Math 650 Fall 2016 Homework #7

Due Dec. 13, Tue in class

1. In what follows, we prove Farkas' Lemma using the separation argument. Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, and let the finitely generated cone $C = \{Ax : x \geq 0\}$. Use the separation argument to show that if $b \notin C$, then there exists $0 \neq y \in \mathbb{R}^m$ such that $A^T y \leq 0$ and $b^T y > 0$.

2. Let $P := \{x \in \mathbb{R}^n \mid Ax \geq b\}$ be a nonempty polyhedron for a matrix A and a vector b.

   (1) Show that a nonempty intersection of P and an affine set in $\mathbb{R}^n$ is a polyhedron.

   (2) Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be an affine transformation defined by $F(x) := Tx + c$, where $T \in \mathbb{R}^{m \times n}$ and $c \in \mathbb{R}^m$. Show that the set $F(P) \subseteq \mathbb{R}^m$ is polyhedral (namely, any affine transformation of P is polyhedral). (*Hint*: use the Minkowski-Weyl Theorem.)

3. Let C_1, C_2 be two convex polyhedral cones in $\mathbb{R}^n$. Show that $C_1 + C_2$ ia also a convex polyhedral cone.

4. Let $P := \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$ be a nonempty convex polyhedron. Show that P is bounded (i.e., it is a polytope) if and only if the linear inequality $Ax = 0,\ x \geq 0$ has the trivial solution $x = 0$ only.

5. Let the polyhedral cone $C = \{x \in \mathbb{R}^n : Ax = 0,\ x \geq 0\}$ for some matrix $A \in \mathbb{R}^{m \times n}$. Show that its dual cone $C^* = \{A^T u + v : u \in \mathbb{R}^m,\ v \in \mathbb{R}^n_+\}$. (*Hint*: convert C to the standard form.)

6. Let
$$A = \begin{bmatrix} 1 & 2 \\ -1 & 2 \\ -1 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}.$$
Show that the inequality system $A^T y \leq 0,\ b^T y > 0$ has a solution. (*Hint*: use Farkas' Lemma.)

7. Let $A \in \mathbb{R}^{m \times n}$. Show that exactly one of the following inequality systems has a solution:
$$\mathbf{I}:\ Ax \leq 0,\ x \geq 0,\ \sum_{i=1}^{n} x_i = 1; \qquad \text{and} \qquad \mathbf{II}:\ A^T y > 0,\ y \geq 0,\ \sum_{i=1}^{m} y_i = 1$$

More practice problems: *Do not submit*

1. Let C be a convex cone in $\mathbb{R}^n$ (containing the zero vector), and let $\mathcal{V} := C \cap (-C)$.

   (1) Show that $\mathcal{V}$ is a subspace of $\mathbb{R}^n$.

   (2) A cone K is called *pointed* if the condition that $x_1 + \cdots + x_k = 0$ with $x_i \in K$, $i = 1, \ldots, k$ implies $x_i = 0$ for all i. Show that the convex cone C is pointed if and only if $\mathcal{V} = \{0\}$. (*Hint*: recall that if $x, y \in C$, then $x + y \in C$.)

   (3) Let C be a polyhedral cone given by $C = \{x : Ax \geq 0\}$. Show that C is pointed if and only if the null space of A is trivial, i.e., $N(A) = \{0\}$.

   (4) Let $\mathcal{K} := C \cap \mathcal{V}^\perp$. Show that $\mathcal{K}$ is a pointed convex cone and $C = \mathcal{K} + \mathcal{V}$ with $\mathcal{K} \perp \mathcal{V}$.

   (5) Suppose the set S has nonempty interior. Show that its dual cone S^* is pointed.