Math 600 Fall 2017 Homework #4

Due Oct. 23, Mon. in class

Note: For the Euclidean space $\mathbb{R}^n$, consider the usual metric induced by the Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^n$, unless otherwise stated.

1. Show that in a metric space, any subset of a totally bounded set is also totally bounded.
2. What are connected *and* compact sets in $\mathbb{R}$?
3. Show that a metric space (M, d) is connected if and only if the only open *and* closed sets in M are M and the empty set.
4. Let (M, d) be a metric space. Fix $z \in M$, and define $f : M \to \mathbb{R}$ by $f(x) := d(z, x)$. Show that f is continuous on M.
5. Solve the following problems.
   (1) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be $f(x_1, x_2) = x_1$. Show that f is continuous on $\mathbb{R}^2$.
   (2) Let A be an open set in $\mathbb{R}$, and $B = \{(x_1, x_2) : x_1 \in A\} \subseteq \mathbb{R}^2$. Use (1) to show that B is open in $\mathbb{R}^2$.
6. Let $f : (M, d) \to (N, \rho)$ be continuous on M, and $B \subseteq M$. Show that $f(\mathrm{cl}(B)) \subseteq \mathrm{cl}(f(B))$. (*Hint*: use the sequential criteria.)

Miscellaneous practice problems: *Do not submit*

1. Which of the following sets are connected in $\mathbb{R}$?

$$\{3, -10\}, \quad [0, 1), \quad (-7, \infty), \quad \mathbb{Z}, \quad \mathbb{Q}, \quad \mathbb{R}.$$

   Here $\mathbb{Z}$ and $\mathbb{Q}$ are the sets of integers and rational numbers, respectively.

2. Define $f : \mathbb{R}^2 \to \mathbb{R}$ as:

$$f(x_1, x_2) := \begin{cases} 0, & \text{if } x_1 \text{ is rational and } x_2 \text{ is irrational} \\ 1, & \text{otherwise} \end{cases}$$

   Show that f is discontinuous at any point of $\mathbb{R}^2$.

3. Let $f, g : (M, d) \to (V, \|\cdot\|)$ be two functions, where (M, d) is a metric space and $(V, \|\cdot\|)$ is a normed space.
   (1) Use the sequential criterion to show that if f and g are continuous at $x_0 \in M$, so is $f + g$;
   (2) Let λ be a scalar. Use the $\varepsilon - \delta$ definition to show that if f is continuous at $x_0 \in M$, so is λf.