Frobenius manifolds, super tau-cover and Virasoro constraints

Jinghao Yu

February 19, 2024

1 Super variables and Hamiltonian structures

Let M be a n-dimensional smooth manifold, the jet bundle $J^\infty(M)$ is the fibre bundle over M with the fibre $\mathbb{R}^\infty$. If $U\times\mathbb{R}^\infty$ and $V\times\mathbb{R}^\infty$ are two local trivializations with charts $(u^\alpha; u^{\alpha,s})$ and $(v^\alpha; v^{\alpha,s})$, where $\alpha = 1,\dots,n$ and $s \geq 1$, then the transition functions are given by

$$v^{\alpha,1} = \frac{\partial v^\alpha}{\partial u^\beta}u^{\beta,1}; v^{\alpha,2} = \frac{\partial v^\alpha}{\partial u^\beta}u^{\beta,2} + \frac{\partial^2 v^\alpha}{\partial u^\beta \partial u^\gamma}u^{\beta,1}u^{\gamma,1},\dots$$

Let $\hat{M}$ be the super manifold obtained by reversing the parity of the fibres of the cotangent bundle T^*M. For the associated infinite jet bundle $J^\infty(\hat{M})$, we can choose $\{u^{\alpha,s}, \theta_\alpha^s | \alpha = 1,\dots,n; s \geq 0\}$ as local coordinates. The super variables θ_α^s admit the following antisymmetric condition:

$$\theta_\alpha^s\theta_\beta^t + \theta_\beta^t\theta_\alpha^s = 0.$$

The ring of differential polynomials $\hat{\mathcal{A}}(M)$ is locally defined

$$C^\infty(U)\otimes\mathbb{C}[u^{\alpha,s}, \theta_\alpha^t | \alpha = 1,\dots,n, s\geq 1, t\geq 0].$$

It has a super gradation defined by the $\deg\theta_\alpha^s = 1$, $\deg u^{\alpha,s} = 0$ and a direct sum decomposition $\hat{\mathcal{A}}(M) = \oplus_{p\geq 0}\hat{\mathcal{A}}^p(M)$. There is a global vector field

$$\partial = \partial_x = \sum_{\alpha,s} u^{\alpha,s+1}\frac{\partial}{\partial u^{\alpha,s}} + \theta_\alpha^{s+1}\frac{\partial}{\partial\theta_\alpha^s}$$

on $J^\infty(\hat{M})$ induces a derivation on $\hat{\mathcal{A}}(M)$. For $f\in\hat{\mathcal{A}}(M)$, $\int f$ denote the local functional of $\hat{M}$. In other words, for any function $\mathbf{u}:\mathbb{R}\to\hat{M}$, we have a functional:

$$\int f : \mathbf{u}\mapsto\int f(u^{\alpha,s}(\mathbf{u}), \theta_\alpha^s(\mathbf{u})).$$

The functional lies in the quotient space

$$\hat{\mathcal{F}}(M) := \hat{\mathcal{A}}(M)/\partial\hat{\mathcal{A}}(M).$$

$\hat{\mathcal{F}}(M) = \oplus_{p\geq 0}\hat{\mathcal{F}}^p(M)$ admits a graded Lie algebra structure by the Schouten-Nijenhuis bracket: $\forall P = \int \tilde{P} \in \hat{\mathcal{F}}^p(M), Q = \int \tilde{Q} \in \hat{\mathcal{F}}^q(M)$

$$[P, Q] = \int \left(\frac{\delta P}{\delta\theta_\alpha}\frac{\delta Q}{\delta u^\alpha} + (-1)^p \frac{\delta P}{\delta u^\alpha}\frac{\delta Q}{\delta\theta_\alpha}\right)$$

Here the variational derivatives are

$$\frac{\delta P}{\delta u^\alpha} = \sum_{s\geq 0}(-\partial)^s \frac{\partial\tilde{P}}{\partial u^{\alpha,s}}, \quad \frac{\delta P}{\delta\theta_\alpha} = \sum_{s\geq 0}(-\partial)^s \frac{\partial\tilde{P}}{\partial\theta^s_\alpha}$$

For each local functional $X = \int X^\alpha\theta_\alpha \in \hat{\mathcal{F}}^1(M)$, we can define a system of evolutionary PDEs of the from

$$\frac{\partial u^\alpha}{\partial t} = X^\alpha, \quad \alpha = 1, \dots, n$$

We call X is a Hamiltonian evolutionary PDE if there exist a Hamiltonian structure $P \in \hat{\mathcal{F}}^2(M)$ and a Hamiltonian $H \in \hat{\mathcal{F}}^0(M)$ such that

$$X = -[P, H], [P, P] = 0.$$

$$P = \frac{1}{2}\int \theta_\alpha\Big(\sum_{s\geq 0} P^{\alpha\beta}_s\theta^s_\beta\Big), \quad H = \int h.$$

The operator $P^{\alpha\beta} = \sum_{s\geq 0} P^{\alpha\beta}_s\partial^s_x$ is antisymmetric in the sense of

$$\sum_{s\geq 0} P^{\alpha\beta}_s\partial^s_x = -\sum_{s\geq 0}(-\partial_x)^s P^{\beta\alpha}_s.$$

Then

$$\frac{\delta P}{\delta\theta_\alpha} = \frac{1}{2}(P^{\alpha\beta}_s\theta^s_\beta - (-\partial)^s(P^{\gamma\alpha}_s\theta_\gamma)) = P^{\alpha\beta}_s\theta^s_\beta$$

$$[P, H] = \int \frac{\delta P}{\delta\theta_\alpha}\frac{\delta H}{\delta u^\alpha} = \int (P^{\alpha\beta}_s\partial^s\theta_\beta)\frac{\delta H}{\delta u^\alpha} = \int \theta_\beta\left(\sum_s(-\partial)^s P^{\alpha\beta}_s\frac{\delta H}{\delta u^\alpha}\right) = -\int \theta_\beta P^{\beta\alpha}_s\partial^s(\frac{\delta H}{\delta u^\alpha})$$

$$u^\alpha_t = X^\alpha = P^{\alpha\beta}\left(\frac{\delta H}{\delta u^\beta}\right)$$

The evolutionary PDE X is called a bihamiltonian system if there exists $P_0, P_1 \in \hat{\mathcal{F}}^2(M)$ and $H_0, H_1 \in \hat{\mathcal{F}}^0(M)$ such that

$$X = -[P_0, H] = -[P_1, G], \quad [P_0, P_0] = [P_1, P_1] = [P_0, P_1] = 0.$$

2 Principal Hierarchy and Tau-cover

Dubrovin-Zhang's approach translates the genus 0 free energy of Frobenius manifold to the language of the principal hierarchy, and view the higher geuns free energy as a topological deformation of genus 0 principal hierarchy. They construct the Virasoro constraints of the partition function from the bihamiltonian structure.

2.1 Principal Hierarchy

The Dubrovin connection is a deformed flat connection $\tilde{\nabla}$ defined by

$$\tilde{\nabla}_u v = \nabla_u v + zu \cdot v, \quad u, v \in TM.$$

There is a system of deformed flat coordinates of the form

$$(\tilde{t}_1(t; z), \ldots, \tilde{t}_n(t; z)) = (\theta_1(t; z), \ldots, \theta_n(t, z))z^\mu z^R,$$

which satisfy the equations

$$\tilde{\nabla} d\tilde{t}_\alpha(t; z) = 0.$$

Here, μ is the diagonal matrix from Hodge grading, $R = R_1 + \cdots + R_m$. They are the monodromy data of the Frobenius manifold at $z = 0$, and $\theta_\alpha(t; z)$ have the expressions

$$\theta_\alpha(t; z) = \sum_{k \geq 0} \theta_{\alpha,k}(t) z^k.$$

In terms of the functions $\theta_{\alpha,p}$, we have

$$\frac{\partial^2 \theta_{\gamma,p+1}}{\partial t^\alpha \partial t^\beta} = c^\xi_{\alpha\beta} \frac{\partial \theta_{\gamma,p}}{\partial t^\xi}, p \geq 0.$$

and we require

$$\theta_\alpha(t; 0) = \eta_{\alpha\beta} t^\beta, \quad \frac{\partial \theta_\alpha(t; z)}{\partial t^1} = z\theta_\alpha(t; z) + \eta_{1\alpha},$$

$$\frac{\partial \theta_\alpha(t; z)}{\partial t^\gamma} \eta^{\gamma\sigma} \frac{\partial \theta_\beta(t; -z)}{\partial t^\sigma} = \eta_{\alpha\beta}$$

$$\partial_E \theta_{\alpha,p}(t) = \big(p + \frac{2-d}{2} + \mu_\alpha\big)\theta_{\alpha,p}(t) + \sum_{k=1}^{p} \theta_{\xi,p-k}(t)(R_k)^\xi_\alpha + \text{constant}.$$

In Gromov-Witten theory, $R_1 = c_1(X) \cup -$, $R_{k\geq 2} = 0$. Let $\mathbf{t}(z) = t^\alpha_k \phi_\alpha z^k$, $t = t^\alpha_0 \phi_\alpha = t^\alpha \phi_\alpha$ define

$$\ll \prod_{i=1}^{n} \tau_{k_i}(\phi_{\alpha_i}) \gg_{g,n} (\mathbf{t}) = \sum_{\beta,n} \frac{q^\beta}{n!} \langle \tau_{k_1}(\phi_{\alpha_1}), \ldots, \tau_{k_n}(\phi_{\alpha_n}), \mathbf{t}(\psi), \ldots, \mathbf{t}(\psi)\rangle^X_{g,n+m,\beta}$$

Then

$$\theta_\alpha(t;z) = \ll 1, \frac{\phi_\alpha}{1-z\psi} \gg_{0,2}(t), \quad \theta_{\alpha,0} = \eta_{\alpha\beta} t^\beta, \quad \theta_{\alpha,p} = \ll 1, \phi_\alpha \psi^p \gg_{0,2}(t)$$

$$d\theta_\alpha = \ll \phi_\gamma, \frac{\phi_\alpha}{z^{-1}-\psi} \gg_{0,2}(t)\phi^\gamma$$

The conditions of $\theta_{\alpha,p}$ are the topological recursion relation, string equation, WDVV equation and homogeneity of S-matrix in GW theory.

The principal hierarchy of M is defined as the following system of evolutionary Hamiltonian PDEs $v^*(t^*_*)$ of hydrodynamic type:

$$\frac{\partial v^\alpha}{\partial t^\beta_p} = \eta^{\alpha\gamma} \partial_x \frac{\partial \theta_{\beta,p+1}(\mathbf{t})}{\partial v^\gamma}, p \geq 0.$$

Here $\theta_{\alpha,p}(\mathbf{t}) = \ll 1, \phi_\alpha \psi^p \gg (\mathbf{t})$. The Hamiltonian is $H_{\alpha,p} = \int \theta_{\alpha,p+1}$. The first Hamiltonian structures are given by

$$P_1 = \int \theta_\alpha \eta^{\alpha\beta} \theta^1_\beta$$

The second Hamiltonian structure is from the metric $g^{\alpha\beta}(t) := E^\epsilon c^{\alpha\beta}_\epsilon$

$$P_2 = \int \theta_\alpha (g^{\alpha\beta}\partial_x + \Gamma^{\alpha\beta}_\gamma t^{\gamma,1})\theta_\beta$$

We have: for $p \geq 1$

$$\frac{\partial v^\alpha}{\partial t^\beta_p} = \eta^{\alpha\gamma}\partial_x \frac{\partial \theta_{\beta,p+1}}{\partial v^\gamma} = P_1^{\alpha\gamma} \frac{\delta H_{\beta,p}}{\delta v^\gamma} = \{v^\alpha, H_{\beta,p}\}_1 = \frac{1}{p+\frac{1}{2}+\mu_\alpha}\{v^\alpha, H_{\beta,p-1}\}_2$$

Proposition 2.1. *In Gromov-Witten theory, consider $\mathcal{F}^0_X(x,\mathbf{t}) = \mathcal{F}^0_X(t^\alpha_k + \delta^\alpha_1 \delta^0_k x)$, let*

$$v_\beta = \frac{\partial^2 \mathcal{F}^0_X}{\partial t^\beta_0 \partial t^1_0} = \ll \phi_\beta, 1 \gg_{0,2}(\mathbf{t}), \quad v^\alpha = \eta^{\alpha\beta} v_\beta = \ll \phi^\alpha, 1 \gg_{0,2}(\mathbf{t})$$

(1) Let $v = v^\gamma \phi_\gamma$

$$\ll \phi_\alpha \psi^p, \phi_\beta \psi^q \gg_{0,2}(\mathbf{t}) = \ll \phi_\alpha \psi^p, \phi_\beta \psi^q \gg_{0,2}(v)$$

(2) View $\theta_{\alpha,p}(\mathbf{t}) = \ll 1, \phi_\alpha\psi^p \gg (\mathbf{t}) = \ll 1, \phi_\alpha \psi^p \gg (v)$ as a function on v, v^α satisfy the Euler-Lagrangian equation:

$$v^\alpha = \sum_{\beta,k} \eta^{\alpha\gamma} t^\beta_k \frac{\partial \theta_{\beta,k}}{\partial v^\gamma}$$

(3) v^α satisfy the principal hierarchy:

$$\frac{\partial v^\alpha}{\partial t_p^\beta} = \eta^{\alpha\gamma}\partial_x \frac{\partial \theta_{\beta,p+1}}{\partial v^\gamma}, p \geq 0.$$

Example 2.2 (dispersionless KdV)**.** *Consider $v = v(t_0 + x, t_1, t_2, \dots)$ satisfying*

$$v_{t_k} = \frac{1}{k!} v^k v_x.$$

For $p \geq 1$, the Hamiltonian is

$$H_p = \int \frac{v^{p+2}}{(p+2)!} dx.$$

The Hamiltonian structures are

$$P_1 = \partial_x \quad P_2 = v\partial_x + \frac{1}{2} v_x$$

$$\frac{\partial v}{\partial t^p} = \{v, H_p\}_1 = P_1 \frac{\delta H_p}{\delta v}$$

Define $\mathcal{R} = P_2 \circ P_1^{-1} = v + \frac{1}{2} v_x \partial_x^{-1}$, then

$$P_2 \frac{\delta H_p}{\delta v} = \mathcal{R} \frac{\partial v}{\partial t^p} = (p + \frac{3}{2}) \frac{\partial v}{\partial t^{p+1}}$$

$$\frac{\partial v}{\partial t^p} = \frac{1}{p + \frac{1}{2}} \{v, H_{p-1}\}_2$$

The Witten-Kontsevich theorem says that 1-point Gromov-Witten partition function is KdV. We can understand this theorem through

$$\theta_{\alpha,p+1} = \ll 1, \psi^{p+1} \gg (v) = \sum_n \frac{v^n}{n!} \int_{\overline{\mathcal{M}}_{0,2+n}} \psi_2^{p+1} \cdot 1^{n+1} = \frac{1}{(p+2)!} v^{p+2}$$

2.2 Tau-Cover

Consider the functions $\Omega_{\alpha,p;\beta,q}(v), \alpha, \beta = 1, \dots, n; p, q \geq 0$ determined by

$$\sum_{p,q \geq 0} \Omega_{\alpha,p;\beta,q}(v) z_1^p z_2^q = \frac{\langle \nabla\theta_\alpha(v, z_1), \nabla\theta_\beta(v, z_2)\rangle - \eta_{\alpha\beta}}{z_1 + z_2}.$$

Let $f_{\alpha,p} = \frac{\partial \mathcal{F}_0}{\partial t^\alpha_p}$ $(\frac{\partial f_{\alpha,p}}{\partial t^1_0} = \theta_{\alpha,p})$, the tau-cover of the principal hierarchy is:

$$\frac{\partial f_{\alpha,p}}{\partial t^\beta_q} = \Omega_{\alpha,p;\beta,q}, \quad \frac{\partial v^\alpha}{\partial t^\beta_q} = \eta^{\alpha\epsilon}(\partial_\epsilon \partial_\gamma \theta_{\beta,q+1}) v^{\gamma,1}.$$

In terms of Gromov-Witten theory:

$$\Omega_{\alpha,p;\beta,q}(v) = \ll \phi_\alpha \psi^p, \phi_\beta \psi^q \gg_{0,2} (v), \quad f_{\alpha,p} = \ll \phi_\alpha \psi^p \gg_{0,1} (\mathbf{t})$$

Tau-cover can be used to construct Virasoro symmetry. Let V be an n-dimensional complex vector space with a fixed basis $\{e_\alpha\}$, and endowed with a symmetric bilinear form $\langle -, - \rangle$ defined by $\eta_{\alpha\beta} = \langle e_\alpha, e_\beta \rangle$. We construct a Heisenberg algebra:

$$a^\alpha_k = \begin{cases} \eta^{\alpha\beta} \dfrac{\partial}{\partial t^\beta_k}, & k \geq 0 \\ (-1)^{k+1} t^\alpha_{-k-1}, k < 0 \end{cases}$$

they satisfy the commutation relations:

$$[a^\alpha_k, a^\beta_l] = (-1)^k \eta^{\alpha\beta} \delta_{k+l+1,0}.$$

The normal ordering is defined by

$$: a^\alpha_k a^\beta_l := \begin{cases} a^\beta_l a^\alpha_k, \text{if } l < 0, k \geq 0 \\ a^\alpha_k a^\beta_l, \text{otherwise.} \end{cases}$$

Introduce vector-valued operators

$$a_k = a^\alpha_k e_\alpha, \quad k \in \mathbb{Z}.$$

Given the monodromy data μ, R of the Frobenius manifold, we define the following matrices for $m \geq 1$:

$$P_m(\mu, R) = \begin{cases} [\exp(R\partial_x) \prod_{j=0}^{m} (x + \mu + j - \frac{1}{2})]_{x=0}, & m \geq 0 \\ 1, \quad m = -1. \end{cases}$$

We assume the antisymmetric linear operator $\mu : V \to V$ is diagonalizable. For any eigenvalue $\lambda \in \mathrm{Spec}\ \mu$, denote V_λ the subspace of V consisting of all eigenvectors with the eigenvalue λ. Let $\pi_\lambda : V \to V_\lambda$ be the projector. For any linear operator $A : V \to V$ and for any integer k, denote

$$A_k := \sum_{\lambda \in \mathrm{Spec}\ \mu} \pi_{\lambda+k} A \pi_\lambda$$

We define the operators of the Virasoro algebra by a Sugawara-type construction

$$L_m^{even} = \frac{1}{2}\sum_{k,l} : \langle a_l, [P_m(\mu - k, R)]_{m-1-l-k} a_k\rangle : + \frac{1}{4}\delta_{m,0}\mathrm{tr}\left(\frac{1}{4} - \mu^2\right), \quad m \geq -1$$

Example 2.3 (Virasoro algebra of KdV hierarchy)**.** *For $n = 1$, it must be $\mu = 0, R = 0$.*

$$L_m^{even} = \frac{1}{2}\sum_k (-1)^{k+1} P_m(-k) : a_k a_{m-1-k} : + \frac{1}{16}\delta_{m,0}$$

where

$$P_m(x) = \begin{cases} \prod_{j=0}^{m}(x + \frac{2j-1}{2}), m \geq 0 \\ 1, m = -1. \end{cases}$$

The operators can be rewritten as

$$L_m^{even} = a_m^{\alpha,p;\beta,q}\frac{\partial^2}{\partial t_p^\alpha \partial t_q^\beta} + b_{m;\alpha,p}^{\beta,q} t_p^\alpha \frac{\partial}{\partial t_q^\beta} + c_{m;\alpha,p;\beta,q} t_p^\alpha t_q^\beta + \frac{1}{4}\delta_{m,0}\mathrm{tr}\left(\frac{1}{4} - \mu^2\right).$$

The infinitesimal transformation of genus zero free energy $\mathcal{F}_0$ is:

$$\frac{\partial \mathcal{F}_0}{\partial s_m} = a_m^{\alpha,p;\beta,q} f_{\alpha,p} f_{\beta,q} + b_{m;\alpha,p}^{\beta,q} t_p^\alpha f_{\beta,q} + c_{m;\alpha,p;\beta,q} t_p^\alpha t_q^\beta$$

We define the Virasoro symmetries of the principal hierarchy as

$$\frac{\partial f_{\alpha,p}}{\partial s_m} := \frac{\partial}{\partial t_p^\alpha}\frac{\partial \mathcal{F}_0}{\partial s_m}; \quad \frac{\partial v^\alpha}{\partial s_m} := \eta^{\alpha\beta}\frac{\partial^2}{\partial t_0^\beta \partial t_0^1}\frac{\partial \mathcal{F}_0}{\partial s_m}, \quad m \geq -1.$$

Theorem 2.4.

1. *The operators L_m^{even} satisfy the Virasoro commutation relations*

$$[L_k^{even}, L_m^{even}] = (k - m)L_{k+m}^{even}, \quad k, m \geq -1.$$

2. *The flows $\frac{\partial}{\partial s_m}$ are symmetries of the tau-cover of the principal hierarchy,*

$$\left[\frac{\partial}{\partial s_m}, \frac{\partial}{\partial t_p^\alpha}\right] = 0.$$

3 Super Tau-Cover

For a bihamiltonian integrable hierarchy with bihamiltonian structures P_0, P_1, we consider the flows

$$\frac{\partial u^\alpha}{\partial \tau_i} = \frac{\delta P_i}{\delta \theta_\alpha}, \quad \frac{\partial \theta_\alpha}{\partial \tau_i} = \frac{\delta P_i}{\delta u^\alpha}, \quad i = 0, 1.$$

We naturally have the relation

$$\frac{\partial u^\alpha}{\partial \tau_1} = P_1^{\alpha\beta}\theta_\beta = \left(P_1 \circ P_0^{-1}\right)^\alpha_\gamma P_0^{\gamma\beta}\theta_\beta = \left(P_1 \circ P_0^{-1}\right)^\alpha_\gamma \frac{\partial u^\gamma}{\partial \tau_0}$$

We want to continue define more flows in a similar way:

$$\frac{\partial u^\alpha}{\partial \tau_p} = \left(P_1 \circ P_0^{-1}\right)^\alpha_\gamma \frac{\partial u^\gamma}{\partial \tau_{p-1}}, p \geq 2.$$

However, the operator $(P_1 \circ P_0^{-1})^p$ contain too many ∂_x^{-1}, which cannot be represented as local functionals of $\hat{M}$ in general. So we need to introduce a new family of super variables

$$\{\sigma^s_{\alpha,k} \mid \alpha = 1, \ldots, n; k \geq 0, s \geq 0\}$$

where $\sigma^s_{\alpha,0} = \theta^s_\alpha$ and $\sigma^0_{\alpha,k} = \sigma_{\alpha,k}$. We enlarge the ring $\hat{\mathcal{A}}(M)$ by these new variables. Locally

$$\hat{\mathcal{A}}(M) = C^\infty(U) \otimes \mathbb{C}[u^{\alpha,s}, \sigma^t_{\alpha,k} | \alpha = 1, \ldots, n, k \geq 0, s \geq 1, t \geq 0].$$

and

$$\partial = \sum_{s\geq 0} u^{\alpha,s+1}\frac{\partial}{\partial u^{\alpha,s}} + \sum_{s,k\geq 0} \sigma^{s+1}_{\alpha,k}\frac{\partial}{\partial \sigma^s_{\alpha,k}}$$

We also require

$$P_0^{\alpha\beta}\sigma_{\beta,k+1} = P_1^{\alpha\beta}\sigma_{\beta,k}, \quad \alpha = 1, \ldots, n.$$

There is a super extension of the principal hierarchy:

Theorem 3.1. *We have the following mutually commuting flows associated with any given Frobenius manifold M:*

$$\frac{\partial v^\alpha}{\partial t^\beta_p} = \eta^{\alpha\gamma}(\partial_\lambda\partial_\gamma h_{\beta,p+1})v^{\lambda,1}, \quad \frac{\partial \sigma_{\alpha,k}}{\partial t^\beta_p} = \eta^{\gamma\epsilon}(\partial_\alpha\partial_\epsilon h_{\beta,p+1})\sigma^1_{\gamma,k},$$

$$\frac{\partial v^\alpha}{\partial \tau_m} = \eta^{\alpha\beta}\sigma^1_{\beta,m}, \quad \frac{\partial \sigma_{\alpha,k}}{\partial \tau_m} = -\frac{\partial \sigma_{\alpha,m}}{\partial \tau_k} = \Gamma^{\gamma\beta}_\alpha \sum_{i=0}^{m-k-1} \sigma_{\beta,k+i}\sigma^1_{\gamma,m-i-1}, \quad 0 \leq k \leq m.$$

Here $\alpha, \beta = 1, \ldots, n$ and $m, p \geq 0$.

Lemma 3.2. *Assume that* $\frac{1-2k}{2} \notin \mathrm{Spec}(\mu)$ *for any* $k = 1, 2, \ldots$. *Then for any* $p, n \geq 0$ *there exists* $\Phi^n_{\alpha,p} \in \hat{\mathcal{A}}(M)$ *such that*

$$\frac{\partial h_{\alpha,p}}{\partial \tau_n} = (\Phi^n_{\alpha,p})'.$$

The super tau-cover of the principal hierarchy associated to the Frobenius manifold M is:

$$\frac{\partial f_{\alpha,p}}{\partial t^\beta_q} = \Omega_{\alpha,p;\beta,q}, \quad \frac{\partial f_{\alpha,p}}{\partial \tau_n} = \Phi^n_{\alpha,p},$$
$$\frac{\partial \Phi^n_{\alpha,p}}{\partial t^\beta_q} = \frac{\partial \Omega_{\alpha,p;\beta,q}}{\partial \tau_n}, \quad \frac{\partial \Phi^n_{\alpha,p}}{\partial \tau_k} = \Delta^{k,n}_{\alpha,p}$$

together with the evolution equations in Theorem 3.1 , where

$$\Delta^{k,n}_{\alpha,p} = -\Delta^{n,k}_{\alpha,p} = \eta^{\gamma\lambda}\partial_\lambda h_{\alpha,p}\Gamma^{\delta\mu}_\gamma \left(\sum_{i=0}^{k-n-1} \sigma_{\mu,n+i}\sigma^1_{\delta,k-i-1}\right), \quad k \geq n.$$

Finally, we can construct the Virasoro symmetries of the super tau-cover of the principal hierarchy:

$$L_m = L^{even}_m + L^{odd}_m, \quad L^{odd}_m = \sum_{k\geq 0}(k + c_0)\tau_k \frac{\partial}{\partial \tau_{k+m}}, \quad m \geq -1.$$

The operators L_m satisfy the commutation relations $[L_m, L_n] = (m - n)L_{m+n}$.

References

[1] Alexandr Buryak, Hessel Posthuma, and Sergey Shadrin. A polynomial bracket for the dubrovin-zhang hierarchies. *JOURNAL OF DIFFERENTIAL GEOMETRY*, 92(1):153–185, SEP 2012.

[2] Boris Dubrovin. Geometry of 2d topological field theories. In *Integrable systems and quantum groups (Montecatini Terme, 1993)*, volume 1620 of *Lecture Notes in Math.*, pages 120–348. Springer, Berlin, 1996.

[3] Boris Dubrovin and Youjin Zhang. Frobenius manifolds and Virasoro constraints. *Selecta Math. (N.S.)*, 5(4):423–466, 1999.

[4] Si-Qi Liu, Zhe Wang, and Youjin Zhang. Super tau-covers of bihamiltonian integrable hierarchies. *JOURNAL OF GEOMETRY AND PHYSICS*, 170, DEC 2021.