Math 650 Fall 2016 Homework #5

Due Nov. 10, Thu in class

1. Let $x_1, \dots, x_k, x_{k+1}$ be $(k+1)$ vectors in $\mathbb{R}^n$. Suppose that x_{k+1} is a convex combination of $x_1, \dots, x_k$. Show that $\text{conv}\{x_1, \dots, x_k, x_{k+1}\} = \text{conv}\{x_1, \dots, x_k\}$.
   (*Note*: this result is an analogue to a well-known fact in linear algebra: if x_{k+1} is a linear combination of $x_1, \dots, x_k$, then $\text{span}\{x_1, \dots, x_k, x_{k+1}\} = \text{span}\{x_1, \dots, x_k\}$.)

2. Show that $f : \mathbb{R}^n \to \mathbb{R}$ is sublinear if and only if $f : \mathbb{R}^n \to \mathbb{R}$ is convex and positively homogeneous.

3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a G-differentiable function. Show that f is convex if and only if $\langle \nabla f(y) - \nabla f(x), y - x \rangle \geq 0$, $\forall\, x, y \in \mathbb{R}^n$. (*Hint*: use the convexity condition $f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle$. For the "if" part, consider the Taylor's formula.)

4. For a nonempty set $S \subseteq \mathbb{R}^n$, define its support function $\sigma_S : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ as $\sigma_S(x) := \sup\{\langle x, z \rangle \mid z \in S\}$.
   (1) Show that σ_S is convex, sublinear, and lower semicontinuous.
   (2) Show that if S is bounded, then σ_S is continuous. And give an example of a set S whose support function is *not* continuous on $\mathbb{R}^n$.

5. Given a function $f : \mathbb{R}^n \to \mathbb{R}$ and a vector $x^* \in \mathbb{R}^n$.
   (1) Show that the subdifferential $\partial f(x^*)$ is a closed, bounded (thus compact) and convex set provided that it is nonempty.
   (2) Show that x^* is a global minimizer of f on $\mathbb{R}^n$ if and only if $0 \in \partial f(x^*)$.
   (3) Show that the subdifferential of the absolute-value function $|\cdot|$ at $x = 0$ is $[-1, 1]$.

6. Let $K \subseteq \mathbb{R}^n$ be a convex cone, and $f : \mathbb{R}^n \to \mathbb{R}$ be a G-differentiable function. Consider the convex optimization $\min_{x \in K} f(x)$, and let $x^* \in K$ be a minimizer.
   (1) Show that if x^* is in the interior of K, then $\nabla f(x^*) = 0$.
   (2) Show that $\nabla f(x^*) \in K^*$, where K^* is the dual cone of K.
   (3) Show that $\nabla f(x^*) \perp x^*$ such that the first order necessary condition (or the variational inequality) is equivalent to the condition (H): $K^* \ni \nabla f(x^*) \perp x^* \in K$.
   (4) Find K^* and simplify the condition (H) for the following three cases: (i) $K = \mathbb{R}^n$; (ii) K is a subspace of $\mathbb{R}^n$ with dimension $1 \leq \dim(K) < n$; and (iii) $K = \mathbb{R}^n_+$.
   (*Note*: these three cases correspond to unconstrained, equality constrained, and inequality constrained optimization problems, respectively.)

More practice problems: *Do not submit*

1. Show that $f : \mathbb{R}^n \to \mathbb{R}$ is sublinear if and only if $f(\lambda x + \mu y) \leq \lambda f(x) + \mu f(y)$ for all $x, y \in \mathbb{R}^n$ and all $\lambda \geq 0, \mu \geq 0$. (*Hint*: for positive homogeneity, show that $f(0) = 0$ and $f(x/\lambda) \leq f(x)/\lambda$ for any $x \in \mathbb{R}^n$ and $\lambda > 0$.)

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be $f(x_1, x_2) := \max(x_1, x_2)$. Find the subdifferential of f at different $x = (x_1, x_2) \in \mathbb{R}^2$.