Math 302/600 Spring 2015 Homework #6

Due March 31, Tue. in class

Note: For any Euclidean space $\mathbb{R}^n$, consider the usual metric induced by the Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^n$, unless otherwise stated.

1. Use the definition of compactness (i.e. the open cover definition) to show that the union of two compact sets in a topological space is compact.

2. Use the definition of compactness (i.e. the open cover definition) to show that the following sets are *not* compact, by exhibiting an open cover with no finite sub-cover:

   (1) The open ball $B(x, 1)$ centered at a fixed $x \in M$ with the radius 1 in the metric space (M, d);

   (2) The set $A = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1,\ x_2 \geq 0\}$ in $\mathbb{R}^2$;

   (3) An infinite set in the metric space (M, d) with the discrete metric d.

3. Which of the following sets are connected in $\mathbb{R}$?

   $$\{3, -10\}, \quad [0, 1), \quad (-7, \infty), \quad \mathbb{Z}, \quad \mathbb{Q}, \quad \mathbb{R}.$$

   Here $\mathbb{Z}$ and $\mathbb{Q}$ are the sets of integers and rational numbers, respectively.

4. What are connected *and* compact sets in $\mathbb{R}$?

The following extra problem is for **Math 600** *students only:*

5. Let A be a compact set and B a closed subset of A in a topological space. Use the definition of compactness to show that B is compact.