The $\boldsymbol{\delta}$-Bounded $\boldsymbol{\varepsilon}$-Non-Dominated Multi-Objective User Equilibrium ($\boldsymbol{\delta}$-EBR-MUE): Model, Theory and Approximation

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Abstract

This paper introduces a novel multi-objective user equilibrium model under bounded rationality, termed the $\boldsymbol{\delta}$-EBR-MUE, which integrates two rationality constraints: (i) weakly $\boldsymbol{\varepsilon}$-non-dominated path choice, and (ii) travel time boundedness relative to the shortest path. We provide a complete mathematical formulation, establish existence via a set-valued fixed-point argument, and propose a support function approximation for the non-dominated frontier. Numerical experiments validate the behavioral implications and demonstrate the diversity of solutions beyond classical DUE or BRUE models.

1 Notation and Preliminaries

Let $\mathscr{W}$ denote the set of origin-destination (OD) pairs. For each OD pair $w \in \mathscr{W}$, let $\mathscr{P}^w$ denote the set of all feasible paths, and d^w the total travel demand.

Let $\mathbf{f} = (f_p^w)_{p\in\mathscr{P}^w, w\in\mathscr{W}}$ denote the path flow vector. The feasible flow set is defined as:

$$\Lambda := \left\{ \mathbf{f} \in \mathbb{R}_+^{|\mathscr{P}|} : \sum_{p\in\mathscr{P}^w} f_p^w = d^w,\ \forall w \in \mathscr{W} \right\}.$$

Each path p is associated with a bi-objective cost vector:

$$\mathbf{C}_p(\mathbf{f}) := (t_p(\mathbf{f}),\ c_p(\mathbf{f})),$$

where $t_p(\mathbf{f})$ and $c_p(\mathbf{f})$ represent travel time and monetary cost under flow $\mathbf{f}$.

Dominance and $\boldsymbol{\varepsilon}$-Non-Dominance

Definition 1.1 (Strict Dominance). Given two paths p and q for OD pair w, path q strictly dominates path p, denoted

$$\mathbf{C}_q(\mathbf{f}) \prec \mathbf{C}_p(\mathbf{f}),$$

if and only if:

$$t_q(\mathbf{f}) < t_p(\mathbf{f}) \quad \text{and} \quad c_q(\mathbf{f}) < c_p(\mathbf{f}).$$

Definition 1.2 (ε-Strict Dominance). For tolerance vector $\boldsymbol{\varepsilon} = (\varepsilon_t, \varepsilon_m)$, path q $\boldsymbol{\varepsilon}$-strictly dominates path p if:

$$t_q(\mathbf{f}) + \varepsilon_t < t_p(\mathbf{f}) \quad \text{and} \quad c_q(\mathbf{f}) + \varepsilon_m < c_p(\mathbf{f}).$$

Definition 1.3 (ε-Weakly Non-Dominated Path). Path p is ε-weakly non-dominated if it is not ε-strictly dominated by any $q \in \mathscr{P}^w$, i.e.,

$$\phi_p^\varepsilon(\mathbf{f}) := \max_{q \in \mathscr{P}^w} \max \left\{ t_q - t_p + \varepsilon_t,\ c_q - c_p + \varepsilon_m \right\} \leq 0.$$

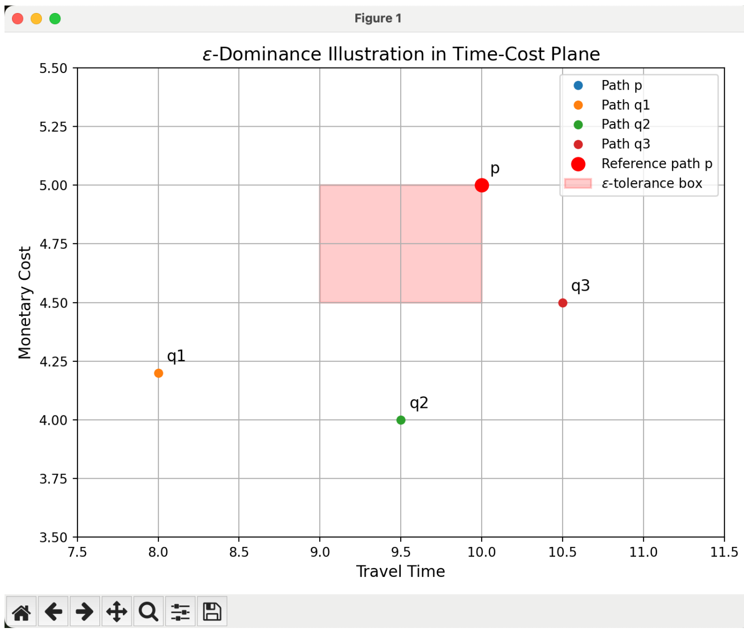


Figure 1: Illustration of ε-dominance in the time-cost objective space. The red box indicates the ε-tolerance area around the reference path p. Paths inside this box are not ε-strictly dominating p. Path q_1 strictly dominates p, q_2 ε-dominates p, and q_3 does not dominate p.

Example: ε-Nondominated Flow Region Between Two Paths

Consider two alternative paths p and q connecting the same origin-destination pair. Let the scalar flow on path p be denoted as f_p, and on path q as f_q. The bi-objective cost vectors for these two paths are given by:

$$\mathbf{C}_p(f) = \begin{pmatrix} C_p^1(f) \\ C_p^2(f) \end{pmatrix} = \begin{pmatrix} 2 + f_p \\ 4 + f_p^2 \end{pmatrix},$$

$$\mathbf{C}_q(f) = \begin{pmatrix} C_q^1(f) \\ C_q^2(f) \end{pmatrix} = \begin{pmatrix} 1 + 2f_q \\ 5 + f_q \end{pmatrix}.$$

We consider an ε-dominance relation with tolerance vector $\boldsymbol{\varepsilon} = (\varepsilon^1, \varepsilon^2) = (0.5, 0.5)$. The cost vector $\mathbf{C}_q(f)$ is said to ε-dominate $\mathbf{C}_p(f)$ if:

$$C_q^1(f) \leq C_p^1(f) - \varepsilon^1 \quad \text{and} \quad C_q^2(f) \leq C_p^2(f) - \varepsilon^2.$$

Therefore, the set of flow pairs $(f_p, f_q) \in \mathbb{R}_+^2$ under which path p is not ε-dominated by path q is:

$$\mathscr{F}_\varepsilon = \left\{(f_p, f_q) \in \mathbb{R}^2_+ \;\middle|\; C^1_q(f) > C^1_p(f) - \varepsilon^1 \quad \text{or} \quad C^2_q(f) > C^2_p(f) - \varepsilon^2\right\}.$$

This region can be visualised in the (f_p, f_q)-space as the set where the ε-dominance condition does not hold. Due to the nonlinear nature of the cost functions, the set $\mathscr{F}_\varepsilon$ is generally non-convex.

2 Mathematical Formulation of the δ-EBR-MUE Model

Let $\mathscr{W}$ be the set of origin-destination (OD) pairs, and $\mathscr{P}^w$ the set of feasible paths for $w \in \mathscr{W}$. For each path $p \in \mathscr{P}^w$:

- f_p^w: flow on path p;
- $\mathbf{C}_p = (t_p(\mathbf{f}), c_p(\mathbf{f}))$: travel time and cost;
- $T_w^{\mathrm{UE}}(\mathbf{f}) := \min_{p \in \mathscr{P}^w} t_p(\mathbf{f})$.

A path p is said to be ε-non-dominated and δ-bounded if:

(i) $\nexists q \in \mathscr{P}^w: \ \mathbf{C}_q(\mathbf{f}) \prec \mathbf{C}_p(\mathbf{f}) - \varepsilon$,
(ii) $t_p(\mathbf{f}) \le T_w^{\mathrm{UE}}(\mathbf{f}) + \delta$.

Let $\phi_p^\varepsilon(\mathbf{f}) := \max_{q \in \mathscr{P}^w} \max\left\{t_q - t_p + \varepsilon_t, c_q - c_p + \varepsilon_m\right\}$ and $\phi_p^\delta(\mathbf{f}) := t_p(\mathbf{f}) - T_w^{\mathrm{UE}}(\mathbf{f}) - \delta$. The δ-EBR-MUE conditions are:

$$\begin{aligned}
&f_p^w \cdot \phi_p^\varepsilon(\mathbf{f}) = 0, \quad \phi_p^\varepsilon(\mathbf{f}) \ge 0,\\
&f_p^w \cdot \phi_p^\delta(\mathbf{f}) = 0, \quad \phi_p^\delta(\mathbf{f}) \le 0,\\
&\sum_{p \in \mathscr{P}^w} f_p^w = d^w, \quad f_p^w \ge 0.
\end{aligned}$$

Interpretation of $\phi_p^\varepsilon(\mathbf{f})$. The expression $\phi_p^\varepsilon(\mathbf{f}) := \max_{q \in \mathscr{P}^w} \max\left\{t_q - t_p + \varepsilon_t,\ c_q - c_p + \varepsilon_m\right\}$ quantifies the worst-case ε-dominance violation for path p.

- The inner $\max$ evaluates whether any path q offers a sufficient improvement in either time or cost to make path p strictly dominated beyond the tolerance ε.
- The outer $\max$ scans all such potential dominating paths q.

If $\phi_p^\varepsilon(\mathbf{f}) > 0$, it means there exists a path q that ε-dominates p in at least one dimension. Therefore, path p is considered infeasible under the δ-EBR-MUE behavioral rule. Conversely, $\phi_p^\varepsilon(\mathbf{f}) = 0$ implies that p is weakly ε-non-dominated.

3 Existence of δ-EBR-MUE

Let the feasible set be:

$$\Lambda := \left\{\mathbf{f} \in \mathbb{R}_+^{|\mathscr{P}|} : \sum_{p \in \mathscr{P}^w} f_p^w = d^w, \forall w \in \mathscr{W}\right\}.$$

Define a set-valued map:

$$\boldsymbol{\Phi}(\mathbf{f}) := \left\{ \tilde{\mathbf{f}} \in \Lambda : \tilde{f}_p^w > 0 \Rightarrow p \in \mathscr{A}_w^{\varepsilon,\delta}(\mathbf{f}) \right\},$$

where $\mathscr{A}_w^{\varepsilon,\delta}(\mathbf{f})$ is the set of all paths satisfying the two criteria above. Under continuity of $\mathbf{C}_p$, compactness of Λ, and closedness of $\boldsymbol{\Phi}$, Kakutani's fixed-point theorem ensures existence of an equilibrium $\mathbf{f}^* \in \boldsymbol{\Phi}(\mathbf{f}^*)$.

4 Existence of δ-EBR-MUE

We now establish the existence of a flow vector $\mathbf{f}^* \in \Lambda$ satisfying the conditions of the δ-EBR-MUE model.

Theorem 4.1 (Existence of δ-EBR-MUE). Suppose that:

1. The feasible set of flows $\Lambda := \left\{ \mathbf{f} \in \mathbb{R}_+^{|\mathscr{P}|} : \sum_{p \in \mathscr{P}^w} f_p^w = d^w,\ \forall w \in \mathscr{W} \right\}$ is nonempty, compact, and convex;
2. The path cost functions $t_p(\mathbf{f})$, $c_p(\mathbf{f})$ are continuous with respect to $\mathbf{f}$;
3. The parameters $\varepsilon_t, \varepsilon_m, \delta \geq 0$ are fixed;

Then there exists a flow vector $\mathbf{f}^* \in \Lambda$ satisfying the following conditions for all OD pairs $w \in \mathscr{W}$ and all $p \in \mathscr{P}^w$:

$$f_p^{w*} > 0 \Rightarrow \begin{cases} \phi_p^\varepsilon(\mathbf{f}^*) = 0, & (\varepsilon\text{-non-dominance}) \\ \\ t_p(\mathbf{f}^*) \leq T_w^{\mathrm{UE}}(\mathbf{f}^*) + \delta, & (\text{time-bounded}) \end{cases}$$

Proof. We reformulate the equilibrium as a fixed point of a set-valued map.

Define the feasible flow set:

$$\Lambda = \left\{ \mathbf{f} \in \mathbb{R}_+^{|\mathscr{P}|} : \sum_{p \in \mathscr{P}^w} f_p^w = d^w, \forall w \in \mathscr{W} \right\},$$

which is nonempty, convex, and compact.

Define for each OD pair w the set of acceptable paths under flow $\mathbf{f}$:

$$\mathscr{A}_w^{\varepsilon,\delta}(\mathbf{f}) := \left\{ p \in \mathscr{P}^w \ : \ \phi_p^\varepsilon(\mathbf{f}) = 0,\ t_p(\mathbf{f}) \leq T_w^{\mathrm{UE}}(\mathbf{f}) + \delta \right\}.$$

Then define the set-valued mapping:

$$\boldsymbol{\Phi}(\mathbf{f}) := \left\{ \tilde{\mathbf{f}} \in \Lambda \ : \ \tilde{f}_p^w > 0 \Rightarrow p \in \mathscr{A}_w^{\varepsilon,\delta}(\mathbf{f}),\ \forall w \in \mathscr{W} \right\}.$$

We claim that this map $\boldsymbol{\Phi} : \Lambda \rightrightarrows \Lambda$ satisfies the conditions of Kakutani's fixed-point theorem:

- The domain Λ is compact and convex (from assumption);
- For each $\mathbf{f}$, the image $\boldsymbol{\Phi}(\mathbf{f})$ is convex: since selecting flows on acceptable paths preserves linearity and the feasible set is convex;
- The graph of $\boldsymbol{\Phi}$ is closed: because both $\phi_p^\varepsilon(\mathbf{f})$ and $t_p(\mathbf{f})$ are continuous in $\mathbf{f}$, the acceptance set $\mathscr{A}_w^{\varepsilon,\delta}(\mathbf{f})$ varies upper hemicontinuously with $\mathbf{f}$.

By Kakutani's theorem, there exists $\mathbf{f}^* \in \Lambda$ such that $\mathbf{f}^* \in \boldsymbol{\Phi}(\mathbf{f}^*)$, i.e., a δ-EBR-MUE exists. $\square$

Algorithm: Iterative Solution for δ-Bounded Rational Multi-Objective Non-Strictly Dominated User Equilibrium (δ-EBR-MUE)

Input:

- OD demand set $\{d_w\}_{w \in W}$
- Path sets $\{\mathscr{P}_w\}_{w \in W}$ for each OD pair
- Time cost tolerance parameter $\delta > 0$
- Path cost function $\mathbf{C}_p(\mathbf{f})$, including time and other objectives
- Initial feasible path flow $\mathbf{f}^{(0)}$ satisfying demand conservation
- Maximum number of iterations $K_{\max}$ and convergence tolerance ε

Output: Path flow vector $\mathbf{f}^*$ satisfying the δ-EBR-MUE equilibrium conditions.

Algorithm Steps:

1. Initialization: Set iteration counter $k = 0$ and initialize path flow $\mathbf{f}^{(0)}$.
2. Compute Path Costs: For current flow $\mathbf{f}^{(k)}$, compute multi-objective path costs:
$$\mathbf{C}_p^{(k)} = \mathbf{C}_p(\mathbf{f}^{(k)}),$$
extracting the time cost component:
$$t_p^{(k)} = \text{time component of } \mathbf{C}_p^{(k)}.$$
3. Determine Minimum Time Cost per OD Pair: For each OD pair w, find the shortest travel time among all paths:
$$t_w^{*(k)} = \min_{p \in \mathscr{P}_w} t_p^{(k)}.$$
4. Filter Feasible Paths: Construct the feasible path set for each OD pair:
$$\mathscr{P}_w^{(k)} = \left\{ p \in \mathscr{P}_w \mid t_p^{(k)} \le t_w^{*(k)} + \delta, \quad p \text{ is non-strictly dominated} \right\}.$$
Here, a path p is non-strictly dominated if there does not exist another path p' such that:
$$\mathbf{C}_{p'}^{(k)} < \mathbf{C}_p^{(k)} \quad \text{(all objectives strictly less)}.$$
5. Flow Reassignment Subproblem: Fixing the feasible path sets $\{\mathscr{P}_w^{(k)}\}$, solve the flow assignment problem:
$$\min_{\mathbf{f} \ge 0} \sum_{w \in W} \sum_{p \in \mathscr{P}_w^{(k)}} \int_0^{f_p} C_p(s)\, ds,$$
subject to:
$$\sum_{p \in \mathscr{P}_w^{(k)}} f_p = d_w, \quad \forall w \in W.$$
Denote the solution by $\mathbf{f}^{(k+1)}$.

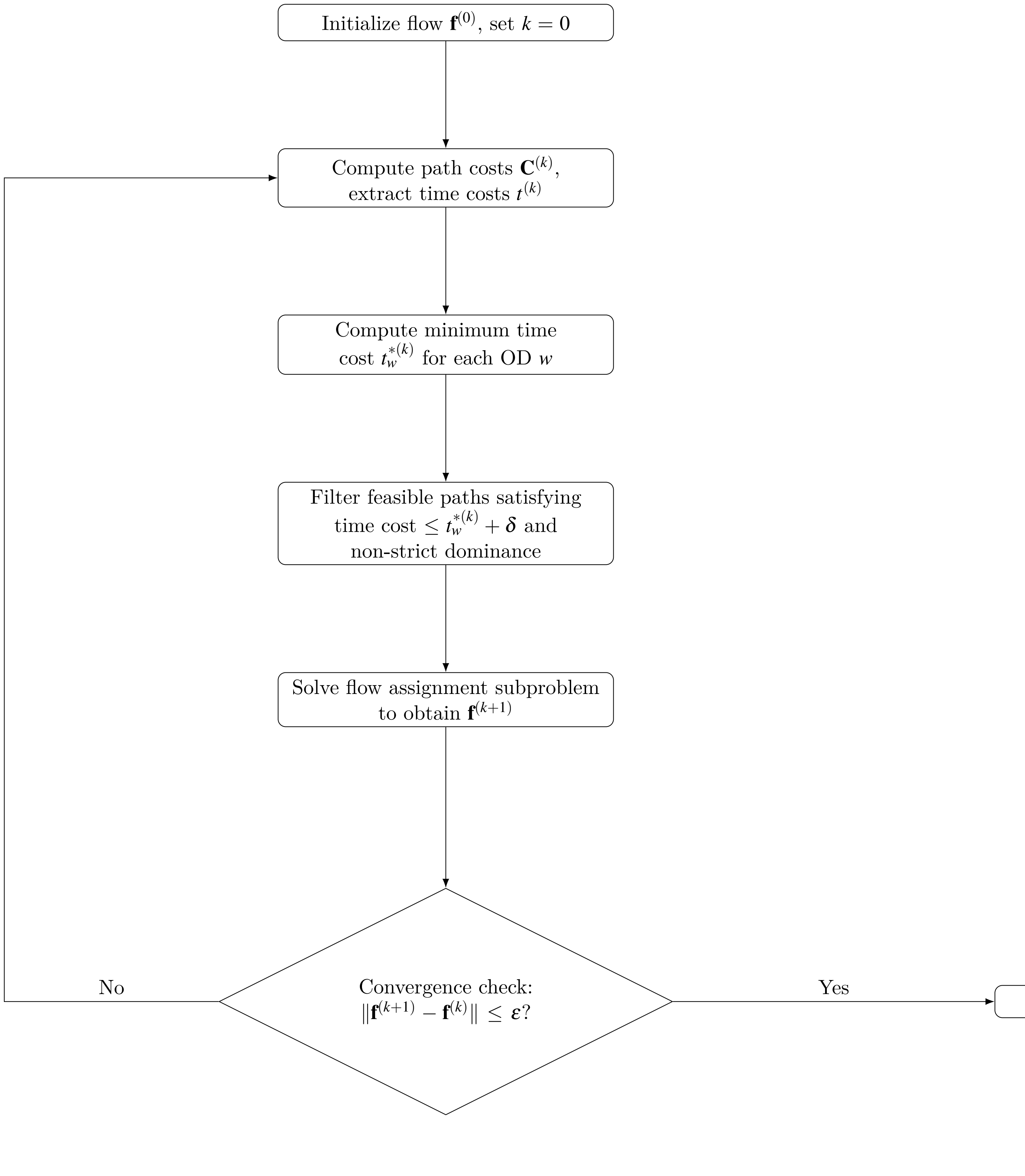
6. Check Convergence: If

$$\|\mathbf{f}^{(k+1)} - \mathbf{f}^{(k)}\| \leq \varepsilon,$$

terminate and set $\mathbf{f}^* = \mathbf{f}^{(k+1)}$.

7. Iteration: Otherwise increment $k \leftarrow k+1$. If $k > K_{\max}$, terminate with current solution; else return to Step 2.

Flowchart summary:


Initialize flow $\mathbf{f}^{(0)}$, set $k = 0$
Compute path costs $\mathbf{C}^{(k)}$, extract time costs $t^{(k)}$
Compute minimum time cost $t_w^{*(k)}$ for each OD w
Filter feasible paths satisfying time cost $\leq t_w^{*(k)} + \delta$ and non-strict dominance
Solve flow assignment subproblem to obtain $\mathbf{f}^{(k+1)}$
Convergence check: $\|\mathbf{f}^{(k+1)} - \mathbf{f}^{(k)}\| \leq \varepsilon$?
No
Yes

Algorithm 1 VI-based Algorithm for Solving δ-Bounded ε-Non-strictly Dominated User Equilibrium (δ-EBR-MUE)

Require: Initial flow $\mathbf{f}^{(0)}$, OD demands $\{d^w\}$, path sets $\{\mathscr{P}_w\}$, tolerance $\delta > 0$, $\varepsilon \in \mathbb{R}^m_+$, maximum iteration $K_{\max}$

Ensure: Equilibrium flow $\mathbf{f}^*$

Set $k \leftarrow 0$

repeat

  Compute the cost vector $\mathbf{C}_p(\mathbf{f}^{(k)}) = (C_p^1, \dots, C_p^m)$ for all p

  for each OD pair $w \in \mathscr{W}$ **do**

    Find minimum time cost: $\mathscr{T}_w = \min_{p\in\mathscr{P}_w} C_p^1(\mathbf{f}^{(k)})$

    Define feasible path set:

$$\hat{\mathscr{P}}_w := \left\{ p \in \mathscr{P}_w : C_p^1(\mathbf{f}^{(k)}) \le \mathscr{T}_w + \delta,\ \nexists\, q \in \mathscr{P}_w \text{ s.t. } \mathbf{C}_q \le \mathbf{C}_p - \varepsilon \right\}$$

  end for

  Define feasible flow set:

$$\mathscr{F}^{\delta,\varepsilon} := \left\{ \mathbf{f} \ge 0 \;\middle|\; \sum_{p\in\mathscr{P}_w} f_p = d^w,\ f_p > 0 \Rightarrow p \in \hat{\mathscr{P}}_w,\ \forall w \right\}$$

  Define operator $F(\mathbf{f}) = (\mathbf{C}_p(\mathbf{f}))_{p\in\cup_w \mathscr{P}_w}$

  Solve the following variational inequality (projected method or proximal point method):

$$\text{Find } \mathbf{f}^{(k+1)} \in \mathscr{F}^{\delta,\varepsilon} \text{ such that } \langle F(\mathbf{f}^{(k+1)}), \mathbf{f} - \mathbf{f}^{(k+1)} \rangle \ge 0, \quad \forall \mathbf{f} \in \mathscr{F}^{\delta,\varepsilon}$$

  $k \leftarrow k+1$

until $\|\mathbf{f}^{(k)} - \mathbf{f}^{(k-1)}\| \le \text{tol}$ or $k \ge K_{\max}$

return $\mathbf{f}^{(k)}$

Algorithm 2 δ-bounded ε-Non-dominated User Equilibrium Solver

Require: Initial flow vector $\mathbf{f}^0$, OD demands $\{d^w\}_{w\in\mathscr{W}}$, path sets $\{\mathscr{P}_w\}$, tolerance parameters $\delta > 0$, $\varepsilon \in \mathbb{R}^m_+$, maximum iterations $K_{\max}$
Ensure: Approximate equilibrium flow $\mathbf{f}^*$

```
Set k ← 0
Initialize f^(0)
repeat
    Compute cost vectors C_p(f^(k)) for all paths p
    for each OD pair w ∈ W do
        Let T_w^(k) = min_{p∈P_w} C_p^1(f^(k))          ▷ Minimum travel time cost
        Let P̂_w^(k) ← ∅
        for each path p ∈ P_w do
            if C_p^1(f^(k)) ≤ T_w^(k) + δ and not strictly ε-dominated by other paths in P_w then
                Add p to P̂_w^(k)
            end if
        end for
    end for
    Assign flow uniformly over P̂_w^(k) for each w, respecting demand d^w, to obtain f^(k+1)
    k ← k + 1
until ||f^(k) − f^(k−1)|| ≤ tol or k ≥ K_max
return f^(k)
```

5 Support Function Approximation of the ε-Nondominated Set

Let $\mathscr{C}^w := \{\mathbf{C}_p : p \in \mathscr{P}^w\}$ and choose $\mathscr{U}_N \subset \mathbb{S}^1$ directions. Define:

$$\rho_{\mathscr{C}^w}(\mathbf{u}) := \min_{p\in\mathscr{P}^w} \mathbf{u}^\top \mathbf{C}_p.$$

Then:

$$\mathscr{E}^w_\varepsilon := \left\{ \mathbf{C}_p : \forall \mathbf{u} \in \mathscr{U}_N,\ \mathbf{u}^\top \mathbf{C}_p \le \rho_{\mathscr{C}^w}(\mathbf{u}) + \varepsilon_{\mathbf{u}} \right\}.$$

As $N \to \infty$, $\mathscr{E}^w_\varepsilon$ converges in Hausdorff distance to the true weakly nondominated frontier.

6 Limiting Case: $\varepsilon = 0$

We consider the limiting case where the dominance tolerance $\varepsilon = (0,0)$. In this case, the δ-EBR-MUE model reduces to a sharper form, where users only accept paths that are not strictly dominated in the cost vector space, while still satisfying the travel time bound.

Definition 6.1 (δ-Bounded Weakly Non-Dominated User Equilibrium (δ-WNUE)). A feasible path flow $\mathbf{f}^*$ is said to satisfy the δ-WNUE condition if:

$$f^w_p > 0 \ \Rightarrow\ \begin{cases} \mathbf{C}_q(\mathbf{f}^*) \not< \mathbf{C}_p(\mathbf{f}^*), & \forall q \in \mathscr{P}^w \\ t_p(\mathbf{f}^*) \le T^{\mathrm{UE}}_w(\mathbf{f}^*) + \delta, \end{cases}$$

Proposition 6.1. Every δ-EBR-MUE with $\varepsilon = 0$ is a δ-WNUE. Conversely, every δ-WNUE is a δ-EBR-MUE with $\varepsilon = 0$.

Proof. Immediate from the definitions: with $\varepsilon = 0$, the ε-non-dominance condition becomes standard weak non-domination. The time-bound condition remains unchanged. □

Remark 6.1. As $\varepsilon \to 0$, the δ-EBR-MUE set converges to the set of δ-WNUE flows. However, for any $\varepsilon > 0$, the equilibrium permits a richer variety of solutions, possibly including near-dominated paths, thereby reflecting behavioural flexibility and tolerance.

7 Numerical Example

Consider a simple network with three parallel paths. Let the travel time on each path be $t_p(f) = a_p + b_p f_p$ and cost c_p be fixed. Set:

- $a = [3,5,8]$, $b = [0.01, 0.01, 0.02]$;
- $c = [2, 1.5, 0.5]$ (monetary units);
- demand $d = 1$, $\varepsilon = (1.5, 0.5)$, $\delta = 2$.

Compute:

1. User Equilibrium (UE);
2. Bounded Rational UE (BRUE);
3. δ-EBR-MUE using fixed-point iteration.

Compare results in terms of used paths, average cost, and Pareto dominance.