

# Gromov-Witten theory and mirror symmetry

Jinghao Yu

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# Chapter 1

## Gromov-Witten invariants

Let  $\overline{\mathcal{M}}_{g,n}$  denote the moduli space (Deligne-Mumford stack) of genus  $g$  curves with  $n$  marked points, and let  $\overline{C}_{g,n}$  be the universal family of  $\overline{\mathcal{M}}_{g,n}$ . The dimension of  $\overline{\mathcal{M}}_{g,n}$  is  $3g - 3 + n$ . Let  $\overline{\mathcal{M}}_{g,n}(X, \beta)$  be the moduli space of  $[f : C \rightarrow X, 1, \dots, n]$ : a map from the genus  $g$  curve  $C$  to the variety  $X$  modulo the automorphism of  $C$ .

The virtual fundamental class of  $\overline{\mathcal{M}}_{g,n}(X, \beta)$  is a cohomological class in  $H^*(\overline{\mathcal{M}}_{g,n}(X, \beta); \mathbb{Q})$ . It plays the role as classical fundamental class in algebraic topology. The virtual dimension of  $\overline{\mathcal{M}}_{g,n}(X, \beta)$  is

$$\text{vdim} \overline{\mathcal{M}}_{g,n}(X, \beta) = \int_{\beta} c_1(T_X) + (\dim X - 3)(1 - g) + n.$$

Roughly speaking, Gromov-Witten invariants is the number of algebraic curves (resp. pseduo-holomorphic curves) of a algebraic variety  $X$  (resp. complex manifold) passing through specific subvarieties (resp. submanifolds) of fixed degree (homology class of the curves) and genus.

**Definition 1.0.1** (Gromov-Witten invariants). *Let  $\gamma_1, \dots, \gamma_n \in H^*(X; \mathbb{Q})$  and let  $\beta \in H^2(X; \mathbb{Q})$ . Let  $ev_i : [\overline{\mathcal{M}}_{g,n}(X, \beta)]^{vir} \rightarrow X$  be the  $i$ th evaluation map*

$$ev_i([f : C \rightarrow X, 1, \dots, n]) = f(i).$$

*The Gromov-Witten invariant of genus  $g$  degree  $\beta$  curves is*

$$\langle \gamma_1 \dots \gamma_n \rangle_{g,n,\beta} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} ev_1^*(\gamma_1) \cup \dots \cup ev_n^*(\gamma_n).$$

There is a generalization of Gromov-Witten, called gravitational descendent invariants

**Definition 1.0.2** (descendent invariants). *Let  $a_0, \dots, a_n \in \mathbb{N}$ , and let  $\psi_i$  be the  $i$ th tautological class of  $\overline{\mathcal{M}}_{g,n}(X, \beta)$ , then the descendent invariant are defined by*

$$\langle \tau_{a_1}(\gamma_1) \dots \tau_{a_n}(\gamma_n) \rangle_{g,n,\beta} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} e\nu_1^*(\gamma_1)\psi_1^{a_1} \cup \dots \cup e\nu_n^*(\gamma_n)\psi_n^{a_n}.$$

## 1.1 String, dilaton, divisor equations

Here are several fundamental relations among these invariants.

I. The string equation. Let  $T_0 \in H^*(X; \mathbb{Q})$  be the unit:

$$\langle \tau_{a_1}(\gamma_1) \dots \tau_{a_n}(\gamma_n) T_0 \rangle_{g,\beta} = \sum_{i=1}^n \langle \tau_{a_1}(\gamma_1) \dots \tau_{a_{i-1}}(\gamma_{i-1}) \tau_{a_i-1}(\gamma_i) \tau_{a_{i+1}}(\gamma_{i+1}) \dots \tau_{a_n}(\gamma_n) \rangle_{g,\beta}$$

II. The dilaton equation.

$$\langle \tau_{a_1}(\gamma_1) \dots \tau_{a_n}(\gamma_n) \tau_1(T_0) \rangle_{g,\beta} = (2g - 2 + n) \langle \tau_{a_1}(\gamma_1) \dots \tau_{a_n}(\gamma_n) \rangle_{g,\beta}.$$

III. divisor equation: Let  $\gamma \in H^2(X; \mathbb{Q})$ . Then

$$\begin{aligned} \langle \tau_{a_1}(\gamma_1) \dots \tau_{a_n}(\gamma_n) \gamma \rangle_{g,\beta} &= \left( \int_{\beta} \gamma \right) \langle \tau_{a_1}(\gamma_1) \dots \tau_{a_n}(\gamma_n) \rangle_{g,\beta} \\ &+ \sum_{i=1}^n \langle \tau_{a_1}(\gamma_1) \dots \tau_{a_{i-1}}(\gamma_{i-1}) \tau_{a_i-1}(\gamma_i \cup \gamma) \tau_{a_{i+1}}(\gamma_{i+1}) \dots \tau_{a_n}(\gamma_n) \rangle_{g,\beta} \end{aligned}$$

If  $X$  is a point, these equation specialized to be equations for  $\overline{\mathcal{M}}_{g,n}$ :

(i) string equation:

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{\beta_1} \dots \psi_n^{\beta_n} = \sum_{i=1}^n \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\beta_1} \dots \psi_i^{\beta_i-1} \dots \psi_n^{\beta_n}$$

(ii) dilaton equation:

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{\beta_1} \dots \psi_n^{\beta_n} \psi_{n+1} = (2g + 2 - n) \sum_{i=1}^n \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\beta_1} \dots \psi_i^{\beta_i-1} \dots \psi_n^{\beta_n}$$

if  $2g + 2 - n > 0$ .

**Example 1.1.1.** *The  $\psi$  class is trivial in  $\overline{\mathcal{M}}_{0,3}$  is trivial. By string equation, if  $\beta_1 + \dots + \beta_n = n - 3$ , then*

$$\int_{\overline{\mathcal{M}}_{0,n}} \psi_1^{\beta_1} \dots \psi_n^{\beta_n} = \binom{n-3}{\beta_1, \dots, \beta_n}$$

## 1.2 Kontsevich's approach

Atiyah-Bott localization formula tells us: if there is a torus action  $\mathbb{T} = (\mathbb{C}^*)^n$  on  $X$ , then the fixed points of torus action could tells us some properties of  $X$ .

By the classifying space theory,  $B\mathbb{T} = (\mathbb{C}P^\infty)^{\times n}$ , so  $H^*(B\mathbb{T}) = \mathbb{Q}[\lambda_1, \dots, \lambda_n]$ . Let  $\mathcal{R}_{\mathbb{T}} = \mathbb{Q}(\lambda_1, \dots, \lambda_n)$ . Let  $X_{\mathbb{T}} = E\mathbb{T} \times_{\mathbb{T}} X$ , the equivariant cohomology of  $X$  is defined by

$$H_{\mathbb{T}}^*(X) := H^*(X_{\mathbb{T}}) = H^*(E\mathbb{T} \times_{\mathbb{T}} X),$$

so naturally,  $H_{\mathbb{T}}^*(X)$  is a  $H_{\mathbb{T}}^*(pt) = H^*(B\mathbb{T})$ -module. The localization of  $H_{\mathbb{T}}^*(X)$  means  $H_{\mathbb{T}}^*(X) \otimes \mathcal{R}_{\mathbb{T}}$ .

**Theorem** (Atiyah-Bott). *Let  $X^{\mathbb{T}}$  be fixed locus of  $\mathbb{T}$ , let  $Z_j$  be a connection component of  $X^{\mathbb{T}}$ , and let  $N_j$  be the normal bundle of  $Z_j$  in  $X$ . Let  $i_j : Z_j \rightarrow X$  and let  $i_{j!} : H_{\mathbb{T}}^*(Z_j) \rightarrow H_{\mathbb{T}}^*(X)$  be the pushforward defined by the Gysin map. Let  $\alpha \in H_{\mathbb{T}}^*(X) \otimes \mathcal{R}_{\mathbb{T}}$ , we have*

$$\alpha = \sum_j \frac{i_{j!} i_j^* \alpha}{\text{Euler}_{\mathbb{T}}(N_j)},$$

In particular,

$$\int_{X_{\mathbb{T}}} \alpha = \sum_j \int_{(Z_j)_{\mathbb{T}}} \frac{i_j^* \alpha}{\text{Euler}_{\mathbb{T}}(N_j)}.$$

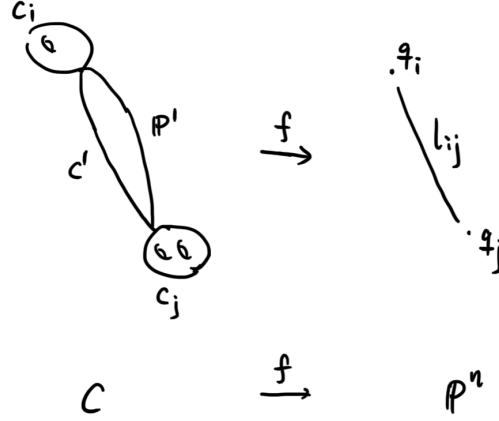
Kontsevich's approach is to apply Atiyah-Bott localization formula in  $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\text{vir}}$  so that we can simplify the computation. We can lift the  $\mathbb{T}$  action on  $X$  to  $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\text{vir}}$  in the following way: let  $t \in \mathbb{T}$ ,  $[f : C \rightarrow X, 1, \dots, n] \in [\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\text{vir}}$ ,  $x \in X$

$$(t \cdot f)(x) = f(t \cdot x).$$

In special, we will consider  $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$  in this section. As claimed before, we need to find  $[f : C \rightarrow X, 1, \dots, n] \in \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^{\mathbb{T}}$ . The fixed points of  $\mathbb{P}^r$  is

$$\{q_i = [0 : 0 : \dots : 1 : 0 : \dots : 0]\}_{0 \leq i \leq r}.$$

The coordinate curve  $l_{ij}$  connecting  $q_i, q_j$  has one dimensional degree of freedom  $\mathbb{C}^*$  (as an invariant component). The curve  $C \in \overline{\mathcal{C}}_{g,n}$  is stable (i.e.  $\text{Aut}(C) < \infty$ ) if and only if  $2g - 2 + n > 0$ . If a components  $C'$  of  $C$  is mapped to  $l_{ij}$ , then  $C'$  has two points mapped to  $q_i, q_j$  respectively (equivalent to with two marked points in


 Figure 1.1:  $f(C_i) = q_i$ ,  $f(C') = l_{ij}$ ,  $f(C_j) = q_j$ 

$C'$ ), so  $2g - 2 + 2 \leq 0$  implies  $g = 0$ , i.e.  $C' \cong \mathbb{P}^1$  (see Fig 1.1). Meanwhile,  $f|_{C'}$  must be uniformly ramified, so  $f|_{C'}(z) = z^e$ ,  $\forall z \in C' \cong \mathbb{P}^1$ , for some  $e \in \mathbb{N}^*$ .

It is convenient to use a decorated graph  $\vec{\Gamma} = (\Gamma, \vec{f}, \vec{d}, \vec{g}, \vec{s})$  (graph, maps, degrees, genus, marked points) to represent  $[f : C \rightarrow X, 1, \dots, n] \in \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^{\mathbb{T}}$ . Let  $\text{val}(v)$ , the valence of  $v$ , be the number of edges connecting vertex  $v$ , and let  $n(v) = |s_v| + \text{val}(v)$ . The stable map  $[f : C \rightarrow X, 1, \dots, n]$  with fixed graph  $\vec{\Gamma} = (\Gamma, \vec{f}, \vec{d}, \vec{g}, \vec{s})$  defines a substack

$$\overline{\mathcal{M}}_{\vec{\Gamma}} \subset \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d).$$

On the other sides, consider

$$\varphi_{\vec{\Gamma}} : \prod_{\dim C_v=1} \overline{\mathcal{M}}_{g_v, n(v)} \rightarrow \overline{\mathcal{M}}_{\vec{\Gamma}}.$$

If  $v, v'$  are connected by an edge  $e$ , then let  $C_v, C_{v'}$  connected by a  $C_e \cong \mathbb{P}^1$  associated with a degree  $d_e$  map to  $\mathbb{P}^r$ . Let  $\overline{\mathcal{M}}_{\Gamma}$  be the product of above  $C_v, C_e$ . There is a group  $\mathbb{A}_{\Gamma}$  acting on  $\overline{\mathcal{M}}_{\Gamma}$ . The group  $\mathbb{A}_{\Gamma}$  is defined by:

$$1 \rightarrow \prod_{\text{edges}} \mathbb{Z}/(d_e) \rightarrow \mathbb{A}_{\Gamma} \rightarrow \text{Aut}(\Gamma) \rightarrow 1.$$

and

$$\overline{\mathcal{M}}_{\vec{\Gamma}} = \overline{\mathcal{M}}_{\Gamma} / \mathbb{A}_{\Gamma}.$$

Therefore, we know the  $\mathbb{T}$ -fixed locus of  $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$  is  $\overline{\mathcal{M}}_{\vec{\Gamma}}$ . Let  $N_{\Gamma}$  be the normal bundle of  $\overline{\mathcal{M}}_{\vec{\Gamma}}$  in  $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ . Then there is an explicit formula for the equivariant Euler class. Before doing that, we define some necessary notations. A flag  $F$  is a pair  $(v, e)$  such that  $e$  is an edge containing the vertex  $v$ . We put  $i(F) = v$ ,  $j(F)$  the vertex of  $e$  different from  $v$ . Let

$$\omega_F = \frac{\lambda_{i(F)} - \lambda_{j(F)}}{d_e} \in H_{\mathbb{T}}^2(pt) = \mathbb{Q}[\lambda_0, \dots, \lambda_r],$$

which is the weight of  $\mathbb{T}$ -action on  $T_{q_{i_v}} C_e$ .

**Theorem 1.2.1** ( $Euler_{\mathbb{T}}(N_{\Gamma})$ ).  $Euler_{\mathbb{T}}(N_{\Gamma}) = e_{\Gamma}^F e_{\Gamma}^v e_{\Gamma}^e$ , where

$$\begin{aligned} e_{\Gamma}^F &= \prod_{n(i(F)) \geq 3} (\omega_F - \psi_F) / \prod_{j \neq i(F)} (\lambda_{i(F)} - \lambda_j), \\ e_{\Gamma}^v &= \prod_v \prod_{j \neq i_v} (\lambda_{i_v} - \lambda_j) \prod_{val(v)=2, s_v=\emptyset} (\omega_{F_1(v)} + \omega_{F_2(v)}) / \prod_{val(v)=1, s_v=\emptyset} \omega_{F(v)} \\ e_{\Gamma}^e &= \prod_e \frac{(-1)^{d_e} (d_e!)^2 (\lambda_i - \lambda_j)^{2d_e}}{d_e^{2d_e}} \prod_{a+b=d_e, k \neq i, j} \left( \frac{a\lambda_i + b\lambda_j}{d_e} - \lambda_k \right) \end{aligned}$$

The proof is partially discussed in section 1.2.

### 1.3 Tangent-obstruction sequence

Consider  $[f : C \rightarrow X, 1, \dots, n] \in \overline{\mathcal{M}}_{\vec{\Gamma}}$ ,  $\vec{\Gamma} = (\Gamma, \vec{f}, \vec{d}, \vec{g}, \vec{s})$ . We put

$$\begin{aligned} V^1(\Gamma) &:= \{v \in V(\Gamma) : g_v = 0, val(v) = 1, |s_v| = 0\} \\ V^2(\Gamma) &:= \{v \in V(\Gamma) : g_v = 0, val(v) = 2, |s_v| = 0\} \\ V^{1,1}(\Gamma) &:= \{v \in V(\Gamma) : g_v = 0, val(v) = 1, |s_v| = 1\} \\ V^s(\Gamma) &:= \{v \in V(\Gamma) : 2g_v - 2 + val(v) + |s_v| > 0\} \\ y(v, e) &:= C_e \cap C_v \end{aligned}$$

The tangent-obstruction sequence is

$$\begin{aligned} &0 \rightarrow Aut(C, 1, \dots, n) \\ &\rightarrow Def(f) \rightarrow Def(C, 1, \dots, n, f) \rightarrow Def(C, 1, \dots, n) \\ &\rightarrow Ob(f) \rightarrow Ob(C, 1, \dots, n, f) \rightarrow 0, \end{aligned}$$

$$\begin{aligned}
 0 &\rightarrow \text{Hom}(\Omega_C(p_1 + \cdots + p_n), \mathcal{O}_C) \\
 &\rightarrow H^0(C, f^*T_X) \rightarrow T^1 \rightarrow \text{Ext}^1(\Omega_C(p_1 + \cdots + p_n), \mathcal{O}_C) \\
 &\rightarrow H^1(C, f^*T_X) \rightarrow T^2 \rightarrow 0.
 \end{aligned}$$

For simplicity:

$$0 \rightarrow B_1 \rightarrow B_2 \rightarrow T^1 \rightarrow B_4 \rightarrow B_5 \rightarrow T^2 \rightarrow 0.$$

The  $N^{\text{vir}} = T^{1,m} - T^{2,m}$  (m means moving part).

$$Euler_{\mathbb{T}}(N^{\text{vir}}) = \frac{Euler_{\mathbb{T}}(B_2^m) Euler_{\mathbb{T}}(B_4^m)}{Euler_{\mathbb{T}}(B_1^m) Euler_{\mathbb{T}}(B_5^m)}.$$

(1)  $Euler_{\mathbb{T}}(B_2^m)/Euler_{\mathbb{T}}(B_5^m)$ . The normalization sequence of  $C$  is:

$$\begin{aligned}
 0 &\rightarrow \mathcal{O}_C \rightarrow \bigoplus_{v \in V^s(\Gamma)} \mathcal{O}_{C_v} \oplus \bigoplus_{e \in E(\Gamma)} \mathcal{O}_{C_e} \\
 &\rightarrow \bigoplus_{v \in V^2(\Gamma)} \mathcal{O}_{y_v} \oplus \bigoplus_{(e,v) \in F^s(\Gamma)} \mathcal{O}_{y(e,v)} \rightarrow 0.
 \end{aligned}$$

Take  $\otimes f^*T_X$ :

$$\begin{aligned}
 0 &\rightarrow H^0(C, f^*T_X) \rightarrow \bigoplus_{v \in V^s(\Gamma)} H^0(C_v, f^*T_X) \oplus \bigoplus_{e \in E(\Gamma)} H^0(C_e, f^*T_X) \\
 &\rightarrow \bigoplus_{v \in V^2(\Gamma)} T_{f(y_v)}X \oplus \bigoplus_{(e,v) \in F^s(\Gamma)} T_{f(y(e,v))}X \\
 &\rightarrow H^1(C, f^*T_X) \rightarrow \bigoplus_{v \in V^s(\Gamma)} H^1(C_v, f^*T_X) \oplus \bigoplus_{e \in E(\Gamma)} H^1(C_e, f^*T_X) \rightarrow 0.
 \end{aligned}$$

$$H^0(C_v, f^*T_X) = T_{f(C_v)}X,$$

$$H^1(C_v, f^*T_X) = H^1(C_v, \mathcal{O}_{C_v}) \otimes T_{f(C_v)}X \cong H^0(C_v, \omega_{C_v})^\vee \otimes T_{f(C_v)}X$$

Here  $H^0(C_v, \omega_{C_v})$  is Hodge bundle  $\mathbb{E}$ . By splitting principle, assume  $\mathbb{E} = L_1 \oplus \cdots \oplus L_g$ , then

$$\begin{aligned}
 e(\mathbb{E}^\vee \otimes \mathbb{C}_1) &= \prod_{i=1}^g c_1(L_i^\vee \otimes \mathbb{C}_1) = \prod_{i=1}^g c_1(L_i^\vee) + c_1(\mathbb{C}_1) \\
 &= \prod_{i=1}^g (-c_1(L_i) + u) = \sum_{k=1}^g (-1)^k c_k(\mathbb{E}) u^{g-k} = \sum_{k=1}^g (-1)^k \lambda_k u^{g-k} =: \Lambda_g^\vee(u)
 \end{aligned}$$



(2)  $Euler_{\mathbb{T}}(B_4^m)/Euler_{\mathbb{T}}(B_1^m)$ .

(2.1)  $B_1 = Aut(C, 1, \dots, n) = Hom(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C)$ : We should classify what is moving and what is fixed. Basically, we have

$$B_1^m = \bigoplus_{v \in V^1(\Gamma), (e,v) \in F(\Gamma)} T_{y(e,v)} C_e.$$

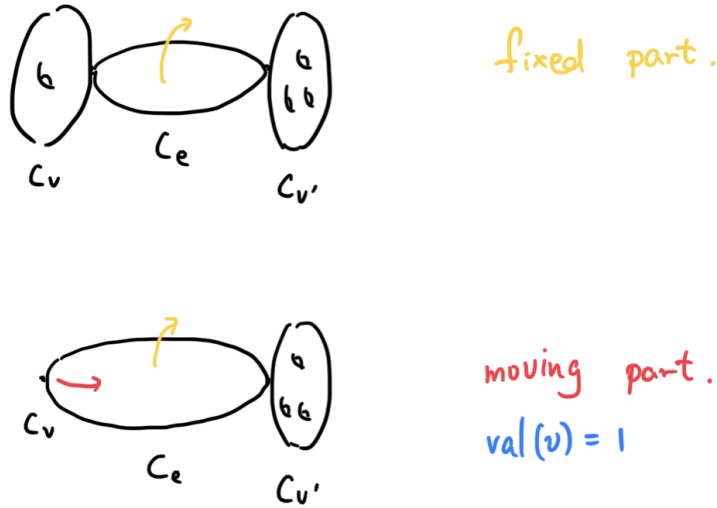


Figure 1.2: automorphism of  $(C, 1, \dots, n)$

(2.2)  $B_4 = Def(C, 1, \dots, n) = Ext^1(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C)$ :  $\mathbb{P}^1$  has just 1 complex structure, so we consider  $g(C) \geq 1$ . If we don't change node  $q$ ,  $C$  will

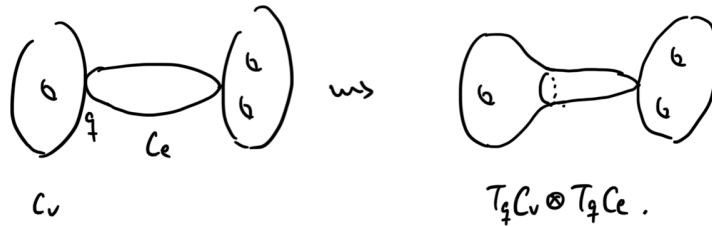


Figure 1.3: deformation of  $(C, 1, \dots, n)$

stay in the same class in  $\overline{\mathcal{M}}_{g,n}$ . Hence we must resolve the node, and geometrically, resolution depends on  $T_q C_v \otimes T_q C_e$ . So basically we have

$$B_4^m = \bigoplus_{v \in V^2(\Gamma), E_v = (e, e')} T_{y_v} C_e \otimes T_{y_v} C_{e'} \oplus \bigoplus_{(e, v) \in F^s(\Gamma)} T_{y(e, v)} C_v \otimes T_{y(e, v)} C_e$$

Returning to the special case  $X = \mathbb{P}^r$ , we can get the theorem 1.1.1.

## 1.4 Aspinwall Morrison formula; Faber Pandaripande formula

In this section, we will use Kontsevich's approach to compute the multiple cover contribution of rigidly embedding curves  $\mathbb{P}^1$  in a Calabi-Yau threefold  $X$ .

The geometry picture is this. The normal bundle  $N$  of  $\mathbb{P}^1 \subset X$  is rank 2 and splits on  $\mathbb{P}^1$ . Because  $X$  is Calabi-Yau and  $c_1(\mathbb{P}^1) = 2$ , the normal bundle is of degree 2. Embedded  $\mathbb{P}^1$ 's in a Calabi-Yau threefold (not necessary lines) with normal bundle  $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$  are called rigid. The degree 2 Gromov-Witten invariant of a generic quintic has two contributions:

- (1) rigid conics curves in  $X$ ;
- (2) lines with double cover, so this part is related to  $\overline{\mathcal{M}}_0(\mathbb{P}^1, 2)$ .

We want to compute the contribution of part (2). This problem finally leads to:

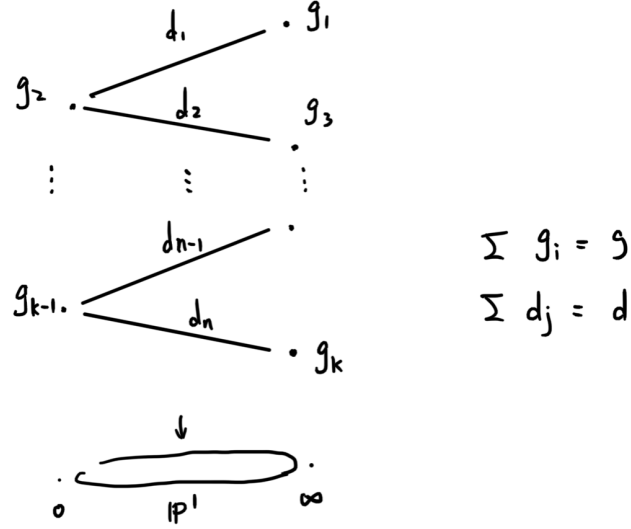
$$N_{g,d} = \int_{\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)} e(R^1 \pi_* f^* N),$$

$$\begin{array}{ccc} \overline{\mathcal{C}}_{g,0}(\mathbb{P}^1, d) & \xrightarrow{f} & \mathbb{P}^1 \\ \downarrow \pi & & \\ \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d) & & \end{array} \quad \text{and } N = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

where

The decorated graphs  $\vec{\Gamma} = (\Gamma, \vec{f}, \vec{d}, \vec{g}, \vec{s})$  in  $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)^{\mathbb{T}}$  are of the type in Figure 1.4. We can choose different lifts on  $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$  so that only  $\vec{\Gamma} = (\Gamma, \vec{f}, \vec{d}, \vec{g}, \vec{s})$  with 1 edge contributing  $N_{g,d}$ .

- (1)  $g = 0$  (Aspinwall Morrison formula):  $N_{0,d} = 1/d^3$ ;

Figure 1.4:  $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)^{\mathbb{T}}$ 

(2)  $g \geq 1$  (Faber-Pandharipande):

$$\begin{aligned}
 N_{g,d} &= \sum_{g_1+g_2=g} \frac{1}{d} \int_{\overline{\mathcal{M}}_{g_1,1}} \lambda_{g_1} \psi^{2g_1-2} d^{2g_1-1} \\
 &\quad \times \int_{\overline{\mathcal{M}}_{g_2,1}} \lambda_{g_2} \psi^{2g_2-2} d^{2g_2-1} = \sum_{g_1+g_2=g} b_{g_1} b_{g_2} d^{2g-3} \\
 b_0 &= 0; \quad b_g = \int_{\overline{\mathcal{M}}_{g,1}} \lambda_{g_2} \psi^{2g-2} \quad (g > 0) \\
 \sum_{g=0}^{\infty} b_g t^{2g} &= \frac{t/2}{\sin t/2}.
 \end{aligned}$$

Then use the Laurent series of  $\cot t$ , we have

$$N_{1,d} = \frac{1}{12d},$$

$$N_{g,d} = d^{2g-3} \frac{|B_{2g}|}{2g \cdot (2g-2)!} = |\chi(\overline{\mathcal{M}}_g)| \frac{d^{2g-3}}{(2g-3)!}, \quad g \geq 2,$$

where  $B_g$  is the Bernoulli number in  $\frac{x}{e^x-1}$ .

# Chapter 2

## Quantum Cohomology

### 2.1 quantum product

The quantum cohomology is a variation of classical cohomology. Let  $T_0 = 1, T_1, \dots, T_p, T_{p+1}, \dots, T_m \in H^*(X)$  be a basis of  $H^*(X)$  as a  $\mathbb{Q}$ -vector space ( $T_1, \dots, T_p \in H^2(X)$ ). Let  $\beta \in H_2(X)$ ,  $\gamma = \sum_{i=0}^m t_i T_i$ , we define quantum potential as

$$\begin{aligned} F_0^X(t_0, \dots, t_m) &= \sum_{n, \beta} \frac{1}{n!} \langle \gamma^n \rangle_{0, n, \beta}^X Q^\beta \\ &= \frac{1}{6} \int_X \left( \sum_{i=0}^m t_i T_i \right)^3 + \sum_{\beta=0, n \geq 4} \langle T_0^{n_0} \dots T_m^{n_m} \rangle_{0, n, 0} \prod_{i=1}^m \frac{t_i^{n_i}}{n_i!} \\ &\quad + \sum_{\beta > 0, n} Q^\beta \langle T_0^{n_0} \dots T_m^{n_m} \rangle_{0, n, \beta} \prod_{i=p+1}^m \frac{t_i^{n_i}}{n_i!}. \end{aligned}$$

By string equations and divisor equations,

$$\begin{aligned} F_0^X(t_0, \dots, t_m) &= \frac{1}{6} \int_X \left( \sum_{i=0}^m t_i T_i \right)^3 + \sum_{\beta=0, n \geq 4} \langle T_1^{n_1} \dots T_m^{n_m} \rangle_{0, n, 0} \prod_{i=1}^m \frac{t_i^{n_i}}{n_i!} \\ &\quad + \sum_{\beta > 0, n} Q^\beta q_1^{\int_\beta T_1} \dots q_p^{\int_\beta T_p} \langle T_{p+1}^{n_{p+1}} \dots T_m^{n_m} \rangle_{0, n, \beta} \prod_{i=p+1}^m \frac{t_i^{n_i}}{n_i!}, \end{aligned}$$

where  $q_i = e^{t_i}$ .

$$F_{ijk} := \frac{\partial^3 F_0^X}{\partial t_i \partial t_j \partial t_k} = \sum_{n, \beta} \frac{1}{n!} \langle T_i T_j T_k \gamma^n \rangle_{0, n+3, \beta}^X Q^\beta$$

$$\begin{aligned}
&= \int_X T_i T_j T_k + \sum_{\beta=0, n \geq 1} \langle T_i T_j T_k T_1^{n_1} \dots T_m^{n_m} \rangle_{0, n+3, 0} \prod_{i=1}^m \frac{t_i^{n_i}}{n_i!} \\
&+ \sum_{\beta > 0, n} Q^\beta q_1^{\int_\beta T_1} \dots q_p^{\int_\beta T_p} \langle T_i T_j T_k T_{p+1}^{n_{p+1}} \dots T_m^{n_m} \rangle_{0, n+3, \beta} \prod_{i=p+1}^m \frac{t_i^{n_i}}{n_i!}, \quad q_i = e^{t_i}.
\end{aligned}$$

Let  $g_{ij} = (T_i, T_j)$  means the Poincare pair of  $T_i, T_j$ . The big quantum product is defined as

$$(T_i *_t T_j, T_k) := F_{ijk},$$

in other words,

$$T_i *_t T_j = \sum_{e, f} F_{ije} g^{ef} T_f.$$

It is known that the quantum product is a generalization of intersection theory: given  $T_i, T_j, T_k$ , they contribute to the quantum product if there exists  $\mathbb{P}^1$  touching their Poincare dual classes at the same time. Extend the  $t_i$  in quantum multiplication linearly, then the  $\mathbb{Q}[[t_0, \dots, t_m]]$ -module  $H^*(X) \otimes_{\mathbb{Q}} \mathbb{Q}[[t_0, \dots, t_m]]$  is the big quantum cohomology  $QH(X)$ .

The associativity of quantum product is formulated as WDVV equation:

$$F_{ija} g^{ab} F_{bkl} = F_{ila} g^{ab} F_{bjk}.$$

It is proved by a forgetful map  $\pi : \overline{\mathcal{M}}_{0,4}(X, \beta) \rightarrow \overline{\mathcal{M}}_{0,4}$ . One should notice that  $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$ , so the boundary divisor  $D(12|34) \sim D(13|24)$  and

$$\begin{aligned}
&\int_{[\overline{\mathcal{M}}_{0,4}(X, \beta)]^{vir} \cap \pi^* D(12|34)} ev_1^*(T_i) ev_2^*(T_j) ev_3^*(T_k) ev_4^*(T_l) \prod_{i=5}^{n+4} ev_i^*(\gamma) \\
&= \int_{[\overline{\mathcal{M}}_{0,4}(X, \beta)]^{vir} \cap \pi^* D(13|24)} ev_1^*(T_i) ev_2^*(T_j) ev_3^*(T_k) ev_4^*(T_l) \prod_{i=5}^{n+4} ev_i^*(\gamma).
\end{aligned}$$

A useful trick is to separate  $[\overline{\mathcal{M}}_{0,4}(X, \beta)]^{vir} \cap \pi^* D(12|34)$  by

$$\coprod_{n_1+n_2=n, \beta_1+\beta_2=\beta} [\overline{\mathcal{M}}_{0, n_1+3}(X, \beta_1) \times \overline{\mathcal{M}}_{0, n_2+3}(X, \beta_2)]^{vir} \cap (ev \times ev)^*[\Delta],$$

$$PD[\Delta] = g^{ab} T_a \otimes T_b,$$

then we get

$$\begin{aligned} & \sum_{n_1+n_2=n} \sum_{\beta_1+\beta_2=\beta} \langle T_i T_j T_a \gamma^{n_1} \rangle_{0,n_1+3,\beta_1} g^{ab} \langle T_b T_k T_l \gamma^{n_2} \rangle_{0,n_2+3,\beta_2} \\ &= \sum_{n_1+n_2=n} \sum_{\beta_1+\beta_2=\beta} \langle T_i T_k T_a \gamma^{n_1} \rangle_{0,n_1+3,\beta_1} g^{ab} \langle T_b T_j T_l \gamma^{n_2} \rangle_{0,n_2+3,\beta_2}. \end{aligned}$$

This is the essential part in the proof of associativity of quantum product.

**Remark 2.1.1.** *It deserves to notice that the quantum product is defined by rational curves, so its usage mainly concentrates in genus 0 GW-invariants. The difficulty to define quantum product via higher genus curves is that there is no so good associativity as the genus 0 case. It must be a good work if we can find a way to give a quantum product via higher genus curves with associativity like now.*

The small quantum product is defined by

$$T_i *_s T_j = T_i *_t T_j|_{t_{p+1}=\dots=t_m=0}, 0 \leq i, j \leq m.$$

Precisely, let

$$\bar{F}_{ijk} = F_{ijk}|_{t_{p+1}=\dots=t_m=0} = \int_X T_i T_j T_k + \sum_{\beta > 0} Q^\beta q_1^{\int_\beta T_1} \dots q_p^{\int_\beta T_p} \langle T_i T_j T_k \rangle_{0,3,\beta},$$

then

$$T_i *_s T_j = \bar{F}_{ije} g^{ef} T_f, \quad 1 \leq e, f \leq m.$$

Extend  $q_i$  linearly, the  $\mathbb{Q}[[q_1, \dots, q_p]]$ -module  $H^*(X) \otimes_{\mathbb{Q}} \mathbb{Q}[[q_1, \dots, q_p]]$  is defined as the small quantum cohomology  $QH^s(X)$ .

**Example 2.1.2.**  $QH^s(\mathbb{P}^m) = \mathbb{Q}[H, q]/(H^{m+1} - q)$ , where  $H \in H^2(\mathbb{P}^m, \mathbb{Q})$ ,  $q = e^{t_1}$ .

## 2.2 quantum differential equation

We can view the vector space  $H(X)$  as a Riemannian manifold  $M$  with standard flat metric  $g_{ij}$  given by Poincare pairing. The quantum product  $*_t$  could be use to define a connection (called Dubrovin connection, or Givental connection)  $\nabla^z$ , which is different from the Levi-Civita connection induced by its Riemannian metric.

**Definition 2.2.1.** (*Dubrovnik connection*) Let  $X, Y \in \Gamma(M, TM)$ ,  $\nabla$  be the Levi-Civita connection w.r.t  $g$ . The Dubrovnik connection  $\nabla^z$  is defined by

$$\nabla_X^z Y := \nabla_X Y - \frac{1}{z} X *_t Y.$$

The WDVV equation shows  $\nabla^z$  is flat. i.e.  $Rm^z = 0$ .

**Definition 2.2.2** (quantum differential equation). Let  $\sigma \in \Gamma(M, TM)$ , the equation  $\nabla^z \sigma = 0$  is the quantum differential equation. The fundamental solution of quantum differential equation is  $(m+1) \times (m+1)$  matrix  $s(z, t)$  ( $t = (t_0, \dots, t_m)$ ) =  $(a_{ij})$ , such that each column defines a solution

$$\sigma_j(t) = \sum_{i=0}^m a_{ij}(t) \frac{\partial}{\partial t_i}.$$

Now we want to find the solution of quantum differential equation. Let  $(S(z)T_a, T_b) = g_{ab} + \langle \frac{T_a}{z - \psi_1}, T_b \rangle_{0,2}$ , where

$$\langle \frac{T_a}{z - \psi_1}, T_b \rangle_{0,2} = \sum_{\substack{n \geq 0, \beta, \\ (n, \beta) \neq (0,0)}} \frac{1}{n!} \langle \frac{T_a}{z - \psi_1} T_b \gamma^n \rangle_{0, n+2, \beta}.$$

**Proposition 2.2.3.** The  $S_a = (S(z)T_a, T_b)g^{bc}\partial_c$  is a flat section with respect to  $\nabla^z$ .

This proposition is proven with the help of topological recursion relation.

**Theorem 2.2.4** (topological recursion relation). Let  $\gamma_i \in H^*(X)$ ,

$$\begin{aligned} & \langle \tau_{a_1+1}(\gamma_1) \tau_{a_2}(\gamma_2) \tau_{a_3}(\gamma_3) \prod_{i=4}^n \tau_{a_i}(\gamma_i) \rangle_{0, n, \beta} \\ &= \sum_{\substack{A \cup B = \{4, \dots, n\} \\ \beta = \beta_1 + \beta_2}} \sum_{a, b=0}^m \langle \tau_{a_1}(\gamma_1) \prod_{i \in A} \tau_{a_i}(\gamma_i) T_a \rangle_{0, |A|+2, \beta_1} g^{ab} \langle T_b \tau_{a_2}(\gamma_2) \tau_{a_3}(\gamma_3) \prod_{j \in B} \tau_{a_j}(\gamma_j) T_a \rangle_{0, |B|+3, \beta_2} \end{aligned}$$

*Proof.* Consider the forgetful map

$$\pi : \overline{\mathcal{M}}_{0,n}(X, \beta) \rightarrow \overline{\mathcal{M}}_{0,3} :$$

$$[f : C \rightarrow X, 1, \dots, n] \mapsto [C, 1, 2, 3].$$

Let  $\mathbb{L}_1, \mathbb{L}'_1$  be the tautological line bundles

$$\begin{array}{ccc} \mathbb{L}_1 & & \mathbb{L}'_1 \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{0,n}(X, \beta) & \xrightarrow{\pi} & \overline{\mathcal{M}}_{0,3} \end{array}$$

There is

$$\begin{array}{ccc} \mathbb{L}_1 \cong \pi^* \mathbb{L}'_1 \otimes \left( \sum_{\substack{\beta_1 + \beta_2 = \beta \\ A \cup B = \{4, \dots, n\}}} D(1, A, \beta_1 | 2, 3, B, \beta_2) \right), & & \\ D(1, A, \beta_1 | 2, 3, B, \beta_2) & \longrightarrow & \overline{\mathcal{M}}_{0,|B|+3}(X, \beta_2) \\ \downarrow & & \downarrow \text{ev}_{node} \\ \overline{\mathcal{M}}_{0,|A|+2}(X, \beta_1) & \xrightarrow{\text{ev}_{node}} & X. \end{array}$$

Because  $\overline{\mathcal{M}}_{0,3}$  is a point,  $\mathbb{L}'_1$  is trivial and

$$\psi_1 = \sum_{\substack{\beta_1 + \beta_2 = \beta \\ A \cup B = \{4, \dots, n\}}} [D(1, A, \beta_1 | 2, 3, B, \beta_2)].$$

Take this formula into LHS, we get the recursion relation.  $\square$

The fundamental solution of small quantum differential equation directly relates to the definition of J-function in mirror symmetry. It is given by

$$\tilde{S}(z) = S(z)|_{t_{p+1}, \dots, t_m = 0}.$$

Specifically, let  $\gamma = \sum_{i=0}^p t_i T_i = t_0 T_0 + \gamma_1$  and  $\gamma_1 = \sum_{i=1}^p t_i T_i$ , then

$$(\tilde{S}(z) T_a, T_b) = g_{ab} + \sum_{\substack{n \geq 0, \beta \\ (n, \beta) \neq (0, 0)}} \frac{1}{n!} \left\langle \frac{T_a}{z - \psi_1} T_b \gamma^n \right\rangle_{0, n+2, \beta}.$$

By string equations and divisor equations,  $\gamma$  can be put out of the bracket, and finally we get

$$(\tilde{S}(z) T_a, T_b) = \int_X e^{\gamma/z} T_a T_b + \sum_{\beta > 0} \left\langle \frac{e^{\gamma/z} T_a}{z - \psi_1} T_b \right\rangle_{0, 2, \beta} e^{\int_{\beta} \gamma_1}.$$



# Chapter 3

## Mirror Symmetry

I plan to follow Givental's approach to give a proof of genus 0 mirror symmetry of hypersurfaces in  $\mathbb{P}^n$ . The key character of Givental's approach is that it uses J-function and I-function to show the mirror symmetry relation. The J-function is defined as follows, which describes the A-model information.

**Definition 3.0.1.** For a complex manifold  $X$ , the  $J^X$  is

$$(T_a, J^X) := (\tilde{S}(z)T_a, 1) = \int_X e^{\gamma/z} T_a + \sum_{\beta > 0} \langle \frac{e^{\gamma/z} T_a}{z - \psi_1} 1 \rangle_{0,2,\beta} e^{\int_{\beta} \gamma_1}.$$

$J^X$  is a  $H^*(X)$ -value function:

$$\begin{aligned} J^X(t_0, t_1, \dots, t_p, z) &= (T_a, J^X) g^{ab} T_b \\ &= e^{(t_0 + \gamma_1)/z} \left( 1 + \sum_{\beta \neq 0} \sum_{a=0}^m q^\beta \langle \frac{T_a}{z - \psi_1} 1 \rangle_{0,\beta} T^a \right), \end{aligned}$$

where  $q^\beta = e^{\int_{\beta} \gamma_1}$ . In this chapter,  $X$  is a hypersurface of degree  $l$  in  $\mathbb{P}^m$ . We assume  $l \leq m + 1$  so  $X$  is either Fano or Calabi-Yau.

At first,  $J^X$  could be pushforwarded to  $i_* J^X$  as a  $H^*(\mathbb{P}^m)$ -valued function. Let  $i : X \hookrightarrow \mathbb{P}^m$ . It induces  $i : \overline{\mathcal{M}}_{0,n}(X, d) \rightarrow \overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)$ . Consider

$$\begin{array}{ccc} \overline{\mathcal{C}}_{0,n}(\mathbb{P}^m, d) & \xrightarrow{F} & \mathbb{P}^m \\ \downarrow \pi & & \\ \overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d) & & \end{array}$$

Let  $E_d := \pi_* F^* \mathcal{O}(l)$  be the obstruction bundle over  $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)$ . The following theorem shows the relationship of virtual fundamental classes:

**Theorem 3.0.2.**

$$i_*[\overline{\mathcal{M}}_{0,n}(X, d)]^{vir} = e(\pi_* F^* \mathcal{O}(l)) \cap [\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)]^{vir}.$$

Let  $ev_1 : \overline{\mathcal{M}}_{0,2}(X, \beta) \rightarrow X$

**Proposition 3.0.3.**

$$J^X = e^{\gamma/z} \left( 1 + \sum_{\beta > 0} e^{\int_{\beta} \gamma_1} (ev_1)_* \left( \frac{1}{z - \psi_1} \right) \right),$$

$$J^{\mathbb{P}^m, \mathcal{O}(l)}(t_0, t_1, z) := i_* J^X = e^{(t_0+t_1 H)/z} \left( e(\mathcal{O}(l)) + \sum_{d > 0} e^{dt_1} (ev_1)_* \left( \frac{e(E_d)}{z - \psi_1} \right) \right),$$

where  $H \in H^2(\mathbb{P}^m, \mathbb{Q})$  is the generator of  $H^2(\mathbb{P}^m, \mathbb{Q})$ ,  $\gamma = t_0 + t_1 H$ .

Let  $0 \rightarrow E'_d \rightarrow E_d \rightarrow ev_1^* \mathcal{O}(l) \rightarrow 0$ , then

$$J^{\mathbb{P}^m, \mathcal{O}(l)}(t_0, t_1, z) = e^{(t_0+t_1 H)/z} lH \left( 1 + \sum_{d > 0} e^{dt_1} (ev_1)_* \left( \frac{e(E'_d)}{z - \psi_1} \right) \right)$$

The I-function is

$$I^{\mathbb{P}^m, \mathcal{O}(l)}(t_0, t_1, z) := e^{(t_0+t_1 H)/z} lH \left( 1 + \sum_{d=1}^{\infty} e^{dt_1} \frac{\prod_{a=1}^d (lH + az)}{\prod_{a=1}^d (H + az)^{m+1}} \right).$$

### 3.1 Fano case

The equivariant cohomology of  $\mathbb{P}^m$  with respect to  $\mathbb{T} = (\mathbb{C}^*)^{m+1}$  is

$$H_{\mathbb{T}}^*(\mathbb{P}^m; \mathbb{Q}) = \mathbb{Q}[H, \lambda_0, \dots, \lambda_m] / \prod_{i=0}^m (H - \lambda_i).$$

The classes  $\phi_i = \prod_{j \neq i} (H - \lambda_j)$ , are a basis of  $H_{\mathbb{T}}^*(\mathbb{P}^m; \mathbb{Q})$ . Moreover, for  $f(H, \lambda) \in H_{\mathbb{T}}^*(\mathbb{P}^m; \mathbb{Q})$ ,  $(\phi_i, f(H, \lambda)) = f(\lambda_i, \lambda)$ . Lifting J-function and I-function to the equivariant classes  $H_{\mathbb{T}}^*(\mathbb{P}^m)$  and define

$$\widehat{J}^{\mathbb{P}^m, \mathcal{O}(l)} := e^{(t_0+t_1 H)/z} lH \left( 1 + \sum_{d > 0} e^{dt_1} (ev_1)_* \left( \frac{e_{\mathbb{T}}(E'_d)}{z - \psi_1} \right) \right);$$

$$\tilde{I}^{\mathbb{P}^m, O(l)} := e^{(t_0+t_1 H)/z} lH \left( 1 + \sum_{d=1}^{\infty} e^{dt_1} \frac{\prod_{r=1}^{ld} (lH + rz)}{\prod_{k=0}^m \prod_{r=1}^d (H - \lambda_k + rz)} \right).$$

If we can show the relationship of  $\tilde{J}$  and  $\tilde{I}$ , then take  $\lambda \rightarrow 0$ , we get a relation between  $J$  and  $I$ . Let  $q = e^{t_1}$  and define

$$S(q, z, \lambda) = 1 + \sum_{d>0} q^d (ev_1)_* \left( \frac{e_{\mathbb{T}}(E'_d)}{z - \psi_1} \right);$$

$$\Psi(q, z, \lambda) = 1 + \sum_{d=1}^{\infty} e^{dt_1} \frac{\prod_{r=1}^{ld} (lH + rz)}{\prod_{k=0}^m \prod_{r=1}^d (H - \lambda_k + rz)};$$

$$S_i(q, z, \lambda) := (\phi_i, S(q, z, \lambda)) = 1 + \sum_{d>0} q^d \int_{\overline{\mathcal{M}}_{0,2}(\mathbb{P}^m, d)} \frac{e_{\mathbb{T}}(E'_d) ev_1^*(\phi_i)}{z - \psi_1};$$

$$\Psi_i(q, z, \lambda) := (\phi_i, \Psi(q, z, \lambda)) = 1 + \sum_{d=1}^{\infty} q^d \frac{\prod_{r=1}^{dl} (l\lambda_i + rz)}{\prod_{k=0}^m \prod_{r=1}^d (\lambda_i - \lambda_k + rz)}.$$

The first step is to use localization formula to compute  $S_i$ . We can classify the fixed locus  $\overline{\mathcal{M}}_{0,2}(\mathbb{P}^m, d)^{\mathbb{T}}$  into three classes:

$G_d^1$  : the first mark point  $x_1$  is mapped to  $p_j$  ( $j \neq i$ );

$G_d^2$  : the first mark point  $x_1$  is mapped to  $p_i$  and the irreducible component  $C_v$  is stable (i.e. not a point);

$G_d^3$  : the first mark point  $x_1$  is mapped to  $p_i$  and the irreducible component  $C_v$  is a single point.

In  $G_d^1$  case,  $ev_1^*(\phi_i)|_{F_{\Gamma}} = 0$ , so only the latter two cases contribute  $S_i$ . It can be expressed as

$$S_i(q, z, \lambda) = 1 + \sum_{\Gamma \in G_d^2 \cup G_d^3} \text{Cont}_{\Gamma}(S_i(q, z, \lambda));$$

$$\text{Cont}_{\Gamma}(S_i(q, z, \lambda)) = \sum_{d \geq 1} q^d \int_{F_{\Gamma}} \frac{e_{\mathbb{T}}(E'_d) ev_1^*(\phi_i)}{(z - \psi_1) e_{\mathbb{T}}(N_{\Gamma}^{\text{vir}})}$$

The following lemma is important in recursion formula of  $S_i(q, z, \lambda)$ .

**Lemma 3.1.1.** (1)  $S_i(q, z, \lambda) \in \mathbb{Q}(\lambda, z)[[q]]$ ;

(2) Let  $S_i(q, z, \lambda) = 1 + \sum_{d>0} q^d \xi_{id}(z, \lambda)$ . Then  $\xi_{id}(z, \lambda)$  are regular at  $z = \frac{\lambda_i - \lambda_j}{n}$  for all  $i \neq j$  and  $n \geq 1$ .

We will compute the contribution of  $G_d^2$  and  $G_d^3$  respectively.

**Theorem 3.1.2.** Let  $C_i(q, z, \lambda) = \sum_{\Gamma \in G_d^2} \text{Cont}_{\Gamma}(S_i(qz^{m+1-l}, z, \lambda))$

$$\text{then } C_i(q, z, \lambda) = \begin{cases} 0, & l < m \\ -1 + \exp(-m!q + \frac{(m\lambda_i)^m}{\prod_{j \neq i} (\lambda_i - \lambda_j)} q), & l = m. \end{cases}$$

As for  $\Gamma \in G_d^3$ , we can split  $\Gamma$  into  $\Gamma_0$  and  $\Gamma_c$ .

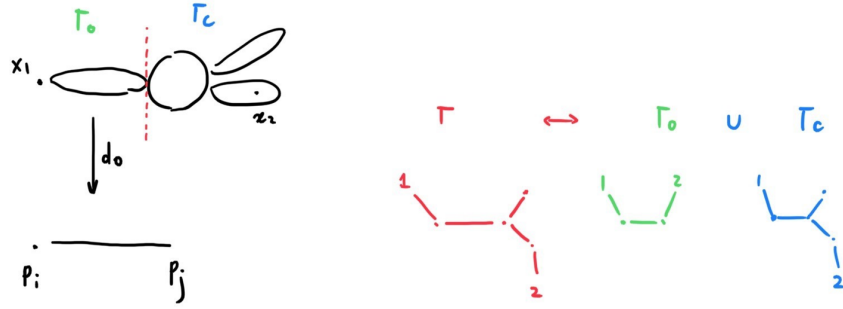


Figure 3.1:  $\Gamma \in G_d^3$

**Theorem 3.1.3.** Let  $\Gamma \in G_d^3$  such that degree of  $C_{ij}$   $d_0$  and  $d_c > 0$ , then

$$\text{Cont}_{\Gamma} S_i(q, z, \lambda) = q^{d_0} \frac{C_i^j}{d_0 z + \lambda_i - \lambda_j} (d_0, \lambda) \text{Cont}_{\Gamma_c} S_j(q, \frac{\lambda_j - \lambda_i}{d_0}, \lambda),$$

$$C_i^j(d, \lambda) = \frac{\prod_{r=1}^{ld} (l\lambda_i + r \frac{\lambda_j - \lambda_i}{d})}{\prod_{k=0}^m \prod_{r=1, (k,r) \neq (j,d)}^d (\lambda_i - \lambda_k + r \frac{\lambda_j - \lambda_i}{d})}.$$

*Proof.* Consider the diagram as Fig 3.1, we have  $F_{\Gamma} = F_{\Gamma_0} \times F_{\Gamma_c}$ . Let  $\pi_0 : F_{\Gamma} \rightarrow F_{\Gamma_0}$  and let  $\pi_c : F_{\Gamma} \rightarrow F_{\Gamma_c}$

$$E'_{d_0+d_c}|_{F_{\Gamma}} = \pi_0^* E'_{d_0} \oplus \pi_c^* E'_{d_c};$$

$$\frac{N_{F_{\Gamma}}}{T_{p_i} \mathbb{P}^m} = \frac{N_{F_{\Gamma_0}}}{T_{p_i} \mathbb{P}^m} \oplus \frac{N_{F_{\Gamma_c}}}{T_{p_j} \mathbb{P}^m} \oplus \pi_0^* \mathbb{L}_2^{\vee} \otimes \pi_c^* \mathbb{L}_1^{\vee};$$

$$e_{\mathbb{T}}(E'_{d_0}) = \prod_{r=1}^{ld_0} (l\lambda_i + r \frac{\lambda_j - \lambda_i}{d_0})$$

$$\begin{aligned} \text{ev}_1^* \phi_i &= \prod_{j \neq i} (\lambda_i - \lambda_j), \quad c_1(\mathbb{L}_2^\vee) = \frac{\lambda_j - \lambda_i}{d_0}; \\ e_{\mathbb{T}}(N_{\Gamma_0}) &= (-1)^{d_0} \prod_{r=1}^{d_0} \left( r \frac{\lambda_j - \lambda_i}{d_0} \right)^2 \prod_{r=0}^{d_0} \prod_{k \neq i, j} (\lambda_i - \lambda_k + r \frac{\lambda_j - \lambda_i}{d_0}). \end{aligned}$$

Hence,

$$\begin{aligned} q^{d_0+d_c} \int_{F_\Gamma} \frac{e_{\mathbb{T}}(E'_{d_0+d_c}) \text{ev}_1^* \phi_i}{(z - \psi) e_{\mathbb{T}}(N_{F_\Gamma})} &= q^{d_0+d_c} \frac{C_i^j(d_0, \lambda)}{d_0 z + \lambda_i - \lambda_j} \int_{F_\Gamma} \frac{e_{\mathbb{T}}(E'_{d_c}) \text{ev}_1^* \phi_i}{(z - \psi) e_{\mathbb{T}}(N_{F_\Gamma})} \Big|_{z=\frac{\lambda_j - \lambda_i}{d_0}}, \\ C_i^j(d_0, \lambda) &= \frac{e_{\mathbb{T}}(E'_{d_0}) \text{ev}_1^* \phi_i}{e_{\mathbb{T}}(N_{\Gamma_0})} = \frac{\prod_{r=1}^{ld_0} (l\lambda_i + r \frac{\lambda_j - \lambda_i}{d_0}) \prod_{k \neq i} (\lambda_i - \lambda_k)}{(-1)^{d_0} \prod_{r=1}^{d_0} \left( r \frac{\lambda_j - \lambda_i}{d_0} \right)^2 \prod_{r=0}^{d_0} \prod_{k \neq i, j} (\lambda_i - \lambda_k + r \frac{\lambda_j - \lambda_i}{d_0})} \\ &= \frac{(\lambda_i - \lambda_j) \prod_{r=1}^{ld_0} (l\lambda_i + r \frac{\lambda_j - \lambda_i}{d_0})}{(-1)^{d_0} \prod_{r=1}^{d_0} \left( r \frac{\lambda_j - \lambda_i}{d_0} \right)^2 \prod_{r=1}^{d_0} \prod_{k \neq i, j} (\lambda_i - \lambda_k + r \frac{\lambda_j - \lambda_i}{d_0})} \\ &= \frac{\prod_{r=1}^{ld_0} (l\lambda_i + r \frac{\lambda_j - \lambda_i}{d_0})}{\prod_{k=0}^m \prod_{r=1, (k,r) \neq (j, d_0)}^{d_0} (\lambda_i - \lambda_k + r \frac{\lambda_j - \lambda_i}{d_0})}. \end{aligned}$$

Finally,

$$\begin{aligned} \text{Cont}_\Gamma S_i(q, z, \lambda) &= \sum_{d_c > 0} q^{d_0+d_c} \int_{F_{\Gamma_c}} \frac{e_{\mathbb{T}}(E'_{d_0+d_c}) \text{ev}_1^* \phi_i}{(z - \psi_1) e_{\mathbb{T}}(N_{F_\Gamma})} \\ &= q^{d_0} \frac{C_i^j(d_0, \lambda)}{d_0 z + \lambda_i - \lambda_j} \sum_{d_c > 0} q^{d_c} \int_{F_{\Gamma_c}} \frac{e_{\mathbb{T}}(E'_{d_c}) \text{ev}_1^* \phi_i}{(z - \psi_1) e_{\mathbb{T}}(N_{F_\Gamma})} \Big|_{z=\frac{\lambda_j - \lambda_i}{d_0}} \\ &= q^{d_0} \frac{C_i^j(d_0, \lambda)}{d_0 z + \lambda_i - \lambda_j} \text{Cont}_{\Gamma_c} S_j(q, \frac{\lambda_j - \lambda_i}{d_0}, \lambda). \quad \square \end{aligned}$$

**Remark 3.1.4.**  $S_j(q, \frac{\lambda_j - \lambda_i}{d_0}, \lambda)$  is well-defined by Lemma 3.1.1.

**Theorem 3.1.5.** The function  $S_i$  satisfies the following recursion formula:

$$S_i(qz^{m+1-l}, z, \lambda) = 1 + C_i(q, z, \lambda) + \sum_{j \neq i} \sum_{d > 0} q^d z^{(m+1-l)d} \frac{C_i^j(d, \lambda)}{dz + \lambda_i - \lambda_j} S_j(qz^{m+1-l}, \frac{\lambda_j - \lambda_i}{d}, \lambda).$$

*Proof.* It directly follows from Theorem 3.1.2 and 3.1.3 and

$$S_i(qz^{m+1-l}, z, \lambda) = 1 + \sum_{\Gamma \in G_d^2 \cup G_d^3} \text{Cont}_{\Gamma} S_i(qz^{m+1-l}, z, \lambda) \quad \square$$

The second step is to check  $\Psi_i$  satisfies the same recursion relation.

**Proposition 3.1.6.** *For  $l < m$ ,  $\Psi_i$  has the recursion relation*

$$\Psi_i(qz^{m+1-l}, z, \lambda) = 1 + \sum_{j \neq i} \sum_{d > 0} q^d z^{(m+1-l)d} \frac{C_i^j(d, \lambda)}{dz + \lambda_i - \lambda_j} \Psi_j(qz^{m+1-l}, \frac{\lambda_j - \lambda_i}{d}, \lambda);$$

for  $l = m$ , they differ a function depending on  $q, \lambda$ .

*Proof.* The hint is to view the formula as meromorphic functions and analyse the simple poles.

$$\text{deg } d \text{ part of LHS} = z^{(m+1-l)d} \frac{\prod_{r=1}^{dl} (l\lambda_i + rz)}{\prod_{k=0}^m \prod_{r=1}^d (\lambda_i - \lambda_k + rz)}$$

has simple poles at  $z = \frac{\lambda_j - \lambda_i}{e}$  with  $j \neq i$ ,  $1 \leq e \leq d$ . The residue is

$$\text{Res}_z \text{ LHS} = \left( \frac{\lambda_j - \lambda_i}{e} \right)^{(m+1-l)d} \frac{\prod_{r=1}^{dl} (l\lambda_i + r \frac{\lambda_j - \lambda_i}{e})}{e \prod_{k=0}^m \prod_{r=1, (k,r) \neq (j,e)}^d (\lambda_i - \lambda_k + r \frac{\lambda_j - \lambda_i}{e})}.$$

$$\begin{aligned} \text{deg } d \text{ part of RHS} &= z^{(m+1-l)d} \sum_{j \neq i} \left( \frac{C_i^j(d, \lambda)}{dz + \lambda_i - \lambda_j} \right. \\ &\quad \left. + \sum_{e=1}^{d-1} \frac{C_i^j(e, \lambda)}{ez + \lambda_i - \lambda_j} \frac{\prod_{r=1}^{l(d-e)} (l\lambda_j + r \frac{\lambda_j - \lambda_i}{e})}{\prod_{k=0}^m \prod_{r=1}^{d-e} (\lambda_j - \lambda_k + r \frac{\lambda_j - \lambda_i}{e})} \right) \end{aligned}$$

The simple poles are also  $z = \frac{\lambda_j - \lambda_i}{e}$  with  $j \neq i$ ,  $1 \leq e \leq d$ .

$$e = d : \text{Res}_z \text{ RHS} = \left( \frac{\lambda_j - \lambda_i}{d} \right)^{(m+1-l)d} C_i^j(d, \lambda) / d = \text{Res}_z \text{ LHS}$$

$e < d :$

$$\text{Res}_z \text{ RHS} = \left( \frac{\lambda_j - \lambda_i}{e} \right)^{(m+1-l)d} \frac{\prod_{r=1}^{le} (l\lambda_i + r \frac{\lambda_j - \lambda_i}{e})}{e \prod_{k=0}^m \prod_{r=1, (k,r) \neq (j,e)}^e (\lambda_i - \lambda_k + r \frac{\lambda_j - \lambda_i}{e})}$$

$$\times \frac{\prod_{r=1}^{l(d-e)} (l\lambda_j + r \frac{\lambda_j - \lambda_i}{e})}{\prod_{k=0}^m \prod_{r=1}^{d-e} (\lambda_j - \lambda_k + r \frac{\lambda_j - \lambda_i}{e})}$$

For numerator, let  $s = le + r$ ,  $1 \leq r \leq l(d - e)$ ,  $le + 1 \leq s \leq ld$ ,

$$l\lambda_j + r \frac{\lambda_j - \lambda_i}{e} = \frac{le + r}{e} \lambda_j - r \frac{\lambda_i}{e} = l\lambda_i + s \frac{\lambda_j - \lambda_i}{e};$$

for denominator, let  $s = e + r$ ,  $1 \leq r \leq d - e$ ,  $e + 1 \leq s \leq d$ , then

$$\lambda_j - \lambda_k + r \frac{\lambda_j - \lambda_i}{e} = \frac{e + r}{e} \lambda_j - \lambda_k - \frac{r}{e} \lambda_i = \lambda_i - \lambda_k + s \frac{\lambda_j - \lambda_i}{e};$$

$$\begin{aligned} \text{Res}_z \text{ RHS} &= \left( \frac{\lambda_j - \lambda_i}{e} \right)^{(m+1-l)d} \frac{\prod_{r=1}^{dl} (l\lambda_i + rz)}{e \prod_{k=0}^m \prod_{r=1, (k,r) \neq (j,e)}^d (\lambda_i - \lambda_j + r \frac{\lambda_j - \lambda_i}{e})} \\ &= \text{Res}_z \text{ LHS}. \end{aligned}$$

If  $l < m$ , we find out that  $\text{LHS}=\text{RHS}=0$  at  $z = 0$ , so we have done.  $\square$

As a result, we show a mirror symmetry of  $l < m$  case:

**Theorem 3.1.7** (Mirror symmetry for  $l < m$ ). *If  $l < m$ , then  $S_i(q, z, \lambda) = \Psi_i(q, z, \lambda)$ . As a corollary,*

$$J^{\mathbb{P}^m, O(l)}(t_0, t_1, z) = I^{\mathbb{P}^m, O(l)}(t_0, t_1, z).$$

We need another recursion relation to prove  $l = m$  case

**Proposition 3.1.8.**  $\Psi_i$  has the recursion relation

$$\begin{aligned} e^{-m!q} \Psi_i(qz, z, \lambda) &= 1 + C_i(q, z, \lambda) \\ &+ \sum_{j \neq i} \sum_{d > 0} q^d z^d \frac{C_i^j(d, \lambda)}{dz + \lambda_i - \lambda_j} e^{-m!q} \Psi_j(qz, \frac{\lambda_j - \lambda_i}{d}, \lambda), \\ C_i(q, z, \lambda) &= -1 + \exp(-m!q + \frac{(m\lambda_i)^m}{\prod_{j \neq i} (\lambda_i - \lambda_j)} q) \end{aligned}$$

*Proof.* It is equivalent to proof

$$\begin{aligned} \Psi_i(qz, z, \lambda) &= \exp\left(\frac{(m\lambda_i)^m}{\prod_{j \neq i}(\lambda_i - \lambda_j)} q\right) \\ &\quad + \sum_{j \neq i} \sum_{d > 0} q^d z^d \frac{C_i^j(d, \lambda)}{dz + \lambda_i - \lambda_j} \Psi_j(qz, \frac{\lambda_j - \lambda_i}{d}, \lambda), \end{aligned}$$

Similar to the proof of Prop 3.1.6.

$$\begin{aligned} \deg d \text{ part of LHS} &= \frac{\prod_{r=1}^{dl} (l\lambda_i + rz)}{d! \prod_{k=0, k \neq i}^m \prod_{r=1}^d (\lambda_i - \lambda_k + rz)} \\ \deg d \text{ part of RHS} &= \frac{(m\lambda_i)^{md}}{d! \prod_{j \neq i} (\lambda_i - \lambda_j)^d} + \sum_{j \neq i} z^d \left( \frac{C_i^j(d, \lambda)}{dz + \lambda_i - \lambda_j} \right. \\ &\quad \left. + \sum_{e=1}^{d-1} \frac{C_i^j(e, \lambda)}{ez + \lambda_i - \lambda_j} \frac{\prod_{r=1}^{m(d-e)} (m\lambda_j + r \frac{\lambda_j - \lambda_i}{e})}{\prod_{k=0}^m \prod_{r=1}^{d-e} (\lambda_j - \lambda_k + r \frac{\lambda_j - \lambda_i}{e})} \right) \end{aligned}$$

As Prop 3.1.6, they have same simple poles and residue numbers. Take  $z = 0$ ,

$$\deg d \text{ part of LHS}(z = 0) = \frac{(m\lambda_i)^{md}}{d! \prod_{j \neq i} (\lambda_i - \lambda_j)^d} = \deg d \text{ part of LHS}(z = 0).$$

Hence two formulas identify.  $\square$

**Remark 3.1.9.**  $\exp(\frac{(m\lambda_i)^m}{\prod_{j \neq i}(\lambda_i - \lambda_j)} q)$  is the function depending on  $q, \lambda$  in Prop 3.1.6.

**Theorem 3.1.10** (Mirror symmetry for  $l = m$ ). *For  $l = m$ ,  $e^{m!q/z} S_i(q, z, \lambda) = \Psi_i(q, z, \lambda)$ . As a corollary,  $e^{m!q/z} S(q, z, \lambda) = \Psi(q, z, \lambda)$  and*

$$J^{\mathbb{P}^m, O(l)}(t_0 + m!e^{t_1}, t_1, z) = I^{\mathbb{P}^m, O(l)}(t_0, t_1, z).$$

## 3.2 Calabi-Yau case



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