Math 650 Fall 2011 Homework #5

Due Nov. 9, Wed. in class

P.1 Let $f : \mathbb{R}^n \to \mathbb{R}$ be a G-differentiable function. Show that f is convex if and only if $\langle \nabla f(y) - \nabla f(x), y - x \rangle \geq 0, \ \forall\ x, y \in \mathbb{R}^n$. (*Hint*: consider the convexity condition $f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle$.)

P.2 Show that the following statements are equivalent:

(1) $f : \mathbb{R}^n \to \mathbb{R}$ is sublinear;

(2) $f : \mathbb{R}^n \to \mathbb{R}$ is convex and positively homogeneous;

(3) $f(\lambda x + \mu y) \leq \lambda f(x) + \mu f(y)$ for all $x, y \in \mathbb{R}^n$ and all $\lambda \geq 0, \mu \geq 0$. (*Hint*: for positive homogeneity, show that $f(x/\lambda) \leq f(x)/\lambda$ for any $x \in \mathbb{R}^n$ and $\lambda > 0$.)

P.3 Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. It has been shown that the one-sided directional derivative $f'_+(x; d)$ exists. Fix $x \in \mathbb{R}^n$, define $h : \mathbb{R}^n \to \mathbb{R}$ as $h(d) := f'_+(x; d)$, where $d \in \mathbb{R}^n$.

(1) Show that h is convex. (*Hint*: use $f(x+t(\lambda d^1+(1-\lambda)d^2)) = f(\lambda(x+td^1)+(1-\lambda)(x+td^2))$ for all $\lambda \in [0, 1]$, $t \in \mathbb{R}$, and $d^1, d^2 \in \mathbb{R}^n$.)

(2) Show that $f'_+(x; d)$ is sublinear in d. (You may assume the positive homogeneity of h proved in HW #1.)

P.4 For a given nonempty set $S \subseteq \mathbb{R}^n$, define its support function $\sigma_S : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ as $\sigma_S(x) := \sup\{\langle x, z \rangle \mid z \in S\}$.

(1) Show that σ_S is convex, sublinear, and lower semicontinuous.

(2) Show that if S is bounded, then σ_S is continuous. And give an example of a set S whose support function is *not* continuous on $\mathbb{R}^n$.

P.5 Given a function $f : \mathbb{R}^n \to \mathbb{R}$ and a vector $x^* \in \mathbb{R}^n$.

(1) Show that the subdifferential $\partial f(x^*)$ is a closed convex set provided that it is nonempty.

(2) Show that x^* is a global minimizer of f on $\mathbb{R}^n$ if and only if $0 \in \partial f(x^*)$.

P.6 Let $K \subseteq \mathbb{R}^n$ be a convex cone, and $f : \mathbb{R}^n \to \mathbb{R}$ be a G-differentiable function. Consider the convex optimization $\min_{x \in K} f(x)$, and let $x^* \in K$ be a minimizer.

(1) Show that if x^* is in the interior of K, then $\nabla f(x^*) = 0$.

(2) Show that $\nabla f(x^*) \in K^*$, where K^* is the dual cone of K.

(3) Show that $\nabla f(x^*) \perp x^*$.

(4) Show that the first order necessary condition (or the variational inequality) is equivalent to the condition (H): $K^* \ni \nabla f(x^*) \perp x^* \in K$.

(5) Find K^* and simplify the condition (H) for the following three cases: (i) $K = \mathbb{R}^n$; (ii) K is a subspace of $\mathbb{R}^n$ with dimension $1 \leq \dim(K) < n$; and (iii) $K = \mathbb{R}^n_+$.
(*Note*: these three cases correspond to unconstrained, equality constrained, and inequality constrained optimization problems, respectively.)