KP hierarchy, quantum curves, and topological recursion

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Abstract

A well-chosen element of the infinite Grassmannian manifold can associate a KP-hierarchy τ-function. In special, we can derive the hypergeometric tau function of 2D Toda type $\tau(\mathbf{t}, \mathbf{s})$, which is the generation function of the weighted Hurwitz numbers. In this special case, the adapted basis $\Psi_k^+(x)$ of $\tau(\mathbf{t}, \mathbf{s})$ satisfies an ODE, called the quantum curve. By a change of variables, the quantum curve transforms into the classical spectral curve $\mathcal{S}$. The multicurrent $\{\tilde{\omega}_{g,n}\}$ of the weighted Hurwitz numbers satisfies the topological recursion of $\mathcal{S}$. The ELSV formula can be derived in this way.

Contents

1 Hurwitz numbers

1.1 Geometric and combinatorial definition of Hurwitz numbers

A partition $\mu \vdash N$ is an array $\mu = (\mu_1, \dots, \mu_l)$ with $|\mu| := \mu_1 + \cdots + \mu_l = N$. The length of μ is denoted by $l(\mu) = l$. Let $\{\mu^{(i)}\}_{i=1,\dots,k}$ be a k-array of N-partition, $N \in \mathbb{N}^+$. The pure Hurwitz number $H(\mu^{(1)}, \dots, \mu^{(k)})$ is the sum of N-sheeted covering C of Riemann sphere $\mathbb{CP}^1$ with k-branch points having ramification profiles $\mu^{(i)}$, normalized by the inverse of the order of automorphism group of covering $|Aut(C)|$:

$$H(\mu^{(1)}, \dots, \mu^{(k)}) = \sum_{[C \to \mathbb{CP}^1]} \frac{1}{|Aut(C)|}$$

The Euler characteristic numbers give the geometric restriction of covering $C \to \mathbb{CP}^1$:

$$\chi(C) - \sum_{i=1}^{k} l(\mu^{(i)}) = N(\chi(\mathbb{CP}^1) - k)$$

$$\chi(C) = 2N - \sum_{i=1}^{k} (|\mu^{(i)}| - l(\mu^{(i)})) = 2N - \sum_{i=1}^{k} l^*(\mu^{(i)})$$

where $l^*(\mu) := |\mu| - l(\mu)$ is the colength of μ.

Combinatorially, $H(\mu^{(1)}, \dots, \mu^{(k)})$ is the number of distinct k-arraies $(h_1, \dots, h_k)$ of elements in $\mathcal{S}_N$ satisfying:

1 $h_i \in cyc(\mu^{(i)})$. i.e. h_i lies in the same conjugacy class as $\mu^{(i)}$.

2 The product of h_i's: $h_1 \dots h_k = I$ is the identity element in $\mathcal{S}_N$.

normalized by the factor $1/N!$. By the Frobenius-Schur formula, it can be expressed by the irreducible characters of $\mathcal{S}_N$:

$$H(\mu^{(1)}, \dots, \mu^{(k)}) = \sum_{|\lambda|=N} h_\lambda^{k-2} \prod_{i=1}^{k} \frac{\chi_\lambda(\mu^{(i)})}{z_{\mu^{(i)}}}.$$

Here, h_λ is the hook length of the Young diagram of λ; z_μ is the order of stabilizer of μ.

$$h_\lambda = \frac{N!}{\prod_{(i,j)\in\lambda} h_\lambda(i,j)}$$

$$z_\mu = \prod_{i=1}^{|\mu|} i^{m_i} (m_i)!, \qquad m_i = \#\{j | \mu_i = j\}$$

Definition 1.1 (simple Hurwitz numbers). *The simple Hurwitz number $H_{g,\mu}$ with just one partition $\mu \vdash N$ is a special case of $H(\mu^{(1)}, \dots, \mu^{(k)})$ by setting*

- $\mu^{(1)} = \mu$,
- *the remaining $2g - 2 + N + l(\mu)$ partitions are simples: $\mu^{(i)} = (2, 1^{N-2})$, $i \geq 2$.*

1.2 Weighted Hurwitz numbers

Let $G(z)$ be a generating series of weight $\{g_i\}_{i\geq 1}$:

$$G(z) = 1 + \sum_{i=1}^{\infty} g_i z^i = \prod_{i=1}^{\infty}(1 + c_i z)$$

The element $g_i = e_i(\mathbf{c})$ is the elementary symmetric functions of $\mathbf{c}$. Fix a pair of partitions $\mu, \nu \vdash N$, a positive integer $d \in \mathbb{N}^+$, The weighted double Hurwitz numbers $H_G^d(\mu, \nu)$ is

$$H_G^d(\mu,\nu) := \sum_{k=0}^{d} \sum_{\substack{\mu^{(1)},\ldots,\mu^{(k)} \\ |\mu^{(i)}|=N \\ \sum_{i=1}^{k} l^*(\mu^{(i)})=d}}' \mathcal{W}_G(\mu^{(1)},\ldots,\mu^{(k)}) H(\mu^{(1)},\ldots,\mu^{(k)},\mu,\nu)$$

Here, $\sum'$ is the sum of k-array of partitions $(\mu^{(1)},\ldots,\mu^{(k)})$ except (1^N). $\mathcal{W}_G$ is the weight parametrized by G, defined as

$$\mathcal{W}_G(\mu^{(1)},\ldots,\mu^{(k)}) := \frac{1}{k!} \sum_{\sigma\in\mathcal{S}_k} \sum_{1\leq b_1<\cdots<b_k} c_{b_{\sigma(1)}}^{l^*(\mu^{(1)})} \ldots c_{b_{\sigma(k)}}^{l^*(\mu^{(k)})}$$

1.3 Hypergeometric tau functions of 2D Toda type

Let β, γ be complex parameters. The hypergeometric τ-functions of 2D Toda type associated to (G, β, γ) is

$$\tau^{(G,\beta,\gamma)}(\mathbf{t},\mathbf{s}) = \sum_{\lambda} \gamma^{|\lambda|} r_\lambda^{(G,\beta)} s_\lambda(\mathbf{t}) s_\lambda(\mathbf{s}) \tag{1}$$

Here $\mathbf{t} = (t_1, t_2, \ldots)$, $\mathbf{s} = (s_1, s_2, \ldots)$ are called flow parameters. $s_\lambda(\mathbf{t})$ is the Schur polynomials: The one-part partition Schur polynomial $s_i(\mathbf{t})$ is given by complete symmetric polynomials $h_i(\mathbf{t})$

$$\exp(\sum_{i=1}^{\infty} t_i z^i) = \sum_{i=0}^{\infty} h_i(\mathbf{t}) z^i, \quad s_i(\mathbf{t}) = h_i(\mathbf{t})$$

As for general λ, the Schur polynomial is given by the determinant

$$s_\lambda(\mathbf{t}) = \det_{i,j\in l(\lambda)} (s_{\lambda_i - i + j})$$

The content product $r_\lambda^{(G,\beta)}$ is given by the weight series $G(z)$

$$r_\lambda^{(G,\beta)} := \prod_{(i,j)\in\lambda} r_{j-i}^G, \quad r_j^{(G,\beta)} := G(j\beta)$$

Two quantities ρ_i, T_i is useful:

$$r_j^{(G,\beta)} = \frac{\rho_j}{\gamma \rho_{j-1}}$$

$$\rho_j = e^{T_j} = \gamma^j \prod_{i=1}^{j} G(i\beta) \quad \rho_0 = 1 \tag{2}$$

$$\rho_{-j} = e^{T_{-j}} = \gamma^{-j} \prod_{i=0}^{j-1} (G(-i\beta))^{-1}, \quad \text{for } j \in \mathbb{N}^+$$

Identifying

$$t_i = \frac{p_i}{i} \quad s_i = \frac{p'_i}{i}$$

$$p_\mu(\mathbf{t}) := \prod_{i=1}^{l(\mu)} p_{\mu_i} \quad p_\nu(\mathbf{s}) := \prod_{i=1}^{l(\nu)} p'_{\nu_i}$$

The hypergeometric τ-function is the generating series of weighted Hurwitz numbers:

Theorem 1.2 ([4] Theorem 2.1)**.**

$$\tau^{(G,\beta,\gamma)}(\boldsymbol{t},\boldsymbol{s}) = \sum_{d=0}^{\infty} \beta^d \sum_{\substack{\mu,\nu \\ |\mu|=|\nu|}} \gamma^{|\mu|} H_G^d(\mu,\nu) p_\mu(\boldsymbol{t}) p_\nu(\boldsymbol{s}),$$

$$\ln \tau^{(G,\beta,\gamma)}(\boldsymbol{t},\boldsymbol{s}) = \sum_{d=0}^{\infty} \beta^d \sum_{\substack{\mu,\nu \\ |\mu|=|\nu|}} \gamma^{|\mu|} h_G^d(\mu,\nu) p_\mu(\boldsymbol{t}) p_\nu(\boldsymbol{s})$$

In geometry definition, $h_G^d(\mu,\nu)$ means the connected Hurwitz numbers. i.e. the N-sheeted covering C is a connected Riemann surface.

Definition 1.3 (multicurrent correlator)**.** *Define*

$$\nabla(x) = \sum_{i=1}^{\infty} x^{i-1} \frac{\partial}{\partial t_i}$$

The multicurrent correlators of $\tau^{(G,\beta,\gamma)}(\boldsymbol{t},\boldsymbol{s})$ *are*

$$W_n(\boldsymbol{s}, x_1, \ldots, x_n) = \left(\left(\prod_{i=1}^{n} \nabla(x_i) \right) \tau^{(G,\beta,\gamma)}(\boldsymbol{t}, \beta^{-1}\boldsymbol{s}) \right) \Big|_{\boldsymbol{t}=0}$$

$$\tilde{W}_n(\boldsymbol{s}, x_1, \ldots, x_n) = \left(\left(\prod_{i=1}^{n} \nabla(x_i) \right) \ln \tau^{(G,\beta,\gamma)}(\boldsymbol{t}, \beta^{-1}\boldsymbol{s}) \right) \Big|_{\boldsymbol{t}=0}$$

Let $\tilde{W}_{g,n}$ denote the coefficient of β^{2g-2+n} of $\tilde{W}_n$

$$\tilde{W}_{g,n}(\mathbf{s}, x_1, \ldots, x_n) = [\beta^{2g-2+n}] \tilde{W}_n(\mathbf{s}, x_1, \ldots, x_n) \tag{3}$$

Let

$$\tilde{F}_{g,n}(\mathbf{s}; x_1, \ldots, x_n) = \sum_{\mu,\nu,l(\mu)=n} \gamma^{|\mu|} h_G^d(\mu,\nu) M_\mu(x_1, \ldots .x_n) p_\mu(\mathbf{s})$$

where $M_\mu(\mathbf{x}) = \sum_{\sigma \in \mathcal{S}_n} \prod_{i=1}^{n} x_{\sigma(i)}^{\mu_i}$.

Theorem 1.4 ([4] Theorem 2.6)**.**

$$\tilde{W}_{g,n}(\boldsymbol{s}; x_1, \ldots, x_n) = \frac{\partial^n}{\partial x_1 \ldots \partial x_n} \tilde{F}_{g,n}(\boldsymbol{s}; x_1, \ldots, x_n)$$

It is worth remarking that [4] uses the normalize monomial symmetric polynomials

$$m_\lambda(\mathbf{c}) = \frac{1}{|\mathrm{aut}(\mu)|} \sum_{\sigma \in S_k} \sum_{1 \le b_1 < \cdots < b_k} c_{b_{\sigma(1)}}^{\lambda_1} \ldots c_{b_{\sigma(k)}}^{\lambda_k},$$

so there is $|\mathrm{aut}(\mu)|$ factor in $\tilde{F}_{g,n}$. It will be shown that $\tilde{W}_{g,n}$ is related to the differential forms $\{\omega_{g,n}\}$ in Section 5.

2 Integrable hierarchy

2.1 Infinite Grassmannian and KP tau-functions

Let $\mathcal{H} = L^2(S^1) = \mathbb{C}[[z, z^{-1}]]$ be the Hilbert space. It is useful to use $e_i := z^{-i-1}$ as a basis of $\mathcal{H}$.

$$\mathcal{H}_+ = \mathrm{span}\{z^i\}_{i \ge 0} = \mathrm{span}\{e_i\}_{i \le -1} \quad \mathcal{H}_- = \mathrm{span}\{z^i\}_{i \le -1} = \mathrm{span}\{e_i\}_{i \ge 0}$$

The infinite Grassmannian $Gr_{\mathcal{H}_+}(\mathcal{H}) = Fr(\mathcal{H}_+, \mathcal{H})/GL(\mathcal{H}_+)$ is a set of subspaces $W \subset \mathcal{H}$ heomorphic to $\mathcal{H}_+$. There are two important abelian groups acting on $Gr_{\mathcal{H}_+}(\mathcal{H})$:

$$\Gamma_+ = \{\gamma_+(\mathbf{t}) = e^{\xi(\mathbf{t}, z)}\} \quad \Gamma_- = \{\gamma_-(\mathbf{s}) = e^{\xi(\mathbf{s}, z^{-1})}\}$$

where

$$\xi(\mathbf{t}, z) = \sum_{i=1}^{\infty} t_i z^i \quad \xi(\mathbf{s}, z^{-1}) = \sum_{i=1}^{\infty} s_i z^{-i}$$

By Sato's work, each subspace W of virtual dimension 0 can deduce a tau-function:

$$\tau_W(\mathbf{t}) = \det(\pi_+ : W(\mathbf{t}) \to \mathcal{H}_+) \tag{4}$$

where $W(\mathbf{t}) = \gamma_+(\mathbf{t})$ and π_+ is the projection operator. The function τ_W satisfies Hirota's bilinear relation:

$$\oint_\infty e^{-\xi(\delta\mathbf{t}, z)} \tau_W(\mathbf{t} + \delta\mathbf{t} + [z^{-1}]) \tau_W(\mathbf{t} - [z^{-1}]) dz = 0$$

where

$$[z] := \frac{1}{i} z^i.$$

It means τ_W is a KP hierarchy tau-function. The Baker functions are

$$\Psi_W^-(z, \mathbf{t}) = e^{\xi(\mathbf{t}, z)} \frac{\tau_W(\mathbf{t} - [z^{-1}])}{\tau_W(\mathbf{t})}, \tag{5}$$

which takes value for all flow parameters $\mathbf{t}$ on the annihilator $W^\perp$ of W with respect to the Hirota bilinear pairing

$$\langle u, v \rangle = \frac{1}{2\pi i} \oint_{z=0} u(z) v(z) dz$$

The dual Baker functions are

$$\Psi^{+}_{W}(z,\mathbf{t}) = e^{-\xi(\mathbf{t},z)}\frac{\tau_W(\mathbf{t}+[z^{-1}])}{\tau_W(\mathbf{t})}, \tag{6}$$

which takes values on W. The Baker function plays a special role in quantum curves.

Remark 2.1. *One can find a representative $g \in GL(\mathcal{H})$ such that $W = g\mathcal{H}_+$. We will use τ_g, $\Psi^{\mp}_{g}$ to represent τ_W, $\Psi^{\mp}_{W}$ respectively.*

2.2 Plücker embedding of $Gr_{\mathcal{H}_+}(\mathcal{H})$

The Fock space is

$$\mathcal{F} = \bigwedge^{\infty/2}\mathcal{H} = \bigoplus_{N\in\mathbb{Z}}\mathcal{F}_N$$

where $\mathcal{F}_N$ is spanned by

$$|\lambda, N\rangle = e_{l_1}\wedge e_{l_2}\wedge\dots \quad l_i = \lambda_i - i + N$$

$$|N\rangle = |\emptyset, N\rangle, \quad |\lambda\rangle = |\lambda, 0\rangle, \quad |0\rangle = e_{-1}\wedge e_{-2}\wedge\dots$$

The Plücker embedding is an embedding of infinite Grassmannian into the projective space:

$$Gr_{\mathcal{H}_+}(\mathcal{H}) \to \mathbb{P}\mathcal{F} : W \mapsto [w_1\wedge w_2\wedge\dots] \quad \{w_i\}\text{is a basis of } W.$$

Expanding w_j under the basis $\{e_i\}_{i\in\mathbb{Z}}$

$$w_j = \sum_i \xi^{i,j}e_i$$

For each inclusion $\sigma : \mathbb{N}^c \to \mathbb{Z}$ such that $\sigma(i) = i$ for $i \ll 0$, the function

$$\xi_\sigma := \det(\xi^{\sigma(i),j})_{\substack{i\in\mathbb{N}^c\\ j\in\mathbb{N}^+}}$$

is called the homogeneous Plücker coordinates of $Gr_{\mathcal{H}_+}(\mathcal{H})$. The Lie algebra $(\mathbf{gl}(\mathbb{Z}), [,])$ is

$$\mathbf{gl}(\mathbb{Z}) := \{X = \sum_{a,b\in\mathbb{Z}} X_{ab}E_{ab} | X_{ab} = 0 \text{ when } |b-a| \gg 0\}$$

where $X_{ab}\in\mathbb{C}$, E_{ab} is the elementary matrix, $[X,Y] = XY - YX$. Let O be the ring of Plücker coordinates. There is a Lie homomorphism

$$\tilde{\delta} : \mathbf{gl}(\mathbb{Z}) \to \mathrm{Der}\, O$$

$$X = \sum_{a,b\in\mathbb{Z}} X_{ab}E_{ab} \mapsto \tilde{\delta}_X : \left(\xi_\sigma \mapsto \sum_{a,b\in\mathbb{Z}} X_{ab}(\delta_{E_{ab}} - \theta_{a<0}\delta_{a,b}M)\xi_\sigma\right)$$

where $M, \delta_{E_{ab}} \in \mathrm{Der}\, O$ and $M(\xi_\sigma) = \xi_\sigma$, $\delta_{E_{ab}}\xi_\sigma = \xi_{\sigma,a\to b}$. Here, $\xi_{\sigma,a\to b}$ means switching row a and b at first, then taking the determinant of the switched matrix ξ-coordinate matrix.

Lemma 2.2. $[\tilde{\delta}_{E_{ab}}, \tilde{\delta}_{E_{mn}}] = \tilde{\delta}_{[E_{ab},E_{mn}]} + \frac{1}{2}(1 - sign(ab))sign(b)\delta_{an}\delta_{bm}M$

It is convenient to define the central extension $(\widehat{\mathbf{gl}(\mathbb{Z})}, \widehat{[,]})$ by $\widehat{\mathbf{gl}(\mathbb{Z})} = \mathbf{gl}(\mathbb{Z}) \oplus \mathbb{C}c$ and

$$\widehat{[E_{ab}, E_{mn}]} = [E_{ab}, E_{mn}] + \frac{1}{2}(1 - \mathrm{sign}(ab))\mathrm{sign}(b)\delta_{an}\delta_{bm}c$$

$\widehat{\mathbf{gl}(\mathbb{Z})}$ is a Lie algebra acting on $Gr_{\mathcal{H}_+}(\mathcal{H})$ as derivatives. Let

$$\Lambda = \sum_{a,b\in\mathbb{Z}} E_{ab}\delta_{a+1,b}, \quad K = \sum_{a,b\in\mathbb{Z}} aE_{ab}\delta_{a-1,b}$$

$$J_n = \Lambda^n, \quad L_n = \Lambda^{n+1}K, \quad n \in \mathbb{Z}$$

$\{J_n, c\}_{n\in\mathbb{Z}}$ is the Heisenberg algebra, with commutative relations

$$\widehat{[J_m, J_n]} = m\delta_{m+n,0}c \tag{7}$$

$\{L_n, c\}_{n\in\mathbb{Z}}$ is the Virasoro algebra, with commutative relations

$$\widehat{[L_m, L_n]} = (m - n)L_{m+n} - \frac{1}{6}(m^3 - m)c\delta_{m+n,0} \tag{8}$$

They are two special subalgebras of $\widehat{\mathbf{gl}(\mathbb{Z})}$ acting on $Gr_{\mathcal{H}_+}(\mathcal{H})$ via Plücker coordinates.

3 Fermionic representation

3.1 Free Fermionic Field

The Clifford algebra Cl is generated by $\{\psi_i, \psi_j^\dagger\}_{i,j\in\mathbb{Z}}$ with generating relations:

$$\{\psi_i, \psi_j^\dagger\} = \delta_{i,j} \quad \{\psi_i, \psi_j\} = \{\psi_i^\dagger, \psi_j^\dagger\} = 0$$

where $\{x, y\}$ = xy+yx is the anticommutator. The Fock space $\mathcal{F}$ can be interpreted as a Clifford module by

$$\psi_i(e_{i_1} \wedge e_{i_2} \wedge \dots) = e_i \wedge e_{i_1} \wedge e_{i_2} \wedge \dots$$

$\psi_j^\dagger(e_{i_1}\wedge e_{i_2}\wedge\dots) = i_{e_j}(e_{i_1}\wedge e_{i_2}\wedge\dots) = \delta_{j,i_1}e_{i_2}\wedge e_{i_3}\wedge\cdots-\delta_{j,i_2}e_{i_1}\wedge e_{i_3}\wedge\cdots+\delta_{j,i_3}e_{i_1}\wedge e_{i_2}\wedge\cdots-\dots$

The Fermionic representation of $|\lambda, N\rangle$ is :

$$|\lambda, N\rangle = (-1)^{\sum_{i=1}^k \beta_i} \prod_{i=1}^k \psi_{N+\alpha_i}\psi^\dagger_{N-\beta_i-1}|N\rangle,$$

where $\lambda = (\alpha_1, \dots, \alpha_k|\beta_1, \dots, \beta_k)$ is the Frobenius notation, $\alpha_i = \lambda_i - i$, $\beta_i = \lambda_i' - i$. λ' is the transpose of λ.

The normal ordering is

$$:\psi_i\psi_j^\dagger: = \begin{cases} \psi_i\psi_j^\dagger & j \geq 0 \\ -\psi_j^\dagger\psi_i & j < 0 \end{cases}$$

There is a Lie homomorphism

$$\hat{\phi} : (\widehat{\mathbf{gl}(\mathbb{Z})}, \widehat{[,]}) \to (Cl, [,]) : E_{ab} \mapsto :\psi_a\psi_b^\dagger: \quad c \mapsto 1 \tag{9}$$

As a result, we have a Clifford representation of the derivative of $Gr_{\mathcal{H}_+}(\mathcal{H})$. For $g = e^A \in GL(\mathcal{H})$ with $A \in \mathbf{gl}(\mathbb{Z})$, define

$$\hat{g} := \hat{\phi}(g) = \exp\Big(\sum_{i.j\in\mathbb{Z}} A_{ij} : \psi_i\psi_j^\dagger :\Big) \tag{10}$$

The Fermionic field is the generating series of $\psi_i, \psi_j^\dagger$:

$$\psi(z) = \sum_{i\in\mathbb{Z}} \psi_i z^i \quad \psi^\dagger(z) = \sum_{j\in\mathbb{Z}} \psi_j^\dagger z^{-j-1}$$

Then the current operator $J(z)$, energy-momentum tensor $T(z)$ can be represented by Fermionic field

$$J(z) =: \psi(z)\psi^\dagger(z) := \sum_{i\in\mathbb{Z}} J_i z^{-i-1} \quad J_i(z) = \sum_{r\in\mathbb{Z}} : \psi_r\psi_{i+r}^\dagger :$$

$$T(z) = \frac{1}{2} : J(z)J(z) := \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

By (7) and (9) $\{J_i\}_{i\in\mathbb{Z}}$ is the Heisenberg algebra:

$$[J_m, J_n] = m\delta_{m+n,0};$$

and $\{L_n\}_{n\in\mathbb{Z}}$ is the Virasoro algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}$$

Remark 3.1. *The energy-momentum tensor can be defined in another way:*

$$T(z) = - : \psi(z)\partial\psi^\dagger(z) := \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

$$L_n = \sum_{r\in\mathbb{Z}} (l + r + 1) : \psi_r\psi_{n+r}^\dagger :$$

In this case, (8) and (9) give

$$[L_m, L_n] = (m-n)L_{m+n} - \frac{1}{6}(m^3 - m)\delta_{m+n,0}$$

The element in $\Gamma_+(\mathbf{t})$, $\Gamma_-(\mathbf{s})$ also has its Fermionic representation via the Lie homomorphism (9):

$$z^i(e_j) = z^{i-j-1} = e_{j-i}.$$

It means that z^i acts on $\mathcal{H}$ as $\Lambda^i = \sum_{j\in\mathbb{Z}} E_{j-i,j}$ and

$$\widehat{z^i} = \sum_j : \psi_{j-i}\psi_j^\dagger := J_i, \quad \widehat{\gamma_+(\mathbf{t})} = e^{\sum_{i\geq 1} t_i J_i}, \quad \widehat{\gamma_-(\mathbf{s})} = e^{\sum_{i\geq 1} s_i J_{-i}}$$

The evaluation of $\widehat{\gamma_+(\mathbf{t})}, \widehat{\gamma_-(\mathbf{s})}$ gives the Schur polynomials:

$$\langle N|\widehat{\gamma_+(\mathbf{t})}|\lambda, N\rangle = \langle \lambda, N|\widehat{\gamma_-(\mathbf{t})}|N\rangle = s_\lambda(\mathbf{t}) \tag{11}$$

3.2 Fermionic construction of tau-function

In this section, following Sato's method, we will construct the tau-function explicitly from an element $W = g\mathcal{H}_+ \in Gr_{\mathcal{H}_+}(\mathcal{H})$. Let $g = e^A \in GL(\mathbb{Z})$, $\hat{A} = \sum_{i,j\in\mathbb{Z}} A_{ij} : \psi_i \psi_j^\dagger :$ and $\hat{g} = \exp(\sum_{i,j\in\mathbb{Z}} A_{ij} : \psi_i\psi_j^\dagger :)$ as before. By anti-commutators in Clifford algebra:

$$[\hat{A}, \psi_n] = \sum_i A_{in}\psi_i \quad [\hat{A}, \psi_n^\dagger] = -\sum_j A_{nj}\psi_j^\dagger$$

$$\hat{g}\psi_n\hat{g}^{-1} = e^{ad_{\hat{A}}}\psi_n = \sum_i g_{in}\psi_i \quad \hat{g}\psi_n^\dagger\hat{g}^{-1} = e^{ad_{\hat{A}}}\psi_n^\dagger = \sum_j (g^{-1})_{nj}\psi_j^\dagger \tag{12}$$

where

$$g(e_i) = \sum_{j\in\mathbb{Z}} e_j g_{ji}$$

As a corollary, $\hat{g}$ satisfies the bilinear commutative relation:

$$\sum_i \psi_i \hat{g} \otimes \psi_i^\dagger \hat{g} = \sum_i \hat{g}\psi_i \otimes \hat{g}\psi_i^\dagger$$

In Sato's approach to KP τ-function:

$$\tau_g(\mathbf{t}) = \langle 0|\widehat{\gamma_+(\mathbf{t})}\hat{g}|0\rangle$$

satisfies the Hirota's bilinear relation, hence τ_g is a KP τ-function.

The above discussion is the Fermionic version of the tau function in (4). The vacuum state $|0\rangle = e_{-1} \wedge e_{-2} \ldots$ is a vector in $\bigwedge^{\text{top}} \mathcal{H}_+$; $\hat{g}|0\rangle$ is the vector in $\bigwedge^{\text{top}} W$; $\widehat{\gamma_+(\mathbf{t})}$ reflects the flow of $W(\mathbf{t}) = \gamma_+(\mathbf{t})W$; the projection π_+ is expressed as $\langle 0|$. The determinant of π_+ is the same as dealing with the top wedge vector.

To find the Fermionic construction of the Baker function, we take $\hat{g} = \widehat{\gamma_+(\mathbf{t})}, \widehat{\gamma_-(\mathbf{s})}$ in (12), then

$$\widehat{\gamma_+(\mathbf{t})}\psi(z)\widehat{\gamma_+(\mathbf{t})}^{-1} = e^{\xi(\mathbf{t},z)}\psi(z), \quad \widehat{\gamma_+(\mathbf{t})}\psi^\dagger(z)\widehat{\gamma_+(\mathbf{t})}^{-1} = e^{-\xi(\mathbf{t},z)}\psi^\dagger(z)$$

$$\widehat{\gamma_-(\mathbf{s})}\psi\widehat{\gamma_-(\mathbf{s})}^{-1} = e^{\xi(\mathbf{s},z^{-1})}\psi(z), \quad \widehat{\gamma_-(\mathbf{s})}\psi^\dagger(z)\widehat{\gamma_-(\mathbf{s})}^{-1} = e^{-\xi(\mathbf{s},z^{-1})}\psi^\dagger(z)$$

The Baker and dual functions in (5), (6) are

$$\Psi_g^-(z,\mathbf{t}) = \frac{\langle 0|\psi_0^\dagger\widehat{\gamma_+(\mathbf{t})}\psi(z)\hat{g}|0\rangle}{\tau_g(\mathbf{t})}$$

$$\Psi_g^-(z,\mathbf{t}) = \frac{\langle 0|\psi_{-1}\widehat{\gamma_+(\mathbf{t})}\psi(z)\hat{g}|0\rangle}{\tau_g(\mathbf{t})}$$

The proof can be found on [1] Appendix corollary A.2

3.3 Special case: Hypergeometric tau function of 2D Toda type

The hypergeometric tau function (1) can be recovered in the framework of Section 3.2 by choosing suitable g.

Let $\rho(z) = \sum_{i\in\mathbb{Z}} \rho_i z^{-i-1}$. The symbols ρ_i, T_i are the same as (2). The convolution C_ρ on $w = \sum_{i\in\mathbb{Z}} w_i z^{-i-1}$ is defined as

$$C_\rho(w)(z) = \frac{1}{2\pi i}\oint_{\xi\in S^1} \rho(\xi) w(\frac{z}{\xi})\frac{d\xi}{\xi} = \sum_{i\in\mathbb{Z}} \rho_i w_i z^{-i-1}.$$

View w as vectors in $\mathcal{H}$, the convolution C_ρ is an element in $GL(\mathcal{H})$

$$[C_\rho]_{ij} = \rho_i \delta_{ij}, \quad \widehat{C_\rho} = \exp\left(\sum_{i\in\mathbb{Z}} T_i : \psi_i \psi_i^\dagger :\right)$$

By (11) and [13] Lemma 3.1, the 2D Toad lattice hypergeometric tau function is

$$\tau^{(G,\beta,\gamma)}(\mathbf{t},\mathbf{s}) = \tau_g(\mathbf{t},\mathbf{s}) := \langle 0|\widehat{\gamma_+(\mathbf{t})}\widehat{C_\rho}\widehat{\gamma_-(\mathbf{s})}|0\rangle$$

The Baker and dual Baker functions associated with 2D Toda lattice are

$$\Psi^-_{(G,\beta,\gamma)}(z,\mathbf{t},\mathbf{s}) = e^{\xi(\mathbf{t},z)}\frac{\tau^{(G,\beta,\gamma)}(\mathbf{t}-[z^{-1}]),\mathbf{s})}{\tau^{(G,\beta,\gamma)}(\mathbf{t},\mathbf{s})} = \frac{\langle 0|\psi_0^\dagger\widehat{\gamma_+(\mathbf{t})}\psi(z)\widehat{C_\rho}\widehat{\gamma_-(\mathbf{s})}|0\rangle}{\tau_g(\mathbf{t},\mathbf{s})}$$

$$\Psi^+_{(G,\beta,\gamma)}(z,\mathbf{t},\mathbf{s}) = e^{-\xi(\mathbf{t},z)}\frac{\tau^{(G,\beta,\gamma)}(\mathbf{t}+[z^{-1}]),\mathbf{s})}{\tau^{(G,\beta,\gamma)}(\mathbf{t},\mathbf{s})} = \frac{\langle 0|\psi_{-1}\widehat{\gamma_+(\mathbf{t})}\psi(z)\widehat{C_\rho}\widehat{\gamma_-(\mathbf{s})}|0\rangle}{\tau_g(\mathbf{t},\mathbf{s})}$$

Let $x = z^{-1}$

$$[x] = (x, \frac{x^2}{2}, \dots, \frac{x^n}{n}, \dots)$$

Usually, we view $\mathbf{t}$ as time parameters and view $\mathbf{s}$ as auxiliary parameters. The initial value of $\Psi^+_{(G,\beta,\gamma)}$, $\Psi^-_{(G,\beta,\gamma)}$ at $\mathbf{t} = \mathbf{0}$ is

$$\Psi^+_{(G,\beta,\gamma)}(\frac{1}{x},\mathbf{0},\beta^{-1}\mathbf{s}) = \gamma\sum_{j=0}^{\infty}\rho_{j-1}h_j(\beta^{-1}\mathbf{s})x^j$$

$$\Psi^-_{(G,\beta,\gamma)}(\frac{1}{x},\mathbf{0},\beta^{-1}\mathbf{s}) = \sum_{j=0}^{\infty}\rho_{-j}^{-1}h_j(-\beta^{-1}\mathbf{s})x^j$$

More generally, the adapted basis $\{\Psi^+_k(x)\}_{k\in\mathbb{Z}}$ and its dual $\{\Psi^-_k(x)\}_{k\in\mathbb{Z}}$ is

$$\Psi^+_k(x) = \gamma\sum_{j=0}^{\infty}\rho_{j+k-1}h_j(\beta^{-1}\mathbf{s})x^{j+k} \tag{13}$$

$$\Psi^-_k(x) = \sum_{j=0}^{\infty}\rho_{-j-k}^{-1}h_j(-\beta^{-1}\mathbf{s})x^{j+k} \tag{14}$$

When $k = 0$, they are the initial values of the Baker functions. The recursion relations of the adapted basis are related to the quantum curves.

4 Quantum curves

In principle, the quantum curve is the differential equation of some generating functions. It could recover some correlation functions in some cases. The following discussion is mainly for the hyper-geometric tau function $\tau^{(G,\beta,\gamma)}(\mathbf{t},\mathbf{s})$.

Define the Euler operator $\mathcal{D}$

$$\mathcal{D} = x\frac{d}{dx},$$

the recursion operator

$$R_{\pm} = \gamma x G(\pm\beta\mathcal{D})$$

Because $\mathcal{D}$ commutes with $G(\pm\beta\mathcal{D})$,

$$[\mathcal{D}, R_{\pm}] = R_{\pm}$$

By direct calculation, the adapted basis in (13) (14) has recursion relations

Proposition 4.1.

$$\Psi^{+}_{k+1}(x) = R^{\pm}_{+}\Psi^{+}_{k}(x)$$
$$\Psi^{-}_{k+1}(x) = R^{\pm}_{-}\Psi^{-}_{k}(x)$$

Define the series

$$S(z) = \sum_{k\geq 1} k s_k z^k$$

where s_k is the flow parameter in (1). According to [1] Prop 4.3, we have

Proposition 4.2.

$$[\beta\mathcal{D} - S(R_{+})]\Psi^{+}_{k}(x) = k\Psi^{+}_{k}(x)$$
$$[\beta\mathcal{D} + S(R_{-})]\Psi^{+}_{k}(x) = k\Psi^{-}_{k}(x)$$

The element $\beta\mathcal{D} - S(R_{+})$ is the quantum curve of the hyper-geometric tau function. The dequantization is $x \mapsto x$, $\beta\frac{d}{dx} \mapsto y$, we get the spectral curve of double weighted Hurwitz numbers in topological recursion:

$$xy = S(\gamma x G(xy)) \tag{15}$$

5 Topological recursion

The topological recursion is a recursion definition of a family of symmetric meromorphic forms $\{\omega_{g,n}\}_{g\in\mathbb{N},n\in\mathbb{N}^{+}}$ from a spectral curve $\mathcal{S}$

Definition 5.1 ([10] Def. 2.1)**.** *A spectral curve $\mathcal{S}$, is the data of :*

$$\mathcal{S} = (C, x, y, B)$$

- *C is a plane complex curve with coordinate (x, y)*
- *x and y are two analytical functions $C \to \mathbb{C}$*

- ♦ *$B(z,z')$ is called the Bergman kernel. It is a symmetric 2nd kind differential on $C\times C$, having a double pole at $z=z'$ and no other pole. It behaves like*

$$B(z,z') \sim_{z\to z'} \frac{dz\otimes dz'}{(z-z')^2} + O(1)$$

in any local parameter z.

The point a such that $dx(a)=0$ is called a branch point. Let's restrict to a specific class of $\mathcal{S}$:

Definition 5.2 ([11] Def. 7.1.3)**.** *A spectral curve $\mathcal{S}$ is called regular if:*

- *the differential form dx has a finite number (non vanishing) of zeros $dx(a_i)=0$, and all zeros of dx are simple zeros.*
- *The differential form dy does not vanish at the zeros of dx, i.e. $dy(a_i)\neq 0$.*

In this case, the spectral curve just has simple branch points. It means that near $x(a_i)$, $x(z)-x(a_i)$ has a double zero, and thus $\zeta(z)=\sqrt{x(z)-x(a_i)}$ is a good local coordinate. Let $\bar{z}$ denote the point corresponding the other sign of $\zeta(z)$, i.e.

$$\zeta(\bar{z}) = -\zeta(z), \quad x(z)=x(\bar{z}).$$

If dy does not vanish it means that

$$y(z)\sim y(a_i)+y'(a_i)\sqrt{x(z)-x(a_i)}+O(x(z)-x(a_i)), \quad y'(a_i)\neq 0.$$

The recursion kernel with z in a vicinity of a branch point a is

$$K_a(z_0,z)=\frac{1}{2}\frac{\int_{z'=\bar{z}}^{z} B(z_0,z')}{(y(z)-y(\bar{z}))dx(z)}$$

$K_a(z_0,z)$ is a meromorphic 1-form in the variable z_0, globally defined on $z_0\in C$. It has a simple pole at $z_0=z$ and $z_0=\bar{z}$. On the other hand, concerning z, $K_a(z_0,z)$ is defined only locally near the branch point a. $K_a(z_0,z)$ is symmetric under involution:

$$K_a(z_0,z)=K_a(z_0,\bar{z})$$

Definition 5.3. *The topological recursion is a recursive definition of a family of meromorphic forms $\omega_{g,n}(\mathcal{S};z_1,\dots,z_n)$ with $g\geq 0$ and $n\geq 1$ via the recursion kernel $K_a(z_0,z)$:*

$$\omega_{0,1}(z)=y(z)dx(z)$$

$$\omega_{0,2}(z_1,z_2)=B(z_1,z_2)$$

For $2g-2+n>0$ and $J=\{z_1,\dots,z_n\}$

$$\omega_{g,n+1}(z_0,J)=\sum_{a=\text{branch points}} Res_{z\to a}K_a(z_0,z)\Big[\omega_{g,n-1}(z,\bar{z},J) \\ + \sum_{h+h'=g,I\uplus I'=J}^{\prime} \omega_{h,1+|I|}(z,I)\omega_{h',1+|I'|}(\bar{z},I')\Big] \tag{16}$$

where $\sum'$ means we exclude the terms $(h,I)=(0,\emptyset),(g,J)$.

The symplectic invariant $\mathcal{F}_g = \omega_{g,0}$ is defined for $g \in \mathbb{N}$. For $g \geq 2$,

$$\mathcal{F}_g = \frac{1}{2-2g} \sum_{a=\text{branch points}} \mathrm{Res}_{z\to a}\omega_{g,1}(z)\Phi(z)$$

where $\varphi(z)$ is any function defined locally near a such that

$$d\Phi = ydx.$$

The definition of $\mathcal{F}_1, \mathcal{F}_0$ can be found on [11] Definition 7.1.8, 7.1.9.

There are some examples of $\mathcal{S}$.

♦ The Airy curve: ref. [8] Lemma 3.3.

$$x(z) = z^2 + a, \quad y(z) = \alpha z \quad and \quad B(z,z') = \beta\frac{dz \otimes dz'}{(z-z')^2}$$

The correlation functions of the Airy curve is

$$\omega_{g,n}(z_1,\ldots,z_n) = \left(-\frac{\beta}{2\alpha}\right)^{2g+n-2} \beta^{g+n-1} \sum_{\alpha_1,\ldots,\alpha_n\geq 0} \langle \tau_{\alpha_1}\ldots\tau_{\alpha_n}\rangle_{g,n} \prod_{i=1}^{n} \frac{(2\alpha_i+1)!!dz_i}{z_i^{2\alpha_i+2}}$$

Use (16), we get

$$\omega_{0,3}(z_1,z_2,z_3) = -\frac{\beta^3}{2\alpha}\prod_{i=1}^{3}\frac{dz_i}{z_i^2}, \quad \omega_{1,1}(z) = -\frac{\beta^2}{2\alpha}\frac{dz}{8z^4}$$

Hence, we verify that

$$\langle\tau_0^3\rangle_{0,3} = 1, \quad \langle\tau_1\rangle_{1,1} = \frac{1}{24}$$

♣ The Weil-Peterson volumes, Mirzakhani's recursion: ref. [11] Section 6.6, [9] Section 8.1

$$C = \{(x,y)|y = \frac{1}{2\pi}\sin(2\pi\sqrt{x})\}$$

$$x(z) = z^2, \quad y(z) = \frac{1}{2\pi}\sin(2\pi z), \quad B(z,z') = \frac{dz\otimes dz'}{(z-z')^2}$$

Let

$$\chi_{g,n} = 2-2g-n, \quad d_{g,n} = 3g-3+n$$

The Weil-Petersson form is

$$2\pi^2\kappa = \sum_i dl_i \wedge d\theta_i$$

The Weil-Petersson volume of the moduli space $\mathcal{M}_{g,n}(L_1,\ldots,L_n)$ is

$$\mathrm{Vol}(\mathcal{M}_{g,n}(L_1,\ldots,L_n)) = \frac{1}{d_{g,n}!}\left\langle (2\pi^2\kappa_1 + \frac{1}{2}\sum_{i=1}^{n} L_i^2\psi_i)^{d_{g,n}} \right\rangle_{\mathcal{M}_{g,n}}$$

The correlation functions are the Laplace transforms of Weil-Petersson volumes:

$$\frac{\omega_{g,n}(z_1,\ldots,z_n)}{(-2)^{\chi_{g,n}}\prod_i dz_i} = \int_0^\infty L_1 dL_1 e^{-z_1L_1}\cdots\int_0^\infty L_n dL_n e^{-z_nL_n}\mathrm{Vol}(\mathcal{M}_{g,n}(L_1,\ldots,L_n))$$

$$= 2^{d_{g,n}} \sum_{d_1,\dots,d_n} \prod_{i=1}^{n} \frac{(2d_i+1)!!}{2^{d_i} z_i^{2d_i+2}} \left\langle e^{\pi^2 \kappa_1} \prod_i \psi_j^{d_j} \right\rangle_{g,n}$$

where the second equality follows from

$$\int_0^\infty LdLL^{2d} e^{-zL} = \frac{(2d+1)!}{z^{2d+2}}$$

♠ The simple Hurwitz numbers: ref. [7] Section 4.3. The spectral curve is called the Lambert curve $\mathcal{S}_{\text{Lambert}}$

$$C = \{(x,y) | e^x = ye^{-y}\} \tag{17}$$

$$x(z) = -z + \ln z, \quad y(z) = z, \quad B(z,z') = \frac{dz \otimes dz'}{(z-z')^2}$$

Define a generating series of connected simple Hurwitz numbers $h_{g,\mu}$:

$$H^{(g)}(x_1,\dots,x_n) = \sum_{l(\mu)=n} \frac{\prod_{i=1}^n \mu_i \cdot M_\mu(x_1,\dots,x_n)}{(2g-2+|\mu|+n)!} h_{g,\mu},$$

where $M_\mu(\mathbf{x}) = \sum_{\sigma \in \mathcal{S}_n} \prod_{i=1}^n x_{\sigma(i)}^{\mu_i}$.

Let $X = e^x = ye^{-y} = ze^{-z}$, $z = L(X)$. By [7], the correlation functions derived from the Lambert curve are the generating series of simple Hurwitz numbers:

$$\frac{\omega_{g,n}(z_1,\dots,z_n)}{dx(z_1)\dots dx(z_n)} = (-1)^n H^{(g)}(X_1,\dots,X_n) \tag{18}$$

where the sign $(-1)^n$ comes from the different signs of $\omega_{0,1}$ and recursion kernel $K_a(z_0,z)$ in [7].

🏆 The weighted double Hurwitz numbers: the spectral curve C is the curve (15) $xy = \gamma S(\gamma x G(xy))$ with rational parametrization and Bergamm kernel:

$$x(z) = \frac{z}{\gamma G(S(z))}, \quad y(z) = \frac{S(z)}{z} \gamma G(S(z)), \quad B(z,z') = \frac{dz \otimes dz'}{(z-z')^2} \tag{19}$$

Let $\phi(z) = G(S(z)) - zG'(S(z))S'(z)$, then

$$x'(z) = \frac{\phi(z)}{\gamma G(S(z))^2}$$

The recursion kernel is

$$\begin{aligned} K(z_0,z) &= \frac{1}{2}\left(\frac{dz_0}{z_0 - z} - \frac{dz_0}{z_0 - \bar{z}}\right) \frac{1}{\frac{\gamma G(S(z))(S(z)-S(\bar{z}))}{z} \cdot \frac{\phi(z)}{\gamma G(S(z))^2} dz} \\ &= \frac{1}{2}\left(\frac{dz_0}{z_0 - z} - \frac{dz_0}{z_0 - \bar{z}}\right) \frac{zG(S(z))}{S(z) - S(\bar{z})\phi(z) dz} \end{aligned}$$

Recall $\tilde{W}_{g,n}$ defined in (3) , we define $\tilde{\omega}_{g,n}$ to be

$$\begin{aligned} \tilde{\omega}_{g,n}(z_1,\dots,z_n) = \tilde{W}_{g,n}(x(z_1),\dots,x(z_n)) x'(z_1)\dots x'(z_n) dz_1 \dots dz_n \\ + \delta_{n,2}\delta_{g,0} \frac{x'(z_1)x'(z_2)}{(x(z_1)-x(z_2))^2} dz_1 dz_2 \end{aligned}$$

Theorem 5.4 ([3] Thm 10.1)**.** *Assume that the curve C is regular. Then $\{\tilde{\omega}_{g,n}\}$ satisfies the topological recursion relation derived form (19):*

$$\tilde{\omega}_{g,n+1}(z_0, J) = \sum_{a=\text{branch points}} Res_{z\to a} K_a(z_0, z) \Big[\tilde{\omega}_{g,n-1}(z, \bar{z}, J) + \sum'_{h+h'=g, I\uplus I'=J} \tilde{\omega}_{h,1+|I|}(z, I)\tilde{\omega}_{h',1+|I'|}(\bar{z}, I')\Big]$$

Remark 5.5. *There may be a $(-1)^n$ sign problem. We will see it in Section 6.*

6 Application: ELSV formula

We will verify Theorem 5.4 in the case of simple Hurwitz numbers by direct computation. Then state a proof of the ELSV formula via topological recursion, which may derive more ELSV-like formulas.

6.1 Example: simple Hurwitz numbers

Let the weight generating series be $G(z) = e^z$. Then the content product is

$$r_j^G(\beta) = e^{j\beta}, \quad r_\lambda^{(G,\beta)} = e^{\sum_{i=1}^{l(\lambda)} \sum_{j=1}^{\lambda_i} (j-i)\beta} = e^{\frac{\beta}{2} \sum_{i=1}^{l(\lambda)} \lambda_i(1+\lambda_i-2i)}$$

By $e^z = \lim_{m\to\infty}(1+\frac{1}{m})^m$, the monomial symmetric function

$$M_\lambda(\mathbf{c}) = \sum_{\sigma\in\mathcal{S}_k} \sum_{1\le b_1<\dots<b_k} c_{b_{\sigma(1)}}^{\lambda_1} \dots c_{b_{\sigma(k)}}^{\lambda_k}$$

Observe that if $\lambda = (1^k)$, $c_j = \frac{1}{m}$ with $j = 1, \dots, m$

$$\lim_{m\to\infty} M_\lambda(\mathbf{c}) = \lim_{m\to\infty} \frac{k!}{m^k}\binom{m}{k} = 1,$$

so in the general case

$$\mathcal{W}_{\exp}(\mu^{(1)}, \dots, \mu^{(k)}) = \frac{1}{k!}\prod_{i=1}^{k} \delta_{l^*(\mu^{(i)}),1} = \frac{1}{k!}\prod_{i=1}^{k} \delta_{\mu^{(i)},(2,1^{N-2})}.$$

Hence the weighted Hurwitz number $H_G^d(\mu,\nu)$ is a weighted counting of double simple Hurwitz numbers. The corresponding spectral curve (19) is

$$xy = \sum_{k\ge1} \gamma^{k+1} k s_k x^k e^{kxy}.$$

In particular, let $\gamma = 1$, $s_1 = 1$, $s_{\ge2} = 0$, we recover the Lambert curve (17)

$$X(z) = ze^{-z}, \quad Y(z) = e^z$$

According to Theorem 1.4, where $p_\nu(\mathbf{s}) = \prod_{i=1}^{l(\nu)} \delta_{\nu_i,1}$, we have

$$\tilde{W}_{g,n}(X_1, \dots, X_n) = \sum_{l(\mu)=n} \frac{h_{g,\mu}}{(2g-2+|\mu|+l(\mu))!} \frac{\partial^n M_\mu(X_1, \dots, X_n)}{\partial X_1 \dots \partial X_n}$$

Setting $X = e^x$, $Y = e^y$, for $2g-2+n>0$

$$\begin{aligned}\tilde{\omega}_{g,n}(x_1,\dots,x_n) &= \sum_{l(\mu)=n} \frac{h_{g,\mu}}{(2g-2+|\mu|+l(\mu))!} \prod_{i=1}^{n} \left(X_i \frac{\partial}{\partial X_i}\right) M_\mu(X_1,\dots,X_n) dx_1 \dots dx_n \\ &= \sum_{l(\mu)=n} \frac{h_{g,\mu}}{(2g-2+|\mu|+l(\mu))!} \prod_{i=1}^{n} \mu_i \cdot M_\mu(X_1,\dots,X_n) dx_1 \dots dx_n\end{aligned}$$

Hence

$$\frac{\tilde{\omega}_{g,n}(x_1,\dots,x_n)}{dx_1 \dots dx_n} = H_g(X_1,\dots,X_n),$$

Compare with (18), we get

$$\tilde{\omega}_{g,n} = (-1)^n \omega_{g,n}(\mathcal{S}_{\text{Lambert}})$$

6.2 ELSV formula

The ELSV formula is a formula of connected simple Hurwitz numbers and Hodge integrals. For $2g-2+l(\mu)>0$,

$$h_{g,\mu} = \frac{(2g-2+|\mu|+l(\mu))!}{|\mathrm{aut}(\mu)|} \prod_{i=1}^{l(\mu)} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\mathcal{M}_{g,l(\mu)}} \frac{\Lambda_g^\vee(1)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)}$$

where

$$\Lambda_g^\vee(u) := \sum_{k=0}^{g} (-1)^k u^{g-k} \lambda_k$$

It can be proved via localization methods, e.g [17], [16], [15]. It can also be derived from topological recursion because we have known that the spectral curve of the Hurwitz numbers is $\mathcal{S}_{\text{Lambert}}$. In [9] (8.39), we know

$$\begin{aligned}\frac{\omega_{g,n}(\mathcal{S}_{\text{Lambert}}; z_1,\dots,z_n)}{dx(z_1)\dots dx(z_n)} =& 2^{d_{g,n}} e^{-\tilde{t}_0 \chi_{g,n}} (-2)^{n/2} \\ & \sum_{l(\mu)=n} \prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!} \mu_i e^{\mu_i x(z_i)} \left\langle \frac{\Lambda_g^\vee(1)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)} \right\rangle_{g,n},\end{aligned}$$

where $\tilde{t}_0$ is given in [9] (8.15)

$$\tilde{t}_0 = -\frac{1}{2}\ln 8 + \frac{i\pi}{2}.$$

By computation, we get

$$2^{d_{g,n}} e^{-\tilde{t}_0 \chi_{g,n}} (-2)^{n/2} = (-1)^n$$

$$H^{(g)}(X_1,\dots,X_n) = \sum_{l(\mu)=n} \frac{|\mathrm{aut}(\mu)|}{(2g-2+|\mu|+n)!} \prod_{i=1}^{n} \mu_i e^{\mu_i x_i} \cdot h_{g,\mu}$$

By (18),

$$h_{g,\mu} = \frac{(2g-2+|\mu|+l(\mu))!}{|\mathrm{aut}(\mu)|} \prod_{i=1}^{l(\mu)} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\mathcal{M}_{g,l(\mu)}} \frac{\Lambda_g^\vee(1)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)}$$

This method can deduce more ELSV-like formulas. If we want to know the ELSV-like formula of a specific kind of Hurwitz number $H(\mu^{(1)},\dots,\mu^{(k)})$, we can:

Step 1 Choose the weight series $G(z)$ good enough so that only $H(\mu^{(1)}, \ldots, \mu^{(k)})$ is alive in $H^d_G(\mu, \nu)$.

Step 2 By Theorem 5.4, the correlators $\{\omega_{g,n}\}$ of the spectral curve (15) are the generating series of $H(\mu^{(1)}, \ldots, \mu^{(k)})$.

Step 3 Write $\{\omega_{g,n}\}$ as a sum of intersection numbers. E.g. [9], [8].

Hence, we find a relation between the weighted Hurwitz numbers and intersection numbers.

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