

Gromov-Witten theory and mirror symmetry

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Chapter 1

Gromov-Witten invariants

Roughly speaking, Gromov-Witten invariants is the number of algebraic curves (resp. pseudo-holomorphic curves) of a algebraic variety X (resp. complex manifold) passing through specific subvarieties (resp. submanifolds) under specific degree (homology class of the curves) and genus.

Definition 1.0.1. Let $\gamma_1, \dots, \gamma_n \in H^*(X; \mathbb{Q})$ and let $\beta \in H^2(X; \mathbb{Q})$. The Gromov-Witten invariant of genus g degree β curves is

$$\langle \gamma_1 \dots \gamma_n \rangle_{g,n,\beta} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} ev_1^*(\gamma_1) \cup \dots \cup ev_n^*(\gamma_n).$$

Here, a point in $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{vir}$ is $[f : C \rightarrow X, 1, \dots, n]$:

a map from the genus g curve C to the variety X modulo the automorphism of C .

The evaluation map $ev_i : [\overline{\mathcal{M}}_{g,n}(X, \beta)]^{vir} \rightarrow X$ is given by

$$ev_i([f : C \rightarrow X, 1, \dots, n]) = f(i).$$

Let $\overline{\mathcal{M}}_{g,n}$ denote the moduli space (Deligne-Mumford stack) of genus g curves with n marked points, and let $\overline{\mathcal{C}}_{g,n}$ be the universal family of $\overline{\mathcal{M}}_{g,n}$.

1.1 Kontsevich's approach

Atiyah-Bott localization formula tells us: if there is a torus action $\mathbb{T} = (\mathbb{C}^*)^n$ on X , then the fixed points of torus action could tells us some properties of X .

By the classifying space theory, $B\mathbb{T} = (\mathbb{C}P^\infty)^{\times n}$, so $H^*(B\mathbb{T}) = \mathbb{C}[\lambda_1, \dots, \lambda_n]$. Let $\mathcal{R}_{\mathbb{T}} = \mathbb{C}(\lambda_1, \dots, \lambda_n)$. Let $X_{\mathbb{T}} = E\mathbb{T} \times_{\mathbb{T}} X$, the equivariant cohomology of X is defined by

$$H_{\mathbb{T}}^*(X) := H^*(X_{\mathbb{T}}) = H^*(E\mathbb{T} \times_{\mathbb{T}} X),$$

so naturally, $H_{\mathbb{T}}^*(X)$ is a $H_{\mathbb{T}}^*(pt) = H^*(B\mathbb{T})$ -module. The localization of $H_{\mathbb{T}}^*(X)$ means $H_{\mathbb{T}}^*(X) \otimes \mathcal{R}_{\mathbb{T}}$.

Theorem (Atiyah-Bott). *Let $X^{\mathbb{T}}$ be fixed locus of \mathbb{T} , let Z_j be a connection component of $X^{\mathbb{T}}$, and let N_j be the normal bundle of Z_j in X . Let $i_j : Z_j \rightarrow X$ and let $i_{j!} : H_{\mathbb{T}}^*(Z_j) \rightarrow H_{\mathbb{T}}^*(X)$ be the pushforward defined by the Gysin map. Let $\alpha \in H_{\mathbb{T}}^*(X) \otimes \mathcal{R}_{\mathbb{T}}$, we have*

$$\alpha = \sum_j \frac{i_{j!} i_j^* \alpha}{Euler_T(N_j)},$$

In particular,

$$\int_{X_{\mathbb{T}}} \alpha = \sum_j \int_{(Z_j)_{\mathbb{T}}} \frac{i_j^* \alpha}{Euler_T(N_j)}.$$

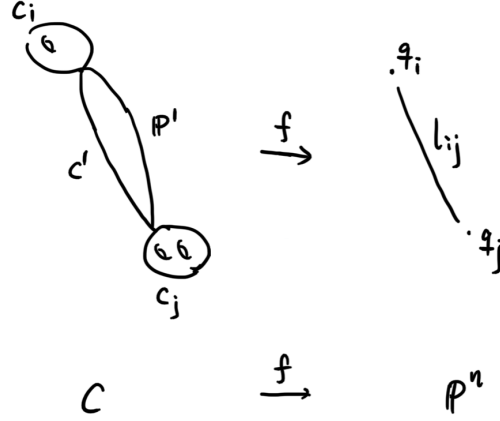
Kontsevich's approach is to apply Atiyah-Bott localization formula in $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{vir}$ so that we can simplify the computation. We can lift the \mathbb{T} action on X to $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{vir}$ in the following way: let $t \in \mathbb{T}$, $[f : C \rightarrow X, 1, \dots, n] \in [\overline{\mathcal{M}}_{g,n}(X, \beta)]^{vir}$, $x \in X$

$$(t \cdot f)(x) = f(t \cdot x).$$

In special, we will consider $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^n, d)$ in this section. As claimed before, we need to find $[f : C \rightarrow X, 1, \dots, n] \in \overline{\mathcal{M}}_{g,n}(\mathbb{P}^n, d)^{\mathbb{T}}$. The fixed points of $\mathbb{C}P^n$ is

$$\{q_i = [0 : 0 : \dots : 1 : 0 : \dots : 0]\}_{0 \leq i \leq n}.$$

The coordinate curve l_{ij} connecting q_i, q_j has one dimensional degree of freedom \mathbb{C}^* (as an invariant component). The curve $C \in \overline{\mathcal{C}}_{g,n}$ is stable (i.e. $\text{Aut}(C) < \infty$) if and only if $2g - 2 + n > 0$. If a components C' of C is mapped to l_{ij} , then C' has two points mapped to q_i, q_j respectively (equivalent to with two marked points in C'), so $2g - 2 + 2 \leq 0$ implies $g = 0$, i.e. $C' \cong \mathbb{P}^1$ (see Fig 1.1). Meanwhile, $f|_{C'}$ must be uniformly ramified, so $f|_{C'}(z) = z^e, \forall z \in C' \cong \mathbb{P}^1$, for some $e \in \mathbb{N}^*$.


 Figure 1.1: $f(C_i) = q_i$, $f(C') = l_{ij}$, $f(C_j) = q_j$

It is convenient to use a decorated graph $\vec{\Gamma} = (\Gamma, \vec{f}, \vec{d}, \vec{g}, \vec{s})$ (graph, maps, degrees, genus, marked points) to represent $[f : C \rightarrow X, 1, \dots, n] \in \overline{\mathcal{M}}_{g,n}(\mathbb{P}^n, d)^{\mathbb{T}}$. Let $\text{val}(v)$, the valence of v , be the number of edges connecting vertex v , and let $n(v) = |s_v| + \text{val}(v)$. The stable map $[f : C \rightarrow X, 1, \dots, n]$ with fixed graph $\vec{\Gamma} = (\Gamma, \vec{f}, \vec{d}, \vec{g}, \vec{s})$ defines a substack

$$\overline{\mathcal{M}}_{\vec{\Gamma}} \subset \overline{\mathcal{M}}_{g,n}(\mathbb{P}^n, d).$$

On the other sides, consider

$$\varphi_{\vec{\Gamma}} : \prod_{\dim C_v=1} \overline{\mathcal{M}}_{g_v, n(v)} \rightarrow \overline{\mathcal{M}}_{\vec{\Gamma}}.$$

If v, v' are connected by an edge e , then let $C_v, C_{v'}$ connected by a $C_e \cong \mathbb{P}^1$ associated with a degree d_e map to \mathbb{P}^n . Let $\overline{\mathcal{M}}_{\Gamma}$ be the product of above C_v, C_e . There is a group \mathbb{A}_{Γ} acting on $\overline{\mathcal{M}}_{\Gamma}$. The group \mathbb{A}_{Γ} is defined by:

$$1 \rightarrow \prod_{\text{edges}} \mathbb{Z}/(d_e) \rightarrow \mathbb{A}_{\Gamma} \rightarrow \text{Aut}(\Gamma) \rightarrow 1.$$

and

$$\overline{\mathcal{M}}_{\vec{\Gamma}} = \overline{\mathcal{M}}_{\Gamma} / \mathbb{A}_{\Gamma}.$$

Therefore, we know the \mathbb{T} -fixed locus of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^n, d)$ is $\overline{\mathcal{M}}_{\vec{\Gamma}}$. Let N_{Γ} be the normal bundle of $\overline{\mathcal{M}}_{\vec{\Gamma}}$ in $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^n, d)$. Then there is an explicit formula for

the equivariant Euler class. Before doing that, we define some necessary notations. A flag F is a pair (v, e) such that e is an edge containing the vertex v . We put $i(F) = v$, $j(F)$ the vertex of e different from v . Let

$$\omega_F = \frac{\lambda_{i(F)} - \lambda_{j(F)}}{d_e} \in H_{\mathbb{T}}^2(pt) = \mathbb{Q}[\lambda_0, \dots, \lambda_r],$$

which is the weight of \mathbb{T} -action on $T_{q_{i_v}} C_e$.

Theorem 1.1.1 ($Euler_{\mathbb{T}}(N_{\Gamma})$). $Euler_{\mathbb{T}}(N_{\Gamma}) = e_{\Gamma}^F e_{\Gamma}^v e_{\Gamma}^e$, where

$$\begin{aligned} e_{\Gamma}^F &= \prod_{n(i(F)) \geq 3} (\omega_F - \psi_F) / \prod_{j \neq i(F)} (\lambda_{i(F)} - \lambda_j), \\ e_{\Gamma}^v &= \prod_v \prod_{j \neq i_v} (\lambda_{i_v} - \lambda_j) \prod_{val(v)=2, s_v=\emptyset} (\omega_{F_1(v)} + \omega_{F_2(v)}) / \prod_{val(v)=1, s_v=\emptyset} \omega_{F(v)} \\ e_{\Gamma}^e &= \prod_e \frac{(-1)^{d_e} (d_e!)^2 (\lambda_i - \lambda_j)^{2d_e}}{d_e^{2d_e}} \prod_{a+b=d_e, k \neq i, j} \left(\frac{a\lambda_i + b\lambda_j}{d_e} - \lambda_k \right) \end{aligned}$$

The proof is partially discussed in section 1.2.

1.2 Tangent-obstruction sequence

Consider $[f : C \rightarrow X, 1, \dots, n] \in \overline{\mathcal{M}}_{\vec{\Gamma}}$, $\vec{\Gamma} = (\Gamma, \vec{f}, \vec{d}, \vec{g}, \vec{s})$. We put

$$V^1(\Gamma) := \{v \in V(\Gamma) : g_v = 0, val(v) = 1, |s_v| = 0\}$$

$$V^2(\Gamma) := \{v \in V(\Gamma) : g_v = 0, val(v) = 2, |s_v| = 0\}$$

$$V^{1,1}(\Gamma) := \{v \in V(\Gamma) : g_v = 0, val(v) = 1, |s_v| = 1\}$$

$$V^s(\Gamma) := \{v \in V(\Gamma) : 2g_v - 2 + val(v) + |s_v| > 0\}$$

$$y(v, e) := C_e \cap C_v$$

The tangent-obstruction sequence is

$$\begin{aligned} &0 \rightarrow Aut(C, 1, \dots, n) \\ &\rightarrow Def(f) \rightarrow Def(C, 1, \dots, n, f) \rightarrow Def(C, 1, \dots, n) \\ &\rightarrow Ob(f) \rightarrow Ob(C, 1, \dots, n, f) \rightarrow 0, \end{aligned}$$

$$\begin{aligned}
 0 &\rightarrow \text{Hom}(\Omega_C(p_1 + \cdots + p_n), \mathcal{O}_C) \\
 &\rightarrow H^0(C, f^*T_X) \rightarrow T^1 \rightarrow \text{Ext}^1(\Omega_C(p_1 + \cdots + p_n), \mathcal{O}_C) \\
 &\rightarrow H^1(C, f^*T_X) \rightarrow T^2 \rightarrow 0.
 \end{aligned}$$

For simplicity:

$$0 \rightarrow B_1 \rightarrow B_2 \rightarrow T^1 \rightarrow B_4 \rightarrow B_5 \rightarrow T^2 \rightarrow 0.$$

The $N^{\text{vir}} = T^{1,m} - T^{2,m}$ (m means moving part).

$$Euler_{\mathbb{T}}(N^{\text{vir}}) = \frac{Euler_{\mathbb{T}}(B_2^m) Euler_{\mathbb{T}}(B_4^m)}{Euler_{\mathbb{T}}(B_1^m) Euler_{\mathbb{T}}(B_5^m)}.$$

(1) $Euler_{\mathbb{T}}(B_2^m)/Euler_{\mathbb{T}}(B_5^m)$. The normalization sequence of C is:

$$\begin{aligned}
 0 &\rightarrow \mathcal{O}_C \rightarrow \bigoplus_{v \in V^s(\Gamma)} \mathcal{O}_{C_v} \oplus \bigoplus_{e \in E(\Gamma)} \mathcal{O}_{C_e} \\
 &\rightarrow \bigoplus_{v \in V^2(\Gamma)} \mathcal{O}_{y_v} \oplus \bigoplus_{(e,v) \in F^s(\Gamma)} \mathcal{O}_{y(e,v)} \rightarrow 0.
 \end{aligned}$$

Take $\otimes f^*T_X$:

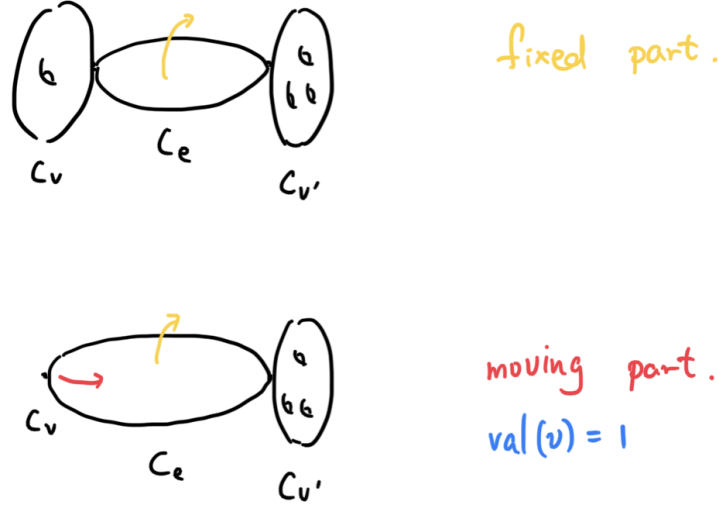
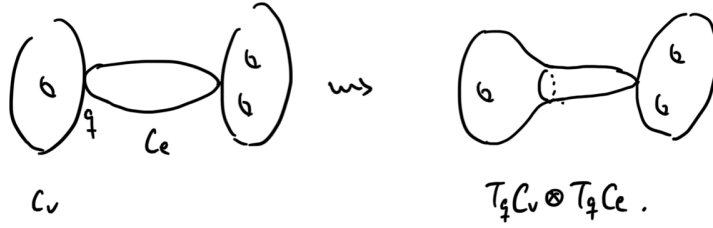
$$\begin{aligned}
 0 &\rightarrow H^0(C, f^*T_X) \rightarrow \bigoplus_{v \in V^s(\Gamma)} H^0(C_v, f^*T_X) \oplus \bigoplus_{e \in E(\Gamma)} H^0(C_e, f^*T_X) \\
 &\rightarrow \bigoplus_{v \in V^2(\Gamma)} T_{f(y_v)}X \oplus \bigoplus_{(e,v) \in F^s(\Gamma)} T_{f(y(e,v))}X \\
 &\rightarrow H^1(C, f^*T_X) \rightarrow \bigoplus_{v \in V^s(\Gamma)} H^1(C_v, f^*T_X) \oplus \bigoplus_{e \in E(\Gamma)} H^1(C_e, f^*T_X) \rightarrow 0.
 \end{aligned}$$

$$H^0(C_v, f^*T_X) = T_{f(C_v)}X,$$

$$H^1(C_v, f^*T_X) = H^1(C_v, \mathcal{O}_{C_v}) \otimes T_{f(C_v)}X \cong H^0(C_v, \omega_{C_v})^\vee \otimes T_{f(C_v)}X$$

Here $H^0(C_v, \omega_{C_v})$ is Hodge bundle \mathbb{E} . By splitting principle, assume $\mathbb{E} = L_1 \oplus \cdots \oplus L_g$, then

$$\begin{aligned}
 e(\mathbb{E}^\vee \otimes \mathbb{C}_1) &= \prod_{i=1}^g c_1(L_i^\vee \otimes \mathbb{C}_1) = \prod_{i=1}^g c_1(L_i^\vee) + c_1(\mathbb{C}_1) \\
 &= \prod_{i=1}^g (-c_1(L_i) + u) = \sum_{k=1}^g (-1)^k c_k(\mathbb{E}) u^{g-k} = \sum_{k=1}^g (-1)^k \lambda_k u^{g-k} =: \Lambda_g^\vee(u)
 \end{aligned}$$


 Figure 1.2: automorphism of $(C, 1, \dots, n)$

 Figure 1.3: deformation of $(C, 1, \dots, n)$

(2) $Euler_{\mathbb{T}}(B_4^m)/Euler_{\mathbb{T}}(B_1^m)$.

(2.1) $B_1 = Aut(C, 1, \dots, n) = Hom(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C)$: We should classify what is moving and what is fixed. Basically, we have

$$B_1^m = \bigoplus_{v \in V^1(\Gamma), (e, v) \in F(\Gamma)} T_{y(e, v)} C_e.$$

(2.2) $B_4 = Def(C, 1, \dots, n) = Ext^1(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C)$: \mathbb{P}^1 has just 1 complex structure, so we consider $g(C) \geq 1$. If we don't change node q , C will stay in the same class in $\overline{\mathcal{M}}_{g, n}$. Hence we must resolve the node, and

geometrically, resolution depends on $T_q C_v \otimes T_q C_e$. So basically we have

$$B_4^m = \bigoplus_{v \in V^2(\Gamma), E_v = (e, e')} T_{y_v} C_e \otimes T_{y_v} C_{e'} \oplus \bigoplus_{(e, v) \in F^s(\Gamma)} T_{y(e, v)} C_v \otimes T_{y(e, v)} C_e$$

Returning to the special case $X = \mathbb{P}^n$, we can get the theorem 1.1.1.

1.3 Aspinwall Morrison formula; Faber Pandaripande formula

In this section, we will use Kontsevich's approach to compute the multiple cover contribution of rigidly embedding curves \mathbb{P}^1 in a Calabi-Yau threefold X .

The geometry picture is this. The normal bundle N of $\mathbb{P}^1 \subset X$ is rank 2 and splits on \mathbb{P}^1 . Because X is Calabi-Yau and $c_1(\mathbb{P}^1) = 2$, the normal bundle is of degree 2. Embedded \mathbb{P}^1 's in a Calabi-Yau threefold (not necessary lines) with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ are called rigid. The degree 2 Gromov-Witten invariant of a generic quintic has two contributions:

- (1) rigid conics curves in X ;
- (2) lines with double cover, so this part is related to $\overline{\mathcal{M}}_0(\mathbb{P}^1, 2)$.

We want to compute the contribution of part (2). This problem finally leads to:

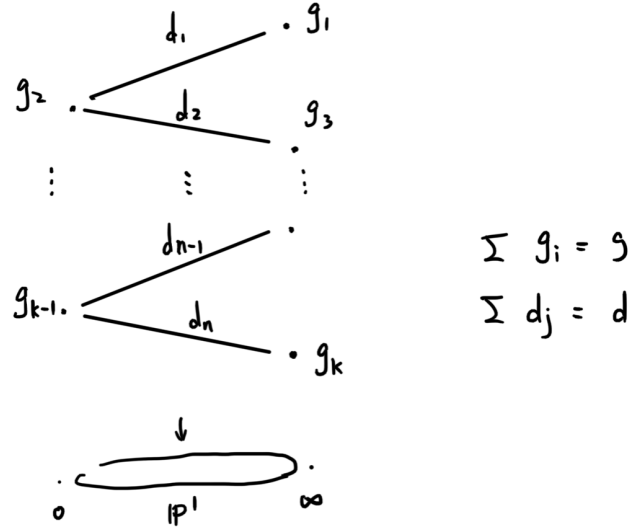
$$N_{g,d} = \int_{\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)} e(R^1 \pi_* f^* N),$$

$$\begin{array}{ccc} \overline{\mathcal{C}}_{g,0}(\mathbb{P}^1, d) & \xrightarrow{f} & \mathbb{P}^1 \\ \downarrow \pi & & \\ \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d) & & \end{array} \quad \text{and } N = \mathcal{O}(-1) \oplus \mathcal{O}(-1).$$

where

The decorated graphs $\vec{\Gamma} = (\Gamma, \vec{f}, \vec{d}, \vec{g}, \vec{s})$ in $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)^{\mathbb{T}}$ are of the type in Figure 1.4. We can choose different lifts on $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ so that only $\vec{\Gamma} = (\Gamma, \vec{f}, \vec{d}, \vec{g}, \vec{s})$ with 1 edge contributing $N_{g,d}$.

- (1) $g = 0$ (Aspinwall Morrison formula): $N_{0,d} = 1/d^3$;


 Figure 1.4: $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)^{\mathbb{T}}$

(2) $g \geq 1$ (Faber-Pandharipande):

$$\begin{aligned}
 N_{g,d} &= \sum_{g_1+g_2=g} \frac{1}{d} \int_{\overline{\mathcal{M}}_{g_1,1}} \lambda_{g_1} \psi^{2g_1-2} d^{2g_1-1} \\
 &\quad \times \int_{\overline{\mathcal{M}}_{g_2,1}} \lambda_{g_2} \psi^{2g_2-2} d^{2g_2-1} = \sum_{g_1+g_2=g} b_{g_1} b_{g_2} d^{2g-3} \\
 b_0 &= 0; b_g = \int_{\overline{\mathcal{M}}_{g,1}} \lambda_{g_2} \psi^{2g-2} \quad (g > 0) \\
 \sum_{g=0}^{\infty} b_g t^{2g} &= \frac{t/2}{\sin t/2}.
 \end{aligned}$$

Then use the Laurent series of $\cot t$, we have

$$N_{1,d} = \frac{1}{12d},$$

$$N_{g,d} = d^{2g-3} \frac{|B_{2g}|}{2g \cdot (2g-2)!} = |\chi(\overline{\mathcal{M}}_g)| \frac{d^{2g-3}}{(2g-3)!}, \quad g \geq 2,$$

where B_g is the Bernoulli number in $\frac{x}{e^x-1}$.

Chapter 2

Quantum Cohomology

Chapter 3

Mirror Symmetry

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