Math 302 Fall 2011 Homework #12

Due Dec. 5, Mon. in class

⋆ *Use the standard Euclidean metric on $\mathbb{R}$, and all x below are in $\mathbb{R}$ unless otherwise indicated.*

1. Textbook, page 317, Exercises 4, 6, 8. (*Hint for Ex. 8*: consider the sequence (f_n) in Problem 1 of HW #11 on the compact set $[0, 2\pi]$.)

2. Let $f_n(x) = (x^2 + n^4)^{-1}$, where $x \in \mathbb{R}$. Use Weierstrass M-test to show uniform convergence of the series $\sum_{n=1}^{\infty} f_n$ on $\mathbb{R}$.

3. Let
$$f_n(x) = \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!}, \quad x \in \mathbb{R}.$$

   (3.1) Let $A = [-a, a]$ with $a > 0$. Use Weierstrass M-test to show uniform convergence of the series $\sum_{n=1}^{\infty} f_n$ on A.

   (3.2) Let f_* be the limiting function of the series on A, i.e., $f_*(x) = \sum_{n=1}^{\infty} f_n(x)$. Is f_* differentiable on $(-a, a)$? If so, is $f_*'(x) = \sum_{n=1}^{\infty} f_n'(x)$ on $(-a, a)$? Prove your answers.

4. Let $f_n : \mathbb{R} \to \mathbb{R}$ be such that the sequence (f_n) converges uniformly on the set A to f_*. Suppose that each f_n is bounded on A, i.e., for each f_n, there exists $M_n > 0$ (dependent on f_n) such that $|f_n(x)| \leq M_n, \forall\, x \in A$. Show that f_* is bounded on A.