Math 600 Fall 2017 Homework #3

Due Oct. 9, Mon. in class

Note: For the Euclidean space $\mathbb{R}^n$, consider the usual metric induced by the Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^n$, unless otherwise stated.

1. Let (x_n) be a Cauchy sequence in the metric space (M, d). Show that for any given $z \in M$, the real sequence $\big(d(x_n, z)\big)$ converges.

2. Let (M, d) be the *discrete* metric space. A sequence (x_n) in (M, d) is said to have a *constant tail* if there exist $K \in \mathbb{N}$ and $c \in M$ such that $x_n = c$, $\forall\, n \geq K$. Prove the following:

   (1) A sequence in (M, d) is convergent if and only if it has a constant tail;

   (2) A Cauchy sequence in (M, d) has a constant tail;

   (3) Show that (M, d) is complete using (1)-(2).

3. Let A be a nonempty set in the metric space (M, d). Show that A is sequentially compact if and only if any infinite subset of A (namely, a subset having infinitely many points) has a limit point that belongs to A.

4. Let (M, d) be a metric space such that M is sequentially compact. Show that (M, d) is complete.

5. (1) Use the sequential argument to show that the union of two sequentially compact sets in a metric space is sequentially compact.

   (2) Use the open cover definition to show that the intersection of two compact sets in a metric space is compact. (*Hint*: note that the intersection of two compact sets is closed (why?).)

Miscellaneous practice problems: *Do not submit*

1. Let (x^k) be a sequence in $\mathbb{R}^2$, where for each k,

   $$x^k = (x_1^k, x_2^k), \qquad \text{with} \qquad x_1^k = (-1)^k + 1/k, \quad x_2^k = \exp(\cos(k)).$$

   Show that (x^k) is divergent. Does (x^k) have a convergent subsequence? Explain why.

2. Determine which of the following sets is sequentially compact:

   (1) On $\mathbb{R}$: $A_1 = \{2/n : n \in \mathbb{N}\}$, $A_2 = \mathbb{Q} \cap [0, 1]$ (where $\mathbb{Q}$ is the set of rational numbers);

   (2) On $\mathbb{R}^2$: $A_3 = \mathbb{Q} \times \mathbb{Q}$, $A_4 = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| + x_2^2 \leq 4\}$.

   If a set is *not* (sequentially) compact, briefly explain why; otherwise, give a proof.

3. Use the definition of compactness (i.e., the open cover definition) to show that the following sets are *not* compact, by exhibiting an open cover with no finite sub-cover:

   (1) The open ball $B(x, 1/2)$ centered at a fixed $x \in \mathbb{R}^n$ with the radius $1/2$ in the Euclidean space $\mathbb{R}^n$;

   (2) The set $A = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1,\, x_2 \geq 0\}$ in $\mathbb{R}^2$;

   (3) An infinite set in the metric space (M, d) with the discrete metric d.