Math 302/401/600 Fall 2010 Homework #3

Due Sept. 27, Mon. in class

1. Consider $\mathbb{R}^2$ and the metric induced by the 1-norm: $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$, $\forall\, x, y \in \mathbb{R}^2$. Let the set $A = \{x = (x_1, x_2) \in \mathbb{R}^2 \,:\, x_1 \geq 1 \text{ and } x_2 \leq 1\}$.

   (1) Prove that the set A is closed via the definition, namely, by showing that the complement of A is open;

   (2) Prove that the set A is closed using sequential criterion.

2. Let (M, d) be a metric space and A, B be two subsets of M. Show the following:

   (1) $\mathrm{cl}(A \cup B) = (\mathrm{cl}A) \cup (\mathrm{cl}B)$;

   (2) $\mathrm{cl}(A \cap B) \subseteq (\mathrm{cl}A) \cap (\mathrm{cl}B)$.

3. Let (M, d) be a metric space and $A \subseteq M$. Show that:

$$M \setminus \mathrm{int}A \;=\; \mathrm{cl}(M \setminus A)$$

4. Let $A \subset \mathbb{R}$ be the set of all irrational numbers. Using the standard metric (induced by the absolute value), find the closure of A and boundary of A.

5. Given a metric space (M, d). Prove the following statements:

   (1) In (M, d), any convergent sequence has a unique limit;

   (2) In (M, d), any convergent sequence is Cauchy;

   (3) In (M, d), any Cauchy sequence is bounded.

The following extra problem is for **Math 401/600** *students only:*

6. Let A be a nonempty set in $\mathbb{R}^n$ and d be the standard Euclidean norm on $\mathbb{R}^n$. Let $z \in \mathbb{R}^n$ be given.

   (1) Show that the infimum of the *real* set $\{d(z, x) : x \in A\}$ exists. In the following, define $d(z, A) := \inf\{d(z, x) : x \in A\}$.

   (2) Show that there exists a sequence $\{x_k\}$ in A such that the *real* sequence $\{d(z, x_k)\}$ converges to $d(z, A)$. Further, show that $\{x_k\}$ converges in $\mathbb{R}^n$.

   (3) Use (2) to show that if the set A is closed, then there exists $x^* \in A$ such that $d(z, A) = d(z, x^*)$.