Math 600 Fall 2017 Homework #2

Due Sept. 25, Mon. in class

Note: For the Euclidean space $\mathbb{R}^n$, consider the usual metric induced by the Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^n$, unless otherwise stated.

1. Let A be a set in the metric space (M, d).

   (1) Show that x is an interior point of A if and only if there exists an ε-ball $B(x, \varepsilon)$ of x (with some $\varepsilon > 0$) such that $B(x, \varepsilon) \subseteq A$;

   (2) Show that the closure of A equals the intersection of all closed sets containing A. (*Hint*: use the DeMorgan's law, $\big(\text{int}A^c\big)^c = \text{cl}(A)$, and the fact that the interior of a set equals the union of all open sets contained in that set.)

2. Let (M, d) be a metric space, and A, B be two subsets of M. Show that (i) $\text{int}(A \cap B) = \text{int}A \cap \text{int}B$; and (ii) $\text{cl}(A \cap B) \subseteq (\text{cl}A) \cap (\text{cl}B)$.

3. Show that any set in the discrete metric space is open *and* closed.

4. Let (x_k) and (y_k) be two sequences in the metric space (M, d) that converge to $x \in M$ and $y \in M$, respectively. Show that the real sequence $\big(d(x_k, y_k)\big)$ converges to $d(x, y)$. (*Hint*: use the reverse triangle inequality of d.)

5. Let (M, d) be a metric space. Prove any two of the following statements (also think about the other but do not submit):

   (1) a convergent sequence in (M, d) is bounded and has a unique limit;

   (2) a convergent sequence in (M, d) is Cauchy;

   (3) a Cauchy sequence in (M, d) is bounded.

Miscellaneous practice problems: *Do not submit*

1. Does a set in a metric space with nonempty interior always have nonempty closure? Does a set in a metric space with nonempty closure always have nonempty interior? For each question, prove your answer if it is yes; otherwise, give a counterexample.

2. Let A be a set in (M, d). Show that $x \in \text{cl}(A)$ if and only if there exists a sequence in A that converges to x.

3. Find all the limit points of each of the following sets (without proof):

   (1) $A = \{x = (x_1, x_2) \in \mathbb{R}^2 : 1 < x_1 \leq 2, \text{ and } x_2 < -2\}$ in $\mathbb{R}^2$;

   (2) $B = \{x = (\frac{1}{n}, 1 - \frac{1}{n}) \in \mathbb{R}^2 : n \in \mathbb{N}\} \cup \{(1, 2)\}$ in $\mathbb{R}^2$;

   (3) $C = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 \text{ is rational, and } x_2 = \frac{1}{n}, n \in \mathbb{N}\}$ in $\mathbb{R}^2$.

4. Let $\mathbb{Q}$ denote the set of rational numbers, and $\mathbb{I}$ denote the set of irrational numbers. Let the set $\mathbb{Q} \times \mathbb{I} := \{(x_1, x_2) : x_1 \in \mathbb{Q},\ x_2 \in \mathbb{I}\} \subseteq \mathbb{R}^2$. Determine the interior, closure, and boundary of $\mathbb{Q} \times \mathbb{I}$ (without proof).

5. Consider the metric induced by the 1-norm on $\mathbb{R}^2$: $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$, $\forall\ x, y \in \mathbb{R}^2$. Let the set $A = \{x = (x_1, x_2) \in \mathbb{R}^2\ :\ x_1 \geq 1 \text{ and } x_2 \leq 1\}$.

   (1) Prove that the set A is closed via the definition, namely, the complement of A is open;

   (2) Prove that the set A is closed using the sequential criterion.