Math 302 Fall 2011 Homework #9

Due Nov. 14, Mon. in class

Use the standard Euclidean metric on each Euclidean space below, unless otherwise specified.

1. Textbook, page 184, Section 4.2, 1.

2. Textbook, page 191, Section 4.4, 3.

3. Let $\|\cdot\|_2$ be the standard Euclidean norm on $\mathbb{R}^n$ and $\|\cdot\|_\alpha$ be an arbitrary norm on $\mathbb{R}^n$. Recall that both $\|\cdot\|_2$ and $\|\cdot\|_\alpha$ are continuous on $\mathbb{R}^n$.

   (1) Let $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n \, : \, \|x\|_2 = 1\}$. Show that $\mathbb{S}^{n-1}$ is closed. (*Hint*: convert $\mathbb{S}^{n-1}$ into the inverse image of a closed set in $\mathbb{R}$ under $\|\cdot\|_2$.)

   (2) Show that $\mathbb{S}^{n-1}$ is compact.

   (3) Show that there exist $x^* \in \mathbb{S}^{n-1}$ and $y^* \in \mathbb{S}^{n-1}$ such that

   $$\|x^*\|_\alpha = \max_{x \in \mathbb{S}^{n-1}} \|x\|_\alpha, \qquad \|y^*\|_\alpha = \min_{x \in \mathbb{S}^{n-1}} \|x\|_\alpha.$$

   Further show that $\|x^*\|_\alpha > 0$ and $\|y^*\|_\alpha > 0$.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the projection function defined by

   $$f(x, y) := x, \qquad \forall \, (x, y) \in \mathbb{R}^2.$$

   (1) Show that f is continuous on $\mathbb{R}^2$.

   (2) Let $A, B \subseteq \mathbb{R}$ be such that $A \times B \subseteq \mathbb{R}^2$ is connected. Show that A is connected.

5. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous on $\mathbb{R}^n$. Define the following function $g : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$:

   $$g(x) := (x, f(x)), \qquad \forall \, x \in \mathbb{R}^n.$$

   (1) Show g is continuous on $\mathbb{R}^n$ (*Hint*: think of sequential criterion).

   (2) For the function f given above, define the following set (called the *graph* of f)

   $$\mathcal{S} := \{(x, f(x)) \, : \, x \in \mathbb{R}^n\} \subseteq \mathbb{R}^n \times \mathbb{R}^m.$$

   Show that $\mathcal{S}$ is connected.