

Gromov-Witten theory and mirror symmetry

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Chapter 1

Gromov-Witten invariants

Roughly speaking, Gromov-Witten invariants is the number of algebraic curves (resp. pseudo-holomorphic curves) of a algebraic variety X (resp. complex manifold) passing through specific subvarieties (resp. submanifolds) under specific degree (homology class of the curves) and genus.

Definition 1.0.1. Let $\gamma_1, \dots, \gamma_n \in H^*(X; \mathbb{Q})$ and let $\beta \in H^2(X; \mathbb{Q})$. The Gromov-Witten invariant of genus g degree β curves is

$$\langle \gamma_1 \dots \gamma_n \rangle_{g,n,\beta} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} ev_1^*(\gamma_1) \cup \dots \cup ev_n^*(\gamma_n).$$

Here, a point in $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{vir}$ is $[f : C \rightarrow X, 1, \dots, n]$:

a map from the genus g curve C to the variety X modulo the automorphism of C .

The evaluation map $ev_i : [\overline{\mathcal{M}}_{g,n}(X, \beta)]^{vir} \rightarrow X$ is given by

$$ev_i([f : C \rightarrow X, 1, \dots, n]) = f(i).$$

Let $\overline{\mathcal{M}}_{g,n}$ denote the moduli space (Deligne-Mumford stack) of genus g curves with n marked points, and let $\overline{\mathcal{C}}_{g,n}$ be the universal family of $\overline{\mathcal{M}}_{g,n}$.

1.1 Kontsevich's approach

Atiyah-Bott localization formula tells us: if there is a torus action $\mathbb{T} = (\mathbb{C}^*)^n$ on X , then the fixed points of torus action could tells us some properties of X .

By the classifying space theory, $B\mathbb{T} = (\mathbb{C}P^\infty)^{\times n}$, so $H^*(B\mathbb{T}) = \mathbb{Q}[\lambda_1, \dots, \lambda_n]$. Let $\mathcal{R}_{\mathbb{T}} = \mathbb{Q}(\lambda_1, \dots, \lambda_n)$. Let $X_{\mathbb{T}} = E\mathbb{T} \times_{\mathbb{T}} X$, the equivariant cohomology of X is defined by

$$H_{\mathbb{T}}^*(X) := H^*(X_{\mathbb{T}}) = H^*(E\mathbb{T} \times_{\mathbb{T}} X),$$

so naturally, $H_{\mathbb{T}}^*(X)$ is a $H_{\mathbb{T}}^*(pt) = H^*(B\mathbb{T})$ -module. The localization of $H_{\mathbb{T}}^*(X)$ means $H_{\mathbb{T}}^*(X) \otimes \mathcal{R}_{\mathbb{T}}$.

Theorem (Atiyah-Bott). *Let $X^{\mathbb{T}}$ be fixed locus of \mathbb{T} , let Z_j be a connection component of $X^{\mathbb{T}}$, and let N_j be the normal bundle of Z_j in X . Let $i_j : Z_j \rightarrow X$ and let $i_{j!} : H_{\mathbb{T}}^*(Z_j) \rightarrow H_{\mathbb{T}}^*(X)$ be the pushforward defined by the Gysin map. Let $\alpha \in H_{\mathbb{T}}^*(X) \otimes \mathcal{R}_{\mathbb{T}}$, we have*

$$\alpha = \sum_j \frac{i_{j!} i_j^* \alpha}{Euler_T(N_j)},$$

In particular,

$$\int_{X_{\mathbb{T}}} \alpha = \sum_j \int_{(Z_j)_{\mathbb{T}}} \frac{i_j^* \alpha}{Euler_T(N_j)}.$$

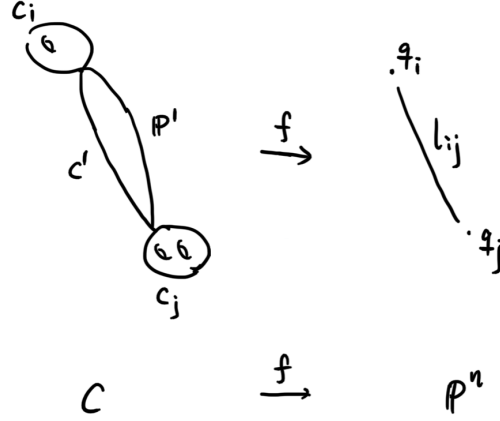
Kontsevich's approach is to apply Atiyah-Bott localization formula in $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{vir}$ so that we can simplify the computation. We can lift the \mathbb{T} action on X to $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{vir}$ in the following way: let $t \in \mathbb{T}$, $[f : C \rightarrow X, 1, \dots, n] \in [\overline{\mathcal{M}}_{g,n}(X, \beta)]^{vir}$, $x \in X$

$$(t \cdot f)(x) = f(t \cdot x).$$

In special, we will consider $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ in this section. As claimed before, we need to find $[f : C \rightarrow X, 1, \dots, n] \in \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^{\mathbb{T}}$. The fixed points of \mathbb{P}^r is

$$\{q_i = [0 : 0 : \dots : 1 : 0 : \dots : 0]\}_{0 \leq i \leq r}.$$

The coordinate curve l_{ij} connecting q_i, q_j has one dimensional degree of freedom \mathbb{C}^* (as an invariant component). The curve $C \in \overline{\mathcal{C}}_{g,n}$ is stable (i.e. $\text{Aut}(C) < \infty$) if and only if $2g - 2 + n > 0$. If a components C' of C is mapped to l_{ij} , then C' has two points mapped to q_i, q_j respectively (equivalent to with two marked points in C'), so $2g - 2 + 2 \leq 0$ implies $g = 0$, i.e. $C' \cong \mathbb{P}^1$ (see Fig 1.1). Meanwhile, $f|_{C'}$ must be uniformly ramified, so $f|_{C'}(z) = z^e, \forall z \in C' \cong \mathbb{P}^1$, for some $e \in \mathbb{N}^*$.


 Figure 1.1: $f(C_i) = q_i$, $f(C') = l_{ij}$, $f(C_j) = q_j$

It is convenient to use a decorated graph $\vec{\Gamma} = (\Gamma, \vec{f}, \vec{d}, \vec{g}, \vec{s})$ (graph, maps, degrees, genus, marked points) to represent $[f : C \rightarrow X, 1, \dots, n] \in \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^{\mathbb{T}}$. Let $\text{val}(v)$, the valence of v , be the number of edges connecting vertex v , and let $n(v) = |s_v| + \text{val}(v)$. The stable map $[f : C \rightarrow X, 1, \dots, n]$ with fixed graph $\vec{\Gamma} = (\Gamma, \vec{f}, \vec{d}, \vec{g}, \vec{s})$ defines a substack

$$\overline{\mathcal{M}}_{\vec{\Gamma}} \subset \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d).$$

On the other sides, consider

$$\varphi_{\vec{\Gamma}} : \prod_{\dim C_v=1} \overline{\mathcal{M}}_{g_v, n(v)} \rightarrow \overline{\mathcal{M}}_{\vec{\Gamma}}.$$

If v, v' are connected by an edge e , then let $C_v, C_{v'}$ connected by a $C_e \cong \mathbb{P}^1$ associated with a degree d_e map to \mathbb{P}^r . Let $\overline{\mathcal{M}}_{\Gamma}$ be the product of above C_v, C_e . There is a group \mathbb{A}_{Γ} acting on $\overline{\mathcal{M}}_{\Gamma}$. The group \mathbb{A}_{Γ} is defined by:

$$1 \rightarrow \prod_{\text{edges}} \mathbb{Z}/(d_e) \rightarrow \mathbb{A}_{\Gamma} \rightarrow \text{Aut}(\Gamma) \rightarrow 1.$$

and

$$\overline{\mathcal{M}}_{\vec{\Gamma}} = \overline{\mathcal{M}}_{\Gamma} / \mathbb{A}_{\Gamma}.$$

Therefore, we know the \mathbb{T} -fixed locus of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ is $\overline{\mathcal{M}}_{\vec{\Gamma}}$. Let N_{Γ} be the normal bundle of $\overline{\mathcal{M}}_{\vec{\Gamma}}$ in $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$. Then there is an explicit formula for

the equivariant Euler class. Before doing that, we define some necessary notations. A flag F is a pair (v, e) such that e is an edge containing the vertex v . We put $i(F) = v$, $j(F)$ the vertex of e different from v . Let

$$\omega_F = \frac{\lambda_{i(F)} - \lambda_{j(F)}}{d_e} \in H_{\mathbb{T}}^2(pt) = \mathbb{Q}[\lambda_0, \dots, \lambda_r],$$

which is the weight of \mathbb{T} -action on $T_{q_{i_v}} C_e$.

Theorem 1.1.1 ($Euler_{\mathbb{T}}(N_{\Gamma})$). $Euler_{\mathbb{T}}(N_{\Gamma}) = e_{\Gamma}^F e_{\Gamma}^v e_{\Gamma}^e$, where

$$\begin{aligned} e_{\Gamma}^F &= \prod_{n(i(F)) \geq 3} (\omega_F - \psi_F) / \prod_{j \neq i(F)} (\lambda_{i(F)} - \lambda_j), \\ e_{\Gamma}^v &= \prod_v \prod_{j \neq i_v} (\lambda_{i_v} - \lambda_j) \prod_{val(v)=2, s_v=\emptyset} (\omega_{F_1(v)} + \omega_{F_2(v)}) / \prod_{val(v)=1, s_v=\emptyset} \omega_{F(v)} \\ e_{\Gamma}^e &= \prod_e \frac{(-1)^{d_e} (d_e!)^2 (\lambda_i - \lambda_j)^{2d_e}}{d_e^{2d_e}} \prod_{a+b=d_e, k \neq i, j} \left(\frac{a\lambda_i + b\lambda_j}{d_e} - \lambda_k \right) \end{aligned}$$

The proof is partially discussed in section 1.2.

1.2 Tangent-obstruction sequence

Consider $[f : C \rightarrow X, 1, \dots, n] \in \overline{\mathcal{M}}_{\vec{\Gamma}}$, $\vec{\Gamma} = (\Gamma, \vec{f}, \vec{d}, \vec{g}, \vec{s})$. We put

$$V^1(\Gamma) := \{v \in V(\Gamma) : g_v = 0, val(v) = 1, |s_v| = 0\}$$

$$V^2(\Gamma) := \{v \in V(\Gamma) : g_v = 0, val(v) = 2, |s_v| = 0\}$$

$$V^{1,1}(\Gamma) := \{v \in V(\Gamma) : g_v = 0, val(v) = 1, |s_v| = 1\}$$

$$V^s(\Gamma) := \{v \in V(\Gamma) : 2g_v - 2 + val(v) + |s_v| > 0\}$$

$$y(v, e) := C_e \cap C_v$$

The tangent-obstruction sequence is

$$\begin{aligned} &0 \rightarrow Aut(C, 1, \dots, n) \\ &\rightarrow Def(f) \rightarrow Def(C, 1, \dots, n, f) \rightarrow Def(C, 1, \dots, n) \\ &\rightarrow Ob(f) \rightarrow Ob(C, 1, \dots, n, f) \rightarrow 0, \end{aligned}$$

$$\begin{aligned}
 0 &\rightarrow \text{Hom}(\Omega_C(p_1 + \cdots + p_n), \mathcal{O}_C) \\
 &\rightarrow H^0(C, f^*T_X) \rightarrow T^1 \rightarrow \text{Ext}^1(\Omega_C(p_1 + \cdots + p_n), \mathcal{O}_C) \\
 &\rightarrow H^1(C, f^*T_X) \rightarrow T^2 \rightarrow 0.
 \end{aligned}$$

For simplicity:

$$0 \rightarrow B_1 \rightarrow B_2 \rightarrow T^1 \rightarrow B_4 \rightarrow B_5 \rightarrow T^2 \rightarrow 0.$$

The $N^{\text{vir}} = T^{1,m} - T^{2,m}$ (m means moving part).

$$\text{Euler}_{\mathbb{T}}(N^{\text{vir}}) = \frac{\text{Euler}_{\mathbb{T}}(B_2^m) \text{Euler}_{\mathbb{T}}(B_4^m)}{\text{Euler}_{\mathbb{T}}(B_1^m) \text{Euler}_{\mathbb{T}}(B_5^m)}.$$

(1) $\text{Euler}_{\mathbb{T}}(B_2^m)/\text{Euler}_{\mathbb{T}}(B_5^m)$. The normalization sequence of C is:

$$\begin{aligned}
 0 &\rightarrow \mathcal{O}_C \rightarrow \bigoplus_{v \in V^s(\Gamma)} \mathcal{O}_{C_v} \oplus \bigoplus_{e \in E(\Gamma)} \mathcal{O}_{C_e} \\
 &\rightarrow \bigoplus_{v \in V^2(\Gamma)} \mathcal{O}_{y_v} \oplus \bigoplus_{(e,v) \in F^s(\Gamma)} \mathcal{O}_{y(e,v)} \rightarrow 0.
 \end{aligned}$$

Take $\otimes f^*T_X$:

$$\begin{aligned}
 0 &\rightarrow H^0(C, f^*T_X) \rightarrow \bigoplus_{v \in V^s(\Gamma)} H^0(C_v, f^*T_X) \oplus \bigoplus_{e \in E(\Gamma)} H^0(C_e, f^*T_X) \\
 &\rightarrow \bigoplus_{v \in V^2(\Gamma)} T_{f(y_v)}X \oplus \bigoplus_{(e,v) \in F^s(\Gamma)} T_{f(y(e,v))}X \\
 &\rightarrow H^1(C, f^*T_X) \rightarrow \bigoplus_{v \in V^s(\Gamma)} H^1(C_v, f^*T_X) \oplus \bigoplus_{e \in E(\Gamma)} H^1(C_e, f^*T_X) \rightarrow 0.
 \end{aligned}$$

$$H^0(C_v, f^*T_X) = T_{f(C_v)}X,$$

$$H^1(C_v, f^*T_X) = H^1(C_v, \mathcal{O}_{C_v}) \otimes T_{f(C_v)}X \cong H^0(C_v, \omega_{C_v})^\vee \otimes T_{f(C_v)}X$$

Here $H^0(C_v, \omega_{C_v})$ is Hodge bundle \mathbb{E} . By splitting principle, assume $\mathbb{E} = L_1 \oplus \cdots \oplus L_g$, then

$$\begin{aligned}
 e(\mathbb{E}^\vee \otimes \mathbb{C}_1) &= \prod_{i=1}^g c_1(L_i^\vee \otimes \mathbb{C}_1) = \prod_{i=1}^g c_1(L_i^\vee) + c_1(\mathbb{C}_1) \\
 &= \prod_{i=1}^g (-c_1(L_i) + u) = \sum_{k=1}^g (-1)^k c_k(\mathbb{E}) u^{g-k} = \sum_{k=1}^g (-1)^k \lambda_k u^{g-k} =: \Lambda_g^\vee(u)
 \end{aligned}$$

(2) $Euler_{\mathbb{T}}(B_4^m)/Euler_{\mathbb{T}}(B_1^m)$.

(2.1) $B_1 = Aut(C, 1, \dots, n) = Hom(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C)$: We should classify what is moving and what is fixed. Basically, we have

$$B_1^m = \bigoplus_{v \in V^1(\Gamma), (e,v) \in F(\Gamma)} T_{y(e,v)} C_e.$$

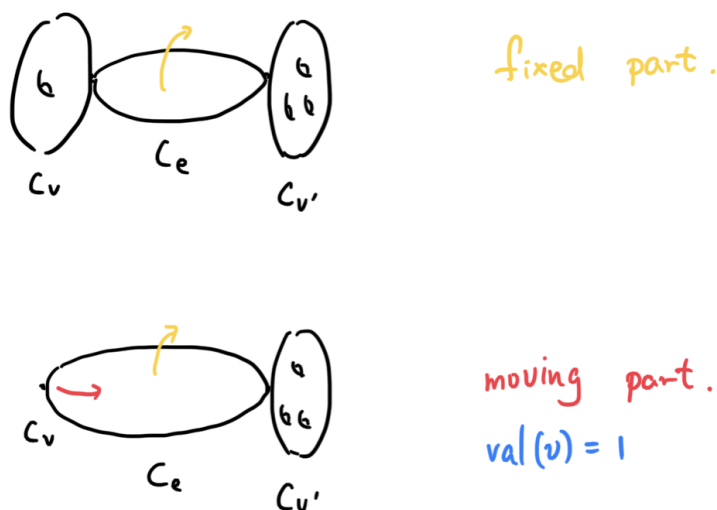


Figure 1.2: automorphism of $(C, 1, \dots, n)$

(2.2) $B_4 = Def(C, 1, \dots, n) = Ext^1(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C)$: \mathbb{P}^1 has just 1 complex structure, so we consider $g(C) \geq 1$. If we don't change node q ,

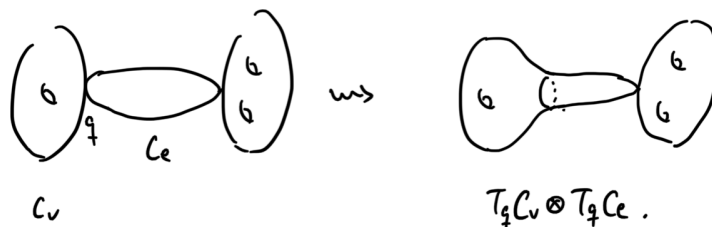


Figure 1.3: deformation of $(C, 1, \dots, n)$

C will stay in the same class in $\overline{\mathcal{M}}_{g,n}$. Hence we must resolve the node, and geometrically, resolution depends on $T_q C_v \otimes T_q C_e$. So basically we have

$$B_4^m = \bigoplus_{v \in V^2(\Gamma), E_v = (e, e')} T_{y_v} C_e \otimes T_{y_v} C_{e'} \oplus \bigoplus_{(e, v) \in F^s(\Gamma)} T_{y(e, v)} C_v \otimes T_{y(e, v)} C_e$$

Returning to the special case $X = \mathbb{P}^r$, we can get the theorem 1.1.1.

1.3 Aspinwall Morrison formula; Faber Pandaripande formula

In this section, we will use Kontsevich's approach to compute the multiple cover contribution of rigidly embedding curves \mathbb{P}^1 in a Calabi-Yau threefold X .

The geometry picture is this. The normal bundle N of $\mathbb{P}^1 \subset X$ is rank 2 and splits on \mathbb{P}^1 . Because X is Calabi-Yau and $c_1(\mathbb{P}^1) = 2$, the normal bundle is of degree 2. Embedded \mathbb{P}^1 's in a Calabi-Yau threefold (not necessary lines) with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ are called rigid. The degree 2 Gromov-Witten invariant of a generic quintic has two contributions:

- (1) rigid conics curves in X ;
- (2) lines with double cover, so this part is related to $\overline{\mathcal{M}}_0(\mathbb{P}^1, 2)$.

We want to compute the contribution of part (2). This problem finally leads to:

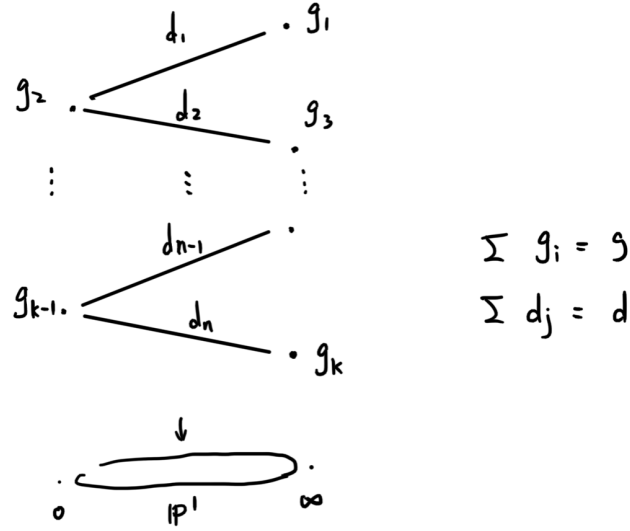
$$N_{g,d} = \int_{\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)} e(R^1 \pi_* f^* N),$$

$$\begin{array}{ccc} \overline{\mathcal{C}}_{g,0}(\mathbb{P}^1, d) & \xrightarrow{f} & \mathbb{P}^1 \\ \downarrow \pi & & \\ \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d) & & \text{and } N = \mathcal{O}(-1) \oplus \mathcal{O}(-1). \end{array}$$

where

The decorated graphs $\vec{\Gamma} = (\Gamma, \vec{f}, \vec{d}, \vec{g}, \vec{s})$ in $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)^{\mathbb{T}}$ are of the type in Figure 1.4. We can choose different lifts on $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ so that only $\vec{\Gamma} = (\Gamma, \vec{f}, \vec{d}, \vec{g}, \vec{s})$ with 1 edge contributing $N_{g,d}$.

- (1) $g = 0$ (Aspinwall Morrison formula): $N_{0,d} = 1/d^3$;


 Figure 1.4: $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)^{\mathbb{T}}$

(2) $g \geq 1$ (Faber-Pandharipande):

$$\begin{aligned}
 N_{g,d} &= \sum_{g_1+g_2=g} \frac{1}{d} \int_{\overline{\mathcal{M}}_{g_1,1}} \lambda_{g_1} \psi^{2g_1-2} d^{2g_1-1} \\
 &\quad \times \int_{\overline{\mathcal{M}}_{g_2,1}} \lambda_{g_2} \psi^{2g_2-2} d^{2g_2-1} = \sum_{g_1+g_2=g} b_{g_1} b_{g_2} d^{2g-3} \\
 b_0 &= 0; b_g = \int_{\overline{\mathcal{M}}_{g,1}} \lambda_{g_2} \psi^{2g-2} \quad (g > 0) \\
 \sum_{g=0}^{\infty} b_g t^{2g} &= \frac{t/2}{\sin t/2}.
 \end{aligned}$$

Then use the Laurent series of $\cot t$, we have

$$N_{1,d} = \frac{1}{12d},$$

$$N_{g,d} = d^{2g-3} \frac{|B_{2g}|}{2g \cdot (2g-2)!} = |\chi(\overline{\mathcal{M}}_g)| \frac{d^{2g-3}}{(2g-3)!}, \quad g \geq 2,$$

where B_g is the Bernoulli number in $\frac{x}{e^x-1}$.

Chapter 2

Quantum Cohomology

2.1 quantum product

The quantum cohomology is a variation of classical cohomology. Let $T_0 = 1, T_1, \dots, T_p, T_{p+1}, \dots, T_m \in H^*(X)$ be a basis of $H^*(X)$ as a \mathbb{Q} -vector space ($T_1, \dots, T_p \in H^2(X)$). Let $\beta \in H^2(X)$, $\gamma = \sum_{i=0}^m t_i T_i$, we define quantum potential as

$$\begin{aligned} F_0^X(t_0, \dots, t_m) &= \sum_{n, \beta} \frac{1}{n!} \langle \gamma^n \rangle_{0, n, \beta}^X Q^\beta \\ &= \frac{1}{6} \int_X \left(\sum_{i=0}^m t_i T_i \right)^3 + \sum_{\beta=0, n \geq 4} \langle T_0^{n_0} \dots T_m^{n_m} \rangle_{0, n, 0} \prod_{i=1}^m \frac{t_i^{n_i}}{n_i!} \\ &\quad + \sum_{\beta > 0, n} Q^\beta \langle T_0^{n_0} \dots T_m^{n_m} \rangle_{0, n, \beta} \prod_{i=p+1}^m \frac{t_i^{n_i}}{n_i!}. \end{aligned}$$

By string equations and divisor equations,

$$\begin{aligned} F_0^X(t_0, \dots, t_m) &= \frac{1}{6} \int_X \left(\sum_{i=0}^m t_i T_i \right)^3 + \sum_{\beta=0, n \geq 4} \langle T_1^{n_1} \dots T_m^{n_m} \rangle_{0, n, 0} \prod_{i=1}^m \frac{t_i^{n_i}}{n_i!} \\ &\quad + \sum_{\beta > 0, n} Q^\beta q_1^{\int_\beta T_1} \dots q_p^{\int_\beta T_p} \langle T_{p+1}^{n_{p+1}} \dots T_m^{n_m} \rangle_{0, n, \beta} \prod_{i=p+1}^m \frac{t_i^{n_i}}{n_i!}, \end{aligned}$$

where $q_i = e^{t_i}$.

$$F_{ijk} := \frac{\partial^3 F_0^X}{\partial t_i \partial t_j \partial t_k} = \sum_{n, \beta} \frac{1}{n!} \langle T_i T_j T_k \gamma^n \rangle_{0, n+3, \beta}^X Q^\beta$$

$$\begin{aligned}
&= \int_X T_i T_j T_k + \sum_{\beta=0, n \geq 1} \langle T_i T_j T_k T_1^{n_1} \dots T_m^{n_m} \rangle_{0, n+3, 0} \prod_{i=1}^m \frac{t_i^{n_i}}{n_i!} \\
&+ \sum_{\beta > 0, n} Q^\beta q_1^{\int_\beta T_1} \dots q_p^{\int_\beta T_p} \langle T_i T_j T_k T_{p+1}^{n_{p+1}} \dots T_m^{n_m} \rangle_{0, n+3, \beta} \prod_{i=p+1}^m \frac{t_i^{n_i}}{n_i!}, \quad q_i = e^{t_i}.
\end{aligned}$$

Let $g_{ij} = (T_i, T_j)$ means the Poincare pair of T_i, T_j . The big quantum product is defined as

$$(T_i *_t T_j, T_k) := F_{ijk},$$

in other words,

$$T_i *_t T_j = \sum_{e, f} F_{ije} g^{ef} T_f.$$

It is known that the quantum product is a generalization of intersection theory: given T_i, T_j, T_k , they contribute to the quantum product if there exists \mathbb{P}^1 touching their Poincare dual classes at the same time. Extend the t_i in quantum multiplication linearly, then the $\mathbb{Q}[[t_0, \dots, t_m]]$ -module $H^*(X) \otimes_{\mathbb{Q}} \mathbb{Q}[[t_0, \dots, t_m]]$ is the big quantum cohomology $QH(X)$.

The associativity of quantum product is formulated as WDVV equation:

$$F_{ija} g^{ab} F_{bkl} = F_{ila} g^{ab} F_{bjk}.$$

It is proved by a forgetful map $\pi : \overline{\mathcal{M}}_{0,4}(X, \beta) \rightarrow \overline{\mathcal{M}}_{0,4}$. One should notice that $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$, so the boundary divisor $D(12|34) \sim D(13|24)$ and

$$\begin{aligned}
&\int_{[\overline{\mathcal{M}}_{0,4}(X, \beta)]^{vir} \cap \pi^* D(12|34)} ev_1^*(T_i) ev_2^*(T_j) ev_3^*(T_k) ev_4^*(T_l) \prod_{i=5}^{n+4} ev_i^*(\gamma) \\
&= \int_{[\overline{\mathcal{M}}_{0,4}(X, \beta)]^{vir} \cap \pi^* D(13|24)} ev_1^*(T_i) ev_2^*(T_j) ev_3^*(T_k) ev_4^*(T_l) \prod_{i=5}^{n+4} ev_i^*(\gamma).
\end{aligned}$$

A useful trick is to separate $[\overline{\mathcal{M}}_{0,4}(X, \beta)]^{vir} \cap \pi^* D(12|34)$ by

$$\coprod_{n_1+n_2=n, \beta_1+\beta_2=\beta} [\overline{\mathcal{M}}_{0, n_1+3}(X, \beta_1) \times \overline{\mathcal{M}}_{0, n_2+3}(X, \beta_2)]^{vir} \cap (ev \times ev)^*[\Delta],$$

$$PD[\Delta] = g^{ab} T_a \otimes T_b,$$

then we get

$$\begin{aligned} & \sum_{n_1+n_2=n} \sum_{\beta_1+\beta_2=\beta} \langle T_i T_j T_a \gamma^{n_1} \rangle_{0, n_1+3, \beta_1} g^{ab} \langle T_b T_k T_l \gamma^{n_2} \rangle_{0, n_2+3, \beta_2} \\ &= \sum_{n_1+n_2=n} \sum_{\beta_1+\beta_2=\beta} \langle T_i T_k T_a \gamma^{n_1} \rangle_{0, n_1+3, \beta_1} g^{ab} \langle T_b T_j T_l \gamma^{n_2} \rangle_{0, n_2+3, \beta_2}. \end{aligned}$$

This is the essential part in the proof of associativity of quantum product.

Remark 2.1.1. *It deserves to notice that the quantum product is defined by rational curves, so its usage mainly concentrates in genus 0 GW-invariants. The difficulty to define quantum product via higher genus curves is that there is no so good associativity as the genus 0 case. It must be a good work if we can find a way to give a quantum product via higher genus curves with associativity like now.*

The small quantum product is defined by

$$T_i *_s T_j = T_i *_t T_j|_{t_{p+1}=\dots=t_m=0}, 0 \leq i, j \leq m.$$

Precisely, let

$$\bar{F}_{ijk} = F_{ijk}|_{t_{p+1}=\dots=t_m=0} = \int_X T_i T_j T_k + \sum_{\beta > 0} Q^\beta q_1^{\int_\beta T_1} \dots q_p^{\int_\beta T_p} \langle T_i T_j T_k \rangle_{0, 3, \beta},$$

then

$$T_i *_s T_j = \bar{F}_{ije} g^{ef} T_f, \quad 1 \leq e, f \leq m.$$

Extend q_i linearly, the $\mathbb{Q}[[q_1, \dots, q_p]]$ -module $H^*(X) \otimes_{\mathbb{Q}} \mathbb{Q}[[q_1, \dots, q_p]]$ is defined as the small quantum cohomology $QH^s(X)$.

Example 2.1.2. $QH^s(\mathbb{P}^m) = \mathbb{Q}[H, q]/(H^{m+1} - q)$, where $H \in H^2(\mathbb{P}^m, \mathbb{Q})$, $q = e^{t_1}$.

2.2 quantum differential equation

We can view the vector space $H(X)$ as a Riemannian manifold M with standard flat metric g_{ij} given by Poincare pairing. The quantum product $*_t$ could be use to define a connection (called Dubrovin connection, or Givental connection) ∇^z , which is different from the Levi-Civita connection induced by its Riemannian metric.

Definition 2.2.1. (*Dubrovin connection*) Let $X, Y \in \Gamma(M, TM)$, ∇ be the Levi-Civita connection w.r.t g . The Dubrovin connection ∇^z is defined by

$$\nabla_X^z Y := \nabla_X Y - \frac{1}{z} X *_t Y.$$

The WDVV equation shows ∇^z is flat. i.e. $Rm^z = 0$.

Definition 2.2.2 (quantum differential equation). Let $\sigma \in \Gamma(M, TM)$, the equation $\nabla^z \sigma = 0$ is the quantum differential equation. The fundamental solution of quantum differential equation is $(m+1) \times (m+1)$ matrix $s(z, t)$ ($t = (t_0, \dots, t_m)$) $= (a_{ij})$, such that each column defines a solution

$$\sigma_j(t) = \sum_{i=0}^m a_{ij}(t) \frac{\partial}{\partial t_i}.$$

Now we want to find the solution of quantum differential equation. Let $(S(z)T_a, T_b) = g_{ab} + \langle \langle \frac{T_a}{z - \psi_1}, T_b \rangle \rangle_{0,2}$, where

$$\langle \langle \frac{T_a}{z - \psi_1}, T_b \rangle \rangle_{0,2} = \sum_{\substack{n \geq 0, \beta, \\ (n, \beta) \neq (0,0)}} \frac{1}{n!} \langle \frac{T_a}{z - \psi_1} T_b \gamma^n \rangle_{0, n+2, \beta}.$$

Proposition 2.2.3. The $S_a = (S(z)T_a, T_b)g^{bc}\partial_c$ is a flat section with respect to ∇^z .

This proposition is proven with the help of topological recursion relation.

Definition 2.2.4. The descendent invariants are defined by

$$\langle \tau_{a_1}(\gamma_1) \dots \tau_{a_n}(\gamma_n) \rangle_{g, n, \beta} = \int_{[\overline{\mathcal{M}}_{g, n}(X, \beta)]^{vir}} ev_1^*(\gamma_1) \psi_1^{a_1} \cup \dots \cup ev_n^*(\gamma_n) \psi_n^{a_n}.$$

Theorem 2.2.5 (topological recursion relation). Let $\gamma_i \in H^*(X)$,

$$\begin{aligned} & \langle \tau_{a_1+1}(\gamma_1) \tau_{a_2}(\gamma_2) \tau_{a_3}(\gamma_3) \prod_{i=4}^n \tau_{a_i}(\gamma_i) \rangle_{0, n, \beta} \\ &= \sum_{\substack{A \cup B = \{4, \dots, n\} \\ \beta = \beta_1 + \beta_2}} \sum_{\substack{a, b=0 \\ a, b=0}}^m \langle \tau_{a_1}(\gamma_1) \prod_{i \in A} \tau_{a_i}(\gamma_i) T_a \rangle_{0, |A|+2, \beta_1} g^{ab} \langle T_b \tau_{a_2}(\gamma_2) \tau_{a_3}(\gamma_3) \prod_{j \in B} \tau_{a_j}(\gamma_j) T_a \rangle_{0, |B|+3, \beta_2} \end{aligned}$$

Proof. Consider the forgetful map

$$\begin{aligned} \pi : \overline{\mathcal{M}}_{0,n}(X, \beta) &\rightarrow \overline{\mathcal{M}}_{0,3} : \\ [f : C \rightarrow X, 1, \dots, n] &\mapsto [C, 1, 2, 3]. \end{aligned}$$

Let $\mathbb{L}_1, \mathbb{L}'_1$ be the tautological line bundles

$$\begin{array}{ccc} \mathbb{L}_1 & & \mathbb{L}'_1 \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{0,n}(X, \beta) & \xrightarrow{\pi} & \overline{\mathcal{M}}_{0,3} \end{array}$$

There is

$$\begin{array}{ccc} \mathbb{L}_1 \cong \pi^* \mathbb{L}'_1 \otimes \left(\sum_{\substack{\beta_1 + \beta_2 = \beta \\ A \cup B = \{4, \dots, n\}}} D(1, A, \beta_1 | 2, 3, B, \beta_2) \right), & & \\ D(1, A, \beta_1 | 2, 3, B, \beta_2) & \longrightarrow & \overline{\mathcal{M}}_{0,|B|+3}(X, \beta_2) \\ \downarrow & & \downarrow \text{ev}_{node} \\ \overline{\mathcal{M}}_{0,|A|+2}(X, \beta_1) & \xrightarrow{\text{ev}_{node}} & X. \end{array}$$

Because $\overline{\mathcal{M}}_{0,3}$ is a point, \mathbb{L}'_1 is trivial and

$$\psi_1 = \sum_{\substack{\beta_1 + \beta_2 = \beta \\ A \cup B = \{4, \dots, n\}}} [D(1, A, \beta_1 | 2, 3, B, \beta_2)].$$

Take this formula into LHS, we get the recursion relation. \square

The fundamental solution of small quantum differential equation directly relates to the definition of J-function in mirror symmetry. It is given by

$$\tilde{S}(z) = S(z)|_{t_{p+1}, \dots, t_m=0}.$$

Specifically, let $\gamma = \sum_{i=0}^p t_i T_i = t_0 T_0 + \gamma_1$ and $\gamma_1 = \sum_{i=1}^p t_i T_i$, then

$$(\tilde{S}(z) T_a, T_b) = g_{ab} + \sum_{\substack{n \geq 0, \beta \\ (n, \beta) \neq (0, 0)}} \frac{1}{n!} \left\langle \frac{T_a}{z - \psi_1} T_b \gamma^n \right\rangle_{0, n+2, \beta}.$$

By string equations and divisor equations, γ can be put out of the bracket, and finally we get

$$(\tilde{S}(z)T_a, T_b) = \int_X e^{\gamma/z} T_a T_b + \sum_{\beta > 0} \left\langle \frac{e^{\gamma/z} T_a}{z - \psi_1} T_b \right\rangle_{0,2,\beta} e^{\int_\beta \gamma_1}.$$

Chapter 3

Mirror Symmetry

I plan to follow Givental's approach to give a proof of genus 0 mirror symmetry of hypersurfaces in \mathbb{P}^n . The key character of Givental's approach is that it uses J-function and I-function to show the mirror symmetry relation. The J-function is defined as follows, which describes the A-model information.

Definition 3.0.1. *For a complex manifold X , the J^X is*

$$(T_a, J^X) := (\tilde{S}(z)T_a, 1) = \int_X e^{\gamma/z} T_a + \sum_{\beta > 0} \langle \frac{e^{\gamma/z} T_a}{z - \psi_1} 1 \rangle_{0,2,\beta} e^{\int_{\beta} \gamma_1}.$$

J^X is a $H^*(X)$ -value function: $J^X(t_0, t_1, \dots, t_p, z) = (T_a, J^X) g^{ab} T_b$. In this chapter, X is a hypersurface of degree l in \mathbb{P}^m . We assume $l \leq m+1$ so X is either Fano or Calabi-Yau.

At first, J^X could be pushforwarded to $i_* J^X$ as a $H^*(\mathbb{P}^m)$ -valued function. Let $i : X \hookrightarrow \mathbb{P}^m$. It induces $i : \overline{\mathcal{M}}_{0,n}(X, d) \rightarrow \overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)$. Consider

$$\begin{array}{ccc} \overline{\mathcal{C}}_{0,n}(\mathbb{P}^m, d) & \xrightarrow{F} & \mathbb{P}^m \\ \downarrow \pi & & \\ \overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d) & & \end{array}$$

Let $E_d := \pi_* F^* \mathcal{O}(l)$ be the obstruction bundle over $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)$. The following theorem shows the relationship of virtual fundamental classes:

Theorem 3.0.2.

$$i_* [\overline{\mathcal{M}}_{0,n}(X, d)]^{vir} = e(\pi_* F^* \mathcal{O}(l)) \cap [\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)]^{vir}.$$

Let $ev_1 : \overline{\mathcal{M}}_{0,2}(X, \beta) \rightarrow X$

Proposition 3.0.3.

$$J^X = e^{\gamma/z} \left(1 + \sum_{\beta > 0} e^{\int_{\beta} \gamma_1} (ev_1)_* \left(\frac{1}{z - \psi_1} \right) \right),$$

$$J^{\mathbb{P}^m, \mathcal{O}(l)}(t_0, t_1, z) := i_* J^X = e^{(t_0+t_1 H)/z} \left(e(\mathcal{O}(l)) + \sum_{d > 0} e^{dt_1} (ev_1)_* \left(\frac{e(E_d)}{z - \psi_1} \right) \right),$$

where $H \in H^2(\mathbb{P}^m, \mathbb{Q})$ is the generator of $H^2(\mathbb{P}^m, \mathbb{Q})$, $\gamma = t_0 + t_1 H$.

Let $0 \rightarrow E'_d \rightarrow E_d \rightarrow ev_1^* \mathcal{O}(l) \rightarrow 0$, then

$$J^{\mathbb{P}^m, \mathcal{O}(l)}(t_0, t_1, z) = e^{(t_0+t_1 H)/z} lH \left(1 + \sum_{d > 0} e^{dt_1} (ev_1)_* \left(\frac{e(E'_d)}{z - \psi_1} \right) \right)$$

The I-function is

$$I^{\mathbb{P}^m, \mathcal{O}(l)}(t_0, t_1, z) := e^{(t_0+t_1 H)/z} lH \left(1 + \sum_{d=1}^{\infty} e^{dt_1} \frac{\prod_{a=1}^{dl} (lH + az)}{\prod_{a=1}^d (H + az)^{m+1}} \right).$$

3.1 Fano case

The equivariant cohomology of \mathbb{P}^m with respect to $\mathbb{T} = (\mathbb{C}^*)^{m+1}$ is

$$H_{\mathbb{T}}^*(\mathbb{P}^m; \mathbb{Q}) = \mathbb{Q}[H, \lambda_0, \dots, \lambda_m] / \prod_{i=0}^m (H - \lambda_i).$$

The classes $\phi_i = \prod_{j \neq i} (H - \lambda_j)$, are a basis of $H_{\mathbb{T}}^*(\mathbb{P}^m; \mathbb{Q})$. Moreover, for $f(H, \lambda) \in H_{\mathbb{T}}^*(\mathbb{P}^m; \mathbb{Q})$, $(\phi_i, f(H, \lambda)) = f(\lambda_i, \lambda)$. Lifting J-function and I-function to the equivariant classes $H_{\mathbb{T}}^*(\mathbb{P}^m)$ and define

$$\tilde{J}^{\mathbb{P}^m, \mathcal{O}(l)} := e^{(t_0+t_1 H)/z} lH \left(1 + \sum_{d > 0} e^{dt_1} (ev_1)_* \left(\frac{e_{\mathbb{T}}(E'_d)}{z - \psi_1} \right) \right);$$

$$\tilde{I}^{\mathbb{P}^m, \mathcal{O}(l)} := e^{(t_0+t_1 H)/z} lH \left(1 + \sum_{d=1}^{\infty} e^{dt_1} \frac{\prod_{r=1}^{ld} (lH + rz)}{\prod_{k=0}^m \prod_{r=1}^d (H - \lambda_k + rz)} \right).$$

If we can show the relationship of \tilde{J} and \tilde{I} , then take $\lambda \rightarrow 0$, we get a relation between J and I . Let $q = e^{t_1}$ and define

$$\begin{aligned}
S(q, z, \lambda) &= 1 + \sum_{d>0} q^d (ev_1)_* \left(\frac{e_{\mathbb{T}}(E'_d)}{z - \psi_1} \right); \\
\Psi(q, z, \lambda) &= 1 + \sum_{d=1}^{\infty} e^{dt_1} \frac{\prod_{r=1}^{ld} (lH + rz)}{\prod_{k=0}^m \prod_{r=1}^d (H - \lambda_k + rz)}; \\
S_i(q, z, \lambda) &:= (\phi_i, S(q, z, \lambda)) = 1 + \sum_{d>0} q^d \int_{\overline{\mathcal{M}}_{0,2}(\mathbb{P}^m, d)} \frac{e_{\mathbb{T}}(E'_d) \text{ev}_1^*(\phi_i)}{z - \psi_1}; \\
\Psi_i(q, z, \lambda) &:= (\phi_i, \Psi(q, z, \lambda)) = 1 + \sum_{d=1}^{\infty} q^d \frac{\prod_{r=1}^{dl} (l\lambda_i + rz)}{\prod_{k=0}^m \prod_{r=1}^d (\lambda_i - \lambda_k + rz)}.
\end{aligned}$$

The first step is to use localization formula to compute S_i . We can classify the fixed locus $\overline{\mathcal{M}}_{0,2}(\mathbb{P}^m, d)^{\mathbb{T}}$ into three classes:

- G_d^1 : the first mark point x_1 is mapped to p_j ($j \neq i$);
- G_d^2 : the first mark point x_1 is mapped to p_i and the irreducible component C_v is stable (i.e. not a point);
- G_d^3 : the first mark point x_1 is mapped to p_i and the irreducible component C_v is a single point.

In G_d^1 case, $\text{ev}_1^*(\phi_i)|_{F_{\Gamma}} = 0$, so only the latter two cases contribute S_i . It can be expressed as

$$\begin{aligned}
S_i(q, z, \lambda) &= 1 + \sum_{\Gamma \in G_d^2 \cup G_d^3} \text{Cont}_{\Gamma}(S_i(q, z, \lambda)); \\
\text{Cont}_{\Gamma}(S_i(q, z, \lambda)) &= \sum_{d \geq 1} q^d \int_{F_{\Gamma}} \frac{e_{\mathbb{T}}(E'_d) \text{ev}_1^*(\phi_i)}{(z - \psi_1) e_{\mathbb{T}}(N_{\Gamma}^{\text{vir}})}
\end{aligned}$$

The following lemma is important in recursion formula of $S_i(q, z, \lambda)$.

Lemma 3.1.1. (1) $S_i(q, z, \lambda) \in \mathbb{Q}(\lambda, z)[[q]]$;
(2) Let $S_i(q, z, \lambda) = 1 + \sum_{d>0} q^d \xi_{id}(z, \lambda)$. Then $\xi_{id}(z, \lambda)$ are regular at $z = \frac{\lambda_i - \lambda_j}{n}$ for all $i \neq j$ and $n \geq 1$.

We will compute the contribution of G_d^2 and G_d^3 respectively.

Theorem 3.1.2. *Let $C_i(q, z, \lambda) = \sum_{\Gamma \in G_d^2} \text{Cont}_{\Gamma}(S_i(qz^{m+1-l}, z, \lambda))$*

$$\text{then } C_i(q, z, \lambda) = \begin{cases} 0, & l < m \\ -1 + \exp(-m!q + \frac{(m\lambda_i)^m}{\prod_{j \neq i} (\lambda_i - \lambda_j)} q), & l = m. \end{cases}$$

As for $\Gamma \in G_d^3$, we can split Γ into Γ_0 and Γ_c .

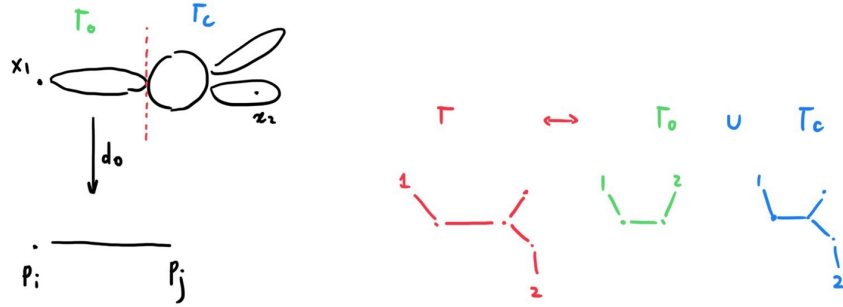


Figure 3.1: $\Gamma \in G_d^3$

Theorem 3.1.3. *Let $\Gamma \in G_d^3$ such that degree of C_{ij} d_0 and $d_c > 0$, then*

$$\text{Cont}_{\Gamma} S_i(q, z, \lambda) = q^{d_0} \frac{C_i^j}{d_0 z + \lambda_i - \lambda_j} (d_0, \lambda) \text{Cont}_{\Gamma_c} S_j(q, \frac{\lambda_j - \lambda_i}{d_0}, \lambda),$$

$$C_i^j(d, \lambda) = \frac{\prod_{r=1}^{ld} (l\lambda_i + r \frac{\lambda_j - \lambda_i}{d})}{\prod_{k=0}^m \prod_{r=1, (k,r) \neq (j,d)}^d (\lambda_i - \lambda_k + r \frac{\lambda_j - \lambda_i}{d})}.$$

Proof. Consider the diagram as Fig 3.1, we have $F_{\Gamma} = F_{\Gamma_0} \times F_{\Gamma_c}$. Let $\pi_0 : F_{\Gamma} \rightarrow F_{\Gamma_0}$ and let $\pi_c : F_{\Gamma} \rightarrow F_{\Gamma_c}$

$$E'_{d_0+d_c}|_{F_{\Gamma}} = \pi_0^* E'_{d_0} \oplus \pi_c^* E'_{d_0};$$

$$\frac{N_{F_{\Gamma}}}{T_{p_i} \mathbb{P}^m} = \frac{N_{F_{\Gamma_0}}}{T_{p_i} \mathbb{P}^m} \oplus \frac{N_{F_{\Gamma_c}}}{T_{p_j} \mathbb{P}^m} \oplus \pi_0^* \mathbb{L}_2^{\vee} \otimes \pi_c^* \mathbb{L}_1^{\vee};$$

$$\text{ev}_1^* \phi_i = \prod_{j \neq i} (\lambda_i - \lambda_j), \quad c_1(\mathbb{L}_2^{\vee}) = \frac{\lambda_j - \lambda_i}{d_0};$$

$$e_{\mathbb{T}}(N_{\Gamma_0}) = (-1)^{d_0} \prod_{r=1}^{d_0} \left(r \frac{\lambda_j - \lambda_i}{d_0} \right)^2 \prod_{r=0}^{d_0} \prod_{k \neq i, j} (\lambda_i - \lambda_k + r \frac{\lambda_j - \lambda_i}{d_0}).$$

Hence,

$$\begin{aligned} q^{d_0+d_c} \int_{F_{\Gamma}} \frac{e_{\mathbb{T}}(E'_{d_0+d_c} \text{ev}_1^* \phi_i)}{(z - \psi) e_{\mathbb{T}}(N_{F_{\Gamma}})} &= q^{d_0+d_c} \frac{C_i^j(d_0, \lambda)}{d_0 z + \lambda_i - \lambda_j} \int_{F_{\Gamma}} \frac{e_{\mathbb{T}}(E'_{d_c} \text{ev}_1^* \phi_i)}{(z - \psi) e_{\mathbb{T}}(N_{F_{\Gamma}})} \Big|_{z = \frac{\lambda_j - \lambda_i}{d_0}}, \\ C_i^j(d_0, \lambda) &= \frac{e_{\mathbb{T}}(E'_{d_0}) \text{ev}_1^* \phi_i}{e_{\mathbb{T}}(N_{\Gamma_0})} = \frac{\prod_{r=1}^{d_0} (l \lambda_i + r \frac{\lambda_j - \lambda_i}{d_0}) \prod_{k \neq i} (\lambda_i - \lambda_k)}{(-1)^{d_0} \prod_{r=1}^{d_0} \left(r \frac{\lambda_j - \lambda_i}{d_0} \right)^2 \prod_{r=0}^{d_0} \prod_{k \neq i, j} (\lambda_i - \lambda_k + r \frac{\lambda_j - \lambda_i}{d_0})} \\ &= \frac{(\lambda_i - \lambda_j) \prod_{r=1}^{d_0} (l \lambda_i + r \frac{\lambda_j - \lambda_i}{d_0})}{(-1)^{d_0} \prod_{r=1}^{d_0} \left(r \frac{\lambda_j - \lambda_i}{d_0} \right)^2 \prod_{r=1}^{d_0} \prod_{k \neq i, j} (\lambda_i - \lambda_k + r \frac{\lambda_j - \lambda_i}{d_0})} \\ &= \frac{\prod_{r=1}^{d_0} (l \lambda_i + r \frac{\lambda_j - \lambda_i}{d_0})}{\prod_{k=0}^m \prod_{r=1, (k, r) \neq (j, d_0)}^{d_0} (\lambda_i - \lambda_k + r \frac{\lambda_j - \lambda_i}{d_0})}. \end{aligned}$$

Finally,

$$\begin{aligned} \text{Cont}_{\Gamma} S_i(q, z, \lambda) &= \sum_{d_c > 0} q^{d_0+d_c} \int_{F_{\Gamma_c}} \frac{e_{\mathbb{T}}(E'_{d_0+d_c} \text{ev}_1^* \phi_i)}{(z - \psi_1) e_{\mathbb{T}}(N_{F_{\Gamma}})} \\ &= q^{d_0} \frac{C_i^j(d_0, \lambda)}{d_0 z + \lambda_i - \lambda_j} \sum_{d_c > 0} q^{d_c} \int_{F_{\Gamma_c}} \frac{e_{\mathbb{T}}(E'_{d_c} \text{ev}_1^* \phi_i)}{(z - \psi_1) e_{\mathbb{T}}(N_{F_{\Gamma}})} \Big|_{z = \frac{\lambda_j - \lambda_i}{d_0}} \\ &= q^{d_0} \frac{C_i^j(d_0, \lambda)}{d_0 z + \lambda_i - \lambda_j} \text{Cont}_{\Gamma_c} S_j(q, \frac{\lambda_j - \lambda_i}{d_0}, \lambda). \quad \square \end{aligned}$$

Remark 3.1.4. $S_j(q, \frac{\lambda_j - \lambda_i}{d_0}, \lambda)$ is well-defined by Lemma 3.1.1.

Theorem 3.1.5. The function S_i satisfies the following recursion formula:

$$S_i(qz^{m+1-l}, z, \lambda) = 1 + C_i(q, z, \lambda) + \sum_{j \neq i} \sum_{d > 0} q^d z^{(m+1-l)d} \frac{C_i^j(d, \lambda)}{dz + \lambda_i - \lambda_j} S_j(qz^{m+1-l}, \frac{\lambda_j - \lambda_i}{d}, \lambda).$$

Proof. It directly follows from Theorem 3.1.2 and 3.1.3 and

$$S_i(qz^{m+1-l}, z, \lambda) = 1 + \sum_{\Gamma \in G_d^2 \cup G_d^3} S_i(qz^{m+1-l}, z, \lambda) \quad \square$$

The second step is to check Ψ_i satisfies the same recursion relation.

Proposition 3.1.6. *For $l < m$, Ψ_i has the recursion relation*

$$\Psi_i(qz^{m+1-l}, z, \lambda) = 1 + \sum_{j \neq i} \sum_{d > 0} q^d \frac{C_i^j(d, \lambda)}{dz + \lambda_i - \lambda_j} \Psi_j(qz^{m+1-l}, \frac{\lambda_j - \lambda_i}{d}, \lambda);$$

for $l = m$, they differ a constant.

Proof. The hint is to view the formula as meromorphic functions and analyse the simple poles.

$$\text{deg } d \text{ part of LHS} = z^{(m+1-l)d} \frac{\prod_{r=1}^{dl} (l\lambda_i + rz)}{\prod_{k=0}^m \prod_{r=1}^d (\lambda_i - \lambda_k + rz)}$$

has simple poles at $z = \frac{\lambda_j - \lambda_i}{e}$ with $j \neq i$, $1 \leq e \leq d$. The residue is

$$\text{Res}_z \text{ LHS} = \left(\frac{\lambda_j - \lambda_i}{e} \right)^{(m+1-l)d} \frac{\prod_{r=1}^{dl} (l\lambda_i + r \frac{\lambda_j - \lambda_i}{e})}{\prod_{k=0}^m \prod_{r=1, (k,r) \neq (j,e)}^d (\lambda_i - \lambda_k + r \frac{\lambda_j - \lambda_i}{e})}.$$

$$\begin{aligned} \text{deg } d \text{ part of RHS} &= z^{(m+1-l)d} \sum_{j \neq i} \left(\frac{C_i^j(d, \lambda)}{dz + \lambda_i - \lambda_j} \right. \\ &\quad \left. + \sum_{e=1}^{d-1} \frac{C_i^j(e, \lambda)}{ez + \lambda_i - \lambda_j} \frac{\prod_{r=1}^{l(d-e)} (l\lambda_j + r \frac{\lambda_j - \lambda_i}{e})}{\prod_{k=0}^m \prod_{r=1}^{d-e} (\lambda_j - \lambda_k + r \frac{\lambda_j - \lambda_i}{e})} \right) \end{aligned}$$

The simple poles are also $z = \frac{\lambda_j - \lambda_i}{e}$ with $j \neq i$, $1 \leq e \leq d$.

$$e = d : \text{Res}_z \text{ RHS} = \left(\frac{\lambda_j - \lambda_i}{d} \right)^{(m+1-l)d} C_i^j(d, \lambda) = \text{Res}_z \text{ LHS}$$

$e < d$:

$$\begin{aligned} \text{Res}_z \text{ RHS} &= \left(\frac{\lambda_j - \lambda_i}{e} \right)^{(m+1-l)d} \frac{\prod_{r=1}^{le} (l\lambda_i + r \frac{\lambda_j - \lambda_i}{e})}{\prod_{k=0}^m \prod_{r=1, (k,r) \neq (j,e)}^e (\lambda_i - \lambda_k + r \frac{\lambda_j - \lambda_i}{e})} \\ &\quad \times \frac{\prod_{r=1}^{l(d-e)} (l\lambda_j + r \frac{\lambda_j - \lambda_i}{e})}{\prod_{k=0}^m \prod_{r=1}^{d-e} (\lambda_j - \lambda_k + r \frac{\lambda_j - \lambda_i}{e})} \end{aligned}$$

For numerator, let $s = le + r$, $1 \leq r \leq l(d-e)$, $le + 1 \leq s \leq ld$,

$$l\lambda_j + r \frac{\lambda_j - \lambda_i}{e} = \frac{le + r}{e} \lambda_j - r \frac{\lambda_i}{e} = l\lambda_i + s \frac{\lambda_j - \lambda_i}{e};$$

for denominator, let $s = e + r$, $1 \leq r \leq d - e$, $e + 1 \leq s \leq d$, then

$$\begin{aligned} \lambda_j - \lambda_k + r \frac{\lambda_j - \lambda_i}{e} &= \frac{e + r}{e} \lambda_j - \lambda_k - \frac{r}{e} \lambda_i = \lambda_i - \lambda_k + s \frac{\lambda_j - \lambda_i}{e}; \\ \text{Res}_z \text{ RHS} &= \left(\frac{\lambda_j - \lambda_i}{e} \right)^{(m+1-l)d} \frac{\prod_{r=1}^{dl} (l\lambda_i + rz)}{\prod_{k=0}^m \prod_{r=1, (k,r) \neq (j,e)}^d (\lambda_i - \lambda_j + r \frac{\lambda_j - \lambda_i}{e})} \\ &= \text{Res}_z \text{ LHS}. \end{aligned}$$

If $l < m$, we find out that $\text{LHS} = \text{RHS} = 0$ at $z = 0$, so we have done. \square

As a result, we show a mirror symmetry of $l < m$ case:

Theorem 3.1.7 (Mirror symmetry for $l < m$). *If $l < m$, then $S_i(q, z, \lambda) = \Psi_i(q, z, \lambda)$. As a corollary,*

$$J^{\mathbb{P}^m, \mathcal{O}(l)}(t_0, t_1, z) = I^{\mathbb{P}^m, \mathcal{O}(l)}(t_0, t_1, z).$$

We need another recursion relation to prove $l = m$ case

Proposition 3.1.8. Ψ_i has the recursion relation

$$\begin{aligned} e^{-m!q} \Psi_i(qz, z, \lambda) &= 1 + C_i(q, z, \lambda) \\ &+ \sum_{j \neq i} \sum_{d > 0} q^d z^d \frac{C_i^j(d, \lambda)}{dz + \lambda_i - \lambda_j} e^{-m!q} \Psi_j(qz, \frac{\lambda_j - \lambda_i}{d}, \lambda), \\ C_i(q, z, \lambda) &= -1 + \exp(-m!q + \frac{(m\lambda_i)^m}{\prod_{j \neq i} (\lambda_i - \lambda_j)} q) \end{aligned}$$

Proof. It is equivalent to proof

$$\begin{aligned} \Psi_i(qz, z, \lambda) &= \exp\left(\frac{(m\lambda_i)^m}{\prod_{j \neq i} (\lambda_i - \lambda_j)} q\right) \\ &+ \sum_{j \neq i} \sum_{d > 0} q^d z^d \frac{C_i^j(d, \lambda)}{dz + \lambda_i - \lambda_j} \Psi_j(qz, \frac{\lambda_j - \lambda_i}{d}, \lambda), \end{aligned}$$

Similar to the proof of Prop 3.1.6.

$$\deg d \text{ part of LHS} = \frac{\prod_{r=1}^{dl} (l\lambda_i + rz)}{d! \prod_{k=0, k \neq i}^m \prod_{r=1}^d (\lambda_i - \lambda_k + rz)}$$

$$\begin{aligned} \deg d \text{ part of RHS} &= \frac{(m\lambda_i)^{md}}{d! \prod_{j \neq i} (\lambda_i - \lambda_j)^d} + \sum_{j \neq i} z^d \left(\frac{C_i^j(d, \lambda)}{dz + \lambda_i - \lambda_j} \right. \\ &\quad \left. + \sum_{e=1}^{d-1} \frac{C_i^j(e, \lambda)}{ez + \lambda_i - \lambda_j} \frac{\prod_{r=1}^{m(d-e)} (m\lambda_j + r \frac{\lambda_j - \lambda_i}{e})}{\prod_{k=0}^m \prod_{r=1}^{d-e} (\lambda_j - \lambda_k + r \frac{\lambda_j - \lambda_i}{e})} \right) \end{aligned}$$

As Prop 3.1.6, they have same simple poles and residue numbers. Take $z = 0$,

$$\deg d \text{ part of LHS}(z = 0) = \frac{(m\lambda_i)^{md}}{d! \prod_{j \neq i} (\lambda_i - \lambda_j)^d} = \deg d \text{ part of LHS}(z = 0).$$

Hence two formulas identify. \square

Remark 3.1.9. $\exp(\frac{(m\lambda_i)^m}{\prod_{j \neq i} (\lambda_i - \lambda_j)})q$ is the constant in Prop 3.1.6.

Theorem 3.1.10 (Mirror symmetry for $l = m$). *For $l = m$, $e^{m!q/z} S_i(q, z, \lambda) = \Psi_i(q, z, \lambda)$. As a corollary, $e^{m!q/z} S(q, z, \lambda) = \Psi(q, z, \lambda)$ and*

$$J^{\mathbb{P}^m, \mathcal{O}(l)}(t_0 + m!e^{t_1}, t_1, z) = I^{\mathbb{P}^m, \mathcal{O}(l)}(t_0, t_1, z).$$

3.2 Calabi-Yau case

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