$P(v_i, v_j) = \begin{cases} 1 - p, & \min\{|v_i - v_j|, 2\pi - |v_i - v_j|\} \leq r \\ p, & \text{otherwise.} \end{cases}$$

Small-World Graph Example

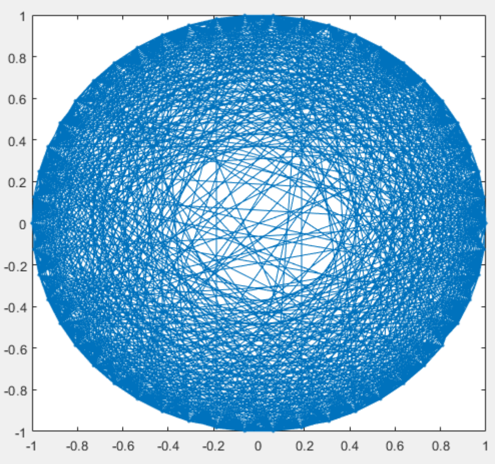

Figure: Small-world graph with $N = 50$ and parameters $r = 0.4, p = 0.2$.

The Kuramoto Model

Definition

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Definition

The *Kuramoto Model with Inertia* for a network of coupled oscillators with adjacency matrix G is given by

$$\ddot{\theta}_i + \alpha\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N} G_{ij}\sin(\theta_j - \theta_i), \quad i = 1, 2, \ldots, N,$$

where $K \in \mathbb{R}$ is the coupling strength between oscillators, and ω_i is the intrinsic frequency of the i^{th} oscillator.

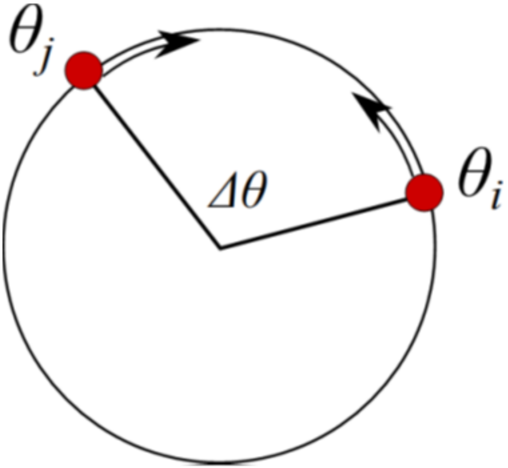

Figure: Sinusoidal coupling visualization.

Numerical Simulations that Follow

The following slides will occasionally show numerical simulations as snapshots of the initial state, an intermediate state, and the final (steady) state.

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We solve the system using ODE45 with time interval $T \in [0, 50]$, which is sufficient to reach stable equilibrium.

The following parameters are taken for all simulations on various K:

- $\alpha = 0.3$
- $G = G_{1000,0.4,0.2}$
- $g(\omega)$ is a standard Gaussian distribution

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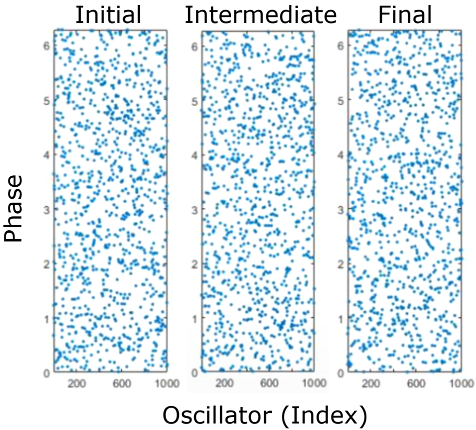

Figure: Evolution of the incoherent state for $K = 0$.

Coherent State

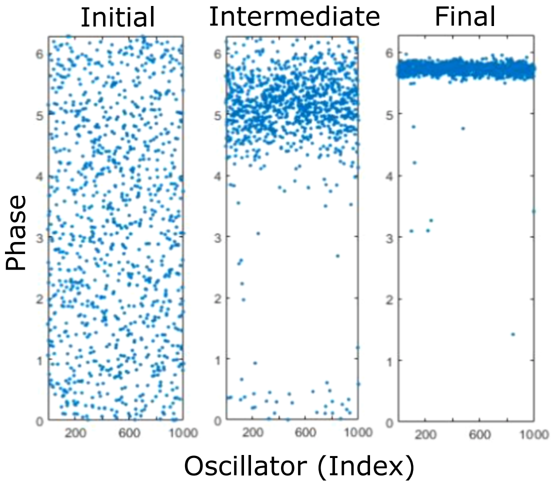

Figure: Formation of the coherent state when starting from the incoherent state for $K = 50$.

Collective Synchronization

Definition

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As K is increased, eventually solutions will bifurcate to the coherent state at some bifurcation point K_c. This is the phenomenon known as collective synchronization, which is the reason the Kuramoto Model is important.

Stability

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Definition

A solution is said to be *stable* if small perturbations to the initial state result in the same solution. A solution is said to be unstable if it is not stable.

Thus the numerical simulations suggest that the incoherent state loses stability at the bifurcation point K_c, wherein the coherent state becomes stable and solutions tend towards the coherent state.

Qualitative Behavior

To summarize, the qualitative behavior of solutions depends on:

- The coupling strength K
- The initial state (Distribution of $\theta_i(0)$)
- The distribution $g(\omega)$

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The magnitude of α can play a role as well if it is sufficiently large. For this work we are interested in small α.

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The *complex order parameter* for a system of N oscillators is given by

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We are effectively averaging the interactions of the oscillators.

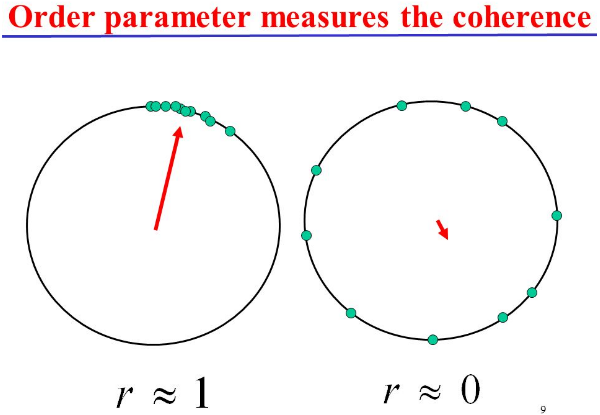

Figure: The magnitude $|r|$ measures the amount of coherence, whereas ψ is the mean phase of the oscillators.

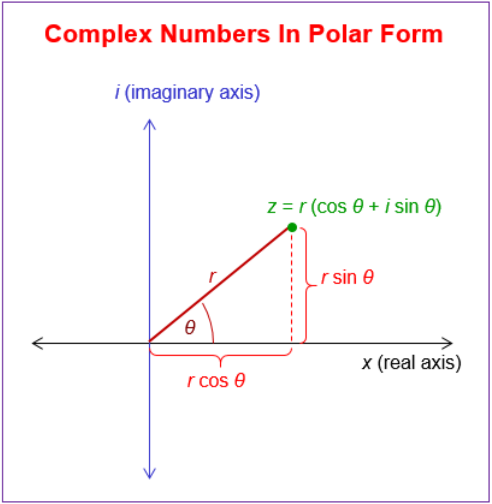

Figure: Polar form of a complex number.

Euler's Formula

Euler's formula is very useful in complex analysis and gives us an identity that relates the polar form of a complex number to its exponential form.

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This will be very useful soon when simplifying the second order KM.

Pendulum Reduction

Express in terms of the interaction of θ_i on ψ

$$re^{i(\psi-\theta_i)} = \frac{1}{N}\sum_{j=1}^{N} e^{i(\theta_j-\theta_i)}$$

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Second order KM reduces to a damped driven pendulum for each oscillator

$$\ddot{\theta}_i + \alpha\dot{\theta}_i = \omega_i + Kr\sin(\psi-\theta_i), \quad i = 1, 2, \ldots, N$$

Bifurcation Diagrams

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We create a *bifurcation diagram* based off numerical data that plots $(K, |r|)$ over some interval $K \in [K_0, K_f]$.

Thus for a given K and initial state, we can visually determine the steady state.

Numerical Results: Coherence

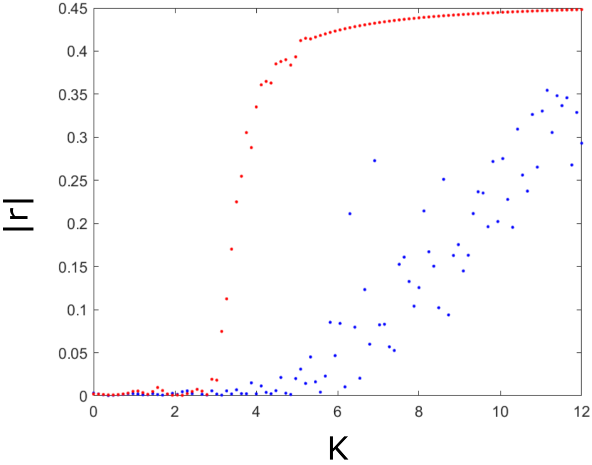

Figure: Bifurcation diagram for $0 \leq K \leq 12$. The blue curve starts from incoherence and the red curve starts from the previous stable coherent state.

Twisted States

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Definition

A *q-twisted state* is a solution of the form

$$\theta_j = \frac{2\pi q j}{N} + C \quad q \in \mathbb{Z}, C \in \mathbb{R},$$

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Note: the time parameter t was omitted since this is a spatial pattern in equilibrium.

2-Twisted State Example

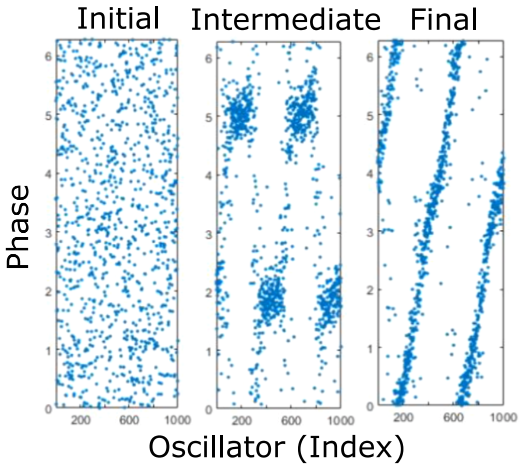

Figure: Formation of a 2-twisted state when starting from incoherence for $K = -6$.

Local Order Parameter

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Definition

The *local order parameter* for a system of N oscillators with adjacency matrix G is given by

$$h = (h_1, h_2, \dots, h_N), \quad h_i = \frac{1}{N} \sum_{j=1}^{N} G_{ij} e^{i\theta_j}, i = 1, 2, \dots, N$$

Relationship to the Standard Order Parameter

Consider an all-to-all coupled network ($G \equiv 1$), each component of the local order parameter is given by

$$h_i = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j} = re^{i\psi}$$

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Thus for an all-to-all coupled network the identity follows,

$$\frac{1}{N} h \cdot \overline{h} = r^2$$

Numerical Results: Twisted States

Since h is a vector, we plot $\sqrt{\frac{1}{N} h \cdot \bar{h}}$ which is a value $\approx |r|$.

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We create bifurcation diagrams using the same way but with the following changes:

- $g(\omega)$ is now a Gaussian distribution with $\mu = 1, \sigma = 0.1$
- For the upper curve we use a 2-twisted state instead of a coherent state as I.C.

Numerical Results: Twisted States

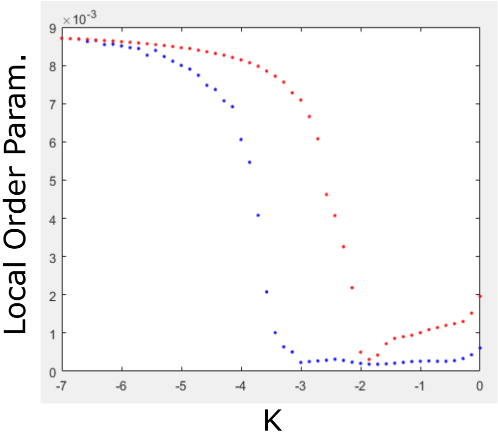

Figure: Bifurcation diagram for $-7 \leq K \leq 0$. The blue curve starts from the incoherent state and the red curve starts from a 2-twisted state.

Conclusion

Through numerical investigation, we see hysteresis between two multistable states for two different K regions:

- incoherent state and a 2-twisted state for $K < 0$.
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Rigorous analysis of the positive K region has been done, but involves very advanced techniques such as asymptotic expansions and perturbation methods.

Therefore a rigorous analysis of twisted states would be difficult for the following reasons:

- There is no obvious simplification of the second order KM involving the local order parameter.
- Accounting for the small-world graph structure is a difficult task.

Acknowledgements

I would like to extend my warmest thanks to my research advisor for the past year, Dr. Mizuhara, and also to Dr. Gevertz for my acquisition of a research position. I also want to thank the committee, Dr. Clarke and Dr. Hagedorn.

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The following professors have also been critical to my development and understanding of higher mathematics:

- Dr. Papantonopoulou
- Dr. Kardos
- Dr. Marcus

Thank you!

Euler's Formula I

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One can verify that $y(x) = Ce^{Ax}$ solves the ODE

The I.C. then gives the particular solution to the I.V.P

$$y(x) = e^{Ax}$$

Euler's Formula II

If $y(x) = \cos(x) + i\sin(x)$, observe the following

- $i \cdot y(x) = i\cos(x) - \sin(x)$
- $\mathrm{y}(0) = 1$

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Therefore Euler's formula can now be obtained

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