Math 600 Fall 2015 Homework #4

Due Oct. 22, Thu. in class

Note: For the Euclidean space $\mathbb{R}^n$, consider the usual metric induced by the Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^n$, unless otherwise stated.

1. What are connected *and* compact sets in $\mathbb{R}$?
2. Let (M,d) be a metric space. Fix $z \in M$, define $f : M \to \mathbb{R}$ by $f(x) := d(z,x)$. Show that f is continuous on M.
3. Define $f : \mathbb{R}^2 \to \mathbb{R}$ as:

$$f(x_1,x_2) := \begin{cases} 0, & \text{if } x_1 \text{ is rational and } x_2 \text{ is irrational} \\ 1, & \text{otherwise} \end{cases}$$

   Show that f is discontinuous at any point of $\mathbb{R}^2$.

4. Solve the following problems.
   (1) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be $f(x_1,x_2) = x_1$. Show that f is continuous on $\mathbb{R}^2$.
   (2) Let A be an open set in $\mathbb{R}$, and $B = \{(x_1,x_2) : x_1 \in A\} \subseteq \mathbb{R}^2$. Use (1) to show that B is open in $\mathbb{R}^2$.
5. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$. Which of the following sets are necessarily open, closed, compact, and connected (no proof is needed)?

$$(1)\ \{x \in \mathbb{R} \mid f(x) = 0\}; \qquad (2)\ \{x \in \mathbb{R} \mid f(x) > 1\}.$$

6. Let $f : (M,d) \to (N,\rho)$ be continuous on M, and $B \subseteq M$. Show that $f(\mathrm{cl}(B)) \subseteq \mathrm{cl}(f(B))$. (*Hint*: use the sequential criteria.)

Miscellaneous practice problems: *Do not submit*

1. Which of the following sets are connected in $\mathbb{R}$?

$$\{3,-10\}, \quad [0,1), \quad (-7,\infty), \quad \mathbb{Z}, \quad \mathbb{Q}, \quad \mathbb{R}.$$

   Here $\mathbb{Z}$ and $\mathbb{Q}$ are the sets of integers and rational numbers, respectively.

2. Let $f,g : (M,d) \to (V,\|\cdot\|)$ be two functions, where (M,d) is a metric space and $(V,\|\cdot\|)$ is a normed space.
   (1) Use the sequential criterion to show that if f and g are continuous at $x_0 \in M$, so is $f+g$;
   (2) Let λ be a scalar. Use the $\varepsilon-\delta$ definition to show that if f is continuous at $x_0 \in M$, so is λf.