Math 302/600 Spring 2015 Homework #10

Due April 30, Thu. in class

Note: For any Euclidean space $\mathbb{R}^n$, consider the usual metric induced by the Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^n$, unless otherwise stated.

1. Let $f_n : \mathbb{R} \to \mathbb{R}$ and $g_n : \mathbb{R} \to \mathbb{R}$ be two sequences of functions that converge uniformly on the set A to f_* and g_*, respectively. Show that $(f_n + g_n)$ converges uniformly on A to $f_* + g_*$.

2. Let a sequence of *continuous* functions (f_n) converge pointwise to f_* on a *compact* set A. If f_* is *continuous* on A, does this imply that (f_n) always converges uniformly to f_* on A? If so, prove it; otherwise, give a counterexample. (*Hint*: consider the sequence (f_n) in Problem 4 of HW#9 on the interval $[0, 2\pi]$.)

3. Let $f_n(x) = x + 1/n,\ x \in \mathbb{R}$.

   (1) Show that (f_n) converges pointwise on $\mathbb{R}$, and find the limit function f_*;

   (2) Show that (f_n) converges uniformly to f_* on $\mathbb{R}$;

   (3) Show that (f_n^2) does *not* converge uniformly to f_*^2 on $\mathbb{R}$.

   $\star$ This problem shows that the product of uniformly convergent sequences of functions need *not* be uniformly convergent.

4. Let the sequence (f_n) converge uniformly to f_* on the set $A \subseteq \mathbb{R}$, where $f_n : \mathbb{R} \to \mathbb{R}$. Suppose that each f_n is bounded on A, i.e., for each f_n, there exists $M_n > 0$ (dependent on f_n) such that $|f_n(x)| \leq M_n, \forall\, x \in A$. Show that f_* is bounded on A.

5. Let $f_n(x) = (x^2 + n^4)^{-1}$, where $x \in \mathbb{R}$. Use the Weierstrass M-test to show uniform convergence of the series $\sum_{n=1}^{\infty} f_n$ on $\mathbb{R}$.

6. Construct a sequence of functions on $[0, 1]$ each of which is discontinuous at every point of $[0, 1]$ and which converges uniformly to a function that is continuous at every point of $[0, 1]$.

The following extra problems are for **Math 600** *students only:*

7. Let $f_n : \mathbb{R} \to \mathbb{R}$ and $g_n : \mathbb{R} \to \mathbb{R}$ be two sequences of *bounded* functions that converge uniformly on the set A to f_* and g_*, respectively. Show that $(f_n g_n)$ converges uniformly to $f_* g_*$ on A.