Divisors and Invertible Sheaves on Noetherian Schemes

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1 INTRODUCTION

The group structure of divisor and invertible sheaf are both important invariant on schemes. There is some well-known result that the cartier class group CaCl X is isomorphic to the group of isomorphic invertible sheaves Pic X on some schemes. This paper is going to illustrate this relationship. The first half of paper, including Sec 2, Sec 3 and Sec 4.1, is mainly from [Hartshorne] chapter 2, which introduces the basic definition of divisors and invertible sheaves. The middle part, Sec 4.2, is mainly from [Vakil] and [Edward]. The latter part, Sec 4.3, is mainly from [Nakai] and [Marta], which is aimed to prove the isomorphism of CaCl X and Pic X on a noetherian projective scheme.

2 PRELIMINARY

Definition 1 (ringed space)**.** *A* ***ringed space*** *is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X. The ringed space is a* ***locally ringed space*** *if for each point $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring.*

Definition 2 (morphism)**.** *A* ***morphism*** *of ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\sharp)$ which consists of*

$$\text{a continuous map: } \quad f : X \to Y$$

$$\text{a map of sheaves of rings on Y: } \quad f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X.$$

A ***morphism*** *of locally ringed spaces is a morphism $(f, f^\sharp)$ of ringed space, such that the induced map on stalk is a local homomorphism of local rings.*

An ***isomorphism*** *of locally ringed spaces is a morphism with a two side inverse. That is a morphism $(f, f^\sharp)$ is an isomorphism if and only if f is a homeomorphism of the underlying topological spaces, and $f^\sharp$ is an isomorphism of sheaves.*

Definition 3 (scheme)**.** *An* ***affine scheme*** *is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic (as a locally ringed space) to the spectrum of some ring.*

A ***scheme*** *is a locally ringed space $(X, \mathcal{O}_X)$ in whiche every point has an open neighborhood U such that the topological space U, together with the restricted sheaf $\mathcal{O}_{X|U}$, is an affine scheme.*

Definition 4 (noetherian scheme)**.** *A scheme X is* ***locally noetherian*** *if for every open affine subset $U =$ Spec A, A is a noetherian ring. X is* ***noetherian*** *if it can be covered by a finite number of open affine subsets Spec A_i, with each A_i is a noetherian ring.*

Definition 5 (integral scheme)**.** *A scheme X is* ***integral*** *if for every open set $U \subset X$, the ring $O_X(U)$ is an integral domain.*

Definition 6 (separated)**.** *Let $f : X \to Y$ be a morphism of schemes. The diagonal morphism is the unique morphism $\Delta : X \to X \times_Y X$ whose composition with both projection maps $p_1, p_2 : X \times_Y X$ is the identity map of $X \to X$. We say that the morphism f is separated if the diagonal morphism Δ is a closed immersion. In that case we also say X is* ***separated*** *over Y. A scheme X is* ***separated*** *if it is separated over Spec $\mathbb{Z}$.*

Definition 7 (regular in codimension one)**.** *We say a scheme X is* ***regular in codimension one*** *(or sometimes* ***nonsingular in codimension one****) if every local ring O_x of X of dimension one is regular.*

Definition 8 (sheaf of $\mathcal{O}_X$-modules)**.** *Let $(X, mathcalO_X)$ be a ringed space. A **sheaf of $\mathcal{O}_X$-modules**(or simply an $\mathcal{O}_X$) is a sheaf $\mathcal{F}$ on X, such that for each open set $U \subset X$, the group $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and for each inclusion of open sets $V \subset U$, the restriction homomorphism $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the module structures via the ring homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$.*

Definition 9 (morphism)**.** *A **morphism** $\mathcal{F} \to \mathcal{G}$ of sheaves of $\mathcal{O}_X$-modules is a morphism of sheaves, such that for each open set $U \subset X$, the map $\mathcal{F}(U) \to \mathcal{G}(U)$ is a homomorphism of $\mathcal{O}_X(U)$-modules.*

Definition 10 (constant sheaf)**.** *Let X be a topological space, and A a commutative ring. We define the **constant sheaf** $\mathcal{A}$ on X as follows. Given A the discrete topology, and for any open set $U \subset X$, let $\mathcal{A}(U)$ be the subring of all continuous maps of U into A. Then with the restriction maps, we obtain a sheaf $\mathcal{A}$.*

Definition 11 ((quasi-)coherent sheaf)**.** *Let $(X, \mathcal{O}_X)$ be a scheme. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is **quasi-coherent** if X can be covered by open affine subsets $U_i = Spec\ A_i$, such that for each i there is an A_i-module M_i with $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$. We say that $\mathcal{F}$ is **coherent** if furthermore each M_i can be taken to be a finitely generated A_i-module.*

Definition 12 (projective scheme)**.** *We call a scheme of the form (i.e., isomorphic to) Proj S, where S is a finitely generated graded ring over A, a **projective scheme over A**, or a **projective** A-scheme. A **quasiprojective** A**-scheme** is a quasicompact open subscheme of a projective A-scheme. The "A" is omitted if it is clear from the context.*

Theorem 1. *Let S be a graded ring, and M a graded S-module. Let $X = Proj\ S$.*

(a) For any $\mathfrak{p} \in X$, the stalk $(\widetilde{M})_{\mathfrak{p}} = M_{(\mathfrak{p})}$.

(b) For any homogeneous $f \in S_+$, we have $\widetilde{M}|_{D_+(f)} \cong \widetilde{M_{(f)}}$ via the isomorphism of $D_+(f)$ with Spec $S_{(f)}$, where $M_{(f)}$ denotes the group of elements of degree 0 in the localized module M_f.

(c) $\widetilde{M}$ is a quasi-coherent $\mathcal{O}_X$-module. If S is noetherian and M is finitely generated, then $\widetilde{M}$ is coherent.

Definition 13 (sheaf associated to M)**.** *Let S be a graded ring and let M be a graded S-module. $\widetilde{M}$ is defined to be the **sheaf associated to M** on Proj S.*

Theorem 2. *Let S be a graded ring and let M, N be two graded S-modules, then $(\widetilde{M \otimes_S N}) \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$.*

3 DIVISORS

The divisors forms an useful tool for the study of the geometry on a algebraic variety or scheme. The definition of divisors, linear equivalent and the divisor class group will be introduced in this section.

Here Weil Divisors, and Cartier Divisors two different divisors. Weil divisors is only established on noetherian integral scheme. Cartier divisors is defined on more general schemes. These two definitions coincide on some special schemes.

3.1 Weil Divisors

In this subsection we consider the schemes satisfying the following condition:

($\star$)X is a noetherian integral separated scheme which is regular in codimension one.

Definition 14 (Weil divisor)**.** *Let X satisfy ($\star$). A* ***prime divisor*** *on X is a closed integral subscheme Y of codimension one. A* ***Weil divisor*** *is an element of the free abelian group Div X generated by the prime divisors. We write a divisor as $D = \sum n_i Y_i$, where the Y_i are prime divisors, the n_i are integers, and only finitely many n_i are different from zero. If all the $n_i \geq 0$, we say that D is* ***effective****. The* ***support*** *of a Weil divisor D, denoted Supp D, is the subset $\cup_{n_i \neq 0} Y_i$.*

To introduce principle Weil divisor and linear equivalence, we need some knowledge about discrete valuation.

Definition 15 (Discrete Valuation Rings)**.** *Let K be a field. A* ***discrete valuation*** *on K is a mapping v of K^* onto $\mathbb{Z}$ such that*

1) $v(xy) = v(x) + v(y)$, i.e., v is a homomorphism;

2) $v(x+y) \geq \min(v(x), v(y))$.

The set consisting of 0 and all $x \in K*$ such that $v(x) \geq 0$ is a ring, called the ***valuation ring*** *of v. It is a valuation of the field K.*

Proposition 1 (Noetherian local domain of dimension one)**.** *Let A be a noetherian local domain of dimension one, m its maximal ideal, k=A/m its residue field. The the following are equivalent:*

1) A is a discrete valuation ring;

2) m is a principal ideal;

3) Every non-zero ideal is a power of m;

4) There exists $x \in A$ such that every non-zero ideal is of the form (x^k), $k \geq 0$.

If Y is a prime divisor on X, let $\eta \in Y$ be the generic point of Y. Then the noetherian local ring $\mathcal{O}_{\eta,X}$ is a discrete valuation ring with quotient field K, the function field of X. Now let $f \in K^*$ be any nonzero rational function on X. Then $v_Y(f)$ is an integer. Since X is separated, for all generic point $\eta \in Y$, $\mathcal{O}_{\eta,X}$ has the same valuation for f. If it is positive, we say f has a **zero** along Y, of that order; if it is negative, we say f has a **pole** along Y, of order $-v_Y(f)$. If we choose different prime divisor Y on X, there should be different valuation on f. Thus we can define a formal sum (f):

$$(f) := \sum v_Y(f) \cdot Y.$$

Lemma 1. *Let X satisfy ($\star$), and let $f \in K*$ be a nonzero function on X. Then $v_Y(f) = 0$ for all but except finitely many prime divisors Y.*

Definition 16 (principle divisor)**.** *Let X satisfy ($\star$) and let $f \in K^*$. There is a well-defined map $K^* \to DivX$ defined by*

$$f \mapsto (f) = \sum v_Y(f) \cdot Y,$$

where (f) is a finite sum by the lemma. Furthermore, we define that any divisor which is equal to divisor of a function is called a ***principle divisor****.*

Remark 1. *Because of the properties of valuation, for any $f, g \in K^*$ (f/g) = (f) - (g). Hence, the above map is a homomorphism.*

Definition 17 (linearly equivalent; divisor class group)**.** *Let X satisfy ($\star$). Two divisors D and D' are said to be **linearly equivalent**, written $D \sim D'$, if $D - D'$ is a principal divisor. The group Div X of all divisors divided by the subgroup of principal divisors is called the **divisor class group**, and is denoted by Cl X.*

3.2 Cartier Divisors

Cartier divisors is a notion to extend the divisors to arbitrary scheme.

Definition 18 (sheaf of total quotient ring of $\mathcal{O}_X$)**.** *Let (X,$\mathcal{O}_X$) be a scheme. For each open affine subset U = Spec A, define the presheaf K:*

$$U \mapsto S(U)^{-1}\mathcal{O}_X(U),$$

where S(U) is the set of elements of $\mathcal{O}_X(U)$ which are not zero divisors.

*Let $\mathcal{K}_X$ be the associated sheaf of ring of K, called the **sheaf of total quotient ring** of $\mathcal{O}_X$.*

Definition 19 (sheaf of invertible elements)**.** *Given a sheaf of ring $\mathcal{R}$, we denote by $\mathcal{R}^*$ the sheaf of multiplicative group in the sheaf of ring $\mathcal{R}$.*

Definition 20 (Cartier divisor)**.** *A **Cartier divisor** on a scheme (X,$\mathcal{O}_X$) is a global section D of the sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$. That is D is given by an open cover $\{U_i\}$ of X, and for each i, section $f_i \in \Gamma(U_i, \mathcal{K}_X^*)$, such that for each i,j, $\frac{f_i}{f_j} \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*)$.*

The set of Cartier divisors is denoted by $Div_C X$ and it is a group with the following operation: given $D_1 = \{(U_i, f_i)\}$ and $D_2 = \{(U_i, g_i)\}$ two divisors, $D_1 + D_2$ is defined as the divisor represented by $\{(U_i, f_i g_i)\}$.

*A Cartier divisor is **principal** if it is the image of a rational function $f \in \Gamma(X, \mathcal{K}_X^*)$ through the canonic homomorphism*

$$div_C : \Gamma(X, \mathcal{K}_X^*) \to \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*).$$

We denote a principal Cartier divisor by $div_C(f)$.

*The group Div_C of Cartier divisors divided by the subgroup of principal Cartier divisors is called the **Cartier divisor class group**, and is denoted by CaCl X.*

Now there are two different definition of divisors on different schemes. They coincide together in some special case.

Proposition 2. *Let X be an integral, separated noetherian scheme, all of whose local rings are unique factorization domains. Then the group Div X of Weil divisors on X is isomorphic to the group of Cartier divisors Div_C X, and furthermore, the principal Weil divisors correspond to the principal Cartier divisors under this isomorphism.*

Remark 2 (regular scheme)**.** *A scheme is **regular** if all of its local rings are regular local rings. There is a proposition that every regular local ring is unique factorization domain, so it implies that on regular, integral, separated noetherian scheme, Weil divisors and Cartier divisors are the same element.*

4 INVERTIBLE SHEAVES

The concept of Picard group, Pic X, defined by invertible sheaf, is another global invariant on a noetherian scheme $(X, \mathcal{O}_X)$. There is an important theorem showing the 1-1 correspondence between divisors and Picard group in some special noetherian scheme.

Sec 4.1 talks about this correspondence in noetherian integral scheme, guided by the illustration of Hartshorne. Sec 4.2 gives an example of Sec 4.1 – the invertible sheaves on $\mathbb{P}^n_k$. Sec 4.3 gives a proof that: on a (non necessarily reduced) projective scheme X over a noetherian ring that the Cartier divisor class group, CaCl X, agrees with the Picard group, Pic X.

Definition 21 (locally free sheaves)**.** *A **locally free sheaf** of rank n on a scheme $(X, \mathcal{O}_X)$ is defined as an $\mathcal{O}_X$-module $\mathcal{E}$ that is locally a free sheaf of rank n. Precisely, there is an open cover $\{U_i\}$ of X such that for each U_i, $\mathcal{E}|_{U_i} \cong \mathcal{O}_{X|U_i}^{\oplus n}$*

Remark 3. *From $\mathcal{E}$ is locally of finite rank, it is a coherent sheaf. Thus we should assume X is a noetherian scheme such that $\mathcal{E}$ is well-behaved.*

Definition 22 (dual sheaf)**.** *Let $(X, \mathcal{O}_X)$ be a noetherian scheme, and let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-module of finite rank. We define the **dual** of $\mathcal{E}$, denoted $\check{\mathcal{E}}$, to be the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$.*

Proposition 3 (dual of dual sheaf)**.** *There are isomorphisms:*

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X), \mathcal{O}_X) \cong \mathcal{E}.$$

Proof. *Define a map $\varphi_U : \mathcal{E}(U) \to \check{\check{\mathcal{E}}}(U) = Hom(\mathcal{H}om(\mathcal{E}, \mathcal{O}_X)|_U, \mathcal{O}_X|_U)$ by sending $e \in \mathcal{E}(U)$ to the collection of maps $\{e_V\}_V$:*

$$Hom(\mathcal{E}, \mathcal{O}_X)(U \cap V) \to \mathcal{O}_X(U \cap V) :$$

$$\sigma \mapsto e_V(\sigma) = \sigma_{U \cap V}(e|_{U \cap V}),$$

where $\sigma : \mathcal{E}|_{U \cap V} \to \mathcal{O}_X|_{U \cap V}$.

Because $\mathcal{E}$ is coherent,

$$\text{the stalk } \mathcal{H}om(\mathcal{E}, \mathcal{O}_X)_P = Hom(\mathcal{E}_P, \mathcal{O}_{X,P}).$$

And that the morphism we have defined induces the stalk morphism

$$\mathcal{E}_P \to Hom(Hom(\mathcal{E}_P, \mathcal{O}_{X,P}), \mathcal{O}_{X,P})$$

given by

$$e_P \mapsto (\sigma_P \mapsto \sigma_P(e_P)).$$

Since $\mathcal{E}$ is locally free of finite rank, the stalk $\mathcal{E}_P$ is free of finite rank, and the stalk map is the canonical isomorphism of a free module of finite rank with its double dual. Since all of the induced stalk maps are isomorphisms, we have $\check{\check{\mathcal{E}}} \cong \mathcal{E}$.

Definition 23 (invertible sheaf)**.** *Let $(X, \mathcal{O}_X)$ be a noetherian scheme, a coherent sheaf $\mathcal{F}$ is **invertible** if it is locally free of rank 1.*

Lemma 2 (stalks of invertible sheaf)**.** *Let $(X, \mathcal{O}_X)$ be a noetherian scheme, and let $\mathcal{F}$ be a coherent sheaf.*

(a) *If the stalk $\mathcal{F}$ is a free stalk $\mathcal{F}_x$ is a free $\mathcal{O}_x$-module for some point $x \in X$, then there is a neighborhood U of x such that $\mathcal{F}|_U$ is free;*

(b) *$\mathcal{F}$ is locally free if and only if its stalks $\mathcal{F}_x$ are free $\mathcal{O}_x$-modules for all $x \in X$.*

(c) *For any $\mathcal{O}_X$-module $\mathcal{F}$, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \cong \check{\mathcal{E}} \otimes \mathcal{F}$.*

Proof.

(a) *For any affine neighborhood U=Spec A of $x \in X$, $x = p$ for some prime ideal $p \in$ Spec A. We know that $\mathcal{F}|_U = \widetilde{M}$ for some finitely generated A-module M, and that $M_p = \widetilde{M}_p = \mathcal{F}|_x = (\mathcal{O}_X^{\oplus n})_x = A_p^{\oplus n}$, so $M_p = A_p^{\oplus n}$.*

Suppose M is finitely generated by $\{m_1, \ldots, m_k\}$. Since $M_p = A_p^{\oplus n}$, there is 1-1 correspondence: $\frac{m_i}{1} \leftrightarrow (\frac{a_{i1}}{s_{i1}}, \ldots, \frac{a_{in}}{s_{in}})$. So define $s = \prod_{i,j} s_{ij}$, we have $m_i|_{D(s)} = (\frac{a_{i1}}{s_{i1}}, \ldots, \frac{a_{in}}{s_{in}}) \in A_s^{\oplus n}$. That means that there is a surjective map $A_s^{\oplus n} \to M_s$. Since A is noetherian, the kernel is finitely presented and that $M_s = \langle m_1, \ldots, m_k | l_1, \ldots, l_j \rangle$ with $m_1, \ldots, m_n, l_1, \ldots, l_j \in A_s$. Because M_p is free, $(n_i)_p = 0$ for all $1 \le i \le j$. i.e. There is $t_i \in A - p$ such that $t_i n_i = 0$ for all i. Let $t = s \prod_{i=1}^{j} t_i$, then $t \in A - p$ and $tn_i = 0$ for all i. That means $U := D(t) \subset D(s)$ and $\mathcal{F}|_U = \widetilde{M}|_{D(t)}$ is free.

(b) *$\Rightarrow$: Suppose the rank of $\mathcal{F}$ is n. For any $x \in X$, there exists an open neighborhood U with $x \in U$ such that $\mathcal{F}|_U \cong \mathcal{O}_{X|U}^{\oplus n}$, then $\mathcal{F}_x = \varinjlim \mathcal{F}|_U(V) = \varinjlim \mathcal{O}_{X|U}^{\oplus n}(V) = \mathcal{O}_x^{\oplus n}$ for all open sets V with $x \in V$.*

$\Leftarrow$: implied from (a).

(c) *Define $\varphi_U : Hom(\mathcal{E}|_U, \mathcal{O}_{X|U}) \otimes \mathcal{F}(U) \to Hom(\mathcal{E}|_U, \mathcal{F}|_U)$ by*

$$(\varphi_U(\phi \otimes f))_V(e) = \phi_V(e) \cdot f|_V \in \mathcal{F}(U \cap V),$$

where $\varphi = \{\varphi_V\}$ and $e \in \mathcal{E}(U \cap V)$. We extend this definition linearly. Observe that the restriction maps are compatible and that the φ_U glue to give φ, since they are all canonically defined. On stalks, φ_P is the map

$$\varphi_P : Hom(\mathcal{E}_P, \mathcal{O}_{X,P}) \otimes \mathcal{F}_P \to Hom(\mathcal{E}_P, \mathcal{F}_P) \text{given by}$$

$$\phi_P \otimes f_P \mapsto (e_P \mapsto \phi_P(e_P) \cdot f_P).$$

By the conclusion in (b), $\mathcal{E}_P$ is a free $\mathcal{O}_{X,P}$-module, so the map φ_P is an isomorphism. Thus the map φ is an isomorphism.

Lemma 3.

(a) *If $\varphi : A \to B$ is a homomorphism of rings, then φ induces a natural morphism of locally ringed spaces*

$$(f, f^\sharp) : (SpecB, \mathcal{O}_{SpecB}) \to (SpecA, \mathcal{O}_{SpecA}).$$

(b) *If A and B are rings, then any morphism of locally ringed spaces from Spec B to Spec A is induced by a homomorphism of rings $\varphi : A \to B$ as in (a).*

Proof.

(a) *Check the map f: Spec B $\to$ Spec A by $f(p) \in \varphi^{-1}(p)$ for any $p \in$ Spec B.*

(b) *Check the map $\varphi : \Gamma(SpecA, \mathcal{O}_{SpecA}) \to \Gamma(SpecB, \mathcal{O}_{SpecB})$ is the desired map.*

Proposition 4 (Picard group)**.** *Let $(X, \mathcal{O}_X)$ be a noetherian scheme.*

(a) *If $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on a noetherian scheme $(X, \mathcal{O}_X)$, $\mathcal{L} \otimes \mathcal{M}$ is also invertible;*

(b) *The multiplication $\otimes$ is associative;*

(c) *$\mathcal{O}_X$ is invertible such that for any invertible sheaf $\mathcal{F}$, $\mathcal{F} \otimes \mathcal{O}_X = \mathcal{F}$;*

(d) *$\mathcal{F}$ is invertible if and only if there is an invertible sheaf $\mathcal{G}$ such that $\mathcal{F} \otimes \mathcal{G} \cong \mathcal{O}_X$*

As a result, we can define the ***Picard group*** *of X, Pic X, to be the group of isomorphism class of invertible sheaves on X, under the operation $\otimes$.*

Proof.

(a) *Let $x \in X$, by lemma 2(b) $\mathcal{L}_x$ and $\mathcal{M}_x$ is isomorphic to $\mathcal{O}_{X,x}$. Then $(\mathcal{L} \otimes \mathcal{M})_x \cong \mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{M}_x \cong \mathcal{O}_{X,x}$, which means stalks $(\mathcal{L} \otimes \mathcal{M})_x$ is free of rank 1 for all $x \in X$. Using lemma 2(b) again, we know $\mathcal{L} \otimes \mathcal{M}$ is invertible.*

(b) *follow from the definition of $\otimes$.*

(c) *follow from the definition of $\otimes$.*

(d) $\Rightarrow$: *Suppose first that $\mathcal{F}$ is locally free of rank 1. Then by lemma 2(c) we have $\mathcal{F} \otimes \check{\mathcal{F}} \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$. We define an isomorphism $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) \cong \mathcal{O}_X$ as follows: cover X by open affine subsets U_i with $\mathcal{F}|_{U_i} \cong \mathcal{O}_X|_{U_i}$ so*

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})(U_i) = Hom_{\mathcal{O}_X}(\mathcal{O}_X|_{U_i}, \mathcal{O}_X|_{U_i}).$$

Then we let $\varphi_{U_i}(\phi) = \phi(U_i)(1)$ with $\phi : \mathcal{O}_X|_{U_i} \to \mathcal{O}_X|_{U_i}$. Since $(U_i, \mathcal{O}_X|_{U_i})$ is a locally ring space, φ is an isomorphism from $Hom_{\mathcal{O}_X}(\mathcal{O}_X|_{U_i}, \mathcal{O}_X|_{U_i})$ to $\mathcal{O}_X|_{U_i}(U_i)$ (follows from lemma 3). This gives a map $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) \to \mathcal{O}_X$ that is an isomorphism on each U_i, hence an isomorphism and $\mathcal{F} \otimes \check{\mathcal{F}} \cong \mathcal{O}_X$.

It remains to check $\check{\mathcal{F}}$ is invertible. This is because on each U_i

$$\check{\mathcal{F}}(U_i) \cong Hom_{\mathcal{O}_X}(\mathcal{F}|_{U_i}, \mathcal{O}_X|_{U_i}) \cong Hom_{\mathcal{O}_X}(\mathcal{O}_X|_{U_i}, \mathcal{O}_X|_{U_i}) \cong \mathcal{O}_X(U_i).$$

$\Leftarrow$: *Suppose there exists $\mathcal{G}$ such that $\mathcal{F} \otimes \mathcal{G} \cong \mathcal{O}_X$. Then $(\mathcal{F} \otimes \mathcal{G})_x = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x \cong \mathcal{O}_{X,x}$ for all $x \in X$. It suffices to show that if M, N are finitely generated A-modules with (A, m, k) a local ring and $M \otimes_A N \cong A$, then $M \cong A$ and $N \cong A$. Indeed we have an isomorphism*

$$M/mM \otimes_k N/mN \cong (M \otimes_A N)/m(M \otimes_A N) \cong k \otimes_A (M \otimes_A N) \cong k,$$

so M/mM (and N/mN) are vector space over k with dimension 1. By Nakayama's lemma, M is rank 1 A-module. Let $a \in AnnM$. Then a annihilates A since $M \otimes_A N \cong A$, and in particular, $a \cdot 1 = 0$ so $a = 0$ and M is free of rank 1.

Remark 4. *Pic X can be expressed as the cohomology group $H^1(X, \mathcal{O}_X^*)$.*

4.1 Noetherian integral scheme

Definition 24 (sheaf associated to Cartier divisor)**.** *Let D be a Cartier divisor on a scheme $(X, \mathcal{O}_X)$, represented by $\{(U_i, f_i)\}$ as above. We define a subsheaf $\mathcal{L}(D)$ of the sheaf of total quotient rings $\mathcal{K}_X$ by taking $\mathcal{L}(D)$ to the sub-$\mathcal{O}_X$-module of $\mathcal{K}_X$ generated by f_i^{-1} on U_i. More precisely,*

$$\begin{aligned}\mathcal{L}(D)(U_i) :&= \{s \in \mathcal{K}_X(X) \big| (s)|_{U_i} + (f_i)|_{U_i} \geq 0\} \\ &= \{s \in \mathcal{K}_X(X) \big| (f_i \cdot s)|_{U_i} \geq 0\} \\ &= \{s \in \mathcal{K}_X(X) \big| f_i \cdot s \in \Gamma(U_i, \mathcal{O}_X|_{U_i})\} \\ &= f_i^{-1} \cdot \Gamma(U_i, \mathcal{O}_X|_{U_i}) \\ &= f_i^{-1} \cdot \mathcal{O}_X|_{U_i}(U_i).\end{aligned}$$

*This is well-defined, since f_i/f_j is invertible on $U_i \cap U_j$, so f_i^{-1} and f_j^{-1} generate the same $\mathcal{O}_X$-module. We call $\mathcal{L}(D)$ the **sheaf associated to D**.*

Proposition 5 (CaCl X $\hookrightarrow$ Pic X)**.** *Let $(X, \mathcal{O}_X)$ be a noetherian scheme. Then:*

(a) for any Cartier divisor D, $\mathcal{L}(D)$ is an invertible sheaf on X. The map $D \mapsto \mathcal{L}(D)$ gives a 1-1 correspondence between Cartier divisors on X and invertible subsheaves of $\mathcal{K}_X$;

(b) $\mathcal{L}(D_1 - D_2) \cong \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1}$;

(c) $D_1 \sim D_2$ if and only if $\mathcal{L}(D_1) \cong \mathcal{L}(D_2)$ as abstract invertible sheaves;

(d) the map $D \mapsto \mathcal{L}(D)$ gives an injective homomorphism of the group CaCl X of Cartier divisors modulo linear equivalence to Pic X.

Proof.

(a) Naturally, by the definition of $\mathcal{L}(D)$, there is

$$Div_C X \subset \{\mathcal{L}(D)\}_D.$$

Given a Cartier divisor $D = \{(U_i, f_i)\}$, $\mathcal{L}(D)|_{U_i} = f_i^{-1}\mathcal{O}_X|_{U_i} \cong \mathcal{O}_X|_{U_i}$. Thus $\mathcal{L}(D)$ is invertible. So

$$\{\mathcal{L}(D)\}_D \subset \textit{invertible subsheaves of } \mathcal{K}_X,$$

and

$$Div_C X \subset \{\textit{invertible subsheaves of } \mathcal{K}_X\}.$$

The Cartier divisor D can be recovered from $\mathcal{L}(D)$ together with its embedding in $\mathcal{K}_X$, by taking f_i on U_i to be the inverse of a local generator of $\mathcal{L}(D)$. Different choice of local generator $\{1/f_i, 1/g_i\}$, there is $f_i/g_i \in \Gamma(U_i, \mathcal{O}_X^*)$, so D is well-defined. i.e.

$$\{\mathcal{L}(D)\}_D \subset Div_C X.$$

This construction is useful for any invertible subsheaf of $\mathcal{K}_X$, so

$$\{\textit{invertible subsheaves of } \mathcal{K}_X\} \subset Div_C X.$$

Hence there is 1-1 correspondence:

$$Div_C X = \{\mathcal{L}(D)\}_D = \{\textit{invertible subsheaves of } \mathcal{K}_X\}.$$

(b) *If D_1 is locally defined by f_i and D_2 is locally defined by g_i, then $\mathcal{L}(D_1 - D_2)$ is locally generated by $f_i^{-1}g_i$, so $\mathcal{L}(D_1 - D_2) = \mathcal{L}(D_1) \cdot \mathcal{L}(D_2)^{-1}$ as subsheaves of $\mathcal{K}_X$. This product is clearly isomorphic to the abstract tensor product $\mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1}$.*

(c) *Using (b), it will be sufficient to show that $D = D_1 - D_2$ is principal if and only if $\mathcal{L}(D) \cong \mathcal{O}_X$. If D is principal, defined by $f \in \Gamma(X, \mathcal{K}_X^*)$, then $\mathcal{L}(D)$ is globally generated by f^{-1}, so sending $1 \mapsto f^{-1}$ gives an isomorphism $\mathcal{O}_X \cong \mathcal{L}(D)$. Conversely, given such an isomorphism, the image of 1 gives an element of $\Gamma(X, \mathcal{K}_X^*)$ whose inverse will define D as a principal divisor.*

(d) *follows from (a)(b)(c).*

Proposition 6 (CaCl X $\cong$ Pic X)**.** *If X is a noetherian integral scheme $(X, \mathcal{O}_X)$, the homomorphism CaCl $X \to$ Pic X is an isomorphism.*

Integral scheme induces a proposition, which helps to prove the isomorphism between CaCl X and Pic X.

Lemma 4 (integral $\Rightarrow$ reduced and irreducible)**.** *A scheme X is integral if and only if it is both reduced and irreducible.*

Proof (proof of lemma)**.** *Clearly an integral scheme is reduced. If X is not irreducible, then one can find two nonempty disjoint open subsets U_1 and U_2. Then $\mathcal{O}(U_1 \cup U_2) = \mathcal{O}(U_1) \times \mathcal{O}(U_2)$ which is not an integral domain. Thus integral implies irreducible.*

Now there are enough tools to prove the isomorphism.

Proof. *It suffices to show that every invertible sheaf is isomorphic to a subsheaf of $\mathcal{K}_X$. Here $\mathcal{K}_X$ is the constant sheaf K, where K is the function field of X. So let $\mathcal{L}$ be any invertible sheaf, and consider the sheaf $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}_X$. On any open set U where $\mathcal{L}|_U \cong \mathcal{O}_X|_U$, we have $\mathcal{L}|_U \otimes_{\mathcal{O}_X|_U} \mathcal{K}_X|_U \cong \mathcal{K}|_U$, so $\mathcal{L}|_U \otimes_{\mathcal{O}_X|_U} \mathcal{K}_X|_U$, and that $\mathcal{L}|_U \otimes_{\mathcal{O}_X|_U} \mathcal{K}_X|_U$ is a constant sheaf on U.*

By the lemma 4 we know X is irreducible, it follows that any sheaf whose restriction to each open set of a covering of X is constant, is in fact a constant sheaf. i.e. $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ is isomorphic to a constant sheaf $\mathcal{K}_X$. Then there is a natural map $\mathcal{L} \to \mathcal{L} \otimes \mathcal{K}_X \cong \mathcal{K}$ express $\mathcal{L}$ as a subsheaf of $\mathcal{K}_X$. Finally, using Proposition 5 (d) there is an isomorphism CaCl $X \cong$ Pic X.

Proposition 7 (Cl $X \cong$ Pic X)**.** *If X is a noetherian, integral, separated locally factorial scheme, then there is a natural isomorphism Cl $X \cong$ Pic X.*

Proof. *The proposition 2 implies Cl $X \cong$ CaCl X. Thus, Cl $X \cong$ CaCl $X \cong$ Pic X.*

4.2 Projective scheme over a field

In this subsection, we are going to find all the invertible sheaf on a projective scheme $X = \mathbb{P}_k^n$, where $\mathbb{P}_k^n$ is Proj $k[x_0, \ldots, x_n]$, a special projective scheme over a field k.

The method to do it is by the isomorphism of group Cl $X \cong$ Pic X on the space $\mathbb{P}_k^n$.

Theorem 3 (Cl X $\cong \mathbb{Z}$)**.** *Let X be the projective space $\mathbb{P}_k^n$ over a field k. For the divisor $D = \sum n_i Y_i$, define the degree of D by deg $D = \sum n_i Y_i$, where deg Y_i, where deg Y_i is the degree of the hypersurface Y_i. Let H be the hyperplane $x_0 = 0$. Then:*

(a) *if D is any divisor of degree d, then $D \sim dH$;*

(b) *for any $f \in K^*$, deg $f = 0$;*

(c) *the degree function gives an isomorphism deg: Cl $X \to \mathbb{Z}$.*

Proof. *We prove (b) at first.*

(b) *Since a rational function f is quotient g/h of homogeneous polynomials of the same degree, and (f) = (g) - (h), we have deg(f)= deg(g)-deg(h) = 0.*

(a) *Let $S = k[x_0, \ldots, x_n]$ be the homogeneous coordinate ring of X. If $g \in S$ is a homogeneous polynomial of degree d, we can factor it into a product of polynomials $g = g_1^{n_1} \ldots g_r^{n_r}$. Note that g_i defines a hypersurface Y_i of degree d_i = deg g_i. The divisor of g is (g) $= \sum n_i Y_i$, which is of degree $d = \sum n_i d_i$.*

If D is a divisor of degree, then we may write it as a difference $D_1 - D_2$ of effective divisors of degree d_1 and d_2 with $d_1 - d_2 = d$. Since any effective divisor $\sum n_i Y_i$ is a divisor of a polynomial $\prod g_i^{n_i}$ where Y_i is defined by g_i, we may write $D_1 = (g_1)$ and $D_2 = (g_2)$. Now D-dH = (f) where $f = g_1 / x_0^d g_2$ is a rational function on X, that is, $D \sim dH$.

(c) *The last statement follows from the fact that deg H = 1.*

Definition 25 (twisting sheaf)**.** *Let S be a graded ring, and let X = Proj S. For any $n \in \mathbb{Z}$, we define S(n) a graded S-module as follows:*

$$S(n) := \oplus_{d \in \mathbb{Z}} S(n)_d, \text{ where } S(n)_d := S_{n+d}.$$

we define the sheaf $\mathcal{O}_X(n)$ to be $\widetilde{S(n)}$. We call $\mathcal{O}_X(1)$ the ***twisting sheaf*** *of Serre.*

Remark 5. *It is not hard to verify that $S(n)$ is a grade S-module: for any $d, k \in \mathbb{Z}$, $S_d \times S(n)_k$ is mapped into S_{d+n+k}, which is equal to $S(n)_{d+k}$.*

Proposition 8 ($\mathcal{O}_X(n)$ is invertible)**.** *Let S be a graded ring and let X = Proj S. Assume that S is generated by S_1 as an S_0-algebra.*

(a) *The sheaf $\mathcal{O}_X(n)$ is an invertible sheaf on X.*

(b) *$\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \cong \mathcal{O}_X(m+n)$*

Proof.

(a) *We assert that Proj $S = \cup_{f \in S_1} D_+(f)$: for any $\mathfrak{p} \in$ Proj S, $\mathfrak{p} \not\supseteq \oplus_{i \geq 1} S_i$. Since S is generated by S_1, $\mathfrak{p} \not\supseteq S_1$. So there is $f \in S_1$ such that $\mathfrak{p} \in D_+(f)$, then we prove Proj $S \subset \cup_{f \in S_1} D_+(f)$. The "$\supset$" follows from the definition of Proj S.*

Thus, it suffices to prove $\mathcal{O}_X(n)|_{D_+(f)}$ is free for all $f \in S_1$. By theorem 1,

$$\mathcal{O}_X(n)|_{D_+(f)} = \widetilde{S(n)}|_{D_+(f)} \cong \widetilde{S(n)_{(f)}}.$$

Furthermore, there is an isomorphism $S(n)_{(f)} \to S_{(f)}$ by setting $\frac{a}{f} \mapsto \frac{1}{f^n} \cdot \frac{a}{f}$. Thus,

$$\mathcal{O}_X(n)|_{D_+(f)} \cong \mathcal{O}_X|_{D_+(f)} \text{ is free on all open subset } D_+(f), \text{ where } f \in S_1.$$

i.e., $\mathcal{O}_X(n)$ is invertible.

(b) Using theorem 2:

$$\mathcal{O}_X(n+m) = \widetilde{S(n+m)} = \widetilde{S(n) \otimes_S S(m)} \cong \widetilde{S(n)} \otimes \widetilde{S(m)} = \mathcal{O}_X(n) \otimes \mathcal{O}_X(m).$$

Lemma 5 ($\check{\mathcal{O}_X}(d) \cong \mathcal{O}_X(-d)$)**.**
Let $d \in \mathbb{Z}$, the dual of twisting sheaf $\mathcal{O}_X(d)$ is $\mathcal{O}_X(-d)$.

Proof.

$$\mathcal{O}_X(-d) \otimes \mathcal{O}_X(d) \cong \mathcal{O}_X(-d+d) \cong \mathcal{O}_X.$$

Then

$$\mathcal{O}_X(-d) \cong \mathcal{O}_X(-d) \otimes (\mathcal{O}_X(d) \otimes \check{\mathcal{O}_X}(d)) \cong (\mathcal{O}_X(-d) \otimes \mathcal{O}_X(d)) \otimes \check{\mathcal{O}_X}(d) \cong \check{\mathcal{O}_X}(d).$$

Proposition 9 (global sections of $\mathcal{O}_X(d)$)**.**
The global sections of the sheaf $\mathcal{O}_X(d)$ on $\mathbb{P}^n_k$ correspond to the homogeneous elements of $k[x_0, \dots, x_n]$ of degree d, which form a vector space of dimension $\binom{d+n}{n}$. In particular, when $d > 0$, this is a polynomial of degree n in d, and when $d < 0$, there are no global sections.

Proof. *$\mathbb{P}^n_k$ consists of n+1 copies of $\mathbb{A}^n$, called $U_0, \dots, U_n$, where*

$$U_i = D_+(x_i).$$

Then $\Gamma(U_i, \mathcal{O}_X(d)) = \Gamma(D_+(x_i), \mathcal{O}_X(d)) = k[x_0, \dots, x_{i-1}, x_{i+1}, x_n](d)_{(0)}$ for all i. Thus, $\Gamma(X, \mathcal{O}_X(d)) = k[x_0, \dots, x_n](d)_{(0)}$ consists of homogeneous polynomial of $k[x_0, \dots, x_n]$ with deg d. By combination, it is a vector space with dimension $\binom{d+n}{n}$.

Proposition 10 (group of $\{\mathcal{O}_X(d)\}_{d \in \mathbb{Z}}$)**.**
If $d_1 \neq d_2$, then $\mathcal{O}_X(d_1) \not\cong \mathcal{O}_X(d_2)$.
Hence conclude that we have an injection of groups $\mathbb{Z} \hookrightarrow$ Pic $\mathbb{P}^n_k$.

Proof. *The dimension of global section is a polynomial of d. i.e. the global section of $\{\mathcal{O}_X(d)\}$ is different for all $d \geq 0$. Secondly, $\mathcal{O}_X(-d)$ is the dual of $\mathcal{O}_X(d)$, so $\{\mathcal{O}_X(d)\}$ is different for all $d \leq 0$. Finally, because there is no global section when $d < 0$, $\mathcal{O}_X(d_1) \not\cong \mathcal{O}_X(d_2)$ if $d_1 \cdot d_2 < 0$. Thus, if $d_1 \neq d_2$, then $\mathcal{O}_X(d_1) \not\cong \mathcal{O}_X(d_2)$.*

Theorem 4. *If $X = \mathbb{P}^n_k$ for some field k, then every invertible sheaf on X is isomorphic to $\mathcal{O}_X(l)$ for some $l \in \mathbb{Z}$.*

Proof. *Using theorem 3 and proposition 7, there exists Pic $X \cong$ Cl $X \cong \mathbb{Z}$. And the map is defined by*

$$Cl\ X \to \{\Gamma(X, \mathcal{O}_X(l))\}_{l \in \mathbb{Z}} \to \{\mathcal{O}_X(l)\}_{l \in \mathbb{Z}} :$$
$$dH = dD_+(x_0) \mapsto x_0^d \mapsto \mathcal{O}_X(d).$$

Here we see the generator $H \in$ Cl X is mapped to $\mathcal{O}_X(1)$ and this map is a monomorphism. Thus, Pic $X = \{\mathcal{O}_X(l)\}_{l \in \mathbb{Z}}$. i.e., every invertible sheaf on X is isomorphic to $\mathcal{O}_X(l)$ for some $l \in \mathbb{Z}$.

4.3 Projective scheme over a noetherian ring

Let X be a projective scheme with coordinates ring $S = \oplus_{i\in\mathbb{N}} S_i$ and $S_0 = A$ a noetherian ring. This subsection is going to prove:

Theorem 5 (CaCl $X \cong$ Pic X)**.** *If X is a projective scheme over a noetherian ring A, then CaCl X and Pic X are naturally isomorphic.*

Let us use some homology language to simplify the proof. (The core of proof does not use cohomology). Using above notation, let X be a projective scheme defined over a noetherian ring A and let $\mathcal{O}_X$ be the structure sheaf of X. Let $\mathcal{K}_X$ be as before the sheaf of total quotient rings of $\mathcal{O}_X$. Let us denote by $\mathcal{O}_X^*$, $\mathcal{K}_X^*$ the multiplicative groups of units in $\mathcal{O}_X$ and $\mathcal{K}_X$, respectively. Then we have an exact sequence

$$0 \to \mathcal{O}_X^* \to \mathcal{K}_X^* \to \mathcal{K}_X^*/\mathcal{O}_X^* \to 0.$$

This short exact sequence induces a long exact sequence

$$0 \to H^0(X, \mathcal{O}_X^*) \to H^0(X, \mathcal{K}_X^*) \to H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \xrightarrow{\delta} H^1(X, \mathcal{O}_X^*) \to H^1(X, \mathcal{K}_X^*).$$

According to homology algebra, there is $div_C\ X = \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \cong H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. Moreover, $Pic\ X \cong H^1(X, \mathcal{O}_X^*)$. In proposition 5, we have known $CaCl\ X \hookrightarrow Pic\ X$ on every arbitrary scheme. As a result, it suffices to the connecting homomorphism δ is surjective when X is projective over a noetherian ring if we want to prove the theorem 5.

Definition 26 (irrelevant coordinate ring)**.** *Let X be a projective scheme over a noetherian ring A and let S be a homogeneous coordinate ring of X, i.e., $X = Proj\ S$. We shall say that S is an **irrelevant coordinate ring** if any element of S_+ is a zero divisor of S.*

Proposition 11. *Let X be a projective scheme over a noetherian ring A. Then it is possible to find a non-irrelevant ring of homogeneous coordinates for X.*

Corollary 1. *Let X be a projective scheme over a noetherian ring A. Then it is possible to find an homogeneous coordinate ring for X with a non zero divisor homogeneous of degree one.*

Definition 27 (associated prime ideals)**.** *Let A be a ring and M an A-module. Define the set of **associated prime ideals of M**, $Ass(M)$ or $Ass_A(M)$, as*

$$Ass(M) := \{\mathfrak{p} \in SpecA | \mathfrak{p} = Ann(s) \text{ for some nonzero } s \in M\}.$$

Remark 6. *When A is noetherian, the associated primes has local property:*

> *If S is a multiplicative subset of A, then the associated primes of $S^{-1}M$ are precisely those associated primes of M that lie in Spec $S^{-1}A$. i.e.,*
>
> $$Ass_A(M) \cap \{\mathfrak{p} \in SpecA | S \cap \mathfrak{p} = \emptyset\} \leftrightarrow Ass_{S^{-1}A}(S^{-1}M),$$
>
> $$\mathfrak{p} \leftrightarrow S^{-1}\mathfrak{p}.$$

Remark 7. *Let A be a noetherian ring and M be a nonzero finitely generated A-module, then Ass(M) is finite.*

Definition 28 (associated points)**.** *Let X be a locally noetherian scheme. The **associated points**, Ass (X), are those points $\mathfrak{p} \in X$ such that, on any affine open set Spec A containing $\mathfrak{p}$, $\mathfrak{p}$ corresponds to an associated prime of A.*

Remark 8. *The good local property makes sure Ass (X) is well-defined:*

> *If $\mathfrak{p}$ has two affine open neighborhoods Spec A and Spec B (say corresponding to prime ideals $\mathfrak{p} \subset A$ and $\mathfrak{q} \subset B$ respectively), then $\mathfrak{p}$ corresponds to an associated prime of A if and only if it corresponds to an associated prime of $A_{\mathfrak{p}} = \mathcal{O}_{X,p} = B_{\mathfrak{q}}$ if and only if it corresponds to an associated prime of B, by remark 6.*

Definition 29 (schematically dense)**.** *Let X be a locally noetherian scheme. An open subset U of X is **schematically dense** if Ass $(X) \subset U$.*

Lemma 6 (existence of schematically dense)**.** *If X is a projective scheme and U is an open set of the form $D_+(f)$, U is schematically dense if f is a non zero divisor.*

The following context follows that idea of [Nakai], which further illustrate the proposition of sheaf associated to total quotient ring of $\mathcal{O}_X$, i.e., $\mathcal{K}_X$.

Proposition 12. *Let A be a noetherian ring and let K be the total quotient ring of A. Let us put $X =$ Spec A and $Y =$ Spec K. Let λ be a morphism $Y \to X$ induced by the injection map $A \to K$. Let U be an open subset of X of the form X_f, $f \in A$. Then we have $\Gamma(U, \mathcal{K}_X) = \Gamma(\lambda^{-1}, \mathcal{O}_Y)$ and $\Gamma(U, \mathcal{K}_X^*) = \Gamma(\lambda^{-1}(U), \mathcal{O}_Y^*)$.*

Proof. *K_f is the total quotient ring of A_f and $\lambda^{-1}(U)$ is isomorphic to Spec K_f. Hence $\Gamma(U, \mathcal{K}_X) = K_f = \Gamma(\lambda^{-1}(U), \mathcal{O}_Y^*)$.*

Now let X be a projective scheme over a noetherian ring A and let S be a homogeneous coordinate ring of X which is not irrelevant and let K be the function ring of X over A. Let us put $Y =$ Spec K. Then we can define a natural morphism

$$\lambda : Y \to X.$$

Proposition 13. *Using X and Y as above, let $U = D_+(f)$ be an open subset of X defined by a homogeneous element $f \in S$, we have $\Gamma(U, \mathcal{K}_X) = \Gamma(\lambda^{-1}(U), \mathcal{O}_Y)$ and $\Gamma(U, \mathcal{K}_X^*) = \Gamma(\lambda^{-1}(U), \mathcal{O}_Y^*)$.*

Proof. *The affine coordinate ring A of $U = D_+(f)$ is $S_{(f)}$. The total quotient ring of $S_{(f)}$ is the quotient ring of K with respect to the multiplicatively closed set $\{(f/g)^n, n = 0, 1, 2, \dots\}$, where g is an arbitrary fixed homogeneous element of S having the same degree as f, and that g is not a zero-divisor of S. Then we have*

$$\Gamma(U, \mathcal{K}_X) = \{\textit{total quotient ring of } S_{(f)}\} = K_{f/g}.$$

On the other hand, $\lambda^{-1}(U) =$ Spec $K_{f/g}$. Hence $\Gamma(U, \mathcal{K}_X) = \Gamma(\lambda^{-1}(U), \mathcal{O}_Y)$ and similarly $\Gamma(U, \mathcal{K}_X^*) = \Gamma(\lambda^{-1}(U), \mathcal{O}_Y^*)$.

Since $D_+(f)$ is a basis of open sets of X, the above proposition implies the equality of cohomology ring/group.

Theorem 6. *Let X, Y be as above. We have $H^q(X, \mathcal{K}_X) = H^q(Y, \mathcal{O}_Y)$ and $H^q(X, \mathcal{K}_X^*) = H^q(Y, \mathcal{O}_Y^*)$*

Proof. *Using Čech cohomology. Given any open covering* $\mathfrak{U}$ *there is*

$$\check{H}^p(\mathfrak{U}, \mathcal{K}_X) = h^p(C^\cdot(\mathfrak{U}, \mathcal{K}_X)) = h^p(C^\cdot(\lambda^{-1}(\mathfrak{U}), \mathcal{O}_Y)) = \check{H}^p(\lambda^{-1}(\mathfrak{U}), \mathcal{O}_Y),$$

so

$$H^p(X, \mathcal{K}_X) = \varinjlim_{\mathfrak{U}} \check{H}^p(\mathfrak{U}, \mathcal{K}_X) = \varinjlim_{\mathfrak{U}} \check{H}^p(\lambda^{-1}(\mathfrak{U}), \mathcal{O}_Y) = H^p(Y, \mathcal{O}_Y).$$

In similar, there is

$$H^q(X, \mathcal{K}_X^*) = H^q(Y, \mathcal{O}_Y^*)$$

Remark 9. *Theorem 6 is the corollary of proposition 13, which is not used in the proof of theorem 5.*

Proposition 14. *Let X be a projective scheme over a noetherian ring A and U a schematically dense open set of the form* $D_+(f)$ *where* f *is a homogeneous element of degree 1 non zero divisor. Then the canonic homomorphism*

$$\Gamma(D_+(g), \mathcal{K}_X) \to \Gamma(D_+(f) \cap D_+(g), \mathcal{K}_X)$$

is an isomorphism, for all homogeneous elements $g \in S_+$.

Proof. *Here* $Y := D_+(f) \cap D_+(g)$, *the map* $\lambda : D_+(f) \cap D_+(g) \to D_+(g)$ *is natural inclusion. Then use proposition 13:*

$$\begin{aligned}\Gamma(D_+(g), \mathcal{K}_X) &= \Gamma(\lambda^{-1}(D_+(g)), \mathcal{O}_Y) \\ &= \Gamma(\lambda^{-1}(D_+(f) \cap D_+(g)), \mathcal{O}_Y) = \Gamma(D_+(f) \cap D_+(g), \mathcal{K}_X).\end{aligned}$$

Theorem 7 (free basis)**.** *Given A a noetherian ring, X = Spec A, M a finitely generated B-module. Let*

$$\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n \in Spec\ A$$

be maximal ideals of A and assume that each $M \otimes A_{\mathfrak{m}_i}$ *is a free* $A_{\mathfrak{m}_i}$*-module of rank 1 (i = 1,2, ..., n). Then there exists an element f in M such that* $f \otimes 1$ *is a free base of* $M \otimes A_{\mathfrak{m}_i}$ *for each i.*

Proof. *Let* f_i *be an element of M such that* $f_i \otimes 1$ *is a base of* $M \otimes_A A_{\mathfrak{m}_i}$ *over* $A_{\mathfrak{m}_i}$. *We can assume without loss of generality that* f_i *is contained in* $\mathfrak{m}_j M (j \neq i)$ *since we can multiply* f_i *by an element not contained in* $\mathfrak{m}_i$ *without changing the property that* $f_i \otimes 1$ *is a free base of* $M \otimes A_{\mathfrak{m}_i}$. *Let us put* $f = f_1 + f_2 + \cdots + f_n$. *Then f will satisfy our requirement. In fact, we have* $f \equiv f_i(\mod \mathfrak{m}_i M)$ *hence* $M \otimes A_{\mathfrak{m}_i} = A_{\mathfrak{m}_i}(f \otimes 1) + \mathfrak{m}_i(M \otimes A_{\mathfrak{m}_i})$. *By our assumption* $M \otimes A_{\mathfrak{m}_i}$ *is a finite* $A_{\mathfrak{m}_i}$*-module and Nakayama's lemma, we have* $M \otimes A_{\mathfrak{m}_i} = A_{\mathfrak{m}_i}(f \otimes 1)$.

Proposition 15. *Let X = Spec A and assume that A is a noetherian affine ring. Then under the same assumptions and notations as in theorem 7, there exists an open subset U of X containing all* $\mathfrak{m}_\alpha$ *such that for any* $x \in U$, *we have* $M_x = M \otimes A_x = A_x(f \otimes 1)$. *i.e.,* $\mathcal{M}|_U = \widetilde{M}|_U$ *is a rank-1 free* $\mathcal{O}_U$*-module.*

Proof. *Continue our argument in theorem 7. Let* $M = Ag_1 + \cdots + Ag_t$. *Since there is* $f \in M$ *such that* $M \otimes A_{\mathfrak{m}_i} = A_{\mathfrak{m}_i}(f \otimes 1)$, *for each* $\alpha(\alpha = 1, \ldots, n)$ *and i (i = 1, 2, ..., t) there is an element* $s_{\alpha i}$ *in A not contained in* $\mathfrak{m}_\alpha$ *such that we have*

$$s_{\alpha i} g_i = a_i f,$$

where a_i are elements of A. i.e., $s_{\alpha i}g_i \in Af$.

Observe that given a subset T_i of A which satisfies $\forall t \in T_i$ there is $tg_i \in Af$, then T_i is a nonempty ideal of A. Thus, T_i is finitely generated. Let $s_{\alpha i}$ be added into the generators set of T_i, and define

$$s = \prod_i \{\textit{generators of } T_i \in A\}.$$

Due to $\prod_{\alpha,i} s_{\alpha i} \mid s$, we have $s \in \mathfrak{m}_\alpha$ for all α. i.e., D(s) is an open set containing all $\mathfrak{m}_\alpha$. For any $\mathfrak{p} \in D(s)$, there is $s \notin \mathfrak{p}$ and that $M_\mathfrak{p} = M_{\mathfrak{m}_i} \otimes_{A_{\mathfrak{m}_i}} A_\mathfrak{p} = A_{\mathfrak{m}_i}(f \otimes 1) \otimes_{A_{\mathfrak{m}_i}} A_\mathfrak{p} = A_\mathfrak{p}(f \otimes 1)$. Thus, $D(s)$ is the open set with the propositions we want.

Now we are going to prove theorem 5. Just as claim before, it suffices to prove the connecting homomorphism δ is surjective.

Theorem 8 (surjectivity of connecting homomorphism)**.** *Let X be a projective scheme over a noetherian ring A. Let $\mathcal{O}_X$ be the structure sheaf of X and let $\mathcal{K}_X$ be the sheaf of total quotient rings of $\mathcal{O}_X$. Let $\mathcal{O}_X^*$, $\mathcal{K}_X^*$ be the sheaves of multiplicative groups of units in $\mathcal{O}_X$ and $\mathcal{K}_X$, respectively. Then the connecting homomorphism*

$$\delta : H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \to H^1(X, \mathcal{O}_X^*)$$

is surjective.

Proof. *By Corollary 1, it is possible to find a coordinate ring S for X such that there exists a homogeneous element $y \in S_+$ of degree one non zero divisor. Then the open set $D_+(y)$ is schematically dense.*

Let $\mathcal{L}$ be an invertible sheaf on X. The sheaf $\mathcal{L}$ is coherent and, then $\mathcal{L}|_{D_+(y)}$ is also a coherent sheaf. If we call $M = \Gamma(D_+(y), \mathcal{L})$, then $\mathcal{L}|_{D_+(y)} \cong \widetilde{M}$. Moreover, M is a finitely generated $S_{(y)}$-module.

On the other hand, the remark 7 said Ass(M) is finite. So given $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n$ the maximal ideals of the set Ass (M) and $p_1, \ldots, p_n \in$Ass(X) the corresponding points of X, by the proposition 15 there is an affine open set $U_1 = D_+(s) \subset D_+(y)$, $s \in S_{(y)}$, such that Ass(X)$\subset U_1$. Moreover there exists an element $f_1 \in M$ with

$$\mathcal{L}_p = \widetilde{M}_p = M_p = M \otimes \mathcal{O}_{X,p} = \mathcal{O}_{X,p}(f_1 \otimes 1) \quad \forall p \in U_1.$$

i.e.,

$$\mathcal{L}|_{U_1} = (f_1 \otimes 1)\mathcal{O}_X|_{U_1} = f_1 \cdot \mathcal{O}_X|_{U_1},$$

so on the subset U_1, define (U_1, f_1^{-1}) as a part of Cartier divisor.

We still need to find the part of Cartier divisor outside U_1. Given an arbitrary point q_1 in $X - U_1$ and U_2 an affine open set of the form $D_+(g)$ such that $\mathcal{L}|_{U_2} \cong \mathcal{O}_X|_{U_2}$ with $q_1 \in U_2$ and $U_2 \nsubseteq X - U_1$. Let $f_2 \in \Gamma(U_2, \mathcal{L})$ such that corresponds to the one through the isomorphism $\mathcal{L}|_{U_2} \cong \mathcal{O}_X|_{U_2}$.

Since f_1 is not zero divisor, $\frac{f_2}{f_1} \in \Gamma(U_1 \cap U_2, K_X^*)$, Moreover, by proposition 14, we know $\Gamma(U_2, \mathcal{K}_X) = \Gamma(D_+(g), \mathcal{K}_X) \cong \Gamma(D_+(s) \cap D_+(g), \mathcal{K}_X) = \Gamma(U_1 \cap U_2, \mathcal{K}_X)$, so we can extend the section $\frac{f_2}{f_1} \in \Gamma(U_1 \cap U_2, K_X^*)$ to a section $a_2 \in \Gamma(U_2, \mathcal{K}_X^*)$.

Using the same argument and by quasicompacity, we find an affine open finite covering $\{U_2, \ldots, U_q\}$ of $X - U_1$ and section $\{a_2, \ldots, a_q\}$ of $\mathcal{K}_X^*$ in any of these open sets such that $a_i a_j^{-1} \in \Gamma(U_i \cap U_j, \mathcal{O}_X)$ is a unit $\forall i, j \in \{2, \ldots, q\}$. Finally the system $\{(U_1, f_1^{-1}), (U_2, a_2), \ldots, (U_q, a_q)\}$ defines a Cartier divisor on X such that $\mathcal{L}_X(D) \cong \mathcal{L}$. Thus, we prove the surjectivity.

Remark 10. *Using theorem 8, we can prove theorem 5, i.e., If X is a projective scheme over a noetherian ring A, then $CaCl\ X \cong Pic\ X$.*

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