Math 302/401/600 Fall 2010 Homework #1

Due Sept. 15, Wed. in class

1. Textbook, page 98, Ex. 12 (a), (b).

2. On the Euclidean space $\mathbb{R}^n$, define for each $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$,

$$\|x\|_\star := c_1|x_1| + c_2|x_2| + \cdots + c_n|x_n|,$$

where $c_i, i = 1, \cdots, n$ are fixed positive real numbers. Show that $\|\cdot\|_\star$ is a norm on $\mathbb{R}^n$. Use this result to deduce that

$$\|x\|_1 := |x_1| + |x_2| + \cdots + |x_n|$$

is a norm on $\mathbb{R}^n$.

3. Recall that $C([0,1])$ is the vector space of all continuous functions $f : [0,1] \to \mathbb{R}$ on the interval $[0,1]$. For any $f \in C([0,1])$, define

$$\|f\| := \max\{|f(t)| : t \in [0,1]\}$$

   (1) Explain why $\|f\|$ exists for any $f \in C([0,1])$. (*hint*: think of Theorem 5.3.4 of Bartle and Sherbert's book.)

   (2) Prove that $\|\cdot\|$ is a norm on $C([0,1])$.

4. Prove that the discrete metric d discussed in class satisfies the triangle inequality, i.e. $d(x,y) \leq d(x,z) + d(y,z), \forall\, x,y,z \in M$.

5. Let d_1 and d_2 be two metrics on a set M. Define the sum $d_1 + d_2$ on M as

$$(d_1 + d_2)(x,y) := d_1(x,y) + d_2(x,y), \quad \forall\ x,y \in M$$

Show that $d_1 + d_2$ is a metric on M.

The following extra problems are for **Math 401/600** *students only:*

6. On the Euclidean space $\mathbb{R}^n$, define for each $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$,

$$\|x\|_\infty := \max(|x_1|, |x_2|, \cdots, |x_n|)$$

Show that $\|\cdot\|_\infty$ is a norm on $\mathbb{R}^n$.

7. Let (M,d) be a metric space. Define

$$\rho(x,y) := \frac{d(x,y)}{1 + d(x,y)}, \quad \forall\ x,y \in M.$$

Show that ρ is a metric on M. (*hint*: observe that $t/(1+t)$ is an increasing function on $\mathbb{R}_+$.)