Residues, Duality and the Riemann-Roch Theorem

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1 Abstract

This paper is going to show the detail proof of the Serre duality on a projective, irreducible, and non-singular algebraic curve. It is shown that the concept about residue is important in the Serre duality. This duality plays a central role in the proof of the definitive form of the Riemann-Roch theorem. Some corollaries about the Riemann-Roch theorem will also be introduced in this paper.

2 Introduction

The Riemann-Roch theorem is a insightful conclusion in algebraic curves. It states the relation between the rational functions on an algebraic curve and the underlying topology of the algebraic curve. It can also be regarded as a fancy equation between the Euler character of a sheaf associated by a divisor and the underlying topology genus. There are several forms of the Riemann-Roch theorems. They are listed as follows.

Theorem 1 (Riemann-Roch theorem – first form)**.** *Let X be a projective, irreducible and non-singular algebraic curve. For every divisor D on X,*

$$l(D) - i(D) = deg\ D + 1 - g$$

Theorem 2 (Riemann-Roch theorem – definitive form)**.** *Let X be a projective, irreducible and non-singular algebraic curve. For every divisor D on X,*

$$l(D) - l(K - D) = deg\ D + 1 - g,$$

where K is a canonical divisor on X.

The proof the first form is relative easy. However, the proof of the definitive form is not so easy. It is shown that the residues and duality play important roles in the proof of the definitive form. This survey paper will show these key steps in detail.

3 Notations and conventions

Let us suppose the base field k is algebraically closed. Let X be an algebraic curve, $\mathcal{O}_X$ be the structure sheaf of X; let us suppose that X is irreducible, nonsingular, and projective. Let $k(X)$ be the field of rational functions on X and let $\mathcal{K}_X$ be the constant sheaf of $k(X)$.

A divisor D is a formal sum of the points $P \in X$:

$$D = \sum_{P \in X} n_P P \quad \text{with } n_P \in \mathbb{Z},$$

and $n_P = 0$ for all but finitely many $P \in X$. The notation $v_P(D)$ means the coefficient of P in D.

The **degree** of D is defined by

$$\deg D = \sum n_P = \sum v_P(D)$$

A divisor D is **effective** (or **positive**) if all the $v_P(D)$ are not less than 0.

Let $f \in k(X)^*$. Because X is a nonsingular algebraic curve, $\mathcal{O}_{X,P}$ is a discrete valuation ring for $\forall P \in X$. Thus, one can define the divisor of (f), denoted by (f), by the formula

$$(f) = \sum_{P \in X} v_P(f) P,$$

where $v_P(f)$ is the valuation of f in the ring $\mathcal{O}_{X,P}$.

Let D be a divisor on X. We define the vector space $L(D)$ to be the set of rational functions f which satisfy $(f) \geq -D$, which means

$$v_P(f) \geq -v_P(D), \quad \forall P \in X.$$

We define the sheaf $\mathcal{L}(D)$ to be a subsheaf of the sheaf of rational functions $\mathcal{K}(X)$:

$$\mathcal{L}(D)(U) := \{f \in \mathcal{K}_X(U) | v_P(f) \geq -v_P(D), \forall P \in U\}, \quad \forall \text{ open } U \subset X.$$

It is easy to see that $L(D)$ is the global section of the sheaf $\mathcal{L}(D)$. As $\mathcal{L}(D)$ is a coherent sheaf, the cohomology groups $H^0(X, \mathcal{L}(D))$ and $H^1(X, \mathcal{L}(D))$ are finite dimensional over k. For $p \geq 2$, $H^p((X, \mathcal{L}(D))$ vanish because the dimension of X is 1.

Just as conventions, we define the following notations:

$$L(D) = \Gamma(X, \mathcal{L}(D)) = H^1(X, \mathcal{L}(D)), \quad l(D) = dim_k L(D),$$

$$I(D) = H^1(X, \mathcal{L}(D)), \quad i(D) = dim_k I(D), \quad g = i(0) = dim_k H^1(X, \mathcal{O}_X).$$

The integer g is the **arithmetic genus** of the curve X.

The first form of the Riemann-Roch theorem states that

$$l(D) - i(D) = \deg D + 1 - g$$

4 Repartitions

The **repartition** in Weil's language is a bridge from $I(D)$ to the set of differential form $\Omega(D)$. The duality theorem is to show $I(D) \cong \Omega(D)$. That is why the classes of repartitions is important here.

4.1 Classes of repartitions

Definition 1. *A* ***repartition*** *r is a family $\{r_P\}_{P \in X}$ of elements of $k(X)$ such that $r_P \in \mathcal{O}_P$ for almost all $P \in X$. Since all local rings $\mathcal{O}_P$ and $k(X)$ are algebra over the base field k, the repartitions form an algebra R over the field k by pointwise addition and multiplication.*

Definition 2. *Given an divisor D, R(D) is defined as the vector subspace of R formed by the $r = \{r_P\}$ such that*

$$v_P(r_P) \geq -v_P(D).$$

Following the definition, $R(D)$ owns the following propositions:

(a) If D_1, D_2 are two divisors on X satisfying $D_1 \leq D_2$, then $R(D_1) \subset R(D_2)$.

(b) The union of all $R(D)$ as D run through the set of all divisors of X is the algebra R.

Proof.

(a) Let $\{r_P\} \in R(D_1)$, then $v_P(r_P) \geq -v_P(D_1) \geq -v_P(D_2)$ for all $P \in X$ since $D_1 \leq D_2$. Thus, $\{r_P\} \in R(D_2)$ and $R(D_1) \subset R(D_2)$.

(b) follows from definition. □

On the other hand, for every $f \in k(X)$ we can associate the repartition $\{r_P\}$ such that $r_P = f$ for every $P \in X$. Hence, there is an injection $k(X) \hookrightarrow R$ and it permits us to identify $k(X)$ as a subring of R. With these notation, we can show:

Proposition 1. *If D is a divisor on X, then the vector space $I(D) = H^1(X, \mathcal{L}(D))$ is canonically isomorphic to $R/(R(D) + k(X))$.*

Proof. *The sheaf $\mathcal{L}(D)$ is a subsheaf of the sheaf of rational functions $\mathcal{K}_X$. Thus there is an exact sequence*

$$0 \to \mathcal{L}(D) \to \mathcal{K}_X \to \mathcal{K}_X/\mathcal{L}(D) \to 0.$$

Since the curve X is irreducible and the sheaf $\mathcal{K}_X$ is constant,

$$H^1(X, \mathcal{K}_X) = 0$$

On the other hand, since X is connected, $H^0(X, \mathcal{K}_X) = k(X)$. Then the long exact sequence of cohomology groups associated to the exact sequence of sheaves implies that

$$k(X) \to H^0(X, \mathcal{K}_X/\mathcal{L}(D)) \to H^1(X, \mathcal{L}(D)) \to 0.$$

The sheaf $\mathcal{A} = \mathcal{K}_X/\mathcal{L}(\mathcal{D})$ is a skyscraper sheaf, so $H^0(X, \mathcal{K}_X/\mathcal{L}(D)) \cong \bigoplus_{P\in X} \mathcal{A}_P$. Each $\mathcal{A}_P = k(X)/L(D)_P$ can be identified as the component of $R/R(D)$ at the point P, so $H^0(X, \mathcal{K}_X/\mathcal{L}(D)) \cong R/R(D)$. Thus, $H^1(X, \mathcal{L}(D)) \cong R/(R(D) + k(X))$. □

As a result, $I(D)$ is identified as $R/(R(D) + k(X))$.

4.2 Dual of the space of classes of repartitions

Let $J(D)$ be the dual space of the vector space $I(D)$; an element in $J(D)$ can be identified as an element in R vanishing in $R(D)$ and $k(X)$. If $D_1 \le D_2$, then $R(D_1) \subset R(D_2)$, which shows $J(D_2) \subset J(D_1)$. Let J denote the union of $J(D)$ for D running through the set of divisors of X.

Let $f \in k(X)$ and $\alpha \in J$. We can identify J as a $k(X)-$ vector space by defining the map $k(X) \times J \to J$ in the following way:

$$k(X) \times J \to J : \quad (f, \alpha) \quad \mapsto \quad f\alpha : r \mapsto \langle \alpha, fr \rangle.$$

Notice that $f\alpha$ is a linear form on R vanishing on $k(X)$, so the map is well-defined. Specifically, if $\alpha \in J(D)$ and $f \in L(\Delta)$, then because

$$r \in R(D - \Delta) \Longrightarrow fr \in R(D) \Longrightarrow f\alpha(r) = 0,$$

we can see $f\alpha$ vanishes on $R(D - \Delta)$, i.e $f\alpha \in J(D - \Delta)$. Thus, J can be endowed with the structure as a vector space over $k(X)$.

Proposition 2. *The dimension of the vector space J over the field $k(X)$ is ≤ 1.*

Proof. *This result can be proved by contradiction. Let α_1 and α_2 be two elements of J which are linearly independent over $k(X)$. Suppose $\alpha_1 \in J(D_1)$ and $\alpha_2 \in J(D_2)$, let D be a new divisor $D = \sum_{P\in X} \min\{v_P(D_1), v_P(D_2)\}P$. Then $D \le D_i$ and $\alpha_i \in J(D_i) \subset J(D)$ for $i = 1, 2$. Let $d = \deg D$.*

For every integer $n \ge 0$, let Δ_n be a divisor of degree n. If $f, g \in L(\Delta_N)$, then $f\alpha_1 + g\alpha_2 \in J(D - \Delta_n)$. Since we assume α_1 and α_2 are linearly independent, $f\alpha_1 + g\alpha_2 = 0$ implies $f = g = 0$. It follows that the map

$$L(\Delta_n) \times L(\Delta_n) \to J(D - \Delta_n) : \quad (f, g) \mapsto f\alpha_1 + g\alpha_2$$

is an injection. Hence we have

$$\dim J(D - \Delta_n) \ge 2 \dim L(\Delta_n) \quad \textit{for all } n. \tag{*}$$

It will be shown that the inequality (*) leads to a contradiction when $n \to +\infty$.

According to theorem 1,

$$\begin{aligned} \dim J(D - \Delta_n) &= \dim I(D - \Delta_n) = i(D - \Delta_n) \\ &= -\deg(D - \Delta_n) + g - 1 + l(D - \Delta_n) \\ &= n + (g - 1 - d) + l(D - \Delta_n). \end{aligned}$$

On the other hand, when $n > d$, $\deg(D - \Delta_n)$, which implies $l(D - \Delta_n)$ (if not, there exists $f \in l(D - \Delta_n)$, $f \neq 0$ such that $\deg(D - \Delta_n) = \deg f = 0$, which is a contradiction). Thus when $n >> 1$ the left hand side of (*) is equal to a linear function $n + (g - 1 + d)$.

As for the right hand side of (*)

$$2 \dim L(\Delta_n) = 2l(\Delta_n) = 2(i(\Delta_n) + n + 1 - g) \geq 2(n + 1 - g).$$

Hence, when n is large, $2 \dim L(\Delta_n) > \dim J(D - \Delta_n)$, which is a contradiction to (*), so the dimension of J over the field $k(X)$ is ≤ 1. □

Remark 1. *It would be easy to shown the dimension of J is exactly 1 because there is a non-zero element in J. More precisely, it would be shown that J is isomorphic to the space of* ***differentials*** *in X.*

5 Differentials, residues

5.1 Differentials

Here is a general definition of a **differential** on an algebraic variety X (not just on a curve).

Definition 3 (differential)**.** *Let F be a commutative algebra over a field k. The* ***module of k-differentials of F****, $D_k(F)$, is a F-module generated by the set $\{d(x) \big| x \in F\}$ with the relations*

$$d(xy) = xd(y) + yd(x) \quad \textit{(Leibniz)}$$

$$d(k_1 x + k_2 y) = k_1 d(x) + k_2 d(y) \quad \textit{(k-linearity)}$$

for all $x, y \in F$, and $k_1, k_2 \in k$. We often write dx instead of $d(x)$. The F-module $D_k(F)$ can be endowed with a k-linear derivation $d : F \to D_k(F)$ defined by $d : x \mapsto dx$, called the ***universal k-linear derivation****.*

Note that the map d has the **universal property**: Given any F-module M and k-linear derivation $e : F \to M$, there is a unique F-linear homomorphism $e' : D_k(F) \to M$ such that $e = e' \circ d$, as in the following figure.

$$\begin{tikzcd}
 & D_k(F) \arrow[dd, "\exists! e'"] \\
F \arrow[ur, "d"] \arrow[dr, "e"'] & \\
 & M
\end{tikzcd}$$

Applying the definition to $\mathcal{O}_P$ and $k(X)$ of an irreducible algebraic variety X (of any dimension r) we have:

Lemma 1.

(a) $\underline{\Omega}_P = D_k(\mathcal{O}_P)$ form a coherent sheaf on X.

(b) $D_k(k(X)) = D_k(\mathcal{O}_P) \otimes_{\mathcal{O}_P} k(X)$.

(c) If P is a nonsingular point of X and if $\{t_1, \ldots, t_r\}$ form a regular system of parameters (i.e. the minimal set of generators of $\mathfrak{m}_P$; the number of this set is exactly the dimension of X) at P, then the set $\{dt_1, \ldots, dt_r\}$ form a basis of $D_k(\mathcal{O}_P)$ over $\mathcal{O}_P$. Thus, on the set of nonsingular points of X, $D_k(\mathcal{O}_P)$ is locally free and is the dual of tangent space.

Now we assume X is an irreducible, nonsingular, projective curve again. In this case, $D_k(k(X))$ is a vector space of dimension 1 over $k(X)$ and that the sheaf $\underline{\Omega}$ defined by $\underline{\Omega}_P$ is a subsheaf of the constant sheaf $D_k(k(X))$. Because X is a nonsingular curve and every $\mathcal{O}_P$ is a discrete valuation ring, any irreducible element of $\mathcal{O}_P$ is a generator for the unique maximal ideal $\mathfrak{m}_P$ and vice versa. Such an element is called a **local uniformizer** at P. If t is such a local uniformizer at P, according to the previous argument, dt is a basis of the $\mathcal{O}_P$-module $\underline{\Omega}_P$ and it is also a basis of the $k(X)$-vector space $D_k(k(X))$. As a result, if $\omega \in D_k(k(X))$ then $\omega = fdt$ for some $f \in k(X)$.

Definition 4 (divisor associated to ω)**.** *Suppose $\omega \neq 0$ and $\omega = fdt$, then define*

$$v_P(\omega) = v_P(f)$$

and put the divisor (ω) to be

$$(\omega) = \sum_{P\in X} v_P(\omega)P = \sum_{P\in X} v_P(f)P = (f).$$

The definition of $v_P(\omega)$ is independent of the choice of t because the valuation of invertible element in $\mathcal{O}_P$ is 0.

5.2 Residues

X is an irreducible, nonsingular curve, so $k(X)$ can be viewed as the fraction field $\mathrm{Frac}(\mathcal{O}_{X,P})$ of $\mathcal{O}_{X,P}$. If t is a uniformizer of $\mathcal{O}_{X,P}$, then we can see that the field $\widehat{k(X)}_P$, the completion of $k(X)$ under the valuation v_P, is isomorphic to $k((T))$ by mapping t to T. Identifying f in $k((T))$, we write

$$f = \sum_{n>>-\infty} a_n T^n, \quad a_n \in k.$$

The symbol $n >> -\infty$ means that n only takes a finite number of values < 0.

Definition 5 (residue)**.** *Let $\omega = fdt \in D_k(k(X))$.*

*The **residue** of $\omega \in D_k(k(X))$ at P, denoted as $Res_P(\omega)$, is the coefficient a_{-1} of T^{-1} in f.*

It can be shown that $\mathrm{Res}_P(\omega)$ is also a local invariant of ω. Hence the residue at a point is well-defined.

Proposition 3 (Invariance of the residue)**.** *The definition of $Res_P(\omega)$ is independent of the choice of the local uniformizer t.*

Remark 2 ($k = \mathbb{C}$)**.** *If $k = \mathbb{C}$ and give X a structure of a compact analytic variety of dimension 1, then $Res_P(\omega) = \frac{1}{2\pi i}\oint_P \omega$, which is independent of the choice of local uniformizer of course and prove a special case of prop 3.*

It would take some pages to prove this proposition.

Here we define some new notations for simplicity: Let K denote the field $\widehat{k(X)}_P$; the choice of a local uniformizer t identifies K with $k((t))$. Let v be the valuation of K. Let $\mathcal{O}$ be its valuation ring under v and $\mathfrak{m}$. One can see that $\mathcal{O}$ is nothing but a new notation for $\widehat{\mathcal{O}}_P$ and $\mathfrak{m}$ is a new notation for $\widehat{\mathfrak{m}}_P$.

Then we can define the module (K-vector space) $D_k(K)$ of differentials of K as the definition 3. Here, it is more convenient to use its associated separated module $D'_k(K)$ for the $\mathfrak{m}$-topology:

$$D'_k(K) = D_k(K)/Q \quad \text{with} \quad Q = \bigcap_{n\geq 0} \mathfrak{m}^n d(\mathcal{O}).$$

The elements in $D'_k(K)$ have the same operation as the differentials in calculus:

Lemma 2. *Let t be a local uniformizer and for every element $f = \sum_{n>>-\infty} a_n t^n$ of K put $f'_t = \sum_{n>>-\infty} n a_n t^{n-1}$. Then $df = f'_t dt$ in $D'_k(K)$ and dt forms a basis of $D'_k(K)$ over K.*

Proof. *To show that* $df = f'_t dt$ *in* $D'_k(K)$ *it suffices to show* $df = f'_t dt \in \mathfrak{m}^N d(\mathcal{O}) \in D_k(K)$ $\forall N \geq 0$. *To do this, we write:*

$$f = f_0 + t^{N+1} f_1 \quad \textit{with} \quad f_0 = \sum_{n \leq N} a_n t^n, f_1 \in \mathcal{O},$$

$$f'_t = (f_0)'_t + t^N g \quad \textit{with} \quad g \in \mathcal{O},$$

and then

$$df - f'_t dt = (N+1) t^N f_1 dt + t^{N+1} df_1 - t^N g dt \in \mathfrak{m}^N d(\mathcal{O}).$$

Thus we know $df = f'_t dt$ *in* $D'_k(K)$ *and that* dt *generates the* K*-vector space* $D'_k(K)$. *Moreover,* $D'_k(K) \neq 0$ *so* dt *is a basis for* $D'_k(K)$ *as a vector space over* k.

Remark 3. *Someone may think it is not necessary to kill* $\bigcap_{n\geq 0} \mathfrak{m}^n d(\mathcal{O})$ *in the first sight of lemma 2, because it seems that* $D_k(K)$ *may also has this property like what* $D'_k(K)$ *owns in lemma 2. But this opinion is incorrect. Although the Leibniz rule in* $D_k(K)$ *promise we can do derivative for a monomial, the linearity in* $D_k(K)$ *could just promise a polynomial, but not a formal series, can split. If we do not eliminate* $\bigcap_{n\geq 0} \mathfrak{m}^n d(\mathcal{O})$, *the best situation is to make* $f = \sum_{n \leq N} a_n t^n + t^{N+1} g$ *for some* $g \in \mathcal{O}$ *so that*

$$df = \sum_{n \leq N} n a_n t^{n-1} dt + (N+1) t^N g dt + t^{N+1} dg.$$

We can see this express is not so perfect because there are some terms not in shape of power series.

Furthermore, $D'_k(K)$ *just help to express the term with higher degree, hence it does not change the coefficient* a_{-1}. *i.e. It does not impact the residue number of* df.

It is reasonable to believe that the module $D'_k(K)$ which owns a better operation property could help us compute residues. From now on, a **differential** in K means an element of $D'_k(K)$. If ω is such a differential in $D'_k(K)$ and t is a local uniformizer of K in $\mathcal{O} = \widehat{\mathcal{O}}_P$, then $\omega = f dt$ for some $f \in K$. If $f = \sum_{n >> \infty} a_n t^n$ under the identification of $K = k((t))$, then the number a_{-1} is called the residue number of ω with respect to t, notated by $Res_t(\omega)$. Just as what we discuss in remark 3, since $\bigcap_{n\geq 0} \mathfrak{m}^n d(\mathcal{O})$ just impacts the higher terms of f, the residue defined here is the same as what we do in definition 5. In particular, the proposition 3 can be reformulated as follows:

Proposition. *3' If* t *and* u *are two local uniformizers of* K *at* $p \in X$, *then* $Res_t(\omega) = Res_u(\omega)$ *for all differential* $\omega \in D'_k(K)$.

We can find out some operations in $D'_k(K)$ related to $Res_t(\omega)$ at first:

(i) $Res_t(\omega)$ is k-linear in ω.

(ii) $Res_t(\omega) = 0$ if $v(\omega) \geq 0$ (i.e., if $\omega \in \mathcal{O}dt$).

(iii) $Res_t(dg) = 0$ for every $g \in K$.

(iv) $Res_t(dg/g) = v(g)$ for every $g \in K^*$.

Proof. (i) Let $\lambda_1, \lambda_2 \in k$ and let $\omega_1 = f_1 dt$, $\omega_2 = f_2 dt$ be differentials with $f_1 = \sum_{n>>-\infty} a_n t^n$ and $f_2 = \sum_{n>>-\infty} b_n t^n$. Then $\lambda_1\omega_1 + \lambda_2\omega_2 = \sum(\lambda_1 a_n + \lambda_2 b_n) t^n dt$. Specifically, $Res_t(\lambda_1\omega_1 + \lambda_2\omega_2) = \lambda_1 a_{-1} + \lambda_2 b_{-1} = \lambda_1 Res_t(\omega_1) + \lambda_2 Res_t(\omega_2)$.

(ii) Let $\omega = f dt$. $v(\omega) \geq 0$ implies $f = \sum_{n\geq 0} a_n t^n$, so $Res_t(w) = 0$.

(iii) Suppose $g = \sum_{n>>-\infty} a_n t^n \in K$. Then $dg = \sum_{\substack{n>>-\infty \\ n\neq -1}} a_n t^n dt$ and $Res_t(dg) = 0$.

(iv) Put $g = t^n f \in K$ with $v(g) = n$ and $v(f) = 0$. Then

$$dg/g = \frac{nt^{n-1} f dt + t^n df}{t^n f} = nt^{-1}dt + df/f,$$

and

$$v(df/f) = v(f') - v(f) \geq 0.$$

According to formula (ii), $Res_t(dg/g) = n + Res_t(df/f) = n = v(g)$.

□

Proof. Now we pass to the proof of prop 3'. We express ω in $k((u))$:

$$\omega = \sum_{n\geq 1} a_n du/u^n + \omega_0 \quad \text{with } v(\omega_0) \geq 0.$$

Then $Res_t(\omega) = \sum_{n\geq 1} a_n Res_t(du/u^n)$. By (iv) we know $Res_t(du/u) = 1$, so if one can show

$$Res_t(du/n^n) = 0 \quad \text{for } n \geq 2, \tag{v}$$

then the prop 3 is done.

When char $k = 0$, $du/u = dg$ with

$$g = -\frac{1}{n-1} u^{n-1},$$

and the formula (v) is a simple consequence of formula (iii). This argument cannot work when char $k \neq 0$ since $n - 1 \equiv 0(\mod p)$ for some n. However, the case of char $k = p \neq 0$ can be reduced to that of char $k = 0$ in the following way:

Since $v(u) = v(t) = 1$,

$$\begin{aligned} u &= a(t + a_2 t^2 + a_3 t^3 + \dots) \\ &= at(1 + a_2 t + a_3 t^2 + \dots) \quad \text{for some } a \in k^*. \end{aligned}$$

Thus, we can do the proof for any fixed n with $n \geq 2$ in two steps:

Step 1: Multiplying u by a scalar factor so that

$$u = t + a_2 t^2 + a_3 t^3 + \dots = t(1 + a_2 t + a_3 t^2 + \dots)$$

and show such u satisfy $Res_t(du/u^n) = 0$ for $n \geq 2$.

Step 2: show that multiplying u by a scalar factor does not impact $Res_t(du/u^n)$ for $n \geq 2$.

In step 1, we deduce that

$$u^n = t^n(1 + na_2 t + \dots),$$

$$\frac{1}{u^n} = \frac{1}{t^n(1 + na_2 t + \dots)} = \frac{1}{t^n}(1 - na_2 t + \dots + b_i t^i),$$

where the b_i are polynomials in $a_2, \dots, a_{i+1}$ with coefficients in $\mathbb{Z}$.

$$du = dt + 2a_2 t dt + \dots + i a_i t^{i-1} dt + \dots,$$

so

$$\begin{aligned} \frac{du}{u^n} &= \frac{dt}{t^n}(1 + 2a_2 t + \dots + i a_i t^{i-1} + \dots)(1 - na_2 t + \dots + b_j t^j) \\ &= \frac{dt}{t^n} \sum_{l=0}^{\infty} c_l t^l, \end{aligned}$$

where the c_l are the polynomials in $a_2, \dots, a_{l+1}$ with coefficients in $\mathbb{Z}$. Moreover, the above arguments illustrates that the polynomials b_i, c_l are independent of the characteristic of k. In particular, $c_{n-1} = Res_t(du/u^n)$ is independent of char k.

Here is a theorem from Bourbaki, Algebra, chap. IV, sec 2, no. 5:

Theorem (Principle of extension of algebraic identities)**.** *Suppose that A is an infinite integral domain and I is an index set. Let $g_1, \dots, g_m, f$ be elements of $A[(X_i)_{i\in I}]$ and assume the following hypotheses:*

(a) $g_1 \neq 0, \dots, g_m \neq 0$;

(b) for all $\boldsymbol{x} \in A^I$ such that $g_1(\boldsymbol{x}) \neq 0, \dots, g_m(\boldsymbol{x}) \neq 0$, we have $f(\boldsymbol{x})=0$.

Then $f = 0$.

In char $k = 0$, as is well-known, $c_{n-1}(a_2, \dots, a_n)$ vanishes for any $a_2, \dots, a_n$. By the virtue of the **principle of extension of algebraic identities**, the polynomial c_{n-1} is identically zero. Thus, no matter what char k is, there is always $Res_t(du/u^n) = c_{n-1} = 0$ for any $n \geq 2$, which finishes step 1.

As for step 2, it follows from the definition: suppose that $u = a\bar{u}$ for $a \in k^*$ such that $\bar{u} = t + a_2 t^2 + \dots$, then

$$\frac{du}{u^n} = \frac{ad\bar{u}}{a^n \bar{u}^n}$$

$$Res_t(\frac{du}{u^n}) = a^{-n+1} Res_t(\frac{d\bar{u}}{\bar{u}^n}) \overset{step1}{=\!=\!=} 0.$$

Up to now, we have finished the proof of prop 3. $\square$

We shall need a more general formula than prop 3. Consider a finite field extension L/K. K is a field of formal power series $k((t))$; L is also a field power series $k((u))$; let m be the degree of L/K, i.e. $m = [L : K]$; t is a non-zero element such that $v_L(t) = m$, i.e. after multiplying t by a constant if necessary, we can write

$$t = u^m + b_1 u^{m+1} + b_2 u^{m+2} + \dots$$
$$= u^m(1 + b_1 u + b_2 u^2 + \dots).$$

Then $\forall f \in k((u))$ can be expressed as

$$f = a_0(t) + a_1(t)u + \dots + a_{m-1}(t)u^{m-1} \quad \text{with } a_i \in k((t)).$$

Furthermore, the elements $1, u, \dots, u^{m-1}$ are linearly independent over $k((t))$. Since $v_L(u) = 1$, $v_K(t) = 1$, one can see that the ramification index is also m, i.e. the degree of L/K.

Let $Tr_{L/K}$, or simply Tr, denote the trace from $k((u))$ to $k((t))$. The following proposition states the relations between the residues taken in K or in L.

Proposition 4. *Let K, L be formal power series as before. Let $\omega = fdt \in D'_k(L)$ then*

$$Res_u(fdt) = Res_t(Tr(f)dt).$$

Proof. $Tr(f)$ can be computed from the matrix representing f with respect to the basis $1, u, \dots, u^{m-1}$. Hence we assume that

$$t = u^m + b_1 u^{m+1} + b_2 u^{m+2} + \dots = u^m(1 + b_1 u + b_2 u^2 + \dots),$$

with $b_v \in k$. One can solve recursively

$$u^m = a_0(t) + a_1(t)u + \dots + a_{m-1}(t)u^{m-1},$$

where $a_i(t)$ are elements of $k((t))$. The coefficients of $a_i(t)$ are the polynomials in $b_1, b_2, \dots$, with coefficients in $\mathbb{Z}$. i.e. each $a_i(t)$ can be written as

$$a_i(t) = \sum P_{il}(b)t^l,$$

where each P_{il} is a polynomial with integer coefficients, involving only finite number of b's.

The matrix representing an arbitrary element $f = g_0(t) + g_1(t)u + \cdots + g_{m-1}(t)u^{m-1} \in k((u))$ is

$$\begin{pmatrix} g_0 & g_{m-1}a_0 & \dots & \dots \\ g_1 & g_0 + g_{m-1}a_1 & \dots & \dots \\ \dots & \dots & \dots & \dots \\ g_{m-1} & g_{m-2} + g_{m-1}a_{m-1} & \dots & \dots \end{pmatrix}.$$

It is of the type

$$\begin{pmatrix} G_{0,0}(t) & G_{0,1}(t) & \dots & G_{0,m-1}(t) \\ \dots & \dots & \dots & \dots \\ G_{m-1,0}(t) & G_{m-1,1}(t) & \dots & G_{m-1,m-1}(t) \end{pmatrix}$$

where $G_{\alpha\beta}(t) \in k((t))$. Suppose $G_{\alpha\beta}(t) = \sum Q_{\alpha\beta,\gamma}t^{\gamma}$. Then the coefficient $Q_{\alpha\beta,\gamma}$ are the polynomials in the b's and g's with coefficients in $\mathbb{Z}$. This means that the formula is a formal identity independent of char k and it suffices to check it when char $k = 0$.

Now assume char $k = 0$. In this case we can write $t = v^m$ where $v = u + c_2u^2 + \dots$ is another parameter of the field $k((u))$. This can be done by taking the binomial expansion for $(1 + b_1u + \dots)^{1/m}$. Prop 3 tells that

$$Res_u(fdt) = Res_v(fdt),$$

so it suffices to show that

$$Res_v(fdt) = Res_t(Tr(f)dt). \tag{**}$$

Since both Res and Tr are k - linear, it suffices to prove this for $f = v^j$, $-\infty < j < \infty$. If we write

$$j = ms + r \quad \text{ with } 0 \le r \le m-1,$$

then $v^j = t^s v^r$ and $Tr(v^j) = t^s Tr(v^r)$. We have trivially

$$Tr(v^r) = \begin{cases} m \text{ if } r = 0 \\ 0 \text{ if } r \neq 0, \end{cases}$$

whence

$$Tr(v^j) = \begin{cases} mt^s \text{ if } j = ms \\ 0 \ \text{ otherwise.} \end{cases}$$

Consequently, we get

$$Res_t(Tr(v^j)dt) = \begin{cases} m \text{ if } j = -m \\ 0 \ \text{ otherwise.} \end{cases}$$

On the other hand,

$$Res_v(v^j dt) = Res_v(v^j \frac{dt}{dv}dv) = Res_v(v^j \cdot mv^{m-1}dv) = \begin{cases} m \text{ if } j = -m \\ 0 \ \text{ otherwise.} \end{cases}$$

This shows (**) holds and proves the proposition. $\square$

In the preceding discussion, we started with a power series field $k((u))$ and a subfield $k((t))$. This situation is a special case of power series field extensions:

Proposition 5. *Let $K = k((t))$ be a power series field over an algebraically closed field k. Then the natural k-valued place of $k((t))$ has a unique extension to any finite algebraic extension of $k((t))$. If L is such an extension and u is an element of order 1 in the extended valuation, then L may be identified with the power series field $k((u))$ and $[L : K] = m$.*

5.3 Residue Formula

Let $\omega \in D_k(k(X))$ and $Res_P(\omega)$ be the residue of ω at P. The formula (ii) in last subsection tells that $Res_t(\omega) = 0$ if $v(\omega) \geq 0$. It is well-known that the set $\{P \in X \big| v_P(\omega) \geq 0\}$ is dense in X. Hence, $Res_P(\omega) = 0$ for almost P and the formula $\sum_{P \in X} Res_P(\omega)$ makes sense. Here is a fundamental result about this formula:

Proposition 6 (Residue formula)**.** *For every differential $\omega \in D_k(k(X))$,*

$$\sum_{P \in X} Res_P(\omega) = 0.$$

Remark 4. *Take the assumption as in remark 2. In this special case:*

$$\sum_{P \in X} Res_P(\omega) = \sum_{P \in X} \oint_P \omega = \oint_{\partial X = \emptyset} \omega = 0.$$

Thus it proves a special case of prop 6.

Before proving this proposition, it is wise to check it in a particular case:

Lemma 3 ($X = \mathbb{P}_1(k)$)**.** *The residue formula is true when the curve X is the projective line $\mathbb{P}_1(k)$.*

Proof. In this case, $k(X) = k(t)$ where $t = T_0/T_1$. Every $\omega \in D_k(k(X))$ can be written as $\omega = f(t)dt$ with $f(t) \in k(t)$. f consists of two simple elements: $f = t^n$ or $f = 1/(t-a)^n$ for $n \geq 1$.

In the first case: if $P = [a_0 : 1]$ then $Res_P(\omega) = 0$; if $P = [1 : 0] = \infty$, take $u = 1/t$ and then $\omega = u^{-n}(-1/u^2 du) = -du/u^{n+2}$. Thus $Res_\infty(\omega) = 0$ and the sum of residues is 0.

In the second case:

- $n = 1$: if $P = [a_0 : 1]$ then $Res_P(\omega) \neq 0 \Leftrightarrow a_0 = a$, and

$$Res_{[a:1]}(\omega) = 1;$$

  if $P = [1 : 0] = \infty$, take $u = 1/t$ and then $\omega = -du/(1-u)u$. Thus $Res_\infty(\omega) = -1$ and the sum of residues is 0.

- $n \geq 2$: if $P \neq \infty$, $Res_P(\omega) = Res_P(1/(t-a)^n) = 0$; if $P = \infty$, take $u = 1/t$ and then $\omega = -u^{n-2}du/(1-u)^n$, so ∞ is not a pole of ω. Thus the sum of residues is 0.

All cases in lemma 3 are checked now. □

Now let X be any curve, $X' = \mathbb{P}_1(k)$ and let $\varphi : X \to X' = \mathbb{P}_1(k)$ be a non-constant regular map. φ is dominating because $\mathbb{P}_1(k)$ is a curve. Then φ induces a field extension $\varphi^* : k(X') \to k(X)$. Since $tr.d.k(X)/k = tr.d.k(X')/k = 1$ and $k(X)/k$ is finitely generated, φ^* is a finite extension. As a conclusion, φ is in fact a surjection because a finite morphism is dominating if and only if it is surjective. Hence the points of X is the disjoint union of fibres $\varphi^{-1}(P)$ on $\forall P \in X'$.

let $E = k(X')$ and $F = k(X)$; let K be the completion of $\widehat{E}_P$ as before; let L be the completion of $\widehat{F}_Q$ with $\varphi(Q) = P$. There are two transcendental element t, u equipped with discrete valuations v_K, v_L such that $K = k((t))$, $L = k((u))$, $v_K(t) = 1$ and $v_L(u) = 1$. Let m_Q be the ramification index such that $v_L(t) = m_Q$. For any $\omega \in D_k(k(X))$, it could be viewed as an element $\omega = f du \in D_k(L)$. We can define its trace $Tr_Q(\omega) \in D_k(K)$ as $Tr_{L/K}(f)du$.

Next, suppose that F/E is a separable field extension, there is a trace $Tr_{F/E}(f) \in E$, or simply $Tr(f)$, for any $f \in F$. There is a relation called trace formula between $Tr(f)$ and $Tr_Q(f)$ with $Q \in \varphi^{-1}(P)$:

Proposition 7 (trace formula)**.** *The notation being as above. Assume $k(X') \to k(X)$ is a separable extension. Let Tr be the trace from $k(X)$ to $k(X')$, then there is a trace formula: for any fixed $P \in X'$ we have*

$$Tr(f) = \sum_{Q \in \varphi^{-1}(P)} Tr_Q(f).$$

for any $f \in k(X)$. Notice that the sum of Q is finite since φ is a finite morphism.

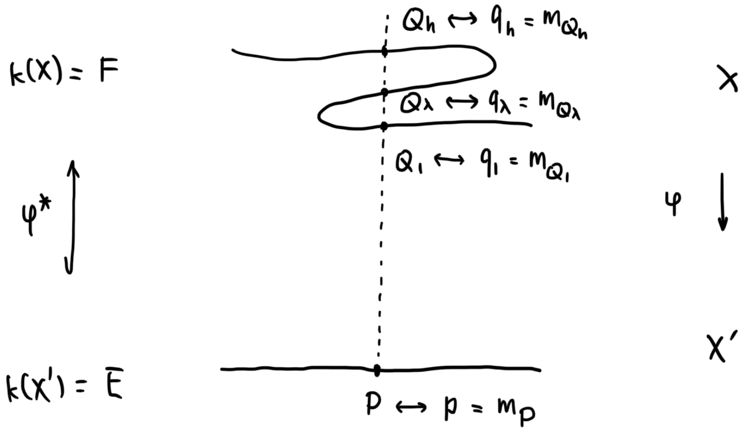

Figure 1: mapping relations

The proof of this proposition depends on a theorem on [Chevalley, page 60, Thm 4]:

Theorem. *Let R and S be fields of algebraic functions of one variable such that S is an overfield of finite degree of R. Let $\mathfrak{p}$ be a place of R, $\bar{R}$ the $\mathfrak{p}$-adic completion of R, and $\mathfrak{q}_1, \dots, \mathfrak{q}_h$ the distinct places of S which lie above $\mathfrak{p}$. Then the product of the $\mathfrak{q}_\lambda$-adic completions of $S (1 \leq \lambda \leq h)$ is an algebra over R which is isomorphic with the algebra $S_{\bar{R}}$ deduced from S (considered as an algebra over R) by extending the basic field from R to $\bar{R}$*

Now a proof of prop 7 could be given:

Proof. Let $\varphi^{-1} = \{Q_1, \dots, Q_h\}$. As shown in the Figure 1: in this case, $R = E = k(X')$; $\mathfrak{p} = \mathfrak{m}_P \subset E$; $\bar{R} = \widehat{E}_P$; $S = F = k(X)$; $\mathfrak{q}_\lambda = \mathfrak{m}_{Q_\lambda} \subset F$; $S_{\bar{R}} = F \otimes_E \widehat{E}_P$. What we get is an isomorphism:

$$F \otimes_E \widehat{E}_P = \prod_{Q_\lambda \in \varphi^{-1}(P)} \widehat{F}_{Q_\lambda}.$$

The trace formula follows immediately from this isomorphism:

$$Tr(f) = \sum_{Q \in \varphi^{-1}(P)} Tr_Q(f), \quad f \in F,$$

where Tr_Q denotes the trace in the extension $\widehat{F}_Q / \widehat{E}_P$. □

The following proposition states the relationship between the residues computed by $Q \in \varphi^{-1}(P)$ and those by P:

Proposition 8. *Let x be a transcendental element such that $E = k(X') = k(x)$; $f \in F = k(X)$. Let P be a point in X'; $Q_\lambda (1 \leq \lambda \leq h) \in X$ lie over P. Then*

$$Res_P(Tr(f)dx) = \sum_{Q \in \varphi^{-1}(P)} Res_Q(fdx),$$

where the sum of Q is finite since φ is a finite morphism.

Proof. Let t be a local uniformizer at P in $k(x)$; u_λ be the local uniformizers at Q_λ respectively.

$$Res_P(Tr(f)dx) = Res_P(\sum_\lambda Tr_{Q_\lambda}(f)dx) \qquad \text{(Prop 7)}$$

$$= \sum_{\lambda} Res_t(Tr_{Q_\lambda}(f)\frac{dx}{dt}dt)$$

$$= \sum_{\lambda} Res_t(Tr_{Q_\lambda}(f\frac{dx}{dt})dt) \qquad \text{(dx/dt at k(x))}$$

$$= \sum_{\lambda} Res_{u_\lambda}(f\frac{dx}{dt}dt) \qquad \text{(Prop 4)}$$

$$= \sum_{\lambda} Res_{Q_\lambda}(fdx).$$

□

Now we have enough tools to give a proof about prop 6:

Proof. A differential form ω can be written as fdx such that $F(f)$ is separable algebraic over $E = k(x)$. Hence by prop 8 we have

$$\sum_{Q\in X} Res_Q(\omega) = \sum_{P\in X'} \sum_{Q\in\varphi^{-1}(P)} Res_Q(fdx)$$

$$= \sum_{P\in X'} Res_P(fdx) \qquad \text{(prop 8)}$$

$$= 0 \qquad \text{(lemma 3)}$$

□

6 Duality theorem

Let ω be a non-zero differential on the curve X. Recall that in def 4 we define

$$(\omega) = \sum_{P\in X} v_P(\omega)P.$$

If D is a divisor, we write $\Omega(D)$ for the $k(X)$-vector space:

$$\{0\} \cup \{\omega \in D_k(k(X))^* \big| (\omega) \geq D\};$$

it is a subspace of the space $D_k(k(X))$ of all differentials on X.

For any $P \in X$, we define $\underline{\Omega}(D)_P$ for the set of differentials

$$\{0\} \cup \{\omega \in D_k(k(X)) * \big| v_P(\omega) \geq v_P(D)\}.$$

The $\underline{\Omega}(D)_P$ form a subsheaf $\underline{\Omega}(D)$ of the constant sheaf $D_k(k(X))$.

Given these definitions, we could define a scalar product $\langle\omega, r\rangle$ between differentials $\omega \in D_k(k(X))$ and repartitions $r \in R$ as follows:

Definition 6 (scalar product)**.**

$$\langle -, -\rangle : D_k(k(X)) \times R \to k :$$

$$(\omega, r) \mapsto \langle\omega, r\rangle := \sum_{P\in X} Res_P(r_P\omega).$$

This definition is legitimate since $r_P\omega \in \underline{\Omega}_P = D_k(\mathcal{O}_P)$ for almost all P and the formula (ii) tells that $Res_P(r_P\omega) = 0$ for such $r_P\omega$.

The scalar product has the following properties:

(a) $\langle \omega, r\rangle = 0$ if $r \in k(X)$;

(b) $\langle \omega, r\rangle = 0$ if $r \in R(D)$ and $\omega \in \Omega(D)$ for then $r_P\omega \in \underline{\Omega}_P$ for every $P \in X$

(c) If $f \in k(X)$, then $\langle f\omega, r\rangle = \langle \omega, fr\rangle$.

Proof. (a) Because of $r\omega \in D_k(k(X))$, by the residue formula (prop 6)

$$\langle \omega, r\rangle = \sum_{P\in X} Res_P(r\omega) = 0.$$

(b) We have

$$v_P(r_P\omega) = v_P(r_P) + v_P(\omega) \geq -v_P(D) + v_P(D) = 0,$$

so by formula (ii),

$$\langle \omega, r\rangle = \sum_{P\in X} Res_P(r_P\omega) = 0.$$

(c)

$$\langle f\omega, r\rangle = \sum_{P\in X} Res_P(r_P f\omega) = \sum_{P\in X} Res_P(f r_P\omega) = \langle \omega, fr\rangle.$$

□

For every differential ω, let $\theta(\omega)$ be the linear form:

$$\theta(\omega) : R \to k : r \mapsto \langle \omega, r\rangle.$$

Properties (a) and (b) mean that, if $\omega \in \Omega(D)$ then $R(D) + k(X) \subset ker(\theta(\omega))$, hence $\theta(\omega) \in J(D)$ which is by definition the dual of $R/(R(D) + k(X))$.

Theorem 3 (Duality theorem)**.** *For every divisor D, the map θ is an isomorphism from $\Omega(D)$ to $J(D)$.*

Remark 5. *The duality theorem also shows that $\Omega(D)$ and $I(D) = R/(R(D) + k(X))$ in duality.*

First we prove a lemma:

Lemma 4. *If ω is a differential such that $\theta(\omega) \in J(D)$, then $\omega \in \Omega(D)$.*

Proof. Indeed, otherwise there would be a point $P \in X$ such that $v_P 9\omega < v_P(D)$. Put $n = v_P(\omega) + 1$, and let r be the repartition whose components are

$$\begin{cases} r_Q = 0 \text{ if } Q \neq P, \\ r_P = 1/t^n, \ t \text{ being a local uniformizer at } P. \end{cases}$$

We have

$$v_P(r_P\omega) = v_P(r_P) + v_P(\omega) = -n + v_P(\omega) = -1,$$

whence $Res_P(r_P\omega) \neq 0$ and $\langle \omega, r\rangle \neq 0$; but since $n \leq v_P(D)$, $r \in R(D)$ and we arrive at a contradiction since $\theta(\omega)$ is assumed to vanish on $R(D)$. □

Now we can prove theorem 3:

Proof. First of all, θ is injective. Indeed, if $\theta(\omega)$, the preceding lemma shows that $\omega \in \Omega(\Delta)$ for every divisor Δ, whence evidently $\omega = 0$. Next, θ is surjective. Indeed, according to property (c), θ is an $k(X)$-linear map from $D_k(k(X))$ to J; as $D_k(k(X))$ has dimension 1, and J has dimension ≤ 1 (prop 2), θ maps $D_k(k(X))$ onto J. Thus if α is any element of $J(D)$, there exists $\omega \in D_k(k(X))$ such that $\theta(\omega) = \alpha$, and the lemma above shows that $\omega \in \Omega(D)$. □

Corollary 1. *We have $i(D) = \dim_k \Omega(D)$. In particular, the genus $g = i(0)$ is equal to the dimension of the vector space of differential forms such that $(\omega) \geq 0$.*

7 Riemann-Roch theorem

In this section, we will deduce the definitive form of the Riemann-Roch theorem.

Let ω and ω' be two differentials $\neq 0$. Since $D_k(k(X))$ has dimension 1 over $k(X)$, we have $\omega' = f\omega$ with $f \in k(X)^*$, whence $(\omega') = (f) + (\omega)$. Thus, all divisors of differential forms are linearly equivalent and form a single class for linear equivalence, called the **canonical class** and written K. By abuse of language, one often writes K for a divisor belonging to this class.

Now let D be any divisor; we seek to determine $\Omega(D)$. If $K = (\omega_0)$ is a canonical divisor, every differential ω can written $\omega = f\omega_0$ and that

$$\omega \in \Omega(D) \Longleftrightarrow (f) + (\omega_0) \geq D$$

$$\Longleftrightarrow (f) \geq -((\omega_0) - D) \Longleftrightarrow (f) \in L(K - D).$$

Thus, we have

Corollary 2. *(a)* $i(D) = \dim_k \Omega(D) = l(K - D)$;

(b) $H^0(X, \underline{\Omega}(D)) \cong H^0(X, \mathcal{L}(K - D))$, *thus* $\underline{\Omega}(D) \cong \mathcal{L}(K - D)$.

Combining this result with theorem 1, we get theorem 2:

Theorem. *Let X be a projective, irreducible and non-singular algebraic curve. For every divisor D on X, we have*

$$l(D) - l(K - D) = deg\ D + 1 - g.$$

We put $D = K$ in this formula. Then $l(K) = i(0) = g$ and $l(0) = 1$, whence

$$g - 1 = degK + 1 - g,$$

and we get

$$degK = 2g - 2.$$

Corollary 3. *If* $\deg D \geq 2g - 1$*, then the complete linear series* $\left|D\right|$ *has dimension deg D - g.*

Proof. $\deg K - D \leq -1$, whence $l(K - D) = 0$ and $l(D) = \deg D + 1 - g$, which deduces that

$$\dim \left|D\right| = l(D) - 1 = \deg D - g.$$

□

Linear series are closely related to maps of X to a projective space. Let $\varphi : X \to \mathbb{P}_r(k)$ be a regular map from X to a projective space. We suppose that $\varphi(X)$ generates (projectively) $\mathbb{P}_r(k)$. With this hypothesis, if H is a hyperplane of $\mathbb{P}_r(k)$, the divisor $\varphi^{-1}(H)$ is well-defined. As H varies, the $\varphi^{-1}(H)$ form a linear series F of dimension r, without base point (i.e., for every $P \in X$ there exists $D \in F$ such that $v_P(D) = 0$); conversely, every linear series without based points arises uniquely (up to an automorphism of $\mathbb{P}_r(k)$) this way.

Corollary 4. *If* $\deg D \geq 2g$*,* $\left|D\right|$ *has no base points.*

Proof. Suppose that $\left|D\right|$ has a base point P. Then there exists a linear series F:

$$\{H \in F \big| P + H \in \left|D\right|\}$$

Thus $\dim F = \dim \left|D\right| = \deg D - g$. But coro 3 claims that $\dim F = \deg F - g = \deg D - 1 - g$, which is a contraction. □

Corollary 5. *If* $\deg D \geq 2g+1$, $|D|$ *is ample - that is to say it defines a biregular embedding of* X *in a projective space.*

Proof. Suppose that $\deg D \geq 2g+1$, and let $P \in X$. According to coro 4, the linear series $|D-P|$ has no base points. Thus there exists $\Delta \in D$ such that $v_P(\Delta) = 1$. If $\varphi : X \to \mathbb{P}_r(k)$ is the map associated to $|D|$, this means that there exists a hyperplane H of $\mathbb{P}_r(k)$ such that $\varphi^{-1}(H)$ contains P with coefficient 1. It follows that the map $\varphi : X \to \varphi(X)$ has degree 1, then that $\varphi(P)$ is a simple point of $\varphi(X)$. The map φ is thus an isomorphism, as was to be shown. □

8 Remark on the duality theorem

Since $i(K) = l(0) = 1$, the vector space $H^1(X, \mathcal{L}(K)) = H^1(X, \underline{\Omega})$ is one dimensional. The scalar product $\langle \omega, r \rangle$ between the elements of $\Omega(D) = H^0(X, \underline{\Omega}(D))$ and $I(D) = H^1(X, \mathcal{L}(D))$ can be interpreted as a **cup product** with values in $H^1(X, \underline{\Omega})$:

$$H^1(X, \mathcal{L}(D)) \times H^0(X, \underline{\Omega}(D)) \to H^1(X, \underline{\Omega}) \cong k$$

$$(r, \omega) \mapsto \langle \omega, r \rangle.$$

Then the duality theorem says that this product puts the two spaces in duality. In this form, the theorem can be extended to an arbitrary coherent algebraic sheaf $\mathcal{F}$: putting $\tilde{\mathcal{F}} = Hom_{\mathcal{O}}(\mathcal{F}, \underline{\Omega})$, the cup product maps

$$H^1(X, \mathcal{F}) \times H^0(X, \tilde{\mathcal{F}}) \to H^1(X, \underline{\Omega})$$

and puts the two spaces in duality.

Moreover, the proof of duality theorem could extend without great modification to normal varieties of any dimension r. The sheaf $\underline{\Omega}$ should then be replaced by the sheaf $\underline{\Omega}^r$ of differential forms of degree r without poles. One proves by induction on r that $H^r(X, \underline{\Omega}^r)$ is canonically isomorphic to k. Given this, the cup product defines a scalar product on $H^r(X, \mathcal{L}(D)) \times H^0(X, \underline{\Omega}^r(D))$, whence a linear map θ from $H^0(X, \underline{\Omega}^r(D))$ to the dual $J(D)$ of $H^r(X, \mathcal{L}(D))$. The duality theorem shows that θ is an isomorphism.

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