Math 302/600 Spring 2017 Homework #5

Due March 9, Thu. in class

Note: For any Euclidean space $\mathbb{R}^n$, consider the usual metric induced by the Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^n$, unless otherwise stated.

1. Let (x_k) be a Cauchy sequence in the metric space (M, d). Show that for any given $z \in M$, the real sequence $\big(d(x_k, z)\big)$ converges. (*Hint*: consider the reverse triangle inequality of d.)

2. Let A be a set in the metric space (M, d), and A' be the set of all accumulation points of A. Suppose a sequence in A' converges to $x^* \in M$. Show that there exists a sequence in A that converges to x^*.

3. Let (M, d) be a metric space, and A, B be two subsets of M. Show the following:

   (1) $\mathrm{cl}(A \cap B) \subseteq (\mathrm{cl}A) \cap (\mathrm{cl}B)$;

   (2) $(\mathrm{int}A)^c = \mathrm{cl}(A^c)$, where $(\cdot)^c$ denotes the complement of a set.

4. Determine which of the following sets is sequentially compact:

   (1) On $\mathbb{R}$: $A_1 = \{2/n : n \in \mathbb{N}\}$, $A_2 = \mathbb{Q} \cap [0, 1]$ (where $\mathbb{Q}$ is the set of rational numbers);

   (2) On $\mathbb{R}^2$: $A_3 = \mathbb{Q} \times \mathbb{Q}$, $A_4 = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| + x_2^2 \leq 4\}$.

   If a set is *not* sequentially compact, briefly explain why; otherwise, give a proof.

The following extra problem(s) are for **Math 600** *students only:*

5. Let (x_k) and (y_k) be two sequences in the metric space (M, d) that converge to $x \in M$ and $y \in M$ respectively. Show that the real sequence $\big(d(x_k, y_k)\big)$ converges to $d(x, y)$.

6. Show via an example that the sum of *infinitely* many convergent sequences in a normed space may not be convergent.