Math 302/401/600 Fall 2010 Homework #8

Due Nov. 3, Wed. in class

1. Textbook, page 184, Section 4.2, 1. (Here use the standard Euclidean metric on $\mathbb{R}$.)

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the projection function defined by

$$f(x, y) := x, \qquad \forall\, (x, y) \in \mathbb{R}^2.$$

   (1) Show that f is continuous on $\mathbb{R}^2$.

   (2) Let $A, B \subseteq \mathbb{R}$ be such that $A \times B \subseteq \mathbb{R}^2$ is connected. Show that A is connected.

   (3) *The following extra problem is for* **Math 401/600** *students only:*
   Let $A, B \subseteq \mathbb{R}$ be such that $A \times B \subseteq \mathbb{R}^2$ is open. Is A open? If so, prove it; otherwise, find a counterexample.

   Use the standard Euclidean metric on each Euclidean space above.

3. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous on $\mathbb{R}^m$. Define the following function $g : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$:

$$g(x) := (x, f(x)), \qquad \forall\, x \in \mathbb{R}^n.$$

   (1) Show g is continuous on $\mathbb{R}^n$ (*Hint*: think of sequential criterion).

   (2) For the function f given above, define the following set (called the *graph* of f)

$$\mathcal{S} := \{(x, f(x)) \,:\, x \in \mathbb{R}^n\} \subseteq \mathbb{R}^n \times \mathbb{R}^m.$$

   Show that $\mathcal{S}$ is connected.

   Use the standard Euclidean metric on each Euclidean space above.

4. Let $f : [a, b] \to [a, b]$ be a continuous function, where the real numbers $a < b$. Show that f has a fixed point on $[a, b]$.