Fano Varieties

Jinghao Yu

January 9, 2022

Abstract

The article is a survey of classification of Fano varieties. Section 2 is about del Pezzo surfaces and Fano 3-folds; section 4 is about the boundness of Fano varieties and BAB theorem.

Contents

1 Basic Definitions

Definition 1.1. *(X, B) is a pair if X is a normal variety, $B = \sum b_i B_i$ is a divisor with $b_i \in [0, 1] \cap \mathbb{Q}$.*

Definition 1.2 (log resolution)**.** *A log resolution is a resolution of singularities of* (X, B)*: a projective birational* $\varphi : W \to X$ *such that* W *is smooth,* $\varphi^{-1} suppB \cup exc(\varphi)$ *is a simple normal crossing divisor. Log resolutions always exist if char* $k = 0$ *by Hironaka.*

Definition 1.3 (singularities)**.** *Let* (X, B) *be a pair,* $\varphi : W \to X$ *be a log resolution. Then we can write* $K_W + B_W = \varphi^*(K_X + B)$*. We say* (X, B) *is*

(1) terminal if each coefficient of B_W *is* ≤ 0 *and* < 0 *for exceptional components.*

(2) canonical if each coefficient of B_W *is* ≤ 0*.*

(3) Kawamata log terminal if each coefficient of B_W *is* < 1*.*

(4) log canonical if each coefficient of B_W *is* ≤ 1*.*

(5) ϵ*-log canonical if each coefficient of* B_W *is* $\leq 1 - \epsilon$*.*

Definition 1.4 (Hyperstandard sets)**.** *Let* $\mathcal{R} \subset [0, 1]$ *be a set of rational numbers. We define*

$$\Phi(\mathcal{R}) = \{1 - \frac{r}{m} | r \in \mathcal{R}, m \in \mathbb{N}\}$$

to be the set of hyperstandard multiplicities associated to $\mathcal{R}$*.*

Definition 1.5. *Let* X *be a projective variety.*

$Z_1(X) := \{\sum n_i Z_i | n_i \in \mathbb{Z}, C_i \subset X curve\}$

$Pic(X)$ *:= the group of Cartior divisor/~*

For each $D \in Pic(X) \otimes \mathbb{R}$ *and* $C \in Z_1(X) \otimes \mathbb{R}$*, can define intersection number* $D \cdot C$*. For each* $D, D' \in Pic(X) \otimes \mathbb{R}$*, we define* $D \equiv D'$ *if*

$$D \cdot C = D' \cdot C \quad \forall C \in Z_1(X) \otimes \mathbb{R}.$$

In similar, For each $C, C' \in Z_1(X) \otimes \mathbb{R}$*, we define* $C \equiv C'$ *if*

$$D \cdot C = D \cdot C' \quad \forall D \in Pic(X) \otimes \mathbb{R}.$$

$N^1(X) := Pic(X) \otimes \mathbb{R}/\equiv$*, called Neron-Severi space;* $N_1(X) := Z_1(X) \otimes \mathbb{R}/\equiv$*, and there is a non-degenerate intersection pair:*

$$N^1(X) \times N_1(X) \to \mathbb{R}.$$

The Picard number of X*, denoted by* $\rho = \rho(X)$*, is the real dimension of* $N^1(X)$ *(equivalently,* $N_1(X)$*).*

Definition 1.6 (Fano variety)**.** *A smooth projective variety X is called a Fano variety if its anticanonical divisor $-K_X$ is ample. If a normal projective variety X has singular points, and some positive integral multiple $-nK_X$, $n \in \mathbb{N}$, of an anticanonical Weil divisor $-K_X$ is an ample Cartier divisor, then X is called a singular Fano variety. A log terminal pair (X, B) is called a log Fano variety if a $\mathbb{Q}$-Cartier divisor $-(K_X + B)$ is ample. If $B = 0$, then X is said to be a log Fano variety.*

Definition 1.7 (Fano index; fundamental system; Fano degree)**.** *For a log Fano vairety X, there exists the greatest rational $r = r(X) > 0$ such that $-K_X \sim_{\mathbb{R}} rH$ for some (ample) Cartier divisor H. The rational number r is called the index of the Fano variety X, and the divisor H (resp. the linear system $|H|$) is called a fundamental divisor (resp. fundamental system) on X. The self-intersection index $d = d(X) = H^{dimX}$ is the degree of the Fano variety X.*

Remark 1.8. *The Fano index of Fano varieties is an integer.*

2 Classification of Fano Varieties

The Fano varieties can be classified by their index, degree, genus, Picard number, Hodge number and so on numerical invariants. Furthermore, it is feasible to find the explicit construction in each classes. In dimension 1, $\mathbb{P}^1$ is a unique Fano variety up to isomorphism. The classification of Fano surfaces (called Del Pezzo surfaces) is a classical result. The classification results of Fano threefolds, fourfolds of index 2 with $\rho \geq 2$, and toric Fano threefolds can be found in [10] Chapter 12.

2.1 Simplest Properties

There are some simplest results about Fano varieties, which are generally from [10] chapter 2.1.

Proposition 2.1. *Let X be a n-dimensional log Fano variety of index r, and H be a fundamental divisor. Then*

$$H^i(X, \mathcal{O}_X(mH)) = 0, \quad \forall i > 0, \quad \forall m > -r.$$

Proof. It shows that $mH = -rH + (m + r)H \sim_{\mathbb{R}} K_X + (m + r)H$, where $(m + r)H$ is ample. Hence, by Kamawata-Viehweg vanishing theorem, we prove the proposition. $\square$

Corollary 2.2. *The index of a log Fano variety X does not exceed* $\dim X + 1$.

Proof. The Riemann-Roch theorem tells us

$$\chi(mH) = \frac{(mH)^{\dim X}}{\dim X!} + \text{lower terms} = \frac{d(X)m^{\dim X}}{\dim X!} + \text{lower terms}.$$

Meanwhile, Prop 2.1 tells that $m = -1, -2, \ldots, 1 - \lceil r \rceil$ are roots of $\chi(mH)$, so $\lceil r \rceil - 1 \leq \dim X$, $r \leq \dim X + 1$. □

Corollary 2.3. *Let X be an n-dimensional log Fano variety of index r with at most canonical Gorenstein singularities. Let H be a fundamental divisor on X, and let $d = H^n$ be the degree of X. Then:*

(i) if $r > n - 2$, then

$$h^0(X, H) = \frac{1}{2}d(r - n + 3) + n - 1,$$

and $d = 1$ for $r = n + 1$, $d = 2$ for $r = n$.

(ii) if $r = n - 2$, then

$$h^0(X, H) = g + n - 1,$$

where $g = \frac{d}{2} + 1 \geq 2$ is an integer.

Proof. The polynomial $P(m) = \chi(mH) =$

$$(m + 1)(m + 2) \ldots (m + r - 1)(a_N m^N + a_{N-1} m^{N-1} + \cdots + a_0),$$

where $N = n - r + 1$. In general, $P(0) = 1$ and by Serre duality,

$$P(m) = \sum (-1)^i h^i(X, mH)$$

$$= \sum (-1)^i h^{n-i}(X, (-r - m)H) = (-1)^n P(-r - m).$$

(i)(a) $r = n + 1$, $N = 0$, $a_0 = 1/n!$, so

$$d = 1, \quad h^0(X, H) = P(1) = n + 1;$$

(i)(b) $r = n$, $N = 1$, $a_0 = 1/(n - 1)!$, $a_1 = 2/n!$, so

$$d = 2, \quad h^0(X, H) = P(1) = n + 2;$$

(i)(c) $r = n - 1$, $N = 2$, $a_0 = 1/(n - 2)!$, $(n - 1)a_2 = a_1$, $a_2 = d/n!$, so

$$h^0(X, H) = P(1) = \frac{d}{n} + \frac{(n - 1)d}{n} + n - 1 = d + n - 1;$$

(ii)$r = n - 2$, $N = 3$, the result is gotten by more calculation processes. □

Remark 2.4. *The $g = \frac{d}{2} + 1$ is the genus of Fano varieties of index $n - 2$.*

2.2 Del Pezzo Surfaces

The Du Val singularity, also called the rational double singularity, A-D-E singularity is the canonical singularity on algebraic surface. A normal Fano surface with the Du Val singularity is called a del Pezzo surface. The classification of del Pezzo surfaces and the analytic properties of Du Val singularities are well-known.

At first, the linear system $|-K_X|$ gives a fine classification of smooth Del Pezzo surface X. By Serre duality and Kodaira's vanishing theorem, we have

$$h^0(X, -nK_X) = 0, \quad \forall n < 0;$$

$$h^1(X, -nK_X) = 0, \quad \forall n \in \mathbb{Z};$$

$$h^2(X, -nK_X) = h^0(X, (n+1)K_X).$$

In particular, $h^1(O_X) = h^2(O_X) = h^0(2K_X) = 0$, then by Castelnuovo's Rationality criterion, X is rational.

Remark 2.5 (Castelnuovo's criterion)**.** *Let X be a nonsingular complete algebraic surface of an algebraically closed field k. Let $p_a(X) = \chi(O_X) - 1$ and let $P_n(X) = h^0(X, nK_X)$. X is rational if and only if $p_a(X) = P_2(X) = 0$.*

The smooth del Pezzo surface has a blow-up model:

Theorem 2.6. *Let X be a smooth del Pezzo surface. Then either $X \cong \mathbb{F}_0$, or $X \cong \mathbb{F}_2$, or X is obtained from $\mathbb{P}^2$ by blowing up $N \leq 8$ points in almost general position.*

Remark 2.7. *$\mathbb{F}_n$ is the Hirzebruch surface, defined as Proj $(O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-n))$. They are rational ruled surfaces. In particular, $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. Every rational surface is the blow-up of $\mathbb{P}^2$ or $\mathbb{F}_n$.

The blow-up case: Suppose X_r is an algebraic surface by blowing up $\mathbb{P}^2$ in points $P_1, \ldots, P_r$ in general position. Let e_i be the exceptional curve above P_i and let l be the pullback of a line on $\mathbb{P}^2$. Then Pic $X \cong \mathbb{Z}^{r+1}$, generated by $l, e_1, \ldots, e_r$. The anticanonical divisor $-K_X \sim 3l - \sum e_i \sim H$. The intersection numbers are $l^2 = 1$, $l \cdot e_i = 0$, $e_i \cdot e_j = -\delta_{ij}$. The degree of X_r given by blowing up r points of $\mathbb{P}^2$ is $9 - r$. Hence at most blow up 8 points.

The $\mathbb{F}_n$ case: let C be a irreducible curve in X with negative self-intersection. Then by adjunction

$$C^2 + C \cdot K_X = \deg K_C = 2g(C) - 2,$$

and $C \cdot K_X \leq 0$ ($-K_X$ is nef), we know $g(C) = 0$ (i.e. $C \cong \mathbb{P}^1$) and $C^2 = -1$ or -2. Meanwhile, the self-intersection of a section of $\mathbb{P}^1$ of $\mathbb{F}_n$ is $-n$, so $0 \leq n \leq 2$. $\mathbb{F}_1$ is isomorphic to $\mathbb{P}^2$ blow-up 1 point, so $n = 0, 2$. □

A concrete description, the anticanonical model, is as follows:

Proposition 2.8. *Let X be a smooth Del Pezzo surface of index r, and let $-K_X \sim rH$. Then*

(i) the linear system $|-K_X|$ has no fixed components or base points except in the case $K_X^2 = 1$, where $|-K_X|$ is a sheaf of irreducible elliptic curves with a single base point;

(ii) if $r = 3$, them $\varphi_{|H|} : X \to \mathbb{P}^2$ is an isomorphism; if $r = 2$, then $\varphi_{|H|} : X \to \mathbb{F}_0 \subset \mathbb{P}^3$ is an isomorphism onto a smooth quadric;

(iii) for $3 \leq K_X^2 \leq 9$ the morphism $\varphi_{|-K_X|} : X \to \mathbb{P}^n$ is an embedding, and the image $X' = \varphi(X)$ is a surface of degree n in $\mathbb{P}^n$, where $n = K_X^2 \in [3, 9]$; the natural homomorphism of graded algebras

$$Sym^* H^0(X', \mathcal{O}_{X'}(1)) \to \oplus_{m \geq 0} H^0(X', \mathcal{O}_{X'}(m)),$$

is surjective; for $n \geq 4$ the surface X' is the intersection of quadrics in $\mathbb{P}^n$ containing X';

(iv) if $K_X^2 = 2$, then $\varphi_{|-K_X|} : X \to \mathbb{P}^2$ is a double covering with a smooth ramification curve of degree 4; the morphism $\varphi_{|-2K_X|} : X \to \mathbb{P}^6$ is an embedding onto a projectively normal surface $X' = \varphi_{|-2K_X|}(X)$ of degree 8; in the weighted projective space $\mathbb{P}(1, 1, 1, 2)$, the surface X can be prescribed by a single homogeneous equation of degree 4;

(v) if $K_X^2 = 1$, then $\varphi_{|-2K_X|} : X \to \mathbb{F}_2^* \subset \mathbb{P}^3$ is a morphism of degree 2 onto a quadric cone $\mathbb{F}_2^*$ in $\mathbb{P}^3$ with ramification at a nonsingular curve $C \subset \mathbb{F}_2^*$, cut out by a cubic in $\mathbb{P}^3$, not passing through the vertex of the cone; the mapping $\varphi_{|-3K_X|} : X \to \mathbb{P}^6$ is an embedding onto a projectively normal surface X' of degree 9; in the weighted projective space $\mathbb{P}(1, 1, 2, 3)$, the surface X can be prescribed by a single equation of degree 6.

proof of (i)(ii). (i) By Riemann-Roch formula:

$$h^0(X, -K_X) = \chi(-K_X) = \chi(\mathcal{O}_X) + K_X^2 = 1 + K_X^2,$$

so $\dim|-K_X| = K_X^2$ and $|-K_X|$ is nonempty. Let $S \in |-K_X|$ then by adjunction formula

$$\deg K_S = S \cdot (K_X + S) = 0,$$

so S is an elliptic curve. The exact sequence

$$0 \to \mathcal{O}_X(-S) \to \mathcal{O}_X \to \mathcal{O}_S \to 0$$

induces an exact sequence of cohomologies

$$0 \to H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X(S)) \to H^0(S, \mathcal{O}_S(S)) \to 0.$$

Because $S^2 = K_X^2$, Riemann-Roch tells that

$$h^0(S, \mathcal{O}_S(S)) = K_X^2,$$

for $K_X^2 > 1$, $|-K_X|$ is base point free. The proves (i).

(ii) A general result is

$$\deg \varphi(X) \geq \text{codim } X + 1.$$

in case of surface,

$$\frac{d}{\deg \varphi_{|H|}} \geq h^0(X, H) - 2.$$

(a) For $r = 3 = \dim X + 1$, by Corollary 2.3 (i), we know $d = 1$, $h^0(X, H) = 3$, $\dim|H| = 2$, so $\varphi_{|H|} : X \longrightarrow \mathbb{P}^2$ and $\deg \varphi_{|H|} = 1$.

(b) For $r = 2 = \dim X$, we have $d = 2$, $h^0(X, H) = 4$, $\dim|H| = 3$, so $\varphi_{|H|} : X \longrightarrow \mathbb{P}^3$ and $\deg \varphi_{|H|} = 1$. Hence the only possible case is that X is isomorphic to quadric hypersurface in $\mathbb{P}^3$. □

The del Pezzo surfaces with singularities are also known:

Proposition 2.9. *If X is a singular del Pezzo surfaces with Du Val singularities, and $K_X^2 = d$ then X is one of the following:*

(i) $d = 8$, $X = Q' \subset \mathbb{P}^3$ is a quadric cone;

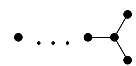
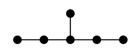
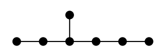
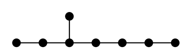
(ii) $3 \leq d \leq 6$, $X = X_d \subset \mathbb{P}^d$ is a projectively normal surface of degree d;

(iii) $d = 2$, X can be represented as a double cover $X \to \mathbb{P}^2$ ramified along a singular curve of degree 4;

(iv) $d = 1$, and X can be represented as a double cover $X \to Q'$ of a quadric cone ramified along a singular curve cut out on Q' by a surface of degree 3.

Remark 2.10. *Any del Pezzo surface with $\rho = 1$ is isomorphic to $\mathbb{P}^2$.*

The following part discusses Du Val singularities. The resolution of X will give some normal crossing exceptional curves. Their intersections just depend on the type of singularities, and more excitingly they are classified by Dynkin diagram. The Du Val singularities are also quotient of $\mathbb{A}^2$ by a finite group action G. See the following table.

Name	Equation	Group	Resolution graph
A_n	$x^2 + y^2 + z^{n+1}$	cyclic $\mathbb{Z}/(n+1)$	
D_n	$x^2 + y^2 z + z^{n-1}$	binary dihedral $BD_{4(n-2)}$	
E_6	$x^2 + y^2 + z^4$	binary tetrahedral	
E_7	$x^2 + y^3 + yz^3$	binary octahedral	
E_8	$x^2 + y^3 + z^5$	binary icosahedral	

2.3 Fano threefolds of $r \geq 2$

Proposition 2.11. *Let X be a Fano threefold. Then*

(i) $h^i(X, -mK_X) = 0$, $\forall m \in \mathbb{Z}$, $i = 1, 2$, *and moreover,* $h^i(X, -mK_X) = 0$, *if* $i > 0$, $m \geq 0$ *and if* $i < 3$, $m < 0$*; in particular,* $h^i(X, \mathcal{O}_X) = 0$ *for* $i > 0$.

(ii)

$$h^0(X, -mK_X) = \frac{m(m+1)(2m+1)}{12}(-K_X)^3 + 2m + 1$$

and in particular

$$h^0(X, -K_X) = \dim|-K_X| + 1 = \frac{-K_X^3}{2} + 3 \geq 4.$$

Proof. By Riemann-Roch theorem, for a divisor D,

$$\begin{aligned}
\chi(D) &= \deg(ch(D)td(X))_3 \\
&= \deg((1 + D + \frac{1}{2}D^2 + \frac{1}{6}D^3)(1 - \frac{1}{2}K_X + \frac{1}{12}(K_X^2 + c_2(X)) - \frac{1}{24}(K_X \cdot c_2(X))))_3 \\
&= \frac{1}{6}D^3 - \frac{1}{4}D^2 \cdot K_X + \frac{1}{12}(K_X^2 + c_2(X)) \cdot D - \frac{1}{24}K_X \cdot c_2(X),
\end{aligned}$$

In particular,

$$K_X \cdot c_2(X) = -24\chi(\mathcal{O}_X) = -24;$$

$$h^0(X, -mK_X) = \chi(-mK_X) = \frac{m(m+1)(2m+1)}{12}(-K_X)^3 + 2m + 1.$$

Then take $m = 1$, we get the special formula. □

Proposition 2.12. *Let X be a Fano 3-fold*

(i) $r \geq 2$*:*

$$h^0(X, H) = \frac{(r+1)(r+2)}{12} H^3 + \frac{2}{r} + 1$$

(ii) $r = 1$*:*

$$h^0(X, H) = \frac{H^3}{2} + 3$$

Proof. Take $D = H$ in the proof of Prop 2.11. □

Definition 2.13. *The integer* $g = g(X) = -\frac{K_X^3}{2} + 1$ *is genus of the Fano variety* X.

Remark 2.14. *If* $-K_X$ *is very ample and* $\varphi_{|-K_X|}$ *define an embeding into* $\mathbb{P}^{\dim|-K_X|} = \mathbb{P}^{g+1}$*, then the intersection of* $\varphi_{|-K_X|}(X)$ *with a codimension 2 linear hypersubspace is a curve with genus* g.

Proposition 2.15. *Let X be a Fano threefold,* $-K_X \sim rH$*. If* $S \in |H|$ *is a smooth surface, then*

(i) S is a Del Pezzo surface if $r \geq 2$.

(ii) S is a surface of type K3 if $r = 1$.

Proof. By adjunction formula

$$K_S \sim S + K_X|_S,$$

$$\deg K_S = S^3 + K_X \cdot S^2 = (1-r)H^3 = (1-r)d$$ □

Remark 2.16. *In a Fano 3-fold of index* $r \geq 1$, $|H|$ *must contains a smooth surface. See Theorem 3.8.*

Proposition 2.17. *Let* $r \geq 2$ *be the index of X, and let* $S \in |H|$ *be a smooth surface, then* *(i)* $2 \leq r \leq 4$*;*

(ii) *if* $r = 2$*, then* $1 \leq d = H^3 \leq 9$*;*

(iii) *if* $r = 3$*, then* $d = H^3 = 2$*;*

(iv) *if* $r = 4$*, then* $d = H^3 = 1$.

Proof. As $r \geq 2$, S is a Del Pezzo surface and $1 \leq K_S^2 \leq 9$. It also known $K_S \sim (1-r)S^2$, so

$$1 \leq (r-1)^2 S^3 \leq 9.$$

This implies (i), (ii), (iv). For (iii), it implies $S^3 = 1$ or 2. However, we know

$$2g - 2 = K_X^3 = -r^3 H^3 \equiv H^3 \mod 2,$$

so $H^3 = 2$. □

Remark 2.18. *(iii) and (iv) are a special case of Coro 2.3.*

The ample sheaf $\mathcal{O}_X(H)$ induces a birational map $\varphi_{|H|} : X \dashrightarrow \mathbb{P}^{\dim |H|}$. If X is a three-dimensional Fano varieties with φ being a morphism, we have

Proposition 2.19. *Let X be a three-dimensional Fano variety of index r with canonical Gorenstein singularities, and let $-K_X = rH$, where H is a fundamental divisor on X. Assume that the linear system $|H|$ is base point free and determines a morphism $\varphi_{|H|} : X \to \varphi_{|H|}(X) \subset \mathbb{P}^{\dim |H|}$. Then degree* $\deg \varphi_{|H|} = 1$ *or* 2*.*

Proof. Let $d = H^3$. A general fact is for an irreducible $X \subset \mathbb{P}^n$ not lying in a hyperplane, we have

$$\deg X \geq \text{codim } X + 1.$$

In this case,

$$\deg \varphi_{|H|}(X) \geq \text{codim } \varphi_{|H|}(X) + 1.$$

By Coro 2.3, we have

(i) if $r \geq 2$, then

$$\frac{d}{\deg \varphi_{|H|}} \geq \frac{1}{2} dr - 1;$$

(ii) if $r = 1$, then

$$\frac{2g-2}{\deg \varphi_{|H|}} \geq g - 1.$$

Hence the possible case is $\deg \varphi_{|H|} = 1$ or 2. □

The following is a list of Fano threefold with index $r \geq 2$, let $\varphi_{|H|}$ be a rational morphism.

Theorem 2.20. *Let X be a Fano threefold of index $r \geq 2$ and let $S \in |H|$ be a smooth surface, $d = H^3$, and $\varphi_{|H|} : X \dashrightarrow \mathbb{P}^{\dim |H|}$ is a map. Then*

(i) if $r \geq 3$, then for

$r = 4$: $\varphi_{|H|} : X \longrightarrow \mathbb{P}^3$ is an isomorphism;

$r = 3$: $\varphi_{|H|} : X \longrightarrow X_2 \subset \mathbb{P}^4$ is an isomorphism onto a smooth quadric hypersurface.

(ii) if $r = 2$, then varieties X exist only for $1 \leq d \leq 7$ and for $d \geq 3$ the morphism $\varphi_{|H|} : X \longrightarrow X_d \subset \mathbb{P}^{d+1}$ is an embedding onto a variety X_d of degree d in $\mathbb{P}^{d+1}$, which for $d \geq 4$ is the intersection of quadrics containing it; conversely, any smooth, projectively normal, three-dimensional variety V_d in $\mathbb{P}^{d+1}$, $3 \leq d \leq 8$, not lying in any hyperplane is a Fano variety which has index 2 except in the case $d = 8$, where $r = 4$ and V_8 is the Veronese image of $\mathbb{P}^3$ in $\mathbb{P}^9$;

(iii) if $r = 2$ and $1 \leq d \leq 7$, then for

d=7 : X_7 is the projection of the Veronese variety $V_8 \subset \mathbb{P}^9$ from one of its points;

d=6 : $X_6 \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is embedded in $\mathbb{P}^7$ in the sense of Serge;

d=5 : X_5 in $\mathbb{P}^6$ is unique up to projective equivalence and can be realized in one of the following two manners:

(a) as a birational image of a smooth quadric $W \subset \mathbb{P}^4$ under the mapping of the linear system $|\mathcal{O}_W(2) - Y|$ of quadrics passing through a spatial cubic curve Y;

(b) as a section of the Grassmannian $G(1, 4)$

d=4 : X_4 is any smooth intersection of two quadrics in $\mathbb{P}^5$;

d=3 : X_3 is any smooth cubic hypersurface in $\mathbb{P}^4$;

d=2 : $\varphi_{|H|} : X \longrightarrow \mathbb{P}^3$ is a double covering with a smooth ramification surface D_4 of degree 4 and any such covering is a Fano variety with $r = 2$ and $d = 2$; it can also be realized as a smooth hypersurface of degree 4 in the weighted projective space $\mathbb{P}(1, 1, 1, 1, 2)$;

d=1 : $\varphi_{|H|} : X \dashrightarrow \mathbb{P}^2$ *is a rational mapping with a single point of indeterminacy and with irreducible elliptic fibres. It can be achieved in one of the following ways:*

(a) *as a double cover* $\varphi_{|-K_X|} : X \to W \subset \mathbb{P}^6$ *of a cone over the Veronese surface in* $\mathbb{P}^5$ *with a smooth ramification divisor* $D \subset W$ *which is cut out on W by a cubic hypersurface not passing through the vertex of W;*

(b) *as a hyper surface of degree 6 in the weighted projective space* $\mathbb{P}(1, 1, 1, 2, 3)$.

partial proof. By Prop 2.12, Prop 2.17 and Prop 2.19, for

$r = 4$, $d = 1$, $\deg \varphi_{|H|} = 1$, $\dim |H| = 3$;

$r = 3$, $d = 2$, $\deg \varphi_{|H|} = 1$, $\dim |H| = 4$.

This proves (i).

For (ii), First, it need to show $\varphi_{|H|}$ is an embedding when $d \geq 3$. By $r = 2$, by Prop 2.12, $\dim |H| = d + 1$. Then by

$$\deg \varphi_{|H|}(X) \geq \text{codim } \varphi_{|H|}(X) + 1,$$

we get $\deg \varphi_{|H|} \leq \frac{d}{d-1}$, so when $d \geq 3$, $\varphi_{|H|}$ is an embedding.

Second, it needs to show $d = 8, 9$ varieties do not exist. For $d = 9$, assume $X \subset \mathbb{P}^{10}$ is a Fano 3-fold of degree 9. Let S be a smooth hyperplane section of X, then S is a del Pezzo surface of degree 9 in $\mathbb{P}^9$. According to Theorem 2.8, S is the Veronese image of the plane $\mathbb{P}^2$ under the imbedding of a complete linear system of cubic curves. By Lefschetz theorem there is a surjection

$$H_2(S, \mathbb{Z}) \xrightarrow{i_*} H_X(X, \mathbb{Z}) \longrightarrow 0,$$

where $i : S \to X$ is the embedding. Since $H_2(S, \mathbb{Z}) \cong \mathbb{Z}$ and $\text{rk} H_2(X, \mathbb{Z}) \neq 0$, it follows that i_* is an isomorphism. By duality, $H^2(X, \mathbb{Z}) \cong H^2(S, \mathbb{Z})$. Hence $\text{Pic}(X) \cong \text{Pic}(S)$ under natural restriction. But we know $\text{Pic}(X) = \mathbb{Z}[H]$ (H is the fundamental divisor of X), and $H|_S$ is divisible in $\text{Pic}(S)$ by 3, so there is a contradiction.

For $d = 8$, let S be a smooth hyperplane section of X, then S is a del Pezzo surface of degree 8 in $\mathbb{P}^8$. By Theorem 2.8, S can be

(a) $r(S) = 2$, S is the embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ under anticanonical divisor;

(b) $r(S) = 1$, S is the embedding of rational ruled surface $\mathbb{F}_1$ under anticanonical divisor.

$\mathrm{Pic}(X) \cong H^2(X, \mathbb{Z})$, H is a primitive (i.e. nondivisible) element in $\mathrm{Pic}(X)$, by Poincare duality, we can find an element $Z \in H_2(X, \mathbb{Z})$ such that $Z \cdot H = 1$. Z can be lifted to be a divisor of S by its rationality. Hence we get a contradiction in case (a), because every divisor in S has even degree.

Case (b): the image of the exceptional curve, called Z, is one and only one line on $\mathbb{F}_1$. It could be shown that this line sweeps a plane P in X. The normal sheaf $\mathcal{N}_Z$ in X can be represented in the expansion

$$0 \to \mathcal{O}_Z(1) \to \mathcal{N}_Z \to \mathcal{O}_Z(-1) \to 0.$$

$\mathcal{N}_Z = \mathcal{O}_Z(1) \oplus \mathcal{O}_Z(-1)$, where $\mathcal{O}_Z(-1)$ is the restriction of normal sheaf of P to Z. Since $\mathcal{O}_P(S+P) \cong \mathcal{O}_P$ and $h^1(\mathcal{O}_X(S)) = 0$, there is an exact sequence

$$0 \to H^0(\mathcal{O}_X(S)) \to H^0(\mathcal{O}_X(S+P)) \to H^0(\mathcal{O}_P) \to 0.$$

Hence P can be contracted to a nonsingular point by $\mathcal{O}_X(S+P)$. Let $X' = \varphi_{|S+P|}(X)$. Because P is contracted, $d(X') = 9$ and

$$\varphi_*(K_X) = K_{X'},$$

$$-K_{X'} \sim 2\varphi_*(S) = 2\varphi_*(S+P).$$

By $-K_{X'}^3 = 2^3 \cdot 9$, we see $r(X') = 2$. It has been shown such X' does not exist, so there does not exist Fano varieties of index $r = 2$ and $d = 8$. □

Remark 2.21 (Fujita)**.** *A n-dimensional log Fano variety X of index $r \geq n-1$ is one of the following:*

(i) $X \cong \mathbb{P}^n$, $r = n+1$;

(ii) $X \cong Q \subset \mathbb{P}^{n+1}$ is a quadric, $r = n$;

(iii) $X \cong X_d \subset \mathbb{P}^{d+n-1}$ is a cone over a rational normal curve $C_d \subset \mathbb{P}^d$ with vertex in $\mathbb{P}^{n-1}$, $r = n - 1 + \frac{2}{d}$;

(iv) $X \cong X_4 \subset \mathbb{P}^{n+3}$ is a cone over the Veronese surface $S_4 \subset \mathbb{P}^5$ with vertex in $\mathbb{P}^{n-3}$, $r = n - \frac{1}{2}$.

Only the first two cases are nonsingular, and they are exactly the cases shown in Theorem 2.20 (i).

Remark 2.22. *A Fano n-fold with $r = n-1$ is called Del Pezzo varieties, which has a classical classification result. It corresponds to $r = 2$ case here.*

Remark 2.23. *Unlike the rationality of Del Pezzo surfaces, Fano threefolds are not necessary rational. For example, a general cubic 3-fold in $\mathbb{P}^4$ (of degree 8 and genus 5) is irrational.*

2.4 Prime Fano threefolds

A prime Fano threefold X is a Fano threefold of $\rho = 1$ and $K_X = -H$, so it is a special Fano threefold of $r = 1$. There are 105 classes of Fano threefolds. 17 families are prime Fano threefolds; 88 families are nonprime Fano threefolds. The genus of a Fano threefold is $\frac{-K_X^3}{2} + 1$. This subsection discusses some classical results relative to the genus of prime Fano threefolds. For convenience, let $d = -K_X^3$, it is the degree of X. The following statement is mainly from [2], [8].

Theorem 2.24 (Iskovskikh)**.** *Let X be a prime Fano threefold of degree d. If $g \geq 4$, the linear system $|-K_X|$ is very ample hence induces an embedding $X \subset \mathbb{P}^{g+1}$ whose image is a smooth threefold of degree $d = (2g-2)$. If moreover $g \geq 5$, the image is an intersection of quadrics.*

Remark 2.25. *Assume $X \subset \mathbb{P}^{g+1}$ is an anticanonically embedded Fano threefold (so that $g \geq 3$) such that $Pic(X) = \mathbb{Z}[H]$. Then $H^3 = 2g - 2 \geq 4$, so X does not contain a plane.*

There is a upper bound of the genus of a prime Fano 3-fold:

Theorem 2.26. *The genus g of a prime Fano 3-fold satisfies $g \leq 10$ or $g = 12$.*

This theorem is proved by Mukai's vector bundle method, see [8]. A general surface section of X is a smooth K3 surface S of degree d and Mukai's idea is to construct a vector bundle on S and then extend it to X.

Consider the 19-dimensional moduli space $\mathcal{F}_g$ of polarized K3 surfaces of genus g, the $(g+19)$-dimensional moduli space $\mathcal{P}_g$ of pairs (S, C) where $S \subset \mathbb{P}^g$ is a K3 surface of genus g and $C \subset S$ a hyperplane section, and $(3g-3)$-dimensional moduli space $\mathcal{M}_g$ of curves of genus g. There is a map:

$$\begin{tikzcd} \mathcal{P}_g \arrow[r, dashed, "\varphi_g"] \arrow[d, dashed] & \mathcal{M}_g \\ \mathcal{F}_g & \end{tikzcd}$$

Proposition 2.27. *If a prime Fano 3-fold of genus g exists, then the rational map $\varphi_g : \mathcal{P}_g \dashrightarrow \mathcal{M}_g$ is not generically finite.*

Proof. Consider a general pencil $P = \{S_t | t \in \mathbb{P}^1\} \subset \mathcal{F}_g$ of hyperplane sections of X. All the K3 surfaces S_t contain a same smooth base curve C, which is a curve of genus g hence P lifts to a curve in $\mathcal{P}_g$. C is also a generic element, so the map φ_g is not generically finite. □

Moreover, [8], Theorem 7 says:

Theorem 2.28. *The rational map φ_g is generically finite if and only if $g = 11$ or $g \geq 13$.*

This complete the proof of Theorem 2.26. [2] says this method can be generalized to arbitrary prime Fano n-fold:

Theorem 2.29. *Let $X \subset \mathbb{P}^{g+n-2}$ be a Fano n-fold with $\rho(X) = 1$ and index $n-2$. Then $g \leq 12$ and $g \neq 11$.*

3 Fundamental Linear System

3.1 Base points of $|H|$

The map $\varphi_{|H|}$ plays an important role in the classification of Fano varieties. It defines a morphism only if $|H|$ is base point free, so it deserves to discuss when $|H|$ has such a property.

Let X be a Fano threefold, $-K_X \sim rH$.

Proposition 3.1. *A linear system $|H|$ on a Fano threefold X has no base points except in the following two cases:*

(a) $r = 2$, $H^3 = 1$; $|H|$ has a unique base point;

(b) $r = 1$; there exist smooth irreducible curves Z and E on a smooth surface $S \in |H|$ such that Z is a curve of genus zero, E is a fibre of an elliptic sheaf $|E|$ on S, $Z \cdot E = 1$, and $H|_S = \mathcal{O}_S(Z + mE)$, where $m \geq 3$ is an integer; $|H|$ has a unique base curve Z and has no other base points.

A lemma about ample line bundle on K3 surface is helpful.

Lemma 3.2. *Let $\mathcal{L}$ be an ample invertible sheaf on a K3 surface F, then $\mathcal{L}$ is one of the following types:*

(a) $\mathcal{L} \cong \mathcal{O}_F(C)$, where C is an irreducible curve with $p_a(C) > 1$;

(b) $\mathcal{L} \cong O_F(mE + Z)$*, where* E *is an elliptic curve, and* Z *is an irreducible rational curve such that* $E \cdot Z = 1$ *and* $m \geq 3$*.*

Remark 3.3. *Recall Francia's lemma: let* X *be a nonsingular surface,* D *an effective divisor on* X*, and* $x \in D$ *a singular point of* D*. Denote by* $f : Y \to X$ *the blow up of the maximal ideal of* $x \in X$ *and* $D' = f^*D - 2E$*. Then* $x \in Bs|K_X + D|$ *if and only if* D' *is disconnected. When* X *is K3,* $K_X \sim 0$*, and* $Bs|K_X + D| = Bs|D|$*, so case (a) is base point free.*

Proof of Prop 3.1. Since $H^1(X, O_X) = 0$, the sequence

$$H^0(X, O_X(S)) \to H^0(S, O_S(S)) \to 0$$

is exact. Hence, each base curve in $|H|$ is a stationary component of $|O_S(S)|$ on S, and conversely, each stationary component in $|O_S(S)|$ is a base curve of $|H|$. If there are no base curves in $|H|$, then the base points of $|H|$ and only these points are base points of $H|_S$.

Suppose $r \geq 2$, then by Prop 2.15 (i) S is a Del Pezzo surface, and by Prop 2.8 (i) $H|_S$ has no base points except $K_S^2 = 1$. Then as the proof of Prop 2.17, by adjunction formula

$$1 = K_S^2 = (1 - r)^2 d,$$

$r = 2$, $d = H^3 = 1$. In this case, $|S|$ has just a single base point, so is $|H|$.

As for $r = 1$, in this case S is a K3 surface by Prop 2.15 (ii). The sheaf $H|_S$ is ample, so the only possible case is Lemma 3.2 (b). ◻

Lemma 3.4. *The* m *in Prop 3.1 cannot exceed 4.*

The structure of X with base points in $|H|$ is as follows:

Theorem 3.5. *Let* X *be a smooth Fano threefold,* $-K_X \sim rH$*.* $|H|$ *is base point free except:*

- *(i)* $r = 2$, $\deg X = 1$*, and* $|H|$ *has a unique simple base point. In such a case* $|H|$ *determines a rational map* $X \dashrightarrow \mathbb{P}^2$ *with elliptic fibres,* $g(X) = 5$*,* $Pic(X) = \mathbb{Z}$*, and the variety* X *can be realized as Theorem 2.20 (iii) d=1 case.*
- *(ii)* $r = 1$, $g(X) = 3$*. In such a case* $m = 3$*,* $Pic(X) = \mathbb{Z} \oplus \mathbb{Z}$*, and the variety* X *is the blow-up of a variety from part* (i) *along a smooth fibre of the rational map* $X \dashrightarrow \mathbb{P}^2$*.*
- *(iii)* $r = 1$, $g(X) = 4$*. In such a case* $m = 4$*,* $Pic(X) = \mathbb{Z}^{10}$*, and the variety* X *is isomorphic to* $F_1 \times \mathbb{P}^1$*, where* F_1 *is a del Pezzo surface of degree* 1*.*

3.2 Existence of good divisors

Prop 2.16 and Prop 2.18 show the existence of good divisors in $|H|$ gives a upper bound of degree of smooth Fano threefold. The smoothness of $S \in |H|$ reminds as a problem. In fact, there are many results about the existence of good divisors in the fundamental system.

Theorem 3.6 (Fujita (1980-84))**.** *Let X be an n-dimensional Fano variety of index $r \geq n-1$, and let $H \in Div(X)$ be a fundamental divisor. Then the linear system $|H|$ contains an irreducible nonsingular divisor.*

Theorem 3.7 (Reid (1983))**.** *Let X be a Gorenstein Fano threefold with at most canonical singularities. Then $|-K_X|$ contains an irreducible surface with at most Du Val singularities.*

Theorem 3.8. *Let X be a Fano threefold of index $r \geq 1$, and let $H \in Pic(X)$ be a fundamental divisor. Then the linear system $|H|$ contains a smooth surface.*

Proof. Via Theorem 3.6 and 3.7, it remains to show the case $r = 1$. Let X be a nonsingular Fano threefold with $r = 1$, and let $S \in |H| = |-K_X|$ be an irreducible surface. If $|H|$ is base point free, then Bertini theorem tells that S is smooth. Also by Bertini theorem, a general surface S from $|H|$ can have singularities only in $Bs|H|$. By Prop 2.15 (ii), S is a K3 surface with at most Du Val singularities. Let $\sigma : \widetilde{S} \to S$ be a minimal resolution of S. Since $Bs|H|$ is nonempty, $\widetilde{S}$ is in the case of Lemma 3.2 (b). i.e. Let $|M| = \sigma^*|H||_S$, then $M \sim Z + mE$, where $Bs|M| = Z \cong \mathbb{P}^1$ is a (-2)-curve on $\widetilde{S}$, and E is an elliptic curve, $Z \cdot E = 1$, $m \geq 3$. Let Z' be the exceptional divisor of σ. Then $\sigma \cdot M = 0$. Since $\sigma(Z') \in Bs|H||_S$ (singular point lies in base points of $|H|$), $Z' \in Bs|M| = Z$. Hence $m = 2$, which contradicts to $m \geq 3$, so $Bs|H||_S = \emptyset$. □

4 Boundedness

4.1 Boundedness of degree

We say a set Q of normal projective varieties is birationally bounded (resp. bounded) if there exist finitely many projective morphisms $V^i \to T^i$ of varieties such that for each $X \in Q$ there exist an i, a closed point $t \in T^i$, and a birational isomorphism (resp. isomorphism) $\phi : V^i_t \dashrightarrow X$ (resp. $\phi : V^i_t \to X$) where V^t_i is the fibre of $V^i \to T^i$ over t.

The first step is to make sure we can use finitely many index to prescribe Fano varieties. In [5], Kollar, Miyaoka and Mori show the boundedness of smooth Fano varieties via rational connectedness.

Definition 4.1. *A variety X is called rationally connected if two general points can be joined by an irreducible rational curve on X.*

Rational connectedness is a birational and deformation invariant, which fits well into the classification theory.

Theorem 4.2 (Kollar-Miyaoka-Mori)**.** *A Fano variety X over an algebraically closed field of characteristic zero is rationally connected. More precisely, any two general points x and y can be joined by an irreducible rational curve C such that $-K_X \cdot C \leq c(\dim X)$, where $c(n)$ is an effectively computable function in n, a positive integer.*

As a corollary, we can find an upper bound for degree of Fano varieties w.r.t anticanonical divisor and this bound just depends on the dimension of varieties.

Corollary 4.3. *For a Fano variety X of dimension n over an algebraically closed field of characteristic zero, the degree of X with respect to the anticanonical divisor is bounded,*

$$(-K_X)^n \leq c(n)^n,$$

where the function $c(n)$ is the same as previous theorem. In particular, by a theorem of Kollar and Matsusaka, the n-dimensional Fano manifolds forms a bounded family; i.e., they are (noneffectively) parametrized by a quasiprojective scheme.

Proof. Riemann-Roch tells that

$$h^0(X, -mK_X) \sim \frac{(-K_X)^n}{n!} m^n \quad m >> 0.$$

Therefore for every $x \in X$, $\epsilon > 0$ and $m >> 0$ there is a divisor $H_{m,x} \in |-mK_X|$ such that

$$\text{mult}_x H_{m,x} \geq m((-K_X)^n)^{1/n} - m\epsilon.$$

$$mc(n) \geq mC \cdot (-K_X) = C \cdot H_{m,x} \geq m((-K_X)^n)^{1/n} - m\epsilon.$$

Let $\epsilon \longrightarrow 0$, we get the result. □

Remark 4.4. *(i) In [5], the function $c(n)$ is very large:*

$$c(n) = (n+1)n^{(2^n-1)(n+1)}(1 + \frac{1}{n} + \frac{n+1}{n(n-1)}).$$

(ii) In the case $Pic(X) \cong \mathbb{Z}$ (shown in [6]), the estimate of $c(n)$ is much sharper:

$$c(n) = n(n+1).$$

4.2 BAB theorem

The Borisov-Alexeev-Borisov (BAB) conjecture is

Conjecture 4.5 (BAB)**.** *Let d be a natural number and ϵ a positive real number. Then the set of ϵ-lc Fano varieties X of dimension d forms a bounded family.*

The requirement of ϵ-lc in the Conj 4.5 is necessary.

Example 4.6 (non-bounded family of singular Fano surfaces)**.**

For $n \geq 2$ consider

$$\begin{array}{ccc} E & \subset W_n & \xrightarrow[f]{} X_n \\ & \downarrow & \\ & \mathbb{P}^1 & \end{array}$$

where X_n is the cone over a rational curve of degree n, f is the blow-up of the vertex, and E is the exceptional divisor. In other words, W_n = Proj $(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$, E is a section of bundle with $E|_E \cong \mathcal{O}_{\mathbb{P}^1}(-n)$.

By adjunction formula we know

$$K_E \sim (K_{W_n} + E)|_E,$$

$$K_{W_n} \cdot E = K_E \cdot E - E \cdot E = -2 + n.$$

Then we can determine the α in the formula of blow-up

$$K_{W_n} + \alpha E = f^* K_X, \quad K_{W_n} \cdot E - n\alpha = 0,$$

$$K_{W_n} + \frac{n-2}{n} E = f^* K_X.$$

In the formula we see the larger n, the deeper singular of X_n. In particular, the set $\{\frac{n-2}{n}\}$ is not a finite set, so X_n does not form a bounded family.

The theory of complements is essential in the study of $|-n(K_X+B)|$, $n \in \mathbb{N}$. Let (X, B) be a log canonical (abbr. lc) pair equipped with a projective morphism $X \to Z$. A strong n-complement of $K_X + B$ over a point $z \in Z$ is of the form $K_X + B^+$ where over some neighbourhood of z we have:

- (X, B^+) is lc;
- $n(K_X + B^+) \sim 0$;
- $B^+ \geq B$.

By definition, $-n(K_X+B) \sim n(B^+ - B) \geq 0$ over some neighbourhood of z, so in particular, it means the linear system $|-n(K_X+B)|$ is nonempty over z. Here is an example for n-complement:

Example 4.7. *When X is a toric variety, Z is a point and $B = 0$, we know $K_X = -\sum D_i$ where D_i are invariant subvarieties under torus action, so we can let $n = 1$, B^+ to be the sum of D_i.*

The existence of n-complement is shown in [1]:

Theorem 4.8 ([1], Theorem 3.3)**.** *Let d be a natural number and $\mathcal{R} \subset [0, 1]$ be a finite set of rational numbers. Then there exists a natural number n depending only on d and $\mathcal{R}$ satisfying the following. Assume (X, B) is a pair and $X \to Z$ a contraction such that*

- *(X, B) is lc of dimension d;*
- *the coefficients of B are in $\Phi(\mathcal{R})$;*
- *X is Fano type over Z;*
- *$-(K_X + B)$ is nef over Z.*

Then for any point $z \in Z$, there is a strong n-complement $K_X + B^+$ of $K_X + B$ over z. Moreover, the complement is also an mn-complement for any $m \in \mathbb{N}$.

Let X be a Fano variety. Theorem 4.8 says that $|-mK_X|$ is nonempty containing a "nice" element for some $m > 0$ depending only on $\dim X$. If we bound the singularities of X, we get a stronger result.

Theorem 4.9 ([1], Theorem 3.5)**.** *Let d be a natural number and $\epsilon > 0$ a real number. Then there is a natural number m depending only on d and ϵ such that if X is any ϵ-lc Fano variety of dimension d, then $|-mK_X|$ defines a birational map.*

It is possible to strengthen

Theorem 4.10 (BAB)**.** *Let d be a natural number, and ϵ and δ be positive real numbers. Consider projective varieties X equipped with a boundary B such that:*

(1) (X, B) is ϵ-lc of dimension d,

(2) $-(K_X + B)$ is nef and big.

Then the set of such X forms a bounded family.

References

[1] Caucher Birkar. Birational geometry of algebraic varieties, 2017.

[2] Olivier Debarre. How to classify fano varieties?

[3] Igor V. Dolgachev. *Classical Algebraic Geometry: A Modern View*. Cambridge University Press, 2012.

[4] V. A. Iskovskih. Anticanonical models of three-dimensional algebraic varieties. In *Current problems in mathematics, Vol. 12 (Russian)*, pages 59–157, 239 (loose errata). VINITI, Moscow, 1979.

[5] János Kollár, Yoichi Miyaoka, and Shigefumi Mori. Rational connectedness and boundedness of Fano manifolds. *J. Differential Geom.*, 36(3):765–779, 1992.

[6] János Kollár, Yoichi Miyaoka, and Shigefumi Mori. Rational curves on Fano varieties. In *Classification of irregular varieties (Trento, 1990)*, volume 1515 of *Lecture Notes in Math.*, pages 100–105. Springer, Berlin, 1992.

[7] William E. Lang. A short proof of Castelnuovo's criterion of rationality. *Trans. Amer. Math. Soc.*, 264(2):579–582, 1981.

[8] Shigeru Mukai. Fano 3-folds. In *Complex projective geometry (Trieste, 1989/Bergen, 1989)*, volume 179 of *London Math. Soc. Lecture Note Ser.*, pages 255–263. Cambridge Univ. Press, Cambridge, 1992.

[9] Miles Reid. The du val singularities an, dn, e6, e7, e8.

[10] I. R. Shafarevich, editor. *Algebraic geometry. V*, volume 47 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 1999. Fano varieties, A translation of ıt Algebraic geometry. 5 (Russian), Ross. Akad. Nauk, Vseross. Inst. Nauchn. i Tekhn. Inform., Moscow, Translation edited by A. N. Parshin and I. R. Shafarevich.