The Ricci flow on closed 3 manifolds with strictly positive Ricci curvature

Jinghao Yu

Southern University of Science and Technology, China

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Background

The survey I do is related to the Poincare conjecture. In dimension 3, Poincare's conjecture states that:

If M is a simply-connected closed smooth manifold of dimension 3, then M and the 3-sphere $\mathbb{S}^3$ are diffeomorphic.

This conjecture has been solved in 2003 by Grigori Perelman. The Ricci flow plays an important role in Perelman proof. The thesis I do is to read R.S. Hamilton's paper in 1982, which is the origin of the Ricci flow.

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In Hamilton's paper, he showed this conclusion:

Theorem (Main theorem)

Let (M, g) be a closed Riemannian manifold of dimension 3 which admits a strictly positive Ricci curvature. Then (M, g) also admits a metric of constant positive sectional curvature.

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Theorem (space form)

Let M^n be a complete Riemannian manifold with constant sectional curvature K. Then the universal covering $\tilde{M}$ of M, with the covering metric, is isometric to:

(a) *hyperbolic space* $\mathbb{H}^n$, *if* $K = -1$,

(b) *Euclidean space* $\mathbb{R}^n$, *if* $K = 0$,

(c) *sphere* $\mathbb{S}^n$, *if* $K = 1$.

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Ricci flow

The **Ricci flow** is a partial differential equation of g in a Riemannian manifold (M, g_0) satisfying

$$\frac{\partial g}{\partial t} = -2Rc,$$

$$g(0) = g_0.$$

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Short time existence

Theorem (Hamilton, DeTurck)

If (M^n, g_0) is a closed Riemannian manifold, there exists a unique solution $g(t)$ to the Ricci flow defined on some positive time interval $[0, \epsilon)$ such that $g(0) = g_0$.

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Curvature estimate

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First, the Riemannian curvature can be expressed as the combination of Ricci curvature:

Theorem

When M is of dimension 3,

$$R_{ijkl} = R_{il}g_{jk} + R_{jk}g_{il} - R_{ik}g_{jl} - R_{jl}g_{ik} - \frac{1}{2}R(g_{il}g_{jk} - g_{ik}g_{jl})$$

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Let λ, μ, υ be the eigenvalues of Ricci curvature, then

Corollary

The sectional curvature is the only possible nonzero component of Riemannian curvature. Its value can be represented by the eigenvalue of Ricci curvature. For example,

$$R_{1221} = \frac{1}{2}(\lambda + \mu - \upsilon).$$

Curvature estimate

Thus, if we want to flow g to be a metric with constant sectional curvature, it suffices to control Ricci curvature. Mainly, we need to do two things:

(1) First, at each point we make λ, μ, υ approach each other as time flows.

(2) Second, we control the global difference of R_{ij}. In some extent, $R_{max} - R_{min}$ could be test this because scalar curvature $R = \lambda + \mu + \upsilon$.

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curvature estimate: first step

The L^2-distance among λ, μ, υ

$$(\lambda-\mu)^2+(\mu-\upsilon)^2+(\lambda-\upsilon)^2$$

is a good quantity to measure pointwise difference.

It could also be expressed as a geometric quantity. Let $S=|Rc|^2$ then

$$S-\frac{1}{3}R^2=\frac{1}{3}[(\lambda-\mu)^2+(\mu-\upsilon)^2+(\lambda-\upsilon)^2]$$

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Theorem

Let M^3 be a closed 3-manifold, with strictly positive Ricci curvature. Under the variation of Ricci flow, $\exists$ constant $\delta > 0$ and $C \in \mathbb{R}_+$ both depending only on the initial metric such that on $0 \leq t < T$ we have

$$S - \frac{1}{3}R^2 \leq CR^{2-\delta}.$$

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Theorem (gradient estimate)

Let M^3 be a closed Riemannian 3 manifold with positive Ricci curvature. For every $\eta > 0$, $\exists$ constant $\mathcal{C} = \mathcal{C}(\eta, g(0))$ depending only on η and the initial value of the metric such that on $0 \le t < T$ we have

$$|\nabla_i R|^2 \le \eta R^3 + \mathcal{C}$$

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Theorem

$R_{max}/R_{min} \to 1$ *as* $t \to T$.

Blow up

Up to now, everything goes right.

- difference of λ, μ, $\upsilon \Longrightarrow R_{max} - R_{min}$.
- R_{max}/R_{min} is also controlled.
- If we can control the value of R, then $R_{max} - R_{min}$ is well-controlled.
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Theorem

If M is a Riemannian 3-manifold and $R \geq \rho > 0$ at $t = 0$, then R goes to infinity before the time $3/2\rho$.

Proof: (Sketch)

$$\frac{\partial R}{\partial t} \geq \Delta R + \frac{2}{3} R^2$$

$$\frac{df}{dt} = \frac{2}{3} f^2, \quad f(0) = \rho$$

Apply comparison principle:

$$R \geq f = \frac{3\rho}{3 - 2\rho t} \to \infty \quad \text{as} \quad t \to 3/2\rho.$$

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It could also be shown that the Riemannian curvature will blow up.

Theorem

Suppose (M, g_0) *is a 3-manifold with strictly positive Ricci curvature. Then its Ricci flow has a unique solution on a maximal time interval* $0 \leq t < T \leq 3/2\rho$, *and* $\max_M |R_{ijkl}| \to \infty$ *as* $t \to T$.

Normalized Ricci flow

To avoid the blow-up, we introduce the normalized Ricci flow.

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Definition (normalized Ricci flow)

The normalized equation of Ricci flow on (M^n, g_0):

$$\frac{\partial}{\partial \tilde{t}} \tilde{g}_{ij} = \frac{2}{n} \tilde{r} \tilde{g}_{ij} - 2\tilde{R}_{ij}$$

$$\tilde{g}(0) = g_0$$

where $\tilde{r} = \int \tilde{R} d\tilde{\mu} / \int d\tilde{\mu}$ is the average of scalar curvature.

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Lemma

It promise $\tilde{T} = \infty$ and $\tilde{R}_{\max}$ could be bounded by a constant C:

$$\tilde{R}_{\max}(\tilde{t}) \leq C < \infty \quad \textit{on} \quad 0 \leq \tilde{t} < \tilde{T}.$$

Curvature estimate: normalized case

An interesting fact is that the normalized Ricci flow could be given by multiplying g with a appropriate coefficient $\varphi(t)$:

$$g \longrightarrow \tilde{g} = \varphi(t)g.$$

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$\exists$ *constants* $C < \infty$ *and* $\delta > 0$ *such that*

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Corollary

$|\tilde{R}_{ij} - \frac{1}{3}\tilde{r}\tilde{g}_{ij}| \leq Ce^{-\delta\tilde{t}}$ *where* $\tilde{r} = \int \tilde{R}d\tilde{\mu} / \int d\tilde{\mu}$.

C^∞-convergence

Hence, if the metric $\tilde{g}$ has good enough convergence such that we can exchange the order of taking limit and taking differential, then $\tilde{g}(\infty)$ has a constant sectional curvature.

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Theorem

As $\tilde{t} \to \infty$ the metrics $\tilde{g}_{ij}(\tilde{t})$ converge to the smooth limit metric $\tilde{g}_{ij}(\infty)$ in C^∞-topology. In special, the curvature $\tilde{R}_{ij}(\tilde{t})$ converge to the curvature $\tilde{R}_{ij}(\infty)$.

The limit Ricci curvature is

$$\tilde{R}_{ij}(\infty) = \frac{1}{3}\tilde{r}(\infty)\tilde{g}_{ij}(\infty)$$

Hence, M has a metric $\tilde{g}(\infty)$ with sectional curvature $\frac{1}{6}\tilde{r}(\infty)$.

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Proof: (Sketch)

- By tensor interpolation inequality we have

$$\int |\tilde{\nabla}^n \tilde{Rc}|^p d\tilde{\mu} \leq Ce^{-\delta\tilde{t}} \quad \forall n, p \in \mathbb{N}_+$$

- By Sobolev's inequality we show: for every $n \in \mathbb{N}_+$ we have

$$\|\tilde{\nabla}^n \tilde{Rc}(\tilde{t})\|_\infty \leq Ce^{-\delta\tilde{t}}$$

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Take this control into the normalized Ricci flow, we can control $\|\partial^n \tilde{g}\|_\infty$:

Corollary

For all $n \in \mathbb{N}_+$,

$$\|\partial^n \tilde{g}\|_\infty \le C e^{-\delta \tilde{t}}$$

for some constant $C < \infty$, $\delta > 0$ *depending on* n.

It implies that $\forall\ \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ we have

$$\frac{\partial^{|\alpha|} g(t)}{\partial x^\alpha} \to \frac{\partial^{|\alpha|} g(\infty)}{\partial x^\alpha} \quad \text{uniformly w.r.t } M \text{ as } t \to \infty.$$

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Corollary (main theorem)

The limit metric $\tilde{g}_{ij}(\infty)$ has constant positive sectional curvature.

Corollary

If M is simply connected smooth manifold endowed with a strictly positive Ricci curvature, then M is diffeomorphic to a 3-sphere.

Thank you!