Math 302/401/600 Fall 2010 Homework #10

Due Nov. 22, Mon. in class

$\star$ Use the standard Euclidean metric for $\mathbb{R}$, and all x in the following are in $\mathbb{R}$.

1. Let $f_n(x) = \sin(nx)/(1 + nx)$, and $A = [0, \infty)$.

   (1.1) Show that (f_n) converges pointwise on A, and find the limiting function f_*;

   (1.2) Let $a > 0$. Show that (f_n) converges uniformly on $[a, \infty)$ to f_*;

   (1.3) Show that (f_n) does not converge uniformly on A to f_*.

2. Let $f_n(x) = x^n/(1 + x^n)$, and $A = [0, \infty)$.

   (2.1) Show that (f_n) converges pointwise on A, and find the limiting function f_*;

   (2.2) Let $a \in (0, 1)$. Show that (f_n) converges uniformly on $[0, a]$ to f_*;

   (2.3) Show that (f_n) does not converge uniformly on $[0, 1]$ to f_*.

3. Let $f_n : \mathbb{R} \to \mathbb{R}$ and $g_n : \mathbb{R} \to \mathbb{R}$ be two sequences of functions that converge uniformly on the set A to f_* and g_*, respectively. Show that $(f_n + g_n)$ converges uniformly on A to $f_* + g_*$.

4. Let $f_n(x) = x + 1/n,\ x \in \mathbb{R}$.

   (4.1) Show that (f_n) converges pointwise on $\mathbb{R}$, and find the limiting function f_*;

   (4.2) Show that (f_n) converges uniformly on $\mathbb{R}$ to f_*;

   (4.3) Show that (f_n^2) does *not* converge uniformly on $\mathbb{R}$ to f_*.

   $\star$ This example shows that the product of uniformly convergent sequences of functions need *not* be convergent uniformly.

5. Let $f_n : \mathbb{R} \to \mathbb{R}$ be such that (f_n) converges pointwise on the set A to f_*. Suppose f_* is bounded on A but each f_n is not. Show that (f_n) does not converge uniformly on A to f_*.
   (*Note*: even if both f_* and each f_n are bounded on A, we still cannot conclude the uniform convergence. Think of a counterexample but do not turn in.)

The following extra problem is for **Math 401/600** *students only:*

6. Let $f_n : \mathbb{R} \to \mathbb{R}$ and $g_n : \mathbb{R} \to \mathbb{R}$ be two sequences of *bounded* functions that converge uniformly on the set A to f_* and g_*, respectively. Show that $(f_n g_n)$ converges uniformly on A to $f_* g_*$.