Math 650 Fall 2011 Homework #3

Due Oct. 10, Mon. in class

P.1 Consider the equality constrained optimization problem on $\mathbb{R}^n$:

$$\min \; -\sum_{i=1}^{n} x_i^3, \qquad \text{subject to} \;\; \sum_{i=1}^{n} x_i^2 = 1,$$

where $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$.

(1) Show that the optimization problem has a global minimizer.

(2) Show that each local minimizer must satisfy the KKT conditions.

(3) Determine a global minimizer (using the 1st and 2nd order optimality conditions).

(4) Use (3) to show the following inequality:

$$\sum_{i=1}^{n} |x_i|^3 \le \Big(\sum_{i=1}^{n} x_i^2 \Big)^{3/2}, \quad \forall\, x \in \mathbb{R}^n.$$

P.2 Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function on $\mathbb{R}^n$. Suppose f is *homogeneous* of degree $\nu \ge 1$, namely, for any real number α, $f(\alpha x) = \alpha^\nu f(x), \forall\, x \in \mathbb{R}^n$. Show that for a given vector norm $\| \cdot \|$ on $\mathbb{R}^n$, there exist $\alpha, \beta \in \mathbb{R}$ such that $\beta \|x\|^\nu \le f(x) \le \alpha \|x\|^\nu, \forall\, x \in \mathbb{R}^n$. Further, if the degree ν is odd and f is not identically zero on $\mathbb{R}^n$, then $\alpha > 0$.

P.3 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be $f(x_1, x_2) = x_1^2 + x_2^4 + c_1 x_1 + c_2 x_2$, where c_1, c_2 are some constants.

(1) Show that f is coercive on $\mathbb{R}^2$.

(2) Show that f has a global minimizer on $\mathbb{R}^2$, and find global minimizer(s) (using the 1st and 2nd order optimality conditions).

P.4 Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be real-valued functions. Prove the following statements:

(1) If f is lower semicontinuous (lsc) at $x^* \in \mathbb{R}^n$, then $-f$ is upper semicontinuous (usc) at x^*.

(2) If f is both lsc and usc at $x^* \in \mathbb{R}^n$, then f is continuous at x^*.

(3) Let (x_k) be a sequence in $\mathbb{R}^n$. Then

$$\liminf f(x_k) + \liminf g(x_k) \le \liminf \big(f(x_k) + g(x_k) \big). \tag{1}$$

Give an example where the strict inequality holds. Further, use the inequality (1) to show that the sum of two lsc functions remains lsc.

P.5 Let $f_\alpha : \mathbb{R}^n \to \mathbb{R}$ be an indexed function that is lower semicontinuous on $\mathbb{R}^n$, where $\alpha \in I$. Let $f(x) := \sup_{\alpha \in I} f_\alpha(x)$ for each $x \in \mathbb{R}^n$. Show that f is lsc on $\mathbb{R}^n$. (*Hint*: use the epigraph argument.)