Gromov-Witten theory and mirror symmetry

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Contents

Chapter 1

Gromov-Witten invariants

Roughly speaking, Gromov-Witten invariants is the number of algebraic curves (resp. pseduo-holomorphic curves) of a algebraic variety X (resp. complex manifold) passing through specific subvarieties (resp. submanifolds) under specific degree (homology class of the curves) and genus.

Definition 1.0.1. *Let $\gamma_1, \ldots, \gamma_n \in H^*(X; \mathbb{Q})$ and let $\beta \in H^2(X; \mathbb{Q})$. The Gromov-Witten invariant of genus g degree β curves is*

$$\langle \gamma_1 \ldots \gamma_n \rangle_{g,n,\beta} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} ev_1^*(\gamma_1) \cup \cdots \cup ev_n^*(\gamma_n).$$

Here, a point in $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}$ is $[f : C \to X, 1, \ldots, n]$:

a map from the genus g curve C to the variety X modulo the automorphism of C.

The evaluation map $ev_i : [\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir} \to X$ is given by

$$ev_i([f : C \to X, 1, \ldots, n]) = f(i).$$

Let $\overline{\mathcal{M}}_{g,n}$ denote the moduli space (Deligne-Mumford stack) of genus g curves with n marked points, and let $\overline{\mathcal{C}}_{g,n}$ be the universal family of $\overline{\mathcal{M}}_{g,n}$.

1.1 Kontsevich's approach

Atiyah-Bott localization formula tells us: if there is a torus action $\mathbb{T} = (\mathbb{C}^*)^n$ on X, then the fixed points of torus action could tells us some properties of X.

By the classifying space theory, $B\mathbb{T} = (\mathbb{C}P^\infty)^{\times n}$, so $H^*(B\mathbb{T}) = \mathbb{C}[\lambda_1, \ldots, \lambda_n]$. Let $\mathcal{R}_\mathbb{T} = \mathbb{C}(\lambda_1, \ldots, \lambda_n)$. Let $X_\mathbb{T} = E\mathbb{T} \times_\mathbb{T} X$, the equivariant cohomology of X is defined by

$$H^*_\mathbb{T}(X) := H^*(X_\mathbb{T}) = H^*(E\mathbb{T} \times_\mathbb{T} X),$$

so naturally, $H^*_\mathbb{T}(X)$ is a $H^*_\mathbb{T}(pt) = H^*(B\mathbb{T})$-module. The localization of $H^*_\mathbb{T}(X)$ means $H^*_\mathbb{T}(X) \otimes \mathcal{R}_\mathbb{T}$.

Theorem (Atiyah-Bott)**.** *Let $X^\mathbb{T}$ be fixed locus of $\mathbb{T}$, let Z_j be a connection component of $X^\mathbb{T}$, and let N_j be the normal bundle of Z_j in X. Let $i_j : Z_j \to X$ and let $i_{j!} : H^*_\mathbb{T}(Z_j) \to H^*_\mathbb{T}(X)$ be the pushforward defined by the Gysin map. Let $\alpha \in H^*_\mathbb{T}(X) \otimes \mathcal{R}_\mathbb{T}$, we have*

$$\alpha = \sum_j \frac{i_{j!} i_j^* \alpha}{Euler_T(N_j)},$$

In particular,

$$\int_{X_\mathbb{T}} \alpha = \sum_j \int_{(Z_j)_\mathbb{T}} \frac{i_j^* \alpha}{Euler_T(N_j)}.$$

Kontsevich's approach is to apply Atiyah-Bott localization formula in $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{vir}$ so that we can simplify the computation. We can lift the $\mathbb{T}$ action on X to $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{vir}$ in the following way: let $t \in \mathbb{T}$, $[f : C \to X, 1, \ldots, n] \in [\overline{\mathcal{M}}_{g,n}(X, \beta)]^{vir}$, $x \in X$

$$(t \cdot f)(x) = f(t \cdot x).$$

In special, we will consider $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^n, d)$ in this section. As claimed before, we need to find $[f : C \to X, 1, \ldots, n] \in \overline{\mathcal{M}}_{g,n}(\mathbb{P}^n, d)^\mathbb{T}$. The fixed points of $\mathbb{C}P^n$ is

$$\{q_i = [0 : 0 : \cdots : 1 : 0 : \cdots : 0]\}_{0 \le i \le n}.$$

The coordinate curve l_{ij} connecting q_i, q_j has one dimensional degree of freedom $\mathbb{C}^*$ (as an invariant component). The curve $C \in \overline{\mathcal{C}}_{g,n}$ is stable (i.e. Aut$(C) < \infty$) if and only if $2g - 2 + n > 0$. If a components C' of C is mapped to l_{ij}, then C' has two points mapped to q_i, q_j respectively (equivalent to with two marked points in C'), so $2g - 2 + 2 \le 0$ implies $g = 0$, i.e. $C' \cong \mathbb{P}^1$ (see Fig 1.1). Meanwhile, $f|_{C'}$ must be uniformly ramified, so $f|_{C'}(z) = z^e, \forall z \in C' \cong \mathbb{P}^1$, for some $e \in \mathbb{N}^*$.

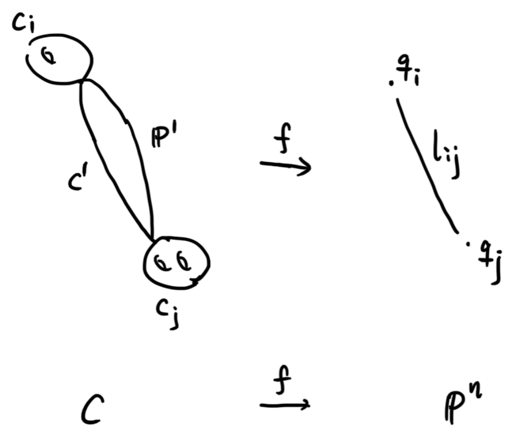

Figure 1.1: $f(C_i) = q_i,\ f(C') = l_{ij},\ f(C_j) = q_j$

It is convenient to use a decorated graph $\vec{\Gamma} = (\Gamma, \vec{f}, \vec{d}, \vec{g}, \vec{s})$ (graph, maps, degrees, genus, marked points) to represent $[f : C \to X, 1, \ldots, n] \in \overline{\mathcal{M}}_{g,n}(\mathbb{P}^n, d)^{\mathbb{T}}$. Let val$(v)$, the valence of v, be the number of edges connecting vertex v, and let $n(v) = |s_v| + val(v)$. The stable map $[f : C \to X, 1, \ldots, n]$ with fixed graph $\vec{\Gamma} = (\Gamma, \vec{f}, \vec{d}, \vec{g}, \vec{s})$ defines a substack

$$\overline{\mathcal{M}}_{\vec{\Gamma}} \subset \overline{\mathcal{M}}_{g,n}(\mathbb{P}^n, d).$$

On the other sides, consider

$$\varphi_{\vec{\Gamma}} : \prod_{\dim C_v = 1} \overline{M}_{g_v, n(v)} \to \overline{\mathcal{M}}_{\vec{\Gamma}}.$$

If v, v' are connected by an edge e, then let $C_v, C_{v'}$ connected by a $C_e \cong \mathbb{P}^1$ associated with a degree d_e map to $\mathbb{P}^n$. Let $\overline{M}_\Gamma$ be the product of above C_v, C_e. There is a group $\mathbb{A}_\Gamma$ acting on $\overline{M}_\Gamma$. The group $\mathbb{A}_\Gamma$ is defined by:

$$1 \to \prod_{edges} \mathbb{Z}/(d_e) \to \mathbb{A}_\Gamma \to Aut(\Gamma) \to 1.$$

and

$$\overline{\mathcal{M}}_{\vec{\Gamma}} = \overline{M}_\Gamma / \mathbb{A}_\Gamma.$$

Therefore, we know the $\mathbb{T}$-fixed locus of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^n, d)$ is $\overline{\mathcal{M}}_{\vec{\Gamma}}$. Let N_Γ be the normal bundle of $\overline{\mathcal{M}}_{\vec{\Gamma}}$ in $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^n, d)$. Then there is an explicit formula for the equivariant Euler class. Before doing that, we define some necessary notations. A flag F is a pair (v, e) such that e is an edge containing the vertex v. We put $i(F) = v$, $j(F)$ the vertex of e different from v. Let

$$\omega_F = \frac{\lambda_{i(F)} - \lambda_{j(F)}}{d_e} \in H^2_{\mathbb{T}}(pt) = \mathbb{Q}[\lambda_0, \ldots, \lambda_r],$$

which is the weight of $\mathbb{T}$-action on $T_{q_{i_v}} C_e$.

Theorem 1.1.1 ($Euler_{\mathbb{T}}(N_\Gamma)$)**.** *$Euler_{\mathbb{T}}(N_\Gamma) = e^F_\Gamma e^v_\Gamma e^e_\Gamma$, where*

$$e^F_\Gamma = \prod_{n(i(F))\geq 3} (\omega_F - \psi_F) / \prod_{j \neq i(F)} (\lambda_{i(F)} - \lambda_j),$$

$$e^v_\Gamma = \prod_v \prod_{j \neq i_v} (\lambda_{i_v} - \lambda_j) \prod_{val(v)=2, s_v=\emptyset} (\omega_{F_1(v)} + \omega_{F_2(v)}) / \prod_{val(v)=1, s_v=\emptyset} \omega_{F(v)}$$

$$e^e_\Gamma = \prod_e \frac{(-1)^{d_e} (d_e!)^2 (\lambda_i - \lambda_j)^{2d_e}}{d_e^{2d_e}} \prod_{a+b=d_e, k\neq i,j} \left(\frac{a\lambda_i + b\lambda_j}{d_e} - \lambda_k\right)$$

The proof is partially discussed in section 1.2.

1.2 Tangent-obstruction sequence

Consider $[f : C \to X, 1, \ldots, n] \in \overline{\mathcal{M}}_{\vec{\Gamma}}$, $\vec{\Gamma} = (\Gamma, \vec{f}, \vec{d}, \vec{g}, \vec{s})$. We put

$$V^1(\Gamma) := \{v \in V(\Gamma) : g_v = 0, val(v) = 1, |s_v| = 0\}$$

$$V^2(\Gamma) := \{v \in V(\Gamma) : g_v = 0, val(v) = 2, |s_v| = 0\}$$

$$V^{1,1}(\Gamma) := \{v \in V(\Gamma) : g_v = 0, val(v) = 1, |s_v| = 1\}$$

$$V^s(\Gamma) := \{v \in V(\Gamma) : 2g_v - 2 + val(v) + n(v) > 0\}$$

$$y(v, e) := C_e \cap C_v$$

The tangent-obstruction sequence is

$$\begin{aligned} & 0 \to Aut(C, 1, \ldots, n) \\ \to Def(f) \to\ & Def(C, 1, \ldots, n, f) \to Def(C, 1, \ldots, n) \\ \to Ob(f) \quad \to\ & Ob(C, 1, \ldots, n, f) \to 0, \end{aligned}$$

$$\begin{aligned} & 0 \to Hom(\Omega_C(p_1+\cdots+p_n),\mathcal{O}_C) \\ \to H^0(C,f^*T_X) \to T^1 \to & Ext^1(\Omega_C(p_1+\cdots+p_n),\mathcal{O}_C) \\ \to H^1(C,f^*T_X) \to T^2 \to & 0. \end{aligned}$$

For simplicity:

$$0 \to B_1 \to B_2 \to T^1 \to B_4 \to B_5 \to T^2 \to 0.$$

The $N^{vir} = T^{1,m} - T^{2,m}$ (m means moving part).

$$Euler_{\mathbb{T}}(N^{vir}) = \frac{Euler_{\mathbb{T}}(B_2^m)Euler_{\mathbb{T}}(B_4^m)}{Euler_{\mathbb{T}}(B_1^m)Euler_{\mathbb{T}}(B_5^m)}.$$

(1) $Euler_{\mathbb{T}}(B_2^m)/Euler_{\mathbb{T}}(B_5^m)$. The normalization sequence of C is:

$$\begin{aligned} 0 \to \mathcal{O}_C \to & \bigoplus_{v\in V^s(\Gamma)} \mathcal{O}_{C_v} \oplus \bigoplus_{e\in E(\Gamma)} \mathcal{O}_{C_e} \\ \to & \bigoplus_{v\in V^2(\Gamma)} \mathcal{O}_{y_v} \oplus \bigoplus_{(e,v)\in F^s(\Gamma)} \mathcal{O}_{y(e,v)} \to 0. \end{aligned}$$

Take $\otimes f^*T_X$:

$$\begin{aligned} 0 \to H^0(C,f^*T_X) \to & \bigoplus_{v\in V^s(\Gamma)} H^0(C_v,f^*T_X) \oplus \bigoplus_{e\in E(\Gamma)} H^0(C_e,f^*T_X) \\ & \to \bigoplus_{v\in V^2(\Gamma)} T_{f(y_v)}X \oplus \bigoplus_{(e,v)\in F^s(\Gamma)} T_{f(y(e,v))}X \\ \to H^1(C,f^*T_X) \to & \bigoplus_{v\in V^s(\Gamma)} H^1(C_v,f^*T_X) \oplus \bigoplus_{e\in E(\Gamma)} H^1(C_e,f^*T_X) \to 0. \end{aligned}$$

$$H^0(C_v,f^*T_X) = T_{f(C_v)}X,$$
$$H^1(C_v,f^*T_X) = H^1(C_v,\mathcal{O}_{C_v}) \otimes T_{f(C_v)}X \cong H^0(C_v,\omega_{C_v})^\vee \otimes T_{f(C_v)}X$$

Here $H^0(C_v,\omega_{C_v})$ is Hodge bundle $\mathbb{E}$. By splitting principle, assume $\mathbb{E} = L_1 \oplus \cdots \oplus L_g$, then

$$e(\mathbb{E}^\vee \otimes \mathbb{C}_1) = \prod_{i=1}^g c_1(L_i^\vee \otimes \mathbb{C}_1) = \prod_{i=1}^g c_1(L_i^\vee) + c_1(\mathbb{C}_1)$$
$$= \prod_{i=1}^g (-c_1(L_i)+u) = \sum_{k=1}^g (-1)^k c_k(\mathbb{E}) u^{g-k} = \sum_{k=1}^g (-1)^k \lambda_k u^{g-k} =: \Lambda_g^\vee(u)$$

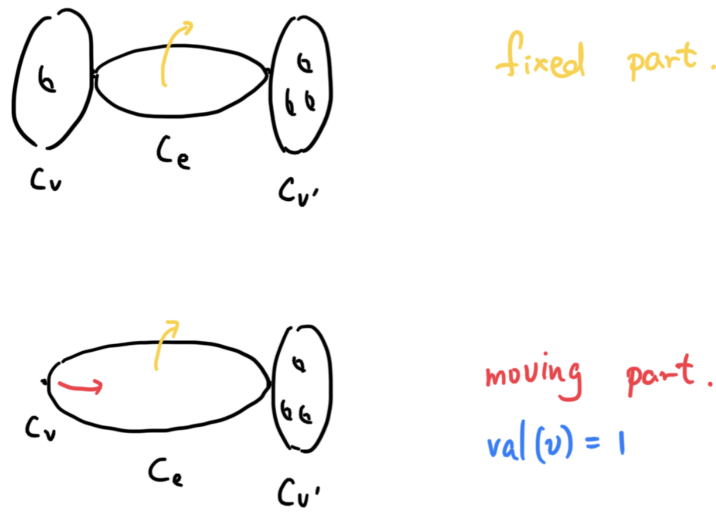

Figure 1.2: automorphism of $(C, 1, \ldots, n)$

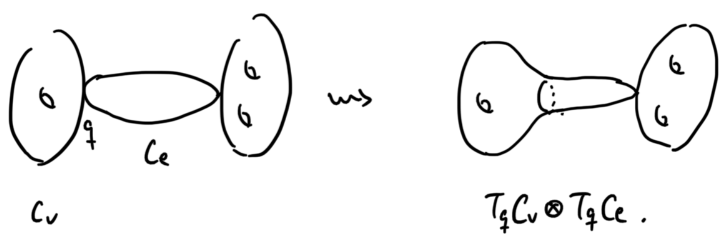

Figure 1.3: deformation of $(C, 1, \ldots, n)$

(2) $Euler_{\mathbb{T}}(B_4^m)/Euler_{\mathbb{T}}(B_1^m)$.

(2.1) $B_1 = Aut(C, 1, \ldots, n) = Hom(\Omega_C(p_1 + \cdots + p_n), \mathcal{O}_C)$: We should classify what is moving and what is fixed. Basically, we have

$$B_1^m = \bigoplus_{v \in V^1(\Gamma), (e,v) \in F(\Gamma)} T_{y(e,v)} C_e.$$

(2.2) $B_4 = Def(C, 1, \ldots, n) = Ext^1(\Omega_C(p_1 + \cdots + p_n), \mathcal{O}_C)$: $\mathbb{P}^1$ has just 1 complex structure, so we consider $g(C) \geq 1$. If we don't change node q, C will stay in the same class in $\overline{\mathcal{M}}_{g,n}$. Hence we must resolve the node, and geometrically, resolution depends on $T_qC_v \otimes T_qC_e$. So basically we have

$$B_4^m = \bigoplus_{v\in V^2(\Gamma), E_v=(e,e')} T_{y_v}C_e \otimes T_{y_v}C_{e'} \oplus \bigoplus_{(e,v)\in F^s(\Gamma)} T_{y(e,v)}C_v \otimes T_{y(e,v)}C_e$$

1.3 Aspinwall Morrison formula; Faber Pandaripande formula

Chapter 2

Quantum Cohomology

Chapter 3

Mirror Symmetry