Quintic threefold and mirror symmetry

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Chapter 1

Differential Topology

1.1 Chern class

Let E be a differentiable complex vector bundle of rank r over a differentiable manifold X, and let $F = dA + A \wedge A$ be the curvature of a connection A on E.

Definition 1.1.1 (total Chern class)**.** *We define the total Chern class of E, $c(E)$, by*

$$\begin{aligned} c(E) &= \det\left(1 + \frac{i}{2\pi}F\right) \\ &= 1 + \frac{i}{2\pi}TrF + \dots \\ &= 1 + c_1(E) + c_2(E) + \dots \in H^0(X;\mathbb{R}) \oplus H^2(X;\mathbb{R}) \oplus \dots \end{aligned}$$

Proposition 1.1.2.

(1) If E, F are two complex vector bundles over X, then $c(E \oplus F) = c(E)c(F)$

(2) If $0 \to A \to B \to C \to 0$ is a short exact sequence of sheaves, then $c(B) = c(A)c(C)$.

Definition 1.1.3 (Chern Character)**.** *Suppose $\exists\ x_i \in H^2(X;\mathbb{R})$ such that $c(E) = \prod_{i=1}^r(1 + x_i)$ $(r \equiv rk(E))$. Then the Chern character class $ch(E)$ is defined by $ch(E) = \sum_i e^{x_i}$ (Taylor expansion). Then we find*

$$ch(E) = r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + \dots$$

Note $ch(E \oplus F) = ch(E) + ch(F)$ and $ch(E \otimes F) = ch(E)ch(F)$.

Definition 1.1.4 (Todd class)**.**

$$td(E) = \prod_{i=1}^r \frac{x_i}{1 - e^{-x_i}} = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \dots$$

Note that $td(E \oplus F) = td(E)td(F)$.

1.2 The Grothendieck-Riemann-Roch formula

Let E be a sheaf or holomorphic vector bundle over some variety X; let $H^k(E)$ be the Cech cohomology group of E over X. Define $\chi(E) = \sum_k (-1)^k \dim H^k(E)$. The Grothendieck-Riemann-Roch formula calculates

$$\chi(E) = \int_X ch(E) \wedge td(X).$$

1.3 Serre Duality

Definition 1.3.1. *For an almost complex manifold X one defines the complex vector bundles*

$$\bigwedge_{\mathbb{C}}^{k} X := \bigwedge^{k} (T_{\mathbb{C}}X)^* \quad \text{and} \quad \bigwedge^{p,q} X := \bigwedge^{p} (T^{1,0}X)^* \otimes_{\mathbb{C}} \bigwedge^{q} (T^{0,1}X)^*.$$

Their sheaves of sections are denoted by $\mathcal{A}^k_{X,\mathbb{C}}$ and $\mathcal{A}^{p,q}_X$, respectively. Elements in $\mathcal{A}^{p,q}(X)$, i.e. global sections of $\mathcal{A}^{p,q}(X)$, are called forms of type (or bidegree) (p,q).

The complex vector bundles Ω^p_X and $\bigwedge^{p,0} X$ of a complex manifold X can be identified.

Corollary 1.3.2.

$$\bigwedge_{\mathbb{C}}^{k} X = \bigoplus_{p+q=k} \bigwedge^{p,q} X \quad \text{and} \quad \mathcal{A}^k_{X,\mathbb{C}} = \bigoplus_{p+q=k} \mathcal{A}^{p,q}_X.$$

Moreover, $\overline{\bigwedge^{p,q} X} = \bigwedge^{q,p} X$ and $\overline{\mathcal{A}^{p,q}_X} = \mathcal{A}^{q,p}_X$.

Definition 1.3.3 (Dolbeault cohomology)**.** *Let X be endowed with an integrable almost complex structure. Then the (p,q)-Dolbeault cohomology is the vector space*

$$H^{p,q}(X) := H^q(\mathcal{A}^{p,-}(X), \bar{\partial}) = \frac{Ker(\bar{\partial} : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q+1}(X))}{Im(\bar{\partial} : \mathcal{A}^{p,q-1}(X) \to \mathcal{A}^{p,q}(X))}$$

Corollary 1.3.4. *The Dolbeault cohomology of X computes the cohomology of the sheaf Ω^p_X, i.e. $H^{p,q}(X) \cong H^q(X, \Omega^p_X)$.*

Definition 1.3.5. *By $\mathcal{A}^{p,q}(E)$ we denote the sheaf*

$$U \longmapsto \mathcal{A}^{p,q}(U, E) := \Gamma(U, \bigwedge^{p,q} X \otimes E).$$

Let α be a section of $\mathcal{A}^{p,q}(E)$. The differential d is not well-defined on α.

Lemma 1.3.6. *If E is a holomorphic vector bundle then there exists a natural $\mathbb{C}$-linear operator $\bar{\partial}_E : \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p,q+1}(E)$ with $\bar{\partial}_E^2 = 0$ which satisfies the Leibniz rule $\bar{\partial}_E(f \cdot \alpha) = \bar{\partial}(f) \wedge \alpha + f\bar{\partial}_E(\alpha)$.*

Proof. Locally $\alpha = \sum \alpha_i \otimes s_i$ with $\alpha_i \in \mathcal{A}_X^{p,q}$ and $s_i \in E$. Then set

$$\bar{\partial}_E \alpha := \sum \bar{\partial}(\alpha_i) \otimes s_i. \qquad \square$$

Definition 1.3.7. *The Dolbeault cohomology of a holomorphic vector bundle E is*

$$H^{p,q}(X,E) := H^q(\mathcal{A}^{p,-}(X,E), \bar{\partial}_E) = \frac{Ker(\bar{\partial}_E : \mathcal{A}^{p,q}(X,E) \to \mathcal{A}^{p,q+1}(X,E))}{Im(\bar{\partial}_E : \mathcal{A}^{p,q-1}(X,E) \to \mathcal{A}^{p,q}(X,E))}.$$

Corollary 1.3.8. $H^{p,q}(X,E) \cong H^q(X, E \otimes \Omega_X^p)$.

Let E be a holomorphic vector bundle over a compact complex manifold X of dimension n and consider the natural pairing

$$H^{p,q}(X,E) \times H^{n-p,n-q}(X,E^*) \longrightarrow \mathbb{C}, \quad (\alpha,\beta) \longmapsto \int_X \alpha \wedge \beta$$

Proposition 1.3.9. *Let X be a compact complex manifold. For any holomorphic vector bundle E on X the natural pairing*

$$H^{p,q}(X,E) \times H^{n-p,n-q}(X,E^*) \longrightarrow \mathbb{C}$$

is non-degenerate.

Corollary 1.3.10. *By Dolbeault isomorphism:*

$$H^q(X, \Omega^p \otimes E) \times H^{n-q}(X, \Omega^{n-p} \otimes E^*) \to \mathbb{C}$$

is non-degenerate. Furthermore, let $p = 0$

$$H^q(X,E) \times H^{n-q}(X, K_X \otimes E^*) \to \mathbb{C}$$

is non-degenerate.

In the special case where X is Calabi-Yau, K_X is trivial and

$$H^q(X,E) \times H^{n-q}(X,E^*) \to \mathbb{C}$$

is non-degenerate.

1.4 Chern class of $\mathbb{P}^n$

Let $H = \mathcal{O}(1)$ be the hyperplane bundle on $\mathbb{P}^n$. Consider homogeneous coordinate $[X_0, \dots, X_n]$. Since $X_0^2 + \cdots + X_n^2 = 1$, differentiate this formula we find $X_i \frac{\partial}{\partial X_i} = 0$. This gives the exact sequence, the Euler sequence:

$$0 \to \mathbb{C} \to H^{\oplus(n+1)} \to T\mathbb{P}^n \to 0$$

$$(a_0 X_0, \dots, a_n X_n) \mapsto a_i X_i \frac{\partial}{\partial X_i}$$

where $a_i \in \mathbb{C}$.

Since $c(\mathbb{C}) = 1$, $c(\mathbb{P}^n) = c(T\mathbb{P}^n) = c(H^{\oplus(n+1)}) = [c(H)]^{n+1}$. Let $x = c_1(H)$. Then

$$c(\mathbb{P}^n) = (1+x)^{n+1} = \sum_{i=0}^{n} \begin{pmatrix} n+1 \\ i \end{pmatrix} x^i$$

It gives an example to check Chern-Gauss-Bonnet formula: $c_n(\mathbb{P}^n) = (n+1)x^n$. The Poincare duality gives that

$$\int_{\mathbb{P}^n} x^n = \#\text{intersection of } n \text{ transverse hyperplane } H\ (\cong \mathbb{P}^{n-1}) = 1$$

$$\int c_n(\mathbb{P}^n) = n+1 = \chi(\mathbb{P}^n)$$

This corresponds to the conclusion in CW-structure of $\mathbb{P}^n$.

1.5 adjunction formulas

Let X be a smooth hypersurface in $\mathbb{P}^n$ defined as the zero-locus of a degree d polynomial, p (so p is a section of $\mathcal{O}_{\mathbb{P}^n}(d)$, or H^d). The normal bundle N_X of X in $\mathbb{P}^n$ is just $\mathcal{O}(d)|_X$. As a result, we have an exact sequence

$$0 \to TX \to T\mathbb{P}^n|_X \to \mathcal{O}(d)|_X \to 0.$$

Now $ch(H) = e^x \Rightarrow ch(H^d) = e^{dx} = 1 + c_1(H^d) + \ldots$, so

$$c(\mathcal{O}(d)) = 1 + c_1 = 1 + dx$$

$$c(X) = \frac{(1+x)^{n+1}}{1+dx}$$

The Euler class $e(X)$ of the normal bundle of a subvariety $X \subset \mathbb{P}^n$ is equal to its Thom class, namely its Poincare dual cohomology cycly. This means

$$\int_X \theta = \int_{\mathbb{P}^n} \theta e(X).$$

In the case of hypersurface, the normal bundle is one-dimensional, so $e(X) = c_{top}(N_{X/\mathbb{P}^4}) = c_1(\mathbb{O}(d)) = d\ x$.

1.6 quintic hypersurface

Now consider the quintic hypersurface in $\mathbb{P}^4$. A quintic hypersurface Q in $\mathbb{P}^4$ has

$$c(Q) = \frac{(1+x)^5}{(1+5x)} = 1 + 10x^2 - 40x^3.$$

Note that $c_1(Q) = 0$, so Q is a Calabi-Yau manifold. Its Euler characteristic is

$$\int_Q -40x^3 = \int_{\mathbb{P}^4} -40x^3(5x) = -200$$

A general formula is given in [3], page 11: If X is a hypersurface in $\mathbb{CP}^n$ with degree d, then its Euler characteristic is

$$\chi(X) = \frac{1}{d} \cdot ((1-d)^{n+1} + d \cdot (n+1) - 1).$$

Chapter 2

Calabi-Yau Manifolds and Mirror Symmetry

2.1 Calabi-Yau manifolds

Definition 2.1.1 (Calabi-Yau manifold 1). *Let $m \geq 2$. A Calabi-Yau m-fold is a quadruple (M, J, g, Ω) such that (M, J) is a compact m-dimensional complex manifold, g a Kahler metric on (M, J) with holonomy group Hol(g)=SU(m), and Ω a nonzero constant (m,0)-form on M called the holomorphic volume form, which satisfies*

$$\omega^m/m! = (-1)^{m(m-1)/2}(i/2)^m \Omega \wedge \bar{\Omega} \tag{*}$$

where ω is the Kahler form of g. The constant factor in (*) is chosen to make Re Ω a calibration.

Definition 2.1.2 (Calabi-Yau manifold 2). *A Calabi-Yau manifold is a compact Kahler manifold X with trivial canonical bundle $\omega_X \cong \mathcal{O}_X$.*

Example 2.1.3. *If X is a simply-connected Calabi-Yau 3-fold, then $H^1(X, \mathcal{O}_X) = 0$.*

$$H^1(X, \mathcal{O}_X) \xlongequal[\text{duality}]{\text{Serre}} H^2(X, \mathcal{O}_X \otimes \omega_X)^* = H^{3,2}(X, \mathbb{C}) = H^{0,1}(X, \mathbb{C}) = 0$$

2.2 Complex structure and Bogomolov-Tian-Todorov Theorem

Definition 2.2.1. *Let X be a differentiable manifold of dimension $2n$. Suppose that J is a differentiable vector bundle isomorphism*

$$J : TX \to TX$$

such that $J^2 = -I$. J is called an almost complex structure for the differentiable manifold X. If X is equipped with an almost complex structure J, then (X, J) is called an almost complex manifold.

In local (real) coordinate $\{\frac{\partial}{\partial x^a}\}_{a=1}^{2n}$ we can write J in terms of a matrix $J^a_{\ b}$, where $J(\frac{\partial}{\partial x^a}) = J^c_{\ a}\frac{\partial}{\partial x^c}$.

Since $P = (1-iJ)/2$ is a projection onto the holomorphic sub-bundle of the tangent bundle (tensor with $\mathbb{C}$) and $\bar{P} = (1+iJ)/2$ is the anti-holomorphic projection, the condition of integrability for finding complex coordinates is

$$\bar{P}[PX, PY] = 0$$

where $X = X^a\frac{\partial}{\partial x^a}$ and $Y = Y^b\frac{\partial}{\partial x^b}$. Define the Nijenhuis tensor N by $N(X,Y) = [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y]$. In local coordinates x^a,

$$N^a_{\ bc} = J^d_{\ b}(\partial_d J^a_{\ c} - \partial_c J^a_{\ d}) - J^d_{\ c}(\partial_d J^a_{\ b} - \partial_b J^a_{\ d}).$$

The integrability condition is equivalent to $N \equiv 0$. It is also equivalent to $\bar{\partial}^2 = 0$.

In complex coordinate, let us fix a complex structure and compatible complex coordinates $z^1, \ldots, z^n$. We use $J^a_{\ b}$, $J^{\bar{a}}_{\ b}$, $J^a_{\ \bar{b}}$ and $J^{\bar{a}}_{\ \bar{b}}$. In fact, because $J^a_{\ b}z^b = iz^a = iz^b\delta^a_{\ b}$, $J^{\bar{a}}_{\ \bar{b}}z^b = iz^{\bar{a}} = iz^{\bar{b}}\delta^{\bar{a}}_{\ \bar{b}}$. J is diagonalized in these coordinates, so that $J^a_{\ b} = i\delta^a_{\ b}$ and $J^{\bar{a}}_{\ \bar{b}} = -i\delta^{\bar{a}}_{\ \bar{b}}$, with mixed component zero.

Now given a smooth manifold X, we try to study all complex structure could be endowed in such a manifold X. At first, one naively define the set

$$\mathcal{A}_c(X) := \{J \in End(TX) | J \text{ is an integrable almost complex structure in } X\}.$$

But that is too redundant. Recall that two complex manifolds (X, J) and (X', J') are isomorphic if there exists a diffeomorphism $F : X \to X'$ such that $dF \circ J = J' \circ dF$. Thus, the set of diffeomorphism classes of complex structures J on a fixed smooth manifold X is the quotient of the set $\mathcal{A}_c(X)$ by the action of the diffeomorphism group

$$\mathrm{Diff}(X) \times \mathcal{A}_c(X) \longrightarrow \mathcal{A}_c(X), (F, J) \longmapsto dF \circ J \circ (dF)^{-1}.$$

Next we define the infinitesimal deformation of a complex structure by its power series expansion.

We start out with the set

$$\mathcal{A}_{ac}(X) := \{J | J^2 = -id\} \subset End(TX)$$

of all almost complex structures on X. It could be shown that $\mathcal{A}_{ac}(X)$ is an infinite dimensional manifold. Moreover, this statement is no longer true for $\mathcal{A}_c(X)$. Let $J(t)$ be a continuous path of almost complex structures with $J(0) = J$. Then one has a continuous family of such decompositions $T_{\mathbb{C}}M = T_t^{1,0} \oplus T_t^{0,1}$ or, equivalently, of subspaces $T_t^{0,1} \subset T_{\mathbb{C}}M$ (retrieve $T_t^{1,0}$ by conjugation).

Thus, for small t the deformation $J(t)$ of J can be encoded by a map

$$\phi(t) : T^{0,1} \longrightarrow T^{1,0} \text{ with } v + \phi(t)(v) \in T_t^{0,1}.$$

We write $T^{1,0}$ and $T^{0,1}$ for subbundles defined by J. Explicitly, one has

$$\phi(t) = -pr_{T_t^{1,0}} \circ j,$$

where $j : T^{0,1} \subset T_{\mathbb{C}}$ and $pr_{T_{\mathbb{C}}^{1,0}} : T_{\mathbb{C}} \to T_t^{1,0}$ are the natural inclusion respectively projection.

Conversely, if $\phi(t)$ is given, then one defines for small t

$$T_t^{0,1} := (id + \phi(t))(T^{0,1}).$$

Let us now consider the power series expansion

$$\phi(t) = \phi_0 + \phi_1 t + \phi_2 t^2 + \ldots$$

Lemma 2.2.2. *The integrability equation $[T_t^{0,1}, T_t^{0,1}] \subset T_t^{0,1}$ is equivalent to the Maurer-Cartan equation*

$$\bar{\partial}\phi(t) + [\phi(t), \phi(t)] = 0 \in \mathcal{A}^{0,2}(T^{1,0}X)$$

This yields a recursive system of equations:

$$\begin{aligned} 0 &= \bar{\partial}\phi_1 \\ 0 &= \bar{\partial}\phi_2 + [\phi_1, \phi_1] \\ &\ldots \\ 0 &= \bar{\partial}\phi_k + \sum_{0<i<k} [\phi_i, \phi_{k-i}]. \end{aligned}$$

The first-order equation $\bar{\partial}\phi_1 = 0$ defines an element $[\phi_1] \in H^1(X, \mathcal{T}_X)$.

Definition 2.2.3 (Kodaira-Spencer class)**.** *The* ***Kodaira-Spencer class*** *of a one-parameter deformation $J(t)$ of the complex structure J is the induced cohomology class $[\phi_1] \in H^1(X, \mathcal{T}_X)$.*

Proposition 2.2.4. *Let X be a complex manifold. There is a natural bijection between all first-order deformations of X and elements of $H^1(X, \mathcal{T}_X)$.*

Corollary 2.2.5. *A first-order deformation $v \in H^1(X, \mathcal{T}_X)$ cannot be integrated if $[v, v] \in H^2(X, \mathcal{T}_X)$ does not vanish.*

Proposition 2.2.6 (Bogomolov-Tian-Todorov unobstructedness theorem)**.** *Let X be a Calabi-Yau manifold and let $v \in H^1(X, \mathcal{T}_X)$. Then there exists a formal power series $\phi_1 t + \phi_2 t^2 + \ldots$ with $\phi_i \in \mathcal{A}^{0,1}(\mathcal{T}_X)$ satisfying the Maurer-Cartan equations*

$$\bar{\partial}\phi_1 = 0 \text{ and } \bar{\partial}\phi_k = -\sum_{0<i<k} [\phi_i, \phi_{k-i}],$$

with $[\phi_1] = v$ and such that

$$\eta(\phi_i) \in \mathcal{A}^{n-1,1}(X) \text{ is } \partial - exact$$

for all $i > 1$.

Remark 2.2.7. *The corollary 2.2.5 states if $H^2(X, \mathcal{T}_X) = 0$, the evolution of the Maurer-Cartan equation has no obstruction. But for a Calabi-Yau manifold X, its $H^2(X, \mathcal{T}_X)$ usually does not vanish, e.g. for a Calabi-Yau quintic 3-fold,*

$$H^2(X, \mathcal{T}_X) = H^2(X, \Omega^2_X) = H^{2,2}(X, \mathbb{C}) = H^{1,1}(X, \mathbb{C}) = \mathbb{C} \neq 0.$$

But even if the second cohomology group does not vanish, the deformation of complex structure can be done in a Calabi-Yau manifold. That is why BTT unobstructedness theorem is important.

2.3 Kahler moduli space

2.4 Pesudo-holomorphic curves

Definition 2.4.1 (J-holomorphic curves)**.** *Let (Σ, j) be a Riemann surface, (X, J) be an almost complex manifold. A smooth map $u : \Sigma \to X$ is called **J-holomorphic** if u_* satisfies*

$$J \circ u_* = u_* \circ j$$

Equivalently, for a map $u : \Sigma \to X$, put

$$\bar{\partial}_J(u) = \frac{1}{2}(u_* + J \circ u_* \circ j) \in \Gamma(\Sigma, \Omega^{0,1}_\Sigma \otimes u^* \mathcal{T}_X),$$

It is clear that u is J-holomorphic if and only if $\bar{\partial}_J(u) = 0$.

Definition 2.4.2. *$u : \Sigma \to X$ is **somewhere injective**, or **simple** if $\exists$ a point $z \in \Sigma$ such that u_* is injective at z and $u^{-1}(u(z)) = \{z\}$.*

For convenience, let us define

- $\mathrm{Map}(\Sigma, X) = \{u : \Sigma \to X | u \text{ is smooth}\}$
- For any $\eta \in H^2(X, \mathbb{Z})$, let
  $\mathrm{Map}(\Sigma, X, \eta) = \{u \in Map(\Sigma, X) |\ u \text{ is a simple map, [im u]=}\eta\}$

What we want is to give a math definition about Gromov-Witten invariant of X, an enumerative invariant associated to the Kahler form ω of X. To accomplish this aim, we use almost complex structure J compatible with ω to define the moduli space of J-holomorphic curves at first. Then we try to show the invariant defined independent to the choice of J.

Definition 2.4.3 (compatible almost complex structure)**.** *Fix a real Kahler form of a Kahler metric (or real symplectic form, or) ω of a Kahler manifold X. We say an almost complex structure J is compatible with ω if*

$$\omega(v, Jv) > 0 \quad \forall v \in \mathcal{T}_X, v \neq 0, \text{and}$$

$$\omega(Jv, Jw) = \omega(v, w) \quad \forall v, w \in \mathcal{T}_X.$$

Let $\mathcal{J}(\omega)$ be the set of almost complex structure compatible with ω.

Definition 2.4.4. *Given a homological class $\eta \in H^2(X, \mathbb{Z})$, an associated almost complex structure J, put*

$$M(\eta, J, \Sigma) = \textit{the zero locus of } \bar{\partial}_J$$

$$= \textit{Moduli space of simple } J\textit{-holomorphic map representing the homology class } \eta$$

We want to say something about the space $M(\eta, J, \Sigma)$. This space has nice properties generically. For each $u \in \mathcal{X} = Map(\Sigma, X)$, define the fibre

$$\mathcal{E}_u = \Gamma(\Sigma, \Omega^{0,1}_\Sigma \otimes u^*\mathcal{T}_X)$$

It gives a vector bundle $\mathcal{E}$ over $\mathcal{X}$. Because $\bar{\partial}_J$ is a smooth section from $\mathcal{X}$ to $\mathcal{E}$, we can define a map:

$$\mathcal{T}_{\mathcal{X},u} \stackrel{(\bar{\partial}_J)_*}{\to} \mathcal{T}_{\mathcal{E},(u,0)} = \mathcal{T}_{\mathcal{X},u} \oplus \mathcal{E}_u \stackrel{\pi}{\to} \mathcal{E}_u.$$

u is called **regular** if $\pi \circ (\bar{\partial}_J)_*$ is surjective.

$$\mathcal{J}_{reg}(\eta, \omega, \Sigma) = \{J \in \mathcal{J}(\omega) | u \text{is regular for all } u \in M(\eta, J, \Sigma)\}.$$

Theorem 2.4.5.

(1) If $J \in \mathcal{J}_{reg}(\eta, \omega, \Sigma)$, then $M(\eta, J, \Sigma)$ is a smooth manifold of real dimension $n(2 - 2g) + 2c_1(X) \cdot \eta$.

(2) $\mathcal{J}_{reg}(\eta, \omega, \Sigma)$ has a second category in $\mathcal{J}(\omega)$.

The following question is to find a criterion to the regularity of u.

Theorem 2.4.6 (Regularity criterion)**.** *If J is an integrable almost complex structure on X, and $u : \mathbb{P}^1 \to X$ is a J-holomorphic curve, then u is regular if in the decomposition $u^*\mathcal{T}_X = \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(a_i)$ we have $a_i \geq -1$ for all i.*

Remark 2.4.7. *In the criterion, we use a classical theorem from Grothendick: any holomorphic vector bundle on $\mathbb{P}^1$ decomposes as a direct sum of line bundles. Any line bundle on $\mathbb{P}^1$ is determined by c_1.*

$$\mathcal{O}_{\mathbb{P}^1}(a) = \textit{the line bundle with } c_1 = a$$

In the special case that $\Sigma = \mathbb{P}^1$, X is a Calabi-Yau 3-fold, e.g. quintic 3-fold, we have $n = 3$, $c_1(X) = 0$, $g(\mathbb{P}^1) = 0$. The regularity criterion in $u : \mathbb{P}^1 \to X$ becomes

Proposition 2.4.8. *u is regular if in the decomposition $u^*\mathcal{T}_X = \mathcal{O}_{\mathbb{P}^1}(a) \oplus_i \mathcal{O}_{\mathbb{P}^1}(b)$ we have $a = b = -1$.*

If $\in \mathcal{J}_{reg}(\eta, \omega, \mathbb{P}^1)$ then by Theorem 2.4.5

$$\dim_{\mathbb{R}} M(\eta, J, \mathbb{P}^1) = 3(2 - 2 \cdot 0) + 2 \cdot 0 \cdot \eta = 6$$

$$Aut(\mathbb{P}^1) = PSL(2, \mathbb{C}), \quad \dim_{\mathbb{R}} Aut(\mathbb{P}^1) = 6$$

$$n_\eta := \#\overline{M(\eta, J, \mathbb{P}^1)/PSL(2,\mathbb{C})} \text{ is finite.}$$

The number n_η is the definition of Gromov-Witten invariant in this special case. It describes the number of J-holomorphic curves with image in the homology class η under the automorphism equivalence of $\mathbb{P}^1$ is generically finite. Since $h^2(X;\mathbb{Z}) = h^{1,1}(X;\mathbb{Z}) = 1$, we use $d \in \mathbb{Z}$ to represent the homology class η in $H^2(X;\mathbb{Z})$, so

$$n_d := \#\overline{M(d, J, \mathbb{P}^1)/PSL(2,\mathbb{C})}.$$

is well-defined.

2.5 Mirror pair of quintic 3-fold

Let $f_\varphi = x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5\varphi x_0 x_1 x_2 x_3 x_4$.

Let X_φ be a smooth hypersurface $f_\varphi = 0$ in $\mathbb{P}^4$. The Hodge diamond of X is

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

There is a $G = (\mathbb{Z}/5\mathbb{Z})^5$ action on $\mathbb{P}^4$:

$$(\mathbb{Z}/5\mathbb{Z})^5 \times \mathbb{P}^4 \longrightarrow \mathbb{P}^4, \quad \lambda = e^{2\pi i/5}.$$

$$(a_0, a_1, a_2, a_3, a_4), [z_0 : z_1 : z_2 : z_3 : z_4] \mapsto [\lambda^{a_0} z_0 : \lambda^{a_1} z_1 : \lambda^{a_2} z_2 : \lambda^{a_3} z_3 : \lambda^{a_4} z_4].$$

For those smooth X_φ, take the quotient of X_φ by $(\mathbb{Z}/5\mathbb{Z})^5$, we get some A_n singularities. Blow-up the singularities of X_φ/G, get a new smooth Calabi-Yau manifold Y_φ with extra 100 divisors $\mathbb{P}^1$. The Hodge diamond of Y_φ is

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 101 & & 0 & \\ 1 & & 1 & & 1 & & 1 \\ & 0 & & 101 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

We can see X_φ and Y_φ has symmetry Hodge diamond over the diagonal line. This is the first (maybe) mirror pair found in history.

2.6 Yukawa coupling and mirror symmetry

In physics(QFT), Yukawa coupling is a quantity to describe the interaction between Neutrino and Higgs field. There are two kinds of Yukawa couplings in physics. Let X be a quintic 3-fold.

The A-model is of the Kahler form of $X = X_\varphi$:

$$\langle h, h, h\rangle_A := 5 + \sum_{d=1}^{\infty} n_d d^3 \frac{q^d}{1-q^d},$$

where n_d is the Gromov-Witten invariant defined in 2.4.

The B-model is of the complex structure of $\check{X} = Y_\varphi$:

$$\langle \partial_z, \partial_z, \partial_z\rangle_B := \int_{\check{X}} \check{\Omega} \wedge \partial_z\partial_z\partial_z\check{\Omega},$$

where $\check{\Omega}$ is the normalized Calabi-Yau 3-form of $\check{X}$. We choose a Calabi-Yau 3-form Ω, the normalized Calabi-Yau 3-form $\check{\Omega}$ is

$$\check{\Omega} = \frac{\Omega}{\int_{\beta_0} \Omega},$$

where β_0 is a three torus by taking limit $\varphi \to \infty$.

The mirror conjecture states that under the coordinate map $q = e^{2\pi i w(z)}$ two Yukawa coupling is equal:

$$\langle h, h, h\rangle_A = \langle \partial_z, \partial_z, \partial_z\rangle_B,$$

where

$$w(z) = \int_{\beta_1} \check{\Omega} = \frac{\int_{\beta_1} \Omega}{\int_{\beta_0} \Omega}$$

for some β_1 in Hodge bundle and $\{\beta_0, \beta_1\}$ is a part of a symplectic basis of Hodge bundle.

Historically, physicists wanted to compute $\langle h, h, h\rangle_A$. But in 1980s the Gromov-Witten invariant is unknown for $n \geq 3$. $n_1 = 2875$ is a classical result, and in 1986 S.Katz computes $n_2 = 609250$. Thus the mirror conjecture gives a way to compute Gromov-Witten invariant by B-model Yukawa coupling. By computation, $\exists$ constant c_1, c_2 such that

$$\langle \partial_z, \partial_z, \partial_z\rangle_B = -c_1 - 575\frac{c_1}{c_2}q - \frac{1950750}{2}\frac{c_1}{c_2^2}q^2 - \frac{10277490000}{6}\frac{c_1}{c_2^3}q^3 + \dots$$

$$\langle h, h, h\rangle_A = 5 + n_1 q + (8n_2 + n_1)q^2 + (27n_3 + n_1)q^3 + \dots$$

$n_1 = 2875$ shows $c_1 = -5$, $c_2 = 1$, and get the Table 2.1.

It is conjectured that n_d is the value as above. The conjecture for all d was proven by Givental in 1996 and Lian, Liu, and Yau in 1997.

degree d	**Gromov-Witten invariant** n_d
1	2875
2	609250
3	317206375
4	242467530000
5	229305888887625
6	248249742118022000
7	295091050570845659250
8	375632160937476603550000
9	503840510416985243645106250
10	704288164978454686113488249750
11	1017913203569692432490203659468875
12	1512323901934139334751675234074638000
13	2299488568136266648325160104772265542625
14	3565959228158001564810294084668822024070250
15	5624656824668483274179483938371579753751395250
16	9004003639871055462831535610291411200360685606000
17	14602074714589033874568888115959699651605558686799250
18	239544445228532694121482634657728114956109652255216482000
19	39701666985451876233836105884497728824100003703180307231625
20	66408603312404471392397268104340892583652834904833089314920000

Table 2.1: computation by B-model

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