Non-Convex Optimisation: Survey & ADAM's Proof
Reinforcement Learning Summer School

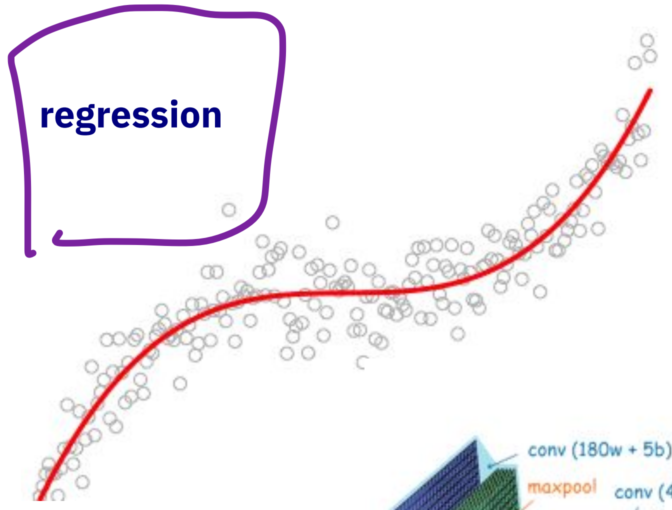
Haitham Bou Ammar

Motivation, Function, and Solution Types

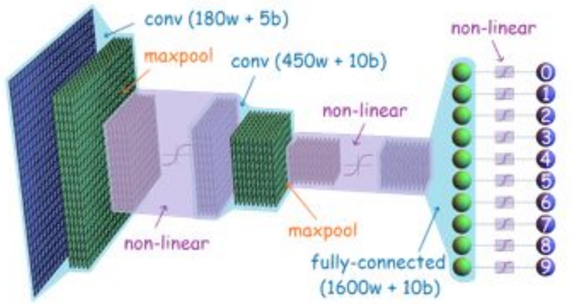
Why Optimisation?

why we research optimization ?

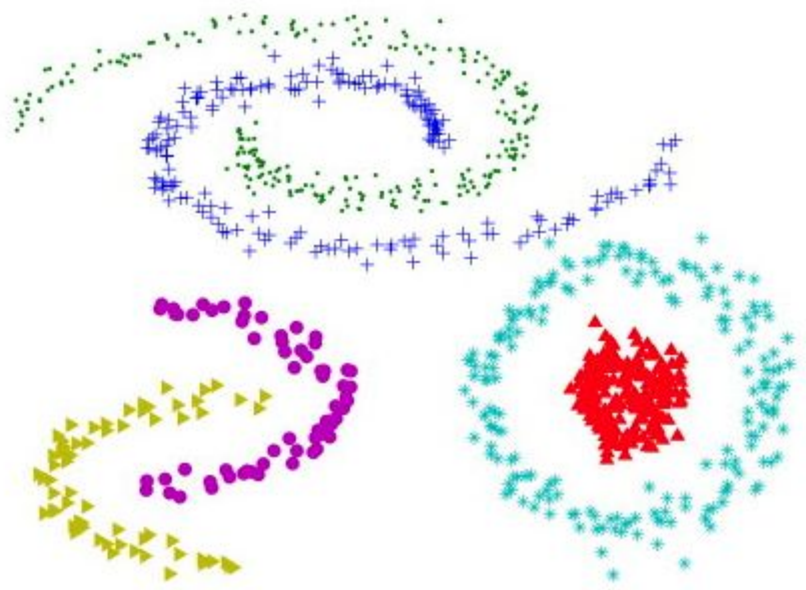
regression



classification



clustering/density estimation



computer games

robotics

Decision

Supervised Learning

$$\min_{\boldsymbol{\theta}} \frac{1}{n} \sum_{j=1}^{n} \mathcal{L}_{\boldsymbol{\theta}} \left(\mathbf{x}^{(i)}, y^{(i)} \right)$$

Unsupervised Learning

$$\min_{\boldsymbol{\theta}} \frac{1}{n} \sum_{j=1}^{n} \mathcal{L}_{\boldsymbol{\theta}} \left(\mathbf{x}^{(i)} \right)$$

Reinforcement Learning

$$\min_{\boldsymbol{\theta}} \mathbb{E}_{\boldsymbol{\tau} \sim p_{\boldsymbol{\theta}}(\boldsymbol{\tau})} \left(\mathcal{R}_{\text{total}}(\boldsymbol{\tau}) \right)$$

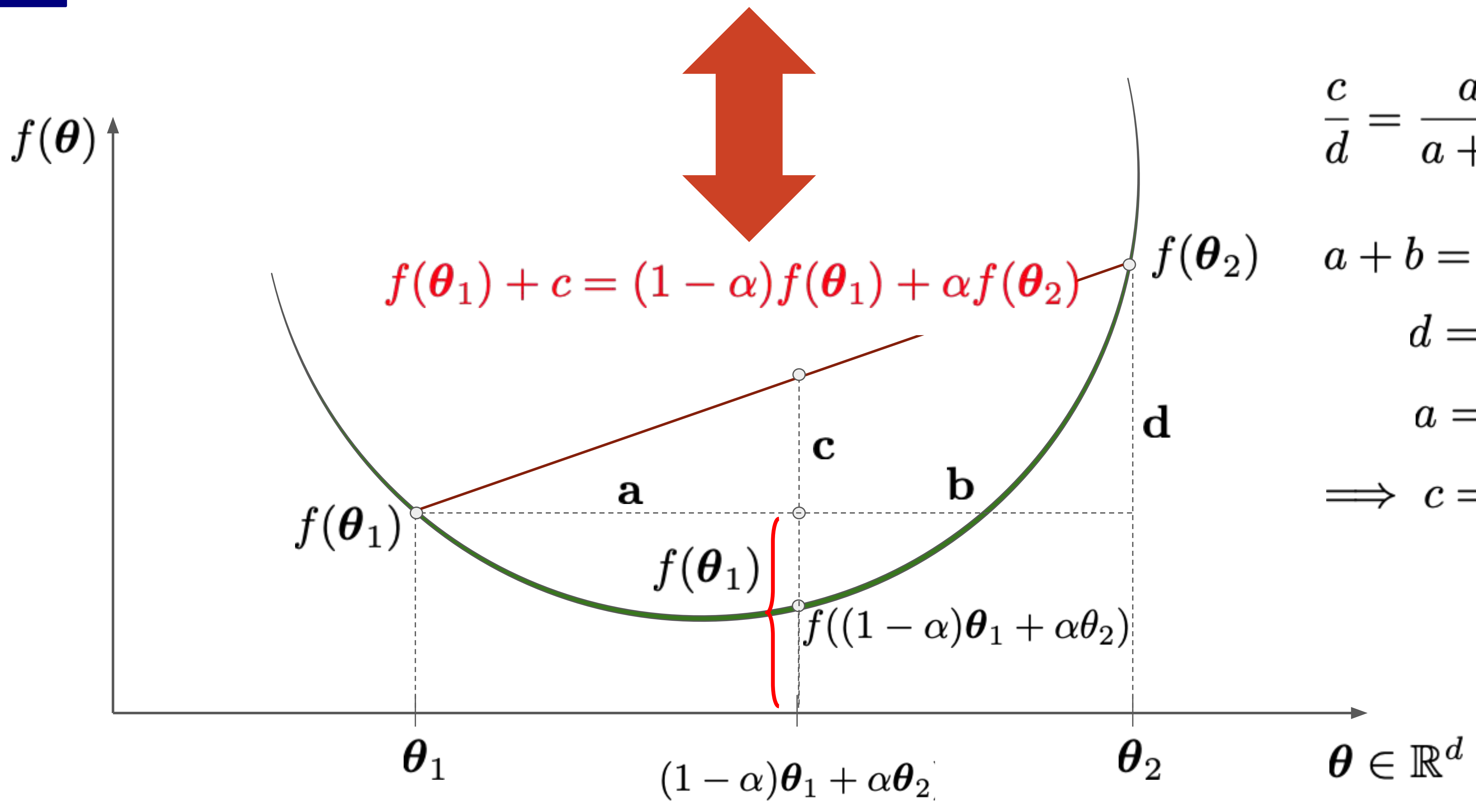
... all these involve a minimisation of some function ...

$$\boxed{\min_{\boldsymbol{\theta} \in \mathbb{R}^d} f(\boldsymbol{\theta})}$$

Function types, and what one can hope for ...

... optimising for unknown parameters depends on the type of function under study ...

Convex $f((1-\alpha)\boldsymbol{\theta}_1 + \alpha\boldsymbol{\theta}_2) \leq (1-\alpha)f(\boldsymbol{\theta}_1) + \alpha f(\boldsymbol{\theta}_2) \quad \forall\alpha \in (0,1) \quad \forall\boldsymbol{\theta}_1,\ \boldsymbol{\theta}_2 \in \mathbb{R}^d$



any point in between

$$\frac{c}{d} = \frac{a}{a+b}$$

$$a + b = ||\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1||_2$$

$$d = f(\boldsymbol{\theta}_2) - f(\boldsymbol{\theta}_1)$$

$$a = \alpha||\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1||_2$$

$$\implies c = \alpha f(\boldsymbol{\theta}_2) - \alpha f(\boldsymbol{\theta}_1)$$

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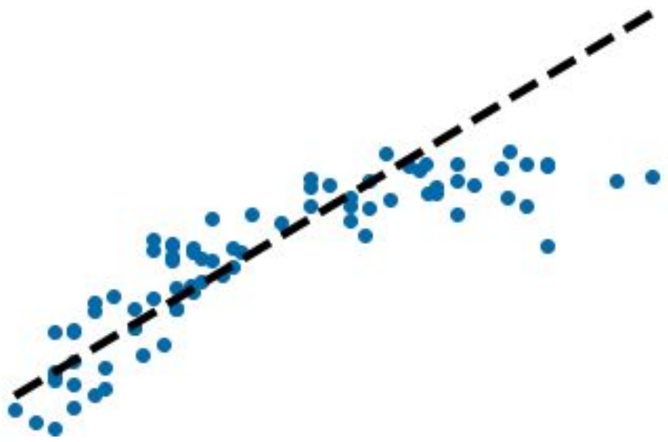
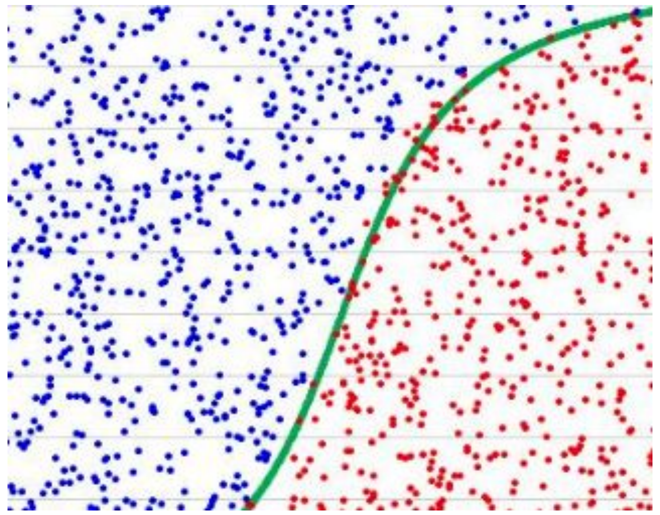
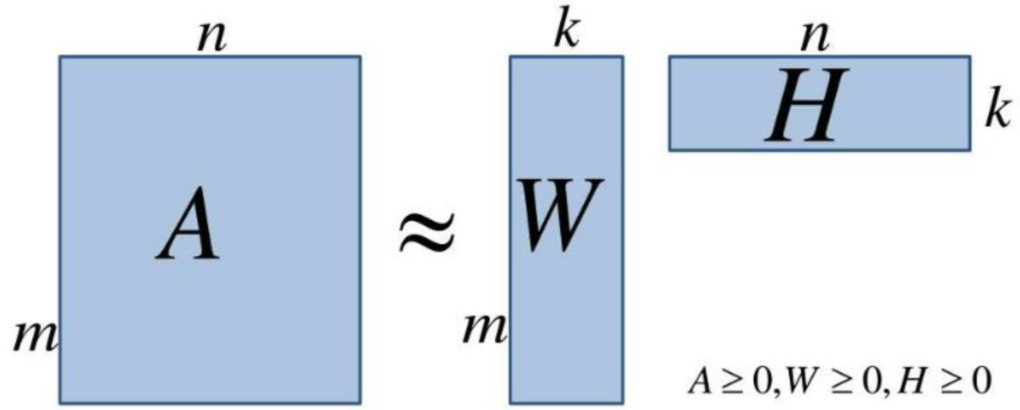


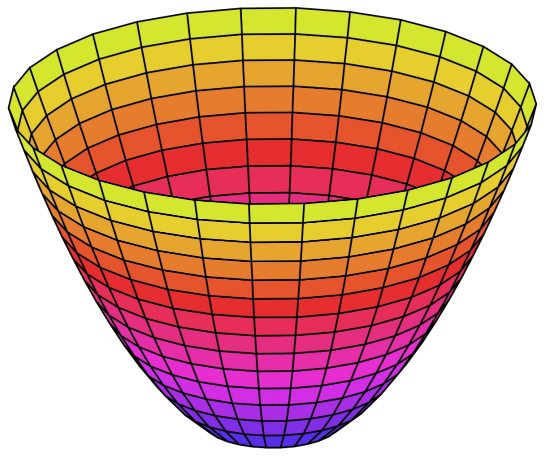
Linear Regression



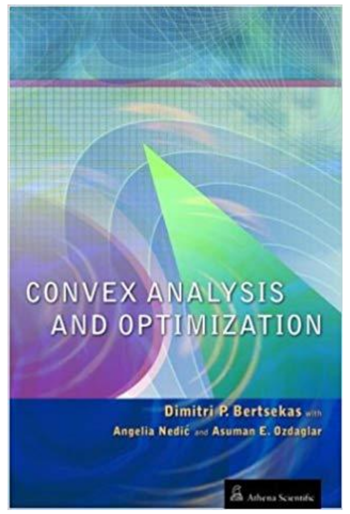
Classification with Hinge Loss



Non-Negative Matrix Factorisation



Unique global minimum



... admits polynomial time algorithms

Function types, and what one can hope for ...

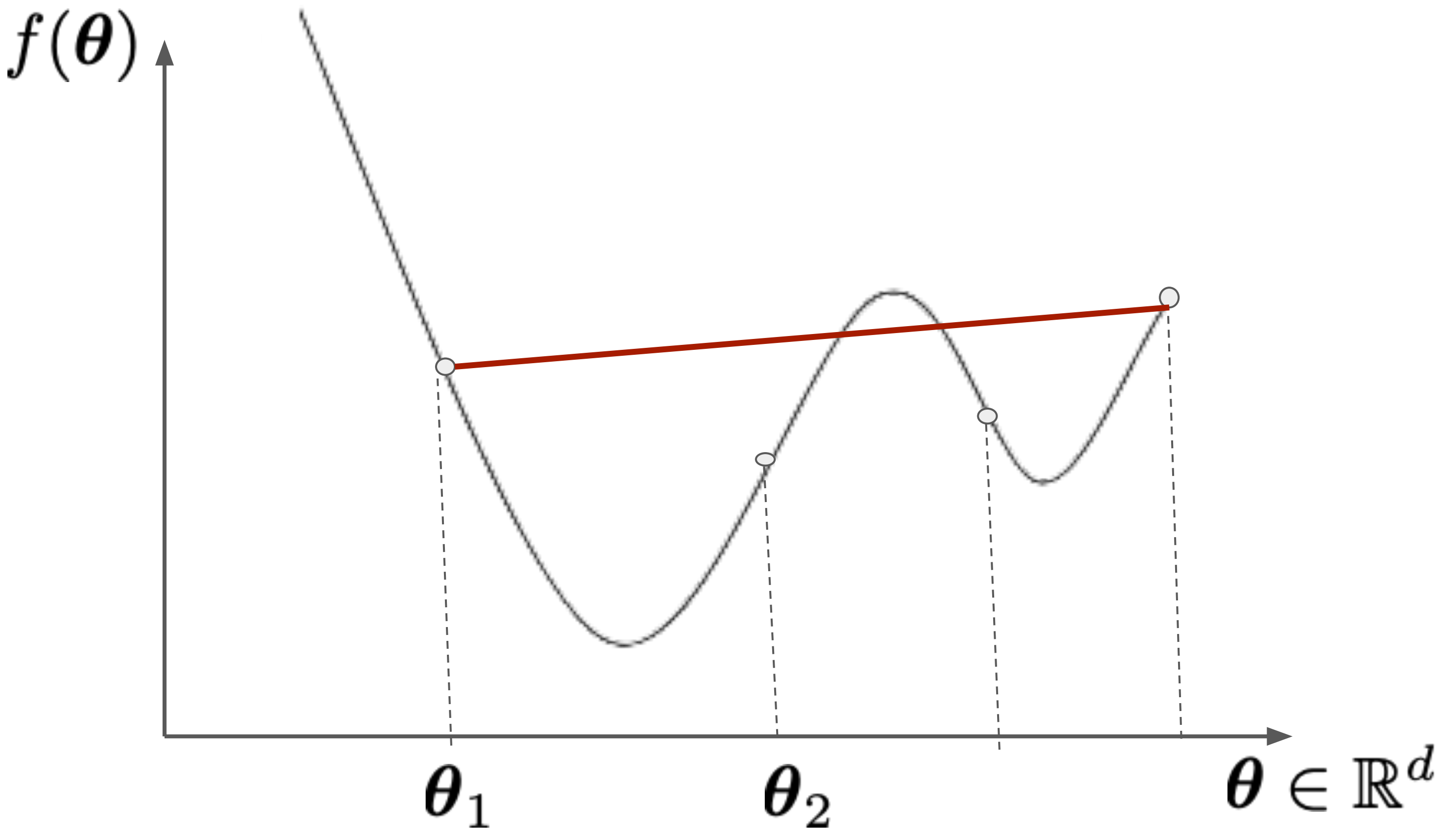
... optimising for unknown parameters depends on the type of function under study ...

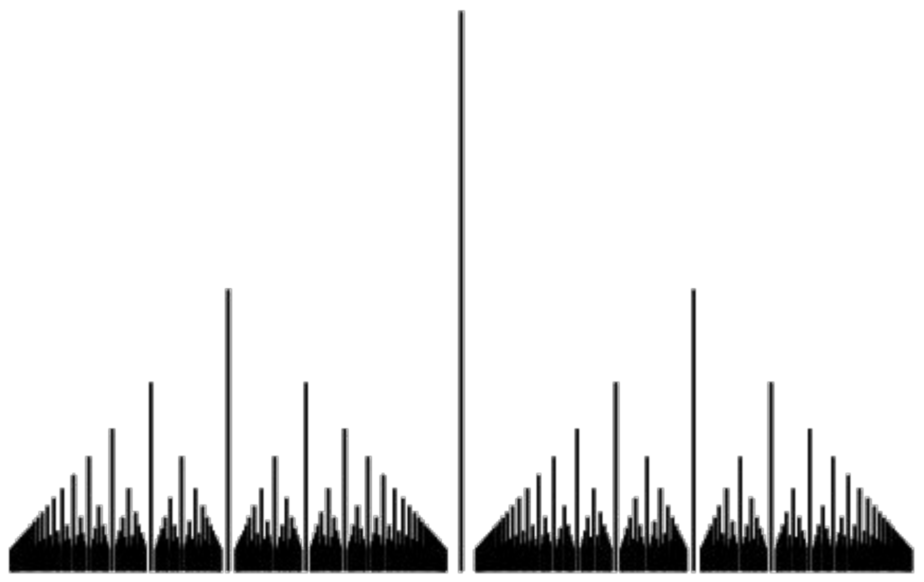
Non-Convex **... we want to negate the convex definition (and avoid concave definition) ...**

$$\exists \;\; \boldsymbol{\theta}_1, \;\; \boldsymbol{\theta}_2, \;\; \text{and} \;\; \alpha \in (0,1) \;\; \text{such that} \;\; f\left((1-\alpha_1)\boldsymbol{\theta}_1 + \alpha\boldsymbol{\theta}_2\right) > (1-\alpha)f(\boldsymbol{\theta}_1) + \alpha f(\boldsymbol{\theta}_2)$$

$$\exists \;\; \tilde{\boldsymbol{\theta}}_1, \;\; \tilde{\boldsymbol{\theta}}_2, \;\; \text{and} \;\; \tilde{\alpha} \in (0,1) \;\; \text{such that} \;\; f\left((1-\tilde{\alpha}_1)\tilde{\boldsymbol{\theta}}_1 + \tilde{\alpha}\tilde{\boldsymbol{\theta}}_2\right) < (1-\tilde{\alpha})f(\tilde{\boldsymbol{\theta}}_1) + \tilde{\alpha} f(\tilde{\boldsymbol{\theta}}_2)$$

&





What happens with a dirichlet function?

Function types, and what one can hope for …

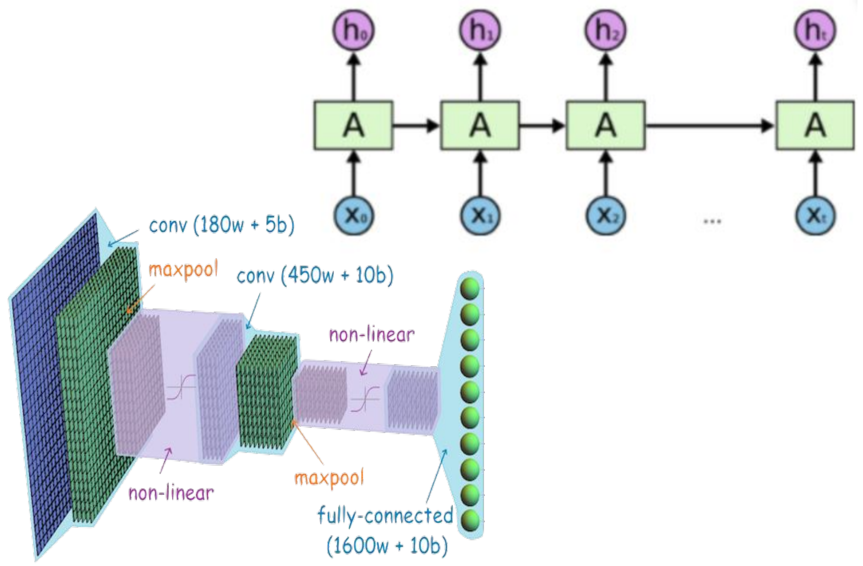
… optimising for unknown parameters depends on the type of function under study …

Non-Convex **… we want to negate the convex definition (and avoid concave definition) …**

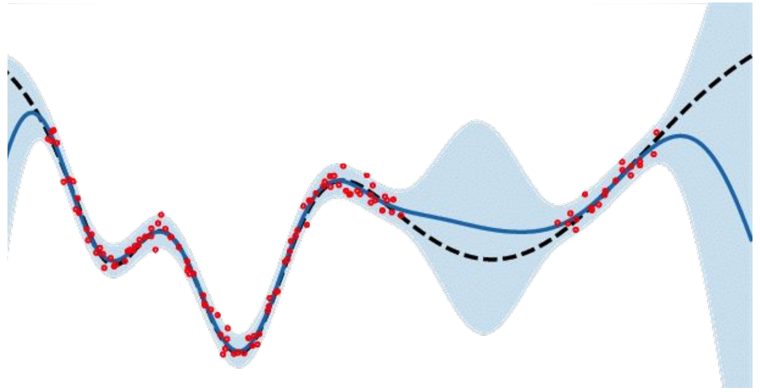
$$\exists \;\; \boldsymbol{\theta}_1, \;\; \boldsymbol{\theta}_2, \;\; \text{and} \;\; \alpha \in (0,1) \;\; \text{such that} \;\; f\left((1-\alpha_1)\boldsymbol{\theta}_1 + \alpha\boldsymbol{\theta}_2\right) > (1-\alpha)f(\boldsymbol{\theta}_1) + \alpha f(\boldsymbol{\theta}_2)$$

&

$$\exists \;\; \tilde{\boldsymbol{\theta}}_1, \;\; \tilde{\boldsymbol{\theta}}_2, \;\; \text{and} \;\; \tilde{\alpha} \in (0,1) \;\; \text{such that} \;\; f\left((1-\tilde{\alpha}_1)\tilde{\boldsymbol{\theta}}_1 + \tilde{\alpha}\tilde{\boldsymbol{\theta}}_2\right) < (1-\tilde{\alpha})f(\tilde{\boldsymbol{\theta}}_1) + \tilde{\alpha} f(\tilde{\boldsymbol{\theta}}_2)$$



Deep Learning



Gaussian Processes & Bayesian Models

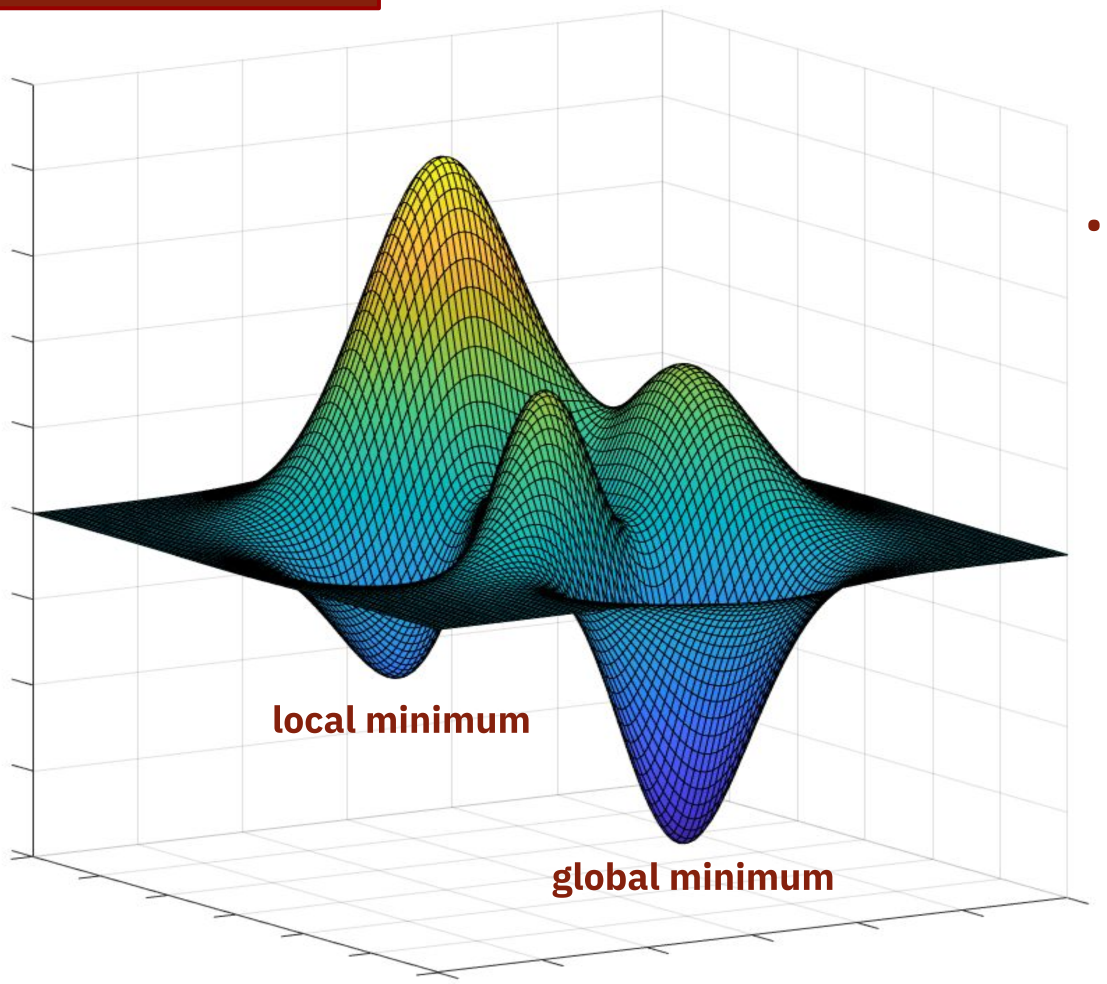
Reinforcement Learning

Function types, and what one can hope for ...

... optimising for unknown parameters depends on the type of function under study ...

Non-Convex

... global and local minima (checking) are NP-Hard, we look for other types of points ...



$$\nabla_{\boldsymbol{\theta}} f(\boldsymbol{\theta}_{\text{stationary}}) = \mathbf{0}$$

... so instead, the community is fetching for stationary points ...

1. ϵ -First-Order-Stationary Point (FOSP): $||\nabla_{\boldsymbol{\theta}} f(\boldsymbol{\theta}_{\text{FOSP}})||_2 \leq \epsilon$

[e.g., all global and local minima, saddle points, plateau points]

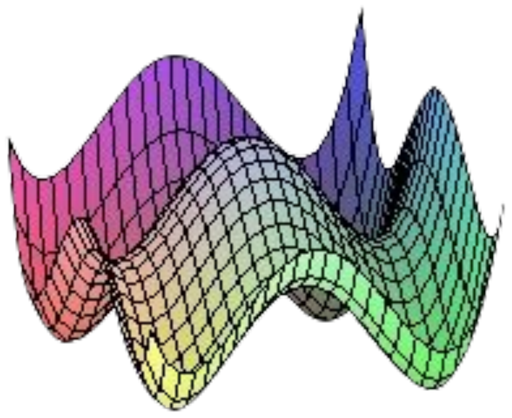
2. ϵ- Second-Order-Stationary Point (SOSP):

$$||\nabla_{\boldsymbol{\theta}} f\left(\boldsymbol{\theta}_{\text{SOSP}}\right)||_2 \leq \epsilon \quad \text{and} \quad \lambda_{\min}\left(\nabla^2_{\boldsymbol{\theta},\boldsymbol{\theta}} f(\boldsymbol{\theta}_{\text{SOSP}})\right) \geq -\sqrt{\epsilon}$$

[e.g., all global and local minima, plateau points]

Brief Survey & ADAM Optimiser

Algorithms vary in type of information used ...



Second-Order Methods

- Newton Method
- Regularised Newton Method
- Stochastic Quasi-Newton

First-Order Methods

adaptive

- GD
- SGD
- ADAM
- AdaGrad
- NAGD
- RMSProp

Momentum

Zero-Order Methods

- Bayesian Optimisation
- StoSOO
- StroquOOL

Non-Convex Optimisation

Let’s Focus on ADAM Optimiser …

Clarified inconsistency with 2015 paper, fixed the proof, and suggested AMSGrad

Corrected Proof for Convex Setting

ON THE CONVERGENCE OF ADAM AND BEYOND

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…

… NADAM

… rectified ADAM

ICLR 2015

ICLR 2018

NeurIPS 2018

ADAM: A METHOD FOR STOCHASTIC OPTIMIZATION

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Conducted proof for non-convex and stochastic settings

Adaptive Methods for Nonconvex Optimization

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Proposed ADAM and demonstrated a proof which was found to have problems that were corrected in 2018

Proved convergence with increasing batch-sizes, and demonstrated a new algorithm (YOGI) with similar convergence guarantees

ADAM’s Proof from NeurIPS 2018

Let's Focus on the 2018's Paper ...

Algorithm's Inputs:

$\underbrace{\boldsymbol{\theta}_1 \in \mathbb{R}^d}_{\text{Initial param value}}$ $\overbrace{\{\eta_t\}_{t=1}^T}^{\text{Learning Rates}}$ $\underbrace{0 \leq \beta_1, \beta_2 \leq 1}_{\text{Decay parameters}}$ $\overbrace{\delta > 0}^{\text{Stability param}}$

Update Procedure:

Set $\mathbf{m}_0 = \mathbf{0}$, and $\mathbf{v}_0 = \mathbf{0}$
for $t = 1$ **to** T **do**
  Draw a sample ξ_t from $\mathbb{P}$
  Compute $\mathbf{g}_t = \nabla \mathcal{L}(\boldsymbol{\theta}_t, \xi_t)$
  Update $\mathbf{m}_t = \beta_1 \mathbf{m}_{t-1} + (1 - \beta_1)\mathbf{g}_t$
  Update $\mathbf{v}_t = \mathbf{v}_{t-1} - (1 - \beta_2)(\mathbf{v}_{t-1} - \mathbf{g}_t^2)$
  Update $\boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}_t - \eta_t \dfrac{\mathbf{g}_t}{(\sqrt{\mathbf{v}_t} + \delta)}$
end for

$$\mathcal{L}(\boldsymbol{\theta}) = \frac{1}{n}\sum_{i=1}^{n}(y_i - f_{\boldsymbol{\theta}}(\mathbf{x}_i))^2$$

Sample $\xi_t = i_t \in \{1, \ldots, n\}$

$$\Longrightarrow \mathcal{L}(\boldsymbol{\theta}, i_t) = (y_{i_t} - f_{\boldsymbol{\theta}}(\mathbf{x}_{i_t}))^2$$

$$\nabla_{\boldsymbol{\theta}} \mathcal{L}(\boldsymbol{\theta}, i_t) = \nabla_{\boldsymbol{\theta}} (y_{i_t} - f_{\boldsymbol{\theta}}(\mathbf{x}_{i_t}))^2$$
$$= -2(y_{i_t} - f_{\boldsymbol{\theta}}(\mathbf{x}_{i_t}))\nabla f_{\boldsymbol{\theta}}(\mathbf{x}_{i_t})$$

From ML to ERM ...

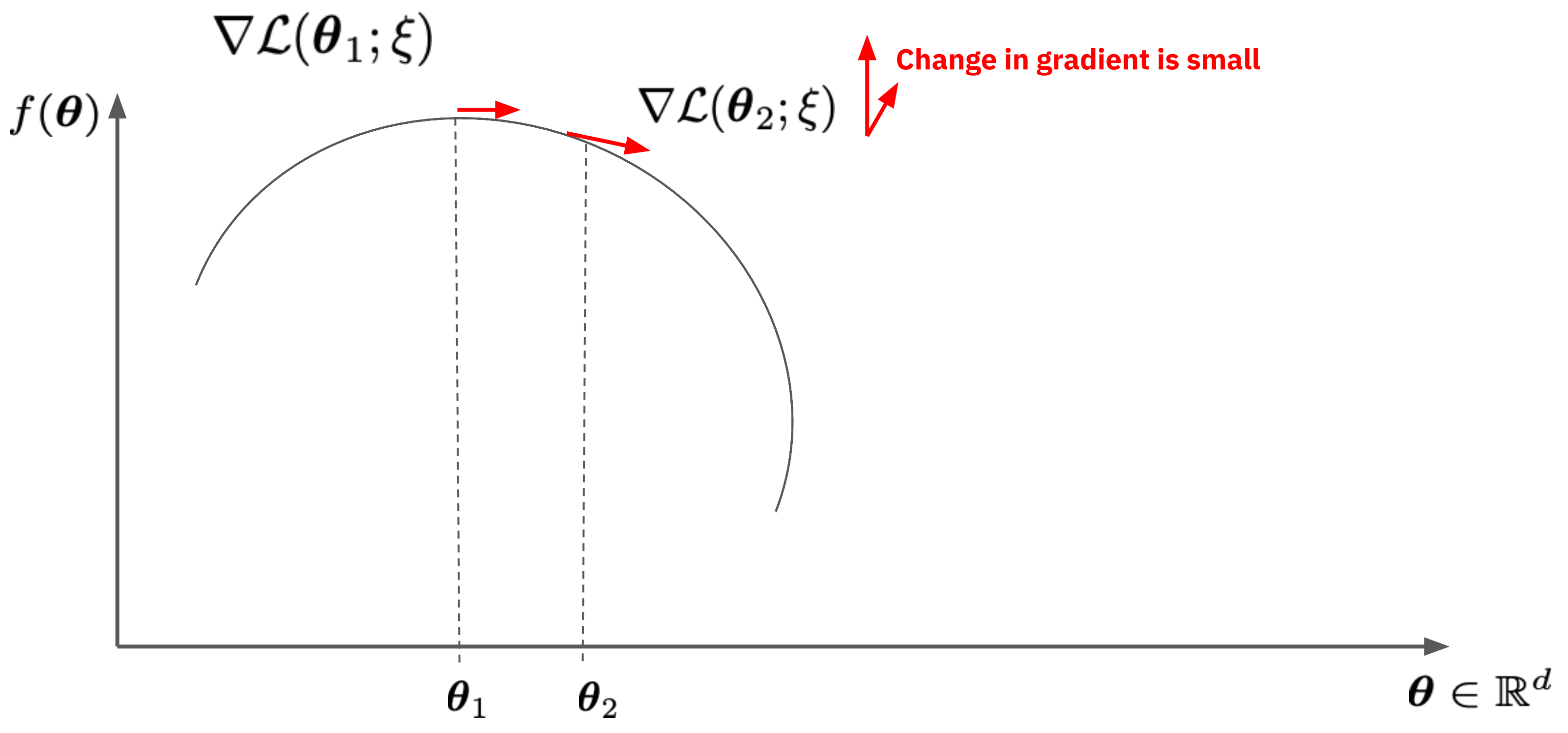
... the authors in the paper, considered the following form of the objective function: $\mathbb{E}_{\xi \sim \mathbb{P}} [\mathcal{L}(\boldsymbol{\theta}; \xi)]$

... for e.g., in regression $\xi \sim \text{Uniform}[1, n]$, then $\mathbb{E}_{\xi \sim \text{Uniform}}[(y_\xi - f_{\boldsymbol{\theta}}(\mathbf{x}_\xi))^2] = \frac{1}{n}\sum_{i=1}^{n}(y_i - f_{\boldsymbol{\theta}}(\mathbf{x}_i))^2$

... now, our goal is to minimise the following $\min_{\boldsymbol{\theta}} \mathbb{E}_{\xi \sim \mathbb{P}} [\mathcal{L}(\boldsymbol{\theta}; \xi)]$ **... using ADAM from the previous slide**

Assumption I -- Loss Function is L-Smooth:

$$||\nabla \mathcal{L}(\boldsymbol{\theta}_1; \xi_1) - \nabla \mathcal{L}(\boldsymbol{\theta}_2; \xi_1)||_2 \leq L||\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1||_2 \;\forall\; \boldsymbol{\theta}_1,\; \boldsymbol{\theta}_2,\; \text{and } \xi$$



Proof Roadmap ...

$$\mathcal{L}(\boldsymbol{\theta}_{t+1}) \leq \mathcal{L}(\boldsymbol{\theta}_t) + \cdots +$$

Objective Func. L-Smoothness

... relation between 2 successive iterations ...

$$\mathbb{E}\left[\mathcal{L}(\boldsymbol{\theta}_{t+1})|\boldsymbol{\theta}_t\right] \leq \mathcal{L}(\boldsymbol{\theta}_t) \cdots + \mathbb{E}\left[\frac{\mathbf{g}_{i,t}}{\sqrt{\mathbf{v}_{i,t}}+\delta}\middle|\boldsymbol{\theta}_t\right]$$

$$\cdots + \mathbb{E}\left[\frac{\mathbf{g}^2_{i,t}}{(\sqrt{\mathbf{v}_{i,t}}+\delta)^2}\middle|\boldsymbol{\theta}_t\right]$$

... we need to bound these ...

True Components to Bound

... consider stochasticity plug-in update rule, and realise terms to bound ...

Bounding the first term

Choose params

Bounding the second term

...

...

...

...

Bound in terms of gradient norm norm

Bound in terms of batch-size

Convergence Proof …

… as in any other optimisation proof, we need to understand the change in function value between two successive iterations of the algorithm:

$$f(\boldsymbol{\theta}_{t+1}) \leq f(\boldsymbol{\theta}_t) - \underbrace{\Delta}_{\text{Some positive value}} \implies \text{convergence to some point if the function is lower-bounded}$$

… now, if we can say that the objective function is L-smooth, then we can have a relation between function values on two successive iterations:

$$\underbrace{\mathcal{L}(\boldsymbol{\theta}_{t+1}) \leq \mathcal{L}(\boldsymbol{\theta}_t)}_{\text{Relation between successive iterations}} + \nabla^{\mathsf{T}}\mathcal{L}(\boldsymbol{\theta}_t)\,(\boldsymbol{\theta}_{t+1} - \boldsymbol{\theta}_t) + \frac{L}{2}\,||\boldsymbol{\theta}_{t+1} - \boldsymbol{\theta}_t||_2^2$$

But how to show that our objective function is L-Smooth

Convergence Proof ...

... let us study the norm of the difference between the gradients of the objective function at any two given input points:

$$
\begin{aligned}
||\nabla\mathcal{L}(\boldsymbol{\theta}_1) - \nabla\mathcal{L}(\boldsymbol{\theta}_2)||_2 &= ||\nabla\mathbb{E}_\xi[\mathcal{L}(\boldsymbol{\theta}_1;\xi)] - \nabla\mathbb{E}_\xi[\mathcal{L}(\boldsymbol{\theta}_2;\xi)]||_2 \\
&= ||\mathbb{E}_\xi[\nabla\mathcal{L}(\boldsymbol{\theta}_1;\xi)] - \mathbb{E}_\xi[\nabla\mathcal{L}(\boldsymbol{\theta}_2;\xi)]||_2 \\
&= ||\mathbb{E}_\xi[\nabla\mathcal{L}(\boldsymbol{\theta}_1;\xi) - \nabla\mathcal{L}(\boldsymbol{\theta}_2;\xi)]||_2 \\
&\leq \mathbb{E}_\xi[||\nabla\mathcal{L}(\boldsymbol{\theta}_1;\xi) - \nabla\mathcal{L}(\boldsymbol{\theta}_2;\xi)||_2] \\
&\leq \mathbb{E}_\xi[L||\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1||_2] \\
&= L||\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1||_2
\end{aligned}
$$

Assumption I -- Loss Function is L-Smooth:

$$||\nabla\mathcal{L}(\boldsymbol{\theta}_1;\xi_1) - \nabla\mathcal{L}(\boldsymbol{\theta}_2;\xi_1)||_2 \leq L||\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1||_2 \ \forall\ \boldsymbol{\theta}_1,\ \boldsymbol{\theta}_2,\ \text{and}\ \xi$$

Objective function is L-Smooth

Convergence Proof ...

... since we just proved that our objective is L-Smooth, now we can write that the objective value between two successive iterations abides by:

$$\mathcal{L}(\boldsymbol{\theta}_{t+1}) \leq \mathcal{L}(\boldsymbol{\theta}_t) + \nabla^\top \mathcal{L}(\boldsymbol{\theta}_t)\,(\boldsymbol{\theta}_{t+1} - \boldsymbol{\theta}_t) + \frac{L}{2}\,||\boldsymbol{\theta}_{t+1} - \boldsymbol{\theta}_t||_2^2$$

... now, remember our update rules from the pseudo-code in the previous slides, we can write:

$$\mathbf{m}_t = \beta_1 \mathbf{m}_{t-1} + (1-\beta_1)\mathbf{g}_t \implies \text{with } \beta_1 = 0, \text{ then } \mathbf{m}_t = \mathbf{g}_t \quad \text{then} \quad \boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}_t - \eta_t \frac{\mathbf{g}_t}{(\sqrt{\mathbf{v}_t} + \delta)}$$

... component-wise update $\boldsymbol{\theta}_{i,t+1} = \boldsymbol{\theta}_{i,t} - \eta_t \frac{\mathbf{g}_{i,t}}{(\sqrt{\mathbf{v}_{i,t}} + \delta)} \quad i \in \{1, \ldots, d\}$

$$\mathcal{L}(\boldsymbol{\theta}_{t+1}) \leq \mathcal{L}(\boldsymbol{\theta}_t) - \eta_t \sum_{i=1}^{d} \left([\nabla \mathcal{L}(\boldsymbol{\theta}_t)]_i \times \frac{\mathbf{g}_{i,t}}{\sqrt{\mathbf{v}_{i,t}} + \delta} \right) + \frac{L\eta_t^2}{2} \sum_{i=1}^{d} \frac{\mathbf{g}_{i,t}^2}{(\sqrt{\mathbf{v}_{i,t}} + \delta)^2}$$

Convergence Proof …

$$\mathcal{L}(\boldsymbol{\theta}_{t+1}) \leq \mathcal{L}(\boldsymbol{\theta}_t) - \eta_t \sum_{i=1}^{d} \left([\nabla \mathcal{L}(\boldsymbol{\theta}_t)]_i \times \frac{\mathbf{g}_{i,t}}{\sqrt{\mathbf{v}_{i,t}} + \delta} \right) + \frac{L\eta_t^2}{2} \sum_{i=1}^{d} \frac{\mathbf{g}_{i,t}^2}{(\sqrt{\mathbf{v}_{i,t}} + \delta)^2}$$

… now, taking the conditional expectation with respect to the sample at iteration t given a fixed random variable $\boldsymbol{\theta}_t$:

$$\mathbb{E}\left[\mathcal{L}(\boldsymbol{\theta}_{t+1}) | \boldsymbol{\theta}_t\right] \leq \underbrace{\mathcal{L}(\boldsymbol{\theta}_t)}_{\text{Fully known}} - \eta_t \sum_{i=1}^{d} \left(\underbrace{[\nabla \mathcal{L}(\boldsymbol{\theta}_t)]_i}_{\text{Fully known}} \times \mathbb{E}\left[\underbrace{\frac{\mathbf{g}_{i,t}}{\sqrt{\mathbf{v}_{i,t}} + \delta}}_{\text{Dependent RVs}} \middle| \boldsymbol{\theta}_t \right] \right) + \frac{L\eta_t^2}{2} \sum_{i=1}^{d} \mathbb{E}\left[\frac{\mathbf{g}_{i,t}^2}{(\sqrt{\mathbf{v}_{i,t}} + \delta)^2} \middle| \boldsymbol{\theta}_t \right]$$

Proof Roadmap ...

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... relation between 2 successive iterations ...

$$\mathbb{E}\left[\mathcal{L}(\boldsymbol{\theta}_{t+1})|\boldsymbol{\theta}_t\right] \leq \mathcal{L}(\boldsymbol{\theta}_t) \cdots + \boxed{\mathbb{E}\left[\frac{\mathbf{g}_{i,t}}{\sqrt{\mathbf{v}_{i,t}} + \delta}\middle|\boldsymbol{\theta}_t\right]}$$

True Components to Bound

... consider stochasticity plug-in update rule, and realise terms to bound ...

... we need to bound these ...

$$\cdots + \boxed{\mathbb{E}\left[\frac{\mathbf{g}_{i,t}^2}{(\sqrt{\mathbf{v}_{i,t}} + \delta)^2}\middle|\boldsymbol{\theta}_t\right]}$$

Bounding the first term

...

...

Bound in terms of gradient norm norm

Convergence Proof ...

$$\mathbb{E}\left[\mathcal{L}(\boldsymbol{\theta}_{t+1})|\boldsymbol{\theta}_t\right] \leq \mathcal{L}(\boldsymbol{\theta}_t) - \eta_t \sum_{i=1}^{d}\left(\left[\nabla\mathcal{L}(\boldsymbol{\theta}_t)\right]_i \times \mathbb{E}\left[\frac{\mathbf{g}_{i,t}}{\sqrt{\mathbf{v}_{i,t}}+\delta}\Big|\boldsymbol{\theta}_t\right]\right) + \frac{L\eta_t^2}{2}\sum_{i=1}^{d}\mathbb{E}\left[\frac{\mathbf{g}_{i,t}^2}{(\sqrt{\mathbf{v}_{i,t}}+\delta)^2}\Big|\boldsymbol{\theta}_t\right]$$

How to deal with such a ratio

$$= \mathcal{L}(\boldsymbol{\theta}_t) - \eta_t \sum_{i=1}^{d}\left(\left[\nabla\mathcal{L}(\boldsymbol{\theta}_t)\right]_i \times \mathbb{E}\left[\frac{\boldsymbol{g}_{i,t}}{\sqrt{\boldsymbol{v}_{i,t}}+\delta} - \frac{\boldsymbol{g}_{i,t}}{\sqrt{\beta_2\boldsymbol{v}_{i,t-1}}+\delta} + \frac{\boldsymbol{g}_{i,t}}{\sqrt{\beta_2\boldsymbol{v}_{i,t-1}}+\delta}\Big|\boldsymbol{\theta}_t\right]\right) + \frac{L\eta_t^2}{2}\sum_{i=1}^{d}\mathbb{E}\left[\frac{\mathbf{g}_{i,t}^2}{(\sqrt{\mathbf{v}_{i,t}}+\delta)^2}\Big|\boldsymbol{\theta}_t\right]$$

... adding and subtracting will allow us to deal with this ...

Convergence Proof …

$$= \mathcal{L}(\boldsymbol{\theta}_t) - \eta_t \sum_{i=1}^{d} \left([\nabla \mathcal{L}(\boldsymbol{\theta}_t)]_i \times \mathbb{E}\left[\underbrace{\frac{\boldsymbol{g}_{i,t}}{\sqrt{\boldsymbol{v}_{i,t}}+\delta}}_{a} - \underbrace{\frac{\boldsymbol{g}_{i,t}}{\sqrt{\beta_2 \boldsymbol{v}_{i,t-1}}+\delta}}_{b} + \underbrace{\frac{\boldsymbol{g}_{i,t}}{\sqrt{\beta_2 \boldsymbol{v}_{i,t-1}}+\delta}}_{c} \middle| \boldsymbol{\theta}_t \right] \right) + \frac{L\eta_t^2}{2} \sum_{i=1}^{d} \mathbb{E}\left[\frac{\mathbf{g}_{i,t}^2}{(\sqrt{\mathbf{v}_{i,t}}+\delta)^2} \middle| \boldsymbol{\theta}_t \right]$$

$\mathbb{E}[a-b+c] = \mathbb{E}[a-b] + \mathbb{E}[c]$

$$\boxed{\mathbb{E}\left[\frac{\mathbf{g}_{i,t}}{\sqrt{\mathbf{v}_{i,t}}+\delta} - \frac{\mathbf{g}_{i,t}}{\sqrt{\beta_2 \mathbf{v}_{i,t-1}}+\delta} \middle| \boldsymbol{\theta}_t \right]} + \mathbb{E}\left[\frac{\mathbf{g}_{i,t}}{\sqrt{\beta_2 \mathbf{v}_{i,t-1}}+\delta} \middle| \boldsymbol{\theta}_t \right]$$

$$\frac{\mathbb{E}[\mathbf{g}_{i,t}|\boldsymbol{\theta}_t]}{\sqrt{\beta_2 \mathbf{v}_{i,t-1}}+\delta} = \boxed{\frac{[\nabla \mathcal{L}(\boldsymbol{\theta})]_i}{\sqrt{\beta_2 \mathbf{v}_{i,t-1}}+\delta}}$$

$$= \mathcal{L}(\boldsymbol{\theta}_t) - \eta_t \sum_{i=1}^{d} \left([\nabla \mathcal{L}(\boldsymbol{\theta}_t)]_i \times \left[\boxed{\frac{[\nabla \mathcal{L}(\boldsymbol{\theta}_t)]_i}{\sqrt{\beta_2 \boldsymbol{v}_{i,t-1}}+\delta}} + \boxed{\mathbb{E}\left[\frac{\boldsymbol{g}_{i,t}}{\sqrt{\boldsymbol{v}_{i,t}}+\delta} - \frac{\boldsymbol{g}_{i,t}}{\sqrt{\beta_2 \boldsymbol{v}_{i,t-1}}+\delta} \middle| \boldsymbol{\theta}_t \right]} \right] \right) + \frac{L\eta_t^2}{2} \sum_{i=1}^{d} \mathbb{E}\left[\frac{\mathbf{g}_{i,t}^2}{(\sqrt{\mathbf{v}_{i,t}}+\delta)^2} \middle| \boldsymbol{\theta}_t \right]$$

Convergence Proof ...

$$\frac{[\nabla \mathcal{L}(\boldsymbol{\theta}_t)]_i^2}{\sqrt{\beta_2 \boldsymbol{v}_{i,t-1}} + \delta}$$

$$= \mathcal{L}(\boldsymbol{\theta}_t) - \eta_t \sum_{i=1}^{d} \left([\nabla \mathcal{L}(\boldsymbol{\theta}_t)]_i \times \left[\frac{[\nabla \mathcal{L}(\boldsymbol{\theta}_t)]_i}{\sqrt{\beta_2 \boldsymbol{v}_{i,t-1}} + \delta} + \mathbb{E}\left[\frac{\boldsymbol{g}_{i,t}}{\sqrt{\boldsymbol{v}_{i,t}} + \delta} - \frac{\boldsymbol{g}_{i,t}}{\sqrt{\beta_2 \boldsymbol{v}_{i,t-1}} + \delta} \middle| \boldsymbol{\theta}_t \right] \right] \right) + \frac{L\eta_t^2}{2} \sum_{i=1}^{d} \mathbb{E}\left[\frac{\mathbf{g}_{i,t}^2}{(\sqrt{\mathbf{v}_{i,t}} + \delta)^2} \middle| \boldsymbol{\theta}_t \right]$$

$$[\nabla \mathcal{L}(\boldsymbol{\theta}_t)]_i \times \mathbb{E}\left[\frac{\boldsymbol{g}_{i,t}}{\sqrt{\boldsymbol{v}_{i,t}} + \delta} - \frac{\boldsymbol{g}_{i,t}}{\sqrt{\beta_2 \boldsymbol{v}_{i,t-1}} + \delta} \middle| \boldsymbol{\theta}_t \right]$$

$$= \mathcal{L}(\boldsymbol{\theta}_t) - \eta_t \boxed{\sum_{i=1}^{d}} \left(\frac{[\nabla \mathcal{L}(\boldsymbol{\theta}_t)]_i^2}{\sqrt{\beta_2 \boldsymbol{v}_{i,t-1}} + \delta} + [\nabla \mathcal{L}(\boldsymbol{\theta}_t)]_i \times \mathbb{E}\left[\frac{\boldsymbol{g}_{i,t}}{\sqrt{\boldsymbol{v}_{i,t}} + \delta} - \frac{\boldsymbol{g}_{i,t}}{\sqrt{\beta_2 \boldsymbol{v}_{i,t-1}} + \delta} \middle| \boldsymbol{\theta}_t \right] \right) + \frac{L\eta_t^2}{2} \sum_{i=1}^{d} \mathbb{E}\left[\frac{\mathbf{g}_{i,t}^2}{(\sqrt{\mathbf{v}_{i,t}} + \delta)^2} \middle| \boldsymbol{\theta}_t \right]$$

$$= \mathcal{L}(\boldsymbol{\theta}_t) - \eta_t \sum_{i=1}^{d} \frac{[\nabla \mathcal{L}(\boldsymbol{\theta}_t)]_i^2}{\sqrt{\beta_2 \boldsymbol{v}_{i,t-1}} + \delta} - \eta_t \sum_{i=1}^{d} [\nabla \mathcal{L}(\boldsymbol{\theta}_t)]_i \times \mathbb{E}\left[\frac{\boldsymbol{g}_{i,t}}{\sqrt{\boldsymbol{v}_{i,t}} + \delta} - \frac{\boldsymbol{g}_{i,t}}{\sqrt{\beta_2 \boldsymbol{v}_{i,t-1}} + \delta} \middle| \boldsymbol{\theta}_t \right] + \frac{L\eta_t^2}{2} \sum_{i=1}^{d} \mathbb{E}\left[\frac{\mathbf{g}_{i,t}^2}{(\sqrt{\mathbf{v}_{i,t}} + \delta)^2} \middle| \boldsymbol{\theta}_t \right]$$

Convergence Proof ...

$$-\eta_t \sum_{i=1}^{d} a_i \times b_i$$

$$= \mathcal{L}(\boldsymbol{\theta}_t) - \eta_t \sum_{i=1}^{d} \frac{[\nabla \mathcal{L}(\boldsymbol{\theta}_t)]_i^2}{\sqrt{\beta_2 \boldsymbol{v}_{i,t-1}} + \delta} - \eta_t \sum_{i=1}^{d} [\nabla \mathcal{L}(\boldsymbol{\theta}_t)]_i \times \mathbb{E}\left[\frac{\boldsymbol{g}_{i,t}}{\sqrt{\boldsymbol{v}_{i,t}} + \delta} - \frac{\boldsymbol{g}_{i,t}}{\sqrt{\beta_2 \boldsymbol{v}_{i,t-1}} + \delta} \Big| \boldsymbol{\theta}_t \right] + \frac{L\eta_t^2}{2} \sum_{i=1}^{d} \mathbb{E}\left[\frac{\mathbf{g}_{i,t}^2}{(\sqrt{\mathbf{v}_{i,t}} + \delta)^2} \Big| \boldsymbol{\theta}_t \right]$$

$$-\eta_t \sum_{i=1}^{d} a_i b_i \leq \left| \eta_t \sum_{I=1}^{d} a_i b_i \right| \leq \eta_t \sum_{I=1}^{d} |a_i||b_i|$$

$$\leq \eta_t \sum_{i=1}^{d} |[\nabla \mathcal{L}(\boldsymbol{\theta}_t)]_i| \left| \mathbb{E}\left[\frac{\mathbf{g}_{i,t}}{\sqrt{\mathbf{v}_{i,t}} + \delta} - \frac{\mathbf{g}_{i,t}}{\sqrt{\beta_2 \mathbf{v}_{i,t-1}} + \delta} \Big| \boldsymbol{\theta}_t \right] \right|$$

$$\mathbb{E}\left[\mathcal{L}(\boldsymbol{\theta}_{t+1}) | \boldsymbol{\theta}_t\right] \leq \mathcal{L}(\boldsymbol{\theta}_t) - \eta_t \sum_{i=1}^{d} \frac{[\nabla \mathcal{L}(\boldsymbol{\theta}_t)]_i^2}{\sqrt{\beta_2 \boldsymbol{v}_{i,t-1}} + \delta} + \eta_t \sum_{i=1}^{d} \left(|[\nabla \mathcal{L}(\boldsymbol{\theta}_t)]_i| \left| \mathbb{E}\left[\frac{\boldsymbol{g}_{i,t}}{\sqrt{\boldsymbol{v}_{i,t}} + \delta} - \frac{\boldsymbol{g}_{i,t}}{\sqrt{\beta_2 \boldsymbol{v}_{i,t-1}} + \delta} \Big| \boldsymbol{\theta}_t \right] \right| \right)$$

... our focus for now..

$$+ \frac{L\eta_t^2}{2} \sum_{i=1}^{d} \mathbb{E}\left[\frac{\mathbf{g}_{i,t}^2}{(\sqrt{\mathbf{v}_{i,t}} + \delta)^2} \Big| \boldsymbol{\theta}_t \right]$$

Convergence Proof ...

$$|\mathbb{E}[x]| \le \mathbb{E}[|x|]$$

$$\left|\mathbb{E}\left[\frac{\boldsymbol{g}_{i,t}}{\sqrt{\boldsymbol{v}_{i,t}}+\delta} - \frac{\boldsymbol{g}_{i,t}}{\sqrt{\beta_2 \boldsymbol{v}_{i,t-1}}+\delta}\middle|\boldsymbol{\theta}_t\right]\right| \le \mathbb{E}\left[\underbrace{\left|\frac{\mathbf{g}_{i,t}}{\sqrt{\mathbf{v}_{i,t}}+\delta} - \frac{\mathbf{g}_{i,t}}{\sqrt{\beta_2 \mathbf{v}_{i,t-1}}+\delta}\right|}_{T_1}\middle|\boldsymbol{\theta}_t\right]$$

... our focus for now..

$$|\sqrt{a}-\sqrt{b}| = \frac{|a-b|}{\sqrt{a}+\sqrt{b}}$$

$$T_1 = \left|\frac{\boldsymbol{g}_{i,t}}{\sqrt{\boldsymbol{v}_{i,t}}+\delta} - \frac{\boldsymbol{g}_{i,t}}{\sqrt{\beta_2 \boldsymbol{v}_{i,t-1}}+\delta}\right| = |\boldsymbol{g}_{i,t}|\left|\frac{1}{\sqrt{\boldsymbol{v}_{i,t}}+\delta} - \frac{1}{\sqrt{\beta_2 \boldsymbol{v}_{i,t-1}}+\delta}\right| = \frac{|\boldsymbol{g}_{i,t}|}{(\sqrt{\boldsymbol{v}_{i,t}}+\delta)(\sqrt{\beta_2 \boldsymbol{v}_{i,t-1}}+\delta)}\overbrace{\left|\sqrt{\boldsymbol{v}_{i,t}} - \sqrt{\beta_2 \boldsymbol{v}_{i,t-1}}\right|}$$

... common denominator ..

$$= \frac{|\boldsymbol{g}_{i,t}|}{(\sqrt{\boldsymbol{v}_{i,t}}+\delta)(\sqrt{\beta_2 \boldsymbol{v}_{i,t-1}}+\delta)} \frac{|\boldsymbol{v}_{i,t} - \beta_2 \boldsymbol{v}_{i,t-1}|}{\sqrt{\boldsymbol{v}_{i,t}} + \sqrt{\beta_2 \boldsymbol{v}_{i,t-1}}}$$

Convergence Proof ...

... update rule ...

$$\mathbf{v}_{i,t} = \beta_2 \mathbf{v}_{i,t-1} + (1-\beta_2)\mathbf{g}_{i,t}^2$$

... plug eq. in ...

$$\frac{|\boldsymbol{g}_{i,t}|}{(\sqrt{\boldsymbol{v}_{i,t}}+\delta)(\sqrt{\beta_2 \boldsymbol{v}_{i,t-1}}+\delta)}\frac{|\boldsymbol{v}_{i,t}-\beta_2 \boldsymbol{v}_{i,t-1}|}{\sqrt{\boldsymbol{v}_{i,t}}+\sqrt{\beta_2 \boldsymbol{v}_{i,t-1}}} = \frac{|\boldsymbol{g}_{i,t}|}{(\sqrt{\boldsymbol{v}_{i,t}}+\delta)(\sqrt{\beta_2 \boldsymbol{v}_{i,t-1}}+\delta)}\frac{(1-\beta_2)\mathbf{g}_{i,t}^2}{\boxed{\sqrt{\mathbf{v}_{i,t}}}+\sqrt{\beta_2 \mathbf{v}_{i,t-1}}}$$

... plug eq. in ...

$$\mathbf{v}_{i,t} = \beta_2 \mathbf{v}_{i,t-1} + (1-\beta_2)\mathbf{g}_{i,t}^2$$

$$= \frac{|\boldsymbol{g}_{i,t}|}{(\sqrt{\boldsymbol{v}_{i,t}}+\delta)(\sqrt{\beta_2 \boldsymbol{v}_{i,t-1}}+\delta)}\frac{(1-\beta_2)\mathbf{g}_{i,t}^2}{\boxed{\sqrt{\beta_2 \mathbf{v}_{i,t-1}+(1-\beta_2)\mathbf{g}_{i,t}^2}}+\sqrt{\beta_2 \mathbf{v}_{i,t-1}}}$$

Now what ...

Convergence Proof ...

$$\frac{1}{a+b} \leq \frac{1}{a} \quad \text{for } a > 0 \text{ and } b \geq 0$$

$$= \frac{|\boldsymbol{g}_{i,t}|}{(\sqrt{\boldsymbol{v}_{i,t}}+\delta)(\sqrt{\beta_2 \boldsymbol{v}_{i,t-1}}+\delta)} \frac{(1-\beta_2)\mathbf{g}_{i,t}^2}{\sqrt{\beta_2 \mathbf{v}_{i,t-1}+(1-\beta_2)\mathbf{g}_{i,t}^2} + \boxed{\sqrt{\beta_2 \mathbf{v}_{i,t-1}}}}$$

non-negative

$$\leq \frac{|\mathbf{g}_{i,t}|}{(\sqrt{\boldsymbol{v}_{i,t}}+\delta)(\sqrt{\beta_2 \boldsymbol{v}_{i,t-1}}+\delta)} \frac{(1-\beta_2)\mathbf{g}_{i,t}^2}{\sqrt{\underbrace{\beta_2 \mathbf{v}_{i,t-1}}_{a}+\underbrace{(1-\beta_2)\mathbf{g}_{i,t}^2}_{b}}}$$

$$\sqrt{a+b} \geq \sqrt{b} \quad \text{if } \begin{matrix} a \geq 0 \\ b > 0 \end{matrix} \implies \frac{1}{\sqrt{a+b}} \leq \frac{1}{\sqrt{b}}$$

$$\leq \frac{|\mathbf{g}_{i,t}|}{(\sqrt{\boldsymbol{v}_{i,t}}+\delta)(\sqrt{\beta_2 \boldsymbol{v}_{i,t-1}}+\delta)} \frac{(1-\beta_2)\mathbf{g}_{i,t}^2}{\sqrt{(1-\beta_2)\mathbf{g}_{i,t}^2}}$$

... remember our focus ...

$$\mathbb{E}\left[\mathcal{L}(\boldsymbol{\theta}_{t+1})|\boldsymbol{\theta}_t\right] \leq \cdots + \eta_t \sum_{i=1}^{d} \left(|[\nabla \mathcal{L}(\boldsymbol{\theta}_t)]_i| \middle| \mathbb{E}\left[\overbrace{\frac{\boldsymbol{g}_{i,t}}{\sqrt{\boldsymbol{v}_{i,t}}+\delta} - \frac{\boldsymbol{g}_{i,t}}{\sqrt{\beta_2 \boldsymbol{v}_{i,t-1}}+\delta}} \middle| \boldsymbol{\theta}_t \right] \right| \right) \cdots$$

Convergence Proof ...

$$T_1 \leq \frac{|\mathbf{g}_{i,t}|}{(\sqrt{\boldsymbol{v}_{i,t}}+\delta)(\sqrt{\beta_2\boldsymbol{v}_{i,t-1}}+\delta)}\frac{(1-\beta_2)\mathbf{g}^2_{i,t}}{\sqrt{(1-\beta_2)\mathbf{g}^2_{i,t}}}$$

$$= \frac{1}{(\underbrace{\sqrt{\boldsymbol{v}_{i,t}}}_{b}+\underbrace{\delta}_{a})(\sqrt{\beta_2\boldsymbol{v}_{i,t-1}}+\delta)}\sqrt{1-\beta_2}\,\mathbf{g}^2_{i,t}$$

...same trick...

$$\frac{1}{a+b} \leq \frac{1}{a} \quad \text{for } a > 0 \text{ and } b \geq 0$$

$$T_1 \leq \frac{\sqrt{1-\beta_2}\,\mathbf{g}^2_{i,t}}{\delta(\sqrt{\beta_2\boldsymbol{v}_{i,t-1}}+\delta)}$$

... now, we'll plug-back in the main bound ...

... remember our focus ...

$$\mathbb{E}\left[\mathcal{L}(\boldsymbol{\theta}_{t+1})|\boldsymbol{\theta}_t\right] \leq \cdots + \eta_t\sum_{i=1}^{d}\left(|[\nabla\mathcal{L}(\boldsymbol{\theta}_t)]_i|\left|\mathbb{E}\left[\frac{\boldsymbol{g}_{i,t}}{\sqrt{\boldsymbol{v}_{i,t}}+\delta} - \frac{\boldsymbol{g}_{i,t}}{\sqrt{\beta_2\boldsymbol{v}_{i,t-1}}+\delta}\middle|\boldsymbol{\theta}_t\right]\right|\right)\cdots$$

Plugging-Back in the main bound ...

$$\mathbb{E}\left[\mathcal{L}(\boldsymbol{\theta}_{t+1})|\boldsymbol{\theta}_t\right] \leq \cdots \; + \eta_t \sum_{i=1}^{d}\left(|[\nabla\mathcal{L}(\boldsymbol{\theta}_t)]_i| \left|\mathbb{E}\left[\frac{\boldsymbol{g}_{i,t}}{\sqrt{\boldsymbol{v}_{i,t}}+\delta} - \frac{\boldsymbol{g}_{i,t}}{\sqrt{\beta_2\boldsymbol{v}_{i,t-1}}+\delta}\middle|\boldsymbol{\theta}_t\right]\right|\right)\cdots$$

$$\leq \cdots + \eta_t \sum_{i=1}^{d}\left(|[\nabla\mathcal{L}(\boldsymbol{\theta})]_i|\,\mathbb{E}\left[\underbrace{\left|\frac{\mathbf{g}_{i,t}}{\sqrt{\mathbf{v}_{i,t}}+\delta} - \frac{\mathbf{g}_{i,t}}{\sqrt{\beta_2\mathbf{v}_{i,t-1}}+\delta}\right|}_{T_1}\middle|\boldsymbol{\theta}_t\right]\right) + \ldots$$

$$T_1 \leq \frac{\sqrt{1-\beta_2}\mathbf{g}_{i,t}^2}{\delta(\sqrt{\beta_2\boldsymbol{v}_{i,t-1}}+\delta)}$$

$$= \cdots + \eta_t \sum_{i=1}^{d}\left(|[\nabla\mathcal{L}(\boldsymbol{\theta})]_i|\,\mathbb{E}\left[T_1|\boldsymbol{\theta}_t\right]\right) + \ldots = \quad \cdot\cdot + \; \eta_t \sum_{i=1}^{d}\left(|[\nabla\mathcal{L}(\boldsymbol{\theta}_t)]_i|\,\frac{\sqrt{1-\beta_2}\mathbb{E}[\mathbf{g}_{i,t}^2|\boldsymbol{\theta}_t]}{\delta(\sqrt{\beta_2\mathbf{v}_{i,t-1}}+\delta)}\right) + \ldots$$

... hence, the overall bound ...

$$\mathbb{E}\left[\mathcal{L}(\boldsymbol{\theta}_{t+1})|\boldsymbol{\theta}_t\right] \leq \mathcal{L}(\boldsymbol{\theta}_t) - \eta_t \sum_{i=1}^{d}\frac{[\nabla\mathcal{L}(\boldsymbol{\theta}_t)]_i^2}{\sqrt{\beta_2\boldsymbol{v}_{i,t-1}}+\delta} + \; \eta_t \sum_{i=1}^{d}\left(|[\nabla\mathcal{L}(\boldsymbol{\theta}_t)]_i|\,\frac{\sqrt{1-\beta_2}\mathbb{E}[\mathbf{g}_{i,t}^2|\boldsymbol{\theta}_t]}{\delta(\sqrt{\beta_2\mathbf{v}_{i,t-1}}+\delta)}\right) + \frac{L\eta_t^2}{2}\sum_{i=1}^{d}\mathbb{E}\left[\frac{\mathbf{g}_{i,t}^2}{(\sqrt{\mathbf{v}_{i,t}}+\delta)^2}\middle|\boldsymbol{\theta}_t\right]$$

Bounding the gradient ...

$$\mathbb{E}\left[\mathcal{L}(\boldsymbol{\theta}_{t+1})|\boldsymbol{\theta}_t\right] \le \mathcal{L}(\boldsymbol{\theta}_t) - \eta_t \sum_{i=1}^{d} \frac{[\nabla\mathcal{L}(\boldsymbol{\theta}_t)]_i^2}{\sqrt{\beta_2 \boldsymbol{v}_{i,t-1}}+\delta} + \eta_t \sum_{i=1}^{d} \left(\boxed{|[\nabla\mathcal{L}(\boldsymbol{\theta}_t)]_i|} \frac{\sqrt{1-\beta_2}\,\mathbb{E}[\mathbf{g}_{i,t}^2|\boldsymbol{\theta}_t]}{\delta(\sqrt{\beta_2 \mathbf{v}_{i,t-1}}+\delta)} \right) + \frac{L\eta_t^2}{2} \sum_{i=1}^{d} \mathbb{E}\left[\frac{\mathbf{g}_{i,t}^2}{(\sqrt{\mathbf{v}_{i,t}}+\delta)^2} \Big| \boldsymbol{\theta}_t \right]$$

.... we can thus say ...

Assumption II -- Loss functions has bounded gradient:

$$||\nabla\mathcal{L}(\boldsymbol{\theta};\boldsymbol{\xi})|| \le G, \quad \forall \boldsymbol{\theta} \in \mathbb{R}^d, \quad \forall \xi$$

$$||\nabla\mathcal{L}(\boldsymbol{\theta})|| = ||\mathbb{E}_\xi\left[\nabla\mathcal{L}(\boldsymbol{\theta};\xi)\right]|| \le \mathbb{E}_\xi\left[||\nabla\mathcal{L}(\boldsymbol{\theta};\xi)||\right] \le G$$

$$\boxed{\implies |[\nabla\mathcal{L}(\boldsymbol{\theta}_t)]_i| \le G}$$

$$\mathbb{E}\left[\mathcal{L}(\boldsymbol{\theta}_{t+1})|\boldsymbol{\theta}_t\right] \le \mathcal{L}(\boldsymbol{\theta}_t) - \eta_t \sum_{i=1}^{d} \frac{[\nabla\mathcal{L}(\boldsymbol{\theta}_t)]_i^2}{\sqrt{\beta_2 \boldsymbol{v}_{i,t-1}}+\delta} + \frac{\eta_t G\sqrt{1-\beta_2}}{\delta} \sum_{i=1}^{d} \mathbb{E}\left[\frac{\mathbf{g}_{i,t}^2}{\sqrt{\beta_2 \mathbf{v}_{i,t-1}}+\delta} \Big| \boldsymbol{\theta}_t \right] + \frac{L\eta_t^2}{2} \sum_{i=1}^{d} \mathbb{E}\left[\frac{\mathbf{g}_{i,t}^2}{(\sqrt{\mathbf{v}_{i,t}}+\delta)^2} \Big| \boldsymbol{\theta}_t \right]$$

... now this ...

Proof Roadmap ...

$$\mathcal{L}(\boldsymbol{\theta}_{t+1}) \leq \mathcal{L}(\boldsymbol{\theta}_t) + \cdots +$$

Objective Func. L-Smoothness

... relation between 2 successive iterations ...

$$\mathbb{E}\left[\mathcal{L}(\boldsymbol{\theta}_{t+1})|\boldsymbol{\theta}_t\right] \leq \mathcal{L}(\boldsymbol{\theta}_t) \cdots + \mathbb{E}\left[\frac{\mathbf{g}_{i,t}}{\sqrt{\mathbf{v}_{i,t}}+\delta}\middle|\boldsymbol{\theta}_t\right] \cdots + \mathbb{E}\left[\frac{\mathbf{g}_{i,t}^2}{(\sqrt{\mathbf{v}_{i,t}}+\delta)^2}\middle|\boldsymbol{\theta}_t\right]$$

... we need to bound these ...

True Components to Bound

... consider stochasticity plug-in update rule, and realise terms to bound ...

Bounding the first term

Choose params

Bounding the second term

...

...

Bound in terms of gradient norm norm

Bounding the 3rd term …

… update rule …

$$\mathbf{v}_{i,t} = \beta_2 \mathbf{v}_{i,t-1} + (1-\beta_2)\mathbf{g}_{i,t}^2$$

… plug eq. in …

$$\mathbb{E}\left[\mathcal{L}(\boldsymbol{\theta}_{t+1})|\boldsymbol{\theta}_t\right] \leq \quad \cdots \quad + \boxed{\frac{L\eta_t^2}{2}\sum_{i=1}^{d}\mathbb{E}\left[\frac{\mathbf{g}_{i,t}^2}{(\sqrt{\mathbf{v}_{i,t}}+\delta)^2}\Big|\boldsymbol{\theta}_t\right]}$$

$$\frac{L\eta_t^2}{2}\sum_{i=1}^{d}\mathbb{E}\left[\frac{\boldsymbol{g}_{i,t}^2}{\left(\sqrt{\beta_2\boldsymbol{v}_{i,t-1}+\underbrace{(1-\beta_2)\boldsymbol{g}_{i,t}^2}_{\text{non-negative}}}+\delta\right)^2}\Big|\boldsymbol{\theta}_t\right]$$

$$\frac{L\eta_t^2}{2}\sum_{i=1}^{d}\mathbb{E}\left[\frac{\boldsymbol{g}_{i,t}^2}{\left(\sqrt{\beta_2\boldsymbol{v}_{i,t-1}}+\delta\right)^2}\Big|\boldsymbol{\theta}_t\right]$$

$$\mathbb{E}\left[\mathcal{L}(\boldsymbol{\theta}_{t+1})|\boldsymbol{\theta}_t\right] \leq \mathcal{L}(\boldsymbol{\theta}_t) - \eta_t\sum_{i=1}^{d}\frac{[\nabla\mathcal{L}(\boldsymbol{\theta}_t)]_i^2}{\sqrt{\beta_2\boldsymbol{v}_{i,t-1}}+\delta} + \frac{\eta_t G\sqrt{1-\beta_2}}{\delta}\sum_{i=1}^{d}\mathbb{E}\left[\frac{\mathbf{g}_{i,t}^2}{\sqrt{\beta_2\mathbf{v}_{i,t-1}}+\delta}\Big|\boldsymbol{\theta}_t\right] + \frac{L\eta_t^2}{2}\sum_{i=1}^{d}\mathbb{E}\left[\frac{\boldsymbol{g}_{i,t}^2}{\left(\sqrt{\beta_2\boldsymbol{v}_{i,t-1}}+\delta\right)^2}\Big|\boldsymbol{\theta}_t\right]$$

Let's continue with the bound ...

$$\mathbb{E}\left[\mathcal{L}(\boldsymbol{\theta}_{t+1})|\boldsymbol{\theta}_t\right] \leq \cdots \quad \frac{\eta_t G\sqrt{1-\beta_2}}{\delta}\sum_{i=1}^{d}\mathbb{E}\left[\frac{\mathbf{g}_{i,t}^2}{\boxed{\sqrt{\beta_2\mathbf{v}_{i,t-1}}+\delta}}\middle|\boldsymbol{\theta}_t\right] + \left[\frac{L\eta_t^2}{2}\sum_{i=1}^{d}\mathbb{E}\left[\frac{\boldsymbol{g}_{i,t}^2}{\left(\sqrt{\beta_2\boldsymbol{v}_{i,t-1}}+\delta\right)^2}\middle|\boldsymbol{\theta}_t\right]\right.$$

$$\left.\leq \frac{L\eta_t^2}{2\delta}\sum_{i=1}^{d}\mathbb{E}\left[\frac{g_{i,t}^2}{\boxed{\sqrt{\beta_2\mathbf{v}_{i,t-1}}+\delta}}\middle|\boldsymbol{\theta}_t\right]\right]$$

... same denominator ...

$$\left(\frac{\eta_t G\sqrt{1-\beta_2}}{\delta}+\frac{L\eta_t^2}{2\delta}\right)\sum_{i=1}^{d}\mathbb{E}\left[\frac{\mathbf{g}_{i,t}^2}{\sqrt{\beta_2\mathbf{v}_{i,t-1}}+\delta}\middle|\boldsymbol{\theta}_t\right] \leq \left(\frac{\eta_t G\sqrt{1-\beta_2}}{\delta}+\frac{L\eta_t^2}{2\delta}\right)\sum_{i=1}^{d}\frac{1}{\delta}\mathbb{E}\left[\mathbf{g}_{i,t}^2\middle|\boldsymbol{\theta}_t\right]$$

$$= \frac{1}{\delta}\left(\frac{\eta_t G\sqrt{1-\beta_2}}{\delta}+\frac{L\eta_t^2}{2\delta}\right)\sum_{i=1}^{d}\mathbb{E}\left[\mathbf{g}_{i,t}^2\middle|\boldsymbol{\theta}_t\right]$$

Let's continue with the bound …

$$\mathbb{E}\left[\mathcal{L}(\boldsymbol{\theta}_{t+1})|\boldsymbol{\theta}_t\right] \leq \; \mathcal{L}(\boldsymbol{\theta}_t) - \eta_t \sum_{i=1}^{d} \frac{[\nabla \mathcal{L}(\boldsymbol{\theta}_t)]_i^2}{\sqrt{\beta_2 \boldsymbol{v}_{i,t-1}} + \delta} + \frac{1}{\delta}\left(\frac{\eta_t G \sqrt{1-\beta_2}}{\delta} + \frac{L\eta_t^2}{2\delta}\right)\mathbb{E}\left[||\mathbf{g}_t||^2 \middle| \boldsymbol{\theta}_t\right]$$

$$\boldsymbol{v}_{i,t} \leq G^2 \quad \forall i,t \;\Rightarrow\; \sqrt{\beta_2 \boldsymbol{v}_{i,t-1}} + \delta \leq \sqrt{\beta_2} G + \delta \;\Rightarrow\; -\eta_t \sum_{i=1}^{d} \frac{[\nabla \mathcal{L}(\boldsymbol{\theta}_t)]_i^2}{\sqrt{\beta_2 \boldsymbol{v}_{i,t}} + \delta} \leq -\frac{\eta_t}{\sqrt{\beta_2} G + \delta} \sum_{i=1}^{d} [\nabla \mathcal{L}(\boldsymbol{\theta}_t)]_i^2 = -\frac{\eta_t}{\sqrt{\beta_2} G + \delta} ||\nabla \mathcal{L}(\boldsymbol{\theta}_t)||_2^2$$

$$\mathbb{E}\left[\mathcal{L}(\boldsymbol{\theta}_{t+1})|\boldsymbol{\theta}_t\right] \leq \; \mathcal{L}(\boldsymbol{\theta}_t) \; \underbrace{-\frac{\eta_t}{\sqrt{\beta_2} G + \delta} ||\nabla \mathcal{L}(\boldsymbol{\theta}_t)||_2^2}_{-\Delta} + \frac{1}{\delta}\left(\frac{\eta_t G \sqrt{1-\beta_2}}{\delta} + \frac{L\eta_t^2}{2\delta}\right)\mathbb{E}\left[||\mathbf{g}_t||^2 \middle| \boldsymbol{\theta}_t\right]$$

Some positive term

Let's continue with the bound ...

$$\mathbb{E}\left[\mathcal{L}(\boldsymbol{\theta}_{t+1})|\boldsymbol{\theta}_t\right] \leq \mathcal{L}(\boldsymbol{\theta}_t) - \frac{\eta_t}{\sqrt{\beta_2}G+\delta}||\nabla\mathcal{L}(\boldsymbol{\theta}_t)||_2^2 + \frac{1}{\delta}\left(\frac{\eta_t G\sqrt{1-\beta_2}}{\delta} + \frac{L\eta_t^2}{2\delta}\right)\mathbb{E}\left[||\mathbf{g}_t||^2\middle|\boldsymbol{\theta}_t\right]$$

Assumption III -- Variance of Loss is Bounded:

$$\mathbb{E}_\xi\left[||\nabla\mathcal{L}(\boldsymbol{\theta};\xi) - \nabla\mathcal{L}(\boldsymbol{\theta})||_2^2\right] \leq \sigma^2, \quad \forall\boldsymbol{\theta}\in\mathbb{R}^d, \quad \forall\xi$$

... if we use a mini-batch, we can write...

$$\boldsymbol{g}_t(\cdot) = \frac{1}{b_t}\sum_{\xi\in\mathcal{B}_t}\nabla\mathcal{L}(\cdot;\xi)$$

... then, we can prove ..

$$\mathbb{E}\left[||\boldsymbol{g}_t||_2^2\middle|\boldsymbol{\theta}_t\right] \leq \frac{1}{b_t}\left(\sigma^2 + ||\nabla\mathcal{L}(\boldsymbol{\theta}_t)||_2^2\right)$$

Can you prove it ?

Proof Roadmap ...

$$\mathcal{L}(\boldsymbol{\theta}_{t+1}) \leq \mathcal{L}(\boldsymbol{\theta}_t) + \cdots +$$

Objective Func. L-Smoothness

... relation between 2 successive iterations ...

$$\mathbb{E}\left[\mathcal{L}(\boldsymbol{\theta}_{t+1}) | \boldsymbol{\theta}_t\right] \leq \mathcal{L}(\boldsymbol{\theta}_t) \cdots + \boxed{\mathbb{E}\left[\frac{\mathbf{g}_{i,t}}{\sqrt{\mathbf{v}_{i,t}} + \delta} \middle| \boldsymbol{\theta}_t\right]}$$

$$\cdots + \boxed{\mathbb{E}\left[\frac{\mathbf{g}_{i,t}^2}{(\sqrt{\mathbf{v}_{i,t}} + \delta)^2} \middle| \boldsymbol{\theta}_t\right]}$$

... we need to bound these ...

True Components to Bound

... consider stochasticity plug-in update rule, and realise terms to bound ...

Bounding the first term

Choose params

Bounding the second term

...

...

...

...

Bound in terms of gradient norm norm

Bound in terms of batch-size

Therefore, we can write ...

$$\mathbb{E}\left[\mathcal{L}(\boldsymbol{\theta}_{t+1})|\boldsymbol{\theta}_t\right] \leq \mathcal{L}(\boldsymbol{\theta}_t) - \frac{\eta_t}{\sqrt{\beta_2}G+\delta}||\nabla\mathcal{L}(\boldsymbol{\theta}_t)||_2^2 + \frac{1}{\delta}\left(\frac{\eta_t G\sqrt{1-\beta_2}}{\delta} + \frac{L\eta_t^2}{2\delta}\right)\frac{1}{b_t}\left(\sigma^2 + ||\nabla\mathcal{L}(\boldsymbol{\theta}_t)||_2^2\right)$$

$$= \mathcal{L}(\boldsymbol{\theta}_t) - ||\nabla\mathcal{L}(\boldsymbol{\theta}_t)||_2^2 \left(\frac{\eta_t}{\sqrt{\beta_2}G+\delta} - \frac{1}{\delta}\left(\frac{\eta_t G\sqrt{1-\beta_2}}{\delta} + \frac{L\eta_t^2}{2\delta}\right)\frac{1}{b_t}\right)$$

$$+ \frac{1}{\delta}\left(\frac{\eta_t G\sqrt{1-\beta_2}}{\delta} + \frac{L\eta_t^2}{2\delta}\right)\frac{1}{b_t}\sigma^2$$

... now, we need to handle each of these constants ...

... has to be a constant ...

$$\mathbb{E}[\mathcal{L}(\boldsymbol{\theta}_{t+1})|\boldsymbol{\theta}_t] \leq \mathcal{L}(\boldsymbol{\theta}) - \Delta + \epsilon_t$$

... we want this to go to zero ...

... let's start choosing free parameters (e.g., batch-sizes, learning rates ...) to get what we want ...

Let's choose free parameters …

We'll make 3 choices:

1. **Batch size:** b_t
2. **Learning rate:** η_t
3. **Free parameter :** β_2

$$\mathbb{E}\left[\mathcal{L}(\boldsymbol{\theta}_{t+1})|\boldsymbol{\theta}_t\right] \leq \mathcal{L}(\boldsymbol{\theta}_t) - \frac{\eta_t}{\sqrt{\beta_2}G+\delta}||\nabla\mathcal{L}(\boldsymbol{\theta}_t)||_2^2 + \frac{1}{\delta}\left(\frac{\eta_t G\sqrt{1-\beta_2}}{\delta} + \frac{L\eta_t^2}{2\delta}\right)\frac{1}{b_t}\left(\sigma^2 + ||\nabla\mathcal{L}(\boldsymbol{\theta}_t)||_2^2\right)$$

$$= \mathcal{L}(\boldsymbol{\theta}_t) - ||\nabla\mathcal{L}(\boldsymbol{\theta}_t)||_2^2 \underbrace{\left(\frac{\eta_t}{\sqrt{\beta_2}G+\delta} - \frac{1}{\delta}\left(\frac{\eta_t G\sqrt{1-\beta_2}}{\delta} + \frac{L\eta_t^2}{2\delta}\right)\frac{1}{b_t}\right)}_{\text{… let's start with … } \mathcal{A}} + \frac{1}{\delta}\left(\frac{\eta_t G\sqrt{1-\beta_2}}{\delta} + \frac{L\eta_t^2}{2\delta}\right)\frac{1}{b_t}\sigma^2$$

Choose $b_t \geq 1$, then we can say that:

$$\frac{1}{\delta b_t}\left(\frac{\eta_t G\sqrt{1-\beta_2}}{\delta} + \frac{L\eta_t^2}{2\delta}\right) \leq \frac{1}{\delta}\left(\frac{\eta_t G\sqrt{1-\beta_2}}{\delta} + \frac{L\eta_t^2}{2\delta}\right) \implies -\frac{1}{\delta b_t}\left(\frac{\eta_t G\sqrt{1-\beta_2}}{\delta} + \frac{L\eta_t^2}{2\delta}\right) \geq -\frac{1}{\delta}\left(\frac{\eta_t G\sqrt{1-\beta_2}}{\delta} + \frac{L\eta_t^2}{2\delta}\right)$$

$$\implies \mathcal{A} \geq \eta_t \underbrace{\left[\frac{1}{\sqrt{\beta_2}G+\delta} - \frac{1}{\delta}\left(\frac{G\sqrt{1-\beta_2}}{\delta} + \frac{L\eta_t}{2\delta}\right)\right]}_{\text{… let's call this … } \mathcal{B}}$$

Let's choose free parameters ...

We'll make 3 choices:

1. **Batch size:** b_t
2. **Learning rate:** η_t
3. **Free parameter :** β_2

Choose $b_t \geq 1$, then we can say that:

$$\frac{1}{\delta b_t}\left(\frac{\eta_t G\sqrt{1-\beta_2}}{\delta}+\frac{L\eta_t^2}{2\delta}\right) \leq \frac{1}{\delta}\left(\frac{\eta_t G\sqrt{1-\beta_2}}{\delta}+\frac{L\eta_t^2}{2\delta}\right) \implies -\frac{1}{\delta b_t}\left(\frac{\eta_t G\sqrt{1-\beta_2}}{\delta}+\frac{L\eta_t^2}{2\delta}\right) \geq -\frac{1}{\delta}\left(\frac{\eta_t G\sqrt{1-\beta_2}}{\delta}+\frac{L\eta_t^2}{2\delta}\right)$$

$$\implies \mathcal{A} \geq \eta_t \underbrace{\left[\frac{1}{\sqrt{\beta_2}G+\delta}-\frac{1}{\delta}\left(\frac{G\sqrt{1-\beta_2}}{\delta}+\frac{L\eta_t}{2\delta}\right)\right]}_{\text{... let's call this ... } \mathcal{B}}$$

Choose $\eta_t = \eta$, such that $\frac{L\eta}{2\delta} \leq \frac{G\sqrt{1-\beta_2}}{\delta}$, i.e., $\eta \leq \frac{2G\sqrt{1-\beta_2}}{L}$:

$$\frac{G\sqrt{1-\beta_2}}{\delta}+\frac{L\eta}{2\delta} \leq \frac{2G\sqrt{1-\beta_2}}{\delta} \implies -\left(\frac{G\sqrt{1-\beta_2}}{\delta}+\frac{L\eta}{2\delta}\right) \geq -\frac{2G\sqrt{1-\beta_2}}{\delta} \implies \mathcal{B} \geq \frac{1}{\sqrt{\beta_2}G+\delta}-\frac{2G\sqrt{1-\beta_2}}{\delta^2}$$

Let's choose free parameters ...

$$\frac{1}{G+\delta} \leq \frac{1}{\sqrt{\beta_2}G+\delta}$$

We'll make 3 choices:

1. **Batch size:** b_t
2. **Learning rate:** η_t
3. **Free parameter :** β_2

Further, choose β_2 such that $\frac{2G\sqrt{1-\beta_2}}{\delta^2} \leq \frac{1}{2}\left(\frac{1}{\sqrt{\beta_2}G+\delta}\right)$, then:

Let us choose β_2 such that: $\frac{2G\sqrt{1-\beta_2}}{\delta^2} = \frac{1}{2}\frac{1}{(G+\delta)}$, then: $\beta_2 = 1 - \frac{\delta^4}{16G^2(G+\delta)}$ **... should be close to one!**

$$\implies \mathcal{B} \geq \frac{1}{2(\sqrt{\beta_2}G+\delta)} \implies \mathcal{A} \geq \eta\mathcal{B} \geq \frac{\eta}{2(\sqrt{\beta_2}G+\delta)} \implies \boxed{-\mathcal{A} \leq -\frac{\eta}{2(\sqrt{\beta_2}G+\delta)}}$$

$$\mathbb{E}\left[\mathcal{L}(\boldsymbol{\theta}_{t+1})|\boldsymbol{\theta}_t\right] \leq \mathcal{L}(\boldsymbol{\theta}_t)\boxed{-}||\nabla\mathcal{L}(\boldsymbol{\theta}_t)||_2^2\boxed{\left(\frac{\eta_t}{\sqrt{\beta_2}G+\delta} - \frac{1}{\delta}\left(\frac{\eta_t G\sqrt{1-\beta_2}}{\delta} + \frac{L\eta_t^2}{2\delta}\right)\frac{1}{b_t}\right)}$$

$$+\frac{1}{\delta}\left(\underbrace{\frac{\eta_t G\sqrt{1-\beta_2}}{\delta} + \frac{L\eta_t^2}{2\delta}}_{\text{... let's call this ... } \mathcal{C}}\right)\frac{1}{b_t}\sigma^2$$

Let's choose free parameters …

We'll make 3 choices:

1. **Batch size:** b_t
2. **Learning rate:** η_t
3. **Free parameter :** β_2

$$\mathbb{E}\left[\mathcal{L}(\boldsymbol{\theta}_{t+1})|\boldsymbol{\theta}_t\right] \leq \mathcal{L}(\boldsymbol{\theta}_t) - ||\nabla\mathcal{L}(\boldsymbol{\theta}_t)||_2^2 \left(\frac{\eta_t}{\sqrt{\beta_2}G+\delta} - \frac{1}{\delta}\left(\frac{\eta_t G\sqrt{1-\beta_2}}{\delta} + \frac{L\eta_t^2}{2\delta}\right)\frac{1}{b_t}\right)$$

$$+ \frac{1}{\delta}\underbrace{\left(\frac{\eta_t G\sqrt{1-\beta_2}}{\delta} + \frac{L\eta_t^2}{2\delta}\right)}_{\text{… let's call this … } \mathcal{C}}\frac{1}{b_t}\sigma^2$$

Note, we chose $\eta_t = \eta$ such that $\frac{L\eta}{2\delta} \leq \frac{G\sqrt{1-\beta_2}}{\delta}$:

… then, we can say that $\mathcal{C} \leq 2\eta\frac{G\sqrt{1-\beta_2}}{\delta}$

$$\mathbb{E}\left[\mathcal{L}(\boldsymbol{\theta}_{t+1})|\boldsymbol{\theta}_t\right] \leq \mathcal{L}(\boldsymbol{\theta}_t) - ||\nabla\mathcal{L}(\boldsymbol{\theta}_t)||_2^2 \frac{\eta}{2(\sqrt{\beta_2}G+\delta)} + \frac{2\eta\sigma^2}{\delta^2 b_t}G\sqrt{1-\beta_2}$$

Let’s finalise the bound ...

$$\mathbb{E}\left[\mathcal{L}(\boldsymbol{\theta}_{t+1})|\boldsymbol{\theta}_t\right] \leq \mathcal{L}(\boldsymbol{\theta}_t) - ||\nabla\mathcal{L}(\boldsymbol{\theta}_t)||_2^2 \frac{\eta}{2(\sqrt{\beta_2}G+\delta)} + \frac{2\eta\sigma^2}{\delta^2 b_t} G\sqrt{1-\beta_2}$$

$$\implies ||\nabla\mathcal{L}(\boldsymbol{\theta}_t)||_2^2 \frac{\eta}{2(\sqrt{\beta_2}G+\delta)} \leq \mathcal{L}(\boldsymbol{\theta}_t) - \mathbb{E}\left[\mathcal{L}(\boldsymbol{\theta}_{t+1})|\boldsymbol{\theta}_t\right] + \frac{2\eta\sigma^2}{\delta^2 b_t} G\sqrt{1-\beta_2}$$

$$\frac{1}{2(\sqrt{\beta_2}G+\delta)}\mathbb{E}_{\text{total}}\left[||\nabla\mathcal{L}(\boldsymbol{\theta}_t)||_2^2\right] \leq \frac{\mathbb{E}_{\text{total}}[\mathcal{L}(\boldsymbol{\theta}_t)] - \mathbb{E}_{\text{total}}[\mathcal{L}(\boldsymbol{\theta}_{t+1})]}{\eta} + \frac{2\sigma^2}{\delta_2 b_t} G\sqrt{1-\beta_2}$$

$$\frac{1}{2(\sqrt{\beta_2}G+\delta)}\sum_{t=1}^{T}\mathbb{E}_{\text{total}}\left[||\nabla\mathcal{L}(\boldsymbol{\theta}_t)||_2^2\right] \leq \frac{\mathbb{E}_{\text{total}}[\mathcal{L}(\boldsymbol{\theta}_1)] - \mathbb{E}_{\text{total}}[\mathcal{L}(\boldsymbol{\theta}_{T+1})]}{\eta} + \frac{2\sigma^2}{\delta_2} G\sqrt{1-\beta_2}\sum_{t=1}^{T}\frac{1}{b_t}$$

$$\frac{c_1}{T}\sum_{t=1}^{T}\mathbb{E}_{\text{total}}\left[||\nabla\mathcal{L}(\boldsymbol{\theta}_t)||_2^2\right] \leq \frac{\mathcal{L}(\boldsymbol{\theta}_1) - \mathcal{L}(\boldsymbol{\theta}_{\text{global-min}})}{T\eta} + c_2\frac{1}{T}\sum_{t=1}^{T}\frac{1}{b_t}$$

Let's finalise the bound ...

$$\frac{c_1}{T}\sum_{t=1}^{T}\mathbb{E}_{\text{total}}\left[||\nabla\mathcal{L}(\boldsymbol{\theta}_t)||_2^2\right] \leq \underbrace{\frac{\mathcal{L}(\boldsymbol{\theta}_1)-\mathcal{L}(\boldsymbol{\theta}_{\text{global-min}})}{T\eta} + c_2\frac{1}{T}\sum_{t=1}^{T}\frac{1}{b_t}}_{\text{we want the RHS to be } \leq \epsilon c_1} \implies \frac{\mathcal{L}(\boldsymbol{\theta}_1)-\mathcal{L}(\boldsymbol{\theta}_{\text{global-min}})}{T\eta} + c_2\frac{1}{T}\sum_{t=1}^{T}\frac{1}{b_t} \leq c_1\epsilon$$

... with a constant batch-size ...

$$b_t = b \implies b = \left\lceil\frac{2c_2}{c_1\epsilon}\right\rceil \implies c_2\frac{1}{T}\sum_{t=1}^{T}\frac{1}{b_t} = \frac{c_2}{b} \leq \frac{c_1\epsilon}{2}$$

$$T = \frac{2(\mathcal{L}(\boldsymbol{\theta})-\mathcal{L}(\boldsymbol{\theta}_{\text{global-min}}))}{\eta c_1\epsilon} \implies \frac{\mathcal{L}(\boldsymbol{\theta})-\mathcal{L}(\boldsymbol{\theta}_{\text{global-min}})}{T\eta} \leq \frac{c_1\epsilon}{2}$$

$$\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}_{\text{total}}\left[||\nabla\mathcal{L}(\boldsymbol{\theta}_t)||_2^2\right] \leq \epsilon$$

... how to fix that...

... but as T grows ...

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}_{\text{total}}\left[||\nabla\mathcal{L}(\boldsymbol{\theta}_t)||_2^2\right] = \frac{c_2}{c_1 b} \neq 0$$

... we don't converge to a stationary point ...

Let's finalise the bound ...

$$\frac{c_1}{T}\sum_{t=1}^{T}\mathbb{E}_{\text{total}}\left[||\nabla\mathcal{L}(\boldsymbol{\theta}_t)||_2^2\right] \leq \underbrace{\frac{\mathcal{L}(\boldsymbol{\theta}_1)-\mathcal{L}(\boldsymbol{\theta}_{\text{global-min}})}{T\eta} + c_2\frac{1}{T}\sum_{t=1}^{T}\frac{1}{b_t}}_{\text{we want the RHS to be } \leq \epsilon c_1} \implies \frac{\mathcal{L}(\boldsymbol{\theta}_1)-\mathcal{L}(\boldsymbol{\theta}_{\text{global-min}})}{T\eta} + c_2\frac{1}{T}\sum_{t=1}^{T}\frac{1}{b_t} \leq c_1\epsilon$$

... with an increasing batch-size ... **... chose T such that...** $\frac{\ln T+\gamma}{T} \leq \frac{\epsilon}{2}$

$$\left.\begin{aligned} &b_t = \lceil\frac{c_2}{c_1}\rceil t \implies c_2\frac{1}{T}\sum_{t=1}^{T}\frac{1}{b_t} \leq \frac{c_1}{T}\sum_{t=1}^{T}\frac{1}{t} = \frac{c_1}{T}(\ln T+\gamma) \leq \frac{c_1\epsilon}{2} \\ &T = \frac{2(\mathcal{L}(\boldsymbol{\theta})-\mathcal{L}(\boldsymbol{\theta}_{\text{global-min}}))}{\eta c_1\epsilon} \implies \frac{\mathcal{L}(\boldsymbol{\theta})-\mathcal{L}(\boldsymbol{\theta}_{\text{global-min}})}{T\eta} \leq \frac{c_1\epsilon}{2} \end{aligned}\right\} \Rightarrow \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}_{\text{total}}\left[||\nabla\mathcal{L}(\boldsymbol{\theta}_t)||_2^2\right] \leq \epsilon$$

... and as T grows ...

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}_{\text{total}}\left[||\nabla\mathcal{L}(\boldsymbol{\theta}_t)||_2^2\right] = 0$$

... we converge to a stationary point ...

Thank you!