

# Math 600 Fall 2015 Homework #1

Due Sept. 15, Tue. in class

1. Recall that  $C([0, 1])$  is the vector space of all continuous functions  $f : [0, 1] \rightarrow \mathbb{R}$  on the interval  $[0, 1]$ . For any  $f \in C([0, 1])$ , define

$$\|f\|_{\infty} := \max \left\{ |f(t)| : t \in [0, 1] \right\}.$$

- (1) Explain why  $\|f\|_{\infty}$  exists for any  $f \in C([0, 1])$ . (*Hint:* think of Theorem 5.3.4 of Bartle and Sherbert's book.)
- (2) Prove that  $\|\cdot\|_{\infty}$  is a norm on  $C([0, 1])$ .
2. Let  $(M, d)$  be a metric space. Show the following inequality (i.e., the reverse triangle inequality):

$$|d(x, y) - d(z, y)| \leq d(x, z), \quad \forall x, y, z \in M.$$

3. Let  $d_1$  and  $d_2$  be two metrics on a set  $M$ . Define the sum  $d_1 + d_2$  on  $M$  as

$$(d_1 + d_2)(x, y) := d_1(x, y) + d_2(x, y), \quad \forall x, y \in M.$$

Show that  $d_1 + d_2$  is a metric on  $M$ .

4. Let  $(M, d)$  be a metric space. Define

$$\rho(x, y) := \frac{d(x, y)}{1 + d(x, y)}, \quad \forall x, y \in M.$$

Show that  $\rho$  is a metric on  $M$ . (*Hint:* observe that  $t/(1+t)$  is an increasing function on  $\mathbb{R}_+$ .)

## Miscellaneous practice problems: Do not submit

1. In an inner product space  $V$  with the induced norm  $\|\cdot\|$ , show that for any  $x, y \in V$ ,

$$(1) \|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2);$$

$$(2) \|x + y\| \cdot \|x - y\| \leq \|x\|^2 + \|y\|^2.$$

2. On the Euclidean space  $\mathbb{R}^n$ , define for each  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ ,

$$\|x\|_{\star} := c_1|x_1| + c_2|x_2| + \dots + c_n|x_n|,$$

where  $c_i$  is a positive real number for each  $i = 1, \dots, n$ . Show that  $\|\cdot\|_{\star}$  is a norm on  $\mathbb{R}^n$ .

3. On the Euclidean space  $\mathbb{R}^n$ , define for each  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ ,

$$\|x\|_{\infty} := \max(|x_1|, |x_2|, \dots, |x_n|)$$

Show that  $\|\cdot\|_{\infty}$  is a norm on  $\mathbb{R}^n$ .