Math 302/600 Spring 2017 Homework #8

Due April 11, Tue in class

Note: For any Euclidean space $\mathbb{R}^n$, consider the usual metric induced by the Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^n$, unless otherwise stated.

1. Which of the following sets are connected in $\mathbb{R}$?

$$\{3, -10\}, \quad [0, 1), \quad (-7, \infty), \quad \mathbb{Z}, \quad \mathbb{Q}, \quad \mathbb{R}.$$

   Here $\mathbb{Z}$ and $\mathbb{Q}$ are the sets of integers and rational numbers, respectively.

2. What are connected *and* compact sets in $\mathbb{R}$?

3. Let (M, d) be a metric space. Fix $z \in M$, define $f : M \to \mathbb{R}$ as $f(x) := d(z, x), \forall\, x \in M$. Show that f is continuous on M.

4. Define $f : \mathbb{R}^2 \to \mathbb{R}$ as:

$$f(x_1, x_2) := \begin{cases} 0, & \text{if } x_1 \text{ is rational and } x_2 \text{ is irrational} \\ 1, & \text{otherwise} \end{cases}$$

   Show that f is discontinuous at each point of $\mathbb{R}^2$.

5. Let $f, g : (M, d) \to (V, \|\cdot\|)$ be two functions, where (M, d) is a metric space and $(V, \|\cdot\|)$ is a normed space.

   (1) Use the sequential criterion to show that if f and g are continuous at $x_0 \in M$, so is $f + g$;

   (2) Let λ be a scalar. Use the $\varepsilon - \delta$ definition to show that if f is continuous at $x_0 \in M$, so is λf.

6. Solve the following problems.

   (1) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be $f(x_1, x_2) = x_1, \forall\, (x_1, x_2) \in \mathbb{R}^2$. Show that f is continuous on $\mathbb{R}^2$.

   (2) Let A be an open set in $\mathbb{R}$, and $B = \{(x_1, x_2) : x_1 \in A\} \subseteq \mathbb{R}^2$. Use (1) to show that B is open in $\mathbb{R}^2$. (This result has been shown using the definition of open sets before.)

The following extra problem(s) are for **Math 600** *students only:*

7. Let $f : (M, d) \to (N, \rho)$ be continuous on M, and the set $A \subseteq M$. Show that $f(\mathrm{cl}(A)) \subseteq \mathrm{cl}(f(A))$. (*Hint*: use the sequential criteria.)