Math 650 Fall 2016 Homework #1

Due Sept. 20, Tue in class

1. Let $A \in \mathbb{R}^{m\times n}$ and $y \in \mathbb{R}^m$ be given, and $\|\cdot\|_p$ denote the p-norm on the Euclidean space. Show each of the following minimization problems has an optimal solution.

   (1) $\min_{x\in P} \|y - Ax\|_2^2$, where the set $P := \{x \in \mathbb{R}^n : \|x\|_1 \leq \alpha\}$ for a positive constant α.

   (2) $\min_{x\in\mathbb{R}^n} \|y - Ax\|_2^2 + \lambda\|x\|_1$, where λ is a positive constant.

   Remark: These minimization problems are related to the Least Absolute Shrinkage and Selection Operator (or LASSO), which is popular in statistics and machine learning nowadays.

2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be $f(x) = x^T Ax + c^T x, \forall\, x \in \mathbb{R}^n$, where A is a positive definite matrix, and $c \in \mathbb{R}^n$ is a given vector.

   (1) Show that f is coercive on $\mathbb{R}^n$;

   (2) Show that for any nonempty closed set P in $\mathbb{R}^n$, $\min_{x\in P} f(x)$ has an optimal solution.

3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function on $\mathbb{R}^n$. Suppose f is *homogeneous* of degree ν with $\nu \in \mathbb{N}$, namely, for any real number λ, $f(\lambda x) = \lambda^\nu f(x), \forall\, x \in \mathbb{R}^n$. Show that for any vector norm $\|\cdot\|$ on $\mathbb{R}^n$, there exist real numbers α and β such that $\beta\|x\|^\nu \leq f(x) \leq \alpha\|x\|^\nu, \forall\, x \in \mathbb{R}^n$. Further, if the degree ν is odd and f is not identically zero on $\mathbb{R}^n$, then $\alpha > 0$.

4. Solve the following problems:

   (1) Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be two lower semicontinuous functions on $\mathbb{R}^n$. Show that $f + g$ is lower semincontinuous.

   (2) Let $f_\alpha : \mathbb{R}^n \to \mathbb{R}$ be an indexed function with index $\alpha \in I$. Let f be the pointwise supremum of f_α's, i.e., $f(x) := \sup_{\alpha\in I} f_\alpha(x), \forall\, x \in \mathbb{R}^n$. Show that $\mathrm{epi}(f) = \cap_{\alpha\in I}\, \mathrm{epi}(f_\alpha)$.

   (3) Use (2) and the epigraph argument to construct an alternative proof of the fact: the pointwise supremum of a family of lower semicontinuous functions is lower semicontinuous. (*Hint*: what can you say about the intersection of a family of closed sets?)

More practice problems: *Do not submit*

Let $\|\cdot\|_0$ be the pseudo 0-norm on $\mathbb{R}^n$ defined as

$$\|x\|_0 := \text{ the number of nonzero elements in } x\ , \quad \forall\, x \in \mathbb{R}^n.$$

(1) Explain why $\|\cdot\|_0$ is *not* a norm on $\mathbb{R}^n$.

(2) Show that pseudo 0-norm on $\mathbb{R}$ is lower semicontinuous.

(3) Use (2) to show that $\|\cdot\|_0$ on $\mathbb{R}^n$ is lower semicontinuous. (*Hint*: for any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\|x\|_0 = |x_1|_0 + |x_2|_0 + \cdots + |x_n|_0$, where $|\cdot|_0$ is the pseudo 0-norm on $\mathbb{R}$.)