MATH 487 Fall 2013 Homework #3

Due Oct. 22, Tue in class

- Textbook, Section 3.6, p.102: 4, 5(a-b).
- Consider the sequence of functions (f_n) in $C^0(\mathbb{R}_+, \mathbb{R})$ with the sup-norm, where $\mathbb{R}_+ := [0, \infty)$:

  $$f_n(x) = \frac{\sin(nx)}{2 + nx}, \quad x \in \mathbb{R}_+, \quad n \in \mathbb{N}.$$

  Solve the following problems:

  - Find the limit function f_* on $\mathbb{R}_+$;
  - Show that (f_n) converges to f_* uniformly on the interval $[a, \infty)$ for any $a > 0$ but not uniformly on $\mathbb{R}_+$;
  - Use the definition of Cauchy sequences to show that (f_n) is Cauchy on the interval $[a, \infty)$ with $a > 0$.
- Consider the operator

  $$T(f)(t) = \sin(2\pi t) + \lambda \int_{-1}^{1} \frac{f(s)}{1 + (t - s)^2} ds$$

  on the function space $C^0([-1, 1], \mathbb{R})$ with the sup-norm.

  - Show if $f \in C^0([-1, 1], \mathbb{R})$, then $T(f) \in C^0([-1, 1], \mathbb{R})$;
  - Show that if $|\lambda| < \frac{2}{\pi}$, then $T(f)$ is a contractive mapping;
  - Show that if $|\lambda| > \frac{2}{\pi}$, then $T(f)$ is *not* contractive. (*Hint*: consider constant functions on $[-1, 1]$.)

  (*Hint*: You may use $\int \frac{1}{1+x^2} dx = \arctan(x) + C$.)