The Ricci flow on a closed 3 manifold

Jinghao Yu

June 2021

Contents

Chapter 1

Introduction

It is known that if a topological manifold M can be equipped with a smooth structure, there is a smooth Riemannian metric associated to this smooth structure such that M is a Riemannian manifold. On the other hand, the curvature tensor, which shows how the manifold warps, is an intrinsic geometric property depending only on metric.

$$\text{smooth structure} \Longrightarrow \text{Riemannian metric} \Longrightarrow \text{curvature}.$$

Now one can ask: if one knows the curvature on a smooth manifold M, what we can say about the smooth structure in M? This question has been responded in many different aspects. The main body of this article is also aimed to give some answers to this question. In fact we are going to show this Main Theorem:

Theorem (Main theorem)**.** *Let (M, g) be a closed Riemannian manifold of dimension 3 which admits a strictly positive Ricci curvature. Then (M, g) also admits a metric of constant positive sectional curvature. In particular, when M is simply connected, M is diffeomorphic to a 3-sphere.*

According to the knowledge of space form, we know the universal covering space of (M, g) is diffeomorphic to $\mathbb{S}^3$. That is why we deduce the particular case. The proof is mainly related to the Ricci flow equation:

$$\frac{\partial g}{\partial t} = -2Rc, \quad g(0) = \text{initial metric on } M. \tag{1}$$

We want to variate the metric on the contrary side of Ricci curvature, so that the curvature could be close to each other and finally has a constant sectional curvature. This idea is natural and we could show the short time existence via a method called DeTurck trick. Many estimates about curvature as time flows could be done as well. However, the solution of (1) always blow up in a finite time. This result pushes us to consider another equation, the normalized Ricci flow:

$$\frac{\partial}{\partial t} g_{ij} = \frac{2}{n} r g_{ij} - 2R_{ij}, \quad g(0) = \text{initial metric on } M, \tag{2}$$

where $n = \dim M$. It could be shown that the solution of (1) and (2) can transform to each other via a factor $\varphi(t)$. The estimates on curvature of (1) could be transformed to (2) and finally prove the main theorem.

Chapter 2

Preliminary

2.1 Riemannian manifold

This section is a review of knowledge about manifold, metric, connection and curvature.

Definition 2.1.1 (smooth manifold)**.** *A smooth manifold of dimension n is a set M and a family of injective mappings $\boldsymbol{x}_\alpha \colon U_\alpha \subset \mathbb{R}^n \to M$ of open sets U_α of $\mathbb{R}^n$ into M such that:*

(1) $\cup_\alpha \boldsymbol{x}_\alpha(U_\alpha) = M$.

(2) for any pair α, β, with $\boldsymbol{x}_\alpha(U_\alpha) \cap \boldsymbol{x}_\beta(U_\beta) = W \neq \emptyset$, the sets $\boldsymbol{x}_\alpha^{-1}(W)$ and $\boldsymbol{x}_\beta^{-1}(W)$ are open sets in $\mathbb{R}^n$ and the mappings $\boldsymbol{x}_\beta^{-1} \circ \boldsymbol{x}_\alpha$ are smooth.

(3) The family $\{(U_\alpha, \boldsymbol{x}_\alpha)\}$ is maximal relative to the conditions (1) and (2).

The pair $(U_\alpha, \mathbf{x}_\alpha)$ (or the mapping $\mathbf{x}_\alpha$) with $p \in \mathbf{x}_\alpha(U_\alpha)$ is called a **parametrization** (or **system of coordinates**) of M at p; $\mathbf{x}_\alpha(U_\alpha)$ is then called a **coordinate neighborhood** at p. A family $\{(U_\alpha, \mathbf{x}_\alpha)\}$ satisfying (1) and (2) is called a **smooth structure** on M.

Definition 2.1.2 (Riemannian metric)**.** *A **Riemannian metric** g on a differentiable manifold M is a correspondence which associates to each point p of M an inner product $g_p(-,-)$ (i.e. a symmetric bilinear, positive-definite form) on the tangent space T_pM, which varies smoothly in the following sense: If $\boldsymbol{x} : U \subset \mathbb{R}^n \to M$ is a system of coordinates around p, with $\boldsymbol{x}(x_1, x_2, \ldots, x_n) = q \in \boldsymbol{x}(U)$ and $\frac{\partial}{\partial x_i}(q) = d\boldsymbol{x}_q(0, \ldots, 1, \ldots, 0)$, then $g_q(\frac{\partial}{\partial x_i}(q), \frac{\partial}{\partial x_j}(q)) = g_{ij}(x_1, \ldots, x_n)$ is a smooth function on U.*

Remark 2.1.3. *The inner product at p could also be denoted by $\langle -,- \rangle_p$ or simply $\langle -,- \rangle$.*

A manifold M equipped with a Riemannian metric g is called a Riemannian manifold, denoted by (M, g).

Theorem 2.1.4 (Levi-Civita connection)**.** *On a Riemannian (M, g), there exists a unique connection on the tangent bundle TM such that:*

(1) $\nabla_X Y - \nabla_Y X = [X, Y]$.

(2) $Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$

*for $\forall X, Y, Z \in \Gamma(TM)$. Here $\Gamma(TM)$ denotes the set of smooth global section of the tangent bundle TM (i.e. the smooth vector fields of M). This connection is called the **Levi-Civita connection** of M, which is defined as follows:*

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2}\{X\langle Z, Y\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle$$
$$-\langle [X, Z], Y\rangle - \langle [Y, Z], X\rangle + \langle [X, Y], Z\rangle\}.$$

Suppose that $p \in M$, $(U, \mathbf{x})$ be a local coordinate for p. Let $\{\partial_i = \frac{\partial}{\partial x_i}\}_{i=1}^n$ be the associated basis of T_pM. Let $\Gamma_{ij}^k \in \mathbb{R}$ such that

$$\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k,$$

then we can deduce that

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \{\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\},$$

where (g^{ij}) is the inverse matrix of (g_{ij}).

Definition 2.1.5 (curvature)**.** *The **Riemannian curvature** Rm of a Riemannian manifold (M, g) is a correspondence that associates to every pair $X, Y \in \Gamma(M)$ a mapping $Rm(X, Y) : \Gamma(TM) \to \Gamma(TM)$ with*

$$Rm(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for all $Z \in \Gamma(TM)$.

Let $(U, \mathbf{x})$ be a local coordinate for $p \in M$; let $\{\partial_1 = \frac{\partial}{\partial x_1}, \ldots, \partial_n = \frac{\partial}{\partial x_n}\}$ denote the associated basis of T_pM. Then R_{ijk}^l, R_{ijks} are tensors of M such that

(1) $Rm(\partial_i, \partial_j)\partial_k = R_{ijk}^l \partial_l$.

(2) $R_{ijks} = \langle Rm(\partial_i, \partial_j)\partial_k, \partial_s\rangle = R_{ijk}^l g_{ls}$.

The notations defined above satisfy the following rules:

(1) $R_{ijk}^l = \partial_i \Gamma_{jk}^l - \partial_j \Gamma_{ik}^l + \Gamma_{ip}^l \Gamma_{jk}^p - \Gamma_{jp}^l \Gamma_{ik}^p$.

(2) $R_{ijks} = R_{ijk}^l g_{ls}$; $\quad R_{ijk}^l = R_{ijks} g^{sl}$.

(3) (first Bianchi identity) $R_{ijks} + R_{jkis} + R_{kijs} = 0$.

(4) $R_{ijks} = -R_{jiks}$; $\quad R_{ijks} = -R_{ijsk}$; $\quad R_{ijks} = R_{ksij}$.

The Riemannian curvature Rm contains nearly all information about the shape change of manifold. But even just a part of this information could have described the shape change of manifold. Hence the Ricci tensor, which is defined as a trace of Rm, could be helpful in our main theorem.

Definition 2.1.6 (Ricci tensor; Ricci curvature; scalar curvature)**.** *The Ricci curvature* $Rc : \Gamma(TM) \times \Gamma(TM) \to \mathbb{R}$ *is :*

$$(Y, Z) \mapsto \text{trace of the map: } X \to Rm(X, Y)Z.$$

In local coordinate, $R_{ij} := Rc(\partial_i, \partial_j) = \sum_l R^l_{lij} = R_{lijs} g^{sl}$*.*

Let v *be a unit tangent vector on* T_pM*, the* ***Ricci curvature*** *in the direction* v *is defined as* $Ric_p(v) := Rc(v, v)$*. Moreover, the* ***scalar curvature*** R *is defined to be the trace of* Rc*, i.e.* $g^{ij} R_{ij}$*.*

Definition 2.1.7 (sectional curvature)**.** *Let* σ *be a plane in* T_pM *spanned by* $X, Y \in T_pM$*. The sectional curvature* $K(\sigma) := \frac{\langle Rm(X,Y)Y,X\rangle}{|X \wedge Y|^2}$*.*

Definition 2.1.8 (Einstein manifold)**.** *A Riemannian manifold* (M, g) *is an* ***Einstein manifold*** *if its Ricci curvature is a constant times its Riemannian metric. i.e.* $\exists$ *a constant* λ *such that for all* X*,* $Y \in \Gamma(TM)$ *we have* $Rc(X, Y) = \lambda g(X, Y)$*.*

Proposition 2.1.9. *If* M *is 3-dimensional Einstein manifold, then* M *has constant sectional curvature.*

Proof. In normal coordinate, we have

$$R_{11} = R_{1221} + R_{1331}, \quad R_{22} = R_{1221} + R_{2332}, \quad R_{33} = R_{1331} + R_{2332}$$

$$R_{1221} = \frac{1}{2}(R_{11} + R_{22} - R_{33}) = \frac{1}{2}\lambda,$$

In similar, $R_{1331} = R_{2332} = \frac{1}{2}\lambda$. Hence, M has constant sectional curvature. □

Example 2.1.10. *The unit n-sphere* $\mathbb{S}^n$ *has a natural smooth Riemannian metric* g *by embedding it into* $\mathbb{R}^{n+1}$*:*

$$g = dx^1 \otimes dx^1 + \cdots + dx^{n+1} \otimes dx^{n+1}.$$

The Riemannian curvature of $\mathbb{S}^n$ is

$$Rm(X,Y,Z,W) = \langle Y,Z\rangle\langle X,W\rangle - \langle X,Z\rangle\langle Y,W\rangle, \quad X,Y,Z,W \in \Gamma(T\mathbb{S}^n);$$

so the sectional curvature unit 3-sphere is 1. Let $\{X_i\}_{i=1}^n$ be a normal coordinate of $\mathbb{S}^n$,

$$\begin{aligned} R_{ij} = Rc(X_i,X_j) &= \sum_k Rm(X_i,X_k,X_k,X_j) \\ &= \sum_k [\langle X_i,X_j\rangle\langle X_k,X_k\rangle - \langle X_i,X_k\rangle\langle X_k,X_j\rangle] = (n-1)\delta_{ij} = (n-1)g_{ij}. \end{aligned}$$

This shows that $\mathbb{S}^n$ is an Einstein manifold. In particular, let $n=3$ by Prop 2.1.9, the sectional curvature is $(3-1)/2 = 1$, which corresponds to previous calculation.

2.2 Covariant derivative

Let $\Gamma(M)$ be the set of smooth functions of smooth manifold M; let $T_s^r M$ denote the vector bundle

$$T_s^r M = \underbrace{TM \otimes \cdots \otimes TM}_{\text{r times}} \otimes \underbrace{T^*M \otimes \cdots \otimes T^*M}_{\text{s times}}.$$

Definition 2.2.1 ((r,s) tensor field)**.** *A smooth (r,s) tensor field T is a smooth section of the vector bundle $T_s^r M$:*

$$T \in \underbrace{\Gamma(TM) \otimes \cdots \otimes \Gamma(TM)}_{r\ times} \otimes \underbrace{\Gamma(T^*M) \otimes \cdots \otimes \Gamma(T^*M)}_{s\ times}.$$

It could be viewed as a $\Gamma(M)$-multilinear function:

$$\underbrace{\Gamma(T^*M) \times \cdots \times \Gamma(T^*M)}_{\text{r times}} \times \underbrace{\Gamma(TM) \times \cdots \times \Gamma(TM)}_{\text{s times}} \to \Gamma(M).$$

In local coordinate $(U, \mathbf{x})$, T could be expressed as:

$$T\Big|_U = \sum_{i,j} T^{1,\dots,r}_{1,\dots,s} \partial_1 \otimes \cdots \otimes \partial_r \otimes dx^1 \otimes \cdots \otimes dx_s,$$

where $T^{1,\dots,r}_{1,\dots,s}$ are all smooth functions of M. The inner product on TM can be naturally generalized to tensor field. For example, if $T, U \in \Gamma(T^2_3 M)$, then

$$\langle T, U \rangle_P := g^{ip} g^{jq} g^{km} g_{lm} g_{ht} T^{lh}_{ijk} T^{nt}_{pqm}.$$

Proposition 2.2.2 (covariant derivative)**.** *Let (M, g) be a Riemannian manifold, $X \in \Gamma(TM)$, then there exists a unique $\Gamma(M)$-module homomorphism $\nabla_X : \Gamma(T^r_s M) \to \Gamma(T^r_s M)$ for every $r, s \in \mathbb{Z}^{\geq 0}$ such that:*

(i) $\nabla_X(T \otimes T') = (\nabla_X T) \otimes T' + T \otimes (\nabla_X T')$ *for any* $T, T' \in \Gamma(T^r_s M)$.

(ii) The contraction $\mathcal{C}$ commutes with ∇_X. i.e. for any $T \in \Gamma(T^r_s M)$, we have $\mathcal{C}(\nabla_X T) = \nabla_X(\mathcal{C}T)$.

(iii) $\nabla_X f = Xf$ *for every* $f \in \Gamma(M)$.

(iv) $\nabla_X : T^1_0(M) = TM \to T^1_0(M) = TM$ *is the Levi-Civita connection associated to* (M, g).

The mapping satisfies these conditions is the covariant derivative on (M, g).

The formula of covariant derivative is: $\forall\ \alpha \in \Gamma(T^r_s M)$, $X, Y_1, \dots, Y_s \in \Gamma(TM)$, $\theta_1, \dots, \theta_s \in \Gamma(T^*M)$, the covariant derivative $\nabla_X \alpha$ satisfies

$$\begin{aligned}(\nabla_X \alpha)(\theta_1, \dots, \theta_r, Y_1, \dots, Y_s) &= X(\alpha(\theta_1, \dots, \theta_r, Y_1, \dots, Y_s)) \\ &- \sum_{i=1}^{r} \alpha(\theta_1, \dots, \nabla_X \theta_i, \dots, \theta_r, Y_1, \dots, Y_s) - \sum_{j=1}^{s} \alpha(\theta_1, \dots, \theta_r, Y_1, \dots, \nabla_X Y_j, \dots, Y_s),\end{aligned}$$

where $\nabla_X \theta$, $\theta \in \Gamma(T^*M)$ is

$$(\nabla_X \theta)(Y) = X(\theta(Y)) - \theta(\nabla_X Y), \quad Y \in \Gamma(TM).$$

It is natural to define the so called **covariant differential** ∇ which maps every (r,s) tensor field T to a (r, s+1) tensor field ∇T:

$$\nabla T : (X, \omega^1, \ldots, \omega^r, X_1, \ldots, X_s) \mapsto \nabla_X T(\omega^1, \ldots, \omega^r, X_1, \ldots, X_s),$$

$$\forall \text{ 1-forms } \omega^i, \forall \text{ vector fields } X_j \text{ and } X.$$

The covariant differential of the metric g is 0:

$$\nabla g(X, Y, Z) = (\nabla_X g)(Y, Z) = \nabla_X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0$$

for every vector fields $X, Y, Z \in \Gamma(TM)$.

In a local coordinate, $\nabla_i := \nabla_{\partial_i}$. For $T \in \Gamma(T^r_s M)$ we have:

$$\begin{aligned}
\nabla_i T^{k_1,\ldots,k_r}_{j_1,\ldots,j_s} &= (\nabla_i T)(dx^{k_1}, \ldots, dx^{k_r}, \partial_{j_1}, \ldots, \partial_{j_s}) \\
&= \mathcal{C}(\nabla_i T \otimes dx^{k_1} \otimes \cdots \otimes dx^{k_r} \otimes \partial_{j_1} \otimes \cdots \otimes \partial_{j_s}) \\
&= \mathcal{C}(\nabla_i (T \otimes dx^{k_1} \otimes \cdots \otimes dx^{k_r} \otimes \partial_{j_1} \otimes \cdots \otimes \partial_{j_s}) \\
&\quad - \sum_l T \otimes dx^{k_1} \otimes \cdots \otimes \nabla_i dx^{k_l} \otimes \cdots \otimes dx^{k_r} \otimes \partial_{j_1} \otimes \cdots \otimes \partial_{j_s} \\
&\quad - \sum_m T \otimes dx^{k_1} \otimes \cdots \otimes dx^{k_r} \otimes \partial_{j_1} \otimes \cdots \otimes \nabla_i \partial_{j_m} \otimes \cdots \otimes \partial_{j_s}) \\
&= \nabla_i (T(dx^{k_1}, \ldots, dx^{k_r}, \partial_{j_1}, \ldots, \partial_{j_s})) \\
&\quad - \sum_l T(dx^{k_1}, \ldots, \nabla_i dx^{k_l}, \ldots, dx^{k_r}, \partial_{j_1}, \ldots, \partial_{j_s}) \\
&\quad - \sum_m T(dx^{k_1}, \ldots, dx^{k_r}, \partial_{j_1}, \ldots, \nabla_i \partial_{j_m}, \ldots, \partial_{j_s})
\end{aligned}$$

$$\Longrightarrow \quad \nabla_i T^{k_1,\ldots,k_r}_{j_1,\ldots,j_s} = \partial_i T^{k_1,\ldots,k_r}_{j_1,\ldots,j_s} - \sum_l \Gamma^p_{ij_k} T^{k_1,\ldots,k_r}_{j_1,\ldots,j_{k-1},p,j_{k+1},\ldots,j_s} + \sum_m \Gamma^{k_l}_{iq} T^{k_1,\ldots,k_{l-1},q,k_{l+1},\ldots,k_r}_{j_1,\ldots,j_s}$$

This calculation implies the following properties:

Proposition 2.2.3.

(1) $\nabla_i T^{k_1,\ldots,k_r}_{j_1,\ldots,j_s} = \partial_i T^{k_1,\ldots,k_r}_{j_1,\ldots,j_s} - \sum_l \Gamma^p_{ij_k} T^{k_1,\ldots,k_r}_{j_1,\ldots,j_{k-1},p,j_{k+1},\ldots,j_s} + \sum_m \Gamma^{k_l}_{iq} T^{k_1,\ldots,k_{l-1},q,k_{l+1},\ldots,k_r}_{j_1,\ldots,j_s}$

(2) $\nabla_i R_{jk} = \partial_i R_{jk} - \Gamma^p_{ij} R_{pk} - \Gamma^p_{ik} R_{jp}$

(3) $\nabla_i R^m_{jkl} = \partial_i R^m_{jkl} - \Gamma^p_{ij} R^m_{pkl} - \Gamma^p_{ik} R^m_{jpl} - \Gamma^p_{il} R^m_{jkp} + \Gamma^m_{ip} R^p_{jkl}$

(4) $\nabla_i R_{jklm} = \partial_i R_{jklm} - \Gamma^p_{ij} R_{pklm} - \Gamma^p_{ik} R_{jplm} - \Gamma^p_{il} R_{jkpm} - \Gamma^p_{im} R_{jklp}$

The Riemannian curvature Rm could be viewed as a (1,3) or a (0,4) tensor field. This viewpoint helps us to implies the **second Bianchi identity**:

Proposition 2.2.4 (second Bianchi identity)**.** *The curvature tensor $Rm(X,Y)Z$ satisfies*

$$(\nabla_X Rm)(Y,Z,W) + (\nabla_Y Rm)(Z,X,W) + (\nabla_Z Rm)(X,Y,W) = 0$$

$$\forall \text{ vector fields } X,Y,Z,W$$

i.e.

$$\nabla_i R_{jklm} + \nabla_j R_{kilm} + \nabla_k R_{ijlm} = 0.$$

Proof. It suffices to prove it when X,Y,Z,W are coordinate basis $\{\partial_i\}_{i=1}^n$. For these X,Y,Z,W we have

$$\nabla_X Y = \nabla_Y X \tag{1}$$

$$Rm(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z \tag{2}$$

View Rm as a $(1,3)$ tensor field then

$$\begin{aligned}(\nabla_X Rm)(Y,Z,W) = \nabla_X(Rm(Y,Z)W) - Rm(\nabla_X Y, Z)W \\ -Rm(Y,\nabla_X Z)W - Rm(Y,Z)\nabla_X W\end{aligned}$$

$\Longrightarrow$

$$(\nabla_X Rm)(Y,Z,W) = \nabla_X\nabla_Y\nabla_Z W - \nabla_X\nabla_Z\nabla_Y W$$
$$-Rm(\nabla_X Y,Z)W - Rm(Y,\nabla_X Z)W - Rm(Y,Z)\nabla_X W$$

$$(\nabla_Y Rm)(Z,X,W) = \nabla_Y\nabla_Z\nabla_X W - \nabla_Y\nabla_X\nabla_Z W$$
$$-Rm(\nabla_Y Z,X)W - Rm(Z,\nabla_Y X)W - Rm(Z,X)\nabla_Y W$$

$$(\nabla_Z Rm)(X,Y,W) = \nabla_Z\nabla_X\nabla_Y W - \nabla_Z\nabla_T\nabla_X W$$
$$-Rm(\nabla_Z X,Y)W - Rm(X,\nabla_Z Y)W - Rm(X,Y)\nabla_Z W$$

Apply (1):

$$\begin{aligned}
&(\nabla_X Rm)(Y,Z,W) + (\nabla_Y Rm)(Z,X,W) + (\nabla_Z Rm)(X,Y,W)\\
&= \nabla_X\nabla_Y\nabla_Z W - \nabla_X\nabla_Z\nabla_Y W - Rm(Y,Z)\nabla_X W\\
&+\nabla_Y\nabla_Z\nabla_X W - \nabla_Y\nabla_X\nabla_Z W - Rm(Z,X)\nabla_Y W\\
&+\nabla_Z\nabla_X\nabla_Y W - \nabla_Z\nabla_T\nabla_X W - Rm(X,Y)\nabla_Z W
\end{aligned}$$

Apply (2):

$$\begin{aligned}
&= \nabla_X\nabla_Y\nabla_Z W - \nabla_X\nabla_Z\nabla_Y W - \nabla_Y\nabla_Z\nabla_X W + \nabla_Z\nabla_Y\nabla_X W\\
&+\nabla_Y\nabla_Z\nabla_X W - \nabla_Y\nabla_X\nabla_Z W - \nabla_Z\nabla_X\nabla_Y W + \nabla_X\nabla_Z\nabla_Y W\\
&+\nabla_Z\nabla_X\nabla_Y W - \nabla_Z\nabla_Y\nabla_X W - \nabla_X\nabla_Y\nabla_Z W + \nabla_Y\nabla_X\nabla_Z W = 0
\end{aligned}$$

□

Corollary 2.2.5 (second Bianchi identity – contract form 1)**.**

$$g^{jl}\nabla_j R_{lmki} = \nabla_i R_{km} - \nabla_k R_{im}$$

Proof.

$$g^{jl}\nabla_j R_{lmki} = -g^{jl}\nabla_i R_{jklm} - g^{jl}\nabla_k R_{ijlm} = \nabla_i R_{km} - \nabla_k R_{im}$$ □

Corollary 2.2.6 (second Bianchi identity – contract form 2)**.**

$$\nabla^j R_{ij} = \frac{1}{2}\nabla_i R \qquad \textit{where } \nabla^j := g^{ij}\nabla_i$$

Proof. Prop 2.2.5 tells that

$$g^{im}g^{jl}(\nabla_i R_{jklm} + \nabla_j R_{kilm} + \nabla_k R_{ijlm}) = 0$$

$$\nabla_k R = g^{im}\nabla_i R_{km} + g^{jl}\nabla_j R_{kl} = 2\nabla^l R_{kl} \qquad \square$$

Proposition 2.2.7 (Ricci identity)**.** *Let* $T \in \Gamma(T^r_s(M))$ *we have*

$$(\nabla_i\nabla_j - \nabla_j\nabla_i)T^{l_1,\ldots,l_r}_{k_1,\ldots,k_s} = \sum_{k=1}^{r} R^{l_k}_{ijp} A^{l_1,\ldots,l_{k-1},p,l_{k+1},\ldots,l_r}_{k_1,\ldots,k_s} - \sum_{l=1}^{s} R^{p}_{ijk_l} A^{l_1,\ldots,l_r}_{k_1,\ldots,k_{l-1},p,k_{l+1},\ldots,k_s}$$

Now consider the issue of higher derivatives.

Consider the operators:

$$\nabla^k : \Gamma(T^r_s M) \to \Gamma(T^r_{s+k} M)$$

defined for $k \geq 1$. If $T \in \Gamma(T^r_s M)$ and $X_1, \ldots, X_k \in \Gamma(TM)$, then inductively define

$$\nabla^k_{X_1,\ldots,X_k} T := (\nabla_{X_1}\nabla^{k-1}T)(X_2, \ldots, X_n).$$

In this notation, Rm could be expressed as follows

$$\begin{aligned}(\nabla^2 Z)(X, Y) - (\nabla^2 Z)(Y, X) &= (\nabla_X \nabla Z)(Y) - (\nabla_Y \nabla Z)(X)\\ &= \nabla_X\nabla_Y Z - \nabla_{\nabla_X Y} Z - \nabla_Y \nabla_X Z + \nabla_{\nabla_Y X} Z = Rm(X, Y)Z\end{aligned}$$

for vector fields $X, Y, Z \in \Gamma(TM)$.

The concepts of gradient, Hessian, divergence, and Laplacian can be generalized to Riemannian manifold (M, g): Suppose $f \in \Gamma(M)$, $X \in \Gamma(TM)$. Let α be a $(0, p)$ tensor field, and let T be a (r, s) tensor field.

(1) The gradient of f, ∇f, is a vector field such that $\langle \nabla f, X\rangle = X(f)$. In local coordinate $(\nabla f)^i = g^{ij}\partial_j(f)$

(2) The Hessian of f is $\nabla^2 f \in \Gamma(T^*M \otimes T^*M)$. Then in local coordinate

$$\nabla^2 f_{ij} = \partial_i \partial_j f - \Gamma_{ij}^k \partial_k f$$

(3) The divergence of α is a $(0, p-1)$ tensor. In local coordinate

$$(\text{div } \alpha)_{i_1,\ldots,i_{p-1}} = g^{jk} \nabla_j \alpha_{k,i_1,\ldots,i_{p-1}}$$

(4) The Laplacian of T, ΔT, is defined as $g^{ij} \nabla_i \nabla_j T$

2.3 First-order differential operators on forms

Let M^n be a smooth manifold of dimension n. A p-form θ is a smooth section of the bundle $\bigwedge^p(T^*M)$, i.e.

$$\theta \in \Omega^p(M) = \Gamma(\bigwedge^p(T^*M)).$$

The **exterior derivative** is the family of operators

$$d \equiv d_p : \Omega^p(M) \to \Omega^{p+1}(M)$$

defined for all p-forms θ and vector fields $Y_1, \ldots, Y_p$ by

$$\begin{aligned} & d\theta(Y_0, \ldots, Y_p) \\ := & \sum_{0 \le i \le p} (-1)^i Y_i(\theta(Y_0, \ldots, Y_{i-1}, \hat{Y}_i, \ldots, Y_p)) \\ + & \sum_{0 \le i < j \le p} (-1)^{i+j} \theta([Y_i, Y_j], Y_0, \ldots, \hat{Y}_i, \ldots, \hat{Y}_j, \ldots, Y_p). \end{aligned}$$

Although $d\theta$ is independent of the Riemannian metric, the Levi-Civita metric could help us compute $d\theta$:

$$d\theta(Y_0, \ldots, Y_p) = \sum_{i=0}^{p} (-1)^i (\nabla_{Y_i} \theta)(Y_0, \ldots, \hat{Y}_i, \ldots, Y_p).$$

2.4 Lie derivative

Let X be a differentiable vector field on a smooth manifold M, i.e. a smooth section in tangent bundle TM, and let $p \in M$. Then there exists a neighbourhood $U \subset M$ at p, an interval $(-\delta, \delta)$, $\delta > 0$, and a differentiable mapping $\varphi : (-\delta, \delta) \times U \to M$ such that the curve $t \mapsto \varphi(t, q)$, $t \in (-\delta, \delta)$, $q \in U$, is the unique curve which satisfies

$$\frac{\partial \varphi}{\partial t} = X(\varphi(t, q)), \quad \varphi(0, q) = q.$$

Let's define the mapping $\varphi_t : U \to M$ by $\varphi_t(q) = \varphi(t, q)$. The map φ_t is called the local flow of X. It is easy to see φ_t is a local diffeomorphism of M.

Proposition 2.4.1. *Let X, Y be differentiable vector fields on a smooth manifold M, let $p \in M$, and let φ_t be the local flow of X in a neighbourhood U of p. Then the Lie bracket satisfies*

$$[X, Y](p) := (XY - YX)(p) = \lim_{t \to 0} \frac{1}{t}[Y - (\varphi_t)_* Y](\varphi_t(p)) \in T_p M$$

Now we can generalize this property to define the Lie derivative of a tensor field. Let α be a tensor field and X be a complete vector field which generates a global 1 - parameter group of diffeomorphisms φ_t. The Lie derivative of α with respect to X is defined by

$$\mathcal{L}_X \alpha := \lim_{t \to 0} \frac{1}{t}(\alpha - (\varphi_t)_* \alpha)$$

In similar to the definition of covariant derivative, the Lie derivative, which measures the infinestimal lack of diffeomorphism invariance of a tensor with respect to a 1-parameter group of diffeomorphisms generated by a vector field, has the following properties:

(1) If f is smooth function, then $\mathcal{L}_X f = Xf$.

(2) If Y is a vector field, then $\mathcal{L}_X Y = [X, Y]$.

(3) If α, β are tensor fields, then $\mathcal{L}_X(\alpha \otimes \beta) = \mathcal{L}_X\alpha \otimes \beta + \alpha \otimes \mathcal{L}_X\beta$.

(4) $\forall\ \alpha \in \Gamma(T^r_s M)$, $X, Y_1, \ldots, Y_s \in \Gamma(TM)$, $\theta_1, \ldots, \theta_s \in \Gamma(T^*M)$, the covariant derivative $\nabla_X\alpha$ satisfies

$$\begin{aligned}
&(\mathcal{L}_X\alpha)(\theta_1, \ldots, \theta_r, Y_1, \ldots, Y_s) = X(\alpha(\theta_1, \ldots, \theta_r, Y_1, \ldots, Y_s))\\
&- \sum_{i=1}^{r} \alpha(\theta_1, \ldots, \mathcal{L}_X\theta_i, \ldots, \theta_r, Y_1, \ldots, Y_s) - \sum_{j=1}^{s} \alpha(\theta_1, \ldots, \theta_r, Y_1, \ldots, \mathcal{L}_X Y_j, \ldots, Y_s)\\
&= X(\alpha(\theta_1, \ldots, \theta_r, Y_1, \ldots, Y_s))\\
&- \sum_{i=1}^{r} \alpha(\theta_1, \ldots, \mathcal{L}_X\theta_i, \ldots, \theta_r, Y_1, \ldots, Y_s) - \sum_{j=1}^{s} \alpha(\theta_1, \ldots, \theta_r, Y_1, \ldots, [X, Y_j], \ldots, Y_s),
\end{aligned}$$

where $\mathcal{L}_X\theta$, $\theta \in \Gamma(T^*M)$ means

$$\begin{aligned}
(\mathcal{L}_X\theta)(Y) &= X(\theta(Y)) - \theta(\mathcal{L}_X Y)\\
&= X(\theta(Y)) - \theta([X, Y])\\
&= d\theta(X, Y) + Y(\theta(X)), \quad Y \in \Gamma(TM).
\end{aligned}$$

Even though the definition of Lie derivative is independent of the Riemannian metric, the metric (and the Levi-Civita connection induced by this metric) could help us compute the Lie derivative. This is because

$$\mathcal{L}_X Y = [X, Y] = \nabla_X Y - \nabla_Y X.$$

Thus if α is a $(0, r)$-tensor field, $Y_1, \ldots, Y_r$ are vector fields, then

$$\begin{aligned}
(\mathcal{L}_X\alpha)(Y_1, \ldots, Y_r) &= X(\alpha(Y_1, \ldots, Y_r)) - \sum_{i=1}^{r} \alpha(Y_1, \ldots, Y_{i-1}, \mathcal{L}_X Y_i, Y_{i+1}, \ldots, Y_r)\\
&= (\nabla_X\alpha)(Y_1, \ldots, Y_r) + \sum_{i=1}^{r} \alpha(Y_1, \ldots, Y_{i-1}, \nabla_{Y_i} X, Y_{i+1}, \ldots, Y_r).
\end{aligned}$$

In particular, if θ is a covector field and X, Y are vector fields, the Lie derivative of θ is given by

$$(\mathcal{L}_X\theta)(Y) = (\nabla_X\theta)(Y) + \theta(\nabla_Y X)$$

Example. *Let $X, Y_1, Y_2 \in \Gamma(TM)$. The Lie derivative of Riemannian metric g is*

$$(\mathcal{L}_X g)(Y_1, Y_2) = g(\nabla_{Y_1}, X) + g(Y_1, \nabla_{Y_2} X),$$

$$(\mathcal{L}_X g)_{ij} = \nabla_i X_j + \nabla_j X_i.$$

In particular, when $X = \nabla f$, $f \in \Gamma(M)$, $(\mathcal{L}_{\nabla f} g)_{ij} = 2\nabla_i \nabla_j f$.

2.5 Space form

There is a standard result about the classification of Riemannian manifold with constant sectional curvature.

Theorem 2.5.1. *Let M^n be a complete Riemannian manifold with constant sectional curvature K. Then the universal covering $\tilde{M}$ of M, with the covering metric, is isometric to:*

(a) hyperbolic space $\mathbb{H}^n$, if $K = -1$,

(b) Euclidean space $\mathbb{R}^n$, if $K = 0$,

(c) sphere $\mathbb{S}^n$, if $K = 1$.

Proof. See [6] Theorem 4.1 page 163. □

If M is a topological space, we say that the group G (of homeomorphisms of M) acts in a totally discontinuous manner if every $x \in M$ has a neighborhood U such that $g(U) \cap U = \emptyset$, for all $g \in G$, $g \neq e$.

Proposition 2.5.2. *Let M be a complete Riemannian manifold with constant sectional curvature $K(1, 0, -1)$. Then M is isometric to $\tilde{M}/\Gamma$, where $\tilde{M}$ is the universal covering space of M, Γ is a subgroup of the group of isometries of $\tilde{M}$ which acts in a totally discontinuous manner on $\tilde{M}$ and the metric on $\tilde{M}/\Gamma$ is induced from the covering $\pi : \tilde{M} \longrightarrow \tilde{M}/\Gamma$.*

Proof. See [6] Prop 4.3 page 165. □

Chapter 3

Evolution equations

In this section, let (M, g) be a compact Riemannian manifold of dimension n. The Ricci flow means the following differential equation:

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} \tag{$*$}$$

Naturally, we should use the evolution equation $(*)$ to find out evolution equations of other variables in M. For convenience, consider the evolution equation

$$\frac{\partial}{\partial t} g_{ij} = h_{ij} \tag{$\star$}$$

where h is a symmetric (0,2) tensor. If we want to get the evolution equations for Ricci flow, it suffices to substitute h for $-2Rc$. Furthermore, it is a trick to do the calculation on a normal coordinate of $p \in M$, where $g = g_{ij} dx^i \otimes dx^j$ and

$$x^i(p) = 0, \quad g_{ij}(p) = \delta_{ij}, \quad dg_{ij}(p) = 0, \quad \Gamma_{ij}^k(p) = 0 \quad \forall i, j, k.$$

That is because tensor field is invariant under coordinate transformation, the result established at a normal coordinate also establishes at other coordinates.

The following results are the evolution equations originate from $(\star)$.

Lemma 3.0.1.
$$\frac{\partial}{\partial t} g^{ij} = -g^{ik} g^{jl} h_{kl}$$

Proof.
$$g^{ij} g_{jl} = \delta^i_l \Longrightarrow \frac{\partial}{\partial t} g^{ij} \cdot g_{jl} + g^{ij} \cdot \frac{\partial}{\partial t} g_{jl} = 0 \Longrightarrow \frac{\partial}{\partial t} g^{ij} \cdot g_{jl} = -g^{ik} h_{kl}$$

$$\text{Multiply } g^{lm} \text{ on both sides: } \frac{\partial}{\partial t} g^{ij} \delta^m_j = -g^{ik} g^{lm} h_{kl}$$

Thus, let $m = j$ we get the desired result. □

Lemma 3.0.2.
$$\frac{\partial}{\partial t} \Gamma^k_{ij} = \frac{1}{2} g^{kl} (\nabla_i h_j l + \nabla_j h_{il} - \nabla_l h_{ij})$$

Proof.
$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \{\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\}$$
$$\Longrightarrow \frac{\partial}{\partial t} \Gamma^k_{ij} = \frac{1}{2} \frac{\partial}{\partial t} g^{kl} \cdot (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) + \frac{1}{2} g^{kl} \cdot (\partial_i h_{jl} + \partial_j h_{il} - \partial_l h_{ij})$$

In normal coordinate of $p \in M$,

$$\partial_i g_{jl}(p) = 0, \quad \partial_i h_{jl}(p) = \nabla_i h_{jl}(p) \quad \forall i, j, k.$$
$$\Longrightarrow \frac{\partial}{\partial t} \Gamma^k_{ij}(p) = \frac{1}{2} g^{kl} (\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij})(p)$$

The difference of two connection is a (1,2) tensor, so at point p, the above formula establishes in every parametrization of p for any $p \in M$. Hence, we get the desired result:

$$\frac{\partial}{\partial t} \Gamma^k_{ij} = \frac{1}{2} g^{kl} (\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij})$$

□

Lemma 3.0.3.

$$\frac{\partial}{\partial t} R^l_{ijk} = \frac{1}{2} g^{lp} \begin{Bmatrix} \nabla_i \nabla_j h_{kp} + \nabla_i \nabla_k h_{jp} - \nabla_i \nabla_p h_{jk} \\ -\nabla_j \nabla_i h_{kp} - \nabla_j \nabla_k h_{ip} + \nabla_j \nabla_p h_{ik} \end{Bmatrix}$$

Proof.

$$R^l_{ijk} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^l_{ip} \Gamma^p_{jk} - \Gamma^l_{jp} \Gamma^p_{ik}$$

In normal coordinate of p:

$$\begin{aligned} \frac{\partial}{\partial t} R^l_{ijk}(p) &= \partial_i (\frac{\partial}{\partial t} \Gamma^l_{jk})(p) - \partial_j (\frac{\partial}{\partial t} \Gamma^l_{ik})(p) \\ &= \partial_i (\frac{1}{2} g^{lp} (\partial_j h_{kp} + \partial_k h_{jp} - \partial_p h_{jk}))(p) \\ &\quad - \partial_j (\frac{1}{2} g^{lp} (\partial_i h_{kp} + \partial_k h_{ip} - \partial_p h_{ik}))(p) \\ &= \frac{1}{2} g^{lp} \begin{Bmatrix} \partial_i \partial_j h_{kp} + \partial_i \partial_k h_{jp} - \partial_i \partial_p h_{jk} \\ -\partial_j \partial_i h_{kp} - \partial_j \partial_k h_{ip} + \partial_j \partial_p h_{ik} \end{Bmatrix} (p) \end{aligned}$$

In normal coordinate $\nabla = \partial$ at p. It is done. □

Remark 3.0.4. *Apply Ricci identity to above formula we get*

$$\frac{\partial}{\partial t} R^l_{ijk} = \frac{1}{2} g^{lp} \begin{Bmatrix} \nabla_i \nabla_k h_{jp} + \nabla_j \nabla_p h_{ik} - \nabla_i \nabla_p h_{jk} \\ -\nabla_j \nabla_k h_{ip} - R^q_{ijk} h_{qp} - R^q_{ijp} h_{kq} \end{Bmatrix}.$$

Lemma 3.0.5.

$$\frac{\partial}{\partial t} R_{jk} = \frac{1}{2} g^{pq} (\nabla_q \nabla_j h_{kp} + \nabla_q \nabla_k h_{jp} - \nabla_q \nabla_p h_{jk} - \nabla_j \nabla_k h_{qp})$$

Proof.

$$\frac{\partial}{\partial t} R_{jk} = \frac{\partial}{\partial t} \sum_i R^i_{ijk} = \frac{1}{2} g^{ip} \begin{Bmatrix} \nabla_i \nabla_j h_{kp} + \nabla_i \nabla_k h_{jp} - \nabla_i \nabla_p h_{jk} \\ -\nabla_j \nabla_i h_{kp} - \nabla_j \nabla_k h_{ip} + \nabla_j \nabla_p h_{ik} \end{Bmatrix}$$

$$\frac{\partial}{\partial t} R_{jk} = \frac{1}{2} g^{ip} (\nabla_i \nabla_j h_{kp} + \nabla_i \nabla_k h_{jp} - \nabla_i \nabla_p h_{jk} - \nabla_j \nabla_k h_{ip}),$$

because h is symmetric. □

Lemma 3.0.6. *Let* $H := g^{jk}h_{jk}$

$$\frac{\partial}{\partial t}R = -\Delta H + \nabla^p\nabla^q h_{pq} - \langle h, Rc\rangle$$

Proof.

$$\begin{aligned}\frac{\partial}{\partial t}R = \frac{\partial}{\partial t}(g^{jk}R_{jk}) &= \frac{\partial}{\partial t}g^{jk}\cdot R_{jk} + g^{jk}\frac{\partial}{\partial t}R_{jk}\\ &= -g^{ji}g^{kl}h_{il}R_{jk} + g^{jk}\frac{1}{2}g^{pq}(\nabla_q\nabla_j h_{kp} + \nabla_q\nabla_k h_{jp} - \nabla_q\nabla_p h_{jk} - \nabla_j\nabla_k h_{qp})\\ &= -\langle h, Rc\rangle + g^{pq}\nabla_q\nabla^j h_{jp} - \Delta H\\ &= -\langle h, Rc\rangle + \nabla^p\nabla^q h_{pq} - \Delta H\end{aligned}$$

□

Remark 3.0.7. *Notice that*

$$div(div\ h) = g^{ij}\nabla_i(div\ h)_j = g^{ij}\nabla_i g^{kl}\nabla_k h_{lj} = \nabla^j\nabla^l h_{lj},$$

$$\implies \quad \frac{\partial}{\partial t}R = -\Delta H + div(div\ h) - \langle h, Rc\rangle.$$

Lemma 3.0.8. *Let* $d\mu = \sqrt{\det g_{ij}}dx^1\wedge\cdots\wedge dx^n$ *be the volume form. Its evolution equation is* $\frac{\partial}{\partial t}d\mu = \frac{H}{2}d\mu$*, where* $H = g^{ij}h_{ij}$*.*

Proof.

$$\frac{\partial}{\partial t}d\mu = \frac{1}{2}(g^{ij}\frac{\partial}{\partial t}g_{ij})\sqrt{\det g_{ij}}dx = \frac{1}{2}(g^{ij}h_{ij})d\mu = \frac{H}{2}d\mu.$$

□

Corollary 3.0.9. *If* (M, g) *is a closed Riemannian* n*-manifold then*

$$\frac{d}{dt}\int_M Rd\mu = \int_M(\frac{1}{2}RH - \langle h, Rc\rangle)d\mu$$

Proof.

$$\begin{aligned}\frac{\partial}{\partial t}(Rd\mu) &= \frac{\partial}{\partial}R\cdot d\mu + R\frac{\partial}{\partial t}(d\mu)\\ &= (-\Delta H + \mathrm{div}(\mathrm{div}\ h) - \langle h, Rc\rangle)d\mu + R\frac{H}{2}d\mu\end{aligned}$$

The divergence theorem tells that

$$\begin{cases} \displaystyle\int_M \operatorname{div}(\operatorname{div}\ h)d\mu = 0 \\ \displaystyle\int_M \Delta H = \int_M \operatorname{div}\ \cdot \nabla H d\mu = 0 \end{cases}$$

$$\Longrightarrow \frac{d}{dt}\int R d\mu = \int \frac{\partial}{\partial t}(R d\mu) = \int (\frac{1}{2}RH - \langle h, Rc\rangle)d\mu \qquad \square$$

Chapter 4

Short time existence

In this section, let (M, g) be a closed Riemannian manifold with dimension n. This section is aim to show the equation of Ricci flow has a short time a unique solution on M.

4.1 Linearization of Ricci flow

Notice that

$$\begin{aligned}
R_{ij} = R^i_{ijk} &= \partial_i \Gamma^i_{jk} - \partial_j \Gamma^i_{ik} + \Gamma^i_{ip}\Gamma^p_{jk} - \Gamma^i_{jp}\Gamma^p_{ik} \\
&= \partial_i[\frac{1}{2}g^{ip}\{\partial_j g_{kp} + \partial_k g_{jp} - \partial_p g_{jk}\}] - \partial_j[\frac{1}{2}g^{ip}\{\partial_i g_{kp} + \partial_k g_{ip} - \partial_p g_{ik}\}] \\
&\qquad\qquad + \text{terms of first derivatives of } g + \text{terms of } g_{ij} \\
&= \frac{1}{2}g^{ip}\{\partial_i\partial_k g_{jp} - \partial_i\partial_p g_{jk} - \partial_j\partial_k g_{ip} + \partial_j\partial_p g_{ik}\} \\
&\qquad\qquad + \text{terms of first derivatives of } g + \text{terms of } g_{ij}
\end{aligned}$$

Hence the Ricci flow could be expressed as

$$\frac{\partial}{\partial t}g_{ij} = g^{ip}\{-\partial_i\partial_k g_{jp} + \partial_i\partial_p g_{jk} + \partial_j\partial_k g_{ip} - \partial_j\partial_p g_{ik}\} + \text{lower order terms},$$

so it is a system of nonlinear parabolic differential equation of g. A useful method to see whether a short time solution exists is to see its parabolicity, which would be defined later.

4.1.1 The symbol of a nonlinear differential operator

Let $\mathcal{E}, \mathcal{F}$ be smooth vector bundles over M. A linear differential operator L of order k is a morphism between vector bundles:

$$L : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{F}),$$

$$\text{written as } L(E) = \sum_{|\alpha| \leq k} L_\alpha \partial^\alpha E,$$

where $L_\alpha \in Hom(\mathcal{E}, \mathcal{F})$ is a bundle homomorphism for each multi-index α. For example, if L is with order 2, let $\{x^i\}$ be a local parametrization of $p \in U$; let $\{e_i\}$, $\{f_j\}$ be basis for $\mathcal{E}, \mathcal{F}$ in local coordinate, then for $u = u_l e_l \in \Gamma(\mathcal{E})$

$$Lu = \{(\lambda_{ij})_l^k \frac{\partial^2 u_l}{\partial x^i \partial x^j} + (\eta_i)_l^k \frac{\partial u_l}{\partial x^i} + \theta_l^k u_l\} f_k,$$

where $(\lambda_{ij})_l^k, (\eta_i)_l^k, \theta_l^k$ are all independent of u.

A total symbol of L in the direction $\xi \in \Gamma(T^*M)$, denoted as $\sigma[L](\xi)$, is a bundle morphism such that

$$\sigma[L](\xi)(E) = \sum_{|\alpha| \leq k} L_\alpha(\Pi_j \xi^{\alpha_j} E), \quad \forall E \in \Gamma(\mathcal{E}).$$

A principle symbol of L in the direction $\xi \in \Gamma(T^*M)$, denoted as $\widehat{\sigma}[L](\xi)$, is a bundle morphism such that

$$\widehat{\sigma}[L](\xi)(E) = \sum_{|\alpha| = k} L_\alpha(\Pi_j \xi^{\alpha_j} E), \quad \forall E \in \Gamma(\mathcal{E}).$$

In previous example

$$\sigma[L](\xi)(u) = \{(\lambda_{ij})_l^k \xi_i \xi_j u_l + (\eta_i)_l^k \xi_i \partial u_l + \theta_l^k u_l\} f_k$$

$$\widehat{\sigma}[L](\xi)(u) = (\lambda_{ij})_l^k \xi_i \xi_j u_l f_k, \quad \forall u \in \Gamma(\mathcal{E}).$$

Suppose M is another linear differential operator, by the rule of derivative, we have

$$\widehat{\sigma}[M \circ L](\xi) = \widehat{\sigma}[M](\xi) \circ \widehat{\sigma}[L](\xi).$$

Let $S_2 T^* M$ be vector bundle of symmetric (0,2) tensor; let $S_2^+ T^* M$ be a subbundle of $S_2 T^* M$ which is positive-definite. In the situation of Ricci flow, we know $Rc : \Gamma(S_2^+ T^* M) \to \Gamma(S_2 T^* M)$ is not a linear differential operator. But taking derivative gives a way to linearize Rc at metric $g \in \Gamma(S_2^+ T^* M)$. The linearization $D(Rc_g) : \Gamma(S_2 T^* M) \to \Gamma(S_2 T^* M)$ is $D(Rc_g)(h) := \frac{\partial}{\partial t}\big|_{s=0} g$, where $g(t) := g + th$, for any $h \in \Gamma(S_2 T^* M)$. It is easy to check $D(Rc_g)(h)$ is a linear differential operator over h. Specifically, by Lemma 3.0.5 we have

$$[D(Rc_g)(h)]_{jk} = \frac{1}{2} g^{pq} (\nabla_q \nabla_j h_{kp} + \nabla_q \nabla_k h_{jp} - \nabla_q \nabla_p h_{jk} - \nabla_j \nabla_k h_{qp}).$$

The principle symbol in the direction ξ of the linear partial differential operator $D(Rc_g)$ is the bundle homomorphism

$$\widehat{\sigma}[D(Rc_g)](\xi) : S_2 T^* M \to S_2 T^* M$$

$$[\widehat{\sigma}[D(Rc_g)](\xi)(h)]_{jk} = \frac{1}{2} g^{pq} \left\{ \xi_q \xi_j h_{kp} + \xi_q \xi_k h_{jp} - \xi_q \xi_p h_{jk} - \xi_j \xi_k h_{qp} \right\}$$

A linear partial differential operator $L : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{F})$ is side to be **elliptic** if its principal symbol $\widehat{\sigma}[L](\xi)$ is an isomorphism whenever $\xi \neq 0$. A nonlinear operator $N : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{F})$ is said to be elliptic if its linearization $D[N]$ is elliptic. A short remark is that for a linear operator L:

$$D[L_g](h) = \frac{\partial}{\partial s}\Big|_{s=0} (\sum L_\alpha \partial^\alpha g(s)) = \sum L_\alpha \partial^\alpha (\frac{\partial}{\partial s}\Big|_{s=0} g) = \sum L_\alpha \partial^\alpha h = Lh,$$

so $D[L_g] = L$ and two definitions about ellipticity coincide. There is a conclusion which states that if Rc is elliptic, then the Ricci flow equation has a short time unique solution. However, it would be shown in next subsection that the kernel of $D[Rc_g]$ is not trivial.

4.1.2 The principal symbol of the differential operator Rc

This subsection is going to show Rc is not elliptic. We will construct a linear partial differential operator δ_g^* with nontrivial image such that $D(Rc_g)\circ \delta_g^*$ is a zero map. In this way, we know $D(Rc_g)$ has a nontrivial kernel, hence not elliptic.

Let $\delta_g = -\mathrm{div}_g : \Gamma(S_2T^*M) \to \Gamma(T^*M)$ such that

$$(\delta_g h)_{jk} = -g^{ij}\nabla_i h_{jk} \quad \forall h \in \Gamma(S_2T^*M).$$

Let δ_g^* be the formal adjoint of δ_g with respect to the L^2 inner product

$$(V, W) = \int_M \langle V, W\rangle d\mu_g$$

for any $V, W \in \Gamma(T_s^r M)$, $\forall r, s \in \mathbb{Z}^{\geq 0}$.

Lemma 4.1.1. *The partial differential operator $\delta_g^* : \Gamma(T^*M) \to \Gamma(S_2T^*M)$ is a map*

$$(\delta_g^*(X))_{jk} = \frac{1}{2}(\nabla_j X_k + \nabla_k X_j) = \frac{1}{2}(\mathcal{L}_{X^\#} g)_{jk} \quad \forall X \in \Gamma(T^*M).$$

Proof. Let $X \in \Gamma(T^*M)$ and $h \in \Gamma(S_2T^*M)$

$$\begin{aligned}
(\delta_g^* X, h) &= \int_M (\delta_g^* X)_{jk} h_{il} g^{ij} g^{kl} d\mu \\
(\delta_g^* X, h) &= (X, \delta_g h) = \int_M \langle X, \delta_g h\rangle d\mu = \int_M -X_l g^{ij}\nabla_i h_{jk} g^{lk} d\mu \\
&= \int_M \{-g^{ij} g^{lk}\nabla_i (X_l h_{jk}) + g^{ij} g^{lk} h_{jk}\nabla_i X_l\} d\mu \\
&= \int_M g^{ij} g^{lk} (\frac{1}{2} h_{jk} + \frac{1}{2} h_{kj}) \nabla_i X_l d\mu
\end{aligned}$$

For every X, h we have this equation, so

$$\begin{aligned}(\delta_g^* X)_{jk} h_{il} g^{ij} g^{kl} &= (\nabla_i X_l) g^{ij} g^{lk} (h_{jk} + h_{kj})/2 \\ &= (\nabla_j X_k) g^{ji} g^{kl} h_{il}/2 + (\nabla_k X_j) g^{kl} g^{ji} h_{il}/2\end{aligned}$$

$$(\delta_g^* X)_{jk} = \frac{1}{2}(\nabla_j X_k + \nabla_k X_j) \in \Gamma(S_2 T^* M) \qquad \square$$

According to Lemma 4.1.1, δ_g^* is a linear differential operator. Its principal symbol is

$$\widehat{\sigma}[\delta_g^*](\xi) : T^*M \to S_2T^*M : X \mapsto (\widehat{\sigma}[\delta_g^*](\xi)(X))_{jk} = \frac{1}{2}(\xi_j X_k + \xi_k X_j).$$

As a result, if $\xi \neq 0$, then dim $im\ \widehat{\sigma}[\delta_g^*](\xi) = n$.

Now consider the differential operator $D(Rc_g) \circ \delta_g^*$

$$\Gamma(T^*M) \xrightarrow{\delta_g^*} \Gamma(S_2T^*M) \xrightarrow{D(Rc_g)} \Gamma(S_2T^*M)$$

$$T^*M \xrightarrow{\widehat{\sigma}[\delta_g^*](\xi)} S_2T^*M \xrightarrow{\widehat{\sigma}[D(Rc_g)](\xi)} S_2T^*M$$

Proposition 4.1.2.

$$(D(Rc_g) \circ \delta_g^*)(X) = \frac{1}{2}\mathcal{L}_{X^\#} Rc_g \quad \textit{where } X \in \Gamma(T^*M)$$

Proof. Let φ_t be the family of diffeomorphisms generated by the vector field $X^\#$.

$$Rc(\varphi_t^* g) = \varphi_t^*(Rc_g)$$

Take the derivative of t at $t = 0$ on both sides we have

$$D(Rc_g)(\mathcal{L}_{X^\#} g) = \mathcal{L}_{X^\#} Rc_g,$$

$$(D(Rc_g) \circ \delta_g^*)(X) = D(Rc_g)(\frac{1}{2}\mathcal{L}_{X^\#} g) = \frac{1}{2}\mathcal{L}_{X^\#} Rc_g \qquad \square$$

Observe that $\mathcal{L}_{X^\#} Rc_g$ just consists of first derivative of X, so $\widehat{\sigma}[D(Rc_g)\circ \delta_g^*](\xi)$, the components of third derivative of X, is a zero map.

$$0 = \widehat{\sigma}[D(Rc_g)\circ \delta_g^*](\xi) = \widehat{\sigma}[D(Rc_g)](\xi)\circ \widehat{\sigma}[\delta_g^*](\xi)$$

$$\text{im } \widehat{\sigma}[\delta_g^*](\xi) \subseteq \ker \widehat{\sigma}[D(Rc_g)](\xi)$$

Thus, if $\xi \neq 0$, $\widehat{\sigma}[D(Rc_g)](\xi)$ has at least an n dimensional kernel in each $n(n+1)/2$-dimensional fibre S_2T^*M. As a result, the differential operator Rc is not elliptic.

4.2 The Ricci-Deturck flow and its parabolicity

As shown in last subsection, the nonlinear differential operator Rc is not elliptic, so we cannot immediately apply standard theory to conclude there exists a unique solution of the Ricci flow for a short time. However, the Ricci flow still has short time existence and uniqueness.

Theorem 4.2.1. *If (M, g_0) is a closed Riemannian manifold, there exists a unique solution $g(t)$ to the Ricci flow defined on some positive time interval $[0,\epsilon)$ such that $g(0) = g_0$.*

This theorem would be proven with help of **Ricci DeTurck flow**, which would be defined soon.

Let $\tilde{\Gamma}$ be a fixed torsion-free connection, i.e. $\tilde{\Gamma}_{ij}^k = \tilde{\Gamma}_{ji}^k$. Let $W = W(g,\tilde{\Gamma})$ denote a vector field

$$W^k = g^{pq}(\Gamma_{pq}^k - \tilde{\Gamma}_{pq}^k).$$

The Ricci DeTurck flow is a differential equation

$$\begin{cases} \frac{\partial}{\partial t} g = -2Rc_g + \nabla_i W_j + \nabla_j W_i \\ g(0) = g_0 \end{cases}$$

We shall show $N := -2Rc + \mathcal{L}_W : \Gamma(S_2^+ T^* M) \to \Gamma(S_2 T^* M) : g \to -2Rc_g + \mathcal{L}_W g$ is a elliptic differential operator of degree 2. At first, we shall linearize N. Let $H := g^{pq} h_{qp}$. Ricci identity states that

$$\begin{aligned} -2[D(Rc_g)(h)]_{jk} &= -g^{pq}\nabla_q\nabla_j h_{kp} - g^{pq}\nabla_q\nabla_k h_{jp} + g^{pq}\nabla_q\nabla_p h_{jk} + g^{pq}\nabla_j\nabla_k h_{qp} \\ &= -g^{pq}\nabla_j\nabla_q h_{kp} + g^{pq}R^l_{qjk}h_{lp} + g^{pq}R^l_{qjp}h_{kl} \\ &\quad - g^{pq}\nabla_k\nabla_q h_{jp} + g^{pq}R^l_{qkj}h_{lp} + g^{pq}R^l_{qkp}h_{jl} + \Delta h_{jk} + \nabla_j\nabla_k H \\ &= \Delta h_{jk} - \nabla_j(g^{pq}\nabla_q h_{pk} - \frac{1}{2}\nabla_k H) - \nabla_k(g^{pq}\nabla_q h_{pj} - \frac{1}{2}\nabla_j H) \\ &\quad + \text{lower derivative of } h \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial t}\Big|_{t=0} W_k &= \frac{\partial}{\partial t}(g_{kr}g^{pq}(\Gamma^r_{pq} - \tilde{\Gamma}^r_{pq})) \\ &= g_{kr}g^{pq}\frac{\partial}{\partial t}\Big|_{t=0}\Gamma^r_{pq} + \text{terms of } h \\ &= g^{pq}g_{kr}\frac{1}{2}g^{rl}(\nabla_p h_{ql} + \nabla_q h_{pl} - \nabla_l h_{pq}) + \text{terms of } h \\ &= g^{pq}\frac{1}{2}(\nabla_p h_{qk} + \nabla_q h_{pk} - \nabla_k h_{pq}) + \text{terms of } h \\ &= g^{pq}\nabla_q h_{pk} - \frac{1}{2}\nabla_k H + \text{terms of } h \end{aligned}$$

$$\frac{\partial}{\partial t}\Big|_{t=0}\nabla_j W_k = \frac{\partial}{\partial t}(\partial_j W_k - \Gamma^l_{jk}W_l) = \partial_j\frac{\partial}{\partial t}W_k - \Gamma^l_{jk}\frac{\partial}{\partial t}W_l - \frac{\partial}{\partial t}\Gamma^l_{jk}W_l$$

$$
\begin{aligned}
&= \nabla_j(\frac{\partial}{\partial t}W_k) + \text{lower derivative of } h \\
[D((\mathcal{L}_W)_g)(h)]_{jk} &= \frac{\partial}{\partial t}\Big|_{t=0}\mathcal{L}_W(g+th) = \frac{\partial}{\partial t}\Big|_{t=0}(\nabla_j W_k + \nabla_k W_j) \\
&= \nabla_j(\frac{\partial}{\partial t}W_k) + \nabla_k(\frac{\partial}{\partial t}W_j) + \text{lower derivative of } h \\
&= \nabla_j(g^{pq}\nabla_q h_{pk} - \frac{1}{2}\nabla_k H) + \nabla_j(g^{pq}\nabla_q h_{pj} - \frac{1}{2}\nabla_j H) \\
&\qquad + \text{lower derivative of } h
\end{aligned}
$$

Thus,

$$
\begin{aligned}
[D(N_g)(h)]_{jk} &= -2[D(Rc_g)(h)]_{jk} + [D((\mathcal{L}_W)_g)(h)]_{jk} \\
&= \Delta h_{jk} + \text{lower derivative of } h \\
\widehat{\sigma}[D(N_g)(\xi)(h)] &= |\xi|^2 h
\end{aligned}
$$

Then we know N is elliptic, because $\widehat{\sigma}[D(N_g)(\xi)(h)]$ is an isomorphism whenever $\xi \neq 0$. It is a standard result that, for any smooth initial metric g_0, there exists $\epsilon > 0$ and a smooth function $g(t)$ defined at $M \times [0, \epsilon)$ such that g is a unique solution to the Ricci-DeTurck flow for a short time $0 \le t < \epsilon$.

Let $\varphi_t : M \to M$ be a one-parameter family of maps such that

$$
\begin{aligned}
\frac{\partial}{\partial t}\varphi_t(p) &= -W(\varphi_t(p), g(t)) \quad && \forall (p,t) \in M \times [0,\epsilon) \\
\varphi_0(p) &= p && \forall p \in M
\end{aligned}
$$

If M is compact, then all φ_t exist and remain diffeomorphisms for as long as the solution of the Ricci-DeTurck flow, $g(t)$, exists. In fact, there is a general result about the existence of this kind of one-parameter family.

Lemma 4.2.2. *If $\{X_t | 0 \le t < T \le \infty\}$ is a continuous time-dependent family of vector fields on a compact manifold M, then there exists a one-parameter family of diffeomorphisms $\{\varphi_t : M \to M | 0 \le t < T \le \infty\}$ such* *that*

$$\frac{\partial}{\partial t}\varphi_t(p) = X_t(\varphi_t(p)) \quad \forall (p,t) \in M \times [0,T)$$
$$\varphi_0(p) = p \qquad\qquad \forall p \in M$$

Proof. We may assume that there is $t_0 \in [0,T)$ such that $\varphi_s(q)$ exists for all $(q,s) \in M \times [0,t_0]$. Let $t_1 \in (t_0,T)$ be given. If we could show φ_t exists for all $t \in [t_0,t_1]$ then we imply the lemma. Given any $p_0 \in M$, choose local coordinate (U, $\mathbf{x}$) and (V, $\mathbf{y}$) such that $p_0 \in U$ and $\varphi_{t_0}(p_0) \in V$. As long as $p \in U$ and $\varphi_t(p) \in V$, the equation of φ_t is equivalent to

$$\begin{aligned}\frac{\partial}{\partial t}\mathbf{y} \circ \varphi_t \circ \mathbf{x}^{-1}(x) &= \mathbf{y}_*(\frac{\partial \varphi_t}{\partial t}(\mathbf{x}^{-1}(x))) \\ &= \mathbf{y}_*[X_t \circ \mathbf{y}^{-1}(\mathbf{y} \circ \varphi_t \circ \mathbf{x}^{-1}(x)]\end{aligned}$$

for any $x \in \mathbf{x}(U)$ such that $\varphi_t(\mathbf{x}^{-1}(p)) \in V$. Setting $z_t = \mathbf{y} \circ \varphi_t \circ \mathbf{x}^{-1}$ and $F_t = \mathbf{y}_*(X_t \circ \mathbf{y}^{-1})$, we get

$$\frac{\partial}{\partial t} z_t = F_t(z_t)$$

where z_t and F_t are time-dependent maps between subsets of $\mathbb{R}^n$. Locally, the equation in lemma 4.2.2 is equivalent to a nonlinear ODE in $\mathbb{R}^n$. The Picard's ODE theorem tells that $\exists!$ solution for a short time $t \in [t_0, t_0+\epsilon)$. Because M is compact, there is a uniform $\epsilon > 0$ such that the solution $\varphi_t(p)$ exists for $t \in [t_0, t_0+\epsilon]$.

On $t_0' = t_0 + \epsilon$, apply the same argument again, then there exists a covering of $M \times [t_0,t_1]$ by $\{M \times [t_0', t_0'+\epsilon']\}$. Since $M \times [t_0,t_1]$ is compact, there exists a finite subcover of $M \times [t_0,t_1]$ and that we can glue finitely many short time solutions to get φ_t on $t \in [t_0,t_1]$. $\square$

A key method to construct a solution to the Ricci flow is to pull-back the solution of Ricci-DeTurck flow by φ_t: one defines

$$\bar{g}(t) = \varphi_t^* g(t) \quad 0 \le t < \epsilon.$$

Then one observes that

$$\begin{aligned}
\frac{\partial}{\partial t}\bar{g}(t) &= \frac{\partial}{\partial t}\varphi_t^* g(t) = \frac{\partial}{\partial s}\Big|_0 (\varphi_{t+s}^* g(t+s)) \\
&= \varphi_t^*(\frac{\partial}{\partial s}\Big|_0 g(t+s)) + \frac{\partial}{\partial s}\Big|_0 (\varphi_{t+s}^* g(t)) \\
&= \varphi_t^*(-2Rc(g(t)) + \mathcal{L}_{W(t)} g(t)) + \frac{\partial}{\partial s}\Big|_0 [(\varphi_t^{-1}\circ\varphi_{t+s})^* \varphi_t^* g(t)] \\
&= -2Rc[\varphi_t^* g(t)] + \varphi_t^*(\mathcal{L}_{W(t)} g(t)) - \mathcal{L}_{(\varphi_t^{-1})_* W(t)}\varphi_t^* g(t) \\
&= -2Rc[\varphi_t^* g(t)] + \mathcal{L}_{\varphi_t^* W(t)}\varphi_t^* g(t) - \mathcal{L}_{\varphi_t^* W(t)}\varphi_t^* g(t) \\
&= -2Rc[\varphi_t^* g(t)] \\
\bar{g}(0) &= \varphi_0^* g(0) = id_M g(0) = g_0
\end{aligned}$$

Based on the computation, we know $\bar{g}(t) = \varphi_t^* g(t)$ is a solution of the Ricci flow for $t \in [0,\epsilon)$. The proof of uniqueness would be proved later with the help of the harmonic map heat flow.

4.3 The harmonic map heat flow

Let (M^m, g), (N^n, h) be two Riemannian manifolds and let $f : M^m \to N^n$ be a smooth map between M and N. The derivative of f is

$$df \equiv f_* \in \Gamma(T^*M^m \otimes f^*TN^n)$$

where f^*TN is the pullback bundle over M. Let $\{x^i\}$ be the local coordinate of M, $\{y^\alpha\}$ be the local coordinate of N. Let Γ_g (or $\Gamma(g)$) be the Levi-Civita connection of g, and let Γ_h (or $\Gamma(h)$) be the Levi-Civita connection of h. Then

$$df = (df)_j^\alpha (dx^j \otimes f^*\frac{\partial}{\partial y^\alpha}) \equiv \frac{\partial f^\alpha}{\partial x^j}(dx^j \otimes f^*\frac{\partial}{\partial y^\alpha})$$

The map f induces a connection $f^*\Gamma$ in the following way:

$$\nabla : \Gamma(T^*M^m \otimes f^*TN^n) \to \Gamma(T^*M^m \otimes T^*M^m \otimes f^*TN^n)$$

$$\begin{aligned}
\nabla_{\partial/\partial x^i} f^* \frac{\partial}{\partial y^\beta} &= f^*(\nabla_{f_*(\partial/\partial x^i)} \frac{\partial}{\partial y^\beta}) = f^* \nabla_{\frac{\partial f^\alpha}{\partial x^i}\frac{\partial}{\partial x^\alpha}} (\frac{\partial}{\partial y^\beta}) \\
&= f^*(\frac{\partial f^\alpha}{\partial x^i}(\nabla_{\partial/\partial x^\alpha} \partial/\partial y^\beta)) = f^*(\frac{\partial f^\alpha}{\partial x^i}(\Gamma_h)^\gamma_{\alpha\beta}\frac{\partial}{\partial y^\gamma}) \\
&= \frac{\partial f^\alpha}{\partial x^i}(\Gamma_h \circ f)^\gamma_{\alpha\beta} f^* \frac{\partial}{\partial y^\gamma}, \\
\nabla_{\partial/\partial x^i} f^* \frac{\partial}{\partial y^\beta} &:= (f^*\Gamma)^\gamma_{i\beta} f^* \frac{\partial}{\partial y^\gamma},
\end{aligned}$$

so we define

$$(f^*\Gamma)^\gamma_{i\beta} = \frac{\partial f^\alpha}{\partial x^i}(\Gamma_h \circ f)^\gamma_{\alpha\beta}.$$

Hence $\nabla(df) = (\nabla df)^\alpha_{ij} dx^i \otimes dx^j \otimes \frac{\partial}{\partial y^\alpha}$ and

$$\begin{aligned}
(\nabla df)^\alpha_{ij} &= \nabla_i (df)^\alpha_j \\
&= \frac{\partial}{\partial x^i}(\frac{\partial f^\alpha}{\partial x^j}) - (\Gamma_g)^l_{ij}\frac{\partial f^\alpha}{\partial x^l} + (f^*\Gamma_h)^\alpha_{i\gamma}\frac{\partial f^\gamma}{\partial x^j} \\
&= \frac{\partial}{\partial x^i}(\frac{\partial f^\alpha}{\partial x^j}) - (\Gamma_g)^l_{ij}\frac{\partial f^\alpha}{\partial x^l} + \frac{\partial f^\beta}{\partial x^i}(\Gamma_h \circ f)^\alpha_{\beta\gamma}\frac{\partial f^\gamma}{\partial x^j}
\end{aligned}$$

The **harmonic map Laplacian** with respect to the domain metric g and codomain metric h is the trace of ∇:

$$\Delta_{g,h} f = tr_g \nabla(df) = g^{ij}\nabla_i (df)^\gamma_j \frac{\partial}{\partial y^\gamma}$$

and

$$(\Delta_{g,h} f)^\gamma = g^{ij}\nabla_i (df)^\gamma_j = g^{ij}[\frac{\partial^2 f^\gamma}{\partial x^i \partial x^j} - (\Gamma_g)^l_{ij}\frac{\partial f^\gamma}{\partial x^l} + (\Gamma_h \circ f)^\gamma_{\alpha\beta}\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}].$$

Given $f_0 : M \to N$, the harmonic map flow with respect to f_0 is

$$\begin{aligned}
\frac{\partial f}{\partial t} &= \Delta_{g,h} f, \\
f(0) &= f_0
\end{aligned}$$

The principal symbol of $\Delta_{g,h}$ in the direction $\xi \in \Gamma(T^*M)$ is

$$\widehat{\sigma}[\Delta_{g,h}(\xi)(f)] = g^{ij}\xi_i\xi_j f = |\xi|^2 f,$$

so the harmonic map flow is a parabolic equation and there exists a unique short time solution.

Theorem 4.3.1. *If $\varphi : (M^n, g) \to (N^n, h)$ is a diffeomorphism of Riemannian manifolds, we have*

$$(\Delta_{g,h}\varphi)^\gamma(x) = [(\varphi^{-1})^* g]^{\alpha\beta}(-\Gamma((\varphi^{-1})^* g)^\gamma_{\alpha\beta} + \Gamma(h)^\gamma_{\alpha\beta})(\varphi(x))$$

Proof. Let $\{x^i\}$, $\{y^\alpha\}$ be a local coordinate of M, N respectively. Let κ be a metric in N. Pullback κ to M, the Levi-Civita connection induced by $\varphi^*\kappa$ satisfies

$$\begin{aligned}\Gamma^k_{ij}(\varphi^*\kappa)\frac{\partial}{\partial x^k} &= \nabla(\varphi^*\kappa)_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = \varphi^*\left(\nabla(\kappa)_{\varphi_*(\frac{\partial}{\partial x^i})}\varphi_*\left(\frac{\partial}{\partial x^j}\right)\right)\\ &= \varphi^*\left(\nabla(\kappa)_{(\frac{\partial\varphi^\alpha}{\partial x^i}\frac{\partial}{\partial y^\alpha})}\left(\frac{\partial\varphi^\beta}{\partial x^j}\frac{\partial}{\partial y^\beta}\right)\right)\end{aligned}$$

$$\begin{aligned}&= \varphi^*\left(\nabla(\kappa)_{(\frac{\partial\varphi^\alpha}{\partial x^i}\frac{\partial}{\partial y^\alpha})}\left(\frac{\partial\varphi^\beta}{\partial x^j}\frac{\partial}{\partial y^\beta}\right)\right)\\ &= (\varphi^{-1})_*\left(\frac{\partial^2\varphi^\beta}{\partial x^i\partial x^j}\frac{\partial}{\partial y^\beta} + \frac{\partial\varphi^\beta}{\partial x^j}\frac{\partial\varphi^\alpha}{\partial x^i}\Gamma(\kappa)^\gamma_{\alpha\beta}\frac{\partial}{\partial y^\gamma}\right)\\ &= \left(\frac{\partial^2\varphi^\beta}{\partial x^i\partial x^j}\frac{\partial(\varphi^{-1})^k}{\partial y^\beta} + \Gamma(\kappa)^\gamma_{\alpha\beta}\frac{\partial\varphi^\alpha}{\partial x^i}\frac{\partial\varphi^\beta}{\partial x^j}\frac{\partial(\varphi^{-1})^k}{\partial y^\gamma}\right)\frac{\partial}{\partial x^k}\end{aligned}$$

Then

$$\begin{aligned}\Gamma^k_{ij}(\varphi^*\kappa)\frac{\partial\varphi^\gamma}{\partial x^k} &= \left(\frac{\partial^2\varphi^\beta}{\partial x^i\partial x^j}\frac{\partial(\varphi^{-1})^k}{\partial y^\beta} + \Gamma(\kappa)^\eta_{\alpha\beta}\frac{\partial\varphi^\alpha}{\partial x^i}\frac{\partial\varphi^\beta}{\partial x^j}\frac{\partial(\varphi^{-1})^k}{\partial y^\eta}\right)\frac{\partial\varphi^\gamma}{\partial x^k}\\ &= \frac{\partial^2\varphi^\gamma}{\partial x^i\partial x^j} + \Gamma(\kappa)^\gamma_{\alpha\beta}\frac{\partial\varphi^\alpha}{\partial x^i}\frac{\partial\varphi^\beta}{\partial x^j}\end{aligned} \tag{1}$$

Notice that

$$\begin{aligned}\kappa^{\alpha\beta} &= \kappa(dy^\alpha, dy^\beta) = (\varphi^*\kappa)(\varphi^* dy^\alpha, \varphi^* dy^\beta)\\ &= (\varphi^*\kappa)\left(\frac{\partial\varphi^\alpha}{\partial x^i}dx^i, \frac{\partial\varphi^\beta}{\partial x^j}dx^j\right) = (\varphi^*\kappa)^{ij}\frac{\partial\varphi^\alpha}{\partial x^i}\frac{\partial\varphi^\beta}{\partial x^j},\end{aligned} \tag{2}$$

so multiply $(\varphi^*\kappa)^{ij}$ on both sides of (1), we get

$$(\varphi^*\kappa)^{ij}\Gamma_{ij}^k(\varphi^*\kappa)\frac{\partial\varphi^\gamma}{\partial x^k} = (\varphi^*\kappa)^{ij}\frac{\partial^2\varphi^\gamma}{\partial x^i\partial x^j} + \Gamma(\kappa)_{\alpha\beta}^\gamma\kappa^{\alpha\beta}$$

$$(\varphi^*\kappa)^{ij}\frac{\partial^2\varphi^\gamma}{\partial x^i\partial x^j} - (\varphi^*\kappa)^{ij}\Gamma_{ij}^k(\varphi^*\kappa)\frac{\partial\varphi^\gamma}{\partial x^k} = -\Gamma(\kappa)_{\alpha\beta}^\gamma\kappa^{\alpha\beta} \tag{3}$$

Take $\kappa = (\varphi^{-1})^*g$, $\varphi^*\kappa = g$ on equation (2) and (3), we get

$$g^{ij}\frac{\partial^2\varphi^\gamma}{\partial x^i\partial x^j} - g^{ij}\Gamma_{ij}^k(g)\frac{\partial\varphi^\gamma}{\partial x^k} = -\Gamma[(\varphi^{-1})^*g]_{\alpha\beta}^\gamma[(\varphi^{-1})^*g]^{\alpha\beta} \tag{3'}$$

$$g^{ij}\frac{\partial\varphi^\alpha}{\partial x^i}\frac{\partial\varphi^\beta}{\partial x^j} = [(\varphi^{-1})^*g]^{\alpha\beta} \tag{2'}$$

Finally, we get

$$\begin{aligned}(\Delta_{g,h}\varphi)^\gamma &= g^{ij}[\frac{\partial^2\varphi^\gamma}{\partial x^i\partial x^j} - (\Gamma_g)_{ij}^l\frac{\partial\varphi^\gamma}{\partial x^l}] + g^{ij}(\Gamma_h\circ\varphi)_{\alpha\beta}^\gamma\frac{\partial\varphi^\alpha}{\partial x^i}\frac{\partial\varphi^\beta}{\partial x^j} \\ &= -\Gamma[(\varphi^{-1})^*g]_{\alpha\beta}^\gamma[(\varphi^{-1})^*g]^{\alpha\beta} + (\Gamma_h\circ\varphi)_{\alpha\beta}^\gamma[(\varphi^{-1})^*g]^{\alpha\beta} \\ &= [(\varphi^{-1})^*g]^{\alpha\beta}\left(-\Gamma[(\varphi^{-1})^*g]_{\alpha\beta}^\gamma + (\Gamma_h\circ\varphi)_{\alpha\beta}^\gamma\right)\end{aligned}$$

Hence we get the theorem. □

Corollary 4.3.2. *Let $M = N$ and φ be the identity, then*

$$(\Delta_{g,h}id)^\gamma = g^{\alpha\beta}\left(-\Gamma(g)_{\alpha\beta}^\gamma + \Gamma(h)_{\alpha\beta}^\gamma\right)$$

4.4 An approach to uniqueness of Ricci flow

This subsection aims to prove the uniqueness of Ricci flow. Let (M, g_0) be a closed Riemannian manifold; $\tilde{g}$ be a fixed background metric on M; $\tilde{\Gamma}$ be the Levi-Civita connection associated to $\tilde{g}$. If $\bar{g}(t)$ is a solution of the Ricci flow:

$$\begin{cases}\dfrac{\partial}{\partial t}\bar{g} = -2Rc(\bar{g}) \\ \bar{g}(0) = g_0\end{cases}$$

then by the ellipticity of harmonic map Laplacian, there exists diffeomorphisms $\varphi_t : (M, \bar{g}(t)) \to (M, \tilde{g})$ to be the unique solution of the harmonic map heat flow as long as $\bar{g}(t)$ exists:

$$\begin{cases} \dfrac{\partial}{\partial t}\varphi_t = \Delta_{g(t),\tilde{g}}\varphi_t \\ \quad \varphi_0 = id_M \end{cases}$$

By Thm 4.3.1,

$$\frac{\partial}{\partial t}\varphi_t = -W \circ \varphi_t$$

where

$$W(t) = [(\varphi_t^{-1})^*\bar{g}]^{pq}\left(\Gamma[(\varphi_t^{-1})^*\bar{g}(t)]_{pq}^k - \widetilde{\Gamma}_{pq}^k\right).$$

Let $g(t) = (\varphi_t)_*\bar{g}(t)$ then $g(t)$ is a solution of the Ricci-DeTurck flow:

$$\begin{cases} \dfrac{\partial}{\partial t}g = -2Rc(g) + \mathcal{L}_W g \\ g(0) = g_0 \end{cases}$$

It is because

$$\begin{aligned} \frac{\partial}{\partial t}g(t) = \frac{\partial}{\partial t}((\varphi_t)_*\bar{g}(t)) &= \frac{\partial}{\partial s}\Big|_{s=0}((\varphi_{t+s})_*\bar{g}(t+s)) \\ &= (\varphi_t)_*\left(\frac{\partial}{\partial s}\Big|_{s=0}\bar{g}(t+s)\right) + \frac{\partial}{\partial s}\Big|_{s=0}(\varphi_{t+s})_*\bar{g}(t) \end{aligned}$$

$$\begin{aligned} &= (\varphi_t)_*(-2Rc(\bar{g})) + (\varphi_t)_*\frac{\partial}{\partial s}\Big|_{s=0}(\varphi_t^{-1}\circ\varphi_{t-s})^*\bar{g}(t) \\ &= -2Rc[(\varphi_t)_*\bar{g}] + (\varphi_t)_*\mathcal{L}_{(\varphi_t^{-1})_*W(t)}\bar{g}(t) \\ &= -2Rc(g) + \mathcal{L}_{W(t)}g(t) \end{aligned}$$

According to the parabolicity of $-2Rc + \mathcal{L}$, if $\bar{g}_1(t)$, $\bar{g}_2(t)$ are both solutions of Ricci flow, then the corresponding $g_1(t) = g_2(t)$. It also deduces that

$$W_i^k = g_i^{pq}\left(\Gamma(g_i)_{pq}^k - \widetilde{\Gamma}_{pq}^k\right), \quad i = 1, 2$$

is uniquely determined, so the corresponding $(\varphi_i)_t$ is unique. Hence

$$\bar{g}_1(t) = (\varphi_1)_t^* g = (\varphi_2)_t^* g = \bar{g}_2(t)$$

and we prove the uniqueness of the Ricci flow.

Chapter 5

Estimate of curvature

After showing short-time existence of Ricci flow, it is time to discuss the evolution of curvature. The estimate based on the evolution equations would help us approach the main theorem. In this section, we assume that (M, g) is a closed Riemannian 3-manifold with a strictly positive Ricci curvature.

5.1 Evolution of curvature

In this subsection, we will replace the symmetric (0,2) tensor h as $-2Rc$ so that we get the evolutions equation of Ricci flow. The estimate of curvature starts from these evolution equations.

Theorem 5.1.1.

$$\frac{\partial}{\partial t}R_{ijks} = \nabla_i\nabla_s R_{jk} + \nabla_j\nabla_k R_{is} - \nabla_i\nabla_k R_{js} - \nabla_j\nabla_s R_{jk} - g^{pq}R_{ijkp}R_{qs} + g^{pq}R_{ijsp}R_{kq}$$

Proof.

$$\frac{\partial}{\partial t}R_{ijks} = \frac{\partial}{\partial t}(R^l_{ijk}g_{sl}) = (\frac{\partial}{\partial t}R^l_{ijk})g_{sl} + R^l_{ijk}\frac{\partial}{\partial t}h_{sl}$$
$$= -\nabla_i\nabla_k R_{js} - \nabla_j\nabla_s R_{ik} + \nabla_i\nabla_s R_{jk} + \nabla_j\nabla_k R_{is} + R^q_{ijk}R_{qs} + R^q_{ijs}R_{kq} - 2R^l_{ijk}R_{sl}$$
$$= \nabla_i\nabla_s R_{jk} + \nabla_j\nabla_k R_{is} - \nabla_i\nabla_k R_{js} - \nabla_j\nabla_s R_{ik} - R^q_{ijk}R_{qs} + R^q_{ijs}R_{kq}$$
$$= \nabla_i\nabla_s R_{jk} + \nabla_j\nabla_k R_{is} - \nabla_i\nabla_k R_{js} - \nabla_j\nabla_s R_{jk} - g^{pq}R_{ijkp}R_{qs} + g^{pq}R_{ijsp}R_{kq}$$

□

Introduce a new tensor B:

$$B_{ijkl} = g^{pq}g^{mn}R_{pijm}R_{qkln}$$

It satisfies the symmetries:

$$B_{ijkl} = B_{jilk} = B_{klij}$$

Lemma 5.1.2.

$$\begin{aligned}\Delta R_{ijks} &= \nabla_i\nabla_s R_{jk} + \nabla_j\nabla_k R_{is} - \nabla_i\nabla_k R_{js} - \nabla_j\nabla_s R_{ik}\\ &\quad + g^{mn}R_{in}R_{mjks} - g^{mn}R_{jn}R_{miks}\\ &\quad - 2(B_{ijsk} + B_{isjk} - B_{ijks} - B_{ikjs})\end{aligned}$$

Proof. The second Bianchi identity states:

$$\nabla_i R_{jklm} + \nabla_j R_{kilm} + \nabla_k R_{ijlm} = 0$$

$$\Delta R_{ijkl} = g^{pq}\nabla_p\nabla_i R_{qjks} - g^{pq}\nabla_p\nabla_j R_{qiks}$$

Consider the first term, $g^{pq}\nabla_p\nabla_i R_{qjks}$. Apply Ricci identity,

$$\begin{aligned}&g^{pq}\nabla_p\nabla_i R_{qjks} - g^{pq}\nabla_i\nabla_p R_{qjks}\\ &\qquad = -g^{pq}\{R^m_{piq}R_{mjks} + R^m_{pij}R_{qmks} + R^m_{pik}R_{qjms} + R^m_{pis}R_{qjkm}\}\\ &\qquad = -g^{pq}g^{mn}\{R_{piqn}R_{mjks} + R_{pijn}R_{qmks} + R_{pikn}R_{qjms} + R_{pisn}R_{qjkm}\}\end{aligned}$$

Its first term contracts to $g^{mn}R_{in}R_{mjks}$; its second term:

$$\begin{aligned}-g^{pq}g^{mn}R_{pijn}R_{qmks} &= g^{pq}g^{mn}R_{pijn}(R_{mkqs}+R_{kqms}) \\ &= g^{pq}g^{mn}R_{pijn}(-R_{qskm}+R_{qksm}) = -B_{ijsk}+B_{ijks};\end{aligned}$$

the last two terms are $B_{ikjs}-B_{isjk}$. By contracted second Bianchi identity

$$g^{pq}\nabla_p R_{qjks} = \nabla_s R_{jk} - \nabla_k R_{js}$$

Thus

$$\begin{aligned}g^{pq}\nabla_p\nabla_i R_{qjks} &= \nabla_i\nabla_s R_{jk} - \nabla_i\nabla_k R_{js} + g^{mn}R_{in}R_{mjks} \\ &\quad - (B_{ijsk}+B_{isjk}-B_{ijks}-B_{ikjs})\end{aligned}$$

Intertwine i, j,

$$\begin{aligned}\Delta R_{ijks} = {}&\nabla_i\nabla_s R_{jk} + \nabla_j\nabla_k R_{is} - \nabla_i\nabla_k R_{js} - \nabla_j\nabla_s R_{ik} \\ &+ g^{mn}R_{in}R_{mjks} - g^{mn}R_{jn}R_{miks} \\ &- 2(B_{ijsk}+B_{isjk}-B_{ijks}-B_{ikjs})\end{aligned}$$

□

Then we have

Corollary 5.1.3.

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijks} &= \Delta R_{ijks} + 2(B_{ijsk}+B_{isjk}-B_{ijks}-B_{ikjs}) \\ &\quad - g^{pq}(R_{pjks}R_{qi}+R_{ipks}R_{qj}+R_{ijps}R_{qk}+R_{ijkp}R_{qs})\end{aligned}$$

Theorem 5.1.4.

$$\frac{\partial}{\partial t}R_{jk} = \Delta R_{jk} + 2g^{pq}g^{rs}R_{qjks}R_{rp} - 2g^{pq}R_{jp}R_{kq}$$

Proof. Use lemma 3.0.5 and Ricci identity:

$$-2\frac{\partial}{\partial t}R_{jk} = \Delta h_{jk} - \nabla_j(g^{pq}\nabla_q h_{pk} - \frac{1}{2}\nabla_k H) - \nabla_k(g^{pq}\nabla_q h_{pj} - \frac{1}{2}\nabla_j H)$$
$$+ g^{pq}R^l_{qjk}h_{lp} + g^{pq}R^l_{qjp}h_{kl} + g^{pq}R^l_{qkj}h_{lp} + g^{pq}R^l_{qkp}h_{jl}$$

Take $h = -2Rc$; apply second Bianchi identity:

$$-2\frac{\partial}{\partial t}R_{jk} = -2\Delta R_{jk} - 4g^{pq}g^{rs}R_{qjks}R_{rp} + 4g^{pq}R_{jp}R_{kq}$$

Divide -2 on both sides, we get the formula. $\square$

Theorem 5.1.5.

$$\frac{\partial}{\partial t}R = \Delta R + 2g^{ij}g^{kl}R_{ik}R_{jl} = \Delta R + 2|Rc|^2$$

Proof. Apply lemma 3.0.6:

$$\frac{\partial}{\partial t}R = -\Delta H + \nabla^p\nabla^q h_{pq} - \langle h, Rc\rangle$$
$$= 2\Delta R - 2\nabla^p\nabla^q R_{pq} + 2|Rc|^2 = \Delta R + 2|Rc|^2$$

$\square$

Corollary 5.1.6. *If $R > 0$ at $t = 0$, then it remains so whenever $t > 0$.*

Proof. Suppose the Ricci flow has solution at $t \in [0, T)$. Notice that

$$\frac{\partial}{\partial t}R - \Delta R = 2|Rc|^2 > 0$$

and M has no boundary, so

$$\min_{M\times[0,T]} R = \min_{M\times\{0\}} R > 0$$

$\square$

The Weyl conformal curvature tensor W on n-dimensional ($n \geq 3$) manifold M is defined as:

$$\begin{aligned} W_{ijkl} = R_{ijkl} - \frac{1}{n-2}(R_{il}g_{jk} + R_{jk}g_{il} - R_{ik}g_{jl} - R_{jl}g_{ik}) \\ + \frac{1}{(n-1)(n-2)}R(g_{il}g_{jk} - g_{ik}g_{jl}) \end{aligned}$$

Lemma 5.1.7. *W is a trace-free tensor with many symmetries.*

(1) $W_{ijkl} = -W_{jikl} = -W_{ijlk} = W_{jilk} = W_{klij}$

(2) $W_{ijkl} + W_{jkil} + W_{kijl} = 0$

(3) $g^{il}W_{ijkl} = 0$

This lemma's proof derives from direct calculation. Then we could show W vanishes when $n = 3$:

Use normal coordinate here. In dimension three, the index repeats at least once, so we can classify two situations as follows:

(1) W_{iijk} or W_{ijkk}: Lemma 5.1.7(1) states that these components vanish.

(2) W_{ijki} with $i \neq j$, $i \neq k$, $j \neq k$. In dimension 3, {i,j,k} transverse all index, so lemma 5.1.7(3) states that:

$$0 = W_{ijki} + W_{jjkj} + W_{kjkk} = W_{ijki}$$

Hence we have:

Theorem 5.1.8. *When M is of dimension 3,*

$$R_{ijkl} = R_{il}g_{jk} + R_{jk}g_{il} - R_{ik}g_{jl} - R_{jl}g_{ik} - \frac{1}{2}R(g_{il}g_{jk} - g_{ik}g_{jl})$$

Because g and Rc are both symmetric real matrix at every point $p \in M$, we can take normal coordinate at first, then diagonalize Rc at p. After two coordinate transformations, g is in normal coordinate and Rc has been diagonal at p. Suppose that at point p

$$Rc = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \upsilon \end{pmatrix} \quad g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In dimension 3, the index of Rm repeats at least once, use symmetry of Rm, $R_{ijkl} \neq 0$ only if $i \neq j$ and $k \neq l$. Under this condition, it remains two cases:

$j = k$ and $i \neq l$:

$$R_{ijjl} = R_{il}g_{jj} + R_{jj}g_{il} - R_{ij}g_{jl} - R_{jl}g_{ij} - \frac{1}{2}R(g_{il}g_{jj} - g_{ij}g_{jl}) = 0$$

$j = k$ and $i = l$:

$$\begin{aligned} R_{ijji} &= R_{ii}g_{jj} + R_{jj}g_{ii} - R_{ij}g_{ji} - R_{ji}g_{ij} - \frac{1}{2}R(g_{ii}g_{jj} - g_{ij}g_{ji}) \\ &= R_{ii} + R_{jj} - \frac{1}{2}R \end{aligned}$$

Thus, we get

Corollary 5.1.9. *R_{ijkl} of the form R_{1221} is the only possible nonzero component where*

$$R_{1221} = \frac{1}{2}(\lambda + \mu - \upsilon).$$

Now define

$$S_{il} = R_{ij}g^{jk}R_{kl}, S = g^{il}S_{il}; T_{in} = R_{ij}g^{jk}R_{kl}g^{lm}R_{mn}, T = g^{in}T_{in}$$

According to theorem 5.1.4, we have

$$\begin{aligned} \frac{\partial}{\partial t}R_{jk} &= \Delta R_{jk} + 2g^{pi}g^{rl}R_{ijkl}R_{rp} - 2g^{pq}R_{jp}R_{kq} \\ &= \Delta R_{jk} + 2g^{pi}g^{rl}R_{rp}(R_{il}g_{jk} + R_{jk}g_{il} - R_{ik}g_{jl} - R_{jl}g_{ik}) \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}R(g_{il}g_{jk}-g_{ik}g_{jl})-2S_{jk}\\
&=\Delta R_{jk}+3RR_{jk}-4S_{jk}+(2S-R^2)g_{jk}-2S_{jk}\\
&=\Delta R_{jk}-(6S_{jk}-3RR_{jk}+(R^2-2S)g_{jk})
\end{aligned}
$$

so we simplify the evolution equation of Rc as follows:

Theorem 5.1.10. *When* $\dim M = 3$*,*

$$\frac{\partial}{\partial t}R_{jk}=\Delta R_{jk}-Q_{ij}$$

where $Q_{ij}=6S_{jk}-3RR_{jk}+(R^2-2S)g_{jk}$

Given the local coordinate

$$R_{ij}=\begin{pmatrix}\lambda&0&0\\0&\mu&0\\0&0&\upsilon\end{pmatrix}\quad S_{ij}=\begin{pmatrix}\lambda^2&0&0\\0&\mu^2&0\\0&0&\upsilon^2\end{pmatrix}\quad T_{ij}=\begin{pmatrix}\lambda^3&0&0\\0&\mu^3&0\\0&0&\upsilon^3\end{pmatrix}$$

$$R=\lambda+\mu+\upsilon,\qquad S=\lambda^2+\mu^2+\upsilon^2,\qquad T=\lambda^3+\mu^3+\upsilon^3$$

Then

$$Q_{ij}=\begin{pmatrix}Q_{11}&0&0\\0&Q_{22}&0\\0&0&Q_{33}\end{pmatrix}$$

where

$$
\begin{aligned}
Q_{11}&=2\lambda^2-\mu^2-\upsilon^2-\lambda\mu-\lambda\upsilon+2\mu\upsilon\\
Q_{22}&=2\mu^2-\lambda^2-\upsilon^2-\mu\lambda-\mu\upsilon+2\lambda\upsilon\\
Q_{33}&=2\upsilon^2-\lambda^2-\mu^2-\upsilon\lambda-\upsilon\mu+2\lambda\mu
\end{aligned}
$$

Theorem 5.1.11. *Let* T *be the maximum existence interval of the Ricci flow. If* M *is a Riemannian 3-manifold and* $R\geq\rho>0$ *at* $t=0$*, then* $T\leq 3/2\rho$*.*

Proof. Because $|Rc|^2 - \frac{1}{3}R^2 = 1/3((\lambda-\mu)^2 + (\lambda-\upsilon)^2 + (\mu-\upsilon)^2) \geq 0$, by Theorem 5.1.5, we know $\frac{\partial R}{\partial t} \geq \Delta R + \frac{2}{3}R^2$. Now consider $f = f(t)$

$$\frac{df}{dt} = \frac{2}{3}f^2 \quad \text{with } f = \rho \text{ at } t = 0,$$

$$\frac{\partial}{\partial t}(R-f) \geq \Delta(R-f) + \frac{2}{3}(R+f)(R-f)$$

with $R - f \geq 0$ at $t = 0$. The maximum principle tells that $R - f \geq 0$ at $[0, T)$. Meanwhile, solve the ordinary differential equation of f, we get

$$f = \frac{3\rho}{3 - 2\rho t}.$$

Because $f \to \infty$ as $t \to 3/2\rho$, we know $T \leq 3/2\rho$. □

5.2 Preserving Positive Ricci Curvature

In this subsection, a maximum principle to tensor would be proven at first. With this principle, some estimate about curvature would be given. In this subsection, A_{ij}, B_{ij} are symmetric tensors on M; we call a tensor $A_{ij} \geq 0$ if $A_{ij}v^iv^j \geq 0$ for all vectors v^i; u^k is a vector field in M. $B = p(A, g)$ is a polynomial in A_{ij}, with coefficient $\Gamma(M)$, formed by contracting products of A_{ij} with itself using the metric g. Moreover, the polynomial satisfies **null-eigenvector condition**: whenever v^i is a null-eigenvector of A_{ij} (i.e. $A_{ij}v^i = 0$, $\forall j$), we have $B_{ij}v^iv^j \geq 0$. Here, A_{ij}, B_{ij}, u^k, g_{ij} may all depend on time t.

Theorem 5.2.1. *Let M^n be a closed manifold. Suppose the following equation*

$$\frac{\partial}{\partial t}A_{ij} = \Delta A_{ij} + u^k \nabla_k A_{ij} + B_{ij}$$

has solution when $0 \leq t \leq T$. At $t \in [0, T]$, $B_{ij} = p(A_{ij}, g_{ij})$ satisfies the null-eigenvector condition. Then if $A_{ij} \geq 0$ at $t = 0$, then it remains so on $0 \leq t \leq T$.

Proof. It is going to show there exists $\delta > 0$ such that $A_{ij} \geq 0$ on $0 \leq t \leq \delta$, where δ is a constant depending on $\max_{M\times[0,T]}|A_{ij}|$, $\max_{M\times[0,T]}|\frac{\partial g}{\partial t}|$. The theorem follows because we can cover $[0,T]$ in finite steps. Let δ chosen later. For every $\epsilon > 0$, define a new (0,2) tensor $A(\epsilon)$:

$$A(\epsilon)_{ij}(x,t) = A_{ij}(x,t) + \epsilon(\delta + t)g_{ij}.$$

It suffices to show that there exists a constant $\delta > 0$ such that $A(\epsilon)_{ij} > 0$ on $0 \leq t \leq \delta$ for any $\epsilon > 0$. Then $A_{ij} \geq 0$ follows as $\epsilon \to 0$. If there does not exist such δ, then $\forall \delta > 0$, $\exists$ some small $\epsilon > 0$ such that at a first time θ with $0 < \theta \leq \delta$ where $A(\epsilon)_{ij}$ acquires a null-eigenvector v^i of unit length under the metric $g_{ij}(\theta)$ at some point $x_0 \in M$. If $B(\epsilon)_{ij} = p(A(\epsilon)_{ij}, g_{ij})$ then $B(\epsilon)_{ij} \geq 0$ at (x_0, θ). Moreover,

$$|B(\epsilon)_{ij} - B_{ij}| = |p(A(\epsilon)_{ij}, g_{ij}) - p(A_{ij}, g_{ij})| \leq C|A(\epsilon)_{ij} - A_{ij}|$$

where C is a constant depending only on $\max_{M\times[0,T]}(|A(\epsilon)_{ij}| + |A_{ij}|)$. If we keep $\epsilon, \delta \leq 1$, then $\max_{M\times[0,T]}|A(\epsilon)_{ij}|$ depends only on $\max_{M\times[0,T]}|A_{ij}|$. Therefore,

$$B_{ij}v^iv^j(x_0,\theta) \geq (B(\epsilon)_{ij} - C|A(\epsilon)_{ij} - A_{ij}|)v^iv^j \geq -C\epsilon(\delta+\theta)|v|^2 \geq -C\epsilon\delta$$

where C depends on $\max_{M\times[0,T]}|M_{ij}|$.

We can parallel translate v^i w.r.t $g_{ij}(\theta)$ to get a vector field in a neighbourhood of x such that $\nabla_j v^i(x) = 0$ with v^i independent of t. Let $f_\epsilon(x,t) = A(\epsilon)_{ij}v^iv^j$. We have

$$\frac{\partial f_\epsilon}{\partial t} = (\frac{\partial A(\epsilon)_{ij}}{\partial t})v^iv^j = (\frac{\partial A_{ij}}{\partial t})v^iv^j + \left\{\epsilon g_{ij} + \epsilon(\delta+t)(\frac{\partial}{\partial t}g_{ij})\right\}v^iv^j$$

$$\nabla_k f_\epsilon = (\nabla_k A(\epsilon)_{ij})v^iv^j = (\nabla_k A_{ij})v^iv^j$$

$$\Delta f_\epsilon = \Delta A(\epsilon)_{ij}v^iv^j = \Delta A_{ij}v^iv^j$$

The evolution equation tells that

$$\left(\frac{\partial}{\partial t}A_{ij}\right)v^iv^j = (\Delta A_{ij})v^iv^j + (u^k\nabla_k A_{ij})v^iv^j + B_{ij}v^iv^j$$
$$\frac{\partial f_\epsilon}{\partial t} - \left\{\epsilon g_{ij} + \epsilon(\delta+t)(\frac{\partial}{\partial t}g_{ij})\right\}v^iv^j = \Delta f_\epsilon + u^k\nabla_k f_\epsilon + B_{ij}v^iv^j$$

Specifically,

$$f_\epsilon \geq 0 \text{ on } 0 \leq t \leq \theta, \forall x \in M$$
$$\frac{\partial f_\epsilon}{\partial t}(x_0,\theta) \leq 0; \quad f_\epsilon(x_0,\theta) = 0;$$
$$\nabla_k f_\epsilon(x_0,\theta) = 0; \quad \Delta f_\epsilon(x_0,\theta) \geq 0.$$

Hence

$$B_{ij}v^iv^j(x_0,\theta) \leq -\left\{\epsilon g_{ij} + \epsilon(\delta+\theta)(\frac{\partial}{\partial t}g_{ij})\right\}v^iv^j$$
$$C\epsilon\delta \geq \left\{\epsilon g_{ij}(\theta) + \epsilon(\delta+\theta)(\frac{\partial}{\partial t}g_{ij}(\theta))\right\}v^iv^j$$

This requires

$$\delta \geq \frac{1+\theta|\frac{\partial}{\partial t}g_{ij}(\theta)|}{C+|\frac{\partial}{\partial t}g_{ij}(\theta)|} \geq \frac{1}{C+\max_{M\times[0,T]}|\frac{\partial}{\partial t}g_{ij}|} =: d$$

If we take $\delta = \frac{1}{2}\min\{d,1\}$, then $A(\epsilon)_{ij} > 0$ at $[0,\delta]$ for any $\epsilon > 0$ where δ is independent of ϵ, a contradiction. The proof is done. □

Remark 5.2.2. *Suppose* $[0,T_0)$ *is the maximum existence interval, then above conclusion keeps at* $[0,T_0)$. *It is because we can use the theorem in every closed subinterval of* $[0,T_0)$.

It is known that the Ricci flow has a short time solution on $[0,T)$.

Corollary 5.2.3. *Let* M^3 *be a closed manifold. If* $R_{ij} \geq 0$ *at* $t = 0$ *then* $R_{ij} \geq 0$ *on* $[0,T)$.

Proof. Apply Theorem 5.1.10 and Theorem 5.2.1, let $A_{ij} = R_{ij}$, $B_{ij} = -Q_{ij}$, $u^k = 0$. Now check B_{ij} satisfies null-eigenvector condition: if $v^i \neq 0$ such that

$$R_{ij}v^i = \begin{pmatrix} \lambda & & \\ & \mu & \\ & & \upsilon \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} = \begin{pmatrix} \lambda v^1 \\ \mu v^2 \\ \upsilon v^3 \end{pmatrix} = 0$$

If $v^1 = 0$ then $B_{11}v^1v^1 = 0$; if $v^1 \neq 0$ then $\lambda = 0$ and $B_{11} = \mu^2 + \upsilon^2 - 2\mu\upsilon \geq 0$ then $B_{11}v^1v^1 \geq 0$; so $B_{ij}v^iv^j = B_{11}v^1v^1 + B_{22}v^2v^2 + B_{33}v^3v^3 \geq 0$. ☐

Lemma 5.2.4. *Let M^3 be a closed 3 manifold. If $R(t) \neq 0$ on $[0, T)$, then*

$$\frac{\partial}{\partial t}\left(\frac{R_{ij}}{R}\right) = \Delta\left(\frac{R_{ij}}{R}\right) + \frac{2}{R}g^{pq}\nabla_p R \nabla_q \left(\frac{R_{ij}}{R}\right) - \frac{RQ_{ij} + 2SR_{ij}}{R^2}$$

Proof.

$$\frac{\partial}{\partial t}R_{ij} = \Delta R_{ij} - Q_{ij}; \quad \frac{\partial}{\partial t}R = \Delta R + 2S;$$

$$\nabla_l\left(\frac{R_{ij}}{R}\right) = \frac{1}{R^2}\left(\nabla_l R_{ij} \cdot R - R_{ij}\nabla_l R\right);$$

$$\frac{\partial}{\partial t}\left(\frac{R_{ij}}{R}\right) = \frac{1}{R^2}\left(\frac{\partial}{\partial t}R_{ij} \cdot R - R_{ij}\frac{\partial R}{\partial t}\right) = \frac{\Delta R_{ij}}{R} - \frac{R_{ij}\Delta R}{R^2} - \frac{RQ_{ij} + 2SR_{ij}}{R^2};$$

$$\begin{aligned}
\Delta\left(\frac{R_{ij}}{R}\right) &= g^{kl}\nabla_l\nabla_k\left(\frac{R_{ij}}{R}\right) = g^{kl}\nabla_l\left\{\frac{1}{R}\nabla_k R_{ij} - \frac{1}{R^2}R_{ij}\nabla_k R\right\} \\
&= g^{kl}\left(-\frac{1}{R^2}\nabla_l R \nabla_k R_{ij} + \frac{1}{R}\nabla_l\nabla_k R_{ij}\right) \\
&\quad - g^{kl}\left\{\frac{1}{R^4}[(\nabla_l R_{ij}\nabla_k R + R_{ij}\nabla_l\nabla_k R)R^2] - [\frac{1}{R^4}R_{ij}\nabla_k R \cdot 2R\nabla_l R]\right\} \\
&= \frac{\Delta R_{ij}}{R} - \frac{2}{R^2}g^{kl}\nabla_l R\nabla_k R_{ij} - \frac{1}{R^2}R_{ij}\Delta R + \frac{2}{R^3}R_{ij}g^{kl}\nabla_k R\nabla_l R \\
&= \frac{\Delta R_{ij}}{R} - \frac{R_{ij}\Delta R}{R^2} - \frac{2}{R}g^{kl}\nabla_k R\nabla_l\left(\frac{R_{ij}}{R}\right)
\end{aligned}$$

Then we have

$$\frac{\partial}{\partial t}\left(\frac{R_{ij}}{R}\right) = \Delta\left(\frac{R_{ij}}{R}\right) + \frac{2}{R}g^{pq}\nabla_p R \nabla_q \left(\frac{R_{ij}}{R}\right) - \frac{RQ_{ij} + 2SR_{ij}}{R^2}$$ ☐

Theorem 5.2.5. *Let M^3 be a closed Riemannian manifold with initial strictly positive Ricci curvature, then $R > 0$ and $R_{ij} \geq \epsilon R g_{ij}$ for some constant $0 < \epsilon \leq \frac{1}{3}$ at $t = 0$. Moreover, under the variation of Ricci flows, both conditions continues to hold on $[0, T)$.*

Proof. $R \geq 0$ follows from taking trace of Rc. $R_{ij} \geq \epsilon R g_{ij}$ follows from the compactness of M^3; we easily know $\epsilon \leq \frac{1}{3}$ by taking trace on both sides again. That $R > 0$ remains at $t \in [0, T)$ has been shown at corollary 5.1.6. To show $R_{ij} \geq \epsilon R g_{ij}$, we make

$$A_{ij} = \frac{R_{ij}}{R} - \epsilon g_{ij}, \quad u^k = \frac{2}{R} g^{kl} \nabla_l R$$

$$B_{ij} = 2\epsilon R_{ij} - \left(\frac{RQ_{ij} + 2SR_{ij}}{R^2} \right)$$

One could check that A_{ij}, B_{ij}, u^k satisfy the evolution equation in theorem 5.2.1. When $A_{ij} v^i = \left(\frac{R_{ij}}{R} - \epsilon g_{ij} \right) v^i = 0$, WLOG assume $v^1 \neq 0$, we have

$$\lambda = \epsilon(\lambda + \mu + \upsilon) \quad \Longrightarrow \quad \mu + \upsilon = (\frac{1}{\epsilon} - 1)\lambda \geq 2\lambda$$

$$\begin{aligned}
R^2 B_{11} &= 2\epsilon R^2 R_{11} - RQ_{11} - 2SR_{11} \\
&= 2\epsilon(\lambda + \mu + \upsilon)^2 \lambda - (\lambda + \mu + \upsilon)(2\lambda^2 - \mu^2 - \upsilon^2 - \lambda\mu - \lambda\upsilon + 2\mu\upsilon) - 2(\lambda^2 + \mu^2 + \upsilon^2)\lambda \\
&= 2\lambda^2(\lambda + \mu + \upsilon) - (\lambda + \mu + \upsilon)(2\lambda^2 - \mu^2 - \upsilon^2 - \lambda\mu - \lambda\upsilon + 2\mu\upsilon) - 2(\lambda^2 + \mu^2 + \upsilon^2)\lambda \\
&= (\lambda + \mu + \upsilon)(\lambda(\mu + \upsilon) + (\mu - \upsilon)^2) - 2(\lambda^2 + \mu^2 + \upsilon^2)\lambda \\
&= \lambda^2(\mu + \upsilon) + \lambda(\mu - \upsilon)^2 + \lambda(\mu + \upsilon)^2 + (\mu + \upsilon)(\mu - \upsilon)^2 - 2\lambda^3 - 2\lambda(\mu^2 + \upsilon^2) \\
&= \lambda^2(\mu + \upsilon - 2\lambda) + (\mu + \upsilon)(\mu - \upsilon)^2 \geq 0
\end{aligned}$$

Thus, B_{ij} satisfies null-eigenvector condition. The theorem follows by theorem 5.2.1. $\square$

Lemma 5.2.6. *If M^n is a Riemannian manifold with $R_{ij} \geq 0$, we have $R_{ij} \leq R g_{ij}$.*

5.3 Pinching the eigenvalues

In this subsection, we shall prove the following theorem

Theorem 5.3.1. *Let M^3 be a closed 3-manifold, with strictly positive Ricci curvature. Under the variation of Ricci flow, $\exists$ constant $\delta > 0$ and $\mathscr{C} \in \mathbb{R}_+$ both depending only on the initial metric such that on $0 \leq t < T$ we have*

$$S - \frac{1}{3}R^2 \leq \mathscr{C} R^{2-\delta}.$$

Here $S - \frac{1}{3}R^2$ is the l^2 the distance of three eigenvalues:

$$S - \frac{1}{3}R^2 = \frac{1}{3}[(\lambda - \mu)^2 + (\mu - \upsilon)^2 + (\lambda - \upsilon)^2]$$

The proof of theorem follows from maximum principle in partial differential equation. Let $\gamma = 2 - \delta$ and

$$f = S/R^\gamma - \frac{1}{3}R^{2-\gamma}.$$

It needs to find the relations in

$$\frac{\partial f}{\partial t} \sim \Delta f + u^k \nabla_k f + c(x) f \quad \text{for some vector } u^k \text{ and function } c \in \Gamma(M).$$

i.e. find the variation equation of S/R^γ, $R^{2-\gamma}$ respectively.

Lemma 5.3.2. *For any constant $1 < \gamma \leq 2$*

$$\begin{aligned} \frac{\partial}{\partial t} R^{2-\gamma} = \Delta R^{2-\gamma} &+ \frac{2(\gamma-1)}{R} g^{pq} \nabla_p R \nabla_q (R^{2-\gamma}) \\ &- \frac{(2-\gamma)(\gamma-1)}{R^{\gamma+2}} R^2 |\nabla_i R|^2 + 2(2-\gamma) R^{1-\gamma} S \end{aligned}$$

Proof.

$$\begin{aligned} \frac{\partial}{\partial t} R^{2-\gamma} &= (2-\gamma) R^{1-\gamma} \frac{\partial R}{\partial t} = (2-\gamma) R^{1-\gamma} (\Delta R + 2S) \\ \Delta R^{2-\gamma} &= g^{ij} \nabla_i \nabla_j R^{2-\gamma} = g^{ij} \nabla_i ((2-\gamma) R^{1-\gamma} \nabla_j R) \\ &= (2-\gamma) g^{ij} \{(1-\gamma) R^{-\gamma} \nabla_i R \nabla_j R + R^{1-\gamma} \nabla_i \nabla_j R\} \\ &= (2-\gamma)(1-\gamma) R^{-\gamma} |\nabla_i R|^2 + (2-\gamma) R^{1-\gamma} \Delta R \\ \frac{\partial}{\partial t} R^{2-\gamma} &= \Delta R^{2-\gamma} + (2-\gamma)(\gamma-1) R^{-\gamma} |\nabla_i R|^2 + (2-\gamma) R^{1-\gamma} \cdot 2S \end{aligned}$$

Notice that

$$\begin{aligned}\frac{2(\gamma-1)}{R}g^{pq}\nabla_p R\nabla_q(R^{2-\gamma}) &= \frac{2(\gamma-1)}{R}g^{pq}\nabla_p R\cdot(2-\gamma)R^{1-\gamma}\nabla_q R\\ &= 2(2-\gamma)(\gamma-1)R^{-\gamma}|\nabla_i R|^2\end{aligned}$$

The lemma follows. □

Lemma 5.3.3.

$$\frac{\partial}{\partial t}S = \Delta S - 2|\nabla_i R_{jk}|^2 + 4(T-C),$$

where

$$\begin{aligned}C &= \frac{1}{2}g^{ik}g^{jl}Q_{ij}R_{kl} = \frac{1}{2}(R^3 - 5RS + 6T)\\ &= (\lambda^3+\mu^3+\upsilon^3) - (\lambda\mu^2+\lambda\upsilon^2+\mu\lambda^2+\mu\upsilon^2+\upsilon\lambda^2+\upsilon\mu^2) + 3\lambda\mu\upsilon\end{aligned}$$

Proof.

$$\begin{aligned}\frac{\partial}{\partial t}S &= \frac{\partial}{\partial t}g^{il}g^{jk}R_{ik}R_{jl} = 4g^{im}g^{lm}R_{mn}g^{jk}R_{ik}R_{jl} + 2g^{il}g^{jk}(\Delta R_{ik} - Q_{ik})R_{jl}\\ &= 4T + 2g^{il}g^{jk}\Delta R_{ik}\cdot R_{jl} - 4C\\ \Delta S &= g^{ij}\nabla_i\nabla_j(g^{tl}g^{sm}R_{ts}R_{lm}) = g^{ij}g^{tl}g^{sm}\nabla_i[(\nabla_j R_{ts})R_{lm} + R_{ts}(\nabla_j R_{lm})]\\ &= g^{ij}g^{tl}g^{sm}\{\nabla_i\nabla_j R_{ts}\cdot R_{lm} + \nabla_j R_{ts}\nabla_i R_{lm} + \nabla_i R_{ts}\nabla_j R_{tm} + R_{ts}\nabla_i\nabla_j R_{lm}\}\\ &= 2g^{ik}g^{jl}\Delta R_{ij}\cdot R_{kl} + 2|\nabla_i R_{jk}|^2.\end{aligned}$$

Thus, we get the desired result. □

Lemma 5.3.4. *For any constant* $1 < \gamma \le 2$

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{S}{R^\gamma}\right) = \Delta\left(\frac{S}{R^\gamma}\right) &+ \frac{2(\gamma-1)}{R}g^{pq}\nabla_p R\nabla_q\left(\frac{S}{R^\gamma}\right)\\ &- \frac{2}{R^{\gamma+2}}|R\nabla_i R_{jk} - \nabla_i R\cdot R_{jk}|^2\\ &- \frac{(2-\gamma)(\gamma-1)}{R^{\gamma+2}}S|\nabla_i R|^2 + \frac{4R(T-C)-2\gamma S^2}{R^{\gamma+1}}\end{aligned}$$

Proof.

$$
\begin{aligned}
\nabla_j\left(\frac{S}{R^\gamma}\right) &= \frac{1}{R^{\gamma+1}}\left(\nabla_j S\cdot R-\gamma S\nabla_j R\right)\\
\Delta\left(\frac{S}{R^\gamma}\right) &= g^{ij}\nabla_i\nabla_j\frac{S}{R^\gamma}=g^{ij}\nabla_i\left[\frac{1}{R^{\gamma+1}}\left(\nabla_j S\cdot R-\gamma S\nabla_j R\right)\right]\\
&= g^{ij}\frac{1}{R^{\gamma+1}}(\nabla_i R\nabla_j S-\gamma\nabla_i S\nabla_j R-\gamma S\nabla_i\nabla_j R-(\gamma+1)\nabla_j S\nabla_i R)\\
&\quad + g^{ij}\frac{1}{R^\gamma}\nabla_i\nabla_j S+g^{ij}\frac{1}{R^{\gamma+2}}\gamma(\gamma+1)S\nabla_j R\nabla_i R\\
&= \frac{1}{R^{\gamma+1}}\langle\nabla_i R,\nabla_i S\rangle-\frac{\gamma}{R^{\gamma+1}}\langle\nabla_i R,\nabla_i S\rangle-\frac{\gamma S}{R^{\gamma+1}}\Delta R\\
&\quad -\frac{\gamma+1}{R^{\gamma+1}}\langle\nabla_i R,\nabla_i S\rangle+\frac{1}{R^\gamma}\Delta S+\frac{\gamma(\gamma+1)S}{R^{\gamma+2}}|\nabla_i R|^2\\
&= \frac{R\Delta S-\gamma S\Delta R}{R^{\gamma+1}}+\frac{\gamma(\gamma+1)S|\nabla_i R|^2}{R^{\gamma+2}}-\frac{2\gamma}{R^{\gamma+1}}\langle\nabla_i R,\nabla_i S\rangle\\
\frac{\partial}{\partial t}\left(\frac{S}{R^\gamma}\right) &= \frac{1}{R^{\gamma+1}}\left(\frac{\partial S}{\partial t}R-\gamma S\frac{\partial R}{\partial t}\right)\\
&= \frac{1}{R^{\gamma+1}}\left\{R(\Delta S-2|\nabla_i R_{jk}|^2+4(T-C))-\gamma S(\Delta R+2S)\right\}\\
&= \frac{R\Delta S-\gamma S\Delta R}{R^{\gamma+1}}+\frac{1}{R^{\gamma+1}}\left\{R(-2|\nabla_i R_{jk}|^2+4(T-C))-2\gamma S^2\right\}\\
&= \Delta\left(\frac{S}{R^\gamma}\right)+\frac{2\gamma}{R^{\gamma+1}}\langle\nabla_i R,\nabla_i S\rangle-\frac{\gamma(\gamma+1)S|\nabla_i R|^2}{R^{\gamma+2}}\\
&\quad +\frac{1}{R^{\gamma+1}}\left\{R(-2|\nabla_i R_{jk}|^2+4(T-C))-2\gamma S^2\right\}\\
&= \Delta\left(\frac{S}{R^\gamma}\right)+\frac{1}{R^{\gamma+1}}(4R(T-C)-2\gamma S^2)+\frac{1}{R^{\gamma+2}}V
\end{aligned}
$$

where $V := 2\gamma R\langle\nabla_i R,\nabla_i S\rangle-\gamma(\gamma+1)S|\nabla_i R|^2-2R^2|\nabla_i R_{jk}|^2$.

Notice that

$$
\begin{aligned}
\langle\nabla_i R,\nabla_i S\rangle &= g^{ij}\nabla_i R\nabla_j S=g^{ij}\nabla_i R\nabla_j(g^{mn}g^{hk}R_{hm}R_{kn})\\
&= 2g^{ij}g^{mn}g^{hk}\nabla_i R\cdot R_{hm}\nabla_j R_{kn}=2\langle\nabla_i R_{jk},\nabla_i R\cdot R_{jk}\rangle
\end{aligned}
$$

$$
\langle\nabla_i R,\nabla_i\left(\frac{S}{R^\gamma}\right)\rangle=\frac{1}{R^\gamma}\langle\nabla_i R,\nabla_i S\rangle-\frac{\gamma}{R^{\gamma+1}}S|\nabla_i R|^2
$$

$$S|\nabla_i R|^2 = g^{kl} g^{mn} R_{km} R_{ln} g^{ij} \nabla_i R \nabla_j R = |\nabla_i R \cdot R_{jk}|^2$$

$$-\gamma(\gamma+1) = -2(\gamma-1)\gamma - (2-\gamma)(\gamma-1) - 2$$

so

$$\begin{aligned}
&R\langle\nabla_i R, \nabla_i S\rangle - S|\nabla_i R|^2 - R^2|\nabla_i R_{jk}|^2 \\
&= 2\langle R\nabla_i R_{jk}, \nabla_i R \cdot R_{jk}\rangle - \langle\nabla_i R \cdot R_{jk}, \nabla_i R \cdot R_{jk}\rangle - \langle R\nabla_i R_{jk}, R\nabla_i R_{jk}\rangle \\
&= -|R\nabla_i R_{jk} - \nabla_i R \cdot R_{jk}|^2
\end{aligned}$$

$$\begin{aligned}
V &= 2(\gamma-1)R\langle\nabla_i R, \nabla_i S\rangle + 2R\langle\nabla_i R, \nabla_i S\rangle \\
&\quad - (2(\gamma-1)\gamma + (2-\gamma)(\gamma-1) + 2)S|\nabla_i R|^2 - 2R^2|\nabla_i R_{jk}|^2 \\
&= 2(\gamma-1)R(\langle\nabla_i R, \nabla_i S\rangle - \gamma S|\nabla_i R|^2) - (2-\gamma)(\gamma-1)S|\nabla_i R|^2 \\
&\quad + 2R\langle\nabla_i R, \nabla_i S\rangle - 2S|\nabla_i R|^2 - 2R^2|\nabla_i R_{jk}|^2 \\
&= 2(\gamma-1)R^{\gamma+1}\langle\nabla_i R, \nabla_i\left(\frac{S}{R^\gamma}\right)\rangle - (2-\gamma)(\gamma-1)S|\nabla_i R|^2 - 2|R\nabla_i R_{jk} - \nabla_i R \cdot R_{jk}|^2
\end{aligned}$$

Substitute V into previous expression, the lemma follows directly. □

Lemma 5.3.5. *For* $f = S/R^\gamma - \frac{1}{3}R^{2-\gamma}$, $1 < \gamma \leq 2$

$$\begin{aligned}
\frac{\partial f}{\partial t} &= \Delta f + \frac{2(\gamma-1)}{R} g^{pq}\nabla_p R \nabla_q f - \frac{2}{R^{\gamma+2}}|R\nabla_i R_{jk} - \nabla_i R \cdot R_{jk}|^2 \\
&\quad - \frac{(2-\gamma)(\gamma-1)}{R^{\gamma+2}}(S - \frac{1}{3}R^2)|\nabla_i R|^2 \\
&\quad + \frac{2}{R^{\gamma+1}}[(2-\gamma)S(S - \frac{1}{3}R^2) - 2P]
\end{aligned}$$

where $P = S^2 + R(C - T)$

Proof. It follows from Lemma 5.3.2 and Lemma 5.3.4. □

Lemma 5.3.6.

$$P = \lambda^2(\lambda-\mu)(\lambda-\upsilon) + \mu^2(\mu-\lambda)(\mu-\upsilon) + \upsilon^2(\upsilon-\lambda)(\upsilon-\mu)$$

Proof.

$$\begin{aligned} P = S^2 + R(C-T) &= (\lambda^2+\mu^2+\upsilon^2)^2 \\ &\quad + (\lambda+\mu+\upsilon)(-\lambda\mu^2-\lambda\upsilon^2-\mu\lambda^2-\mu\upsilon^2-\upsilon\lambda^2-\upsilon\mu^2+3\lambda\mu\upsilon) \\ &= \lambda^4+\mu^4+\upsilon^4-\mu\lambda^3-\upsilon\lambda^3-\lambda\mu^3-\upsilon\mu^3-\lambda\upsilon^3-\mu\upsilon^3+\lambda^2\mu\upsilon+\lambda\mu^2\upsilon+\lambda\mu\upsilon^2 \\ &= \lambda^2(\lambda^2-\mu\lambda-\upsilon\lambda+\mu\upsilon)+\mu^2(\mu^2-\lambda\mu-\upsilon\mu+\lambda\upsilon)+\upsilon^2(\upsilon^2-\lambda\upsilon-\mu\upsilon+\lambda\mu) \end{aligned}$$

The result follows. □

Lemma 5.3.7. *If $R > 0$ and $R_{ij} \geq \epsilon R g_{ij}$ then $P \geq \epsilon^2 S(S - \frac{1}{3}R^2)$.*

Proof. LHS and RHS are both homogeneous polynomials of degree 4. We may assume $S = \lambda^2+\mu^2+\upsilon^2 = 1$ here, then it remains to show $P \geq \epsilon^2(S - \frac{1}{3}R^2)$. Assume that $\lambda \geq \mu \geq \upsilon > 0$.

$$\begin{aligned} P &= (\lambda-\mu)(\lambda^2(\lambda-\upsilon)-\mu^2(\mu-\upsilon))+\upsilon^2(\upsilon-\lambda)(\upsilon-\mu) \\ &= (\lambda-\mu)(\lambda^2(\lambda-\mu)+\lambda^2(\mu-\upsilon)-\mu^2(\mu-\upsilon))+\upsilon^2(\upsilon-\lambda)(\upsilon-\mu) \\ &= (\lambda-\mu)(\lambda^2(\lambda-\mu)+(\lambda+\mu)(\lambda-\mu)(\mu-\upsilon))+\upsilon^2(\upsilon-\lambda)(\upsilon-\mu) \\ &= (\lambda-\mu)^2(\lambda^2+(\lambda+\mu)(\mu-\upsilon))+\upsilon^2(\upsilon-\lambda)(\upsilon-\mu) \\ &\geq (\lambda-\mu)^2\lambda^2+\upsilon^2(\mu-\upsilon)^2 \end{aligned}$$

Observe that

$$(\lambda+\mu+\upsilon)^2 \geq \lambda^2+\mu^2+\upsilon^2 = 1 \quad \Longrightarrow \quad \lambda+\mu+\upsilon \geq 1$$

By $R_{ij} \geq \epsilon R g_{ij}$, $\lambda \geq \upsilon \geq \epsilon(\lambda+\mu+\upsilon) \geq \epsilon$. Thus, $P \geq \epsilon^2((\lambda-\mu)^2+(\mu-\upsilon)^2)$.

On the other hand,

$$(\lambda-\upsilon)^2 = (\lambda-\mu+\mu-\upsilon)^2 \leq 2(\lambda-\mu)^2+2(\mu-\upsilon)^2$$

$$S - \frac{1}{3}R^2 = \frac{1}{3}((\lambda-\mu)^2+(\lambda-\upsilon)^2)+(\mu-\upsilon)^2) \leq (\lambda-\mu)^2+(\mu-\upsilon)^2$$

Hence, $P \geq \epsilon^2(S - \frac{1}{3}R^2)$. □

Lemma 5.3.8. *If $\delta > 0$ is chosen so small that $\delta \leq 2\epsilon^2$, then with $\gamma = 2-\delta$ and $f = S/R^\gamma - \frac{1}{3}R^{2-\gamma}$ we have*

$$\frac{\partial f}{\partial t} \leq \Delta f + u^k \nabla_k f.$$

where $u^k = \frac{2(\gamma-1)}{R} g^{kl} \nabla_l R$.

Proof. When $\delta \leq 2\epsilon^2$

$$(2-\gamma)S(S - \frac{1}{3}R^2) - 2P \leq (\delta - 2\epsilon^2)S(S - \frac{1}{3}R^2) \leq 0$$

Substitute it into Lemma 5.3.5, the conclusion follows. □

Now we could prove Theorem 5.3.1:

Proof. By Theorem 5.2.5, there exists a constant $\epsilon \geq 0$ such that $R_{ij} \geq \epsilon R g_{ij}$ for all $t \in [0, T)$. Let $\delta \leq 2\epsilon^2$, then Lemma 5.3.8 gives that

$$\frac{\partial f}{\partial t} \leq \Delta f + u^k \nabla_k f \quad \text{for} \quad f = S/R^{2-\delta} - \frac{1}{3}R^{2-\gamma}$$

Because M^3 is compact, $\exists \mathscr{C} < \infty$, which just depends on the initial metric $g(0)$ such that $f \leq \mathscr{C}$ at $t = 0$. Then maximum principle tells that $f \leq \mathscr{C}$ at $t \in [0, T)$, so $S - \frac{1}{3}R^2 \leq \mathscr{C} R^{2-\delta}$ as desired. □

5.4 The gradient of the scalar curvature

In this subsection, the upper bound of $|\nabla_i R|$ will be given as follows:

Theorem 5.4.1. *Let M^3 be a closed Riemannian 3 manifold with positive Ricci curvature. For every $\eta > 0$, $\exists$ constant $\mathcal{C} = \mathcal{C}(\eta, g(0))$ depending only on η and the initial value of the metric such that on $0 \leq t < T$ we have*

$$|\nabla_i R|^2 \leq \eta R^3 + \mathcal{C}$$

The proof would be given in several steps. What we shall do is to find the variation formula of $F = |\nabla_i R|^2/R - \eta R^2 + N(S - \frac{1}{3}R^2)$ with $N \in \mathbb{R}$.

Lemma 5.4.2.

$$\frac{\partial}{\partial t}|\nabla_i R|^2 = \Delta|\nabla_i R|^2 + 4g^{ij}\nabla_i S\nabla_j R - 2|\nabla_i\nabla_j R|^2$$

Proof. $|\nabla_i R|^2 = g^{ij}\nabla_i R\nabla_j R$

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla_i|^2 &= 2g^{ik}g^{jl}R_{kl}\nabla_i R\nabla_j R + 2g^{ij}\nabla_i(\Delta R + 2S)\cdot\nabla_j R\\ &= 2g^{ik}g^{jl}R_{kl}\nabla_i R\nabla_j R + 2g^{ij}(\nabla_i\Delta R)\nabla_j R + 4g^{ij}\nabla_i S\nabla_j R\\ \Delta|\nabla_i R|^2 &= g^{kl}\nabla_k\nabla_l(g^{ij}\nabla_i R\nabla_j R)\\ &= 2g^{kl}g^{ij}\nabla_k((\nabla_l\nabla_i R)\cdot\nabla_j R)\\ &= 2g^{kl}g^{ij}\nabla_k\nabla_l\nabla_i R\cdot\nabla_j R + 2g^{kl}g^{ij}(\nabla_l\nabla_i R)\cdot(\nabla_k\nabla_j R)\\ &= 2g^{ij}(\Delta\nabla_i R)\nabla_j R + 2|\nabla_i\nabla_j R|^2\\ \Delta\nabla_i R &= \nabla_i\Delta R + R_{ij}\nabla_j R = \nabla_i\Delta R + g^{jk}R_{ij}\nabla_k R\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla_i|^2 - \Delta|\nabla_i R|^2 &= 2g^{ik}g^{jl}R_{kl}\nabla_i R\nabla_j R\\ &\quad + 2g^{ij}(\nabla_i\Delta R - \Delta\nabla_i R)\nabla_j R + 4g^{ij}\nabla_i S\nabla_j R - 2|\nabla_i\nabla_j R|^2\\ &= 2g^{ik}g^{jl}R_{kl}\nabla_i R\nabla_j R\\ &\quad - 2g^{ij}g^{lk}R_{il}\nabla_k R\nabla_j R + 4g^{ij}\nabla_i S\nabla_j R - 2|\nabla_i\nabla_j R|^2\\ &= 4g^{ij}\nabla_i S\nabla_j R - 2|\nabla_i\nabla_j R|^2\end{aligned}$$

□

Lemma 5.4.3.

$$\frac{\partial}{\partial t}\left(\frac{|\nabla_i R|^2}{R}\right) = \Delta\left(\frac{|\nabla_i R|^2}{R}\right) - \frac{2S}{R^2}|\nabla_i R|^2 + \frac{4}{R}\langle\nabla_i R, \nabla_i S\rangle - \frac{2}{R^3}|R\nabla_i\nabla_j R - \nabla_i R\nabla_j R|^2$$

Proof.

$$\frac{\partial}{\partial t}\left(\frac{|\nabla_i R|^2}{R}\right) = -\frac{1}{R^2}(\Delta R + 2S)|\nabla_i R|^2 + \frac{1}{R}(\Delta|\nabla_i R|^2 + 4g^{ij}\nabla_i S\nabla_j R - 2|\nabla_i\nabla_j R|^2)$$

$$\begin{aligned}\Delta\left(\frac{|\nabla_i R|^2}{R}\right) &= g^{kl}\nabla_k\nabla_l\left(\frac{|\nabla_i R|^2}{R}\right) = g^{kl}\nabla_k\left\{-\frac{1}{R^2}\nabla_l R\cdot|\nabla_i R|^2 + \frac{1}{R}\nabla_l(|\nabla_i R|^2)\right\}\\ &= g^{kl}\frac{2}{R^3}\nabla_k R\nabla_l R|\nabla_i R|^2 - g^{kl}\frac{1}{R^2}\nabla_k\nabla_l R|\nabla_i R|^2 - g^{kl}\frac{1}{R^2}\nabla_l R\nabla_k|\nabla_i R|^2\\ &\quad - g^{kl}\frac{1}{R^2}\nabla_k R\nabla_l|\nabla_i R|^2 + g^{kl}\frac{1}{R}\nabla_k\nabla_l(|\nabla_i R|^2)\end{aligned}$$

$$\begin{aligned}&= \frac{2}{R^3}|\nabla_i R|^2|\nabla_j R|^2 - \frac{1}{R^2}\Delta R|\nabla_i R|^2 - \frac{1}{R^2}\langle\nabla_l R, \nabla_l|\nabla_i R|^2\rangle\\ &\quad - \frac{1}{R^2}\langle\nabla_l R, \nabla_l|\nabla_i R|^2\rangle + \frac{1}{R}\Delta|\nabla_i R|^2\end{aligned}$$

Observe that:

$$\nabla_l|\nabla_i R|^2 = \nabla_l(g^{ij}\nabla_i R\nabla_j R) = 2g^{ij}\nabla_l\nabla_i R\cdot\nabla_j R$$

$$\langle\nabla_l R, \nabla_l|\nabla_i R|^2\rangle = g^{kl}\nabla_k R\cdot 2g^{ij}\nabla_l\nabla_i R\cdot\nabla_j R = 2\langle\nabla_l\nabla_i R, \nabla_l R\cdot\nabla_i R\rangle$$

We have

$$\Delta\left(\frac{|\nabla_i R|^2}{R}\right) = \frac{2}{R^3}|\nabla_i R\cdot\nabla_j R|^2 - \frac{\Delta R}{R^2}|\nabla_i R|^2 - \frac{4}{R^2}\langle\nabla_l\nabla_i R, \nabla_l R\cdot\nabla_i R\rangle + \frac{1}{R}\Delta|\nabla_i R|^2$$

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{|\nabla_i R|^2}{R}\right) &= \Delta\left(\frac{|\nabla_i R|^2}{R}\right) - 2\frac{S}{R^2}|\nabla_i R|^2 + \frac{4\langle\nabla_i S, \nabla_i R\rangle}{R} - \frac{2|\Delta R|^2}{R}\\ &\quad - \frac{2}{R^3}|\nabla_i R\cdot\nabla_j R|^2 + \frac{4}{R^2}\langle\nabla_i\nabla_j R, \nabla_i R\cdot\nabla_j R\rangle\\ &= \Delta\left(\frac{|\nabla_i R|^2}{R}\right) - \frac{2S}{R^2}|\nabla_i R|^2 + \frac{4}{R}\langle\nabla_i S, \nabla_i R\rangle - \frac{2}{R^3}|R\nabla_i\nabla_j R - \nabla_i R\cdot\nabla_j R|^2\end{aligned}$$

□

Lemma 5.4.4.

$$\frac{\partial}{\partial t}R^2 = \Delta R^2 - 2|\nabla_i R|^2 + 4RS$$

Proof.

$$\frac{\partial}{\partial t}R^2 = 2R\frac{\partial R}{\partial t} = 2R(\Delta R + 2S) = 2R\Delta R + 4RS$$

$$\begin{aligned}\Delta R^2 &= g^{ij}\nabla_i\nabla_j R^2 = 2g^{ij}\nabla_i(R\nabla_j R) = 2g^{ij}\nabla_i R\nabla_j R + 2g^{ij}R\nabla_i\nabla_j R \\ &= 2|\nabla_i R|^2 + 2R\Delta R\end{aligned}$$

The lemma follows directly. □

Lemma 5.4.5. *Let $U = T - \frac{1}{3}RS - C$, then*

$$\frac{\partial}{\partial t}(S - \frac{1}{3}R^2) = \Delta(S - \frac{1}{3}R^2) - 2(|\nabla_i R_{jk}|^2 - \frac{1}{3}|\nabla_i R|^2) + 4U$$

Proof. It follows from Lemma 5.3.3 and Lemma 5.4.4. □

Lemma 5.4.6. $U \leq R(S - \frac{1}{3}R^2)$

Proof. Recall that $P = S^2 + R(C - T) \geq \epsilon^2 S(S - \frac{1}{3}R^2) \geq 0$ for some $\epsilon > 0$ by Lemma 5.3.7. Then

$$UR \leq P + UR = S(S - \frac{1}{3}R^2) \leq R^2(S - \frac{1}{3}R^2)$$

The lemma follows directly. □

By $\nabla_i R = g^{jk}\nabla_i R_{jk}$, in normal coordinate, $\nabla_i R = \nabla_i R_{11} + \nabla_i R_{22} + \nabla_i R_{33}$; for any i we have

$$(\nabla_i R)^2 = (\nabla_i R_{11} + \nabla_i R_{22} + \nabla_i R_{33})^2 \leq 3((\nabla_i R_{11})^2 + (\nabla_i R_{22})^2 + (\nabla_i R_{33})^2),$$

so $|\nabla_i R|^2 \leq 3|\nabla_i R_{jk}|^2$ in arbitrary Riemannian 3 manifold. This estimate could be optimized so we get:

Lemma 5.4.7. $|\nabla_i R|^2 \leq \frac{20}{7}|\nabla_i R_{jk}|^2$ *in any Riemannian 3 manifold M, .*

Proof. The second Bianchi identity tells that: $g^{ij}\nabla_i R_{jk} = \frac{1}{2}\nabla_k R$. Decompose $\nabla_i R_{jk} = E_{ijk} + F_{ijk}$ where

$$E_{ijk} = \frac{1}{20}(g_{ij}\nabla_k R + g_{ik}\nabla_j R) + \frac{3}{10}g_{jk}\nabla_i R$$

It shows that $|E_{ijk}|^2 = \frac{7}{20}|\nabla_i R|^2$: let $g_{ij} = \delta_{ij}$, then

$$\begin{aligned}
&E_{ijk} = 0 \quad \text{if } i \neq j,\ i \neq k \text{ and } j \neq k;\\
&E_{iji} = \frac{1}{20}\nabla_j R \quad \text{if } i \neq j; \qquad E_{iik} = \frac{1}{20}\nabla_k R \quad \text{if } i \neq k;\\
&E_{ijj} = \frac{3}{10}\nabla_j R \quad \text{if } i \neq j; \qquad E_{iii} = \frac{2}{5}\nabla_i R
\end{aligned}$$

$$\begin{aligned}
|E_{ijk}|^2 &= \sum_i\sum_j\sum_k E_{ijk}^2 = \sum_i\sum_j E_{iji}^2 + \sum_i\sum_j\sum_{k\neq i} E_{ijk}^2\\
&= \sum_i E_{iii}^2 + \sum_i\sum_{j\neq i} E_{iji}^2 + \sum_i\sum_{k\neq i} E_{iik}^2 + \sum_i\sum_{j\neq i}\sum_{k\neq i} E_{ijk}^2\\
&= \sum_i \frac{4}{25}(\nabla_i R)^2 + \sum_i\sum_{j\neq i}\frac{1}{400}(\nabla_j R)^2 + \sum_i\sum_{k\neq i}\frac{1}{400}(\nabla_k R)^2 + \sum_i\sum_{j\neq i}\frac{9}{100}(\nabla_j R)^2\\
&= \sum_i \frac{4}{25}(\nabla_i R)^2 + \sum_i \frac{1}{200}(\nabla_i R)^2 + \sum_i \frac{1}{200}(\nabla_i R)^2 + \sum_i \frac{9}{50}(\nabla_i R)^2\\
&= \frac{35}{100}\sum_i (\nabla_i R)^2 = \frac{7}{20}|\nabla_i R|^2
\end{aligned}$$

It shows that $\langle E_{ijk}, F_{ijk}\rangle = 0$:

$$\langle E_{ijk}, F_{ijk}\rangle = \langle E_{ijk}, \nabla_i R_{jk} - E_{ijk}\rangle = \langle E_{ijk}, \nabla_i R_{jk}\rangle - \frac{7}{20}|\nabla_i R|^2$$

$$\begin{aligned}
\langle E_{ijk}, \nabla_i R_{jk}\rangle &= g^{il}g^{jm}g^{kn}E_{ijk}\nabla_l R_{mn}\\
&= g^{il}g^{jm}g^{kn}\frac{1}{20}(g_{ij}\nabla_k R + g_{ik}\nabla_j R)\nabla_l R_{mn} + g^{il}g^{jm}g^{kn}\frac{3}{10}g_{jk}\nabla_i R\nabla_l R_{mn}\\
&= \frac{1}{20}g^{jm}g^{kn}\nabla_k R\nabla_j R_{mn} + \frac{1}{20}g^{jm}g^{kn}\nabla_k R_{mn}\nabla_j R + \frac{3}{10}g^{il}g^{kn}\nabla_i R\nabla_l R_{kn}\\
&= \frac{1}{20}g^{kn}\nabla_k R\nabla_n R + \frac{3}{10}g^{il}\nabla_i R\nabla_l R = \frac{7}{20}|\nabla_i R|^2
\end{aligned}$$

Thus,

$$|\nabla_i R_{jk}|^2 = |E_{ijk}|^2 + |F_{ijk}|^2 \geq |E_{ijk}|^2 = \frac{7}{20}|\nabla_i R|^2$$

The conclusion follows. □

With this estimate we have:

$$\begin{aligned}
&\frac{\partial}{\partial t}(S - \frac{1}{3}R^2) - \Delta(S - \frac{1}{3}R^2) = -2|\nabla_i R_{jk}|^2 + \frac{2}{3}|\nabla_i R|^2 + 4U \\
&\quad \leq -2|\nabla_i R_{jk}|^2 + \frac{40}{21}|\nabla_i R_{jk}|^2 + 4R(S - \frac{1}{3}R^2) \\
&\quad = -\frac{2}{21}|\nabla_i R_{jk}|^2 + 4R(S - \frac{1}{3}R^2)
\end{aligned}$$

Lemma 5.4.8. $\frac{\partial}{\partial t}(S - \frac{1}{3}R^2) \leq \Delta(S - \frac{1}{3}R^2) - \frac{2}{21}|\nabla_i R_{jk}|^2 + 4R(S - \frac{1}{3}R^2)$

Lemma 5.4.9. $\langle \nabla_i R, \nabla_i S \rangle \leq 4R|\nabla_i R_{jk}|^2$

Proof. The Cauchy-Schwartz inequality states:

$$\langle \nabla_i R, \nabla_i S \rangle = 2\langle \nabla_i R_{jk}, \nabla_i R \cdot R_{jk} \rangle \leq 2|\nabla_i R||R_{jk}||\nabla_i R_{jk}|.$$

Observe that $|R_{jk}|^2 = S \leq R^2$, so

$$\langle \nabla_i R, \nabla_i S \rangle \leq 2\sqrt{3}R|\nabla_i R_{jk}|^2 \leq 4R|\nabla_i R_{jk}|^2.$$ □

Lemma 5.4.10. *For* $0 \leq \eta \leq \frac{1}{3}$

$$\frac{\partial}{\partial t}\left(\frac{|\nabla_i R|^2}{R} - \eta R^2\right) \leq \Delta\left(\frac{|\nabla_i R|^2}{R} - \eta R^2\right) + 16|\nabla_i R_{jk}|^2 - \frac{4}{3}\eta R^3$$

Proof. $S - \frac{1}{3}R^2 \geq 0 \Longrightarrow S/R^2 \geq \frac{1}{3}$, then combine Lemma 5.4.3 and Lemma 5.4.9 we get the lemma. □

Lemma 5.4.11. *Let* $F = |\nabla_i R|^2/R - \eta R^2 + N(S - \frac{1}{3}R^2)$ *where* $N \in R$. *For* $N \geq 168$, *there exists a constant* $\mathcal{C}_0 = \mathcal{C}_0(\eta, g(0))$ *depending only on* η *and initial metric* $g(0)$ *such that*

$$\frac{\partial F}{\partial t} \leq \Delta F + \mathcal{C}_0(\eta, g(0))$$

Proof. If $N \geq 168$

$$\begin{aligned}\frac{\partial F}{\partial t} - \Delta F &\leq 16|\nabla_i R_{jk}|^2 - \frac{4}{3}\eta R^3 - \frac{2}{21}N|\nabla_i R_{jk}|^2 + 4RN(S - \frac{1}{3}R^2) \\ &\leq 4RN(S - \frac{1}{3}R^2) - \frac{4}{3}\eta R^3\end{aligned}$$

By Theorem 5.3.1, we know $\exists \delta, \mathscr{C}$ depending only on initial metric such that

$$\frac{\partial F}{\partial t} - \Delta F \leq 4N\mathscr{C}R^{3-\delta} - \frac{4}{3}\eta R^3 \leq \mathcal{C}_0,$$

where $\mathcal{C}_0$ is the upper bound of RHS depending on $(\delta, \mathscr{C}, \eta)$. i.e. $\mathcal{C}_0$ depends only on η and initial metric $g(0)$. □

Now we could give a proof of Theorem 5.4.1. By previous lemma, take $N = 168$,

$$\frac{\partial (F - \mathcal{C}_0 t)}{\partial t} \leq \Delta(F - \mathcal{C}_0 t)$$

The maximum principle tells that

$$\max_{M\times\{t\}} F - \mathcal{C}_0 t \leq \max_{X\times\{0\}} F$$

$$\max_{M\times\{t\}} F \leq \max_{X\times\{0\}} F + \mathcal{C}_0 t$$

By Theorem 5.1.11, we know T is limited, so at $M \times [0, T)$

$$|\nabla_i R|^2/R - \eta R^2 \leq F \leq \mathcal{C}_1 \quad \text{with } \mathcal{C}_1 = \max_{X\times\{0\}} F + \mathcal{C}_0 T.$$

$$|\nabla_i R|^2 \leq \eta R^3 + \mathcal{C}_1 R \leq 2\eta R^3 + \mathcal{C}_2 \quad \text{for some constant } \mathcal{C}_2.$$

$\mathcal{C}_2$ depends only on $\mathcal{C}_0$, initial metric $g(0)$. Hence we have proved the theorem when η is small. We can enlarge η so that the result keeps for arbitrary $\eta > 0$.

5.5 Controlling R_{max}/R_{min}

In this subsection, let M^3 be a closed Riemannian 3 manifold with strictly positive Ricci flow; Let $R_{max}(t) := \max\limits_{M\times\{t\}} R$; $R_{min}(t) := \min\limits_{M\times\{t\}} R$.

Theorem 5.5.1. *$R_{max}/R_{min} \to 1$ as $t \to T$.*

Proof. Before prove this theorem, we first recall Myers theorem:

Theorem (Myers)**.** *Let M be a Riemannian manifold with dimension m. If $R_{ij} \geq (m-1)Hg_{ij}$ along a geodesic of length at least $\pi H^{-1/2}$ then the geodesic has conjugate point.*

It is known that $R_{ij} \geq \epsilon R g_{ij}$ under the variation of Ricci flow at any time for some $\epsilon > 0$. Hence along a geodesic of length at least $l := \frac{\pi\sqrt{2}}{\sqrt{\epsilon R_{min}(t)}}$ has a conjugate point at any $t \in [0, T)$.

For every $\eta > 0$ and suitable constant $\mathcal{C}(\eta) = \mathcal{C}(\eta, g(0))$

$$|\nabla_i R| \leq [\frac{1}{4}\eta^4 R^3 + \mathcal{C}^2(\eta, g(0))]^{1/2} \leq \frac{1}{2}\eta^2 R^{3/2} + \mathcal{C}(\eta).$$

Since $R_{max} \to +\infty$ as $t \to T$, $\exists\ \theta < T$ such that $\mathcal{C}(\eta) \leq \frac{1}{2}\eta^2 R_{max}^{3/2}$ for $\theta \leq t < T$. Then $|\nabla_i R| \leq \eta^2 R_{max}^{3/2}$ for $t \geq \theta$. For any $t \in [\theta, T)$, fix a point $x \in M$ such that $R(x,t) = R_{max}(t)$. Then on any geodesic out of x of length at most $s = \frac{1}{\eta\sqrt{R_{max}(t)}}$ we have $R \geq R_{max} - \eta R_{max} = (1-\eta)R_{max}$, so

$$s = \frac{1}{\eta\sqrt{R_{max}(t)}} \geq \frac{\sqrt{1-\eta}}{\eta\sqrt{R_{min}(t)}} \geq l$$

when $\eta \in (0, \frac{1}{2}(-\frac{\epsilon}{2\pi^2} + \sqrt{\frac{\epsilon^2}{4\pi^4} + 4\frac{\epsilon}{2\pi^2}})]$. In conclusion, for such small η, $\exists$ $\theta \in (0, T)$ such that $R_{min}(t) \geq (1-\eta)R_{max}(t)$ for all $t \in [\theta, T)$. □

Theorem 5.5.2. $\int_0^T R_{max} dt = \infty$

Proof. Let $f(t) : [0, T) \to \mathbb{R}$ such that

$$\begin{cases} \dfrac{df}{dt} = 2R_{max} f \\ f(0) = R_{max}(0) \end{cases}$$

As for R:

$$\frac{\partial R}{\partial t} = \Delta R + 2S \leq \Delta R + 2R_{max} R.$$

$$\frac{\partial}{\partial t}(R-f) \leq \Delta(R-f) + 2R_{max}(R-f).$$

By maximum principle, $R - f \leq 0$ on $0 \leq t < T$. Since $R \to \infty$ as $t \to T$, $f \to +\infty$ too. On the other hand, $\frac{df}{f} = 2R_{max}dt$ gives that

$$\ln f(t)/f(0) = 2\int_0^t R_{max}(\tau)d\tau \to \infty \quad \text{as } t \to T. \qquad \square$$

Corollary 5.5.3. *Let* $r = \frac{\int R d\mu}{\int d\mu}$, *then* $\int_0^T r dt = \infty$

Proof. Because $R_{max}/R_{min} \to 1$ and $R \geq 0$, $\int_0^T r dt$ and $\int_0^T R_{max}dt$ have the same convergence. $\square$

Theorem 5.5.4. $S/R^2 - \frac{1}{3} \to 0$ *as* $t \to T$ *for* $\forall x \in M^3$.

Proof. By Theorem 5.3.1:

$$S/R^2 - \frac{1}{3} \leq \mathscr{C} R^{-\delta},$$

and $R_{min} \to \infty$ because $R_{max} \to \infty$ and $R_{max}/R_{min} \to 1$. $\square$

Remark 5.5.5. *Someone may think Theorem 5.1.11 has stated that* $R_{min} \to \infty$ *as* $t \to T$. *But it has not, because* T *may strictly smaller than* $3/2\rho$ *and* R_{min} *may not have gone to* ∞ *yet. Therefore, it is reasonable to estimate* R_{max}/R_{min} *at first.*

Chapter 6

Long time existence

In Theorem 5.1.11, it has been known that the Ricci flow has finite maximum existence interval because of the blow-up of scalar curvature. This section will give another conclusion to describe the behavior of Rm when $t \to T$. This conclusion is based on the estimate to $\nabla^n Rm$. The required estimates, interpolation inequalities for tensors, will be given at first subsection. The special case of Rm will be discussed in the second subsection. After doing these estimates, the blow-up of Rm will be discussed.

6.1 Interpolation inequalities for tensors

Let M^m be a closed Riemannian manifold of dimension m; $T_{ij\ldots k}$ be any tensor on M^m.

Theorem 6.1.1. *Suppose $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ with $r \geq 1$. Then*

$$\left\{\int |\nabla T|^{2r} d\mu\right\}^{1/r} \leq (2r-2+m)\left\{\int |\nabla^2 T|^p d\mu\right\}^{1/p}\left\{\int |T|^q d\mu\right\}^{1/q}$$

Proof. For simplicity we take $T_J = T_{ij\dots k}$.

$$\begin{aligned}\int |\nabla T|^{2r} d\mu &= \int g^{ij}\nabla_i T \nabla_k T |\nabla T|^{2(r-1)} d\mu \\ &= -\int T g^{ij}\nabla_i\nabla_j T|\nabla T|^{2(r-1)}d\mu - \int g^{ij} T\nabla_j T \cdot \nabla_i |\nabla T|^{2(r-1)} d\mu\end{aligned}$$

$$\nabla_i |\nabla T|^{2(r-1)} = \nabla_i (g^{kl}\nabla_k T\nabla_l T)^{r-1} = 2(r-1) g^{kl}\nabla_i\nabla_k T \cdot \nabla_l T \cdot |\nabla T|^{2(r-2)}$$

$$\begin{aligned}\int |\nabla T|^{2r} d\mu &= -\int T g^{ij}\nabla_i\nabla_j T|\nabla T|^{2(r-1)}d\mu \\ &\quad - 2(r-1)\int g^{ij} T\nabla_j T \cdot g^{kl}\nabla_i\nabla_k T\cdot\nabla_l T \cdot |\nabla T|^{2(r-2)} d\mu \\ &= -\int T g^{ij}\nabla_i\nabla_j T|\nabla T|^{2(r-1)}d\mu \\ &\quad - 2(r-1)\int \langle T\nabla_i\nabla_j T, \nabla_i T\cdot\nabla_j T\rangle |\nabla T|^{2(r-2)} d\mu\end{aligned}$$

Because

$$|TJ\nabla_i\nabla_k T| \le m|T||\nabla^2 T|$$

$$\langle T\nabla_i\nabla_j T, \nabla_i T\cdot\nabla_j T\rangle \le |T||\nabla^2 T||\nabla T|^2$$

$$\begin{aligned}\int |\nabla T|^{2r} d\mu &\le \int n|T||\nabla^2 T||\nabla T|^{2r-2}d\mu + \int 2(r-1)|T||\nabla^2 T||\nabla T|^{2r-2} d\mu \\ &= (2r-2+m)\int |T||\nabla^2 T||\nabla T|^{2r-2}d\mu\end{aligned}$$

The Holder's inequality w.r.t

$$\frac{1}{p} + \frac{1}{q} + \frac{r-1}{r} = 1$$

gives

$$\int |\nabla T|^{2r} d\mu \le (2r-2+m)\left(\int |\nabla^2 T|^p d\mu\right)^{1/p}\left(\int |T|^q d\mu\right)^{1/q}\left(\int |\nabla T|^{2r} d\mu\right)^{1-1/r}$$

Hence

$$\left\{\int |\nabla T|^{2r} d\mu\right\}^{1/r} \le (2r-2+m)\left\{\int |\nabla^2 T|^p d\mu\right\}^{1/p}\left\{\int |T|^q d\mu\right\}^{1/q} \quad \square$$

Corollary 6.1.2. *If $p \geq 1$, we have*

$$\left\{\int |\nabla T|^{2p} d\mu\right\}^{1/p} \leq (2p-2+m)\left\{\int |\nabla^2 T|^p d\mu\right\}^{1/p} \max_M |T|.$$

Proof. Recall that for a $L^p(1 \leq p \leq \infty)$ measurable function F with compact support, we have: $\lim_{p\to\infty} \|F\|_p = \|F\|_\infty$. In this case, the corollary follows by taking $q = \infty$. □

Lemma 6.1.3. *Let f be a function: $\mathbb{N} \cap [0,n] \to \mathbb{R}$. If $f(k) \leq \frac{1}{2}[f(k-1) + f(k+1)]$ $\forall k = 1, \ldots, n-1$, then $f(k) \leq (1-\frac{k}{n})f(0) + \frac{k}{n}f(n)$ $\forall k = 0, 1, \ldots, n$.*

Proof. Let $\widetilde{f}(k) = f(k) - (1-\frac{k}{n})f(0) - \frac{k}{n}f(n)$, $k = 0, 1, \ldots, n$; let $g(k) = \widetilde{f}(k) - \widetilde{f}(k-1)$ for $1 \leq k \leq n$. Then $g(k+1) - g(k) = \widetilde{f}(k+1) + \widetilde{f}(k-1) - 2\widetilde{f}(k) \geq 0$. Then for some $m \in \mathbb{N}_+$:

$$g(1) \leq \cdots \leq g(m) \leq 0 \leq g(m+1) \leq \cdots \leq g(n)$$

$$\sum_{k=1}^{n} g(k) = \sum_{k=1}^{n}\{\widetilde{f}(k) - \widetilde{f}(k-1)\} = \widetilde{f}(n) - \widetilde{f}(0) = 0$$

For any $k \geq 1$, $\widetilde{f}(k) = \sum_{i=1}^{k} g(i) = -\sum_{i=k+1}^{n} g(i)$. When $k \leq m$: $\sum_{i=1}^{k} g(i) \leq 0$; when $k \geq m$: $-\sum_{i=k+1}^{n} g(i) \leq 0$. $\Longrightarrow \widetilde{f}(k) \leq 0$, $k = 0, 1, \ldots, n$. Thus, $f(k) \leq (1-\frac{k}{n})f(0) + \frac{k}{n}f(n)$, $k = 0, 1, \ldots, n$. □

Corollary 6.1.4. *If*

$$f(k) \leq \frac{1}{2}(f(k-1) + f(k+1)) + C \quad \forall 1 \leq k \leq n-1$$

for some constant C, then

$$f(k) \leq (1-\frac{k}{n})f(0) + \frac{k}{n}f(n) + Ck(n-k) \quad \forall 0 \leq k \leq n.$$

Proof. Consider $h(k) = f(k) + Ck^2$, $k = 0, 1, \ldots, n$. It is shown that $h(k) \leq \frac{1}{2}(h(k-1) + h(k+1))$, $k = 1, \ldots, n-1$. By previous lemma, $h(k) \leq (1 - \frac{k}{n})h(0) + \frac{k}{n}h(n) \Longrightarrow$

$$f(k) \leq (1 - \frac{k}{n})f(0) + \frac{k}{n}f(n) + Ck(n-k)$$

□

Corollary 6.1.5. *If*

$$f(k) \leq Cf(k-1)^{1/2}f(k+1)^{1/2} \quad \forall 1 \leq k \leq n-1$$

then

$$f(k) \leq C^{k(n-k)}f(0)^{1-k/n}f(n)^{k/n}$$

Proof. Take $h(k) = \ln f(k)$, we get the conclusion. □

Let $\nabla^n T$ denote the tensor $\nabla_{i_1} \ldots \nabla_{i_n} T_{j \ldots k}$

Corollary 6.1.6. *If T is any tensor and if $1 \leq i \leq n$ then $\exists$ a constant $\mathcal{C} = \mathcal{C}(n, m)$ depending only on n and $m = \dim M$ and independent of the metric g_{ij} or the connection Γ_{ij}^k such that*

$$\int |\nabla^i T|^{2n/i} d\mu \leq \mathcal{C} \max_M |T|^{2(n/i-1)} \int |\nabla^n T|^2 d\mu$$

Proof. When $2 \leq i \leq n-1$, use Theorem 6.1.1 to tensor $\nabla^{i-1}T$ we have:

$$\left\{\int |\nabla^i T|^{2r} d\mu\right\}^{1/r} \leq (2r - 2 + m)\left\{\int |\nabla^{i+1}T|^p d\mu\right\}^{1/p}\left\{\int |\nabla^{i-1}T|^q d\mu\right\}^{1/q}$$

Let

$$p = \frac{2n}{i+1}, q = \frac{2n}{i-1}, r = \frac{n}{i} > 1$$

$$\left\{\int |\nabla^i T|^{2n/i} d\mu\right\}^{i/n} \leq (2\frac{n}{i} - 2 + n)\left\{\int |\nabla^{i+1}T|^{2n/(i+1)} d\mu\right\}^{\frac{i+1}{2n}}\left\{\int |\nabla^{i-1}T|^{2n/(i-1)} d\mu\right\}^{\frac{i-1}{2n}}$$

When $i = 1$, by Corollary 6.1.2:

$$\left\{\int |\nabla T|^{2n} d\mu\right\}^{1/n} \leq (2p - 2 + m)\max_M |T|\left\{\int |\nabla^2 T|^n d\mu\right\}^{1/n}$$

Let $f(0) = \max_M |T|$, $f(i) = \left(\int |\nabla^i T|^{2n/i}\right)^{i/2n}$, $1 \le i \le n$. Then $\exists\, \mathcal{C}_1$ depending only on n and m such that

$$f(i) \le \mathcal{C}_1 f(i+1)^{1/2} f(i-1)^{1/2} \quad \text{where } 1 \le i \le n-1.$$

The previous lemma tells that

$$f(i) \le \mathcal{C}_2 f(0)^{1-i/n} f(n)^{i/n} \quad \text{where } 0 \le i \le n$$

where $\mathcal{C}_2 = \max\{C_1^{i(n-i)} | 0 \le i \le n\}$. The conclusion follows. □

Corollary 6.1.7. *If T is any tensor and if $0 \le i \le n$ then $\exists$ a constant $\mathcal{C} = \mathcal{C}(n, m)$ depending only on n and $m = \dim M$ and independent of the metric g_{ij} or the connection Γ^k_{ij} such that*

$$\int |\nabla^i T|^2 d\mu \le \mathcal{C} \left\{\int |\nabla^n T|^2 d\mu\right\}^{i/n} \left\{\int |T|^2 d\mu\right\}^{1-i/n}$$

Proof. Applying Theorem 6.1.1 to $\nabla^{i-1} T$ with $p = q = 2$, $r = 1$: for $1 \le i \le n-1$

$$\int |\nabla^i T|^2 d\mu \le m \left\{\int |\nabla^{i+1} T|^2 d\mu\right\}^{1/2} \left\{\int |\nabla^{i-1} T|^2 d\mu\right\}^{1/2},$$

Let $f(0) = \int |T|^2 d\mu$, $f(i) = \int |\nabla^i T|^2 d\mu$

$$f(i) \le \mathcal{C} f(i+1)^{1/2} f(i-1)^{1/2}$$

$$f(i) \le \mathcal{C}' f(0)^{1-i/n} f(n)^{i/n}$$

The corollary follows directly. □

6.2 Higher derivatives of the Curvature

If A, B are two tensors. Let's define $A * B$ to be the linear combination of tensor formed by contraction on $A_{i\dots j} B_{k\dots l}$ using g^{ik}.

Lemma 6.2.1. *Let M^m be a Riemannian manifold with dimension m. If A, B are tensors satisfying the evolution equation*

$$\frac{\partial}{\partial t}A = \Delta A + B,$$

then

$$\frac{\partial}{\partial t}\nabla A = \Delta\nabla A + Rm * \nabla A + A * \nabla Rm + \nabla B.$$

In particular, when $m = 3$ we have

$$\frac{\partial}{\partial t}\nabla A = \Delta\nabla A + Rc * \nabla A + A * \nabla Rc + \nabla B.$$

Proof. Since $\nabla = \partial + \Gamma$

$$\frac{\partial}{\partial t}(\nabla A) = \nabla\frac{\partial A}{\partial t} + \frac{\partial \Gamma}{\partial t} * A.$$

Remind that $\frac{\partial}{\partial t}\Gamma^i_{jk} = -g^{il}\{\nabla_j R_{kl} + \nabla_k R_{jl} - \nabla_l R_{jk}\}$, so

$$\frac{\partial}{\partial t}(\nabla A) = \nabla\frac{\partial A}{\partial t} + \nabla Rc * A = \nabla\Delta A + \nabla B + \nabla Rc * A.$$

Because

$$\begin{aligned}
\nabla\Delta A &= \nabla\nabla_i\nabla_i A = \nabla_i\nabla\nabla_i A + Rm * \nabla A \\
&= \nabla_i(\nabla_i\nabla A + Rm * A) + Rm * \nabla A \\
&= \Delta\nabla A + \nabla Rm * A + Rm * \nabla A, \\
\frac{\partial}{\partial t}\nabla A &= \Delta\nabla A + Rm * \nabla A + A * \nabla Rm + \nabla B
\end{aligned}$$

□

Theorem 6.2.2. *$\nabla^n Rm$ satisfies*

$$\frac{\partial}{\partial t}\nabla^n Rm = \Delta(\nabla^n Rm) + \sum_{\substack{i+j=n \\ 0\le i,j\le n}} \nabla^i Rm * \nabla^j Rm$$

Proof. If $n = 0$, Corollary 5.1.3 tells that

$$\frac{\partial}{\partial t} R_{ijkl} = \Delta Rm + Rm * Rm$$

Proceed by induction on n: let $A = \nabla^n Rm$, $B = \sum_{i+j=n} \nabla^i Rm * \nabla^j Rm$. Suppose we have

$$\frac{\partial}{\partial t} A = \Delta A + B,$$

then by Lemma 6.2.1

$$\begin{aligned}
\frac{\partial}{\partial t} \nabla^{n+1} Rm &= \Delta \nabla^{n+1} Rm + Rm * \nabla^{n+1} Rm + \nabla^n * \nabla Rm \\
&\quad + \sum_{i+j=n} \{\nabla^{i+1} Rm * \nabla^j Rm + \nabla^i Rm * \nabla^{j+1}\} \\
&= \Delta \nabla^{n+1} Rm + \sum_{\substack{i+j=n+1 \\ 0 \le i,j \le n+1}} \nabla^i Rm * \nabla^j Rm
\end{aligned}$$

□

Corollary 6.2.3. $\forall\ n \ge 0$, *we have*

$$\frac{\partial}{\partial t} |\nabla^n Rm|^2 = \Delta |\nabla^n Rm|^2 - 2|\nabla^{n+1} Rm|^2 + \sum_{i+j=n} \nabla^i Rm * \nabla^j Rm * \nabla^n Rm$$

Proof.

$$\frac{\partial}{\partial t} |\nabla^n Rm|^n = 2\langle \nabla^n Rm, \frac{\partial}{\partial t} \nabla^n Rm \rangle + Rm * \nabla^n Rm * \nabla^n Rm,$$

where the second term is from the derivative of g.

$$\begin{aligned}
\Delta |\nabla^n Rm|^2 &= 2\nabla_i \langle \nabla^n Rm, \nabla_i \nabla^n Rm \rangle \\
&= 2\langle \nabla^n Rm, \Delta \nabla^n Rm \rangle + 2\langle \nabla_i \nabla^n Rm, \nabla_i \nabla^n Rm \rangle \\
&= 2\langle \nabla^n Rm, \Delta \nabla^n Rm \rangle + 2|\nabla^{n+1} Rm|^2
\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^n Rm|^2 &= \Delta|\nabla^n Rm|^2 + 2\langle\nabla^n Rm, \frac{\partial}{\partial t}\nabla^n Rm - \Delta\nabla^n Rm\rangle \\ &\quad - 2|\nabla^{n+1}Rm|^2 + Rm * \nabla^n Rm * \nabla^n Rm \\ &= \Delta|\nabla^n Rm|^2 - 2|\nabla^{n+1}Rm|^2 + \sum_{i+j=n}\nabla^i Rm * \nabla^j Rm * \nabla^n Rm\end{aligned}$$

□

Theorem 6.2.4. *In any closed Riemannian manifold M^m, for any $n \geq 0$ we have the estimate*

$$\frac{d}{dt}\int_M |\nabla^n Rm|^2 d\mu + 2\int_M |\nabla^{n+1}Rm|^2 d\mu \leq C \max_M |Rm| \int_M |\nabla^n Rm|^2 d\mu$$

where C is a constant independent of the metric, depending only on the number n of derivatives and the dimension m of M.

Proof.

$$\begin{aligned}&\frac{d}{dt}\int_M |\nabla^n Rm|^2 d\mu + 2\int_M |\nabla^{n+1}Rm|^2 d\mu \\ &= \int_M (\frac{d}{dt}\int_M |\nabla^n Rm|^2 d\mu + 2\int_M |\nabla^{n+1}Rm|^2) d\mu \\ &= \int_M (\Delta|\nabla^n Rm|^2 - 2|\nabla^{n+1}Rm|^2 + \sum_{i+j=n}\nabla^i Rm * \nabla^j Rm * \nabla^n Rm) d\mu \\ &= C(n)\int_M |\nabla^i Rm||\nabla^j Rm||\nabla^n Rm| d\mu, \quad \text{for some constant } C(n)\end{aligned}$$

Use Holder's inequality twice:

$$\begin{aligned}&\int_M |\nabla^i Rm||\nabla^j Rm||\nabla^n Rm| d\mu \\ &\qquad \leq \left(\int_M |\nabla^i Rm|^2|\nabla^j Rm|^2 d\mu\right)^{1/2}\left(\int_M |\nabla^n Rm|^2 d\mu\right)^{1/2} \\ &\qquad \leq \left(\int_M |\nabla^i Rm|^{2n/i} d\mu\right)^{i/2n}\left(\int_M |\nabla^j Rm|^{2n/j} d\mu\right)^{j/2n}\left(\int_M |\nabla^n Rm|^2 d\mu\right)^{1/2}\end{aligned}$$

Apply Corollary 6.1.6

$$\left(\int |\nabla^i Rm|^{2n/i} d\mu\right)^{i/2n} \leq \mathcal{C}(n,m) \max_M |Rm|^{1-i/n} \left(\int |\nabla^n T|^2 d\mu\right)^{i/2n}$$

$$\frac{d}{dt}\int_M |\nabla^n Rm|^2 d\mu + 2\int_M |\nabla^{n+1} Rm|^2 d\mu \leq C \max_M |Rm| \int_M |\nabla^n Rm|^2 d\mu$$

where C depends only on n, m. □

If the dimension of M is 3, then Rm could be replaced by Rc. Furthermore, previous arguments set up by replacing Rm by Rc. Thus we have

Corollary 6.2.5. *In any closed Riemannian manifold M of dimension 3, for any $n \geq 0$ we have the estimate*

$$\frac{d}{dt}\int_M |\nabla^n Rc|^2 d\mu + 2\int_M |\nabla^{n+1} Rc|^2 d\mu \leq C \max_M |Rc| \int_M |\nabla^n Rc|^2 d\mu$$

where C is a constant independent of the metric, depending only on the number n of derivatives.

6.3 Finite time blow-up

Let M^m be a closed Riemannian manifold of dimension m. We want to show:

Theorem 6.3.1. *Suppose the Ricci flow has a unique solution on a maximal time interval $0 \leq t < T \leq \infty$. If $T < \infty$, then $\max_M |R_{ijkl}| \to \infty$ as $t \to T$.*

Let's prove by contradiction. Suppose $|Rm| \leq \mathcal{C} < \infty$ as $t \to T$. If we can show g converges to a smooth metric as $t \to T$, then by Theorem 4.2.1 the maximal time interval will be larger then $[0, T)$, a contradiction.

At first, observe that if Rm is uniformly bounded, then $\exists$ constant C such that

$$\int_0^T \max_M |g'_{ij}| dt \leq 2 \int_0^T \max_M |R_{ij}| dt \leq C.$$

This is where we start our deduction.

Lemma 6.3.2. *Let g_{ij} be a time-dependent metric on M for $0 \leq t < T \leq \infty$. Suppose*

$$\int_0^T \max_M |g'_{ij}| dt \leq C < \infty$$

Then the metrics $g_{ij}(t)$ for all different times are equivalent, and they converge as $t \to T$ uniformly to a positive-definite tensor $g_{ij}(T)$ which is continuous and also equivalent to $g_{ij}(t)$ with $0 \leq t < T$.

Proof. Fix a tangent vector $v \neq 0 \in TM$ at a point $x \in M$ and let

$$|v|_t^2 = g_{ij}(t) v^i v^j$$

Then we take

$$\frac{d}{dt} |v|_t^2 = g'_{ij} v^i v^j$$

and it follows by Cauchy-Schwartz inequality that

$$|g'_{ij} v^i v^j| \leq |g'_{ij}| |v|_t^2$$

$$\frac{d}{dt} \ln |v|_t^2 = \frac{1}{|v|_t^2} \frac{d}{dt} |v|_t^2 = \frac{1}{|v|_t^2} \cdot g'_{ij} v^i v^j$$

$\Longrightarrow \left| \frac{d}{dt} \ln |v|_t^2 \right| \leq |g'_{ij}|$. Then for $0 \leq \tau \leq \theta < T$ we have

$$|\ln |v|_\theta^2 / |v|_\tau^2| \leq \int_\tau^\theta |g'_{ij}| dt \leq C < \infty \qquad (\star)$$

The formula $(\star)$ could give several important conclusions:

(a) $\forall\ \epsilon > 0$, $\exists\ \delta > 0$ independent to $v \in TM$ such that $\forall\ \theta, \tau \in (T-\delta, T)$ we have $|\ln |v|_\theta^2/|v|_\tau^2| < \epsilon$. Then $\ln |v|_t^2$ converge uniformly as $t \to T$. Let's define

$$|v|_T^2 := \exp(\lim_{t\to T} \ln |v|_t^2).$$

Because $|v|_t^2\ \forall\ t \in [0, T)$ are all norm function on $T_xM\ \forall\ x \in M$, $|v|_T^2$ is a norm too. Then we get a limit metric $g_{ij}(T)$.

(b) $e^{-C}|v|_\tau^2 \le |v|_\theta^2 \le e^C |v|_\tau^2\ \forall\ \tau, \theta \in [0, T], \forall\ v \in TM$. All metric $g_{ij}(t)$ for $0 \le t < T \le \infty$ are equivalent. □

Lemma 6.3.3. *If $|Rm| \le \mathcal{C}$ on $0 \le t < T < \infty$, then for any n we can find a constant C_n such that for any $t \in [0, T]$ we have*

$$\int_M |\nabla^n Rm|^2 d\mu \le C_n$$

Proof. Let $f(t) = \int_M |\nabla^n Rm|^2 d\mu$. Theorem 6.2.4 tells that $\exists\ C(n, m)$ such that

$$\frac{df}{dt} \le C(n, m)\mathcal{C} f.$$

$$f(t) \le f(0) \exp\{C(n, m)\mathcal{C} t\}$$

We also know that $f(0)$ is bounded because M is compact. Let $C_n := f(0) \exp\{C(n, m)\mathcal{C} T\}$. The lemma follows. □

Lemma 6.3.4. *Assume $|Rm| \le \mathcal{C}$ on $0 \le t < T < \infty$ as before, for all $n \in \mathbb{N}$, $\exists\ \widetilde{C}_n$ such that for any $t \in [0, T)$ we have*

$$\|\nabla^n Rm\|_\infty \le \widetilde{C}_n$$

In particular, $\exists\ \widetilde{D}_n$

$$\|\nabla^n Rc\|_\infty \le \widetilde{D}_n$$

Proof. By Corollary 6.1.6, $\exists$ constant $\mathcal{B} = \mathcal{B}(N, m)$ such that $\forall\ 1 \leq n \leq N$

$$\int |\nabla^n Rm|^{2N/n} d\mu \leq \mathcal{B} \max_M |Rm|^{2(N/n-1)} \int |\nabla^N Rm|^2 d\mu \leq \mathcal{B}\mathcal{C}C_N$$

Take $N = np$ for some $m < p < \infty$:

$$\int |\nabla^n Rm|^{2p} d\mu \leq \mathcal{B}\mathcal{C}C_{np} =: C_{n,p}.$$

Take $f = |\nabla^n Rm|^{2p} \in \Gamma(M)$:

$$\int \{f + |\nabla f|\} d\mu \leq \widetilde{C}_{n,p}$$

Now by Sobolev's inequality (see Appendix A), $\exists$ constant $C(t)$ depending on t such that

$$\|\nabla^n Rm\|_\infty^{2p} = \|f\|_\infty \leq C(t) \int \{f + |\nabla f|\} d\mu \leq C(t)\widetilde{C}_{n,p},$$

Here the constant $C(t)$ depends on (ω^n, t), hence depending on $(g_{ij}(t), d\mu = \sqrt{\det g}dx, t)$. Lemma 6.3.2 tells that $g_{ij}(t)$ could be controlled by $\mathcal{C}$. Finally, $\exists$ unified constant $\widetilde{C}_n$ such that

$$|\nabla^n Rm| \leq \widetilde{C}_n, \quad \forall n \in \mathbb{N},$$

where $\widetilde{C}_n$ just depends on the initial value of the metric and the constant $\mathcal{C}$ which bounds $|Rm|$ $\forall\ 0 \leq t < T$. In particular, because $g(t)$ is equivalent as shown in 6.3.2, $\exists$ constant E such that

$$\|\nabla^n Rc\|_\infty \leq C\|g(0)\|_\infty |\nabla^n Rm| =: \widetilde{D}_n$$

□

Lemma 6.3.5. *Assume $|Rm| \leq \mathcal{C}$ on $0 \leq t < T < \infty$ as before, for all $n \in \mathbb{N}$, $\exists$ C_n such that for any $t \in [0, T)$ we have*

$$\|\partial^n g\|_\infty \leq C_n$$

Proof. Let C be a generic constant depending just on $m, n, \mathcal{C}, g(0), T$.

If $n = 1$: $\forall s \in [0, T)$

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x_k} g_{ij}(s)\right) = \frac{\partial}{\partial x_k}\frac{\partial}{\partial t} g_{ij}(s) = -2\frac{\partial}{\partial x_k} R_{ij}(s)$$
$$= -2(\nabla_k R_{ij} + \Gamma^l_{ki} R_{lj} + \Gamma^l_{kj} R_{il})$$

$$|\frac{\partial}{\partial t}\partial_k g_{ij}(s)| \leq 2|\nabla_k R_{ij}| + 2|\Gamma^l_{ki} R_{lj}| \leq 2|\nabla_k R_{ij}| + 2m|\Gamma(s)||R_{ij}|$$

Notice that

$$\frac{\partial}{\partial t}\Gamma^k_{ij} = -\frac{1}{2} g^{kl}\{\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij}\}$$

$$|\frac{\partial}{\partial t}\Gamma| \leq C|\nabla_i R_{ij}|$$

$$|\frac{\partial}{\partial t}\partial_k g_{ij}(s)| \leq 2\widetilde{D}_1 + 2mC\widetilde{D}_1\widetilde{D}_0 =: B$$

$$|\partial g(s)| \leq \|\partial g(0)\|_\infty + BT =: C_1$$

One could show the general case by induction. The reader can find the detail proof in [4] The Ricci flow: An introduction pp206-207. □

Here is a complete proof of the Theorem 6.3.1. Assume $|Rm|$ is bounded by $\mathcal{C}$. Let C be a generic constant depending just on $m, n, \mathcal{C}, g(0), T$. Fix a local coordinate patch U around an arbitrary point $x \in M^m$, and let $\tau \in (0, T)$ be arbitrary as well. Then by Lemma 6.3.2, a continuous limit metric $g_{ij}(T)$ exists and is given as

$$g_{ij}(x, T) = g_{ij}(x, \tau) - 2\int_\tau^T R_{ij}(x, t)dt.$$

Let $\alpha = (a_1, \dots, a_r)$ be any multi-index with $|\alpha| = n \in \mathbb{N}$. By Lemma 6.3.4 and Lemma 6.3.5, both $\frac{\partial^n}{\partial x^\alpha} g_{ij}$ and $\frac{\partial^n}{\partial x^\alpha} R_{ij}$ are uniformly bounded on $U \times [0, T)$. Thus we can write

$$\left(\frac{\partial^n}{\partial x^\alpha} g_{ij}\right)(x, T) = \left(\frac{\partial^n}{\partial x^\alpha} g_{ij}\right)(x, \tau) - 2\int_\tau^T \left(\frac{\partial^n}{\partial x^\alpha} R_{ij}\right)(x, t)dt,$$

which shows that $|\partial^\alpha g(T)| \leq C$ for some positive constant C, hence that $g_{ij}(T)$ is a smooth metric, and also

$$\left|\left(\frac{\partial^n}{\partial x^\alpha} g_{ij}\right)(x,T) - \left(\frac{\partial^n}{\partial x^\alpha} g_{ij}\right)(x,\tau)\right| \leq C(T-\tau),$$

which shows that $g(\tau) \to g(T)$ uniformly in any C^n norm as $t \to T$, and

$$\left(\frac{\partial^n}{\partial x^\alpha} g_{ij}\right)(x,T) = \lim_{t\to T}\left(\frac{\partial^n}{\partial x^\alpha} g_{ij}\right)(x,t).$$

The smoothness of $g(T)$ implies that $[0,T)$ is not the maximal existence interval of Ricci flow, a contradiction to the definition of T.

Remark 6.3.6. *Recall that Theorem 5.1.11 tells that $T < \infty$ for compact 3-manifold with strictly positive Ricci curvature. Thus, the Rm in such M^3 will blow up as the variation of Ricci flow.*

Chapter 7

The normalized equation

As shown in previous section, the Ricci flow on closed M^3 with strictly positive Rc always blow-up, so we hope there exist some flows with better behaviors. Let's consider the normalized equation of Ricci flow on M^n:

$$\frac{\partial}{\partial t} g_{ij} = \frac{2}{n} r g_{ij} - 2R_{ij}$$

where $r = \int R d\mu / \int d\mu$ is the average of scalar curvature. It will be shown that such flow owns better properties than the origin one.

7.1 Estimating the normalized equation

We will focus on the estimates of the normalized equation on M^3. Let

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} \tag{$*$}$$

$$\frac{\partial}{\partial \tilde{t}} \tilde{g}_{ij} = \frac{2}{3} \tilde{r} \tilde{g}_{ij} - 2\tilde{R}_{ij} \tag{$**$}$$

For convenience we let t, g_{ij}, R_{ij}, R, r denote the variables for the unnormalized equation $(*)$ and $\tilde{t}, \tilde{g}_{ij}, \tilde{R}_{ij}, \tilde{R}, \tilde{r}$ the corresponding variables for the normalized equation (∗∗). At first, for (∗∗) $\tilde{h}_{ij} = \frac{2}{3}\tilde{r}\tilde{g}_{ij} - 2\tilde{R}_{ij}$, Lemma 3.0.8 gives

$$\tilde{H} = \tilde{g}_{ij}\tilde{h}_{ij} = 2(\tilde{r} - \tilde{R})$$

$$\frac{d}{d\tilde{t}}\int d\tilde{\mu} = \int \frac{\partial}{\partial t} d\tilde{\mu} = \int (\tilde{r} - \tilde{R}) d\tilde{\mu} = 0$$

Hence, an important observation is that the volume of M^3 is invariant under the variation of $\tilde{g}(t)$. On the other hand, let $\psi = \psi(t)$ be the normalization factor such that $\tilde{g}_{ij}(t) = \psi(t)g_{ij}(t)$ and $\int d\tilde{\mu} = 1$. The geometry of g and $\tilde{g}$ is connected by following (under the same time scalar t):

$$\tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k \quad \tilde{R}_{ijk}^l = R_{ijk}^l \quad \tilde{R}_{ijkl} = \psi R_{ijkl}$$

$$\tilde{R}_{ij} = R_{ij} \quad \tilde{R} = \psi^{-1}R \quad \tilde{r} = \psi^{-1}r \quad d\tilde{\mu} = \psi^{n/2}d\mu$$

Moreover we choose a new time scalar $\tilde{t} = \int \psi(t)dt$,

$$\frac{d\tilde{t}}{dt} = \psi(t).$$

Under the new time scalar $\tilde{t}$, one could show the normalized Ricci flow again:

$$\frac{\partial}{\partial \tilde{t}}\tilde{g}_{ij} = \frac{2}{n}\tilde{r}\tilde{g}_{ij} - 2\tilde{R}_{ij} \qquad (**)$$

Let (∗) have a solution on a maximal time interval $0 \le t < T$ and let (∗∗) have a corresponding solution on $0 \le \tilde{t} < \tilde{T}$ as the transformation above.

Remark 7.1.1. *In fact, we can discuss the geometry of $\tilde{g}$ under old time scalar t. Here we use the new scalar $\tilde{t}$ for several reasons. One is that the normalized Ricci flow has invariant volume, a special invariant. Second, the normalized Ricci flow also obeys the variation equations found in Section 3. Third, $\tilde{t}$ follows the origin paper [H].*

Lemma 7.1.2. $\tilde{R}_{\max}(\tilde{t})/\tilde{R}_{\min}(\tilde{t}) \to 1$ *as* $\tilde{t} \to \tilde{T}$.

Proof. $\tilde{R}_{\max}$ and $\tilde{R}_{\min}$ are dilated by a same constant, the ratio is unchanged with respect to Theorem 5.5.1. □

Lemma 7.1.3. *$\tilde{R}_{ij}(\tilde{t}) \geq \epsilon \tilde{R}(\tilde{t})\tilde{g}_{ij}(\tilde{t})$ for some ϵ for any $\tilde{t} \in [0, \tilde{T})$.*

Proof. Theorem 5.2.5 and the transform laws in old time scalar t give that: $\tilde{R}_{ij}(t) = \epsilon \tilde{R}(t)\tilde{g}_{ij}(t)$. Thus, under new time scalar $\tilde{t}$ the result remains. □

Recall the Bishop-Gunther-Cheeger-Gromov volume comparison theorem:

Theorem (Bishop-Gunther-Cheeger-Gromov)**.** *We denote by M a complete Riemannian manifold of dimension n, and by M_κ the model space of constant curvature κ. Let $B_p(r)$ (resp. $B_p^\kappa(r)$) be a geodesic ball in M (resp. M_κ). i.e.*

$$B_p(r) = \{x = \exp_p(t\theta) | \theta \in S^{n-1}, 0 \leq t \leq r\} \text{ for } p \in M \text{ and arbitrary } r$$

and let $vol(B_p(r))$ be its Riemannian volume. Then if $Rc \geq (n-1)\kappa$, then

$$r \to \frac{vol(B_p(r))}{vol(B_p^\kappa(r))}$$

is a non-increasing function, which tends to 1 as r goes to 0. In particular, $vol(B_p(r)) \leq vol(B_p^\kappa(r))$.

Proof. see [1] page 21, theorem 4.7 □

This theorem is helpful to bound the scalar curvature $\tilde{R}$.

Lemma 7.1.4. *$\tilde{R}_{\max}(\tilde{t}) \leq C < \infty$ on $0 \leq \tilde{t} < \tilde{T}$*

Proof. Let $\tilde{L}(\tilde{t})$ and $\tilde{V}(\tilde{t})$ denote the diameter and volume of $\tilde{g}(\tilde{t})$ respectively. Since $\tilde{Rc} > 0$, the Bishop-Gunther-Cheeger-Gromov volume comparison theorem implies that

$$1 \equiv \tilde{V} \leq \frac{1}{6}\pi \tilde{L}^3.$$

On the other hand, Lemma 7.1.3 shows there is a positive constant β depending only on g_0 such that

$$\tilde{Rc} \geq 2\beta^2 \tilde{R}_{\min}\tilde{g}.$$

So by Theorem 5.5,

$$\tilde{L} \leq \frac{\pi}{\beta\sqrt{\tilde{R}_{\min}}}$$

Since $\tilde{R}_{\max}/\tilde{R}_{\min} \to 1$ as $\tilde{t} \to \tilde{T}$, there exists a positive number A such that

$$\frac{\tilde{R}_{\max}}{\tilde{R}_{\min}} \geq \frac{1}{A}.$$

Thus,

$$\tilde{R}_{\max} \leq A\tilde{R}_{\min} \leq A\left(\frac{\pi}{\beta\tilde{L}}\right)^2 \leq A\left(\frac{\pi}{\beta}\right)\left(\frac{\pi}{6}\right)^{2/3} \qquad \square$$

Lemma 7.1.5. $\tilde{T} = \infty$

Proof. Since $d\tilde{t}/dt = \psi$ and $\psi\tilde{r} = r$ we have

$$\int_0^{\tilde{T}} \tilde{r}d\tilde{t} = \int_0^T rdt = \infty$$

by Corollary 5.5.3. While $\tilde{r} \leq \tilde{R}_{\max} \leq C$, so $\tilde{T} = \infty$. $\square$

Lemma 7.1.6. $\tilde{S}/\tilde{R}^2 - \frac{1}{3} \to 0$ *as* $\tilde{t} \to \infty$.

Proof. Apply Theorem 5.5.4 and the transform law between g and $\tilde{g}$. $\square$

In normal coordinate e_i, the section curvature in closed $(M^3, g(t))$

$$K(e_i, e_j) = \frac{Rm(e_i, e_j, e_j, e_i)}{g_{ii}g_{jj} - g_{ij}^2} = R_{ijji}$$

is of the form $\frac{1}{2}(\lambda + \mu - \upsilon)$ (see in Corollary 5.1.9). Transform it into normalized Ricci flow, it becomes

$$\tilde{K}(e_i, e_j) = \tilde{R}_{ijji} = \frac{1}{2}(\tilde{\lambda} + \tilde{\mu} - \tilde{\upsilon})$$

In each $p \in M^3$, $\lambda(p,\tilde{t}), \mu(p,\tilde{t}), \upsilon(p,\tilde{t})$ approach each other as $\tilde{t} \to \infty$ by Lemma 7.1.6, so $K(p,\tilde{t})$ can be controlled by half of the scalar curvature at

$R(p,\tilde{t})$ as $\tilde{t} \to \infty$. In global, $\tilde{R}_{\max}/\tilde{R}_{\min} \to 1$ tells that the control of sectional curvature is uniformly. Thus $\forall\ \delta \in (0,1)\ \exists\ C > 0$ such that $\forall\ t \geq C$ we have

$$0 \leq (1-\delta)K_{\max}(t) < K(t) \leq K_{max}(t).$$

This formula may remind you the famous sphere theorem (see in [6] page 265):

Theorem (sphere theorem)**.** *Let M^n be a closed simply connected, Riemannian manifold, whose sectional curvature K satisfies*

$$0 < hK_{\max} < K \leq K_{\max}.$$

Then if $h = 1/4$, M is homeomorphic to a sphere.

Then we know

Theorem 7.1.7. *Let (M,g) be a closed Riemannian manifold of dimension 3 which admits a strictly positive Ricci curvature. Then $\forall\ \epsilon \in (0,1)$ (M,g) also admits a metric such that*

$$0 \leq (1-\epsilon)K_{\max} < K \leq K_{max}.$$

In particular, when M is simply connected, M is homeomorphic to a 3-sphere.

This has been a very good result. But it could be optimized further as what the Main Theorem said. To accomplish this point, one borrows the following lemma which is not so strong compared with sphere theorem:

Lemma 7.1.8 (Klingenberg)**.** *Let $\widetilde{M}$ be a simply connected manifold of dimension 3. The sectional curvature satisfies:*

$$0 < \frac{1}{4}K_{\max} < K \leq K_{\max}.$$

Then the injectivity radius of M is at least $\pi/\sqrt{K_{\max}}$.

Proof. See [3] Theorem 5.10. □

Recall the Bonnet-Myers theorem:

Theorem. *Let (M, g) be a complete Riemannian manifold with $Rc(g) \geq k > 0$ for some constant $k > 0$. Then M is compact and $\pi_1(M)$ is finite.*

Lemma 7.1.9. $\exists\ \epsilon > 0$ *such that* $\tilde{R}_{\min} \geq \epsilon$ *on* $0 \leq \tilde{t} < \infty$.

Proof. Let $T \in (0, \infty)$ such that $\forall\ t \geq T$ the sectional curvature is $1/4$ pinched. For $t \in [0, T]$, $\tilde{R}_{\min} : [0, T] \to \mathbb{R}_+$ is a continuous function with positive value. For this reason, $\tilde{R}_{\min} \geq \epsilon_1 > 0$ for some constant ϵ_1. As for $t \in [T, \infty)$, let $\widetilde{M}$ be the universal covering of M. Notice that $\widetilde{M}$ inherits the metric and curvature in M, so its sectional curvature is $1/4$ pinched. Apply above lemma:

$$vol(\widetilde{M}) \geq C \left(\frac{\pi}{\sqrt{K}} \right)^3 \quad \text{for some constant } C$$

Since T is large enough, $\exists\ C'$ such that $K(t) \leq C' R_{\min}(t)$ for any $t \in [T, \infty)$; so for some other constant C

$$vol(\widetilde{M}) \geq C \tilde{R}_{\min}^{-3/2}(t) \quad \forall t \in [T, \infty).$$

But we know $vol(\widetilde{M}) = |\pi_1(M)| vol(M) = |\pi_1(M)|$ (we assume $vol(M) = 1$ before). We use Bonnet-Myers theorem to $(M, \tilde{g}(0))$, we know $\pi_1(M)$ is finite (normalized Ricci flow just changes metric but not topology). Hence

$$C \tilde{R}_{\min}^{-3/2}(t) \leq |\pi_1(M)| < \infty \quad \forall t \in [T, \infty).$$

Finally, we can find a constant $\epsilon > 0$ such that $R_{\min} \geq \epsilon > 0$ for all $0 \leq t < \infty$. □

In conclusion, we know:

Theorem 7.1.10. *The normalized Ricci flow has a solution on* $0 \leq t < \infty$ *with*

$$0 < \epsilon \leq \tilde{R}_{\min}(t) \leq \tilde{R}_{\max}(t) \leq C \quad \forall t \in [0, \infty);$$

$$\epsilon \tilde{R} \tilde{g}_{ij} \leq \tilde{R}_{ij} \leq \tilde{R} \tilde{g}_{ij} \quad \forall t \in [0, \infty);$$

$$\tilde{R}_{\max}/\tilde{R}_{\min} \to 1 \quad \textit{and} \quad \tilde{S}/\tilde{R}^2 - \frac{1}{3} \to 0 \quad \textit{as} \quad t \to \infty$$

7.2 Exponential convergence

In this subsection, our goal is to show: $\tilde{g}(\tilde{t})$ will converge to a smooth metric with constant positive curvature under the variation of normalized Ricci flow. If we can do it, then the main theorem in Section 1 could have been proved. It is shown that the method used here is similar to what we did in Section 6.3. The only difference is that the estimates here are done in normalized Ricci flow. So we need some normalized evolution equations at first. That is where we begins the proof.

Let ϵ C has the same meaning as in Theorem 7.1.10. Let $\mathscr{C}$ be a generic constant here.

Suppose M^3 is a closed Riemannian manifold. Let P and Q be two expressions formed from the metric and curvature tensors under the flow

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij};$$

and let $\tilde{P}$ and $\tilde{Q}$ be the corresponding expressions under the normalized flow

$$\frac{\partial}{\partial t} g_{ij} = \frac{2}{3} r g_{ij} - 2R_{ij}.$$

We say P has degree n if $\tilde{P} = \psi^n P$ under parameter t.

Lemma 7.2.1. *Suppose P satisfies*

$$\frac{\partial P}{\partial t} = \Delta P + Q$$

for the unnormalized equation, and P has degree n. Then Q has degree $n-1$ and for the normalized equation

$$\frac{\partial \tilde{P}}{\partial \tilde{t}} = \tilde{\Delta}\tilde{P} + \tilde{Q} + \frac{2}{3}\tilde{r}\tilde{P}.$$

Proof. Since $\frac{d\tilde{t}}{dt} = \psi$ and $\tilde{g}_{ij} = \psi g_{ij}$, we have

$$\frac{\partial}{\partial t} = \psi \frac{\partial}{\partial \tilde{t}}$$

and

$$\tilde{\Delta} = \tilde{g}^{ij}\tilde{\nabla}_i\tilde{\nabla}_j = \psi^{-1}g^{ij}\nabla_i\nabla_j = \psi^{-1}\Delta.$$

It has provide one ψ before unnormalized equation, so Q has degree $n-1$. Then

$$\psi\frac{\partial}{\partial \tilde{t}}\psi^{-n}\tilde{P} = \psi\tilde{\Delta}(\psi^{-n}\tilde{P}) + \psi^{-n+1}\tilde{Q}$$

$$-n\psi^{-n}\frac{\partial \psi}{\partial \tilde{t}}\tilde{P} + \psi^{-n+1}\frac{\partial \tilde{P}}{\partial \tilde{t}} = \psi^{-n+1}\tilde{\Delta}\tilde{P} + \psi^{-n+1}\tilde{Q}$$

$$\frac{\partial \tilde{P}}{\partial \tilde{t}} = \tilde{\Delta}\tilde{P} + \tilde{Q} + n\frac{1}{\psi}\frac{\partial \psi}{\partial \tilde{t}}\tilde{P}$$

Recall that $1 = \int d\tilde{\mu} = \psi^{3/2}\int d\mu$. Differentiate both sides:

$$0 = \frac{3}{2}\psi^{1/2}\frac{\partial \psi}{\partial t}\int d\mu - \psi^{3/2}\int R d\mu$$

$$0 = \frac{3}{2}\psi^{1/2}\frac{\partial \psi}{\partial t}\int d\mu - \psi^{3/2}r\int d\mu$$

$$\frac{1}{\psi}\frac{\partial \psi}{\partial t} = \frac{2r}{3}$$

$$\frac{1}{\psi}\frac{\partial \psi}{\partial \tilde{t}} = \frac{1}{\psi^2}\frac{\partial \psi}{\partial t} = \frac{2r}{3}\frac{1}{\psi} = \frac{2\tilde{r}}{3}$$

This prove the lemma. □

With this transform formula, it is not difficult to prove exponential convergence of geometric qualities.

Lemma 7.2.2. $\exists$ *constants* $C < \infty$ *and* $\delta > 0$ *such that*

$$\tilde{S} - \frac{1}{3}\tilde{R}^2 \leq \mathscr{C} e^{-\delta\tilde{t}}$$

Proof. Let $f = S/R^2 - \frac{1}{3}$, $\tilde{f} = \tilde{S}/\tilde{R}^2 - \frac{1}{3}$. Note that f has degree 0. By Lemma 5.3.5 with $\gamma = 2$ we have

$$\frac{\partial \tilde{f}}{\partial \tilde{t}} = \tilde{\Delta}\tilde{f} + \frac{2}{\tilde{R}}\tilde{g}^{pq}\nabla_p \tilde{R}\nabla_q \tilde{f} - \frac{4\tilde{P}}{\tilde{R}^3}$$

By Lemma 5.3.7 we have

$$\tilde{P} \geq \epsilon^2 \tilde{S}(\tilde{S} - \frac{1}{3}\tilde{R}^2) \geq \frac{1}{3}\epsilon^2 \tilde{R}^2(\tilde{S} - \frac{1}{3}\tilde{R}^2)$$

$$\frac{4\tilde{P}}{\tilde{R}^3} \geq \frac{4}{3}\epsilon^3 f$$

Now let $\delta = \frac{4}{3}\epsilon^3$, $\tilde{u}^q = 2\tilde{g}^{pq}\nabla_q \tilde{R}/\tilde{R}$ we have

$$\frac{\partial \tilde{f}}{\partial \tilde{t}} \leq \tilde{\Delta}\tilde{f} + \tilde{u}^k \nabla_k \tilde{f} - \delta \tilde{f}$$

Thus we get

$$\begin{aligned}
\frac{\partial}{\partial \tilde{t}}(e^{\delta\tilde{t}}\tilde{f}) &\leq \delta e^{\delta\tilde{t}}\tilde{f} + e^{\delta\tilde{t}}\left(\tilde{\Delta}\tilde{f} + \tilde{u}^k\nabla_k\tilde{f} - \delta\tilde{f}\right) \\
&= e^{\delta\tilde{t}}\tilde{\Delta}\tilde{f} + e^{\delta\tilde{t}}\tilde{u}^k\nabla_k\tilde{f} \\
&= \tilde{\Delta}(e^{\delta\tilde{t}}\tilde{f}) + \tilde{u}^k\nabla_k(e^{\delta\tilde{t}}\tilde{f})
\end{aligned}$$

Then by maximum principle $\exists\, \mathscr{C} \in \mathbb{R}$ such that $e^{\delta\tilde{t}}\tilde{f} \leq \mathscr{C}$. i.e. $\tilde{f} \leq \mathscr{C}e^{-\delta\tilde{t}}$. Recall that $\tilde{R}$ has upper and positive lower bound, the lemma follows. $\square$

Corollary 7.2.3. $|\tilde{R}_{ij} - \frac{1}{3}\tilde{R}\tilde{g}_{ij}| \leq \mathscr{C}e^{-\delta\tilde{t}}$. *In particular,* $\tilde{R}c$ *is uniformly bounded on* $[0, \infty)$.

Proof. Take eigenvalues $\tilde{\lambda}, \tilde{\mu}, \tilde{v}$ of the $\tilde{R}_{ij}$ then

$$|\tilde{R}_{ij} - \frac{1}{3}\tilde{R}\tilde{g}_{ij}|^2 = \left(\tilde{\lambda} - \frac{1}{3}(\tilde{\lambda} + \tilde{\mu} + \tilde{v})\right)^2 \leq \frac{2}{9}[(\tilde{\lambda} - \tilde{\mu})^2 + (\tilde{\lambda} - \tilde{v})^2].$$

In comparison,

$$\tilde{S} - \frac{1}{3}\tilde{R}^2 = \frac{1}{3}[(\tilde{\lambda} - \tilde{\mu})^2 + (\tilde{\lambda} - \tilde{v})^2 + (\tilde{\mu} - \tilde{v})^2]$$

The estimate follows by Lemma 7.2.2. □

Lemma 7.2.4. *$\exists$ constant $\mathscr{C} < \infty$ and $\delta > 0$ such that*

$$\tilde{R}_{\max} - \tilde{R}_{\min} \leq \mathscr{C} e^{-\delta \tilde{t}}.$$

Proof. Let $F = |\nabla_i R|^2/R + 168(S - \frac{1}{3}R^2)$. Then F has degree -2. From Lemma 5.4.11 (and its proof) with $\eta = 0$ we get

$$\frac{\partial}{\partial t}F \leq \Delta F + 672R(S - \frac{1}{3}R^2)$$

$$\frac{\partial}{\partial \tilde{t}}\tilde{F} \leq \tilde{\Delta}\tilde{F} + 672\tilde{R}(\tilde{S} - \frac{1}{3}\tilde{R}^2) - \frac{4}{3}\tilde{r}\tilde{F}$$

The estimates in Lemma 7.1.4, Lemma 7.1.9 and Lemma 7.2.2 tell that we can find some constant $C < \infty$, $\delta > 0$ and $\epsilon > 0$ such that

$$\frac{\partial}{\partial \tilde{t}} \leq \tilde{\Delta}\tilde{F} + Ce^{-\delta\tilde{t}} - \epsilon\tilde{F}$$

$$\frac{\partial}{\partial \tilde{t}}(e^{\delta\tilde{t}}\tilde{F} - C\tilde{t}) \leq \tilde{\Delta}(e^{\delta\tilde{t}}\tilde{F} - C\tilde{t}).$$

The maximum principle gives that $\exists$ constant $\mathscr{C}$ such that

$$e^{\delta\tilde{t}}\tilde{F} - C\tilde{t} \leq \mathscr{C} \quad \forall \tilde{t} \in [0, \infty)$$

$$\tilde{F} \leq (\mathscr{C} + C\tilde{t})e^{-\delta\tilde{t}}.$$

The exponential function decays faster than linear function, so taking a slightly smaller δ we have

$$|\nabla_i \tilde{R}|^2/\tilde{R} \leq \mathscr{C} e^{-\delta\tilde{t}}.$$

$\tilde{R}$ has upper bound:

$$|\nabla_i \tilde{R}| \leq \mathscr{C} e^{-\delta\tilde{t}}$$

Recall Lemma 7.1.4: let $\beta > 0$ such that $\tilde{R}c \geq 2\beta^2 \tilde{R}_{\min}\tilde{g}$ and $2\beta^2 = \epsilon$; $\tilde{L}$ is the diameter of M^3. Then in the proof of Lemma 7.1.4 we show

$$\tilde{L} \leq \frac{\pi}{\beta\sqrt{\tilde{R}_{\min}}} = \frac{\pi\sqrt{2}}{2\beta^2},$$

and

$$|\tilde{R}_{\max} - \tilde{R}_{\min}| \leq |\nabla_i \tilde{R}|\tilde{L} \leq \mathscr{C}\frac{\pi\sqrt{2}}{2\beta^2}e^{-\delta\tilde{t}}$$

□

Corollary 7.2.5. $|\tilde{R}_{ij} - \frac{1}{3}\tilde{r}\tilde{g}_{ij}| \leq \mathscr{C} e^{-\delta\tilde{t}}$

Proof. There is

$$\begin{aligned} |\tilde{R}_{ij} - \frac{1}{3}\tilde{r}\tilde{g}_{ij}| &\leq |\tilde{R}_{ij} - \frac{1}{3}\tilde{R}\tilde{g}_{ij}| + |\frac{1}{3}\tilde{R}\tilde{g}_{ij} - \frac{1}{3}\tilde{r}\tilde{g}_{ij}| \\ &= |\tilde{R}_{ij} - \frac{1}{3}\tilde{R}\tilde{g}_{ij}| + |\tilde{R} - \tilde{r}|/\sqrt{3}. \end{aligned}$$

The proof follows by Corollary 7.2.3 and Lemma 7.2.4. □

Theorem 7.2.6. *The metrics $\tilde{g}_{ij}(\tilde{t})$ are all equivalent, and converge as $\tilde{t} \to \infty$ uniformly to a continuous positive-definite metric $\tilde{g}(\infty)$.*

Proof. Observe that

$$\int_0^\infty \max_M |\tilde{g}'_{ij}| d\tilde{t} = 2\int_0^\infty \max_M |\tilde{R}_{ij} - \frac{1}{3}\tilde{r}\tilde{g}| d\tilde{t} \leq 2\int_0^\infty \mathscr{C} e^{-\delta\tilde{t}} d\tilde{t} = 2\mathscr{C}/\delta < \infty,$$

so this theorem follows by Lemma 6.3.2. □

The corollary 6.2.5 gives that

$$\frac{d}{dt}\int_M |\nabla^n Rc|^2 d\mu + 2\int_M |\nabla^{n+1} Rc|^2 d\mu \le \mathscr{C} \max_M |Rc| \int_M |\nabla^n Rc|^2 d\mu.$$

In the case of normalized Ricci flow, we have

$$\frac{d}{dt}\int_M |\tilde{\nabla}^n \tilde{Rc}|^2 d\tilde{\mu} + 2\int_M |\tilde{\nabla}^{n+1} \tilde{Rc}|^2 d\tilde{\mu} \le \mathscr{C} \max_M |\tilde{Rc}| \int_M |\tilde{\nabla}^n \tilde{Rc}|^2 d\tilde{\mu}.$$

Furthermore, since Rc is bounded:

$$\frac{d}{dt}\int_M |\tilde{\nabla}^n \tilde{Rc}|^2 d\tilde{\mu} + 2\int_M |\tilde{\nabla}^{n+1} \tilde{Rc}|^2 d\tilde{\mu} \le \mathscr{C}| \int_M |\tilde{\nabla}^n \tilde{Rc}|^2 d\tilde{\mu}.$$

Let's define a now tensor $\tilde{E}$=$\tilde{E}_{ij}$ by

$$\tilde{E}_{ij} = \tilde{R}_{ij} - \frac{1}{3}\tilde{r}\tilde{g}_{ij}.$$

An observation is that $\forall\ n \in \mathbb{Z}_{>0}$ we have

$$\tilde{\nabla}^n \tilde{E} = \tilde{\nabla}^n \tilde{Rc}. \qquad (\tilde{r} \text{ is a constant})$$

Apply Corollary 6.1.7 to the tensor $\tilde{E}$: $\forall\ n \in \mathbb{Z}_{>0}$

$$\int |\tilde{\nabla}^n \tilde{Rc}|^2 d\tilde{\mu} \le \mathscr{C} \left\{ \int |\tilde{\nabla}^{n+1} \tilde{Rc}|^2 d\tilde{\mu} \right\}^{n/(n+1)} \left\{ \int |\tilde{E}|^2 d\tilde{\mu} \right\}^{1/(n+1)}$$

Lemma 7.2.7. *For every n, $\exists$ a constant C_n such that*

$$\int_M |\tilde{\nabla}^n \tilde{Rc}|^2 d\tilde{\mu} \le C_n$$

Proof. Let $A_n = \int |\tilde{\nabla}^n \tilde{Rc}|^2 d\tilde{\mu}$, $B = \int |\tilde{E}|^2 d\tilde{\mu}$. Then

$$\begin{aligned} \frac{d}{d\tilde{t}} A_n &\le -2A_{n+1} + \mathscr{C} A_n \\ &\le -2A_{n+1} + \mathscr{C} A_{n+1}^{n/(n+1)} B^{1/(n+1)} \\ &\le -2A_{n+1} + \mathscr{C}\eta A_{n+1} + \mathscr{C}\eta^{-n} B, \qquad (\forall\ \eta > 0) \end{aligned}$$

where the third inequality is because $t^n \leq t^{n+1} + 1 \ \forall\ \eta \geq 0$ and let $t = \eta A_{n+1}/B$ then $\forall\ \eta > 0$ we have

$$A_{n+1}^n B \leq \eta A_{n+1}^{n+1} + \eta^{-n} B$$

When η is so small that $\mathscr{C}\eta \leq 2$, then

$$\frac{d}{d\tilde{t}} A_n \leq \mathscr{C} B$$

i.e.

$$\frac{d}{d\tilde{t}} \int |\tilde{\nabla}^n \tilde{R}c|^2 d\tilde{\mu} \leq \mathscr{C} \int |\tilde{E}|^2 d\tilde{\mu} \leq \mathscr{C} e^{-\delta\tilde{t}} \qquad \text{(for some } \delta > 0\text{)}$$

Then the lemma follows. □

Lemma 7.2.8. *For every $n, p \in \mathbb{N}_+$ we have*

$$\int |\tilde{\nabla}^n \tilde{R}c|^p d\tilde{\mu} \leq \mathscr{C} e^{-\delta\tilde{t}}$$

for $\mathscr{C}$ and $\delta > 0$ depending only on n and p.

Proof. By Corollary 6.1.6, $\exists$ constant $\mathscr{C}$ such that $\forall\ 1 \leq n \leq N$

$$\int |\tilde{\nabla}^n \tilde{R}c|^{2N/n} d\tilde{\mu} \leq \mathcal{C} \max_M |\tilde{E}|^{2(N/n-1)} \int |\tilde{\nabla}^N \tilde{R}c|^2 d\tilde{\mu}$$

Take $N = np$:

$$\int |\tilde{\nabla}^n \tilde{R}c|^{2p} d\tilde{\mu} \leq \mathcal{C} \max_M |\tilde{E}|^{2(p-1)} \int |\tilde{\nabla}^N \tilde{R}c|^2 d\tilde{\mu}.$$

The RHS converges exponentially as shown before. The lemma follows. □

Theorem 7.2.9. *For every $n \in \mathbb{N}_+$ we have*

$$\|\tilde{\nabla}^n \tilde{R}c(\tilde{t})\|_\infty \leq \mathscr{C} e^{-\delta\tilde{t}}$$

for some constant $\mathscr{C} < \infty$, $\delta > 0$ depending on n.

Proof. The argument is similar to the proof in Lemma 6.3.4: let $\tilde{f} = |\tilde{\nabla}^n \tilde{R}c|^{2p}$, $3 < p < \infty$. The Sobolev's inequality gives

$$\|\tilde{\nabla}^n \tilde{R}c\|_\infty^{2p} = \|\tilde{f}\|_\infty \leq \mathscr{C}(\tilde{t}) \int \{|\tilde{f}| + |\nabla \tilde{f}|\} d\tilde{\mu},$$

where $\mathscr{C}(\tilde{t})$ is also uniformly bounded by Theorem 7.2.6. Thus $\|\tilde{\nabla}^n \tilde{R}c\|_\infty$ is exponentially decreasing. □

Corollary 7.2.10. *For every $n \in \mathbb{N}_+$ we have*

$$\|\tilde{\nabla}^n \tilde{r}(\tilde{t})\|_\infty \leq \mathscr{C} e^{-\delta \tilde{t}}$$

for some constant $\mathscr{C} < \infty$, $\delta > 0$ depending on n.

Proof.

$$|\tilde{\nabla}^n \tilde{r}(\tilde{t})| \leq \int |\tilde{\nabla}^n \tilde{R}(\tilde{t})| d\tilde{\mu} \leq \int |\tilde{g}^{ij}(\tilde{t})| \|\tilde{\nabla}^n \tilde{R}_{ij}(\tilde{t})\|_\infty d\tilde{\mu}$$

Because $\tilde{g}(\tilde{t})$ is uniformly bounded,

$$|\tilde{\nabla}^n \tilde{r}(\tilde{t})| \leq \mathscr{C} \|Rc\|_\infty \leq \mathscr{C} e^{-\delta \tilde{t}}$$

is exponentially convergent. □

Corollary 7.2.11. *For all $n \in \mathbb{N}_+$,*

$$\|\partial^n \tilde{g}\|_\infty \leq \mathscr{C} e^{-\delta \tilde{t}}$$

for some constant $\mathscr{C} < \infty$, $\delta > 0$ depending on n.

Proof. The proof is similar to that of Lemma 6.3.5. □

Theorem 7.2.12. *As $\tilde{t} \to \infty$ the metrics $\tilde{g}_{ij}(\tilde{t})$ converge to the smooth limit metric $\tilde{g}_{ij}(\infty)$ in C^∞-topology. In special, the curvature $\tilde{R}_{ij}(\tilde{t})$ converge to the curvature $\tilde{R}_{ij}(\infty)$.*

Proof. By Weierstrass discriminance, $\forall\ \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$,

$$\int_0^\infty \frac{\partial^{|\alpha|}\tilde{g}_{ij}}{\partial x^\alpha}(p,\tilde{t})d\tilde{t} \quad \text{and} \quad \int_0^\infty \frac{\partial^{|\alpha|}\tilde{R}_{ij}}{\partial x^\alpha}(p,\tilde{t})d\tilde{t}$$

are uniformly convergent on $p \in M^3$. Thus $\forall$ fixed $\tilde{\tau} \in [0,\infty)$

$$\begin{aligned}\tilde{g}_{ij}(\infty) := \lim_{\tilde{t}\to\infty} \tilde{g}_{ij}(\tilde{t}) &= \tilde{g}_{ij}(\tilde{\tau}) + \int_{\tilde{\tau}}^\infty \frac{\partial \tilde{g}_{ij}}{\partial \tilde{t}} d\tilde{t} \\ &= \tilde{g}_{ij}(\tilde{\tau}) + \int_{\tilde{\tau}}^\infty \{\frac{2}{3}\tilde{r}\tilde{g}_{ij} - 2\tilde{R}_{ij}\} d\tilde{t}\end{aligned}$$

is differentiable in M^3, and

$$\frac{\partial^{|\alpha|}\tilde{g}_{ij}(\infty)}{\partial x^\alpha} = \frac{\partial^{|\alpha|}\tilde{g}_{ij}(\tilde{\tau})}{\partial x^\alpha} + \int_{\tilde{\tau}}^\infty \{\frac{2}{3}\tilde{r}\frac{\partial^{|\alpha|}\tilde{g}_{ij}}{\partial x^\alpha} - 2\frac{\partial^{|\alpha|}\tilde{R}_{ij}}{\partial x^\alpha}\} d\tilde{t}.$$

Hence

$$\frac{\partial^{|\alpha|}\tilde{g}_{ij}(\infty)}{\partial x^\alpha} = \lim_{\tilde{t}\to\infty} \frac{\partial^{|\alpha|}\tilde{g}_{ij}(\tilde{t})}{\partial x^\alpha}$$ □

Remark 7.2.13. *A family of smooth functions $h(t) : M^n \to \mathbb{R}$ converges to $h(T) : M \to \mathbb{R}$ in C^∞-topology as $t \to T$ means: $\forall\ \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ we have*

$$\frac{\partial^{|\alpha|}h(t)}{\partial x^\alpha} \to \frac{\partial^{|\alpha|}h(T)}{\partial x^\alpha} \quad \textit{uniformly w.r.t } M \textit{ as } t \to T.$$

Remark 7.2.14. *In the proof of above theorem, we use the lemma citing from [11] theorem 20.17 page 359.*

Corollary 7.2.15. *The limit metric $\tilde{g}_{ij}(\infty)$ has constant positive curvature.*

Proof. By Corollary 7.2.5:

$$\lim_{\tilde{t}\to\infty} \tilde{R}_{ij} - \frac{1}{3}\tilde{r}\tilde{g}_{ij} = 0$$

and $\tilde{r}(\infty) \geq \tilde{R}_{\min}(\infty) \geq \epsilon > 0$. Thus

$$\tilde{Rc}(\infty) = \frac{1}{3}\tilde{r}(\infty)\tilde{g}(\infty)$$

is a constant positive Ricci curvature. □

Up to now, we show M^3 with strictly positive Ricci curvature could be equipped with a Einstein metric such that $Rc = \frac{1}{3}rg$, $r > 0$. As shown in Proposition 2.1.9, M^3 has constant sectional curvature $\frac{1}{6}r > 0$. Theorem 2.5.1 shows that the universal covering of M^3 is $\mathbb{S}^3$, where the smooth structure on $\mathbb{S}^3$ is inherited from the standard smooth structure on Euclidean space $\mathbb{R}^4$. If M^3 is simply-connected, then M^3 is diffeomorphic to $\mathbb{S}^3$ with canonical Riemannian metric. This finishes the proof of main theorem.

Appendix A

Sobolev's inequality

In this appendix, we will prove the Sobelev's inequality of complete Riemannian manifold M^m. Let $C_0^\infty(M)$ be the set of smooth function with compact support in M; $W^{1,p}(M) = \{f \in L^1_{loc}(M) | f, \nabla f \text{ are } L^p \text{ measurable}\}$, $1 \le p \le \infty$. Let ω^n be the area of unit sphere S^n.

Theorem. *Let M^m be a complete Riemannian manifold with injectivity radius $\delta_0 > 0$ and sectional curvature K satisfying the bound $K \le b^2$. If $p > m$, there exists a constant $C(p)$ depending only on p such that for all $f \in C_0^\infty(M)$:*

$$\|f\|_\infty \le C(p)\|f\|_{W^{1,p}} = C(p)(\int |f|^p + |\nabla f|^p d\mu)^{1/p}$$

Proof. Let $\varphi(p) \in C^\infty(\mathbb{R})$ which satisfies: $\varphi(t) = 1$ in a neighbourhood of 0; $\varphi(t) = 0$ for $t \ge \delta$, $\delta \le \min\{\delta_0, \frac{\pi}{2b}\}$. Let x be a given point of M then under normal polar coordinate (r, θ)

$$f(x) = f(0,\theta)\varphi(0) - f(\delta,\theta)\varphi(\delta) = -\int_0^\delta \partial_r(f(r,\theta)\varphi(r))dr$$

$$|f(x)| \le \int_0^\delta |\nabla[f(r,\theta)\varphi(r)]|dr$$

Let $d\sigma = r^{m-1} dr d\theta$

$$
\begin{aligned}
\omega^{m-1}|f(x)| &\leq \int_{B_\delta(x)} |\nabla[f(r,\theta)\varphi(r)]| r^{1-m} d\sigma \\
&\leq \left(\int_{B_\delta(x)} |\nabla[f(r,\theta)\varphi(r)]|^p d\sigma \right)^{1/p} \left(\int_{B_\delta(x)} r^{(1-m)q} d\sigma \right)^{1/q} \\
&=: I_1 I_2 \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1
\end{aligned}
$$

I_1: $g = drdr + g_{\theta^i\theta^i} d\theta^i d\theta^i$, $2 \leq i \leq m$. Because $K \leq b^2$,

$$
\det g = \prod g_{\theta^i\theta^i} \geq \left(\frac{\sin br}{b} \right)^{2(m-1)}.
$$

If $r \leq \pi/2b$, then $\frac{br}{\sin br} \geq \frac{2}{\pi}$

$$
r^{m-1} dr d\theta = \frac{r^{m-1}}{\sqrt{\det g}} d\mu \leq \left(\frac{br}{\sin br} \right)^{m-1} d\mu \leq \left(\frac{\pi}{2} \right)^{m-1} d\mu
$$

$$
I_1 \leq \left(\frac{\pi}{2} \right)^{(m-1)/p} \|\nabla f \cdot \varphi + f \cdot \nabla \varphi\|_{L^p(B_\delta(x))}
$$

I_2:

$$
\begin{aligned}
I_2 &= (\omega^{m-1})^{1/q} \left(\int_0^\delta r^{(m-1)(1-q)} dr \right)^{1/q} \\
&= (\omega^{m-1})^{1/q} \left(\frac{p-1}{p-m} \delta^{\frac{p-m}{p-1}} \right)^{1/q}
\end{aligned}
$$

Thus,

$$
|f(x)| \leq C(\omega, \delta_0, \varphi, \nabla\varphi) \|f\|_{W^{1,p}}
$$

where C is a constant depending only on p finally. □

Appendix B

Poincare conjecture

The basic usage of Ricci flow is to solve three dimensional smooth Poincare conjecture:

Theorem B.0.1 (3-dimensional smooth Poincare conjecture)**.** *If Σ is a simply-connected closed smooth manifold of dimension 3, then Σ and the 3-sphere $\mathbb{S}^3$ are diffeomorphic.*

This conjecture is solved in 2003 by Grigori Perelman. The main theorem of this article is just a special case of the theorem.

Historically, Poincare first conjectured that if a 3-manifold Σ has the same homology groups as $\mathbb{S}^3$, then Σ and $\mathbb{S}^3$ are homeomorphic. However, Poincare discovered an example Σ_P, called the Poincare homology sphere, such that Σ_P and $\mathbb{S}^3$ are not homeomorphic. Thus, Poincare added the requirement that Σ should be simple-connected. The **Whitehead** theorem could be used to generalize this statement.

Theorem (Whitehead theorem)**.** *A map $f : X \to Y$ between simply-connected CW complexes is a homotopy equivalence if $f_* : H_n(X;\mathbb{Z}) \to H_n(Y;\mathbb{Z})$ is an isomorphism for each n.*

Proposition B.0.2. *Let Σ^n be a simply-connected smooth manifold with the same homology group as $\mathbb{S}^n$, $n \geq 2$, then Σ is homotopic to $\mathbb{S}^n$.*

Proof. Since Σ^n is smooth, by Morse's function theorem, Σ^n has a CW-structure. Let Σ^k be the k-skeleton of the CW-structure of Σ^n. Consider the homology sequence of the cofibration

$$\Sigma^{n-1} \to \Sigma^n \to \mathbb{S}^n,$$

we have isomorphism $q_* : H_i(\Sigma; \mathbb{Z}) \to H_i(\mathbb{S}^n; \mathbb{Z})$, $\forall i \geq 0$. Hence, Σ is homotopic to $\mathbb{S}^n$. □

Thus, the key point here is homotopic-equivalence between Σ^3 and $\mathbb{S}^3$. Many different versions of Poincare conjecture is based on the assumption of homotopic-equivalence:

Theorem B.0.3 (Higher dimensional $n \geq 5$)**.** *If Σ^n $(n \geq 5)$ is a smooth n-manifold homotopic-equivalent to $\mathbb{S}^n$, then Σ^n is homeomorphic to $\mathbb{S}^n$.*

The higher dimensional Poincare conjecture is proved by Stephen Smale in 1961. The basic tool is the **h-cobordism** theorem.

Definition B.0.4. *Let W^{n+1} be the cobordism of M^n and N^n where W, M, N are all smooth manifolds. M and N is called a h-cobordism if W is homotoptic equivalent to the trivial cobordism $M \times [0, 1]$.*

Theorem B.0.5 (h-cobordism theorem)**.** *Let M^m and N^m be compact simply-connected oriented smooth n-manifolds that are h-cobordant through the simply-connected (n+1)-manifold W^{n+1}. If $n \geq 5$, then there is a diffeomorphism*

$$W \cong M \times [0, 1],$$

which can be chosen to be the identity from $M \subset W$ to $M \times 0 \subset M \times [0, 1]$. In particular, M and N are diffeomorphic.

Furthermore, when $n = 5, 6$, one can show Σ^n is diffeomorphic to $\mathbb{S}^n$. When $n = 7$, Milnor's exotic sphere shows that the homeomorphism could not be improved to diffeomorphism.

Smale's proof fails when $n = 3, 4$ because it needs Whitney's trick to cancel intersection points. The Whitney's trick requires us to find an embedding disk in the manifolds M^n. Also, the embedding map $f : P^m \to Q^{2m+1}$ is dense in all differentiable map from P to Q. Thus this embedding is promised when $n \geq 5$ and it fails in lower dimensions.

The story in dimension 3 has been claimed before.

In dimension 4, the topological Poincare conjecture is solved by proving a topological 4-dimensional h-cobordism theotrem. This work is done by Casson, Wall and Freedman.

Theorem B.0.6 (Wall's Theorem on h-cobordisms)**.** *If M and N are smooth, simply-connected, and have isomorphic intersection forms, then M and N must be h-cobordant.*

Theorem B.0.7 (topological 4-dimensional h-cobordism theotrem)**.** *Let M^4 and N^4 be compact simply-connected oriented smooth 4-manifolds that are h-cobordant through the simply-connected 5-manifold W^5. Then there is a homeomorphism*

$$W \cong M \times [0, 1],$$

which can be chosen to be the identity from $M \subset W$ to $M \times 0 \subset M \times [0, 1]$. In particular, M and N are homeomorphic.

Theorem B.0.8 (topological 4-dimensional Poincare conjecture)**.** *If Σ^4 is a smooth 4-manifold homotopic-equivalent to $\mathbb{S}^4$, then Σ^4 is homeomorphic to $\mathbb{S}^4$.*

Remark B.0.9. *This result can be generalized to the case that Σ^4 is just a topological 4-manifold.*

However, the smooth version of 4-dimensional Poincare conjecture is still open up to now:

Conjecture B.0.10. *If Σ^4 is a smooth 4-manifold homotopic-equivalent to $\mathbb{S}^4$, then Σ^4 is diffeomorphic to $\mathbb{S}^4$.*

Bibliography

[1] Comparison methods in riemannian geometry. https://www.math.u-psud.fr> breuilla> Essayb.

[2] Thierry Aubin. *Some nonlinear problems in Riemannian geometry*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

[3] Jeff Cheeger and David G. Ebin. *Comparison theorems in Riemannian geometry*. North-Holland Publishing Co., Amsterdam-Oxford; American Elsevier Publishing Co., Inc., New York, 1975. North-Holland Mathematical Library, Vol. 9.

[4] Bennett Chow and Dan Knopf. *The Ricci flow: an introduction*, volume 110 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2004.

[5] Bennett Chow, Peng Lu, and Lei Ni. *Hamilton's Ricci flow*, volume 77 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI; Science Press Beijing, New York, 2006.

[6] Manfredo Perdigao do Carmo. *Riemannian geometry*. Mathematics: Theory and Applications. Birkhauser Boston, Inc., Boston, MA, 1992. Translated from the second Portuguese edition by Francis Flaherty.

[7] Richard S. Hamilton. Three-manifolds with positive Ricci curvature. *J. Differential Geometry*, 17(2):255–306, 1982.

[8] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[9] Alexandru Scorpan. *The wild world of 4-manifolds.* American Mathematical Society, Providence, RI, 2005.

[10] 伍鸿熙，沈纯理，虞言林. 黎曼几何初步. 高等教育出版社, 1985.

[11] 常庚哲，史济怀. 数学分析教程. 高等教育出版社, 2002.

[12] 梅加强. 流形与几何初步. 科学出版社, 2013.

[13] 陈维桓，李兴校. 黎曼几何引论. 北京大学出版社, 2002.