

# Gromov-Witten theory and mirror symmetry

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# Chapter 1

## Gromov-Witten invariants

Roughly speaking, Gromov-Witten invariants is the number of algebraic curves (resp. pseudo-holomorphic curves) of a algebraic variety  $X$  (resp. complex manifold) passing through specific subvarieties (resp. submanifolds) under specific degree (homology class of the curves) and genus.

**Definition 1.0.1.** Let  $\gamma_1, \dots, \gamma_n \in H^*(X; \mathbb{Q})$  and let  $\beta \in H^2(X; \mathbb{Q})$ . The Gromov-Witten invariant of genus  $g$  degree  $\beta$  curves is

$$\langle \gamma_1 \dots \gamma_n \rangle_{g,n,\beta} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} ev_1^*(\gamma_1) \cup \dots \cup ev_n^*(\gamma_n).$$

Here, a point in  $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{vir}$  is  $[f : C \rightarrow X, 1, \dots, n]$ :

a map from the genus  $g$  curve  $C$  to the variety  $X$  modulo the automorphism of  $C$ .

The evaluation map  $ev_i : [\overline{\mathcal{M}}_{g,n}(X, \beta)]^{vir} \rightarrow X$  is given by

$$ev_i([f : C \rightarrow X, 1, \dots, n]) = f(i).$$

Let  $\overline{\mathcal{M}}_{g,n}$  denote the moduli space (Deligne-Mumford stack) of genus  $g$  curves with  $n$  marked points, and let  $\overline{\mathcal{C}}_{g,n}$  be the universal family of  $\overline{\mathcal{M}}_{g,n}$ .

### 1.1 Kontsevich's approach

Atiyah-Bott localization formula tells us: if there is a torus action  $\mathbb{T} = (\mathbb{C}^*)^n$  on  $X$ , then the fixed points of torus action could tells us some properties of  $X$ .

By the classifying space theory,  $B\mathbb{T} = (\mathbb{C}P^\infty)^{\times n}$ , so  $H^*(B\mathbb{T}) = \mathbb{Q}[\lambda_1, \dots, \lambda_n]$ . Let  $\mathcal{R}_{\mathbb{T}} = \mathbb{Q}(\lambda_1, \dots, \lambda_n)$ . Let  $X_{\mathbb{T}} = E\mathbb{T} \times_{\mathbb{T}} X$ , the equivariant cohomology of  $X$  is defined by

$$H_{\mathbb{T}}^*(X) := H^*(X_{\mathbb{T}}) = H^*(E\mathbb{T} \times_{\mathbb{T}} X),$$

so naturally,  $H_{\mathbb{T}}^*(X)$  is a  $H_{\mathbb{T}}^*(pt) = H^*(B\mathbb{T})$ -module. The localization of  $H_{\mathbb{T}}^*(X)$  means  $H_{\mathbb{T}}^*(X) \otimes \mathcal{R}_{\mathbb{T}}$ .

**Theorem** (Atiyah-Bott). *Let  $X^{\mathbb{T}}$  be fixed locus of  $\mathbb{T}$ , let  $Z_j$  be a connection component of  $X^{\mathbb{T}}$ , and let  $N_j$  be the normal bundle of  $Z_j$  in  $X$ . Let  $i_j : Z_j \rightarrow X$  and let  $i_{j!} : H_{\mathbb{T}}^*(Z_j) \rightarrow H_{\mathbb{T}}^*(X)$  be the pushforward defined by the Gysin map. Let  $\alpha \in H_{\mathbb{T}}^*(X) \otimes \mathcal{R}_{\mathbb{T}}$ , we have*

$$\alpha = \sum_j \frac{i_{j!} i_j^* \alpha}{Euler_T(N_j)},$$

In particular,

$$\int_{X_{\mathbb{T}}} \alpha = \sum_j \int_{(Z_j)_{\mathbb{T}}} \frac{i_j^* \alpha}{Euler_T(N_j)}.$$

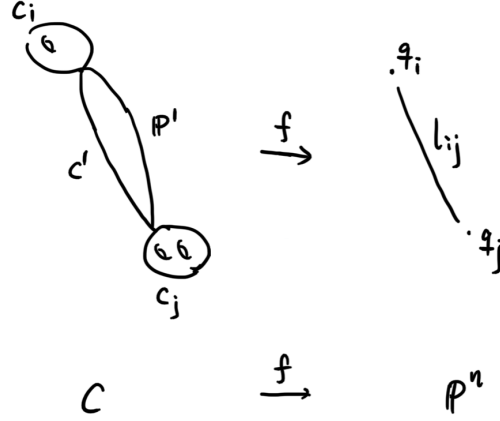
Kontsevich's approach is to apply Atiyah-Bott localization formula in  $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{vir}$  so that we can simplify the computation. We can lift the  $\mathbb{T}$  action on  $X$  to  $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{vir}$  in the following way: let  $t \in \mathbb{T}$ ,  $[f : C \rightarrow X, 1, \dots, n] \in [\overline{\mathcal{M}}_{g,n}(X, \beta)]^{vir}$ ,  $x \in X$

$$(t \cdot f)(x) = f(t \cdot x).$$

In special, we will consider  $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$  in this section. As claimed before, we need to find  $[f : C \rightarrow X, 1, \dots, n] \in \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^{\mathbb{T}}$ . The fixed points of  $\mathbb{P}^r$  is

$$\{q_i = [0 : 0 : \dots : 1 : 0 : \dots : 0]\}_{0 \leq i \leq r}.$$

The coordinate curve  $l_{ij}$  connecting  $q_i, q_j$  has one dimensional degree of freedom  $\mathbb{C}^*$  (as an invariant component). The curve  $C \in \overline{\mathcal{C}}_{g,n}$  is stable (i.e.  $\text{Aut}(C) < \infty$ ) if and only if  $2g - 2 + n > 0$ . If a components  $C'$  of  $C$  is mapped to  $l_{ij}$ , then  $C'$  has two points mapped to  $q_i, q_j$  respectively (equivalent to with two marked points in  $C'$ ), so  $2g - 2 + 2 \leq 0$  implies  $g = 0$ , i.e.  $C' \cong \mathbb{P}^1$  (see Fig 1.1). Meanwhile,  $f|_{C'}$  must be uniformly ramified, so  $f|_{C'}(z) = z^e, \forall z \in C' \cong \mathbb{P}^1$ , for some  $e \in \mathbb{N}^*$ .


 Figure 1.1:  $f(C_i) = q_i$ ,  $f(C') = l_{ij}$ ,  $f(C_j) = q_j$ 

It is convenient to use a decorated graph  $\vec{\Gamma} = (\Gamma, \vec{f}, \vec{d}, \vec{g}, \vec{s})$  (graph, maps, degrees, genus, marked points) to represent  $[f : C \rightarrow X, 1, \dots, n] \in \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^{\mathbb{T}}$ . Let  $\text{val}(v)$ , the valence of  $v$ , be the number of edges connecting vertex  $v$ , and let  $n(v) = |s_v| + \text{val}(v)$ . The stable map  $[f : C \rightarrow X, 1, \dots, n]$  with fixed graph  $\vec{\Gamma} = (\Gamma, \vec{f}, \vec{d}, \vec{g}, \vec{s})$  defines a substack

$$\overline{\mathcal{M}}_{\vec{\Gamma}} \subset \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d).$$

On the other sides, consider

$$\varphi_{\vec{\Gamma}} : \prod_{\dim C_v=1} \overline{\mathcal{M}}_{g_v, n(v)} \rightarrow \overline{\mathcal{M}}_{\vec{\Gamma}}.$$

If  $v, v'$  are connected by an edge  $e$ , then let  $C_v, C_{v'}$  connected by a  $C_e \cong \mathbb{P}^1$  associated with a degree  $d_e$  map to  $\mathbb{P}^r$ . Let  $\overline{\mathcal{M}}_{\Gamma}$  be the product of above  $C_v, C_e$ . There is a group  $\mathbb{A}_{\Gamma}$  acting on  $\overline{\mathcal{M}}_{\Gamma}$ . The group  $\mathbb{A}_{\Gamma}$  is defined by:

$$1 \rightarrow \prod_{\text{edges}} \mathbb{Z}/(d_e) \rightarrow \mathbb{A}_{\Gamma} \rightarrow \text{Aut}(\Gamma) \rightarrow 1.$$

and

$$\overline{\mathcal{M}}_{\vec{\Gamma}} = \overline{\mathcal{M}}_{\Gamma} / \mathbb{A}_{\Gamma}.$$

Therefore, we know the  $\mathbb{T}$ -fixed locus of  $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$  is  $\overline{\mathcal{M}}_{\vec{\Gamma}}$ . Let  $N_{\Gamma}$  be the normal bundle of  $\overline{\mathcal{M}}_{\vec{\Gamma}}$  in  $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ . Then there is an explicit formula for

the equivariant Euler class. Before doing that, we define some necessary notations. A flag  $F$  is a pair  $(v, e)$  such that  $e$  is an edge containing the vertex  $v$ . We put  $i(F) = v$ ,  $j(F)$  the vertex of  $e$  different from  $v$ . Let

$$\omega_F = \frac{\lambda_{i(F)} - \lambda_{j(F)}}{d_e} \in H_{\mathbb{T}}^2(pt) = \mathbb{Q}[\lambda_0, \dots, \lambda_r],$$

which is the weight of  $\mathbb{T}$ -action on  $T_{q_{i_v}} C_e$ .

**Theorem 1.1.1** ( $Euler_{\mathbb{T}}(N_{\Gamma})$ ).  $Euler_{\mathbb{T}}(N_{\Gamma}) = e_{\Gamma}^F e_{\Gamma}^v e_{\Gamma}^e$ , where

$$\begin{aligned} e_{\Gamma}^F &= \prod_{n(i(F)) \geq 3} (\omega_F - \psi_F) / \prod_{j \neq i(F)} (\lambda_{i(F)} - \lambda_j), \\ e_{\Gamma}^v &= \prod_v \prod_{j \neq i_v} (\lambda_{i_v} - \lambda_j) \prod_{val(v)=2, s_v=\emptyset} (\omega_{F_1(v)} + \omega_{F_2(v)}) / \prod_{val(v)=1, s_v=\emptyset} \omega_{F(v)} \\ e_{\Gamma}^e &= \prod_e \frac{(-1)^{d_e} (d_e!)^2 (\lambda_i - \lambda_j)^{2d_e}}{d_e^{2d_e}} \prod_{a+b=d_e, k \neq i, j} \left( \frac{a\lambda_i + b\lambda_j}{d_e} - \lambda_k \right) \end{aligned}$$

The proof is partially discussed in section 1.2.

## 1.2 Tangent-obstruction sequence

Consider  $[f : C \rightarrow X, 1, \dots, n] \in \overline{\mathcal{M}}_{\vec{\Gamma}}$ ,  $\vec{\Gamma} = (\Gamma, \vec{f}, \vec{d}, \vec{g}, \vec{s})$ . We put

$$V^1(\Gamma) := \{v \in V(\Gamma) : g_v = 0, val(v) = 1, |s_v| = 0\}$$

$$V^2(\Gamma) := \{v \in V(\Gamma) : g_v = 0, val(v) = 2, |s_v| = 0\}$$

$$V^{1,1}(\Gamma) := \{v \in V(\Gamma) : g_v = 0, val(v) = 1, |s_v| = 1\}$$

$$V^s(\Gamma) := \{v \in V(\Gamma) : 2g_v - 2 + val(v) + |s_v| > 0\}$$

$$y(v, e) := C_e \cap C_v$$

The tangent-obstruction sequence is

$$\begin{aligned} &0 \rightarrow Aut(C, 1, \dots, n) \\ &\rightarrow Def(f) \rightarrow Def(C, 1, \dots, n, f) \rightarrow Def(C, 1, \dots, n) \\ &\rightarrow Ob(f) \rightarrow Ob(C, 1, \dots, n, f) \rightarrow 0, \end{aligned}$$

$$\begin{aligned}
0 &\rightarrow \text{Hom}(\Omega_C(p_1 + \cdots + p_n), \mathcal{O}_C) \\
&\rightarrow H^0(C, f^*T_X) \rightarrow T^1 \rightarrow \text{Ext}^1(\Omega_C(p_1 + \cdots + p_n), \mathcal{O}_C) \\
&\rightarrow H^1(C, f^*T_X) \rightarrow T^2 \rightarrow 0.
\end{aligned}$$

For simplicity:

$$0 \rightarrow B_1 \rightarrow B_2 \rightarrow T^1 \rightarrow B_4 \rightarrow B_5 \rightarrow T^2 \rightarrow 0.$$

The  $N^{\text{vir}} = T^{1,m} - T^{2,m}$  (m means moving part).

$$Euler_{\mathbb{T}}(N^{\text{vir}}) = \frac{Euler_{\mathbb{T}}(B_2^m) Euler_{\mathbb{T}}(B_4^m)}{Euler_{\mathbb{T}}(B_1^m) Euler_{\mathbb{T}}(B_5^m)}.$$

(1)  $Euler_{\mathbb{T}}(B_2^m)/Euler_{\mathbb{T}}(B_5^m)$ . The normalization sequence of  $C$  is:

$$\begin{aligned}
0 &\rightarrow \mathcal{O}_C \rightarrow \bigoplus_{v \in V^s(\Gamma)} \mathcal{O}_{C_v} \oplus \bigoplus_{e \in E(\Gamma)} \mathcal{O}_{C_e} \\
&\rightarrow \bigoplus_{v \in V^2(\Gamma)} \mathcal{O}_{y_v} \oplus \bigoplus_{(e,v) \in F^s(\Gamma)} \mathcal{O}_{y(e,v)} \rightarrow 0.
\end{aligned}$$

Take  $\otimes f^*T_X$ :

$$\begin{aligned}
0 &\rightarrow H^0(C, f^*T_X) \rightarrow \bigoplus_{v \in V^s(\Gamma)} H^0(C_v, f^*T_X) \oplus \bigoplus_{e \in E(\Gamma)} H^0(C_e, f^*T_X) \\
&\rightarrow \bigoplus_{v \in V^2(\Gamma)} T_{f(y_v)}X \oplus \bigoplus_{(e,v) \in F^s(\Gamma)} T_{f(y(e,v))}X \\
&\rightarrow H^1(C, f^*T_X) \rightarrow \bigoplus_{v \in V^s(\Gamma)} H^1(C_v, f^*T_X) \oplus \bigoplus_{e \in E(\Gamma)} H^1(C_e, f^*T_X) \rightarrow 0.
\end{aligned}$$

$$H^0(C_v, f^*T_X) = T_{f(C_v)}X,$$

$$H^1(C_v, f^*T_X) = H^1(C_v, \mathcal{O}_{C_v}) \otimes T_{f(C_v)}X \cong H^0(C_v, \omega_{C_v})^\vee \otimes T_{f(C_v)}X$$

Here  $H^0(C_v, \omega_{C_v})$  is Hodge bundle  $\mathbb{E}$ . By splitting principle, assume  $\mathbb{E} = L_1 \oplus \cdots \oplus L_g$ , then

$$\begin{aligned}
e(\mathbb{E}^\vee \otimes \mathbb{C}_1) &= \prod_{i=1}^g c_1(L_i^\vee \otimes \mathbb{C}_1) = \prod_{i=1}^g c_1(L_i^\vee) + c_1(\mathbb{C}_1) \\
&= \prod_{i=1}^g (-c_1(L_i) + u) = \sum_{k=1}^g (-1)^k c_k(\mathbb{E}) u^{g-k} = \sum_{k=1}^g (-1)^k \lambda_k u^{g-k} =: \Lambda_g^\vee(u)
\end{aligned}$$

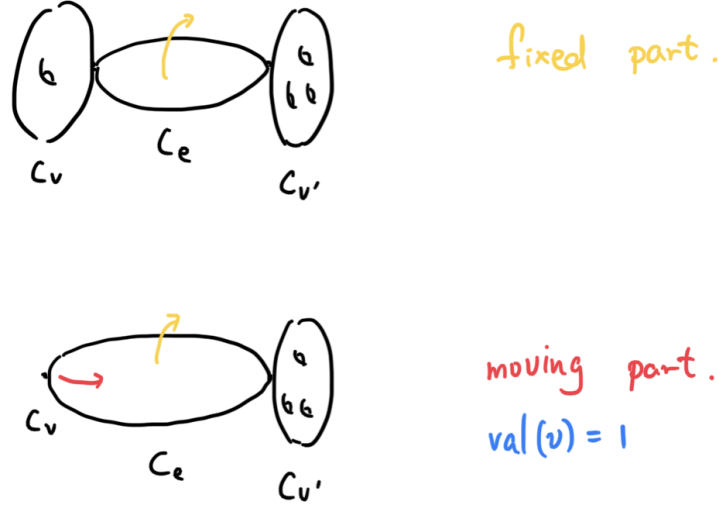

 Figure 1.2: automorphism of  $(C, 1, \dots, n)$ 

 Figure 1.3: deformation of  $(C, 1, \dots, n)$ 

(2)  $Euler_{\mathbb{T}}(B_4^m)/Euler_{\mathbb{T}}(B_1^m)$ .

(2.1)  $B_1 = Aut(C, 1, \dots, n) = Hom(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C)$ : We should classify what is moving and what is fixed. Basically, we have

$$B_1^m = \bigoplus_{v \in V^1(\Gamma), (e, v) \in F(\Gamma)} T_{y(e, v)} C_e.$$

(2.2)  $B_4 = Def(C, 1, \dots, n) = Ext^1(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C)$ :  $\mathbb{P}^1$  has just 1 complex structure, so we consider  $g(C) \geq 1$ . If we don't change node  $q$ ,  $C$  will stay in the same class in  $\overline{\mathcal{M}}_{g, n}$ . Hence we must resolve the node, and



geometrically, resolution depends on  $T_q C_v \otimes T_q C_e$ . So basically we have

$$B_4^m = \bigoplus_{v \in V^2(\Gamma), E_v = (e, e')} T_{y_v} C_e \otimes T_{y_v} C_{e'} \oplus \bigoplus_{(e, v) \in F^s(\Gamma)} T_{y(e, v)} C_v \otimes T_{y(e, v)} C_e$$

Returning to the special case  $X = \mathbb{P}^r$ , we can get the theorem 1.1.1.

### 1.3 Aspinwall Morrison formula; Faber Pandaripande formula

In this section, we will use Kontsevich's approach to compute the multiple cover contribution of rigidly embedding curves  $\mathbb{P}^1$  in a Calabi-Yau threefold  $X$ .

The geometry picture is this. The normal bundle  $N$  of  $\mathbb{P}^1 \subset X$  is rank 2 and splits on  $\mathbb{P}^1$ . Because  $X$  is Calabi-Yau and  $c_1(\mathbb{P}^1) = 2$ , the normal bundle is of degree 2. Embedded  $\mathbb{P}^1$ 's in a Calabi-Yau threefold (not necessary lines) with normal bundle  $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$  are called rigid. The degree 2 Gromov-Witten invariant of a generic quintic has two contributions:

- (1) rigid conics curves in  $X$ ;
- (2) lines with double cover, so this part is related to  $\overline{\mathcal{M}}_0(\mathbb{P}^1, 2)$ .

We want to compute the contribution of part (2). This problem finally leads to:

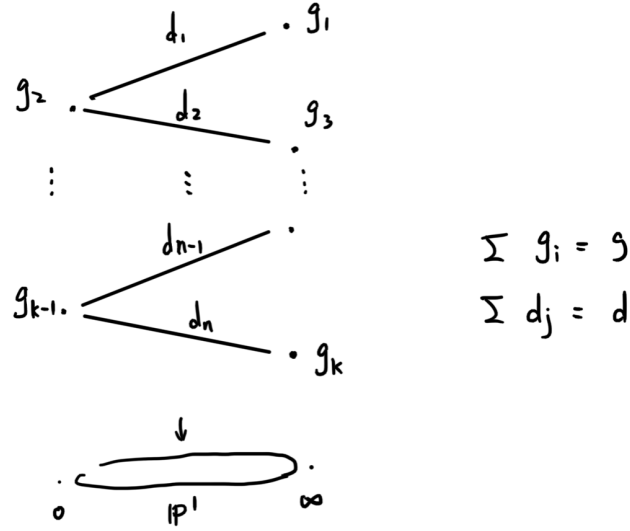
$$N_{g,d} = \int_{\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)} e(R^1 \pi_* f^* N),$$

$$\begin{array}{ccc} \overline{\mathcal{C}}_{g,0}(\mathbb{P}^1, d) & \xrightarrow{f} & \mathbb{P}^1 \\ \downarrow \pi & & \\ \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d) & & \end{array} \quad \text{and } N = \mathcal{O}(-1) \oplus \mathcal{O}(-1).$$

where

The decorated graphs  $\vec{\Gamma} = (\Gamma, \vec{f}, \vec{d}, \vec{g}, \vec{s})$  in  $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)^{\mathbb{T}}$  are of the type in Figure 1.4. We can choose different lifts on  $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$  so that only  $\vec{\Gamma} = (\Gamma, \vec{f}, \vec{d}, \vec{g}, \vec{s})$  with 1 edge contributing  $N_{g,d}$ .

- (1)  $g = 0$  (Aspinwall Morrison formula):  $N_{0,d} = 1/d^3$ ;


 Figure 1.4:  $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)^{\mathbb{T}}$ 

(2)  $g \geq 1$  (Faber-Pandharipande):

$$\begin{aligned}
 N_{g,d} &= \sum_{g_1+g_2=g} \frac{1}{d} \int_{\overline{\mathcal{M}}_{g_1,1}} \lambda_{g_1} \psi^{2g_1-2} d^{2g_1-1} \\
 &\quad \times \int_{\overline{\mathcal{M}}_{g_2,1}} \lambda_{g_2} \psi^{2g_2-2} d^{2g_2-1} = \sum_{g_1+g_2=g} b_{g_1} b_{g_2} d^{2g-3} \\
 b_0 &= 0; b_g = \int_{\overline{\mathcal{M}}_{g,1}} \lambda_{g_2} \psi^{2g-2} \quad (g > 0) \\
 \sum_{g=0}^{\infty} b_g t^{2g} &= \frac{t/2}{\sin t/2}.
 \end{aligned}$$

Then use the Laurent series of  $\cot t$ , we have

$$N_{1,d} = \frac{1}{12d},$$

$$N_{g,d} = d^{2g-3} \frac{|B_{2g}|}{2g \cdot (2g-2)!} = |\chi(\overline{\mathcal{M}}_g)| \frac{d^{2g-3}}{(2g-3)!}, \quad g \geq 2,$$

where  $B_g$  is the Bernoulli number in  $\frac{x}{e^x-1}$ .

# Chapter 2

## Quantum Cohomology

### 2.1 quantum product

The quantum cohomology is a variation of classical cohomology. Let  $T_0 = 1, T_1, \dots, T_p, T_{p+1}, \dots, T_m \in H^*(X)$  be a basis of  $H^*(X)$  as a  $\mathbb{Q}$ -vector space ( $T_1, \dots, T_p \in H^2(X)$ ). Let  $\beta \in H^2(X)$ ,  $\gamma = \sum_{i=0}^m t_i T_i$ , we define quantum potential as

$$F_0^X(t_0, \dots, t_m) = \sum_{n, \beta} \frac{1}{n!} \langle \gamma^n \rangle_{0, n, \beta}^X Q^\beta,$$

$$F_{ijk} := \frac{\partial^3 F_0^X}{\partial t_i \partial t_j \partial t_k} = \sum_{n, \beta} \frac{1}{n!} \langle T_i T_j T_k \gamma^n \rangle_{0, n+3, \beta}^X Q^\beta$$

Let  $g_{ij} = (T_i, T_j)$  means the Poincare pair of  $T_i, T_j$ . The big quantum product is defined as

$$(T_i *_t T_j, T_k) := F_{ijk},$$

in other words,

$$T_i *_t T_j = \sum_{e, f} F_{ije} g^{ef} T_f.$$

It is known that the quantum product is a generalization of intersection theory: given  $T_i, T_j, T_k$ , they contribute to the quantum product if there exists  $\mathbb{P}^1$  touching their Poincare dual classes at the same time.

The associativity of quantum product is formulated as WDVV equation:

$$F_{ija} g^{ab} F_{bkl} = F_{ila} g^{ab} F_{bjk}.$$

It is proved by a forgetful map  $\pi : \overline{\mathcal{M}}_{0,4}(X, \beta) \rightarrow \overline{\mathcal{M}}_{0,4}$ . One should notice that  $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$ , so the boundary divisor  $D(12|34) \sim D(13|24)$  and

$$\begin{aligned} & \int_{\overline{\mathcal{M}}_{0,4}(X, \beta)} ev_1^*(T_i) ev_2^*(T_j) ev_3^*(T_k) ev_4^*(T_l) \prod_{i=5}^{n+4} ev_i^*(\gamma) \pi^* D(12|34) \\ &= \int_{\overline{\mathcal{M}}_{0,4}(X, \beta)} ev_1^*(T_i) ev_2^*(T_j) ev_3^*(T_k) ev_4^*(T_l) \prod_{i=5}^{n+4} ev_i^*(\gamma) \pi^* D(13|24). \end{aligned}$$

A useful trick is to separate  $\overline{\mathcal{M}}_{0,4}(X, \beta) \cap \pi^* D(12|34)$  by

$$[\overline{\mathcal{M}}_{0,n_1+3}(X, \beta_1) \times \overline{\mathcal{M}}_{0,n_2+3}(X, \beta_2)]^{vir} \cap [g^{ab} T_a \otimes T_b],$$

then we get

$$\begin{aligned} & \sum_{n_1+n_2=n} \sum_{\beta_1+\beta_2=\beta} \langle T_i T_j T_a \gamma^{n_1} \rangle_{0, n_1+3, \beta_1} g^{ab} \langle T_b T_k T_l \gamma^{n_2} \rangle_{0, n_2+3, \beta_2} \\ &= \sum_{n_1+n_2=n} \sum_{\beta_1+\beta_2=\beta} \langle T_i T_k T_a \gamma^{n_1} \rangle_{0, n_1+3, \beta_1} g^{ab} \langle T_b T_j T_l \gamma^{n_2} \rangle_{0, n_2+3, \beta_2}. \end{aligned}$$

This is the essential part in the proof of associativity of quantum product.

**Remark 2.1.1.** *It deserves to notice that the quantum product is defined by rational curves, so its usage mainly concentrates in genus 0 GW-invariants. The difficulty to define quantum product via higher genus curves is that there is no so good associativity as the genus 0 case. It must be a good work if we can find a way to give a quantum product via higher genus curves with associativity like now.*

## 2.2 quantum differential equation

## Chapter 3

# Mirror Symmetry

I plan to follow Givental's approach to give a proof of genus 0 mirror symmetry of hypersurfaces in  $\mathbb{P}^n$ .

# Bibliography

- [1] David A. Cox and Sheldon Katz. *Mirror symmetry and algebraic geometry*, volume 68 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.
- [2] W. Fulton and R. Pandharipande. Notes on stable maps and quantum cohomology. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 45–96. Amer. Math. Soc., Providence, RI, 1997.
- [3] Alexander Givental. A mirror theorem for toric complete intersections. In *Topological field theory, primitive forms and related topics (Kyoto, 1996)*, volume 160 of *Progr. Math.*, pages 141–175. Birkhäuser Boston, Boston, MA, 1998.
- [4] T. Graber and R. Pandharipande. Localization of virtual classes. *Invent. Math.*, 135(2):487–518, 1999.
- [5] Kentaro Hori, Sheldon Katz, Albrecht Klemm, Rahul Pandharipande, Richard Thomas, Cumrun Vafa, Ravi Vakil, and Eric Zaslow. *Mirror symmetry*, volume 1 of *Clay Mathematics Monographs*. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2003. With a preface by Vafa.
- [6] Maxim Kontsevich. Enumeration of rational curves via torus actions. In *The moduli space of curves (Texel Island, 1994)*, volume 129 of *Progr. Math.*, pages 335–368. Birkhäuser Boston, Boston, MA, 1995.
- [7] Jun Li and Gang Tian. Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties. *J. Amer. Math. Soc.*, 11(1):119–174, 1998.

- [8] Bong H. Lian, Kefeng Liu, and Shing-Tung Yau. Mirror principle. I [MR1621573 (99e:14062)]. In *Surveys in differential geometry: differential geometry inspired by string theory*, volume 5 of *Surv. Differ. Geom.*, pages 405–454. Int. Press, Boston, MA, 1999.
- [9] Chiu-Chu Melissa Liu. Localization in Gromov-Witten theory and orbifold Gromov-Witten theory. In *Handbook of moduli. Vol. II*, volume 25 of *Adv. Lect. Math. (ALM)*, pages 353–425. Int. Press, Somerville, MA, 2013.
- [10] Zhengyu Zong. Gw theory and mirror symmetry. 2016.