Math 600 Fall 2015 Homework #1

Due Sept. 14, Tue. in class

1. Recall that $C([0,1])$ is the vector space of all continuous functions $f : [0,1] \to \mathbb{R}$ on the interval $[0,1]$. For any $f \in C([0,1])$, define

$$\|f\|_\infty := \max\Big\{|f(t)| \,:\, t \in [0,1]\Big\}.$$

   (1) Explain why $\|f\|_\infty$ exists for any $f \in C([0,1])$. (*Hint*: think of Theorem 5.3.4 of Bartle and Sherbert's book.)

   (2) Prove that $\|\cdot\|_\infty$ is a norm on $C([0,1])$.

2. Let (M,d) be a metric space. Show the following inequality (i.e., the reverse triangle inequality):

$$|d(x,y) - d(z,y)| \le d(x,z), \quad \forall\, x,y,z \in M.$$

3. Let d_1 and d_2 be two metrics on a set M. Define the sum $d_1 + d_2$ on M as

$$(d_1 + d_2)(x,y) := d_1(x,y) + d_2(x,y), \quad \forall\, x,y \in M.$$

   Show that $d_1 + d_2$ is a metric on M.

4. Let (M,d) be a metric space. Define

$$\rho(x,y) := \frac{d(x,y)}{1 + d(x,y)}, \quad \forall\, x,y \in M.$$

   Show that ρ is a metric on M. (*Hint*: observe that $t/(1+t)$ is an increasing function on $\mathbb{R}_+$.)

Miscellaneous practice problems: *Do not submit*

1. In an inner product space V with the induced norm $\|\cdot\|$, show that for any $x,y \in V$,

   (1) $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$;

   (2) $\|x+y\| \cdot \|x-y\| \le \|x\|^2 + \|y\|^2$.

2. On the Euclidean space $\mathbb{R}^n$, define for each $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$\|x\|_\star := c_1|x_1| + c_2|x_2| + \cdots + c_n|x_n|,$$

   where c_i is a positive real number for each $i = 1, \ldots, n$. Show that $\|\cdot\|_\star$ is a norm on $\mathbb{R}^n$.

3. On the Euclidean space $\mathbb{R}^n$, define for each $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$\|x\|_\infty := \max(|x_1|, |x_2|, \ldots, |x_n|)$$

   Show that $\|\cdot\|_\infty$ is a norm on $\mathbb{R}^n$.