Math 302/600 Spring 2017 Homework #11

Due May 2, Tue in class

Note: For any Euclidean space $\mathbb{R}^n$, consider the usual metric induced by the Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^n$, unless otherwise stated.

1. Let $f_n(x) = x^n/(1+x^n)$, and $A = [0, \infty)$.

   (1) Show that (f_n) converges pointwise on A, and find the limit function f_*;

   (2) Let $a \in (0, 1)$. Show that (f_n) converges uniformly on $[0, a]$ to f_*;

   (3) Show that (f_n) does not converge uniformly on $[0, 1]$ to f_*.

2. Let $f_n(x) = \sin(nx)/(1+nx)$, and $A = [0, \infty)$.

   (1) Show that (f_n) converges pointwise on A, and find the limit function f_*;

   (2) Let $a > 0$. Show that (f_n) converges uniformly on $[a, \infty)$ to f_*;

   (3) Show that (f_n) does not converge uniformly on A to f_*.

3. Let a sequence of *continuous* functions (f_n) converge pointwise to f_* on a *compact* set A. If f_* is *continuous* on A, does it imply that (f_n) always converges uniformly to f_* on A? If so, prove it; otherwise, give a counterexample. (*Hint*: consider the sequence (f_n) in Problem 2 on the interval $[0, 2\pi]$.)

4. Let $f_n : \mathbb{R} \to \mathbb{R}$ and $g_n : \mathbb{R} \to \mathbb{R}$ be two sequences of functions that converge uniformly on the set A to f_* and g_*, respectively. Show that $(f_n + g_n)$ converges uniformly on A to $f_* + g_*$.

5. Let $f_n(x) = x + 1/n,\ x \in \mathbb{R}$.

   (1) Show that (f_n) converges pointwise on $\mathbb{R}$, and find the limit function f_*;

   (2) Show that (f_n) converges uniformly to f_* on $\mathbb{R}$;

   (3) Show that (f_n^2) does *not* converge uniformly to f_*^2 on $\mathbb{R}$.

   $\star$ This problem shows that the product of uniformly convergent sequences of functions need *not* be uniformly convergent.

The following extra problem(s) are for **Math 600** *students only:*

6. Let $f_n : \mathbb{R} \to \mathbb{R}$ and $g_n : \mathbb{R} \to \mathbb{R}$ be two sequences of *bounded* functions that converge uniformly on the set A to f_* and g_*, respectively. Show that $(f_n \cdot g_n)$ converges uniformly to $f_* \cdot g_*$ on A.