

Quintic threefold and mirror symmetry

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Chapter 1

Differential Topology

1.1 Chern class

Let E be a differentiable complex vector bundle of rank r over a differentiable manifold X , and let $F = dA + A \wedge A$ be the curvature of a connection A on E .

Definition 1.1.1 (total Chern class). *We define the total Chern class of E , $c(E)$, by*

$$\begin{aligned} c(E) &= \det \left(1 + \frac{i}{2\pi} F \right) \\ &= 1 + \frac{i}{2\pi} \text{Tr} F + \dots \\ &= 1 + c_1(E) + c_2(E) + \dots \in H^0(X; \mathbb{R}) \oplus H^2(X; \mathbb{R}) \oplus \dots \end{aligned}$$

Proposition 1.1.2.

- (1) *If E, F are two complex vector bundles over X , then $c(E \oplus F) = c(E)c(F)$*
- (2) *If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a short exact sequence of sheaves, then $c(B) = c(A)c(C)$.*

Definition 1.1.3 (Chern Character). *Suppose $\exists x_i \in H^2(X; \mathbb{R})$ such that $c(E) = \prod_{i=1}^r (1 + x_i)$ ($r \equiv \text{rk}(E)$). Then the Chern character class $ch(E)$ is defined by $ch(E) = \sum_i e^{x_i}$ (Taylor expansion). Then we find*

$$ch(E) = r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + \dots$$

Note $ch(E \oplus F) = ch(E) + ch(F)$ and $ch(E \otimes F) = ch(E)ch(F)$.

Definition 1.1.4 (Todd class).

$$td(E) = \prod_{i=1}^r \frac{x_i}{1 - e^{-x_i}} = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \dots$$

Note that $td(E \oplus F) = td(E)td(F)$.

1.2 The Grothendieck-Riemann-Roch formula

Let E be a sheaf or holomorphic vector bundle over some variety X ; let $H^k(E)$ be the Čech cohomology group of E over X . Define $\chi(E) = \sum_k (-1)^k \dim H^k(E)$. The Grothendieck-Riemann-Roch formula calculates

$$\chi(E) = \int_X ch(E) \wedge td(X).$$

1.3 Serre Duality

Definition 1.3.1. For an almost complex manifold X one defines the complex vector bundles

$$\bigwedge_{\mathbb{C}}^k X := \bigwedge^k (T_{\mathbb{C}}X)^* \quad \text{and} \quad \bigwedge^{p,q} X := \bigwedge^p (T^{1,0}X)^* \otimes_{\mathbb{C}} \bigwedge^q (T^{0,1}X)^*.$$

Their sheaves of sections are denoted by $\mathcal{A}_{X,\mathbb{C}}^k$ and $\mathcal{A}_X^{p,q}$, respectively. Elements in $\mathcal{A}^{p,q}(X)$, i.e. global sections of $\mathcal{A}^{p,q}(X)$, are called forms of type (or bidegree) (p,q) .

The complex vector bundles Ω_X^p and $\bigwedge^{p,0} X$ of a complex manifold X can be identified.

Corollary 1.3.2.

$$\bigwedge_{\mathbb{C}}^k X = \bigoplus_{p+q=k} \bigwedge^{p,q} X \quad \text{and} \quad \mathcal{A}_{X,\mathbb{C}}^k = \bigoplus_{p+q=k} \mathcal{A}_X^{p,q}.$$

Moreover, $\overline{\bigwedge^{p,q} X} = \bigwedge^{q,p} X$ and $\overline{\mathcal{A}_X^{p,q}} = \mathcal{A}_X^{q,p}$.

Definition 1.3.3 (Dolbeault cohomology). Let X be endowed with an integrable almost complex structure. Then the (p,q) -Dolbeault cohomology is the vector space

$$H^{p,q}(X) := H^q(\mathcal{A}^{p,-}(X), \bar{\partial}) = \frac{\text{Ker}(\bar{\partial} : \mathcal{A}^{p,q}(X) \rightarrow \mathcal{A}^{p,q+1}(X))}{\text{Im}(\bar{\partial} : \mathcal{A}^{p,q-1}(X) \rightarrow \mathcal{A}^{p,q}(X))}$$

Corollary 1.3.4. The Dolbeault cohomology of X computes the cohomology of the sheaf Ω_X^p , i.e. $H^{p,q}(X) \cong H^q(X, \Omega_X^p)$.

Definition 1.3.5. By $\mathcal{A}^{p,q}(E)$ we denote the sheaf

$$U \mapsto \mathcal{A}^{p,q}(U, E) := \Gamma(U, \bigwedge^{p,q} X \otimes E).$$

Let α be a section of $\mathcal{A}^{p,q}(E)$. The differential d is not well-defined on α .

Lemma 1.3.6. If E is a holomorphic vector bundle then there exists a natural \mathbb{C} -linear operator $\bar{\partial}_E : \mathcal{A}^{p,q}(E) \rightarrow \mathcal{A}^{p,q+1}(E)$ with $\bar{\partial}_E^2 = 0$ which satisfies the Leibniz rule $\bar{\partial}_E(f \cdot \alpha) = \bar{\partial}(f) \wedge \alpha + f \bar{\partial}_E(\alpha)$.

Proof. Locally $\alpha = \sum \alpha_i \otimes s_i$ with $\alpha_i \in \mathcal{A}_X^{p,q}$ and $s_i \in E$. Then set

$$\bar{\partial}_E \alpha := \sum \bar{\partial}(\alpha_i) \otimes s_i. \quad \square$$

Definition 1.3.7. *The Dolbeault cohomology of a holomorphic vector bundle E is*

$$H^{p,q}(X, E) := H^q(\mathcal{A}^{p,-}(X, E), \bar{\partial}_E) = \frac{\text{Ker}(\bar{\partial}_E : \mathcal{A}^{p,q}(X, E) \rightarrow \mathcal{A}^{p,q+1}(X, E))}{\text{Im}(\bar{\partial}_E : \mathcal{A}^{p,q-1}(X, E) \rightarrow \mathcal{A}^{p,q}(X, E))}.$$

Corollary 1.3.8. $H^{p,q}(X, E) \cong H^q(X, E \otimes \Omega_X^p)$.

Let E be a holomorphic vector bundle over a compact complex manifold X of dimension n and consider the natural pairing

$$H^{p,q}(X, E) \times H^{n-p,n-q}(X, E^*) \longrightarrow \mathbb{C}, \quad (\alpha, \beta) \longmapsto \int_X \alpha \wedge \beta$$

Proposition 1.3.9. *Let X be a compact complex manifold. For any holomorphic vector bundle E on X the natural pairing*

$$H^{p,q}(X, E) \times H^{n-p,n-q}(X, E^*) \longrightarrow \mathbb{C}$$

is non-degenerate.

Corollary 1.3.10. *By Dolbeault isomorphism:*

$$H^q(X, \Omega^p \otimes E) \times H^{n-q}(X, \Omega^{n-p} \otimes E^*) \rightarrow \mathbb{C}$$

is non-degenerate. Furthermore, let $p = 0$

$$H^q(X, E) \times H^{n-q}(X, K_X \otimes E^*) \rightarrow \mathbb{C}$$

is non-degenerate.

In the special case where X is Calabi-Yau, K_X is trivial and

$$H^q(X, E) \times H^{n-q}(X, E^*) \rightarrow \mathbb{C}$$

is non-degenerate.

1.4 Chern class of \mathbb{P}^n

Let $H = \mathcal{O}(1)$ be the hyperplane bundle on \mathbb{P}^n . Consider homogeneous coordinate $[X_0, \dots, X_n]$. Since $X_0^2 + \dots + X_n^2 = 1$, differentiate this formula we find $X_i \frac{\partial}{\partial X_i} = 0$. This gives the exact sequence, the Euler sequence:

$$\begin{aligned} 0 \rightarrow \mathbb{C} \rightarrow H^{\oplus(n+1)} \rightarrow T\mathbb{P}^n \rightarrow 0 \\ (a_0 X_0, \dots, a_n X_n) \mapsto a_i X_i \frac{\partial}{\partial X_i} \end{aligned}$$

where $a_i \in \mathbb{C}$.

Since $c(\mathbb{C}) = 1$, $c(\mathbb{P}^n) = c(T\mathbb{P}^n) = c(H^{\oplus(n+1)}) = [c(H)]^{n+1}$. Let $x = c_1(H)$. Then

$$c(\mathbb{P}^n) = (1 + x)^{n+1} = \sum_{i=0}^n \binom{n+1}{i} x^i$$

It gives an example to check Chern-Gauss-Bonnet formula: $c_n(\mathbb{P}^n) = (n+1)x^n$. The Poincare duality gives that

$$\int_{\mathbb{P}^n} x^n = \# \text{intersection of } n \text{ transverse hyperplane } H (\cong \mathbb{P}^{n-1}) = 1$$

$$\int c_n(\mathbb{P}^n) = n+1 = \chi(\mathbb{P}^n)$$

This corresponds to the conclusion in CW-structure of \mathbb{P}^n .

1.5 adjunction formulas

Let X be a smooth hypersurface in \mathbb{P}^n defined as the zero-locus of a degree d polynomial, p (so p is a section of $\mathcal{O}_{\mathbb{P}^n}(d)$, or H^d). The normal bundle N_X of X in \mathbb{P}^n is just $\mathcal{O}(d)|_X$. As a result, we have an exact sequence

$$0 \rightarrow TX \rightarrow T\mathbb{P}^n|_X \rightarrow \mathcal{O}(d)|_X \rightarrow 0.$$

Now $ch(H) = e^x \Rightarrow ch(H^d) = e^{dx} = 1 + c_1(H^d) + \dots$, so

$$c(\mathcal{O}(d)) = 1 + c_1 = 1 + dx$$

$$c(X) = \frac{(1+x)^{n+1}}{1+dx}$$

The Euler class $e(X)$ of the normal bundle of a subvariety $X \subset \mathbb{P}^n$ is equal to its Thom class, namely its Poincare dual cohomology cycle. This means

$$\int_X \theta = \int_{\mathbb{P}^n} \theta e(X).$$

In the case of hypersurface, the normal bundle is one-dimensional, so $e(X) = c_{top}(N_{X/\mathbb{P}^4}) = c_1(\mathcal{O}(d)) = d \cdot x$.

1.6 quintic hypersurface

Now consider the quintic hypersurface in \mathbb{P}^4 . A quintic hypersurface Q in \mathbb{P}^4 has

$$c(Q) = \frac{(1+x)^5}{(1+5x)} = 1 + 10x^2 - 40x^3.$$

Note that $c_1(Q) = 0$, so Q is a Calabi-Yau manifold. Its Euler characteristic is

$$\int_Q -40x^3 = \int_{\mathbb{P}^4} -40x^3(5x) = -200$$

A general formula is given in [3], page 11: If X is a hypersurface in \mathbb{CP}^n with degree d , then its Euler characteristic is

$$\chi(X) = \frac{1}{d} \cdot ((1-d)^{n+1} + d \cdot (n+1) - 1).$$

Chapter 2

Calabi-Yau Manifolds and Mirror Symmetry

2.1 Calabi-Yau manifolds

Definition 2.1.1 (Calabi-Yau manifold 1). *Let $m \geq 2$. A Calabi-Yau m -fold is a quadruple (M, J, g, Ω) such that (M, J) is a compact m -dimensional complex manifold, g a Kahler metric on (M, J) with holonomy group $\text{Hol}(g) = \text{SU}(m)$, and Ω a nonzero constant $(m, 0)$ -form on M called the holomorphic volume form, which satisfies*

$$\omega^m / m! = (-1)^{m(m-1)/2} (i/2)^m \Omega \wedge \bar{\Omega} \quad (*)$$

where ω is the Kahler form of g . The constant factor in $(*)$ is chosen to make $\text{Re } \Omega$ a calibration.

Definition 2.1.2 (Calabi-Yau manifold 2). *A Calabi-Yau manifold is a compact Kahler manifold X with trivial canonical bundle $\omega_X \cong \mathcal{O}_X$.*

Example 2.1.3. *If X is a simply-connected Calabi-Yau 3-fold, then $H^1(X, \mathcal{O}_X) = 0$.*

$$H^1(X, \mathcal{O}_X) \xrightarrow[\text{duality}]{\text{Serre}} H^2(X, \mathcal{O}_X \otimes \omega_X)^* = H^{3,2}(X, \mathbb{C}) = H^{0,1}(X, \mathbb{C}) = 0$$

2.2 Complex structure and Bogomolov-Tian-Todorov Theorem

Definition 2.2.1. *Let X be a differentiable manifold of dimension $2n$. Suppose that J is a differentiable vector bundle isomorphism*

$$J : TX \rightarrow TX$$

such that $J^2 = -I$. J is called an almost complex structure for the differentiable manifold X . If X is equipped with an almost complex structure J , then (X, J) is called an almost complex manifold.

In local (real) coordinate $\{\frac{\partial}{\partial x^a}\}_{a=1}^{2n}$ we can write J in terms of a matrix J^a_b , where $J(\frac{\partial}{\partial x^a}) = J^c_a \frac{\partial}{\partial x^c}$.

Since $P = (1 - iJ)/2$ is a projection onto the holomorphic sub-bundle of the tangent bundle (tensor with \mathbb{C}) and $\bar{P} = (1 + iJ)/2$ is the anti-holomorphic projection, the condition of integrability for finding complex coordinates is

$$\bar{P}[PX, PY] = 0$$

where $X = X^a \frac{\partial}{\partial x^a}$ and $Y = Y^b \frac{\partial}{\partial x^b}$. Define the Nijenhuis tensor N by $N(X, Y) = [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y]$. In local coordinates x^a ,

$$N_{bc}^a = J_b^d (\partial_d J_c^a - \partial_c J_d^a) - J_c^d (\partial_d J_b^a - \partial_b J_d^a).$$

The integrability condition is equivalent to $N \equiv 0$. It is also equivalent to $\bar{\partial}^2 = 0$.

In complex coordinate, let us fix a complex structure and compatible complex coordinates z^1, \dots, z^n . We use J_b^a , $J_{\bar{b}}^{\bar{a}}$, $J_{\bar{b}}^a$ and $J_b^{\bar{a}}$. In fact, because $J_b^a z^b = iz^a = iz^b \delta_b^a$, $J_{\bar{b}}^{\bar{a}} z^{\bar{b}} = iz^{\bar{a}} = iz^{\bar{b}} \delta_{\bar{b}}^{\bar{a}}$. J is diagonalized in these coordinates, so that $J_b^a = i\delta_b^a$ and $J_{\bar{b}}^{\bar{a}} = -i\delta_{\bar{b}}^{\bar{a}}$, with mixed component zero.

Now given a smooth manifold X , we try to study all complex structure could be endowed in such a manifold X . At first, one naively define the set

$$\mathcal{A}_c(X) := \{J \in \text{End}(TX) | J \text{ is an integrable almost complex structure in } X\}.$$

But that is too redundant. Recall that two complex manifolds (X, J) and (X', J') are isomorphic if there exists a diffeomorphism $F : X \rightarrow X'$ such that $dF \circ J = J' \circ dF$. Thus, the set of diffeomorphism classes of complex structures J on a fixed smooth manifold X is the quotient of the set $\mathcal{A}_c(X)$ by the action of the diffeomorphism group

$$\text{Diff}(X) \times \mathcal{A}_c(X) \longrightarrow \mathcal{A}_c(X), (F, J) \longmapsto dF \circ J \circ (dF)^{-1}.$$

Next we define the infinitesimal deformation of a complex structure by its power series expansion.

We start out with the set

$$\mathcal{A}_{ac}(X) := \{J | J^2 = -id\} \subset \text{End}(TX)$$

of all almost complex structures on X . It could be shown that $\mathcal{A}_{ac}(X)$ is an infinite dimensional manifold. Moreover, this statement is no longer true for $\mathcal{A}_c(X)$. Let $J(t)$ be a continuous path of almost complex structures with $J(0) = J$. Then one has a continuous family of such decompositions $T_{\mathbb{C}}M = T_t^{1,0} \oplus T_t^{0,1}$ or, equivalently, of subspaces $T_t^{0,1} \subset T_{\mathbb{C}}M$ (retrieve $T_t^{1,0}$ by conjugation).

Thus, for small t the deformation $J(t)$ of J can be encoded by a map

$$\phi(t) : T^{0,1} \longrightarrow T^{1,0} \text{ with } v + \phi(t)(v) \in T_t^{0,1}.$$

We write $T^{1,0}$ and $T^{0,1}$ for subbundles defined by J . Explicitly, one has

$$\phi(t) = -pr_{T_t^{1,0}} \circ j,$$

where $j : T^{0,1} \subset T_{\mathbb{C}}$ and $pr_{T_{\mathbb{C}}^{1,0}} : T_{\mathbb{C}} \rightarrow T_t^{1,0}$ are the natural inclusion respectively projection.

Conversely, if $\phi(t)$ is given, then one defines for small t

$$T_t^{0,1} := (id + \phi(t))(T^{0,1}).$$

Let us now consider the power series expansion

$$\phi(t) = \phi_0 + \phi_1 t + \phi_2 t^2 + \dots$$

Lemma 2.2.2. *The integrability equation $[T_t^{0,1}, T_t^{0,1}] \subset T_t^{0,1}$ is equivalent to the Maurer-Cartan equation*

$$\bar{\partial}\phi(t) + [\phi(t), \phi(t)] = 0 \in \mathcal{A}^{0,2}(T^{1,0}X)$$

This yields a recursive system of equations:

$$\begin{aligned} 0 &= \bar{\partial}\phi_1 \\ 0 &= \bar{\partial}\phi_2 + [\phi_1, \phi_1] \\ &\dots \\ 0 &= \bar{\partial}\phi_k + \sum_{0 < i < k} [\phi_i, \phi_{k-i}]. \end{aligned}$$

The first-order equation $\bar{\partial}\phi_1 = 0$ defines an element $[\phi_1] \in H^1(X, \mathcal{T}_X)$.

Definition 2.2.3 (Kodaira-Spencer class). *The **Kodaira-Spencer class** of a one-parameter deformation $J(t)$ of the complex structure J is the induced cohomology class $[\phi_1] \in H^1(X, \mathcal{T}_X)$.*

Proposition 2.2.4. *Let X be a complex manifold. There is a natural bijection between all first-order deformations of X and elements of $H^1(X, \mathcal{T}_X)$.*

Corollary 2.2.5. *A first-order deformation $v \in H^1(X, \mathcal{T}_X)$ cannot be integrated if $[v, v] \in H^2(X, \mathcal{T}_X)$ does not vanish.*

Proposition 2.2.6 (Bogomolov-Tian-Todorov unobstructedness theorem). *Let X be a Calabi-Yau manifold and let $v \in H^1(X, \mathcal{T}_X)$. Then there exists a formal power series $\phi_1 t + \phi_2 t^2 + \dots$ with $\phi_i \in \mathcal{A}^{0,1}(\mathcal{T}_X)$ satisfying the Maurer-Cartan equations*

$$\bar{\partial}\phi_1 = 0 \text{ and } \bar{\partial}\phi_k = - \sum_{0 < i < k} [\phi_i, \phi_{k-i}],$$

with $[\phi_1] = v$ and such that

$$\eta(\phi_i) \in \mathcal{A}^{n-1,1}(X) \text{ is } \partial - \text{exact}$$

for all $i > 1$.

Remark 2.2.7. *The corollary 2.2.5 states if $H^2(X, \mathcal{T}_X) = 0$, the evolution of the Maurer-Cartan equation has no obstruction. But for a Calabi-Yau manifold X , its $H^2(X, \mathcal{T}_X)$ usually does not vanish, e.g. for a Calabi-Yau quintic 3-fold,*

$$H^2(X, \mathcal{T}_X) = H^2(X, \Omega_X^2) = H^{2,2}(X, \mathbb{C}) = H^{1,1}(X, \mathbb{C}) = \mathbb{C} \neq 0.$$

But even if the second cohomology group does not vanish, the deformation of complex structure can be done in a Calabi-Yau manifold. That is why BTT unobstructedness theorem is important.

2.3 Kahler moduli space

2.4 Pesudo-holomorphic curves

Definition 2.4.1 (J-holomorphic curves). *Let (Σ, j) be a Riemann surface, (X, J) be an almost complex manifold. A smooth map $u : \Sigma \rightarrow X$ is called **J-holomorphic** if u_* satisfies*

$$J \circ u_* = u_* \circ j$$

Equivalently, for a map $u : \Sigma \rightarrow X$, put

$$\bar{\partial}_J(u) = \frac{1}{2}(u_* + J \circ u_* \circ j) \in \Gamma(\Sigma, \Omega_{\Sigma}^{0,1} \otimes u^* \mathcal{T}_X),$$

It is clear that u is J-holomorphic if and only if $\bar{\partial}_J(u) = 0$.

Definition 2.4.2. $u : \Sigma \rightarrow X$ is **somewhere injective**, or **simple** if \exists a point $z \in \Sigma$ such that u_* is injective at z and $u^{-1}(u(z)) = \{z\}$.

For convenience, let us define

- $\text{Map}(\Sigma, X) = \{u : \Sigma \rightarrow X \mid u \text{ is smooth}\}$
 - For any $\eta \in H^2(X, \mathbb{Z})$, let
- $$\text{Map}(\Sigma, X, \eta) = \{u \in \text{Map}(\Sigma, X) \mid u \text{ is a simple map, } [\text{im } u] = \eta\}$$

What we want is to give a math definition about Gromov-Witten invariant of X , an enumerative invariant associated to the Kahler form ω of X . To accomplish this aim, we use almost complex structure J compatible with ω to define the moduli space of J-holomorphic curves at first. Then we try to show the invariant defined independent to the choice of J .

Definition 2.4.3 (compatible almost complex structure). *Fix a real Kahler form of a Kahler metric (or real symplectic form, or) ω of a Kahler manifold X . We say an almost complex structure J is compatible with ω if*

$$\omega(v, Jv) > 0 \quad \forall v \in \mathcal{T}_X, v \neq 0, \text{ and}$$

$$\omega(Jv, Jw) = \omega(v, w) \quad \forall v, w \in \mathcal{T}_X.$$

Let $\mathcal{J}(\omega)$ be the set of almost complex structure compatible with ω .

Definition 2.4.4. *Given a homological class $\eta \in H^2(X, \mathbb{Z})$, an associated almost complex structure J , put*

$$M(\eta, J, \Sigma) = \text{the zero locus of } \bar{\partial}_J$$

= Moduli space of simple J-holomorphic map representing the homology class η

We want to say something about the space $M(\eta, J, \Sigma)$. This space has nice properties generically. For each $u \in \mathcal{X} = \text{Map}(\Sigma, X)$, define the fibre

$$\mathcal{E}_u = \Gamma(\Sigma, \Omega_{\Sigma}^{0,1} \otimes u^* \mathcal{T}_X)$$

It gives a vector bundle \mathcal{E} over \mathcal{X} . Because $\bar{\partial}_J$ is a smooth section from \mathcal{X} to \mathcal{E} , we can define a map:

$$\mathcal{T}_{\mathcal{X},u} \xrightarrow{(\bar{\partial}_J)^*} \mathcal{T}_{\mathcal{E},(u,0)} = \mathcal{T}_{\mathcal{X},u} \oplus \mathcal{E}_u \xrightarrow{\pi} \mathcal{E}_u.$$

u is called **regular** if $\pi \circ (\bar{\partial}_J)_*$ is surjective.

$$\mathcal{J}_{reg}(\eta, \omega, \Sigma) = \{J \in \mathcal{J}(\omega) \mid u \text{ is regular for all } u \in M(\eta, J, \Sigma)\}.$$

Theorem 2.4.5.

- (1) If $J \in \mathcal{J}_{reg}(\eta, \omega, \Sigma)$, then $M(\eta, J, \Sigma)$ is a smooth manifold of real dimension $n(2 - 2g) + 2c_1(X) \cdot \eta$.
- (2) $\mathcal{J}_{reg}(\eta, \omega, \Sigma)$ has a second category in $\mathcal{J}(\omega)$.

The following question is to find a criterion to the regularity of u .

Theorem 2.4.6 (Regularity criterion). *If J is an integrable almost complex structure on X , and $u : \mathbb{P}^1 \rightarrow X$ is a J -holomorphic curve, then u is regular if in the decomposition $u^*\mathcal{T}_X = \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(a_i)$ we have $a_i \geq -1$ for all i .*

Remark 2.4.7. *In the criterion, we use a classical theorem from Grothendick: any holomorphic vector bundle on \mathbb{P}^1 decomposes as a direct sum of line bundles. Any line bundle on \mathbb{P}^1 is determined by c_1 .*

$$\mathcal{O}_{\mathbb{P}^1}(a) = \text{the line bundle with } c_1 = a$$

In the special case that $\Sigma = \mathbb{P}^1$, X is a Calabi-Yau 3-fold, e.g. quintic 3-fold, we have $n = 3$, $c_1(X) = 0$, $g(\mathbb{P}^1) = 0$. The regularity criterion in $u : \mathbb{P}^1 \rightarrow X$ becomes

Proposition 2.4.8. *u is regular if in the decomposition $u^*\mathcal{T}_X = \mathcal{O}_{\mathbb{P}^1}(a) \oplus_i \mathcal{O}_{\mathbb{P}^1}(b)$ we have $a = b = -1$.*

If $\in \mathcal{J}_{reg}(\eta, \omega, \mathbb{P}^1)$ then by Theorem 2.4.5

$$\dim_{\mathbb{R}} M(\eta, J, \mathbb{P}^1) = 3(2 - 2 \cdot 0) + 2 \cdot 0 \cdot \eta = 6$$

$$\text{Aut}(\mathbb{P}^1) = PSL(2, \mathbb{C}), \quad \dim_{\mathbb{R}} \text{Aut}(\mathbb{P}^1) = 6$$

$$n_{\eta} := \# \overline{M(\eta, J, \mathbb{P}^1) / PSL(2, \mathbb{C})} \text{ is finite.}$$

The number n_{η} is the definition of Gromov-Witten invariant in this special case. It describes the number of J -holomorphic curves with image in the homology class η under the automorphism equivalence of \mathbb{P}^1 is generically finite. Since $h^2(X; \mathbb{Z}) = h^{1,1}(X; \mathbb{Z}) = 1$, we use $d \in \mathbb{Z}$ to represent the homology class η in $H^2(X; \mathbb{Z})$, so

$$n_d := \# \overline{M(d, J, \mathbb{P}^1) / PSL(2, \mathbb{C})}.$$

is well-defined.

2.5 Mirror pair of quintic 3-fold

Let $f_\varphi = x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5\varphi x_0 x_1 x_2 x_3 x_4$.

Let X_φ be a smooth hypersurface $f_\varphi = 0$ in \mathbb{P}^4 . The Hodge diamond of X is

$$\begin{array}{ccccc}
 & & 1 & & \\
 & & 0 & & 0 \\
 & 0 & & 1 & & 0 \\
 1 & & 101 & & 101 & & 1 \\
 & 0 & & 1 & & 0 \\
 & & 0 & & 0 \\
 & & 1 & &
 \end{array}$$

There is a $G = (\mathbb{Z}/5\mathbb{Z})^5$ action on \mathbb{P}^4 :

$$(\mathbb{Z}/5\mathbb{Z})^5 \times \mathbb{P}^4 \longrightarrow \mathbb{P}^4, \quad \lambda = e^{2\pi i/5}.$$

$$(a_0, a_1, a_2, a_3, a_4), [z_0 : z_1 : z_2 : z_3 : z_4] \mapsto [\lambda^{a_0} z_0 : \lambda^{a_1} z_1 : \lambda^{a_2} z_2 : \lambda^{a_3} z_3 : \lambda^{a_4} z_4].$$

For those smooth X_φ , take the quotient of X_φ by $(\mathbb{Z}/5\mathbb{Z})^5$, we get some A_n singularities. Blow-up the singularities of X_φ/G , get a new smooth Calabi-Yau manifold Y_φ with extra 100 divisors \mathbb{P}^1 . The Hodge diamond of Y_φ is

$$\begin{array}{ccccc}
 & & 1 & & \\
 & & 0 & & 0 \\
 & 0 & & 101 & & 0 \\
 1 & & 1 & & 1 & & 1 \\
 & 0 & & 101 & & 0 \\
 & & 0 & & 0 \\
 & & 1 & &
 \end{array}$$

We can see X_φ and Y_φ has symmetry Hodge diamond over the diagonal line. This is the first (maybe) mirror pair found in history.

2.6 Yukawa coupling and mirror symmetry

In physics(QFT), Yukawa coupling is a quantity to describe the interaction between Neutrino and Higgs field. There are two kinds of Yukawa couplings in physics. Let X be a quintic 3-fold.

The A-model is of the Kahler form of $X = X_\varphi$:

$$\langle h, h, h \rangle_A := 5 + \sum_{d=1}^{\infty} n_d d^3 \frac{q^d}{1 - q^d},$$

where n_d is the Gromov-Witten invariant defined in 2.4.

The B-model is of the complex structure of $\check{X} = Y_\varphi$:

$$\langle \partial_z, \partial_z, \partial_z \rangle_B := \int_{\check{X}} \check{\Omega} \wedge \partial_z \partial_z \partial_z \check{\Omega},$$

where $\check{\Omega}$ is the normalized Calabi-Yau 3-form of \check{X} . We choose a Calabi-Yau 3-form Ω , the normalized Calabi-Yau 3-form $\check{\Omega}$ is

$$\check{\Omega} = \frac{\Omega}{\int_{\beta_0} \Omega},$$

where β_0 is a three torus by taking limit $\varphi \rightarrow \infty$.

The mirror conjecture states that under the coordinate map $q = e^{2\pi i w(z)}$ two Yukawa coupling is equal:

$$\langle h, h, h \rangle_A = \langle \partial_z, \partial_z, \partial_z \rangle_B,$$

where

$$w(z) = \int_{\beta_1} \check{\Omega} = \frac{\int_{\beta_1} \Omega}{\int_{\beta_0} \Omega}$$

for some β_1 in Hodge bundle and $\{\beta_0, \beta_1\}$ is a part of a symplectic basis of Hodge bundle.

Historically, physicists wanted to compute $\langle h, h, h \rangle_A$. But in 1980s the Gromov-Witten invariant is unknown for $n \geq 3$. $n_1 = 2875$ is a classical result, and in 1986? S.Katz computes $n_2 = 609250$. Thus the mirror conjecture gives a way to compute Gromov-Witten invariant by B-model Yukawa coupling. By computation, \exists constant c_1, c_2 such that

$$\langle \partial_z, \partial_z, \partial_z \rangle_B = -c_1 - 575 \frac{c_1}{c_2} q - \frac{1950750}{2} \frac{c_1}{c_2^2} q^2 - \frac{10277490000}{6} \frac{c_1}{c_2^3} q^3 + \dots$$

$$\langle h, h, h \rangle_A = 5 + n_1 q + (8n_2 + n_1) q^2 + (27n_3 + n_1) q^3 + \dots$$

$n_1 = 2875$ shows $c_1 = -5$, $c_2 = 1$, and get the Table 2.1.

degree d	Gromov-Witten invariant n_d
1	2875
2	609250
3	317206375
4	242467530000
5	229305888887625
6	248249742118022000
7	295091050570845659250
8	375632160937476603550000
9	503840510416985243645106250
10	704288164978454686113488249750

Table 2.1: computation by B-model

It is conjectured that n_d is the value as above. The conjecture for all d was proven by Givental in 1996 and Lian, Liu, and Yau in 1997.

Chapter 3

Toric Geometry

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