Gromov-Witten theory and mirror symmetry

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Contents

Chapter 1

Gromov-Witten invariants

Roughly speaking, Gromov-Witten invariants is the number of algebraic curves (resp. pseduo-holomorphic curves) of a algebraic variety X (resp. complex manifold) passing through specific subvarieties (resp. submanifolds) under specific degree (homology class of the curves) and genus.

Definition 1.0.1. *Let $\gamma_1, \ldots, \gamma_n \in H^*(X;\mathbb{Q})$ and let $\beta \in H^2(X;\mathbb{Q})$. The Gromov-Witten invariant of genus g degree β curves is*

$$\langle \gamma_1 \ldots \gamma_n \rangle_{g,n,\beta} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} ev_1^*(\gamma_1) \cup \cdots \cup ev_n^*(\gamma_n).$$

Here, a point in $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}$ is $[f : C \to X, 1, \ldots, n]$:

a map from the genus g curve C to the variety X modulo the automorphism of C.

The evaluation map $ev_i : [\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir} \to X$ is given by

$$ev_i([f : C \to X, 1, \ldots, n]) = f(i).$$

Let $\overline{\mathcal{M}}_{g,n}$ denote the moduli space (Deligne-Mumford stack) of genus g curves with n marked points, and let $\overline{\mathcal{C}}_{g,n}$ be the universal family of $\overline{\mathcal{M}}_{g,n}$.

1.1 Kontsevich's approach

Atiyah-Bott localization formula tells us: if there is a torus action $\mathbb{T} = (\mathbb{C}^*)^n$ on X, then the fixed points of torus action could tells us some properties of X.

By the classifying space theory, $B\mathbb{T} = (\mathbb{C}P^\infty)^{\times n}$, so $H^*(B\mathbb{T}) = \mathbb{Q}[\lambda_1, \ldots, \lambda_n]$. Let $\mathcal{R}_\mathbb{T} = \mathbb{Q}(\lambda_1, \ldots, \lambda_n)$. Let $X_\mathbb{T} = E\mathbb{T} \times_\mathbb{T} X$, the equivariant cohomology of X is defined by

$$H^*_\mathbb{T}(X) := H^*(X_\mathbb{T}) = H^*(E\mathbb{T} \times_\mathbb{T} X),$$

so naturally, $H^*_\mathbb{T}(X)$ is a $H^*_\mathbb{T}(pt) = H^*(B\mathbb{T})$-module. The localization of $H^*_\mathbb{T}(X)$ means $H^*_\mathbb{T}(X) \otimes \mathcal{R}_\mathbb{T}$.

Theorem (Atiyah-Bott)**.** *Let $X^\mathbb{T}$ be fixed locus of $\mathbb{T}$, let Z_j be a connection component of $X^\mathbb{T}$, and let N_j be the normal bundle of Z_j in X. Let $i_j : Z_j \to X$ and let $i_{j!} : H^*_\mathbb{T}(Z_j) \to H^*_\mathbb{T}(X)$ be the pushforward defined by the Gysin map. Let $\alpha \in H^*_\mathbb{T}(X) \otimes \mathcal{R}_\mathbb{T}$, we have*

$$\alpha = \sum_j \frac{i_{j!} i_j^* \alpha}{Euler_T(N_j)},$$

In particular,

$$\int_{X_\mathbb{T}} \alpha = \sum_j \int_{(Z_j)_\mathbb{T}} \frac{i_j^* \alpha}{Euler_T(N_j)}.$$

Kontsevich's approach is to apply Atiyah-Bott localization formula in $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}$ so that we can simplify the computation. We can lift the $\mathbb{T}$ action on X to $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}$ in the following way: let $t \in \mathbb{T}$, $[f : C \to X, 1, \ldots, n] \in [\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}$, $x \in X$

$$(t \cdot f)(x) = f(t \cdot x).$$

In special, we will consider $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ in this section. As claimed before, we need to find $[f : C \to X, 1, \ldots, n] \in \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^\mathbb{T}$. The fixed points of $\mathbb{P}^r$ is

$$\{q_i = [0 : 0 : \cdots : 1 : 0 : \cdots : 0]\}_{0 \le i \le r}.$$

The coordinate curve l_{ij} connecting q_i, q_j has one dimensional degree of freedom $\mathbb{C}^*$ (as an invariant component). The curve $C \in \overline{\mathcal{C}}_{g,n}$ is stable (i.e. Aut$(C) < \infty$) if and only if $2g - 2 + n > 0$. If a components C' of C is mapped to l_{ij}, then C' has two points mapped to q_i, q_j respectively (equivalent to with two marked points in C'), so $2g - 2 + 2 \le 0$ implies $g = 0$, i.e. $C' \cong \mathbb{P}^1$ (see Fig 1.1). Meanwhile, $f|_{C'}$ must be uniformly ramified, so $f|_{C'}(z) = z^e, \forall z \in C' \cong \mathbb{P}^1$, for some $e \in \mathbb{N}^*$.

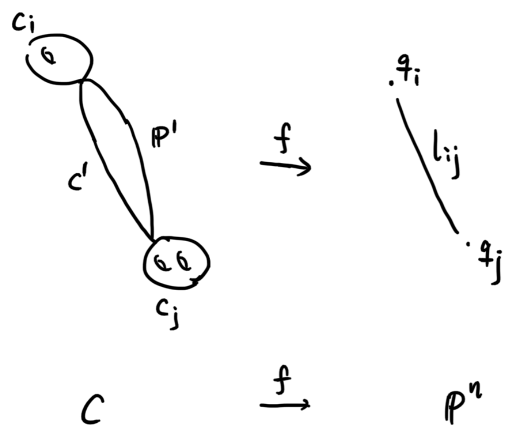

Figure 1.1: $f(C_i) = q_i,\ f(C') = l_{ij},\ f(C_j) = q_j$

It is convenient to use a decorated graph $\vec{\Gamma} = (\Gamma, \vec{f}, \vec{d}, \vec{g}, \vec{s})$ (graph, maps, degrees, genus, marked points) to represent $[f : C \to X, 1, \ldots, n] \in \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^{\mathbb{T}}$. Let val$(v)$, the valence of v, be the number of edges connecting vertex v, and let $n(v) = |s_v| + val(v)$. The stable map $[f : C \to X, 1, \ldots, n]$ with fixed graph $\vec{\Gamma} = (\Gamma, \vec{f}, \vec{d}, \vec{g}, \vec{s})$ defines a substack

$$\overline{\mathcal{M}}_{\vec{\Gamma}} \subset \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d).$$

On the other sides, consider

$$\varphi_{\vec{\Gamma}} : \prod_{\dim C_v = 1} \overline{M}_{g_v, n(v)} \to \overline{\mathcal{M}}_{\vec{\Gamma}}.$$

If v, v' are connected by an edge e, then let $C_v, C_{v'}$ connected by a $C_e \cong \mathbb{P}^1$ associated with a degree d_e map to $\mathbb{P}^r$. Let $\overline{M}_\Gamma$ be the product of above C_v, C_e. There is a group $\mathbb{A}_\Gamma$ acting on $\overline{M}_\Gamma$. The group $\mathbb{A}_\Gamma$ is defined by:

$$1 \to \prod_{edges} \mathbb{Z}/(d_e) \to \mathbb{A}_\Gamma \to Aut(\Gamma) \to 1.$$

and

$$\overline{\mathcal{M}}_{\vec{\Gamma}} = \overline{M}_\Gamma / \mathbb{A}_\Gamma.$$

Therefore, we know the $\mathbb{T}$-fixed locus of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ is $\overline{\mathcal{M}}_{\vec{\Gamma}}$. Let N_Γ be the normal bundle of $\overline{\mathcal{M}}_{\vec{\Gamma}}$ in $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$. Then there is an explicit formula for the equivariant Euler class. Before doing that, we define some necessary notations. A flag F is a pair (v, e) such that e is an edge containing the vertex v. We put $i(F) = v$, $j(F)$ the vertex of e different from v. Let

$$\omega_F = \frac{\lambda_{i(F)} - \lambda_{j(F)}}{d_e} \in H^2_{\mathbb{T}}(pt) = \mathbb{Q}[\lambda_0, \ldots, \lambda_r],$$

which is the weight of $\mathbb{T}$-action on $T_{q_{i_v}} C_e$.

Theorem 1.1.1 ($Euler_{\mathbb{T}}(N_\Gamma)$)**.** *$Euler_{\mathbb{T}}(N_\Gamma) = e^F_\Gamma e^v_\Gamma e^e_\Gamma$, where*

$$e^F_\Gamma = \prod_{n(i(F))\geq 3} (\omega_F - \psi_F) / \prod_{j \neq i(F)} (\lambda_{i(F)} - \lambda_j),$$

$$e^v_\Gamma = \prod_v \prod_{j \neq i_v} (\lambda_{i_v} - \lambda_j) \prod_{val(v)=2, s_v=\emptyset} (\omega_{F_1(v)} + \omega_{F_2(v)}) / \prod_{val(v)=1, s_v=\emptyset} \omega_{F(v)}$$

$$e^e_\Gamma = \prod_e \frac{(-1)^{d_e} (d_e!)^2 (\lambda_i - \lambda_j)^{2d_e}}{d_e^{2d_e}} \prod_{a+b=d_e, k \neq i,j} (\frac{a\lambda_i + b\lambda_j}{d_e} - \lambda_k)$$

The proof is partially discussed in section 1.2.

1.2 Tangent-obstruction sequence

Consider $[f : C \to X, 1, \ldots, n] \in \overline{\mathcal{M}}_{\vec{\Gamma}}$, $\vec{\Gamma} = (\Gamma, \vec{f}, \vec{d}, \vec{g}, \vec{s})$. We put

$$V^1(\Gamma) := \{v \in V(\Gamma) : g_v = 0, val(v) = 1, |s_v| = 0\}$$

$$V^2(\Gamma) := \{v \in V(\Gamma) : g_v = 0, val(v) = 2, |s_v| = 0\}$$

$$V^{1,1}(\Gamma) := \{v \in V(\Gamma) : g_v = 0, val(v) = 1, |s_v| = 1\}$$

$$V^s(\Gamma) := \{v \in V(\Gamma) : 2g_v - 2 + val(v) + |s_v| > 0\}$$

$$y(v, e) := C_e \cap C_v$$

The tangent-obstruction sequence is

$$\begin{aligned} & 0 \to Aut(C, 1, \ldots, n) \\ \to Def(f) \to {} & Def(C, 1, \ldots, n, f) \to Def(C, 1, \ldots, n) \\ \to Ob(f) \quad \to {} & Ob(C, 1, \ldots, n, f) \to 0, \end{aligned}$$

$$\begin{aligned} & 0 \to Hom(\Omega_C(p_1+\cdots+p_n),\mathcal{O}_C) \\ \to H^0(C,f^*T_X) \to T^1 \to & Ext^1(\Omega_C(p_1+\cdots+p_n),\mathcal{O}_C) \\ \to H^1(C,f^*T_X) \to T^2 \to 0. & \end{aligned}$$

For simplicity:

$$0 \to B_1 \to B_2 \to T^1 \to B_4 \to B_5 \to T^2 \to 0.$$

The $N^{vir} = T^{1,m} - T^{2,m}$ (m means moving part).

$$Euler_{\mathbb{T}}(N^{vir}) = \frac{Euler_{\mathbb{T}}(B_2^m)Euler_{\mathbb{T}}(B_4^m)}{Euler_{\mathbb{T}}(B_1^m)Euler_{\mathbb{T}}(B_5^m)}.$$

(1) $Euler_{\mathbb{T}}(B_2^m)/Euler_{\mathbb{T}}(B_5^m)$. The normalization sequence of C is:

$$\begin{aligned} 0 \to \mathcal{O}_C \to \bigoplus_{v\in V^s(\Gamma)} \mathcal{O}_{C_v} \oplus \bigoplus_{e\in E(\Gamma)} \mathcal{O}_{C_e} \\ \to \bigoplus_{v\in V^2(\Gamma)} \mathcal{O}_{y_v} \oplus \bigoplus_{(e,v)\in F^s(\Gamma)} \mathcal{O}_{y(e,v)} \to 0. \end{aligned}$$

Take $\otimes f^*T_X$:

$$\begin{aligned} 0 \to H^0(C,f^*T_X) \to \bigoplus_{v\in V^s(\Gamma)} H^0(C_v,f^*T_X) \oplus \bigoplus_{e\in E(\Gamma)} H^0(C_e,f^*T_X) \\ \to \bigoplus_{v\in V^2(\Gamma)} T_{f(y_v)}X \oplus \bigoplus_{(e,v)\in F^s(\Gamma)} T_{f(y(e,v))}X \\ \to H^1(C,f^*T_X) \to \bigoplus_{v\in V^s(\Gamma)} H^1(C_v,f^*T_X) \oplus \bigoplus_{e\in E(\Gamma)} H^1(C_e,f^*T_X) \to 0. \end{aligned}$$

$$H^0(C_v,f^*T_X) = T_{f(C_v)}X,$$
$$H^1(C_v,f^*T_X) = H^1(C_v,\mathcal{O}_{C_v})\otimes T_{f(C_v)}X \cong H^0(C_v,\omega_{C_v})^\vee \otimes T_{f(C_v)}X$$

Here $H^0(C_v,\omega_{C_v})$ is Hodge bundle $\mathbb{E}$. By splitting principle, assume $\mathbb{E} = L_1\oplus\cdots\oplus L_g$, then

$$\begin{aligned} e(\mathbb{E}^\vee\otimes\mathbb{C}_1) = \prod_{i=1}^g c_1(L_i^\vee\otimes\mathbb{C}_1) = \prod_{i=1}^g c_1(L_i^\vee)+c_1(\mathbb{C}_1) \\ = \prod_{i=1}^g(-c_1(L_i)+u) = \sum_{k=1}^g(-1)^k c_k(\mathbb{E})u^{g-k} = \sum_{k=1}^g(-1)^k\lambda_k u^{g-k} =: \Lambda_g^\vee(u) \end{aligned}$$

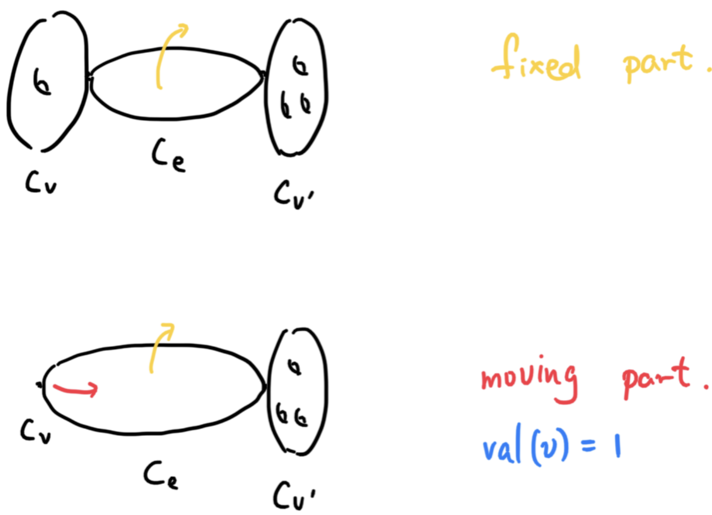

Figure 1.2: automorphism of $(C, 1, \dots, n)$

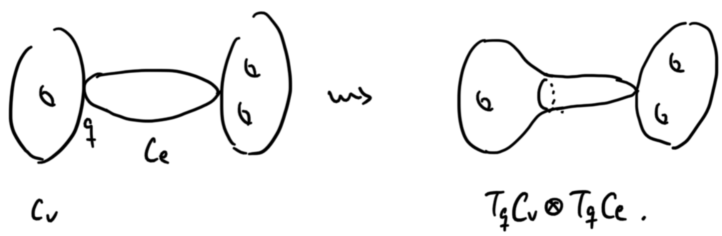

Figure 1.3: deformation of $(C, 1, \dots, n)$

(2) $Euler_{\mathbb{T}}(B_4^m)/Euler_{\mathbb{T}}(B_1^m)$.

(2.1) $B_1 = Aut(C, 1, \dots, n) = Hom(\Omega_C(p_1 + \cdots + p_n), \mathcal{O}_C)$: We should classify what is moving and what is fixed. Basically, we have

$$B_1^m = \bigoplus_{v \in V^1(\Gamma), (e,v) \in F(\Gamma)} T_{y(e,v)} C_e.$$

(2.2) $B_4 = Def(C, 1, \dots, n) = Ext^1(\Omega_C(p_1 + \cdots + p_n), \mathcal{O}_C)$: $\mathbb{P}^1$ has just 1 complex structure, so we consider $g(C) \geq 1$. If we don't change node q, C will stay in the same class in $\overline{\mathcal{M}}_{g,n}$. Hence we must resolve the node, and geometrically, resolution depends on $T_qC_v \otimes T_qC_e$. So basically we have

$$B_4^m = \bigoplus_{v\in V^2(\Gamma), E_v=(e,e')} T_{y_v}C_e \otimes T_{y_v}C_{e'} \oplus \bigoplus_{(e,v)\in F^s(\Gamma)} T_{y(e,v)}C_v \otimes T_{y(e,v)}C_e$$

Returning to the special case $X = \mathbb{P}^r$, we can get the theorem 1.1.1.

1.3 Aspinwall Morrison formula; Faber Pandaripande formula

In this section, we will use Kontsevich's approach to compute the multiple cover contribution of rigidly embedding curves $\mathbb{P}^1$ in a Calabi-Yau threefold X.

The geometry picture is this. The normal bundle N of $\mathbb{P}^1 \subset X$ is rank 2 and splits on $\mathbb{P}^1$. Because X is Calabi-Yau and $c_1(\mathbb{P}^1) = 2$, the normal bundle is of degree 2. Embedded $\mathbb{P}^1$'s in a Calabi-Yau threefold (not necessary lines) with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ are called rigid. The degree 2 Gromov-Witten invariant of a generic quintic has two contributions:

(1) rigid conics curves in X;

(2) lines with double cover, so this part is related to $\overline{\mathcal{M}}_0(\mathbb{P}^1, 2)$.

We want to compute the contribution of part (2). This problem finally leads to:

$$N_{g,d} = \int_{\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,d)} e(R^1\pi_* f^*N),$$

where
$$\begin{array}{ccc} \overline{\mathcal{C}}_{g,0}(\mathbb{P}^1, d) & \xrightarrow[f]{} & \mathbb{P}^1 \\ \downarrow{\scriptstyle \pi} & & \\ \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d) & & \end{array}$$
and $N = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

The decorated graphs $\vec{\Gamma} = (\Gamma, \vec{f}, \vec{d}, \vec{g}, \vec{s})$ in $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)^{\mathbb{T}}$ are of the type in Figure 1.4. We can choose different lifts on $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ so that only $\vec{\Gamma} = (\Gamma, \vec{f}, \vec{d}, \vec{g}, \vec{s})$ with 1 edge contributing $N_{g,d}$.

(1) $g = 0$ (Aspinwall Morrison formula): $N_{0,d} = 1/d^3$;

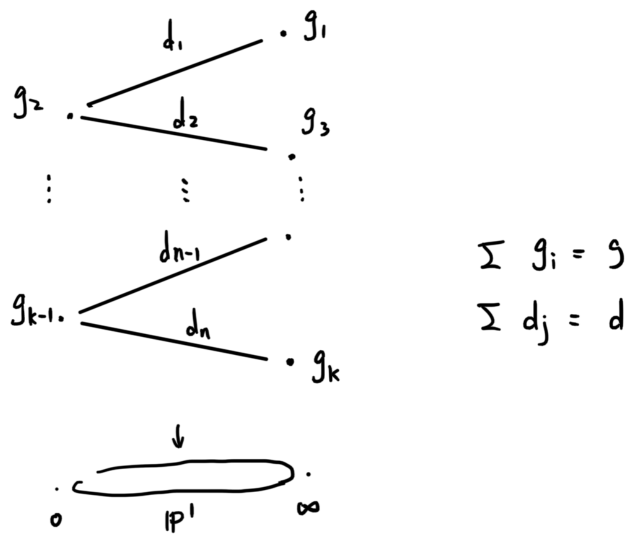

Figure 1.4: $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)^{\mathbb{T}}$

(2) $g \geq 1$ (Faber-Pandharipande):

$$N_{g,d} = \sum_{g_1+g_2=g} \frac{1}{d} \int_{\overline{\mathcal{M}}_{g_1,1}} \lambda_{g_1} \psi^{2g_1-2} d^{2g_1-1} \times \int_{\overline{\mathcal{M}}_{g_2,1}} \lambda_{g_2} \psi^{2g_2-2} d^{2g_2-1} = \sum_{g_1+g_2=g} b_{g_1} b_{g_2} d^{2g-3}$$

$$b_0 = 0; b_g = \int_{\overline{\mathcal{M}}_{g,1}} \lambda_{g_2} \psi^{2g-2} \quad (g > 0)$$

$$\sum_{g=0}^{\infty} b_g t^{2g} = \frac{t/2}{\sin t/2}.$$

Then use the Laurent series of $\cot t$, we have

$$N_{1,d} = \frac{1}{12d},$$

$$N_{g,d} = d^{2g-3} \frac{|B_{2g}|}{2g \cdot (2g-2)!} = |\chi(\overline{\mathcal{M}}_g)| \frac{d^{2g-3}}{(2g-3)!}, \quad g \geq 2,$$

where B_g is the Bernoulli number in $\frac{x}{e^x-1}$.

Chapter 2

Quantum Cohomology

2.1 quantum product

The quantum cohomology is a variation of classical cohomology. Let $T_0 = 1, T_1, \dots, T_p, T_{p+1}, \dots, T_m \in H^*(X)$ be a basis of $H^*(X)$ as a $\mathbb{Q}$-vector space ($T_1, \dots, T_p \in H^2(X)$). Let $\beta \in H^2(X)$, $\gamma = \sum_{i=0}^m t_i T_i$, we define quantum potential as

$$\begin{aligned} F_0^X(t_0, \dots, t_m) &= \sum_{n,\beta} \frac{1}{n!} \langle \gamma^n \rangle_{0,n,\beta}^X Q^\beta \\ &= \frac{1}{6} \int_X (\sum_{i=0}^m t_i T_i)^3 + \sum_{\beta=0, n\geq 4} \langle T_0^{n_0} \dots T_m^{n_m} \rangle_{0,n,0} \prod_{i=1}^m \frac{t_i^{n_i}}{n_i!} \\ &+ \sum_{\beta>0,n} Q^\beta \langle T_0^{n_0} \dots T_m^{n_m} \rangle_{0,n,\beta} \prod_{i=p+1}^m \frac{t_i^{n_i}}{n_i!}. \end{aligned}$$

By string equations and divisor equations,

$$\begin{aligned} F_0^X(t_0, \dots, t_m) &= \frac{1}{6} \int_X (\sum_{i=0}^m t_i T_i)^3 + \sum_{\beta=0, n\geq 4} \langle T_1^{n_1} \dots T_m^{n_m} \rangle_{0,n,0} \prod_{i=1}^m \frac{t_i^{n_i}}{n_i!} \\ &+ \sum_{\beta>0,n} Q^\beta q_1^{\int_\beta T_1} \dots q_p^{\int_\beta T_p} \langle T_{p+1}^{n_{p+1}} \dots T_m^{n_m} \rangle_{0,n,\beta} \prod_{i=p+1}^m \frac{t_i^{n_i}}{n_i!}, \end{aligned}$$

where $q_i = e^{t_i}$.

$$F_{ijk} := \frac{\partial^3 F_0^X}{\partial t_i \partial t_j \partial t_k} = \sum_{n,\beta} \frac{1}{n!} \langle T_i T_j T_k \gamma^n \rangle_{0,n+3,\beta}^X Q^\beta$$

$$= \int_X T_i T_j T_k + \sum_{\beta=0, n\geq 1} \langle T_i T_j T_k T_1^{n_1} \dots T_m^{n_m} \rangle_{0,n+3,0} \prod_{i=1}^{m} \frac{t_i^{n_i}}{n_i!}$$

$$+ \sum_{\beta>0,n} Q^\beta q_1^{\int_\beta T_1} \dots q_p^{\int_\beta T_p} \langle T_i T_j T_k T_{p+1}^{n_{p+1}} \dots T_m^{n_m} \rangle_{0,n+3,\beta} \prod_{i=p+1}^{m} \frac{t_i^{n_i}}{n_i!}, \quad q_i = e^{t_i}.$$

Let $g_{ij} = (T_i, T_j)$ means the Poincare pair of T_i, T_j. The big quantum product is defined as

$$(T_i *_t T_j, T_k) := F_{ijk},$$

in other words,

$$T_i *_t T_j = \sum_{e,f} F_{ije} g^{ef} T_f.$$

It is known that the quantum product is a generalization of intersection theory: given T_i, T_j, T_k, they contribute to the quantum product if there exists $\mathbb{P}^1$ touching their Poincare dual classes at the same time. Extend the t_i in quantum multiplication linearly, then the $\mathbb{Q}[[t_0, \dots, t_m]]$-module $H^*(X) \otimes_{\mathbb{Q}} \mathbb{Q}[[t_0, \dots, t_m]]$ is the big quantum cohomology $QH(X)$.

The associativity of quantum product is formulated as WDVV equation:

$$F_{ija} g^{ab} F_{bkl} = F_{ila} g^{ab} F_{bjk}.$$

It is proved by a forgetful map $\pi : \overline{\mathcal{M}}_{0,4}(X, \beta) \to \overline{\mathcal{M}}_{0,4}$. One should notice that $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$, so the boundary divisor $D(12|34) \sim D(13|24)$ and

$$\int_{[\overline{\mathcal{M}}_{0,4}(X,\beta)]^{vir} \cap \pi^* D(12|34)} ev_1^*(T_i) ev_2^*(T_j) ev_3^*(T_k) ev_4^*(T_l) \prod_{i=5}^{n+4} ev_i^*(\gamma)$$

$$= \int_{[\overline{\mathcal{M}}_{0,4}(X,\beta)]^{vir} \cap \pi^* D(13|24)} ev_1^*(T_i) ev_2^*(T_j) ev_3^*(T_k) ev_4^*(T_l) \prod_{i=5}^{n+4} ev_i^*(\gamma).$$

A useful trick is to separate $[\overline{\mathcal{M}}_{0,4}(X, \beta)]^{vir} \cap \pi^* D(12|34)$ by

$$\coprod_{n_1+n_2=n, \beta_1+\beta_2=\beta} [\overline{\mathcal{M}}_{0,n_1+3}(X, \beta_1) \times \overline{\mathcal{M}}_{0,n_2+3}(X, \beta_2)]^{vir} \cap (ev \times ev)^*[\Delta],$$

$$PD[\Delta] = g^{ab} T_a \otimes T_b,$$

then we get

$$\sum_{n_1+n_2=n}\sum_{\beta_1+\beta_2=\beta}\langle T_iT_jT_a\gamma^{n_1}\rangle_{0,n_1+3,\beta_1}g^{ab}\langle T_bT_kT_l\gamma^{n_2}\rangle_{0,n_2+3,\beta_2}$$
$$=\sum_{n_1+n_2=n}\sum_{\beta_1+\beta_2=\beta}\langle T_iT_kT_a\gamma^{n_1}\rangle_{0,n_1+3,\beta_1}g^{ab}\langle T_bT_jT_l\gamma^{n_2}\rangle_{0,n_2+3,\beta_2}.$$

This is the essential part in the proof of associativity of quantum product.

Remark 2.1.1. *It deserves to notice that the quantum product is defined by rational curves, so its usage mainly concentrates in genus 0 GW-invariants. The difficulty to define quantum product via higher genus curves is that there is no so good associativity as the genus 0 case. It must be a good work if we can find a way to give a quantum product via higher genus curves with associativity like now.*

The small quantum product is defined by

$$T_i *_s T_j = T_i *_t T_j|_{t_{p+1}=\cdots=t_m=0}, 0\le i,j\le m.$$

Precisely, let

$$\overline{F}_{ijk}=F_{ijk}|_{t_{p+1}=\cdots=t_m=0}=\int_X T_iT_jT_k+\sum_{\beta>0}Q^\beta q_1^{\int_\beta T_1}\dots q_p^{\int_\beta T_p}\langle T_iT_jT_k\rangle_{0,3,\beta},$$

then

$$T_i *_s T_j=\overline{F}_{ije}g^{ef}T_f,\quad 1\le e,f\le m.$$

Extend q_i linearly, the $\mathbb{Q}[[q_1,\dots,q_p]]$-module $H^*(X)\otimes_{\mathbb{Q}}\mathbb{Q}[[q_1,\dots,q_p]]$ is defined as the small quantum cohomology $QH^s(X)$.

Example 2.1.2. $QH^s(\mathbb{P}^m)=\mathbb{Q}[H,q]/(H^{m+1}-q)$, *where* $H\in H^2(\mathbb{P}^m,\mathbb{Q})$, $q=e^{t_1}$.

2.2 quantum differential equation

We can view the vector space $H(X)$ as a Riemannian manifold M with standard flat metric g_{ij} given by Poincare pairing. The quantum product $*_t$ could be use to define a connection (called Dubrovin connection, or Givental connection) ∇^z, which is different from the Levi-Civita connection induced by its Riemannian metric.

Definition 2.2.1. *(Dubrovin connection) Let $X, Y \in \Gamma(M, TM)$, ∇ be the Levi-Civita connection w.r.t g. The Dubrovin connection ∇^z is defined by*

$$\nabla^z_X Y := \nabla_X Y - \frac{1}{z} X *_t Y.$$

The WDVV equation shows ∇^z is flat. i.e. $Rm^z = 0$.

Definition 2.2.2 (quantum differential equation)**.** *Let $\sigma \in \Gamma(M, TM)$, the equation $\nabla^z \sigma = 0$ is the quantum differential equation. The fundamental solution of quantum differential equation is $(m+1) \times (m+1)$ matrix $s(z, t)$ $(t = (t_0, \ldots, t_m)) = (a_{ij})$, such that each column defines a solution*

$$\sigma_j(t) = \sum_{i=0}^{m} a_{ij}(t) \frac{\partial}{\partial t_i}.$$

Now we want to find the solution of quantum differential equation. Let $(S(z)T_a, T_b) = g_{ab} + \langle\langle \frac{T_a}{z-\varphi_1}, T_b \rangle\rangle_{0,2}$, where

$$\langle\langle \frac{T_a}{z-\psi_1}, T_b \rangle\rangle_{0,2} = \sum_{\substack{n \geq 0, \beta, \\ (n,\beta) \neq (0,0)}} \frac{1}{n!} \langle \frac{T_a}{z-\psi_1} T_b \gamma^n \rangle_{0,n+2,\beta}.$$

Proposition 2.2.3. *The $S_a = (S(z)T_a, T_b) g^{bc} \partial_c$ is a flat section with respect to ∇^z.*

This proposition is proven with the help of topological recursion relation.

Definition 2.2.4. *The descendent invariants are defined by*

$$\langle \tau_{a_1}(\gamma_1) \ldots \tau_{a_n}(\gamma_n) \rangle_{g,n,\beta} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} ev_1^*(\gamma_1)\psi_1^{a_1} \cup \cdots \cup ev_n^*(\gamma_n)\psi_n^{a_n}.$$

Theorem 2.2.5 (topological recursion relation)**.** *Let $\gamma_i \in H^*(X)$,*

$$\langle \tau_{a_1+1}(\gamma_1) \tau_{a_2}(\gamma_2) \tau_{a_3}(\gamma_3) \prod_{i=4}^{n} \tau_{a_i}(\gamma_i) \rangle_{0,n,\beta}$$

$$= \sum_{\substack{A \cup B = \{4,\ldots,n\} \\ \beta = \beta_1 + \beta_2}} \sum_{a,b=0}^{m} \langle \tau_{a_1}(\gamma_1) \prod_{i \in A} \tau_{a_i}(\gamma_i) T_a \rangle_{0,|A|+2,\beta_1} g^{ab} \langle T_b \tau_{a_2}(\gamma_2) \tau_{a_3}(\gamma_3) \prod_{j \in B} \tau_{a_j}(\gamma_j) T_a \rangle_{0,|B|+3,\beta_2}$$

Proof. Consider the forgetful map

$$\pi : \overline{\mathcal{M}}_{0,n}(X,\beta) \to \overline{\mathcal{M}}_{0,3} :$$

$$[f : C \to X, 1, \ldots, n] \mapsto [C, 1, 2, 3].$$

Let $\mathbb{L}_1, \mathbb{L}_1'$ be the tautological line bundles

$$\begin{tikzcd} \mathbb{L}_1 \arrow[d] & \mathbb{L}_1' \arrow[d] \\ \overline{\mathcal{M}}_{0,n}(X,\beta) \arrow[r, "\pi"] & \overline{\mathcal{M}}_{0,3} \end{tikzcd}$$

There is

$$\mathbb{L}_1 \cong \pi^*\mathbb{L}_1' \otimes \Big(\sum_{\substack{\beta_1+\beta_2=\beta \\ A\cup B=\{4,\ldots,n\}}} D(1, A, \beta_1|2, 3, B, \beta_2)\Big),$$

$$\begin{tikzcd} D(1, A, \beta_1|2, 3, B, \beta_2) \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{0,|B|+3}(X, \beta_2) \arrow[d, "ev_{node}"] \\ \overline{\mathcal{M}}_{0,|A|+2}(X, \beta_1) \arrow[r, "ev_{node}"] & X. \end{tikzcd}$$

Because $\overline{\mathcal{M}}_{0,3}$ is a point, $\mathbb{L}_1'$ is trivial and

$$\psi_1 = \sum_{\substack{\beta_1+\beta_2=\beta \\ A\cup B=\{4,\ldots,n\}}} [D(1, A, \beta_1|2, 3, B, \beta_2)].$$

Take this formula into LHS, we get the recursion relation. $\square$

The fundamental solution of small quantum differential equation directly relates to the definition of J-function in mirror symmetry. It is given by

$$\tilde{S}(z) = S(z)|_{t_{p+1},\ldots,t_m=0}.$$

Specifically, let $\gamma = \sum_{i=0}^{p} t_i T_i = t_0 T_0 + \gamma_1$ and $\gamma_1 = \sum_{i=1}^{p} t_i T_i$, then

$$(\tilde{S}(z)T_a, T_b) = g_{ab} + \sum_{\substack{n\geq 0,\beta \\ (n,\beta)\neq(0,0)}} \frac{1}{n!}\langle \frac{T_a}{z-\psi_1} T_b \gamma^n \rangle_{0,n+2,\beta}.$$

By string equations and divisor equations, γ can be put out of the bracket, and finally we get

$$(\tilde{S}(z)T_a, T_b) = \int_X e^{\gamma/z} T_a T_b + \sum_{\beta>0} \langle \frac{e^{\gamma/z} T_a}{z - \varphi_1} T_b \rangle_{0,2,\beta} e^{\int_\beta \gamma_1}.$$

Chapter 3

Mirror Symmetry

I plan to follow Givental's approach to give a proof of genus 0 mirror symmetry of hypersurfaces in $\mathbb{P}^n$. The key character of Givental's approach is that it uses J-function and I-function to show the mirror symmetry relation. The J-function is defined as follows, which describes the A-model information.

Definition 3.0.1. *For a complex manifold X, the J^X is*

$$(T_a, J^X) := (\tilde{S}(z)T_a, 1) = \int_X e^{\gamma/z} T_a + \sum_{\beta > 0} \langle \frac{e^{\gamma/z} T_a}{z - \varphi_1} 1 \rangle_{0,2,\beta} e^{\int_\beta \gamma_1}.$$

J^X is a $H^*(X)$-value function: $J^X(t_0, t_1, \ldots, t_p, z^{-1}) = (T_a, J^X) g^{ab} T_b$. In this chapter, X is a hypersurface of degree l in $\mathbb{P}^m$. We assume $l \leq m+1$ so X is either Fano or Calabi-Yau.

At first, J^X could be pushforwarded to $i_* J^X$ as a $H^*(\mathbb{P}^m)$-valued function. Let $i : X \hookrightarrow \mathbb{P}^m$. It induces $i : \overline{\mathcal{M}}_{0,n}(X, d) \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)$. Consider

$$\begin{array}{ccc} \overline{\mathcal{C}}_{0,n}(\mathbb{P}^m, d) & \xrightarrow[F]{} & \mathbb{P}^m \\ \downarrow{\scriptstyle\pi} & & \\ \overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d) & & \end{array}$$

Let $E_d := \pi_* F^* \mathcal{O}(l)$ be the obstruction bundle over $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)$. The following theorem shows the relationship of virtual fundamental classes:

Theorem 3.0.2.

$$i_* [\overline{\mathcal{M}}_{0,n}(X, d)]^{vir} = e(\pi_* F^* \mathcal{O}(l)) \cap [\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)]^{vir}.$$

Let $ev_1 : \overline{\mathcal{M}}_{0,2}(X, \beta) \to X$

Proposition 3.0.3.

$$J^X = e^{\gamma/z}\left(1 + \sum_{\beta>0} e^{\int_\beta \gamma_1}(ev_1)_*\left(\frac{1}{z-\psi_1}\right)\right),$$

$$J^{\mathbb{P}^m,\mathcal{O}(l)}(t_0,t_1,z^{-1}) := i_* J^X = e^{(t_0+t_1H)/z}\left(e(\mathcal{O}(l)) + \sum_{d>0} e^{dt_1}(ev_1)_*\left(\frac{e(E_d)}{z-\psi_1}\right)\right),$$

where $H \in H^2(\mathbb{P}^m, \mathbb{Q})$ *is the generator of* $H^2(\mathbb{P}^m, \mathbb{Q})$, $\gamma = t_0 + t_1 H$.

Let $0 \to E'_d \to E_d \to ev_1^*\mathcal{O}(l) \to 0$, then

$$J^{\mathbb{P}^m,\mathcal{O}(l)}(t_0,t_1,z^{-1}) = e^{(t_0+t_1H)/z} lH\left(1 + \sum_{d>0} e^{dt_1}(ev_1)_*\left(\frac{e(E'_d)}{z-\psi_1}\right)\right)$$

The I-function is

$$I^{\mathbb{P}^m,\mathcal{O}(l)}(t_0,t_1,z^{-1}) := e^{(t_0+t_1H)/z} lH\left(1 + \sum_{d=1}^{\infty} e^{dt_1}\frac{\prod_{a=1}^{dl}(lH+az)}{\prod_{a=1}^{d}(H+az)^{m+1}}\right).$$

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