Math 302/600 Spring 2015 Homework #9

Due April 23, Thu. in class

Note: For any Euclidean space $\mathbb{R}^n$, consider the usual metric induced by the Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^n$, unless otherwise stated.

1. Show that the following sets in $\mathbb{R}^2$ are path connected:

   (1) $\{(x, \sin(x)) \,|\, x \geq 0\}$ (where you may assume that $\sin(\cdot)$ is continuous on $\mathbb{R}$);

   (2) $\{(x_1, x_2) \,|\, 0 < x_2 < 1\}$.

2. Let A be a path connected set in a metric space (M, d), and f be a continuous function on M. Show that $f(A)$ is path connected.

3. Solve the following problems.

   (1) Let f be a continuous function on a metric space (M, d) and A be a nonempty set in M. Show that if f is uniformly continuous on the closure of A, so is on A.

   (2) Let $g : \mathbb{R}^2 \to \mathbb{R}$ be continuous on $\mathbb{R}^2$. Let $(a, b]$ and (c, d) be two intervals in $\mathbb{R}$. Use (1) to show that g is uniformly continuous on $(a, b] \times (c, d)$.

4. Let $f_n(x) = \sin(nx)/(1 + nx)$, and $A = [0, \infty)$.

   (1) Show that (f_n) converges pointwise on A, and find the limit function f_*;

   (2) Let $a > 0$. Show that (f_n) converges uniformly on $[a, \infty)$ to f_*;

   (3) Show that (f_n) does not converge uniformly on A to f_*.

5. Let $f_n(x) = x^n/(1 + x^n)$, and $A = [0, \infty)$.

   (1) Show that (f_n) converges pointwise on A, and find the limit function f_*;

   (2) Let $a \in (0, 1)$. Show that (f_n) converges uniformly on $[0, a]$ to f_*;

   (3) Show that (f_n) does not converge uniformly on $[0, 1]$ to f_*.