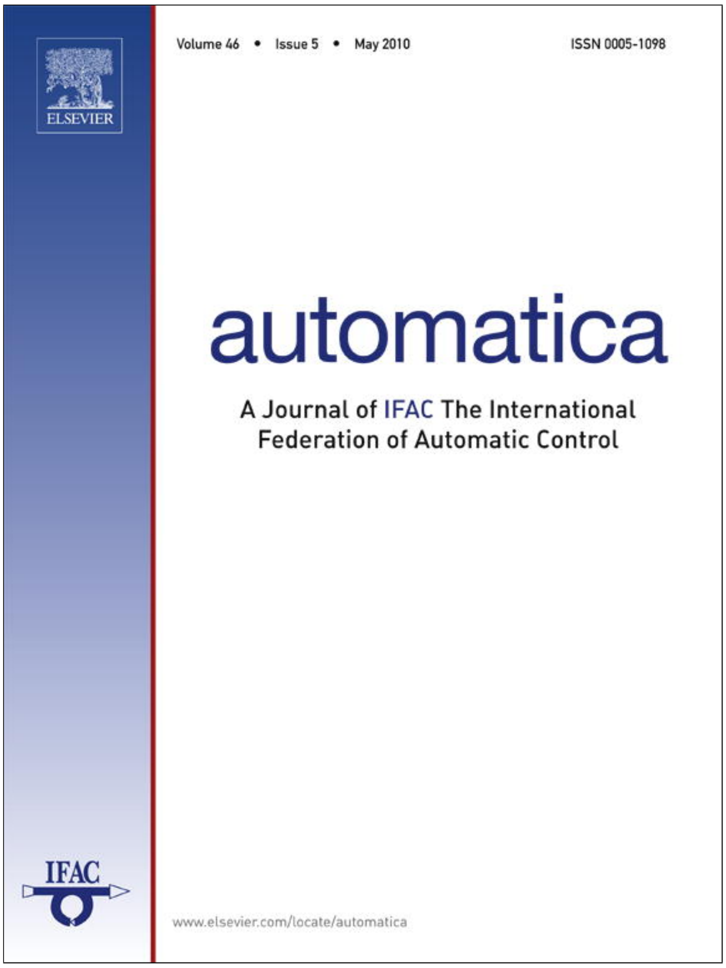
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Observability analysis of conewise linear systems via directional derivative and positive invariance techniques[☆]

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ABSTRACT

Belonging to the broad framework of hybrid systems, conewise linear systems (CLSs) form a class of Lipschitz piecewise linear systems subject to state triggered mode switchings. Motivated by state estimation of nonsmooth switched systems, this paper exploits directional derivative and positive invariance techniques to characterize finite-time and long-time local observability of a general CLS. For the former observability notion, directional derivative results are developed from the simple switching property, and they yield improved observability conditions. For the latter notion, we focus on the case where a nominal trajectory has finitely many switchings. In order to characterize long-time behaviors of the CLS, necessary and sufficient conditions are obtained for the interior of a positively invariant cone. By employing these conditions, we establish connections between finite-time and long-time local observability; underlying positive invariance properties are unveiled.

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1. Introduction

Introduced for modeling Lipschitz piecewise linear systems, the conewise linear system (CLS) constitutes an important class of linear hybrid systems. Such a system consists of a finite family of linear dynamical systems active on polyhedral cones that partition the state space. Each linear system with its associated polyhedral cone is a mode of the system; mode transitions occur along a state trajectory. See Çamlibel, Heemels, and Schumacher (2008), Çamlibel, Pang, and Shen (2006) and Shen and Pang (2007) and the references therein for various discussions. An important feature of the CLS is that it is subject to state-dependent mode switchings with implicit transition times and implicit mode selection at switching times. The state-dependent switchings render many dynamical and control issues rather complicated, albeit critical in applications.

The notion of observability is fundamental and profound in systems and control theory and has been treated in great depth for smooth systems (Hermann & Krener, 1977) with important applications in observer design and asymptotic stability analysis (Hespanha, Liberzon, Angeli, & Sontag, 2005). Observability of hybrid and switched systems, particularly linear hybrid systems and piecewise affine systems (PASs), has received growing interest. State and mode observability of discrete-time switched linear systems is addressed in Babaali and Egerstedt (2004); extension to continuous-time dynamics is made in Babaali and Pappas (2005). Observability tests and observer design of PASs are discussed in Collins and van Schuppen (2004). Bemporad et al. study discrete-time PASs with control inputs and logic-based mode switchings in Bemporad, Ferrari-Trecate, and Morari (2000). The paper (Vidal, Chiuso, Soatto, & Sastry, 2003) establishes necessary and sufficient conditions for the observability of jump linear systems. Other related results include the observability and detectability of jump Markov linear systems (Costa & do Val, 2002) and the observability of discrete-event states of hybrid systems (del Vecchio, Murray, & Klavins, 2006). However, most of these papers assume state-irrelevant arbitrary mode switchings, and much less attention has been paid to state-dependent switchings. An exception is Çamlibel et al. (2006), which initiated an extensive study of observability of the CLS. Also see Pang and Shen (2007) for observability of nonlinear complementarity systems.

Inspired by state estimation of switched and hybrid systems in applications, the present paper carries out further observability analysis, especially for finite-time and long-time local observability. For the former observability notion, we develop directional derivative results and exploit them to obtain improved conditions. For the latter, we focus on the case where a nominal trajectory eventually remains in a polyhedral cone of the CLS, and we address the question of whether the two observability notions are equivalent, particularly whether long-time observability implies T-time observability for some large $T > 0$. We show that the answer is negative in general unless certain positive invariance conditions

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are imposed. The main contributions of the paper are threefold: (1) new directional derivative results are established based on the simple switching property; (2) necessary and sufficient conditions for the interior of positively invariant cones of the CLS are obtained; (3) finite-time and long-time observability conditions are developed for a general CLS, with the aid of the directional derivative and positive invariance results developed. It is noted that partial results are obtained for finite-time and long-time observability in Çamlibel et al. (2006), especially for the bimodal CLS. However, further investigation for a general CLS is stalled due to the lack of the simple switching property and tools to handle long-time dynamics. Recently established in Shen, Han, and Pang (2009), the simple switching property turns out to a cornerstone for rigorous study of switchings of a general CLS. It leads to new directional derivative conditions for finite-time observability analysis. In addition, positive invariance analysis provides a major technique to deal with long-time dynamics. By extending the recent positive invariance results (Shen, submitted for publication; Shen et al., 2009), the current paper generalizes previous observability results and reveals the underlying positive invariance properties ignored before.

The paper is organized as follows. In Section 2, we introduce the CLS and establish new mode switching and directional derivative results that are crucial for observability analysis. Section 3 treats the positively invariant cone associated with each mode; necessary and sufficient conditions are developed to characterize the interior of such the cone. We then address finite-time and long-time local observability of the CLS in Section 4 via the directional derivative and positive invariance results. The paper concludes with final remarks in Section 5.

2. Conewise linear system: mode switching and directional derivative

A *conewise linear system* (CLS) on $\mathbb{R}^n$ is a time-invariant ODE system defined by a continuous and piecewise linear right-hand side. Specifically, it can be written as

$$\dot{x} = A_i x, \quad \forall x \in \mathcal{X}_i \equiv \{x \mid C_i x \geq 0\}, \ i = 1, \ldots, m, \tag{1}$$

where $A_i \in \mathbb{R}^{n\times n}$, $C_i \in \mathbb{R}^{m_i\times n}$, the family of the solid polyhedral cones $\{\mathcal{X}_i\}_{i=1}^m$ forms a conic subdivision of $\mathbb{R}^n$ (Scholtes, 1994), and the continuity condition holds:

$$x \in \mathcal{X}_i \cap \mathcal{X}_j \Longrightarrow A_i x = A_j x. \tag{2}$$

The right-hand side of (1) is globally Lipschitz, albeit non-differentiable, in x and hence the CLS has a unique C^1 state trajectory, denoted by $x(t, x^0)$, for any initial state x^0 and all t. Note that $x(t, x^0)$ is generally only once time differentiable and not differentiable in x^0. We assume, without losing generality, that each C_i has no zero rows. The interior of each polyhedral cone $\mathcal{X}_i$ is thus given by $\operatorname{int}\mathcal{X}_i = \{x \mid C_i x > 0\}$. Associated with the "forward-time" system (1) is a *backward-time* (or *reverse-time*) system, which remains a CLS.

A CLS may admit multiple conic subdivisions. Moreover, the converse of (2) may fail for a general conic subdivision (see Example 2). If the converse implication holds for a conic subdivision, i.e., $x \in \mathcal{X}_i \cap \mathcal{X}_j \Leftrightarrow A_i x = A_j x$, we call this conic subdivision *simple*. If a CLS is described by a simple conic subdivision, then the CLS is called *simple*. We illustrate these notions as follows.

Example 1. A bimodal CLS is referred to as a CLS with two modes only (Çamlibel et al., 2006). Each polyhedral cone is a half-space of $\mathbb{R}^n$ such that $\mathcal{X}_1 \equiv \{x \mid c^T x \geq 0\}$ and $\mathcal{X}_2 \equiv \{x \mid c^T x \leq 0\}$, where $c \in \mathbb{R}^n$, and A_1 and A_2 satisfy $A_1 - A_2 = bc^T$ for $b \in \mathbb{R}^n$. Letting $A_2 \equiv A$, then $A_1 = A + bc^T$ and the bimodal CLS can be put in the compact form: $\dot{x} = Ax + b\max(0, c^T x)$. To avoid triviality, we assume that the vectors b and c are nonzero. In this case, $A_1 x = A_2 x$ if and only if $c^T x = 0$; the latter is equivalent to $x \in \mathcal{X}_1 \cap \mathcal{X}_2$. This gives rise to a simple CLS.

Example 2. Consider the CLS on $\mathbb{R}^2$ with four modes, where the defining matrices of the linear dynamics are $A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $A_2 = A_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $A_4 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, and the corresponding matrices for the polyhedral cones are $C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $C_2 = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}$, $C_3 = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$, $C_4 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$. Hence the condition (2) holds. As $A_2 = A_3$, the CLS possesses multiple conic subdivisions of $\mathbb{R}^2$ with different partitions of the union $\mathcal{X}_2 \cup \mathcal{X}_3$. However, since $\mathcal{X}_2 \cup \mathcal{X}_3$ is non-convex, any conic subdivision must have at least two solid (convex) polyhedral cones, on which the linear functions coincide with $A_2 x$, to cover $\mathcal{X}_2 \cup \mathcal{X}_3$. Obviously, the converse of (2) fails for any such conic subdivision. Thus this CLS is not simple.

The significance of a simple CLS is demonstrated in the following lemma; see Proposition 18 and Corollary 19 in Section 4 for applications.

Lemma 3. *Consider a simple CLS on $\mathbb{R}^n$ and $x^0 \in \mathbb{R}^n$. Then $x(t, x^0) \in \mathcal{X}_i$ on $[0, T]$ with $T > 0$ if and only if $x(t, x^0) = e^{A_i t}x^0$, $\forall t \in [0, T]$.*

Proof. Consider the "if" part only. Note that $\dot{x}(t, x^0) = A_i x(t, x^0)$, $\forall t \in [0, T)$. It follows from Çamlibel et al. (2006, Lemma 3.3 and Theorem 3.5) that, $\forall t_* \in [0, T)$, there exist $\varepsilon_{t_*} > 0$ and an index j (possibly different from i) such that $t_* + \varepsilon_{t_*} \leq T$ and that $x(t, x^0) = e^{A_j(t-t_*)}x(t_*, x^0)$, $\forall t \in [t_*, t_* + \varepsilon_{t_*}]$; the latter shows $\dot{x}(t, x^0) = A_j e^{A_j(t-t_*)}x(t_*, x^0)$, $\forall t \in [t_*, t_* + \varepsilon_{t_*})$. By uniqueness of the CLS trajectories and their time derivatives, $x(t, x^0) = e^{A_j(t-t_*)}x(t_*, x^0) = e^{A_i(t-t_*)}x(t_*, x^0)$, so $A_i x(t, x^0) = A_j x(t, x^0)$, $\forall t \in [t_*, t_* + \varepsilon_{t_*})$. Since the conic subdivision is simple, $x(t, x^0) \in \mathcal{X}_i \cap \mathcal{X}_j$, $\forall t \in [t_*, t_* + \varepsilon_{t_*})$. Hence, $x(t, x^0) \in \mathcal{X}_i$, $\forall t \in [0, T)$. Finally, it follows from the continuity of $x(t, x^0)$ and the closedness of $\mathcal{X}_i$ that $x(T, x^0) \in \mathcal{X}_i$. $\square$

If the CLS is not simple, then the "if" part of Lemma 3 may fail. Consider Example 2 with $x^0 = (0, -1)^T$. Thus $x(t, x^0) = e^{A_3 t}x^0 = (-\sin t, -\cos t)^T$, $\forall t \in [0, \pi]$. Hence $x(t, x^0) \in \mathcal{X}_3$ on $[0, \pi/4]$ but leaves $\mathcal{X}_3$ afterwards.

2.1. Mode switching properties of the CLS

In this section, we develop mode switching properties of the CLS. In particular, we show the finite occurrence of critical times on a compact time interval (see Proposition 7). This result is important for the finite-time observability analysis performed in Section 4.

We introduce more notation and relevant results. An ordered real ℓ-tuple $a = (a_1, \ldots, a_\ell)$ is lexicographically nonnegative if either $a = 0$ or its first nonzero element (from the left) is positive, and we write $a \succcurlyeq 0$. If $a \succcurlyeq 0$ and $a \neq 0$, then a is lexicographically positive, and we write $a \succ 0$. Given two ℓ-tuples a and b, we write $a \succcurlyeq (\succ) b$ if $a - b \succcurlyeq (\succ) 0$. An n-dimensional vector tuple $(x^1, \ldots, x^\ell) \succcurlyeq (\succ) 0$ if each $(x_i^1, \ldots, x_i^\ell) \succcurlyeq (\succ) 0$ for all $i = 1, \ldots, n$. For each $i = 1, \ldots, m$, let $\mathcal{Y}_i \equiv \{x \in \mathbb{R}^n \mid (C_i x, C_i A_i x, \ldots, C_i A_i^{n-1} x) \succcurlyeq 0\}$ be the semiobservable cone associated with the pair (C_i, A_i) (Çamlibel et al., 2006). For a pair (C_i, A_i), let $\overline{O}(C_i, A_i)$ be its unobservable subspace. Given $\xi \in \mathbb{R}^n$, define two index sets $\mathcal{I}(\xi) \equiv \{i \mid \xi \in \mathcal{X}_i\}$ and $\mathcal{J}(\xi) \equiv \{i \mid \xi \in \mathcal{Y}_i\}$. It is obvious that $\mathcal{J}(\xi) \subseteq \mathcal{I}(\xi)$, $\forall \xi \in \mathbb{R}^n$. Similarly, we can define $\mathcal{J}^r(\xi)$ for the associated reverse-time system.

Definition 4. We say that a time instant $t_* \geq 0$ is *not* a switching time along a trajectory $x(t, x^0)$ if there exist $i \in \{1, \ldots, m\}$ and $\varepsilon > 0$ such that $x(t, x^0) \in \mathcal{X}_i$, $\forall t \in [t_* - \varepsilon, t_* + \varepsilon]$; otherwise, t_* is a *switching time* along $x(t, x^0)$, and the CLS has a *mode transition* or *mode switching* along $x(t, x^0)$ at t_*.

Switching and non-switching times along $x(t, x^0)$ can be characterized by the index sets $\mathcal{J}$ and $\mathcal{J}^r$. In fact, a time $t_* > 0$

is a switching time along $x(t, x^0)$ if and only if $\mathcal{J}(x(t_*, x^0)) \cap \mathcal{J}^r(x(t_*, x^0)) = \emptyset$ (Çamlibel et al., 2006, Proposition 3.11). It is further shown in Shen et al. (2009, Proposition 2) that $t_* > 0$ is a non-switching time along $x(t, x^0)$ if and only if $\mathcal{J}(x(t_*, x^0)) = \mathcal{J}^r(x(t_*, x^0))$. This result is the so-called *simple switching property*. While seemingly intuitive and straightforward, this property leads to various important consequences. For example, in light of this property, we have the following lemma for a non-switching trajectory.

Lemma 5. *If there is no switching along $x(t, x^0)$ for all $t \in [0, T]$ with $0 < T \le \infty$, then $\mathcal{J}(x(t, x^0)) = \mathcal{J}(x^0)$, $\forall t \in [0, T]$.*

Proof. Define $t_* \equiv \sup\{\bar{t} \ge 0 \mid \mathcal{J}(x(t, x^0)) = \mathcal{J}(x^0), \forall t \in [0, \bar{t}]\}$. It follows from Statement (a) of Çamlibel et al. (2006, Proposition 3.9) that $t_* > 0$. Now suppose that $t_* < T$ (if $T = \infty$, this implies that t_* is finite). Then we have $\mathcal{J}(x(t, x^0)) = \mathcal{J}(x^0)$, $\forall t \in [0, t_*)$ and $\mathcal{J}(x(t_*, x^0)) \ne \mathcal{J}(x^0)$. On the other hand, we deduce from Statement (b) of Çamlibel et al. (2006, Proposition 3.9) that there exists $\varepsilon > 0$ such that $\mathcal{J}^r(x(t_*, x^0)) = \mathcal{J}(x(t, x^0))$ for all $t \in [t_* - \varepsilon, t_*)$. Since $\mathcal{J}(x(t_* - \varepsilon, x^0)) = \mathcal{J}(x^0)$, we obtain $\mathcal{J}^r(x(t_*, x^0)) = \mathcal{J}(x^0)$. Therefore, $\mathcal{J}(x(t_*, x^0)) \ne \mathcal{J}^r(x(t_*, x^0))$. In view of the simple switching property, we conclude that $t_* \in (0, T)$ is a switching time, a contradiction. $\square$

If $\mathcal{J}(x(t', x^0)) \ne \mathcal{I}(x(t', x^0))$ at some t', then we call t' a *critical time* along $x(t, x^0)$ (and its corresponding state $x(t', x^0)$ is a critical state). It can be shown that a switching time t_* must be a critical time. In fact, suppose that this is not the case, i.e., $\mathcal{J}(x^*) = \mathcal{I}(x^*)$ where $x^* \equiv x(t_*, x^0)$. Since t_* is a switching time, $\mathcal{J}^r(x^*) \cap \mathcal{J}(x^*) = \emptyset$. Hence, $\mathcal{J}^r(x^*) \cap \mathcal{I}(x^*) = \emptyset$. However, there exists $\varepsilon > 0$ such that $x(t, x^0) \in \cup_{i \in \mathcal{I}(x^*)} \mathcal{X}_i$ for all $t \in [t_* - \varepsilon, t_* + \varepsilon]$. Thus we have $\mathcal{J}^r(x^*) \subseteq \mathcal{I}(x^*)$, so $\mathcal{J}^r(x^*) \cap \mathcal{I}(x^*) = \mathcal{J}^r(x^*)$. This leads to a contradiction as $\mathcal{J}^r(x^*) \ne \emptyset$.

For a non-switching trajectory $x(t, x^0)$ on $[0, T]$ with $T > 0$, notice that $\mathcal{J}(x(t, x^0))$ may not be equal to $\mathcal{I}(x(t, x^0))$ at each $t \in [0, T]$. The following lemma asserts that there are only finitely many critical times along $x(t, x^0)$.

Lemma 6. *Consider a non-switching trajectory $x(t, x^0)$ on $[0, T]$ for some $T > 0$. Then there exist finitely many times t_i satisfying $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$ such that $\mathcal{I}(x(t, x^0)) = \mathcal{J}(x(t, x^0))$, $\forall t \in (t_i, t_{i+1})$, $i = 0, \ldots, N-1$.*

Proof. For each $t' \in [0, T]$, let $\widehat{x} \equiv x(t', x^0)$ and consider either of the following two cases.

(i) $\mathcal{J}(\widehat{x}) = \mathcal{I}(\widehat{x})$. In this case, there exists a neighborhood $\mathcal{N}$ of $\widehat{x}$ such that $\mathcal{N} \subseteq \cup_{j \in \mathcal{I}(\widehat{x})} \mathcal{X}_j$ (Çamlibel et al., 2006, Lemma 2.5). Hence, by the continuity of $x(\cdot, x^0)$, we have $x(t, x^0) \in \mathcal{N}$, $\forall t \in [t' - \varepsilon, t' + \varepsilon]$ for some $\varepsilon > 0$. Therefore, $\mathcal{I}(x(t, x^0)) \subseteq \mathcal{I}(\widehat{x})$, $\forall t \in [t' - \varepsilon, t' + \varepsilon]$. Moreover, it follows from Çamlibel et al. (2006, Proposition 3.9) that $\mathcal{J}(x(t, x^0)) = \mathcal{J}(\widehat{x})$, $\forall t \in [t' - \varepsilon, t' + \varepsilon]$ by appropriately restricting $\varepsilon > 0$. Since $\mathcal{J}(\widehat{x}) = \mathcal{I}(\widehat{x})$ and $\mathcal{J}(x(t, x^0)) \subseteq \mathcal{I}(x(t, x^0))$, $\mathcal{I}(x(t, x^0)) = \mathcal{J}(x(t, x^0))$ for all $t \in [t' - \varepsilon, t' + \varepsilon]$.

(ii) $\mathcal{J}(\widehat{x}) \ne \mathcal{I}(\widehat{x})$. Let $j \in \mathcal{I}(\widehat{x}) \setminus \mathcal{J}(\widehat{x})$. Thus, for an $i \in \mathcal{J}(\widehat{x})$, $(C_j\widehat{x}, C_jA_i\widehat{x}, \ldots, C_jA_i^{n-1}\widehat{x}) \not\succcurlyeq 0$. This shows that some row of C_j, say the ℓth row denoted by $(C_j)_{\ell\bullet}$, satisfies $(C_j)_{\ell\bullet}x(t, x^0) = (C_j)_{\ell\bullet}e^{A_i(t-t')}\widehat{x} < 0$, $\forall t \in (t', t' + \varepsilon]$ for some $\varepsilon > 0$. Hence $x(t, x^0) \notin \mathcal{X}_j$, or equivalently $j \notin \mathcal{I}(x(t, x^0))$, for all $t \in (t', t' + \varepsilon]$. This shows that $\mathcal{I}(x(t, x^0)) \subseteq \mathcal{J}(\widehat{x})$, $\forall t \in (t', t' + \varepsilon]$. Since $\mathcal{J}(x(t, x^0)) = \mathcal{J}(\widehat{x})$ for all $t > t_*$ sufficiently close to t_*, we have, by appropriately refining $\varepsilon > 0$, $\mathcal{I}(x(t, x^0)) = \mathcal{J}(x(t, x^0)) = \mathcal{J}(\widehat{x})$, $\forall t \in (t', t' + \varepsilon]$. Similarly, using the reverse-time argument and Çamlibel et al. (2006, Proposition 3.9), we deduce that $\mathcal{I}(x(t, x^0)) = \mathcal{J}(x(t, x^0)) = \mathcal{J}^r(\widehat{x})$, $\forall t \in [t' - \varepsilon, t')$ for some $\varepsilon > 0$. Hence, via the simple switching property, we obtain $\varepsilon > 0$ such that $\mathcal{I}(x(t, x^0)) = \mathcal{J}(x(t, x^0))$, $\forall t \in [t' - \varepsilon, t') \cup (t', t' + \varepsilon]$.

Consequently, for each $t \in [0, T]$, there exists $\varepsilon_t > 0$ such that $\mathcal{I}(x(\tau, x^0)) = \mathcal{J}(x(\tau, x^0))$, $\forall \tau \in [t - \varepsilon_t, t) \cup (t, t + \varepsilon_t]$. Since the family $\{(t - \varepsilon_t, t + \varepsilon_t) : t \in [0, T]\}$ constitutes an open cover of the compact interval $[0, T]$, it thus follows from the Heine–Borel Theorem that there is a finite sub-cover of $[0, T]$. Hence there exist finitely many time instants $\{t'_0, t'_1, \ldots, t'_\ell\} \subset [0, T]$ such that $[0, T] \subset \bigcup_{j=0}^{\ell}[t'_j, t'_j + \varepsilon_{t'_j}]$. By appropriately refining the partition on the right, we obtain finitely many times $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$ such that $\mathcal{I}(x(t, x^0)) = \mathcal{J}(x(t, x^0))$, $\forall t \in (t_i, t_{i+1})$ for each $i = 0, \ldots, N-1$. $\square$

Combining the above results and non-Zenoness of the CLS, the following proposition establishes the finite occurrence of critical times on a compact interval.

Proposition 7. *Consider a trajectory $x(t, x^0)$ on $[0, T]$ with $T > 0$. Then there are finitely many critical times on $[0, T]$. Specifically, there exists a partition $0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = T$ such that $\mathcal{I}(x(t, x^0)) = \mathcal{J}(x(t, x^0)) = \mathcal{J}(x(t', x^0))$ for all $t \in (t_i, t_{i+1})$ and any $t' \in (t_i, t_{i+1})$ for each $i = 0, \ldots, M-1$.*

Proof. Since the CLS is non-Zeno, there are finitely many switching times along $x(t, x^0)$ (Çamlibel et al., 2006, Theorem 3.7), i.e., there exists a partition $0 = \widetilde{t}_0 < \widetilde{t}_1 < \cdots < \widetilde{t}_{N-1} < \widetilde{t}_N = T$ of $[0, T]$ such that there is no switching on a subinterval $(\widetilde{t}_i, \widetilde{t}_{i+1})$ for every $i = 0, \ldots, N-1$. It follows from Lemma 6 that there are a finite number of critical times on $[0, T]$. Finally, since there is no switching on a subinterval (t_i, t_{i+1}), where t_i and t_{i+1} are two neighboring critical times on $[0, T]$, we conclude via Lemmas 5 and 6 that $\mathcal{I}(x(t, x^0)) = \mathcal{J}(x(t, x^0)) = \mathcal{J}(x(t', x^0))$ for all $t \in (t_i, t_{i+1})$ and any $t' \in (t_i, t_{i+1})$. $\square$

Proposition 7 is instrumental to finite-time sensitivity and observability analysis. In fact, given a nominal trajectory and an interval $[0, T]$, one can divide $[0, T]$ into finitely many subintervals defined by consecutive critical times of the nominal trajectory on $[0, T]$. This enables one to focus on each subinterval, which is relatively easier to handle; see Theorem 9 and Proposition 18. Then one can combine obtained sensitivity or observability conditions for these subintervals, together with those at critical times, to establish desired conditions for the entire interval $[0, T]$; see Corollary 19 in Section 4.

2.2. Directional derivative of the CLS

Sensitivity analysis of the CLS with respect to its initial conditions is essential to study various dynamic and numerical properties of the CLS, e.g., stability, robustness, observability and numerical resolution of the systems (Çamlibel et al., 2006; Clarke, 1990; Hu, Ma, & Lin, 2008; Pang & Shen, 2007; Pang & Stewart, 2009). In particular, the first-order variation of a system trajectory with respect to its initial condition is perhaps the most important and the best studied. However, unlike an ODE with a C^1 right-hand side, a trajectory of the CLS is not differentiable in its initial condition. On the other hand, it is shown in Pang and Stewart (2009) that $x(t, x^0)$ is B(ouligand)-differentiable in x^0 at each t, i.e., $x(t, \cdot)$ is locally Lipschitz continuous and directional differentiable at each t (Facchinei & Pang, 2003). For a given t, the directional derivative of $x(t, x^0)$ along a direction vector $v \in \mathbb{R}^n$ is defined by $x'(t, x^0; v) \equiv \lim_{\tau \downarrow 0} \frac{x(t, x^0 + \tau v) - x(t, x^0)}{\tau}$, which is positively homogenous. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ denote the right-hand side of the CLS (1). It is known that $f(x)$ is globally Lipschitz and directionally differentiable (thus B-differentiable). Indeed, it can be shown via Çamlibel et al. (2006, Lemma 3.4) that, for any x and v,

$$f'(x; v) = A_i v \tag{3}$$

for some $i \in \mathcal{I}(x)$ (such an i depends on v). Hence, it follows from Pang and Stewart (2009, Theorem 7) that, for any given initial condition x^0 and direction vector v, the directional derivative $x'(t, x^0; v)$ is the unique solution of the following time-varying differential system:

$$\dot{z} = f'(x(t, x^0); z), \qquad z(0) = v, \tag{4}$$

where $f'(x(t, x^0); z)$ denotes the directional derivative of f at $x(t, x^0)$ along z at each t. Using (3), we can further write the right-hand side of (4) as $f'(x(t, x^0); z) = B(x(t, x^0), z)z$, where $B : \mathbb{R}^n \times \mathbb{R}^n \to \{A_i \mid i = 1, \ldots, m\}$. Hence, the system (4) becomes a time-varying piecewise linear system whose right-hand side $f'(x(t, x^0); z(t))$ is generally discontinuous in t. However, it shall be shown as follows that, on a compact time interval, $f'(x(t, x^0); z)$ is discontinuous only at finitely many time instants. To establish this result, we need a technical lemma that asserts the persistence of the duration of trajectories in a mode under small perturbations on initial conditions.

Lemma 8. *Given $x^* \in \mathbb{R}^n$ and $u \in \mathbb{R}^n$ such that $x^* \in \mathcal{Y}_i$ and $(x^* + u) \in \mathcal{Y}_i$ for some $i \in \{1, \ldots, m\}$, then there exist $\varepsilon_0 > 0$ and $\tau_0 > 0$ such that $x(t, x^* + \tau u) = e^{A_i t}(x^* + \tau u) \in \mathcal{X}_i$ for all $(t, \tau) \in [0, \varepsilon_0] \times [0, \tau_0]$.*

Proof. Let $C_i \in \mathbb{R}^{m_i \times n}$. For each $\ell \in \{1, \ldots, m_i\}$, if $((C_i x^*)_\ell, (C_i A_i x^*)_\ell, \ldots, (C_i A_i^{n-1} x^*)_\ell) \succ 0$, then let μ_ℓ be the first nonnegative integer k such that $(C_i A_i^k x^*)_\ell > 0$; otherwise, let $\mu_\ell \equiv n$. In the first case, $0 \le \mu_\ell < n$, and in the latter, we must have $x^* \in \overline{O}((C_i)_{\ell\bullet}, A_i)$, where $(C_i)_{\ell\bullet}$ denotes the ℓth row of C_i. It is easy to see that, for each ℓ, $\left((C_i u)_\ell, (C_i A_i u)_\ell, \ldots, (C_i A_i^{\mu_\ell - 1} u)_\ell\right) \succcurlyeq 0$. Moreover, for every ℓ, we have (i) if $\mu_\ell < n$, then $\left(C_i e^{A_i t}[x^* + \tau u]\right)_\ell = \tau \sum_{s=0}^{\mu_\ell - 1} \frac{(C_i A_i^s u)_\ell}{s!} t^s + \sum_{j=\mu_\ell}^{\infty} \frac{(C_i A_i^j [x^* + \tau u])_\ell}{j!} t^j$; and (ii) if $\mu_\ell = n$, then $(C_i e^{A_i t}[x^* + \tau u])_\ell = \tau \sum_{s=0}^{n-1} \frac{(C_i A_i^s u)_\ell}{s!} t^s$. In case (i), there exists $\varepsilon'_\ell > 0$, depending on C_i, A_i and u only, such that the first summation on the right is nonnegative for all $(t, \tau) \in [0, \varepsilon'_\ell] \times [0, \infty)$. Since $(C_i A_i^{\mu_\ell} x^*)_\ell > 0$, there exist positive numbers ε''_ℓ and τ''_ℓ, depending on C_i, A_i, x^* and u only, such that the second summation $\sum_{j=\mu_\ell}^{\infty} \frac{(C_i A_i^j [x^* + \tau u])_\ell}{j!} t^j = t^{\mu_\ell} \left\{ \frac{(C_i A_i^{\mu_\ell} x^*)_\ell}{j!} + O(t) + \tau \left[\frac{(C_i A_i^{\mu_\ell} u)_\ell}{j!} + O(t) \right] \right\} \ge 0$ for all $(t, \tau) \in [0, \varepsilon''_\ell] \times [0, \tau''_\ell]$. Letting $\varepsilon_\ell = \min(\varepsilon'_\ell, \varepsilon''_\ell)$ and $\tau_\ell = \min(\tau'_\ell, \tau''_\ell)$, we have $(C_i e^{A_i t}[x^* + \tau u])_\ell \ge 0$ for all $(t, \tau) \in [0, \varepsilon_\ell] \times [0, \tau_\ell]$. The case where $\mu_\ell = n$ also holds true by a similar argument. Finally, letting $\varepsilon_0 = \min_\ell \varepsilon_\ell$ and $\tau_0 = \min_\ell \tau_\ell$, we obtain the desired result. □

Theorem 9. *Let t_i and t_{i+1} be two consecutive critical times along $x(t, x^0)$ and $v \in \mathbb{R}^n$ be a direction vector. Then the following hold.*

(a) *For any interval $[T_1, T_2] \subset (t_i, t_{i+1})$, there exists a partition $T_1 = \widehat{t}_1 < \widehat{t}_2 < \cdots < \widehat{t}_{p-1} < \widehat{t}_p = T_2 < \widehat{t}_{p+1}$ of $[T_1, T_2]$ such that $x'(t, x^0; v)$ satisfies the linear system $\dot{z}(t) = A_k z(t)$ for some $k \in \mathcal{J}(x(T_1, x^0))$ on $(\widehat{t}_i, \widehat{t}_{i+1})$ for each $i = 1, \ldots, p$.*

(b) *For any interval $[T_1, T_2] \subset (t_i, t_{i+1})$, let $\xi \equiv x(T_1, x^0)$ and $\eta \equiv x'(T_1, x^0; v)$. Then, for all $\tau > 0$ sufficiently small and all $t \in [T_1, T_2]$,*

$$x(t - T_1, \xi + \tau\eta) = x(t, x^0) + \tau x'(t, x^0; v). \tag{5}$$

(c) *$f'(x(t, x^0); x'(t, x^0; v))$ is continuous (with respect to t) on (t_i, t_{i+1}).*

Proof. (a) Consider an arbitrary $t_* \in (t_i, t_{i+1})$. For notational convenience, we let $x^* \equiv x(t_*, x^0)$ and $\eta \equiv x'(t_*, x^0; v)$. Since $x'(t, x^0; v)$ is the solution of the differential system (4), it follows from the semi-group property that $x'(t, x^0; v) = x'(t - t_*, x^*; \eta)$ for all $t \ge t_*$ (this can also be shown via the B-differentiability of $x(t, x^0)$), namely, $\forall t \ge t_*$,

$$x'(t, x^0; v) = \lim_{\tau \downarrow 0} \frac{x(t - t_*, x^* + \tau\eta) - x(t - t_*, x^*)}{\tau}. \tag{6}$$

Since there exists a neighborhood $\mathcal{N}$ of x^* such that $\mathcal{N} \subseteq \cup_{i \in \mathcal{I}(x^*)} \mathcal{X}_i$, we obtain a positive number $\widetilde{\tau}$ such that $(x^* + \widetilde{\tau}\eta) \in \mathcal{N}$. This further implies that $\mathcal{I}(x^* + \widetilde{\tau}\eta) \subseteq \mathcal{I}(x^*)$. Since t_* is not a critical time, $\mathcal{I}(x^*) = \mathcal{J}(x^*)$. We thus have $\mathcal{J}(x^* + \widetilde{\tau}\eta) \subseteq \mathcal{I}(x^* + \widetilde{\tau}\eta) \subseteq \mathcal{J}(x^*) = \mathcal{J}(x(T_1, x^0))$ by Proposition 7. Since $\mathcal{J}(x^* + \widetilde{\tau}\eta)$ is nonempty (Çamlibel et al., 2006, Lemma 3.3), there exists an index $k \in \mathcal{J}(x^* + \widetilde{\tau}\eta) \subseteq \mathcal{J}(x(T_1, x^0))$ such that $(x^* + \widetilde{\tau}\eta) \in \mathcal{Y}_k$ and $x^* \in \mathcal{Y}_k$. It hence follows from Lemma 8 that $\varepsilon_0 > 0$ and $\tau_0 > 0$ exist such that

$$x(t - t_*, x^* + \tau\eta) = e^{A_k(t - t_*)}[x^* + \tau\eta] \tag{7}$$

for each pair $(t - t_*, \tau) \in [0, \varepsilon_0] \times [0, \tau_0]$. Therefore, we deduce via (6) that, for all $t \in [t_*, t_* + \varepsilon_0]$, $x'(t, x^0; v) = e^{A_k(t - t_*)}\eta$, which is the unique solution of the linear system $\dot{z} = A_k z$. Hence, $x'(t, x^0; v)$ satisfies $\dot{z} = A_k z$ for all $t \in (t_*, t_* + \varepsilon_0)$. This further implies that, for each $t \in (t_i, t_{i+1})$, there exist $\varepsilon_t > 0$ and A_{k_t} with $k_t \in \mathcal{J}(x(T_1, x^0))$ such that $x'(t, x^0; v)$ satisfies the linear ODE $\dot{z} = A_{k_t} z$ on the open interval $(t, t + \varepsilon_t)$. Since the collection $\{(t, t + \varepsilon_t) : t \in (t_i, t_{i+1})\}$ forms an open cover of the compact interval $[T_1, T_2]$, there exist finitely many times $\{t'_0, t'_1, \ldots, t'_\ell\} \subset (t_i, t_{i+1})$ with $t'_0 < T_1$ such that $[T_1, T_2] \subset \bigcup_{j=0}^{\ell} [t'_j, t'_j + \varepsilon_{t'_j}]$. By refining these intervals, we obtain finitely many times $T_1 = \widehat{t}_1 < \widehat{t}_2 < \cdots < \widehat{t}_{p-1} < \widehat{t}_p = T_2 < \widehat{t}_{p+1}$ such that $x'(t, x^0; v)$ satisfies a linear system on each $(\widehat{t}_i, \widehat{t}_{i+1})$. Hence, the desired partition holds.

(b) Recall that $\xi \equiv x(T_1, x^0)$ and $\eta \equiv x'(T_1, x^0; v)$. Since there is no critical time on $[T_1, T_2]$, $\mathcal{I}(x(t, \xi)) = \mathcal{J}(x(t, \xi)) = \mathcal{J}(\xi) = \mathcal{I}(\xi)$ for all $t \in [T_1, T_2]$. Hence, it follows from the Lipschitz property that there exists $\tau_0 > 0$ such that $\mathcal{I}(x(t, \xi + \tau\eta)) \subseteq \mathcal{I}(\xi)$ for all $t \in [T_1, T_2]$ and all $\tau \in (0, \tau_0]$, which yields $\mathcal{J}(x(t, \xi + \tau\eta)) \subseteq \mathcal{J}(\xi)$, $\forall (t, \tau) \in [T_1, T_2] \times (0, \tau_0]$. Using the partition obtained in (a) and letting $\widehat{x}^i \equiv x(\widehat{t}_i, x^0)$ and $\widehat{\eta}^i = x'(\widehat{t}_i, x^0; v)$, we deduce, in light of (7), that, for each subinterval $[\widehat{t}_i, \widehat{t}_{i+1}]$, there exists $\tau_i > 0$ such that $x(t - \widehat{t}_i, \widehat{x}^i + \tau\widehat{\eta}^i) = e^{A_k(t - \widehat{t}_i)}\widehat{x}^i + \tau x'(t, x^0; v)$ for some A_k with $k \in \mathcal{J}(\xi)$ for all $t \in [\widehat{t}_i, \widehat{t}_{i+1}]$ and all $\tau \in [0, \tau_i]$, $i = 1, \ldots, p-1$. Let $\widehat{\tau} \equiv \min_{i \in \{0, \ldots, p-1\}} \tau_i$. In what follows, we prove (5) by induction. Consider the first subinterval $[\widehat{t}_1, \widehat{t}_2]$. Since $\widehat{t}_1 = T_1$, $\widehat{x}^1 = \xi$ and $\widehat{\eta}^1 = \eta$, we have $x(t - T_1, \xi + \tau\eta) = e^{A_k(t - T_1)}\xi + \tau x'(t, x^0; v)$ for all $t \in [\widehat{t}_1, \widehat{t}_2]$ and all $\tau \in (0, \widehat{\tau}]$. Hence, (5) holds on $[\widehat{t}_1, \widehat{t}_2]$ because $e^{A_k(t - T_1)}\xi = x(t, x^0)$ on $[T_1, T_2]$. Now assume that (5) holds on the subintervals $[\widehat{t}_i, \widehat{t}_{i+1}]$ for all $i = 1, \ldots, \ell$, where $1 \le \ell \le p - 2$. Consider the subinterval $[\widehat{t}_{\ell+1}, \widehat{t}_{\ell+2}]$. Thus $x(t - \widehat{t}_{\ell+1}, \widehat{x}^{\ell+1} + \tau\widehat{\eta}^{\ell+1}) = e^{A_k(t - \widehat{t}_{\ell+1})}\widehat{x}^{\ell+1} + \tau x'(t, x^0; v)$ for some A_k with $k \in \mathcal{J}(\xi)$ for all $t \in [\widehat{t}_{\ell+1}, \widehat{t}_{\ell+2}]$ and all $\tau \in (0, \widehat{\tau}]$. Note that $\widehat{x}^{\ell+1} + \tau\widehat{\eta}^{\ell+1} = x(\widehat{t}_{\ell+1} - T_1, \xi) + \tau x'(\widehat{t}_{\ell+1}, x^0; v)$. It follows from the induction hypothesis that the latter equals $x(\widehat{t}_{\ell+1} - T_1, \xi + \tau\eta)$. Therefore $x(t - \widehat{t}_{\ell+1}, \widehat{x}^{\ell+1} + \tau\widehat{\eta}^{\ell+1}) = x(t - \widehat{t}_{\ell+1}, x(\widehat{t}_{\ell+1} - T_1, \xi + \tau\eta)) = x(t - T_1, \xi + \tau\eta)$ for all $t \in [\widehat{t}_{\ell+1}, \widehat{t}_{\ell+2}]$. Consequently, (5) holds on $[\widehat{t}_{\ell+1}, \widehat{t}_{\ell+2}]$, since $e^{A_k(t - \widehat{t}_{\ell+1})}\widehat{x}^{\ell+1} = x(t, x^0)$ on the subinterval in consideration. By the induction principle, we see that (5) holds on $[T_1, T_2]$.

(c) Given any $t' \in (t_i, t_{i+1})$, it belongs to the interior of some compact interval $[T_1, T_2]$ contained in (t_i, t_{i+1}). Hence (5) holds true on a small open interval $I \subseteq [T_1, T_2]$ containing t'. Therefore $x'(t, x^0; v) = \frac{x(t - T_1, \xi + \tau\eta) - x(t, x^0)}{\tau}$, $\forall t \in I$ for a fixed small $\tau > 0$. Since both $x(t - T_1, \xi + \tau\eta)$ and $x(t, x^0)$ have continuous time derivatives on I, so does $x'(t, x^0; v)$. This shows that $f'(x(t, x^0); x'(t, x^0; v))$ is continuous at t'. Since t' is arbitrary in (t_i, t_{i+1}), we obtain (c). □

3. Positive invariance of the CLS

The concept of positive invariance plays a crucial role in the asymptotic analysis of dynamical systems and Lyapunov stability theory. Roughly speaking, a set S is positively invariant if each trajectory starting from S remains in S for all positive times. In the realm of switched and hybrid systems, positive invariance also sheds light on reachability analysis and control of hybrid systems (Abate, Tiwari, & Sastry, 2007; Belta & Habets, 2006; Tiwari, Fung, Bhattacharya, & Murray, 2004). Pertaining to the CLS, it is shown recently in Shen et al. (2009) that the long-time dynamics are closely related to the positively invariant cone of each mode. In this section, we establish necessary and sufficient conditions for the interior of a positively invariant cone and use them to characterize the long-time observability.

3.1. Preliminary discussions

The positively invariant cone associated with the ith mode of the CLS (1) is defined by $\mathcal{A}_i \equiv \{x \in \mathbb{R}^n \mid C_i e^{A_i t} x \geq 0, \ \forall t \geq 0\}$. Clearly, $\mathcal{A}_i$ is closed and convex and is the largest positively invariant set contained in $\mathcal{X}_i$. While the above formulation is simple and neat, an explicit characterization of $\mathcal{A}$ in terms of x only is highly challenging due to the difficulty of removing the quantifier t. In the following, we focus on each of these cones and drop the subscripts in $\mathcal{A}_i$, C_i and A_i for notational simplicity. We also let $\mathcal{X} \equiv \{x \in \mathbb{R}^n \mid Cx \geq 0\}$.

Since $\mathcal{X}$ is polyhedral, it is natural to ask whether its positively invariant cone $\mathcal{A}$ is also polyhedral. However, the following example shows that the answer is negative in general; this demonstrates another difficulty in analyzing a positively invariant cone.

Example 10. Let $A \in \mathbb{R}^{n\times n}$ be nilpotent and C be an n-row, i.e., $C = c^T \in \mathbb{R}^{1\times n}$. In this case, $\mathcal{X} = \{x \mid c^T x \geq 0\}$ becomes a half-space of $\mathbb{R}^n$. Without loss of generality, we assume that A is in the Jordan canonical form, i.e., $A = \text{diag}(J_1, \ldots, J_\ell)$, where J_i is a Jordan block associated with the zero eigenvalue, and we partition c accordingly as $c^T = (c_1^T, \ldots, c_\ell^T)$. Besides, we assume, without losing generality, that each pair (c_i, J_i) is of the observable canonical form, i.e., $c_i^T = (0, \widetilde{c}_i^T), J_i = \begin{bmatrix} J_{iu} & J_{i2} \\ 0 & \widetilde{J}_i \end{bmatrix}$, such that $(\widetilde{c}_i^T, \widetilde{J}_i)$ is an observable pair. Note that each $\widetilde{J}_i$ remains a Jordan block. Let v be an eigenvector of A. Thus $\text{sgn}(c^T v)v$ is always in the half-space $\mathcal{X}$. Hence $\mathcal{A}$ contains nonzero vectors. We claim that $\mathcal{A}$ is polyhedral if and only if each Jordan block $\widetilde{J}_i$ is at most of order 2. To show this, we partition $x \in \mathbb{R}^n$ as $x^T = ((x^1)^T, \ldots, (x^\ell)^T)$, where each x^i corresponds to the block J_i. Therefore $c^T e^{At} x = \sum_{i=1}^{\ell} c_i^T e^{J_i t} x^i = \sum_{i=1}^{\ell} \widetilde{c}_i^T e^{\widetilde{J}_i t} \widetilde{x}^i$, where $\widetilde{x}^i$ is a sub-vector of x^i corresponding to the observable pair $(\widetilde{c}_i^T, \widetilde{J}_i)$ for each i. Since the sub-vector $x^i \setminus \widetilde{x}^i$ plays no role in determining the polyhedrality of $\mathcal{A}$, we thus assume that each pair (c_i^T, J_i) is observable. The proof for sufficiency is trivial and is thus omitted. We prove necessity by contradiction. Suppose that $\mathcal{A}$ is polyhedral but that one of the Jordan blocks, say J_1, is of order greater than 2. Let x_j denote the jth entry of x and define the polyhedral cone $P_1 \equiv \{x \in \mathbb{R}^n \mid x_j = 0, j \geq 4\}$. Furthermore, let $P_2 \equiv \mathcal{A} \cap P_1 = \{x \in \mathbb{R}^n \mid c^T e^{At} x \geq 0, \forall t \geq 0, x_j = 0, j \geq 4\}$, which is polyhedral. Therefore $c^T e^{At} x = a_0(x) + a_1(x)t + a_2(x)t^2/2$ for $x \in P_1$, where $a_k(x) = c_1^T (J_1)^k x$ with $k = 0, 1, 2$. Since the pair (c_1^T, J_1) is observable, $\{c_1, J_1 c_1, (J_1)^2 c_1\}$ is linearly independent. This leads to a linear transformation $z = Tx$ with an invertible $T \in \mathbb{R}^{n\times n}$ such that $\widetilde{P}_2 = TP_2 = \{z \in \mathbb{R}^n \mid z_1 + z_2 t + z_3 t^2 \geq 0, \forall t \geq 0, z_j = 0, j \geq 4\}$. Clearly, $\widetilde{P}_2$ is also polyhedral. Let $\gamma < 0$ be a given scalar. Therefore $\widetilde{P}_3 \equiv \widetilde{P}_2 \cap \{z \mid z_2 = \gamma\}$ is a polyhedron (provided that it is nonempty). Since a quadratic polynomial $a_0 + a_1 t + a_2 t^2 \geq 0, \forall t \geq 0$ if and only if $a_0, a_2 > 0$ and $a_1 + 2\sqrt{a_0 a_2} \geq 0$, we deduce $\widetilde{P}_3 = \{z \in \mathbb{R}^n \mid z_1 \geq 0, z_3 \geq 0, z_1 z_3 \geq \gamma^2/4, z_2 = \gamma, z_j = 0, j \geq 4\}$, which is clearly nonempty. This further implies that the convex set $\{(z_1, z_3)^T \in \mathbb{R}^2 \mid z_1 \geq 0, z_3 \geq 0, z_1 z_3 \geq \gamma^2/4\}$ is polyhedral, a contradiction. Thus the claim holds.

3.2. Interior of a positively invariant cone

For a given x^* to be in the interior of $\mathcal{A}$, it is easy to expect that $Ce^{At}x^* > 0, \forall t \geq 0$. However, the following example shows that this condition alone is not enough; some additional condition related to the "largest mode" defined by the pair (C, A) is needed.

Example 11. Let $C = (1, 1)$, $A = \text{diag}(1, 2) \in \mathbb{R}^{2\times 2}$, and $x^* = (1, 0)^T$. Hence, $Ce^{At}x^* = e^t > 0, \forall t \geq 0$ and $x^* \in \mathcal{A}$. Consider $\widehat{x} = (1, -\varepsilon)^T$ with $\varepsilon > 0$. Clearly, $\widehat{x} \to x^*$ as $\varepsilon \downarrow 0$. However, for any $\varepsilon > 0$, $Ce^{At}\widehat{x} = e^t - \varepsilon e^{2t} < 0$, for all $t \geq 0$ sufficiently large. Thus $x^* \notin \text{int}\,\mathcal{A}$.

To characterize the interior of a positively invariant cone, we need certain technical results. The next two lemmas provide major tools to treat oscillatory modes corresponding to complex eigenvalues of the defining matrix in positive invariance analysis of linear dynamics.

Lemma 12 (*Shen et al., 2009, Corollary 15*). *Let $f(t) \equiv \sum_{i=1}^{m} [\alpha_i \cos(\omega_i t) + \beta_i \sin(\omega_i t)]$, where $\omega_i > 0$, $\omega_i \neq \omega_j$ for $i \neq j$, and $|\alpha_i| + |\beta_i| \neq 0$ for all i. Then there exist two scalars $\gamma_1 > 0$ and $\gamma_2 < 0$ such that, for any t_*, $t_1, t_2 \in [t_*, \infty)$ exist satisfying $f(t_1) \geq \gamma_1$ and $f(t_2) \leq \gamma_2$.*

More properties for the above $f(t)$ are presented as follows. For notational simplicity, let $d_i(t) \equiv \alpha_i \cos(\omega_i t) + \beta_i \sin(\omega_i t)$. By considering the rationality of ratios of the frequencies, we obtain the collection of equivalent classes $E_{\omega_j} = \{d_i(t) \mid \omega_i/\omega_j \text{ is rational}\}$. Note that each equivalent class E_{ω_j} attains a basis frequency $\widetilde{\omega}_s > 0$, namely, $\omega_i/\widetilde{\omega}_s$ is a positive integer for any frequency ω_i associated with $d_i(t) \in E_{\omega_j}$. Let $E_{\widetilde{\omega}_s}$ denote the equivalent class, and let $q_{\widetilde{\omega}_s}(t) := \sum_{d_i \in E_{\widetilde{\omega}_s}} d_i(t)$. Then the following statements hold: (1) $q_{\widetilde{\omega}_s}(\cdot)$ is a real-valued smooth and periodic function with the frequency $\widetilde{\omega}_s$; (2) if $q_{\widetilde{\omega}_s}(\cdot)$ is not identically zero, then it attains the maximal and minimal values $\sigma_{\widetilde{\omega}_s} > 0$ and $\nu_{\widetilde{\omega}_s} < 0$ on $(-\infty, \infty)$ respectively; (3) $q_{\widetilde{\omega}_s}(\cdot)$ is onto $[\nu_{\widetilde{\omega}_s}, \sigma_{\widetilde{\omega}_s}]$; and (4) the ratio of any two basis frequencies associated with distinct equivalent classes is irrational. Suppose that there are k equivalent classes $E_{\widetilde{\omega}_s}$, i.e., $s = 1, \ldots, k$. Hence, $f(t) = \sum_{s=1}^{k} q_{\widetilde{\omega}_s}(t)$. Then we have:

Lemma 13 (*Shen, submitted for publication, Lemma 5*). *Let $\sigma_{\widetilde{\omega}_s}$ and $\nu_{\widetilde{\omega}_s}$ be defined as above for the function f. Then $\sum_{s=1}^{k} \sigma_{\widetilde{\omega}_s} = \sup_{[t_*,\infty)} f(t)$ and $\sum_{s=1}^{k} \nu_{\widetilde{\omega}_s} = \inf_{[t_*,\infty)} f(t)$ for any t_*.*

We introduce more notions for the following development. We assume that A has the real Jordan canonical form via a real similarity transformation $A = \text{diag}(\widetilde{J}_1, \widetilde{J}_2, \ldots, \widetilde{J}_p)$, where each submatrix $\widetilde{J}_i$ contains all the Jordan blocks associated with a real eigenvalue λ_i or a complex eigenvalue and its conjugate (i.e., μ_i and $\bar{\mu}_j$) whose real parts are $\widetilde{\lambda}$, i.e., $\lambda_i = \widetilde{\lambda}$ and $\mu_i = \widetilde{\lambda} + \mathrm{i}\omega_i$. Without loss of generality, we also assume that the $\widetilde{J}_i$'s are ordered in a way such that the real parts of their corresponding eigenvalues are strictly decreasing. Furthermore, each $\widetilde{J}_i$ is given by $\widetilde{J}_i = \text{diag}(J_{i1}, J_{i2}, \ldots, J_{ir(i)})$. Here $r(i)$ is the number of Jordan blocks in $\widetilde{J}_i$ and J_{ij} is the jth real Jordan block given by (i) $J_{ij} = \begin{bmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$ corresponding to the real eigenvalue λ_i, or (ii)

$$J_{ij} = \begin{bmatrix} D_i & I_2 & & \\ & \ddots & \ddots & \\ & & \ddots & I_2 \\ & & & D_i \end{bmatrix}$$ corresponding to the complex eigenvalue

pair $\lambda_i \pm \mathrm{i}\omega_i$, where $\lambda_i, \omega_i \in \mathbb{R}$ with $\omega_i > 0$, I_2 is the 2×2 identity matrix, and $D_i = \begin{bmatrix} \lambda_i & \omega_i \\ -\omega_i & \lambda_i \end{bmatrix}$. Accordingly, the matrix C can be partitioned as $C = \begin{bmatrix} \widetilde{C}_1 & \widetilde{C}_2 & \cdots & \widetilde{C}_p \end{bmatrix}$ with $\widetilde{C}_i = \begin{bmatrix} C_{i1} & C_{i2} & \cdots & C_{ir(i)} \end{bmatrix}$. Obviously, C_{ij} has at least one column (resp. two columns) if it corresponds to a real eigenvalue (resp. a complex eigenvalue pair). For each $\widetilde{J}_i$, define the index set $\mathcal{K}_i^r \equiv \{j \in \{1,\ldots,r(i)\} \mid J_{ij}$ corresponds to the real eigenvalue $\lambda_i\}$, and $\mathcal{K}_i^c \equiv \{1,\ldots,r(i)\} \setminus \mathcal{K}_i^r$. Hence, for each $j \in \mathcal{K}_i^c$, the Jordan block J_{ij} corresponds to a complex eigenvalue pair with the real part λ_i.

Let $(\widetilde{C}_a)_{\ell\bullet}$ be the first nonzero block in $(C)_{\ell\bullet}$, i.e., $(\widetilde{C}_i)_{\ell\bullet} = 0$ for $i = 1,\ldots,a-1$ and $(\widetilde{C}_a)_{\ell\bullet} \neq 0$. Notice that $(\widetilde{C}_a)_{\ell\bullet}e^{\widetilde{J}_a t}$ is a finite sum of the terms of the form $\kappa e^{(\lambda_a \pm i\omega_s)t}t^{b_s}$, where b_s is a nonnegative integer. Let b denote the largest such b_s. We call $e^{\lambda_a t}t^b$ the *principal mode* associated with the pair $((C)_{\ell\bullet}, A)$. It is noted that $(C)_{\ell\bullet}e^{At}x = \mu_\ell^0(t,x)e^{\lambda_a t}t^b + \sum_{k\geq 1}\mu_\ell^k(t,x)e^{\lambda_k t}t^{b_k}$, where $(\lambda_a, b) \succ (\lambda_k, b_k)$ for all k. Here, for each $i \geq 0$, $\mu_\ell^i(t,x)$ takes the form $c_\ell^i(x) + \sum_s (g_{\ell,\omega_s}^i(x)\cos(\omega_s t) + h_{\ell,\omega_s}^i(x)\sin(\omega_s t))$, where $c_\ell^i(x)$, $g_{\ell,\omega_s}^i(x)$ and $h_{\ell,\omega_s}^i(x)$ are all linear. In particular, we show as follows how to determine $c_\ell^0(x), g_{\ell,\omega_s}^0(x)$, and $h_{\ell,\omega_s}^0(x)$; this eventually leads to the concept of "principal coefficient".

(1) For each $j \in \mathcal{K}_a^r$, let $J_{aj} \in \mathbb{R}^{m_j \times m_j}$ and k_j be the index corresponding to the first nonzero number in $(C_{aj})_{\ell\bullet}$ (from the left), i.e., $(C_{aj})_{\ell k_j} \neq 0$. It is easy to verify that the dominating mode in $(C_{aj})_{\ell\bullet}e^{J_{aj}t}$ for large $t \geq 0$ is given by $\frac{1}{(m_j-k_j)!}(C_{aj})_{\ell k_j}e^{\lambda_a t}t^{(m_j-k_j)}$. In other words, the principal mode associated with $((C_{aj})_{\ell\bullet}, J_{aj})$ is $e^{\lambda_a t}t^{(m_j-k_j)}$. Define the index set $\mathcal{L}_a^r \equiv \{j \in \mathcal{K}_a^r \mid m_j - k_j = b\}$ (note that by the definition of the principal mode associated with the pair $((C)_{\ell\bullet}, A)$, $m_j - k_j \leq b$ for all $j \in \mathcal{K}_a^r$). Therefore, writing $x \in \mathbb{R}^n$ as $((x^{11})^T, \ldots, (x^{ij})^T, \ldots)^T$, where x^{ij} is the sub-vector corresponding to the Jordan block J_{ij}, we see that $c_\ell^0(x) = \frac{1}{b!}\sum_{j\in\mathcal{L}_a^r}(C_{aj})_{\ell k_j}x_{m_j}^{aj}$, where each $x_{m_j}^{aj}$ is the last element of the sub-vector x^{aj}.

(2) For each $s \in \mathcal{K}_a^c$, let $J_{as} \in \mathbb{R}^{2m_s\times 2m_s}$, and we write the row $(C_{as})_{\ell\bullet} \in \mathbb{R}^{1\times 2m_s}$ as $(C_{as})_{\ell\bullet} = ((C_{as})_{\ell 1}, \ldots, (C_{as})_{\ell m_s})$, where each $(C_{as})_{\ell j}$ is a sub-row of two entries. Let $(C_{as})_{\ell k_s}$ be the first nonzero sub-row (from the left) such that $m_s - k_s = b$. For each (distinct) imaginary part ω_j of the complex eigenvalues $\lambda_a \pm \omega_j$ of A, define the index set $\mathcal{L}_{a,\omega_j}^c \equiv \{s \in \mathcal{K}_a^c \mid m_s - k_s = b$ and J_{as} corresponds to $\lambda_a \pm i\omega_j\}$ and $\mathcal{L}_a^c \equiv \cup_{\omega_j}\mathcal{L}_{a,\omega_j}^c$. Let $x_{2m_j-1}^{as}$ and $x_{2m_j}^{as}$ denote the last two elements of the sub-vector x^{as} corresponding to the Jordan block J_{as}. It can be shown that $g_{\ell,\omega_j}^0(x) = \frac{1}{b!}\sum_{s\in\mathcal{L}_{a,\omega_j}^c}(C_{as})_{\ell k_s}(x_{2m_s-1}^{as}, x_{2m_s}^{as})^T$ and $h_{\ell,\omega_j}^0(x) = \frac{1}{b!}\sum_{s\in\mathcal{L}_{a,\omega_j}^c}(C_{as})_{\ell k_s}(x_{2m_s}^{as}, -x_{2m_s-1}^{as})^T$. Moreover, $\mathcal{L}_a^r \cup \mathcal{L}_a^c$ is nonempty by the definition of the principal mode.

We comment on $\sum_s(g_{\ell,\omega_s}^0(x)\cos(\omega_s t) + h_{\ell,\omega_s}^0(x)\sin(\omega_s t))$ more as follows. Let $p_{\ell,\omega_j}(t,x) \equiv g_{\ell,\omega_j}^0(x)\cos(\omega_j t) + h_{\ell,\omega_j}^0(x)\sin(\omega_j t)$. We thus obtain the collection of equivalent classes $E_{\omega_j} = \{p_{\ell,\omega_i}(t,x) \mid \omega_i/\omega_j$ is rational$\}$ (which is independent of x). Let $\widetilde{\omega}_s > 0$ be the basis frequency associated with each equivalent class E_{ω_j} and denote it by $E_{\widetilde{\omega}_s}$. Let $q_{\ell,\widetilde{\omega}_s}(t,x) \equiv \sum_{p_{\ell,\omega_i}\in E_{\widetilde{\omega}_s}} p_{\ell,\omega_i}(t,x)$. Then we obtain the similar properties for $q_{\ell,\widetilde{\omega}_s}(\cdot,x)$ as shown before Lemma 13. For example, for a fixed $x \in \mathbb{R}^n$, (i) $q_{\ell,\widetilde{\omega}_s}(\cdot,x)$ is a periodic function with the frequency $\widetilde{\omega}_s$; (ii) if $q_{\ell,\widetilde{\omega}_s}(\cdot,x)$ is not identically zero, then it attains the maximal and minimal values $\sigma_{\ell,\widetilde{\omega}_s}(x) > 0$ and $\nu_{\ell,\widetilde{\omega}_s}(x) < 0$ on $(-\infty,\infty)$, respectively; (iii) $q_{\ell,\widetilde{\omega}_s}(\cdot,x)$ is onto $[\nu_{\ell,\widetilde{\omega}_s}(x), \sigma_{\ell,\widetilde{\omega}_s}(x)]$; and (iv) the ratio of any two basis frequencies associated with distinct equivalent classes is irrational. Suppose that there are k equivalent classes $E_{\widetilde{\omega}_s}$, i.e., $s = 1,\ldots,k$. We have $\mu_\ell^0(t,x) = c_\ell^0(x) + \sum_{s=1}^k q_{\ell,\widetilde{\omega}_s}(t,x)$. Define $\varphi_\ell(x) \equiv [c_\ell^0(x) + \sum_{s=1}^k \nu_{\ell,\widetilde{\omega}_s}(x)]$. For a given x, we call $\varphi_\ell(x)$ the *principal coefficient* associated with the tuple $((C)_{\ell\bullet}, A, x)$. It can be shown, via the above properties (i)–(iv) and Lemma 13, that, for any x, $\varphi_\ell(x) = \inf_{[t_*,\infty)}\mu_\ell^0(\cdot,x)$ for any $t_* \geq 0$.

Lemma 14. *The function $\varphi_\ell(x)$ is Lipschitz continuous for each ℓ.*

Proof. For each ℓ, $\varphi_\ell(x) \equiv c_\ell^0(x) + \sum_{s=1}^k \nu_{\ell,\widetilde{\omega}_s}(x)$. Since $c_\ell^0(x)$ is linear and thus Lipschitz continuous, it is sufficient to show that each $\nu_{\ell,\widetilde{\omega}_s}(x)$ is Lipschitz continuous. Recall that $\nu_{\ell,\widetilde{\omega}_s}(x) = \min q_{\ell,\widetilde{\omega}_s}(\cdot,x)$, where $q_{\ell,\widetilde{\omega}_s}(t,x)$ is smooth and periodic in t for any fixed x. In addition, since $q_{\ell,\widetilde{\omega}_s}(t,x)$ is the summation of finitely many sinusoidal functions whose coefficient is linear in x, we have (i) for any $x, y \in \mathbb{R}^n$, $q_{\ell,\widetilde{\omega}_s}(t,x) = q_{\ell,\widetilde{\omega}_s}(t,y) + q_{\ell,\widetilde{\omega}_s}(t,x-y)$, and (ii) there exists $\rho > 0$ such that $|q_{\ell,\widetilde{\omega}_s}(t,x)| \leq \rho\|x\|, \forall t$. Notice that, for any $x, y \in \mathbb{R}^n$, $q_{\ell,\widetilde{\omega}_s}(t,x) \leq q_{\ell,\widetilde{\omega}_s}(t,y) + |q_{\ell,\widetilde{\omega}_s}(t,x-y)| \leq q_{\ell,\widetilde{\omega}_s}(t,y) + \rho\|x-y\|$, $\forall t \in \mathbb{R}$. Let t' be a minimum of $q_{\ell,\widetilde{\omega}_s}(\cdot,y)$. Therefore, $\min q_{\ell,\widetilde{\omega}_s}(\cdot,x) \leq q_{\ell,\widetilde{\omega}_s}(t',x) \leq q_{\ell,\widetilde{\omega}_s}(t',y) + \rho\|x - y\| = \min q_{\ell,\widetilde{\omega}_s}(\cdot,y) + \rho\|x-y\|$. This shows that $\nu_{\ell,\widetilde{\omega}_s}(x) \leq \nu_{\ell,\widetilde{\omega}_s}(y) + \rho\|x-y\|$. Similarly, $\nu_{\ell,\widetilde{\omega}_s}(y) \leq \nu_{\ell,\widetilde{\omega}_s}(x) + \rho\|x-y\|$. Consequently, $|\nu_{\ell,\widetilde{\omega}_s}(x) - \nu_{\ell,\widetilde{\omega}_s}(y)| \leq \rho\|x-y\|$. Hence, $\nu_{\ell,\widetilde{\omega}_s}$ is Lipschitz continuous. $\square$

The following theorem provides necessary and sufficient conditions for the interior of $\mathcal{A}$. Note that the two conditions given below are independent, i.e., they do not imply each other in general. See Example 11 for illustration.

Theorem 15. *Let $x^* \in \mathbb{R}^n$. Then $x^* \in \text{int}\,\mathcal{A}$ if and only if both the following conditions hold:* (a) $Ce^{At}x^* > 0, \forall t \geq 0$; *and* (b) *for each $\ell \in \{1,\ldots,m\}$, the principal coefficient associated with $((C)_{\ell\bullet}, A, x^*)$ is positive.*

Proof. "Necessity". We show this by contradiction. Suppose that $x^* \in \text{int}\mathcal{A}$ but (a) does not hold; then there exists $t_* \geq 0$ such that $(C)_{\ell\bullet}e^{At_*}x^* = 0$ for some ℓ. Since $(C)_{\ell\bullet} \neq 0$ (recall that C has no zero rows), we have $(C)_{\ell\bullet}e^{At_*}[x^* - \varepsilon e^{-At_*}((C)_{\ell\bullet})^T] = -\varepsilon(C)_{\ell\bullet}((C)_{\ell\bullet})^T < 0$ for all $\varepsilon > 0$. Hence, $x^* \notin \text{int}\,\mathcal{A}$, a contradiction. Now we assume that (b) does not hold. Then there is an ℓ such that the principal coefficient associated with $((C)_{\ell\bullet}, A, x^*)$ is non-positive, i.e., $\varphi_\ell(x^*) \equiv c_\ell^0(x^*) + \sum_{s=1}^k \nu_{\ell,\widetilde{\omega}_s}(x^*) \leq 0$. We consider a vector of the form $\widehat{x} = x^* + \varepsilon v$, where the scalar $\varepsilon > 0$, and the vector $v \neq 0$ is chosen in the following two cases:

(1) $\mathcal{L}_a^r$ is nonempty. Let $j \in \mathcal{L}_a^r$ and v^{aj} be a sub-vector corresponding to J_{aj} whose last element is chosen as $-(C_{aj})_{\ell k_j} \neq 0$, i.e., $v_{m_j}^{aj} = -(C_{aj})_{\ell k_j}$. Define $v = (0,\ldots,0,(v^{aj})^T, 0,\ldots,0)^T$. Note that this choice of $\widehat{x}$ does not affect $g_{\ell,\omega_j}^0(x)$ and $h_{\ell,\omega_j}^0(x)$ (see the expressions of these functions above), and hence does not change $\nu_{\ell,\widetilde{\omega}_s}$. Therefore, $\varphi_\ell(\widehat{x}) = c_\ell^0(\widehat{x}) + \sum_{s=1}^k \nu_{\ell,\widetilde{\omega}_s}(x^*)$. Moreover, since $c_\ell^0(x)$ is linear, we have $c_\ell^0(\widehat{x}) = c_\ell^0(x^*) - \frac{1}{b!}\varepsilon[(C_{aj})_{\ell k_j}]^2$. Thus the principal coefficient $\varphi_\ell(\widehat{x}) = \varphi_\ell(x^*) - \frac{1}{b!}\varepsilon[(C_{aj})_{\ell k_j}]^2 < 0$ for all $\varepsilon > 0$.

(2) $\mathcal{L}_a^r$ is empty but $\mathcal{L}_a^c$ is nonempty. Therefore $c_\ell^0(x) = 0$ for all x such that $\varphi_\ell(x) = \sum_{s=1}^k \nu_{\ell,\widetilde{\omega}_s}(x)$, where $\nu_{\ell,\widetilde{\omega}_s}(x) \leq 0, \forall s$. Consider two subcases: (2.1) $(g_{\ell,\omega_j}^0(x^*), h_{\ell,\omega_j}^0(x^*)) \neq (0,0)$ for some $j \in \mathcal{L}_a^c$, and (2.2) $(\widetilde{g}_{\ell,\omega_j}^0(x^*), \widetilde{h}_{\ell,\omega_j}^0(x^*)) = (0,0)$ for all $j \in \mathcal{L}_a^c$. In subcase (2.1), let $\omega_j \in E_{\widetilde{\omega}_s}$ for some s. Then $\nu_{\ell,\widetilde{\omega}_s}(x^*) < 0$ and thus $\varphi(x^*) < 0$. Let $v \neq 0$ be arbitrary. It follows from the continuity of φ_ℓ (see Lemma 14) that $\varphi(\widehat{x}) < 0$ for all $\varepsilon > 0$ sufficiently small. For subcase (2.2), $\nu_{\ell,\widetilde{\omega}_s}(x^*) = 0, \forall s$, and thus $\varphi(x^*) = 0$. Let $s \in \mathcal{L}_{\omega_j}^c$ for some ω_j and $v = (0,\ldots,0,(v^{as})^T, 0,\ldots,0)^T$, where v^{as} is the sub-vector corresponding to J_{as} whose last two elements satisfy $(v_{2m_s-1}^{as}, v_{2m_s}^{as}) = -(C_{as})_{\ell k_s} \in \mathbb{R}^{1\times 2}$. Since $(C_{as})_{\ell k_s} \neq 0$, it can be shown via the formulation of $g_{\ell,\omega_j}^0(x)$ and $h_{\ell,\omega_j}^0(x)$ that $g_{\ell,\omega_j}^0(\widehat{x}) = -\frac{1}{b!}\varepsilon\|(C_{as})_{\ell k_s}\|_2^2 < 0$ and $h_{\ell,\omega_j}^0(\widehat{x}) = 0$. This implies that, for all $\varepsilon > 0$, $\nu_{\ell,\widetilde{\omega}_{s'}}(\widehat{x}) < 0$ for some s'. Therefore $\varphi_\ell(\widehat{x}) < 0$ for all $\varepsilon > 0$.

Noting that $\varphi_\ell(\widehat{x}) < 0$ for all $\varepsilon > 0$ sufficiently small in both of the above cases, we see via Lemma 12 that, for any small

$\varepsilon > 0$ and any $t_* \geq 0$, there is $t' \in [t_*, \infty)$ such that $\mu^0_\ell(t', \widehat{x}) \leq \varphi_\ell(\widehat{x})/2 < 0$. Furthermore, since $C_{\ell\bullet} e^{At}\widehat{x}$ tends to $\mu^0_\ell(t, \widehat{x}) e^{\lambda_a t} t^b$ as $t \to +\infty$ (i.e., for any $\epsilon > 0$, there is $t_\epsilon \geq 0$ such that $|C_{\ell\bullet} e^{At}\widehat{x} - \mu^0_\ell(t, \widehat{x}) e^{\lambda_a t} t^b| \leq \epsilon, \forall t \geq t_\epsilon$), we deduce that, for any $\varepsilon > 0$ sufficiently small, $C_{\ell\bullet} e^{A\widehat{t}}\widehat{x} < 0$ for some large $\widehat{t} \geq 0$. Thus $x^* \notin \operatorname{int} \mathcal{A}$, a contradiction.

"Sufficiency". Consider an $\ell \in \{1, \ldots, m\}$. For each $\mu^k_\ell(t, x) \equiv c^k_\ell(x) + \sum_s g^k_{\ell,\omega_s}(x) \cos(\omega_s t) + h^k_{\ell,\omega_s}(x) \sin(\omega_s t)$ with $k \geq 1$, define $d^k_\ell(x) \equiv |c^k_\ell(x)| + \sum_s (|g^k_{\ell,\omega_s}(x)| + |h^k_{\ell,\omega_s}(x)|)$. Hence, $d^k_\ell(x)$ is continuous and provides an upper bound for $\mu^k_\ell(\cdot, x)$ on $\mathbb{R}$ for any given x, i.e., $d^k_\ell(x) \geq \max |\mu^k_\ell(\cdot, x)|$. In view of the continuity of φ_ℓ (see Lemma 14), we deduce that there exists a neighborhood $\mathcal{N}_1$ of x^* such that, for each ℓ, $d^k_\ell(x)$ is bounded on $\mathcal{N}_1$ for all k and that $\varphi_\ell(x) \geq \frac{2\varphi_\ell(x^*)}{3} > 0$ for all $x \in \mathcal{N}_1$. Note that, for each ℓ, $C_{\ell\bullet} e^{At} x \geq \varphi_\ell(x) e^{\lambda_a t} t^b - \sum_{k \geq 1} d^k_\ell(x) e^{\lambda_k t} t^{b_k}$ for all $t \geq 0$ and that $(\lambda_a, b) \succ (\lambda_k, b_k)$ for each k. Consequently, we obtain a scalar $t_* > 0$, via the bounds for $d^k_\ell(x)$ and $\varphi_\ell(x)$, such that, for each ℓ and all $x \in \mathcal{N}_1$, $C_{\ell\bullet} e^{At} x \geq \frac{\varphi_\ell(x^*)}{2} e^{\lambda_a t} t^b > 0, \forall t \geq t_*$. Furthermore, since $Ce^{At}x^* > 0$ on the compact time interval $[0, t_*]$, we claim that there exists a neighborhood $\mathcal{N}_2$ of x^* such that $Ce^{At}x > 0, \forall (t, x) \in [0, t_*] \times \mathcal{N}_2$. To see this, define $r_\ell(x) \equiv \min_{[0,t_*]} C_{\ell\bullet} e^{At} x$. It follows from the compactness of $[0, t_*]$ and the similar argument of Lemma 14 that each r_ℓ is Lipschitz continuous. Since $r_\ell(x^*) > 0$ for each ℓ, we deduce, via the continuity of r_ℓ, that there exists a neighborhood $\mathcal{N}_2$ of x^* such that $r_\ell(x) > 0, \forall x \in \mathcal{N}_2$ for each ℓ, or equivalently, $Ce^{At}x > 0, \forall (t, x) \in [0, t_*] \times \mathcal{N}_2$. Finally, letting $\mathcal{N} \equiv \mathcal{N}_1 \cap \mathcal{N}_2$, we have $Ce^{At}x > 0, \forall t \geq 0$ for any x in the neighborhood $\mathcal{N}$ of x^*. Therefore, $x^* \in \operatorname{int} \mathcal{A}$. $\square$

Note that the principal mode associated with (C, A) in Example 11 is e^{2t} and for the given $x^* = (1, 0)^T$, the principal coefficient associated with (C, A, x^*) is 0. Hence, $x^* \notin \operatorname{int} \mathcal{A}$. It can be shown that $x^* = (0, 1)^T \in \operatorname{int} \mathcal{A}$. The interior conditions established in Theorem 15 are important for understanding the connection between finite-time and long-time observability in Section 4; see Theorem 20 and Example 22 for more details.

4. Finite-time and long-time observability

Throughout this section, let $H \in \mathbb{R}^{r \times n}$ be a given matrix that defines the linear output Hx for the CLS (1). We recall some observability notions (Çamlibel et al., 2006) as follows. A state pair $(\xi, \eta) \in \mathbb{R}^{n+n}$ is called *short-time indistinguishable* if $\varepsilon > 0$ exists such that $Hx(t, \xi) = Hx(t, \eta)$ for all $t \in [0, \varepsilon]$. Similarly, the pair is called *T-time (resp. long-time) indistinguishable* for a given $T > 0$ (resp. $T = \infty$) if $Hx(t, \xi) = Hx(t, \eta)$ for all $t \in [0, T]$.

Definition 16. A state $\xi \in \mathbb{R}^n$ is called

- *short-time* (*resp. T-time/long-time*) *locally observable* if there exists a neighborhood $\mathcal{N}$ of ξ such that no pair (ξ, η) with $\eta \in \mathcal{N} \setminus \{\xi\}$ is short-time (resp. T-time/long-time) indistinguishable;
- *finite-time locally observable* if there exists a $T > 0$ such that ξ is T-time locally observable.

To unify the notation, we allow $T = \infty$. In such a case, T-time observability means long-time observability.

While short-time observability has been extensively studied in Çamlibel et al. (2006), much less is known about finite-time and long-time observability. In this section, we exploit the results of mode switching, directional derivative and positive invariance in the prior sections to establish concrete observability conditions. We begin with T-time local observability for a given finite $T > 0$.

A neat sufficient condition for a state ξ to be T-time locally observable is given in term of directional derivatives (Pang & Shen, 2007, Theorem 10):

$$\left[Hx'(t, \xi; \eta) = 0, \ \forall t \in [0, T]\right] \Longrightarrow \eta = 0. \tag{8}$$

It turns out that, if there is no critical time on $[0, T]$ along $x(t, \xi)$, then this condition is also necessary, as indicated in the following theorem. It is worth mentioning that although the nominal trajectory $x(t, \xi)$ has no critical time, a perturbed trajectory may have critical times and even mode switchings on $[0, T]$. This substantially complicates the observability analysis.

Theorem 17. *Given $\xi \in \mathbb{R}^n$ and $T > 0$. Suppose that there is no critical time on $[0, T]$ along $x(t, \xi)$. Then ξ is T-time locally observable if and only if the condition (8) holds.*

Proof. We prove the necessity only. In light of (b) of Theorem 9, we see that, for all $\tau > 0$ sufficiently small, $x(t, \xi + \tau\eta) = x(t, \xi) + \tau x'(t, \xi; \eta)$ on $[0, T]$. Suppose that $\eta \neq 0$ but $Hx'(t, \xi; \eta) = 0, \ \forall t \in [0, T]$. Then $Hx(t, \xi + \tau\eta) = Hx(t, \xi)$ on $[0, T]$ for all $\tau > 0$ sufficiently small. Noticing $\eta \neq 0$, we deduce that ξ is not T-time locally observable. This is a contradiction. $\square$

Theorem 17 completely characterizes T-time local observability of a state whose trajectory does not have critical times, which in turn rely on critical time and mode switching results from Section 2. Admittedly, the closed-form expression of directional derivatives is generally difficult to obtain. In the following, we provide more explicit sufficient and necessary conditions; the necessary condition even holds for $T = \infty$ and a nominal non-switching trajectory (possibly with critical times).

Proposition 18. *Consider a state $\xi \in \mathbb{R}^n$ such that $x(t, \xi)$ has no mode switching on $[0, T]$ with $0 < T \leq \infty$. Then ξ is T-time locally observable only if, $\forall i \in \mathcal{J}(\xi)$,*

$$\overline{O}(H, A_i) \cap \left\{v \mid C_i e^{A_i t} v \geq 0, \forall t \in [0, T]\right\} = \{0\}. \tag{9}$$

Moreover, suppose that $T > 0$ is finite and that $x(t, \xi)$ has no critical time on $[0, T]$. Then ξ is T-time locally observable if both the following conditions hold: $\forall i \in \mathcal{J}(\xi)$,

$$\overline{O}(H, A_i) \cap \left\{v \mid x(t, \xi + v) = e^{A_i t}(\xi + v), \forall t \in [0, T]\right\} = \{0\}; \tag{10}$$

and

$$\overline{O}(H, A_i) \cap \overline{O}(H, A_j) = \{0\}, \quad \forall i, j \in \mathcal{J}(\xi) \text{ with } A_i \neq A_j. \tag{11}$$

Furthermore, if the CLS is simple, then the condition (10) is equivalent to $\overline{O}(H, A_i) \cap \{v \mid C_i e^{A_i t}(\xi + v) \geq 0, \forall t \in [0, T]\} = \{0\}, \forall i \in \mathcal{J}(\xi)$.

Proof. Note that, if (9) fails, then there exist $i \in \mathcal{J}(\xi)$ and $v \neq 0$ such that $v \in \overline{O}(H, A_i)$ and $C_i e^{A_i t} v \geq 0$ on $[0, T]$. Consider the sequence $\{\eta^k\}$, where $\eta^k = \xi + v/k, \forall k \in \mathbb{N}$. Hence, $\{\eta^k\}$ converges to ξ with $\eta^k \neq \xi$ for all k. Further, it follows from Lemma 5 that $C_i e^{A_i t}\xi \geq 0, \forall t \in [0, T]$. Thus $C_i e^{A_i t}\eta^k = C_i e^{A_i t}[\xi + v/k] \geq 0, \forall t \in [0, T]$ for each k. Hence, $x(t, \eta^k) = e^{A_i t}\eta^k$ on $[0, T]$. However, each pair (ξ, η^k) is T-time indistinguishable as $Hx(t, \xi) = Hx(t, \eta^k)$ on $[0, T]$. This is contradictory to the T-time local observability at ξ.

We prove the second statement by contradiction. Suppose that ξ is not T-time locally observable. It follows from Theorem 17 that there exists a nonzero η such that $Hx'(t, \xi; \eta) = 0$ for all $t \in [0, T]$. Moreover, we deduce via Theorem 9 that there exist $0 = \widehat{t}_0 < \widehat{t}_1 < \cdots < \widehat{t}_{p-1} < \widehat{t}_p = T$ and matrices A_{k_i} with $k_i \in \mathcal{J}(\xi)$ such that, for all $t \in [\widehat{t}_i, \widehat{t}_{i+1}]$, $x'(t, \xi; \eta) = e^{A_{k_i}(t - \widehat{t}_i)} x'(\widehat{t}_i, \xi; \eta)$, $i = 0, \ldots, p-1$. Without loss of generality, we assume that the A_{k_i}'s associated with two neighboring intervals are distinct. We consider two cases as follows: (i) $p = 1$, and (ii) $p \geq 2$. For the first case, $x'(t, \xi; \eta) = e^{A_{k_1} t}\eta, \forall t \in [0, T]$. Hence, $He^{A_{k_1} t}\eta = 0, \forall t \in [0, T]$, which implies that $\eta \in \overline{O}(H, A_{k_1})$. Moreover, we have, via statement (b) of Theorem 9, $x(t, \xi + \tau\eta) = x(t, \xi) + \tau e^{A_{k_1} t}\eta = e^{A_{k_1} t}(\xi + \tau\eta)$ on $[0, T]$ for all $\tau > 0$ sufficiently small. This contradicts the condition (10). For the second case, it is known from Theorem 9 that $x'(t, \xi; \eta) = e^{A_{k_1} t}\eta, \forall t \in [0, \widehat{t}_1]$

and $x'(t, \xi; \eta) = e^{A_{k_2}(t-\widehat{t}_1)}x'(\widehat{t}_1, \xi; \eta), \forall t \in [\widehat{t}_1, \widehat{t}_2]$, where $k_1, k_2 \in \mathcal{J}(\xi)$ and $A_{k_1} \neq A_{k_2}$. Since $Hx'(t, \xi; \eta) = 0, \forall t \in [0, T]$, we have $x'(\widehat{t}_1, \xi; \eta) \in \overline{O}(H, A_{k_1}) \cap \overline{O}(H, A_{k_2})$. On the other hand, notice that $x'(\widehat{t}_1, \xi; \eta) \neq 0$ as $\eta \neq 0$. This is contradictory to (11). Finally, the equivalence of two conditions, under the assumption that the CLS is simple, follows from Lemma 3. □

By virtue of this result and Proposition 7, we combine the observability conditions for each subintervals defined by consecutive critical times to obtain the following corollary pertaining to an arbitrary nominal trajectory without further proof; the conditions obtained can be further simplified if the CLS is simple.

Corollary 19. *Consider a state $\xi \in \mathbb{R}^n$ and $T > 0$. Let $t_i \in [0, T]$, $i = 1, \ldots, p-1$ be the critical times such that $0 = t_0 < t_1 < \cdots < t_{p-1} < t_p = T$ and the interval (t_i, t_{i+1}) does not contain a critical time for each $i = 0, \ldots, p-1$. If, for some interval (t_j, t_{j+1}) with $j \in \{0, \ldots, p-1\}$, there exists a compact interval $[\widehat{t}_1, \widehat{t}_2] \subset (t_j, t_{j+1})$ such that $\overline{O}(H, A_i) \cap \{v \mid x(t, x^* + v) = e^{A_i t}(x^* + v), \ \forall t \in [0, \widehat{t}_2 - \widehat{t}_1]\} = \{0\}$, $\forall i \in \mathcal{J}(x^*)$ and $\overline{O}(H, A_i) \cap \overline{O}(H, A_j) = \{0\}$, $\forall i, j \in \mathcal{J}(x^*)$ with $A_i \neq A_j$, where $x^* \equiv x(\widehat{t}_1, \xi)$, then ξ is T-time locally observable.*

In the following, we establish subtle connections between finite-time and long-time observability. Especially we address the question of whether long-time observability implies finite-time observability, since the former is much more difficult to check in general. We focus on the case where a nominal trajectory eventually remains in a polyhedral cone of the CLS. It should be pointed out that a (locally) perturbed trajectory may not stay in the same polyhedral cone for a long time unless certain positive invariance conditions are imposed for the nominal trajectory (see Theorem 20). This is a major difficulty in large-time observability analysis and a motivation for employing positive invariance.

Theorem 20. *Given a state $\xi \in \mathbb{R}^n$, suppose that there exists $t_* \geq 0$ such that $x(t_*, \xi) \in \operatorname{int} \mathcal{A}_i$, where $\mathcal{A}_i$ is the positively invariant cone of the ith mode. Then the following are equivalent:* (a) *ξ is long-time locally observable;* (b) *ξ is finite-time locally observable;* (c) *ξ is t_*-time locally observable.*

Proof. The implications (c) $\Rightarrow$ (b) $\Rightarrow$ (a) are obvious. To prove the rest, it suffices to consider (a) $\Rightarrow$ (c) to be proved by contradiction as follows. Suppose that ξ is not t_*-time locally observable. Then, there exists a sequence $\{\eta^\nu\}$ with $\eta^\nu \neq \xi, \forall \nu \in \mathbb{N}$ such that $\{\eta^\nu\}$ converges to ξ and (ξ, η^ν) is t_*-time indistinguishable for all ν, i.e., $Hx(t, \xi) = H(x, \eta^\nu)$ for all $t \in [0, t_*]$. Since the CLS is globally Lipschitz, $\|x(t_*, \xi) - x(t_*, \eta^\nu)\| \leq e^{Lt_*}\|\xi - \eta^\nu\|$ for all ν, where $L > 0$ is the Lipschitz constant. This, together with the assumption that $x(t_*, \xi)$ is in the interior of $\mathcal{A}_i$, implies that there exist a neighborhood $\mathcal{U}$ of $x(t_*, \xi)$ and a subsequence $\{\widetilde{\eta}^\nu\}$ of $\{\eta^\nu\}$ such that $x(t_*, \widetilde{\eta}^\nu) \in \mathcal{U} \subseteq \operatorname{int} \mathcal{A}_i$ for all ν. Hence, $x(t, \widetilde{\eta}^\nu) \in \mathcal{X}_i$ for all $t \geq t_*$ and all ν due to positive invariance. It further follows from the continuity of the CLS trajectories that, for each pair $(\xi, \widetilde{\eta}^\nu)$, there exists $\varepsilon_\nu > 0$ such that $x(t, \xi) \in \mathcal{A}_i$ and $x(t, \widetilde{\eta}^\nu) \in \mathcal{A}_i, \forall t \in [t_* - \varepsilon_\nu, t_*]$. Since each pair $(\xi, \widetilde{\eta}^\nu)$ is t_*-time indistinguishable, $Hx(t, \xi) = Hx(t, \widetilde{\eta}^\nu), \forall t \in [t_* - \varepsilon_\nu, t_*]$. Note that $\mathcal{A}_i \subseteq \mathcal{X}_i$ and $x(t, \xi) = e^{A_i(t-t_*+\varepsilon_\nu)}x(t_* - \varepsilon_\nu, \xi)$ and $x(t, \widetilde{\eta}^\nu) = e^{A_i(t-t_*+\varepsilon_\nu)}x(t_* - \varepsilon_\nu, \widetilde{\eta}^\nu)$ for all $t \geq (t_* - \varepsilon_\nu)$. In view of $He^{A_i(t-t_*+\varepsilon_\nu)}x(t_* - \varepsilon_\nu, \xi) = He^{A_i(t-t_*+\varepsilon_\nu)}x(t_* - \varepsilon_\nu, \widetilde{\eta}^\nu)$ for all $t \in [t_* - \varepsilon_\nu, t_*]$, we deduce that $[x(t_* - \varepsilon_\nu, \widetilde{\eta}^\nu) - x(t_* - \varepsilon_\nu, \xi)] \in \overline{O}(H, A_i)$. Thus $[x(t_*, \widetilde{\eta}^\nu) - x(t_*, \xi)] \in \overline{O}(H, A_i)$ also holds such that $He^{A_i(t-t_*)}x(t_*, \xi) = He^{A_i(t-t_*)}x(t_*, \widetilde{\eta}^\nu)$ for all $t \geq t_*$, or equivalently $Hx(t, \xi) = Hx(t, \widetilde{\eta}^\nu)$ for all $t \geq t_*$. This thus shows that $Hx(t, \xi) = Hx(t, \widetilde{\eta}^\nu)$ for all $t \geq 0$. Consequently, the pair $(\xi, \widetilde{\eta}^\nu)$ is long-time indistinguishable for each ν. Therefore ξ is not long-time locally observable. This is a contradiction. Hence (a) $\Rightarrow$ (c) holds, and so does (a) $\Rightarrow$ (b). □

The interior condition in the above theorem can be verified using Theorem 15. While this condition seems strict, it is shown via Proposition 21 and Example 22 below that, if this condition fails, then long-time local observability may not give rise to finite-time observability, even if a nominal trajectory will eventually remain in the interior of a polyhedral cone. Indeed, it is revealed in Proposition 21 and Example 22 that, for a given state ξ and a time $t_* > 0$, even if $x(t_*, \xi)$ satisfies condition (a) of Theorem 15, the failure of condition (b) of Theorem 15 at $x(t_*, \xi)$ may lead to the non-equivalence between finite-time and long-time local observability of ξ. In other words, without the interior condition, a state may be long-time locally observable even though it is *not* T-time locally observable for any $T > 0$. To further elaborate on this, we need the following result.

Proposition 21. *Consider a state $\xi \in \mathbb{R}^n$. Suppose that there exist $t_* > 0$ and a cone $\mathcal{X}_i$ such that $x(t, \xi) \in \operatorname{int} \mathcal{X}_i$ for all $t \geq t_*$. Then ξ is finite-time locally observable if and only if either* (a) *ξ is t_*-time locally observable or* (b) *(H, A_i) is an observable pair.*

Proof. The "if" part is trivial under condition (a); we only need to consider condition (b). Since $x(t_*, \xi)$ is in the interior of $\mathcal{X}_i$, we deduce, using the global Lipschitz property of the CLS, that there exists a neighborhood $\mathcal{N}$ of ξ such that $x(t_*, x^0)$ is in the interior of $\mathcal{X}_i$ for all $x^0 \in \mathcal{N}$. This implies that $x(t, x^0) \in \mathcal{X}_i$ for all t sufficiently close to t_*. Hence, in view of (b), we conclude that $x(t_*, \xi)$ is small-time locally observable and, further, ξ is finite-time locally observable, following from Çamlibel et al. (2006, Proposition 4.11).

We prove the "only if" part by contraposition. Suppose that both (a) and (b) fail. As ξ is not t_*-time locally observable, it is obvious that ξ is not T-time locally observable for any $T \in (0, t_*]$. Moreover, the former implies that there exists a sequence $\{\eta^\nu\}$ converging to ξ such that, for each $\nu \in \mathbb{N}$, $\eta^\nu \neq \xi$ and $Hx(t, \xi) = Hx(t, \eta^\nu)$ for all $t \in [0, t_*]$. By appropriately restricting the sequence $\{\eta^\nu\}$, we may assume, without loss of generality, that $x(t_*, \eta^\nu) \in \mathcal{U} \subseteq \operatorname{int} \mathcal{X}_i$ for some neighborhood $\mathcal{U}$ of $x(t_*, \xi)$ and each ν. We further deduce, via an argument similar to Theorem 20, that $[x(t_*, \eta^\nu) - x(t_*, \xi)] \in \overline{O}(H, A_i)$. (Indeed, we only need to replace $\operatorname{int} \mathcal{A}_i$ by $\operatorname{int} \mathcal{X}_i$ in order to obtain this result.) Define $\tau_\nu \equiv \sup\{\tau \mid x(t, \eta^\nu) \in \mathcal{X}_i, \forall t \in [t_*, \tau)\}$. Notice that $\tau_\nu > t_*$ and $Hx(t, \xi) = Hx(t, \eta^\nu)$ on $[t_*, \tau_\nu]$. Since $x(t, \xi) \in \operatorname{int} \mathcal{X}_i$ for all $t \geq t_*$, $C_i x(t, \xi) > 0, \forall t \geq t_*$ (see Section 2). Moreover, for any $T > t_*$, let $\rho_T \equiv \min_{t \in [t_*, T], \ell \in \{1, \ldots, m_i\}} (C_i)_{\ell \bullet} x(t, \xi)$, where $(C_i)_{\ell \bullet}$ denotes the ℓth row of $C_i \in \mathbb{R}^{m_i \times n}$. Therefore, $\rho_T > 0$. It follows from $\|C_i(x(t, \eta^\nu) - x(t, \xi))\| \leq e^{L(T-t_*)}\|C_i\| \, \|x(t_*, \eta^\nu) - x(t_*, \xi)\| \leq e^{LT}\|C_i\| \, \|\eta^\nu - \xi\|, \forall t \in [t_*, T]$ that there exists $K_T \in \mathbb{N}$ such that, for all $\nu \geq K_T$, $|(C_i)_{\ell \bullet} x(t, \eta^\nu) - (C_i)_{\ell \bullet} x(t, \xi)| \leq \rho_T/2$, for all $t \in [t_*, T]$ and all $\ell \in \{1, \ldots, m_i\}$. Hence, $C_i x(t, \eta^\nu) > 0$ on $[t_*, T]$ for all $\nu \geq K_T$. This yields $x(t, \eta^\nu) \in \mathcal{X}_i$ on $[t_*, T]$ for each $\nu \geq K_T$. This thus implies that $\tau_\nu \geq T$ and $Hx(t, \eta^\nu) = Hx(t, \xi), \forall t \in [0, T]$ for each $\nu \geq K_T$. In other words, $\{\eta^\nu\}_{\nu=K_T}^{\infty}$ converges to ξ such that $\eta^\nu \neq \xi$ and the pair (η^ν, ξ) is T-time locally indistinguishable for each $\nu \geq K_T$. Hence, ξ is not T-time locally observable. Since $T \geq t_*$ is arbitrary, ξ is not finite-time locally observable. □

In what follows, we revisit the bimodal CLS that first appears in Çamlibel et al. (2006, Example 5.14) to illustrate the above observability results and positive invariance conditions. We emphasize the underlying positive invariance property neglected in the original example.

Example 22. Consider the bimodal CLS in $\mathbb{R}^3$ with $A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \alpha & \omega \\ 0 & -\omega & \alpha \end{bmatrix}$, $b = \begin{pmatrix} b_1 & 0 & 0 \end{pmatrix}^T$, $c = \begin{pmatrix} c_1 & c_2 & 0 \end{pmatrix}^T$, and $H = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$, where $\lambda < 0, \alpha > 0, \omega > 0, b_1 \neq 0, c_1 \neq 0, c_2 \neq 0$. Let $\xi = (\xi_1, 0, 0)^T$ with $c_1 \xi_1 < 0$. It is easy to verify that $c^T x(t, \xi) < 0, \forall t \geq 0$ such that $x(t, \xi)$ remains in the interior of $\mathcal{X}_2 \equiv \{x \mid c^T x \leq 0\}$ for all $t \geq 0$. The principal mode associated with $(-c^T, A)$ is $e^{\alpha t}$ and the principal coefficient associated with $(-c^T, A, \xi)$ is zero. Hence, we deduce via Theorem 15 that ξ is not in the interior of $\mathcal{A}_2$ associated with $\mathcal{X}_2$. The same holds true for $x(t, \xi)$ for any $t > 0$. Let $t_* > 0$ be given. Since $x(t, \xi) \in \text{int}\, \mathcal{X}_2$ on $[0, t_*]$ and (H, A) is not an observable pair, ξ is not t_*-time locally observable. It thus follows from Proposition 21 that ξ is not finite-time locally observable. While this observation is given via elementary computations in Çamlibel et al. (2006, Example 5.14), Proposition 21 generalizes it to a broader setting. More importantly, Proposition 21 unveils the critical positive invariance issue for the failure of finite-time observability. Another interesting observation is that ξ is long-time locally observable (Çamlibel et al., 2006, Example 5.14). Hence, this example shows that without the interior condition requested in Theorem 20, long-time and finite-time observability are generally not equivalent.

5. Conclusions

We have investigated finite-time observability via directional derivative techniques and addressed long-time observability from the positive invariance perspective. These perspectives yield new observability conditions for a general CLS. Nevertheless, there remain many open issues. For example, long-time observability of a state whose corresponding trajectory has infinitely many switchings is largely unknown. Moreover, a further study is warranted to understand positively invariant cones and the long-time dynamics of the CLS. Another interesting issue, different from the analytic perspective of the current paper, is how to design an observer for state estimation (namely, observer synthesis). Certain topological techniques may be invoked with suitable stability assumptions, e.g. Xiao (2006).

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