Math 302/600 Spring 2017 Homework #12

Due May 9, Tue in class

Note: For any Euclidean space $\mathbb{R}^n$, consider the usual metric induced by the Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^n$, unless otherwise stated.

1. Let the sequence (f_n) converge uniformly to f_* on the set $A \subseteq \mathbb{R}$, where $f_n : \mathbb{R} \to \mathbb{R}$. Suppose that each f_n is bounded on A, i.e., for each f_n, there exists $C_n > 0$ (depending on n) such that $|f_n(x)| \le C_n, \forall\, x \in A$. Show that f_* is bounded on A. Is this result still true if the assumption of uniform convergence is removed? If so, prove it; otherwise give a counterexample.

2. Construct a sequence of real-valued functions (f_n) on the interval $[0,1]$ such that each $f_n : [0,1] \to \mathbb{R}$ is discontinuous at every point of $[0,1]$ and (f_n) converges uniformly to a function that is continuous at every point of $[0,1]$.

3. Let $f_n(x) = (x^2 + n^4)^{-1}$, where $x \in \mathbb{R}$. Use the Weierstrass M-test to show the uniform convergence of the series $\sum_{n=1}^{\infty} f_n(x)$ on $\mathbb{R}$.

4. Let $f_n : [1,2] \to \mathbb{R}$ be given by $f_n(x) = \frac{x}{(x+1)^n}$.
   (1) Determine if $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent on $A = [1,2]$.
   (2) Determine if $\int_1^2 \left(\sum_{n=1}^{\infty} f_n(x)\right) dx = \sum_{n=1}^{\infty} \int_1^2 f_n(x)dx$.

5. Let $A = [-a, a] \subset \mathbb{R}$ with $a > 0$, and let
$$f_n(x) = \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!}, \quad x \in \mathbb{R}.$$
   (1) Use the Weierstrass M-test to show uniform convergence of the series $\sum_{n=1}^{\infty} f_n$ on A.
   (2) Let f_* be the limit function of the series on A, i.e., $f_*(x) = \sum_{n=1}^{\infty} f_n(x)$. Is f_* differentiable on $(-a, a)$? If so, is $f_*'(x) = \sum_{n=1}^{\infty} f_n'(x)$ on $(-a, a)$? Prove your answers.

The following extra problem(s) are for **Math 600** *students only:*

6. Let each $f_n : \mathbb{R} \to \mathbb{R}$ be
$$f_n(x) = \frac{(-1)^{n+1} x}{n}.$$
   Let A be a bounded set in $\mathbb{R}$. Show that the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on A. (*Hint*: use the Cauchy criterion.)

   $\star$ This problem shows that the Weierstrass M-test is sufficient for uniform convergence but not necessary.

7. Suppose that each $f_n : \mathbb{R} \to \mathbb{R}$ is continuous on the set A, and (f_n) converges to f_* uniformly on A. Let (x_n) in A converge to $x_* \in A$. Show that $(f_n(x_n))$ converges to $f_*(x_*)$.