Strongly Regular Differential Variational Systems

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***Abstract*—A differential variational system is defined by an ordinary differential equation (ODE) parameterized by an algebraic variable that is required to be a solution of a finite-dimensional variational inequality containing the state variable of the system. This paper addresses two system-theoretic topics for such a nontraditional nonsmooth dynamical system; namely, (non-)Zenoness and local observability of a given state satisfying a blanket strong regularity condition. For the former topic, which is of contemporary interest in the study of hybrid systems, we extend the results in our previous paper, where we have studied Zeno states and switching times in a linear complementarity system (LCS). As a special case of the differential variational inequality (DVI), the LCS consists of a linear, time-invariant ODE and a linear complementarity problem. The extension to a nonlinear complementarity system (NCS) with analytic inputs turns out to be non-trivial as we need to use the Lie derivatives of analytic functions in order to arrive at an expansion of the solution trajectory near a given state. Further extension to a differential variational inequality is obtained via its equivalent Karush–Kuhn–Tucker formulation. For the second topic, which is classical in system theory, we use the non-Zenoness result and the recent results in a previous paper pertaining to the B-differentiability of the solution operator of a nonsmooth ODE to obtain a sufficient condition for the short-time local observability of a given strongly regular state of an NCS. Refined sufficient conditions and necessary conditions for local observability of the LCS satisfying the P-property are obtained.**

***Index Terms*—Complementarity systems, differential variational inequalities (DVIs), hybrid systems, observability, Zeno behavior.**

I. Introduction

THE past few years have witnessed a surge in the interest in differential variational systems, which provide a novel mathematical paradigm for the unified study of a wide range of nonsmooth dynamical systems that contain the distinguished features of inequalities (modeling unilateral constraints) and disjunctions (modeling mode switches). Formally, a differential variational inequality (DVI) consists of an ordinary differential equation whose right-hand function is parameterized by an algebraic variable that is required to be a solution of a finite-dimensional VI containing the state variable of the system. A comprehensive study of such a differential system can be found in [21] which covers many of its fundamental aspects; see also [22]. A special DVI is the linear complementarity system (LCS), in which the VI [10] is a linear complementarity problem (LCP) [9]. Due to its versatility in the modeling of many hybrid engineering systems, the LCS is the subject of many recent papers [1], [3]–[6], [8], [12]–[14], [24], [26], [31], [32]. In particular, the survey [24] gives an excellent documentation of the fundamental role of the LCS in optimization and in nonsmooth system theory.

As with all switched dynamical systems, a fundamental issue associated with an LCS is the "Zeno" phenomenon, which pertains to the infinite number of mode switches in a finite time duration. The Zeno phenomenon in hybrid systems was first introduced by Johannson, Lygeros, Sastry, and their collaborators [17], [27], [33], though the issue on boundedness of the number of switchings has been studied for piecewise analytic systems by Brunovsky and Sussmann in a different setting about two decades earlier [2], [29]. Adding to the previous studies of the Zeno issue [8] for complementarity systems, the recent paper [26] formally defines the notions of strong and weak Zenoness and non-Zeno states of an LCS. Roughly speaking, for a given solution trajectory, weak non-Zenoness means that the system remains in one "mode" on an open time interval, i.e., the system is defined by a smooth differential algebraic system in that interval. Strong non-Zenoness is a strengthening of weak non-Zenoness and means that a fundamental triple of index sets (see (7) for definition) remains constant in an open time interval. Such detailed classifications of a rather seemingly intuitive concept turn out to be essential because of the mathematical subtleties that are associated with the key complementarity condition, which is the key distinguishing feature of the LCS from an ordinary linear dynamical system. There are two key contributions in [26]: one is an expansion of the solution trajectory near a given state; the other contribution is a consequence of the expansion, which allows us to show that all states of an LCS satisfying the "P-property" are strongly non-Zeno. The first major goal of the present paper is to extend these two results to a nonlinear complementarity system (NCS) satisfying the strong regularity condition [23], [10]. Such an extension turns out to be non-trivial as we need to rely on the Lie derivatives of smooth functions and use them to obtain an analogous expansion for a solution trajectory to the nonlinear system. Further extension of the results to a DVI using the Karush–Kuhn–Tucker (KKT) formulation of the finite-dimensional VI will also be obtained.

State observability is a fundamental system-theoretic concept that has been studied extensively for smooth dynamical systems; see e.g., [15] and [19]. In particular, certain matrix rank conditions are known to be sufficient for a given state to be "locally observable" relative to a smooth output function. Nevertheless, such a result is obtained under some key differentiability properties of the latter function and the right-hand side of the ODE, which are not valid for the DVI. Indeed, even under the favorable assumption of strong regularity of the given state, which turns the DVI locally near the state into an equivalent ODE with a Lipschitz continuous, albeit nondifferentiable, right-hand side, the solution trajectory of the DVI is only once differentiable in time and not differentiable in the initial condition. Such weak differentiability

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is not enough for the applicability of the classical observability results for smooth systems to the DVI. Consequently, to analyze such an issue for the DVI, new arguments have to be made. The cornerstone of our analysis is the non-Zenoness result established in the first part of the paper together with a directional differentiability result of the solution operator to a nonsmooth ODE as a function of the initial condition; the latter result was proved in [28] using the theory of differential inclusions and recently expanded in [22] using elementary arguments. Armed with these basic results, we postulate a uniqueness condition [cf. (31)] on a homogeneous equation obtained from the "first variational system" of the equivalent (nonsmooth) ODE formulation of the DVI, and show that this postulate is sufficient for the local observability of the given state. This result is applicable in general to a "B-differentiable" [10], [20] (in particular, piecewise smooth) system; i.e., an ODE whose right-hand function and output function are both locally Lipschitz continuous and directionally differentiable. In the case of the linear and nonlinear complementarity systems (or more generally, the KKT system), the non-Zenoness of these special systems allows us to obtain some sharp necessary and sufficient conditions for local observability.

In the next section, we formally define the NCS and review the blanket condition of strong regularity for a solution to a finite-dimensional nonlinear complementarity problem (NCP). For simplicity, we deliberately omit the formal definition of a strongly non-Zeno state; instead, this property is made precise in the statement of Theorem 2. For details on related concepts, see [26]. After establishing some preliminary lemmas in Section III, including the key expansion of a solution trajectory in Lemma 8, we present the detailed proof of Theorem 2 in Section IV. An extension of this theorem to the DVI via the KKT formulation of the VI is presented in Section V. Section VI addresses the local observability issue for a B-differentiable ODE; a sufficient condition is obtained for such a differential system (see Theorem 11) which is then specialized to the NCS. The local observability issue for the LCS satisfying the P-property and with a linear output function is addressed in the last Section VII, where necessary and sufficient conditions are derived. It should be mentioned that subsequent to this work, we have undertaken a comprehensive study of the observability issue for the LCS, where we introduce and examine in detail several types of observability properties for a general "conewise linear system" [7]. The approach taken there is different from the "derivative-based" approach adopted herein. For one thing, we obtain our results in Section VII by specializing their counterparts for the NCS. In contrast, the approach in [7] is applicable basically only to linear systems; its generalization to nonlinear systems (such as the NCS) is not straightforward and will likely have to make use of Theorem 11 in some way.

II. Non-Zenoness Under Strong Regularity

Given vector-valued functions $(F, H) : \Re^{n+m} \to \Re^{n+m}$, the NCS is to determine time-dependent functions $x(t) \in \Re^n$ and $u(t) \in \Re^m$ satisfying

$$\dot{x}(t) = F(x(t), u(t))$$

$$0 \le u(t) \perp H(x(t), u(t)) \ge 0 \tag{1}$$

where $\dot{x} \equiv dx/dt$ denotes the time derivative of the trajectory $x(t)$, and the notation $a \perp b$ means that the two vectors a and b are perpendicular. We make the *blanket assumption* on a state $x^0 \equiv x(t_0)$ at time t_0, which postulates that the NCP:

$$0 \le u \perp H(x^0, u) \ge 0 \tag{2}$$

has a *strongly regular* solution u^0 [23], [10]. We further assume throughout this section that the functions F and H are analytic in a certain neighborhood $\mathcal{Z} \subset \Re^{n+m}$ of the pair (x^0, u^0). Under this assumption, we show in Theorem 2 below that x^0 is a strongly non-Zeno state [26] of (1).

Strong regularity can be characterized in a number of ways. For our purpose, we present a matrix-theoretic characterization of the condition. Define three fundamental index sets $(\alpha_0, \beta_0, \gamma_0)$ corresponding to the pair (x^0, u^0)

$$\begin{aligned} \alpha_0 &= \{i : u_i^0 > 0 = H_i(x^0, u^0)\} && \text{inactive } u- \text{ indices} \\ \beta_0 &= \{i : u_i^0 = 0 = H_i(x^0, u^0)\} && \text{degenerate indices} \\ \gamma_0 &= \{i : u_i^0 = 0 < H_i(x^0, u^0)\} && \text{strongly active } u- \text{ indices.} \end{aligned}$$

According to these sets, we can partition the (partial) Jacobian matrix $J_u H(x^0, u^0)$ as follows:

$$\begin{bmatrix} J_{u_{\alpha_0}} H_{\alpha_0}(x^0, u^0) & J_{u_{\beta_0}} H_{\alpha_0}(x^0, u^0) & J_{u_{\gamma_0}} H_{\alpha_0}(x^0, u^0) \\ J_{u_{\alpha_0}} H_{\beta_0}(x^0, u^0) & J_{u_{\beta_0}} H_{\beta_0}(x^0, u^0) & J_{u_{\gamma_0}} H_{\beta_0}(x^0, u^0) \\ J_{u_{\alpha_0}} H_{\gamma_0}(x^0, u^0) & J_{u_{\beta_0}} H_{\gamma_0}(x^0, u^0) & J_{u_{\gamma_0}} H_{\gamma_0}(x^0, u^0) \end{bmatrix}$$

where $J_{u_\alpha} H_\beta$ denotes the matrix of partial derivatives $(\partial H_j / \partial u_i)_{(i,j) \in \alpha \times \beta}$. It is known that u^0 is a strongly regular solution of the NCP (2) if and only if: a) the principal submatrix $J_{u_{\alpha_0}} H_{\alpha_0}(x^0, u^0)$ is nonsingular, and b) the Schur complement D, which is equal to (replacing the arguments (x^0, u^0) by the superscript "0")

$$J_{u_{\beta_0}} H^0_{\beta_0} - J_{u_{\alpha_0}} H^0_{\beta_0} \left[J_{u_{\alpha_0}} H^0_{\alpha_0} \right]^{-1} J_{u_{\beta_0}} H^0_{\alpha_0} \tag{3}$$

is a P-matrix (i.e., all its principal minors are positive). Before going into the technical details, we present an example from contact mechanics to illustrate the nonlinear complementarity system satisfying the strong regularity condition and interpret the main non-Zeno result.

Example 1: Consider a (frictionless) multibody mechanical system with n degrees of freedom and m possible contacts. Each body is assumed to have a rigid core with a thin elastic shell. Let the system configuration be $q \in \Re^n$ and the velocity be $v \in \Re^n$. The dynamics and kinematics are described by

$$M(q)\dot{v} = D(q, v) + F(q)u \tag{4}$$

$$\dot{q} = G(q)v \tag{5}$$

$$0 \le u \perp \Psi(q) + \delta \ge 0 \tag{6}$$

where $u \in \Re^m$ denotes the (normal) contact force on the system, $\delta \in \Re^m$ describes the local body deformation (due to the contact) in the normal direction, $M(q) \in \Re^{n \times n}$ is a positive definite mass-inertia matrix, $D(q, v)$ summarizes the Coriolis force, potential forces and other external forces/moments except the contact force, $F(q)$ is the Jacobian of geometric constraint functions, $G(q)$ denotes the parametrization matrix describing the body orientation, and $\Psi(q)$ represents the displacement function of the system in the normal direction. Moreover, each contact between the compliant bodies is modeled as the interaction of nonlinear springs such that the contact force u equals to some continuously differentiable function $L(q,\delta)$ with a positive definite partial Jacobian $J_\delta L(q,\delta)$. (As the matter of fact, for the systems subject to small-deformation, $L(q,\delta)$ can be approximated as

$$L(q,\delta) \approx K(q)\delta + \hat{K}(q)o(\|\delta\|)$$

for small $\|\delta\|$, where $K(q)$ is an $m \times m$ positive definite stiffness matrix for all q.) For a given triple $(q^0, v^0, \delta^0) = (q(t_0), v(t_0), \delta(t_0))$ at some time t_0, let $u^0 \equiv L(q^0,\delta^0)$. By the implicit function theorem, there exist neighborhoods $\mathcal{W} \subseteq \Re^m$ of δ^0 and $\mathcal{V} \subseteq \Re^{m+n}$ of (u^0, q^0) and a continuously differentiable function $\hat{L}$ defined on $\mathcal{V}$ such that for all $(u,q) \in \mathcal{V}, \delta = \hat{L}(u,q)$ is the unique vector in $\mathcal{W}$ satisfying $u = L(q,\delta)$. Therefore, the complementarity condition (6) becomes

$$0 \le u \perp \Psi(q) + \hat{L}(u,q) \ge 0$$

for all $(u,q) \in \mathcal{V}$. Furthermore, it is easy to see that $J_u\hat{L}(u^0,q^0) = (J_\delta L(q^0,\delta^0))^{-1}$ is positive definite and thus is a P-matrix. This implies that the system has a strongly regular solution at (q^0,δ^0). With this property and Theorem 2 shown below, we conclude that the system is strongly non-Zeno near the given state (q^0,v^0,δ^0). Physically, this implies that a scalar $\varepsilon > 0$ exists such that all the possible contact points of the system are one of the following three types: a) subject to an elastic contact force, i.e., $u_i(t) > 0$, throughout $(t_0, t_0+\varepsilon]$, b) no contact, i.e., $[\Psi(q(t)) + \delta(t)]_j > 0$, throughout $(t_0, t_0+\varepsilon]$, or c) touch but without causing deformation on the bodies, i.e., $u_k(t) = [\Psi(q(t)) + \delta(t)]_k = 0$ throughout $(t_0, t_0+\varepsilon]$. A similar time interval can also be obtained in the backward-time direction. □

Returning to the general discussion, we note an important consequence of the strong regularity condition; namely, there exist neighborhoods $\mathcal{U}_0$ of u^0 and $\mathcal{V}_0$ of x^0 and a Lipschitz continuous function $u : \mathcal{V}_0 \to \mathcal{U}_0$ such that for every $x \in \mathcal{V}_0, u(x)$ is the only vector u in $\mathcal{U}_0$ satisfying the NCP

$$0 \le u \perp H(x,u) \ge 0$$

with $u_i(x) = 0 < H_i(x,u(x))$ for all $i \in \gamma_0$ and $H_j(x,u(x)) = 0 < u_j(x)$ for all $j \in \alpha_0$. Furthermore, the neighborhood $\mathcal{V}_0$ can be restricted so that $\mathcal{V}_0 \times \mathcal{U}_0 \subset \mathcal{Z}$. As explained in [21], a neighborhood $\mathcal{T} \equiv [t_0-\varepsilon_0, t_0+\varepsilon_0]$ of t_0 exists for some $\varepsilon_0 > 0$ such that the NCS (1) has a unique pair of solution trajectories $(x^*(t), u^*(t))$ passing through (x^0,u^0) at time t_0 with $x^*(t)$ continuously differentiable and $u^*(t)$ Lipschitz continuous on $\mathcal{T}$ and $(x^*(t),u^*(t)) \in \mathcal{V}_0 \times \mathcal{U}_0$ for all $t \in \mathcal{T}$. Extending the definitions of the index sets, $(\alpha_0,\beta_0,\gamma_0)$, we write, for all $t \in \mathcal{T}$

$$\begin{aligned}\alpha(t) &= \{i : u_i^*(t) > 0 = H_i(x^*(t),u^*(t))\}\\ \beta(t) &= \{i : u_i^*(t) = 0 = H_i(x^*(t),u^*(t))\}\\ \gamma(t) &= \{i : u_i^*(t) = 0 < H_i(x^*(t),u^*(t))\}.\end{aligned} \tag{7}$$

Notice that with ε_0 sufficiently small, we have, by continuity of the trajectories, for all $t \in \mathcal{T}$

$$\alpha_0 \subseteq \alpha(t) \subseteq \alpha_0 \cup \beta_0,\ \beta_0 \supseteq \beta(t),\ \gamma_0 \subseteq \gamma(t) \subseteq \gamma_0 \cup \beta_0. \tag{8}$$

We now present the following main non-Zenoness result.

Theorem 2: In the above setting and for the given t_0, there exist a scalar $\varepsilon \in (0,\varepsilon_0)$ and two triples of index sets, $(\alpha_+,\beta_+,\gamma_+)$ and $(\alpha_-,\beta_-,\gamma_-)$ such that

$$\begin{aligned}(\alpha(t),\beta(t),\gamma(t)) &= (\alpha_-,\beta_-,\gamma_-) \qquad \forall t \in [t_0-\varepsilon, t_0)\\ (\alpha(t),\beta(t),\gamma(t)) &= (\alpha_+,\beta_+,\gamma_+) \qquad \forall t \in (t_0, t_0+\varepsilon].\end{aligned}$$

The proof of the theorem is postponed until Section IV. In what follows, we state a corollary of the theorem in terms of switch times in a compact interval.

Corollary 3: Let $(x^*(t),u^*(t))$ be a solution trajectory of (1) in an open interval $\mathcal{I}$. Assume that F and H are analytic functions in a neighborhood of this trajectory. For every compact subinterval $[a,b]$ of $\mathcal{I}$, if $u^*(t)$ is a strongly regular solution of the NCP: $0 \le u \perp H(x^*(t),u) \ge 0$ for all $t \in [a,b]$, then there exist finitely many subintervals

$$a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b \tag{9}$$

and finitely many triples of index sets $\{(\alpha_j,\beta_j,\gamma_j)\}_{j=1}^N$ such that, for all $j = 1,\ldots,N, (\alpha(t),\beta(t),\gamma(t)) = (\alpha_j,\beta_j,\gamma_j)$, for all $t \in (t_{j-1},t_j)$.

Proof: This follows from a standard covering argument [26, Prop. 8]. Indeed, for every $t \in [a,b]$, there exists an open neighborhood of t and two triples of index sets with the property stated in Theorem 2. The corollary follows easily by the fact that every open covering of a compact set contains a finite subcover. The details are omitted. □

Inherited from (8), the pair of triples of index sets $(\alpha_\pm,\beta_\pm,\gamma_\pm)$ satisfies the following connections to the initial triple: $(\alpha_0,\beta_0,\gamma_0)$, namely,

$$\alpha_0 \subseteq \alpha_\pm \subseteq \alpha_0 \cup \beta_0,\ \beta_0 \supseteq \beta_\pm,\ \gamma_0 \subseteq \gamma_\pm \subseteq \gamma_0 \cup \beta_0. \tag{10}$$

It can easily be seen that $\beta_+ = \beta_0$ if and only if $(\alpha_+,\beta_+,\gamma_+) = (\alpha_0,\beta_0,\gamma_0)$. A similar statement holds for the "$-$" triple. We say that the time t_0 is *forward (backward) switching of the first kind* relative to the trajectory $(x^*(t),u^*(t))$ if $\beta_+(\beta_-, \text{resp.}) \neq \beta_0$; negating this definition, we say that the time t_0 is *forward (backward) strongly non-switching* if $\beta_+(\beta_-, \text{resp.}) = \beta_0$. Roughly speaking, Corollary 3 says that if the trajectory $(x^*(t),u^*(t))$ is "time-wise strongly regular" on a compact time interval, then there are only finitely many switch times in the interval. It is worth noting that if u^0 is a *nondegenerate* solution of the NCP $0 \le u \perp H(x^0,u) \ge 0$, then $\beta_\pm = \beta_0 = \emptyset$ and the time t_0 is both forward and backward strongly nonswitching.

An important consequence of Corollary 3 is the piecewise analyticity of the solution trajectory $(x^*(t),u^*(t))$ on compact intervals under the strong regularity assumption.

Corollary 4: In the setting of Corollary 3, the partition (9) can be restricted so that the functions $(x^*(t),u^*(t))$ are analytic at all $t \in (t_{j-1},t_j)$, for $j = 1,\ldots,N$.

Proof: On the interval (t_{j-1},t_j), the pair $(x^*(t),u^*(t))$ is a solution of the differential algebraic system

$$\dot{x} = F(x,u) \quad 0 = H_{\alpha_j}(x,u) \quad 0 = u_{\beta_j} \quad 0 = u_{\gamma_j}.$$

With the partition (9) appropriately restricted, the matrix $J_{u_{\alpha_j}}H(x^*(t),u^*(t))$ is nonsingular for all t in such a subinterval. This means that we can solve for the variable $u_{\alpha_j}^*(t)$ on this interval as an analytic function of $x^*(t)$, which we can then substitute into the differential equation: $\dot{x} = F(x,u_{\alpha_j})$. The desired analyticity of $(x^*(t), u^*(t))$ for all t in each subinterval therefore follows readily. □

III. Preliminary Lemmas

Although the general idea of the proof of Theorem 2 is similar to that for the LCS with the P-property [26], there is significant difference in the technical details. For one thing, the observability degree that has played an important role in the linear case needs to be generalized; indeed nonlinear theory has to be used. In what follows, we take $t_0 = 0$ for simplicity of notation and only treat the "forward-time" case, i.e., $t > 0$, since "backward-time" results follow easily from their forward-time counterparts via a simple change of variable that amounts to time reversal. In fact, letting $x^{\rm r}(t) \equiv x(-t)$ and $u^{\rm r}(t) \equiv u(-t)$, we see that the pair $(x^{\rm r}(t), u^{\rm r}(t))$ satisfies

$$\begin{gathered}\dot{x}^{\rm r}(t) = -\dot{x}(-t) = -F(x(-t), u(-t)) = -F(x^{\rm r}(t), u^{\rm r}(t)) \\ 0 \leq u^{\rm r}(t) \perp H(x^{\rm r}(t), u^{\rm r}(t)) \geq 0.\end{gathered}$$

With a forward-time analysis applied to $(x^{\rm r}(t), u^{\rm r}(t))$, the derived results can easily be translated to the original $(x(t), u(t))$ trajectory, yielding the corresponding backward-time results for the latter.

A. Lie Derivatives

For an analytic vector field $f(x)$ in a neighborhood of $x^* \in \Re^n$, and for a corresponding scalar $\tilde{\tau}_f > 0$, the ODE

$$\dot{\tilde{x}}(t) = f(\tilde{x}(t)) \quad \tilde{x}(0) = x^* \tag{11}$$

has a unique solution that is analytic in the interval $[0, \tilde{\tau}_f]$; see [16, Th. 2.5.2] and [11, Ex. 3.3]. Corresponding to this ODE, we define a sequence of functions $\hat{x}^0(t), \hat{x}^1(t), \hat{x}^2(t), \ldots$ iteratively by: for $t \in [0, \tilde{\tau}_f]$,

$$\begin{aligned}&\hat{x}^0(t) \equiv x^* \\ &\hat{x}^k(t) \equiv x^* + \int_0^t f\left(\hat{x}^{k-1}(s)\right) ds, \qquad k \geq 1\end{aligned} \tag{12}$$

which is the well-known *Picard iteration* for approximating the solution $\tilde{x}(t)$ to (11). It is further known that each $\hat{x}^k(t)$ is analytic for $t \in [0, \tilde{\tau}_f]$ for all $k \geq 1$, provided that $\tilde{\tau}_f$ is suitably restricted. In the following, we use $L_f^k c(x)$ to denote the kth *Lie derivative* of a smooth scalar-valued function $c(x)$ with respect to the vector field $f(x)$ [18], [19], that is

$$\begin{aligned}&L_f^0 c(x) \equiv c(x) \\ &L_f^k c(x) \equiv [\nabla(L_f^{k-1} c(x))]^T f(x), \qquad k \geq 1\end{aligned}$$

where ∇ denotes the gradient vector. Note that $d^j c(\tilde{x}(t))/dt^j = L_f^k c(\tilde{x}(t))$ for all $k \geq 0$ and all $t \in [0, \tilde{\tau}_f]$. Hence, the following expansion holds:

$$c(\tilde{x}(t)) = \sum_{j=0}^{\infty} \left.\frac{d^j c(\tilde{x}(t))}{dt^j}\right|_{t=0} \frac{t^j}{j!} = \sum_{j=0}^{\infty} L_f^j c(x^*) \frac{t^j}{j!}. \tag{13}$$

Lemma 5 extends (13) to the function $c(\hat{x}^k(t))$ by identifying a basic connection between the Picard iteration functions $\hat{x}^k(t)$ defined in (12) and the Lie derivatives. The first assertion of the lemma is a standard result in ODE theory; the second part of the lemma yields

$$c(\hat{x}^k(t)) = \sum_{j=0}^{k} L_f^j c(x^*) \frac{t^j}{j!} + O(t^{k+1}) \tag{14}$$

where $O(t^{k+1})$ denotes a function that is bounded in absolute value by a multiplicative constant times t^{k+1} for all $t > 0$ sufficiently small. The expansion (14) can be compared to (13), revealing the coincidence between the two expansions up to order t^{k+1}.

Lemma 5: Let the functions f and c be analytic in a neighborhood of x^*. The following two statements hold for every integer $k \geq 1$.

a) There exist scalars $\eta > 0$ and $p_j \geq 0$ such that $\|\hat{x}^j(t) - \hat{x}^{j-1}(t)\| \leq p_j t^j$ for all $j = 1, \ldots, k$ and all $t \in [0, \eta]$.

b) $(d^i c(\hat{x}^k(t)))/(dt^i)|_{t=0} = L_f^i c(x^*)$, for all $i = 1, \ldots, k$.

Proof: Since $f(x)$ is analytic near x^*, it follows that there exist a ball $\mathcal{B}(x^*, \delta)$ in $\Re^n$ with center at x^* and radius $\delta > 0$ and a scalar $\rho_f > 0$ such that $\|f(x^1) - f(x^2)\| \leq \rho_f \|x^1 - x^2\|$ for all x^1 and x^2 in $\mathcal{B}(x^*, \delta)$. Now define

$$p_1 \equiv \|f(x^*)\| \quad p_{j+1} \equiv \rho_f p_j/(j+1) \qquad \forall j = 1, \ldots, k-1$$

and choose $\eta > 0$ such that $\sum_{i=1}^k p_i \eta^i < \delta$. We show by induction that $\hat{x}^j(t) \in \mathcal{B}(x^*, \delta)$ for all $t \in [0, \eta]$ and all $1 \leq j \leq k$ and that these p_j's and η satisfy the desired properties in assertion a).

Consider $j = 1$. It is clear that $\hat{x}^1(t) = x^* + f(x^*)t$; thus $\|\hat{x}^1(t) - \hat{x}^0(t)\| \leq \|f(x^*)\| t = p_1 t$ for all $t \geq 0$ and $\|\hat{x}^1(t) - x^*\| \leq p_1 \eta < \delta$ for all $t \in [0, \eta]$. This proves the first step. Suppose that the claim holds for all $j = 1, \cdots, \ell$ for some $1 \leq \ell \leq k-1$. Consider $j = \ell + 1$. By the induction hypothesis, both $\hat{x}^\ell(t)$ and $\hat{x}^{\ell-1}(t)$ belong to $\mathcal{B}(x^*, \delta)$ for all $t \in [0, \eta]$, which implies that

$$\|f(\hat{x}^\ell(t)) - f(\hat{x}^{\ell-1}(t))\| \leq \rho_f \|\hat{x}^\ell(t) - \hat{x}^{\ell-1}(t)\| \leq \rho_f p_\ell t^\ell$$

for all $t \in [0, \eta]$. Thus, for all such t

$$\begin{aligned}\|\hat{x}^{\ell+1}(t) - \hat{x}^\ell(t)\| &\leq \int_0^t \|f(\hat{x}^\ell(s)) - f(\hat{x}^{\ell-1}(s))\| ds \\ &\leq \rho_f \int_0^t \|\hat{x}^\ell(s) - \hat{x}^{\ell-1}(s)\| ds \\ &\leq \rho_f \int_0^t p_\ell s^\ell ds \leq p_{\ell+1} t^{\ell+1}.\end{aligned}$$

Hence, for all $t \in [0, \eta]$

$$\|\hat{x}^{\ell+1}(t) - x^*\| \leq \sum_{i=1}^{\ell+1} \|\hat{x}^i(t) - \hat{x}^{i-1}(t)\| \leq \sum_{i=1}^{\ell+1} p_i \eta^i < \delta.$$

Consequently, $\hat{x}^{\ell+1}(t) \in \mathcal{B}(x^*, \delta)$ for all $t \in [0, \eta]$. This completes the proof of assertion a).

To prove assertion b), we restrict $\eta > 0$ in a) (if necessary) such that $\tilde{x}(t) \in \mathcal{B}(x^*, \delta)$ for all $t \in [0, \eta]$. Therefore, for each $t \in [0, \eta]$, we have

$$\begin{aligned}\|\hat{x}^k(t) - \tilde{x}(t)\| &\leq \int_0^t \|f(\hat{x}^{k-1}(s_1)) - f(\tilde{x}(s_1))\| ds_1 \\ &\leq \rho_f \int_0^t \|\hat{x}^{k-1}(s_1) - \tilde{x}(s_1)\| ds_1 \\ &\leq \rho_f \int_0^t \int_0^{s_1} \|f(\hat{x}^{k-2}(s_2)) - f(\tilde{x}(s_2))\| ds_2 ds_1 \\ &\leq \rho_f^2 \int_0^t \int_0^{s_1} \|\hat{x}^{k-2}(s_2) - \tilde{x}(s_2)\| ds_2 ds_1 \\ &\;\;\vdots\end{aligned}$$

$$\le \rho_f^k \underbrace{\int_0^t \cdots \int_0^{s_{k-1}}}_{k-\text{ fold}} \|\hat{x}^0(s_k) - \tilde{x}(s_k)\| \underbrace{ds_k \cdots ds_1}_{k-\text{ fold}}.$$

Note that $\hat{x}^0(s_k) \equiv x^*$ and $\|\tilde{x}(s) - x^*\| \le \int_0^s \|f(\tilde{x}(\tau))\| d\tau \le M_f s$, where $M_f \equiv \sup_{x \in \mathcal{B}(x^*,\delta)} \|f(x)\|$. Consequently, it follows that $\|\hat{x}^k(t) - \tilde{x}(t)\| \le (\rho_f^k M_f)/((k+1)!) t^{k+1}$ and

$$\|c(\hat{x}^k(t)) - c(\tilde{x}(t))\| \le m_c \rho_f^k M_f t^{k+1}/(k+1)! \tag{15}$$

for all $t \in [0, \eta]$, where $m_c > 0$ is a Lipschitz constant of $c(x)$ in the ball $\mathcal{B}(x^*, \delta)$ (the existence of m_c is due to analyticity of $c(x)$). Furthermore, since $\hat{x}^k(t)$ is analytic in time, so is $c(\hat{x}^k(t))$. Hence, we have

$$c(\hat{x}^k(t)) = \sum_{j=0}^{\infty} \left. \frac{d^j c(\hat{x}^k(t))}{dt^j} \right|_{t=0} \frac{t^j}{j!} \tag{16}$$

for all $t > 0$ sufficiently small. Now let $j \ge 1$ be the first integer such that $d^j c(\hat{x}^k(t))/dt^j|_{t=0} \ne L_f^j c(x^*)$. From this and the two series expansions (13) and (14), it follows that

$$\lim_{t \downarrow 0} \frac{\|c(\hat{x}^k(t)) - c(\tilde{x}(t))\|}{t^j} = k_c$$

for some constant $k_c > 0$. Consequently, from (15), we deduce that $j \ge k+1$. This establishes b). □

B. Solution Expansion

We return to the setting of Theorem 2. Since $H_{\alpha^0}(x^0, u^0) = 0$ and the matrix $J_{u_{\alpha_0}} H_{\alpha_0}(x^0, u^0)$ is nonsingular, it follows from the implicit function theorem that there exist neighborhoods, $\mathcal{V}'$ of x^0, $\mathcal{U}'_{\alpha_0}$ of $u^0_{\alpha_0}$, and $\mathcal{U}'_{\beta_0}$ of $u^0_{\beta_0} = 0$, and an analytic function $\hat{u}_{\alpha_0} : \mathcal{V}' \times \mathcal{U}'_{\beta_0} \to \mathcal{U}'_{\alpha_0}$ such that for every (x, u_{β_0}) in $\mathcal{V}' \times \mathcal{U}'_{\beta_0}, \hat{u}_{\alpha_0}(x, u_{\beta_0})$ is the unique vector in $\mathcal{U}'_{\alpha_0}$ satisfying $H_{\alpha_0}(x, \hat{u}_{\alpha_0}(x, u_{\beta_0}), u_{\beta_0}, 0) = 0$. We can restrict the neighborhoods $\mathcal{V}'$ and $\mathcal{U}'_{\beta_0}$ so that $H_{\gamma_0}(x, \hat{u}_{\alpha_0}(x, u_{\beta_0}), u_{\beta_0}, 0) > 0$ for all (x, u_{β_0}) in $\mathcal{V}' \times \mathcal{U}'_{\beta_0}$. Hence, by restricting the scalar ε_0, we can ensure that $(x^*(t), u^*_{\alpha_0}(t), u^*_{\beta_0}(t)) \in \mathcal{V}' \times \mathcal{U}'_{\alpha_0} \times \mathcal{U}'_{\beta_0}$ and $u^*_{\gamma_0}(t) = 0$ for all $t \in [0, \varepsilon_0]$. Since $H_{\alpha_0}(x^*(t), u^*_{\alpha_0}(t), u^*_{\beta_0}(t), 0) = 0$, it follows that $u^*_{\alpha_0}(t) = \hat{u}_{\alpha_0}(x^*(t), u^*_{\beta_0}(t))$ for all $t \in [0, \varepsilon_0]$. Consequently, in the interval $t \in [0, \varepsilon_0]$ (with a further reduced $\varepsilon_0 > 0$ if necessary), the NCS (1) is equivalent to

$$\begin{aligned} \dot{x}(t) &= \hat{F}(x(t), u_{\beta_0}(t)), \quad x(0) = x^0 \\ 0 &\le u_{\beta_0}(t) \perp \hat{H}_{\beta_0}(x(t), u_{\beta_0}(t)) \ge 0 \end{aligned} \tag{17}$$

where the algebraic variable is u_{β_0}, and

$$\hat{F}(x, u_{\beta_0}) \equiv F(x, \hat{u}_{\alpha_0}(x, u_{\beta_0}), u_{\beta_0}, 0)$$
$$\hat{H}_{\beta_0}(x, u_{\beta_0}) \equiv H_{\beta_0}(x, \hat{u}_{\alpha_0}(x, u_{\beta_0}), u_{\beta_0}, 0)$$

are analytic functions in a neighborhood of $(x, u_{\beta_0}) = (x^0, 0)$. Clearly, $u_{\beta_0} = 0$ is a solution of the NCP

$$0 \le u_{\beta_0} \perp \hat{H}_{\beta_0}(x^0, u_{\beta_0}) \ge 0. \tag{18}$$

The next lemma shows that this solution is strongly regular.

Lemma 6: The zero solution is strongly regular for (18).

Proof: It suffices to show that the matrix $J_{u_{\beta_0}} \hat{H}_{\beta_0}(x^0, 0)$ is a P-matrix. By the chain rule and the definition of the matrix D in (3), it is easy to see that $J_{u_{\beta_0}} \hat{H}_{\beta_0}(x^0, 0)$ is equal to

$$\begin{aligned} & J_{u_{\alpha_0}} H_{\beta_0}(x^0, \hat{u}_{\alpha_0}(x, u_{\beta_0}), u_{\beta_0}, 0) J_{u_{\beta_0}} \hat{u}_{\alpha_0}(x^0, u_{\beta_0})\big|_{u_{\beta_0}=0} \\ &+ J_{u_{\beta_0}} H_{\beta_0}(x^0, \hat{u}_{\alpha_0}(x, u_{\beta_0}), u_{\beta_0}, 0)\big|_{u_{\beta_0}=0} \\ &= J_{u_{\beta_0}} H_{\beta_0}(x^0, u^0) - J_{u_{\alpha_0}} H_{\beta_0}(x^0, u^0) \\ &\quad \times \left[J_{u_{\alpha_0}} H_{\alpha_0}(x^0, u^0)\right]^{-1} J_{u_{\beta_0}} H_{\alpha_0}(x^0, u^0) = D \end{aligned}$$

where the second equality is obtained by differentiating the equation $H_{\alpha_0}(x^0, \hat{u}_{\alpha_0}(x^0, u_{\beta_0}), u_{\beta_0}, 0) = 0$ with respect to u_{β_0} and then substituting $u_{\beta_0} = 0$, yielding

$$J_{u_{\alpha_0}} H_{\alpha_0}(x^0, u^0) J_{u_{\beta_0}} \hat{u}_{\alpha_0}(x^0, 0) + J_{u_{\beta_0}} H_{\alpha_0}(x^0, u^0) = 0.$$

Since D is a P-matrix, the lemma follows readily. □

If β_0 is a proper subset of $\{1, \ldots, m\}$, by an inductive hypothesis on the dimension of the algebraic variable u in the NCS, it follows that the conclusion of Theorem 2 holds for the trajectory $(x^*(t), u^*_{\beta_0}(t))$ as a solution to the system (17) for $t > 0$ sufficiently small. The proof of Theorem 2 is therefore complete in this case.

Consider the case where $\beta_0 = \{1, \ldots, m\}$, which is equivalent to $u^0 = H(x^0, 0) = 0$. The matrix D defined in (3) becomes $J_u H(x^0, 0)$, which is a P-matrix by the strong regularity of $u^0 = 0$. We will obtain a reduced NCS by eliminating the situation in Lemma 7, which makes use of the Lie derivatives applied to the following pair of functions:

$$f(x) \equiv F(x, 0) \quad \text{and} \quad C(x) \equiv H(x, 0). \tag{19}$$

Since F and H are analytic in a neighborhood of x^0, so are f and C; hence $L_f^i C_j(x)$ is well defined for all $i \ge 0$ and $j = 1, \ldots, m$. Let $L_f^i C(x)$ denote the m-vector with components $L_f^i C_j(x)$ for $j = 1, \ldots, m$. Inductively, we have $L_f^0 C(x) = C(x)$ and, for $j \ge 1, L_f^j C(x) = (J L_f^{j-1} C(x)) f(x)$, where Jg denote the Jacobian of the (vector) function g.

Since $D \equiv J_u H(x^0, 0)$ is a P-matrix, the LCP: $0 \le v \perp L_f^i C(x^0) + Dv \ge 0$ has a unique solution for every $i \ge 0$, which we denote v^{i*}. Moreover, a constant $\rho_D > 0$ exists such that

$$\|v(q') - v(q)\| \le \rho_D \|q' - q\| \qquad \forall q, q' \in \Re^m \tag{20}$$

where $v(p)$ denotes the unique solution to the LCP: $0 \le v \perp p + Dv \ge 0$. For details of these results which we use freely in what follows, see [9]. Corresponding to the function f defined in (19), let $\tilde{x}^*(t)$ be the unique analytic solution of the ODE (11) for all $t \in [0, \tilde{\tau}]$ for some $\tilde{\tau} > 0$.

The following lemma shows that the NCS (1) has a trivial solution if the vectors $L_f^i C(x^0)$ are all zero.

Lemma 7: Suppose $u^0 = H(x^0, 0) = 0$ and $J_u H(x^0, 0)$ is a P-matrix. If $L_f^i C(x^0) = 0$ for all $i \ge 0$, then $(\tilde{x}^*(t), 0)$ is the unique solution of the NCS (1) for all $t > 0$ sufficiently small.

Proof: It suffices to show that $H(\tilde{x}^*(t), 0) = C(\tilde{x}^*(t)) \ge 0$ for all $t > 0$ sufficiently small. Since $C(\tilde{x}^*(t))$ is an analytic function for $t > 0$ sufficiently small, we have, for every $j = 1, \ldots, m$

$$\begin{aligned} C_j(\tilde{x}^*(t)) &= \sum_{k=0}^{\infty} \left. \frac{d^k C_j(\tilde{x}^*(t))}{dt^k} \right|_{t=0} \frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty} L_f^k C_j(x^0) \frac{t^k}{k!} = 0 \end{aligned}$$

for all $t > 0$ sufficiently small. □

The next lemma deals with the case where there is at least one i such that $L_f^i C(x^0)$ is not zero. In this case, we establish an expansion of the solution trajectory to (1) near the given state $x(0) = x^0$. In the lemma, we write $o(\psi(t))$ to mean a function such that $\lim_{t \downarrow 0} (o(\psi(t)))/(\psi(t)) = 0$.

Lemma 8: Suppose $u^0 = H(x^0, 0) = 0$ and $J_u H(x^0, 0)$ is a P-matrix. Let $(x^*(t), u^*(t))$ be the unique trajectory satisfying the NCS (1) for all $t > 0$ sufficiently small and passing through

$(x^0, 0)$ at time 0. Let $f(x) \equiv F(x, 0)$ and $C(x) \equiv H(x, 0)$. If $L_f^i C(x^0) = 0$ for all $i = 0, \cdots, k-1$, then for all $t > 0$ sufficiently small,

$$x^*(t) = x^0 + \sum_{j=0}^{k} L_f^j f(x^0) \frac{t^{j+1}}{(j+1)!} + \frac{t^{k+1}}{(k+1)!} J_u F(x^0, u^0) v^{k*} + o(t^{k+1}) \qquad (21)$$

$$u^*(t) = \frac{t^k}{k!} v^{k*} + o(t^k).$$

Proof: Let $\hat{x}^\ell(t)$ denote the successive approximations defined in (12) corresponding to $x^* = x^0$. We first show that

$$x^*(t) = \hat{x}^{k+1}(t) + \frac{t^{k+1}}{(k+1)!} J_u F(x^0, u^0) v^{k*} + o(t^{k+1}). \qquad (22)$$

In the proof, we will also establish the desired formula for $u^*(t)$. Defining $z(t) \equiv x^*(t) - \hat{x}^k(t)$, we have $z(0) = 0$ and

$$\begin{aligned}\dot{z}(t) &= F(x^*(t), u^*(t)) - f(\hat{x}^{k-1}(t)) \\ &= f(x^*(t)) - f(\hat{x}^{k-1}(t)) + F(x^*(t), u^*(t)) - F(x^*(t), 0) \\ &= f(x^*(t)) - f(\hat{x}^{k-1}(t)) + e_x(t) + J_u F(x^0, 0) u^*(t)\end{aligned}$$

where, by the integral form of the mean-value theorem for a vector-valued function

$$\begin{aligned}e_x(t) &\equiv F(x^*(t), u^*(t)) - F(x^*(t), 0) - J_u F(x^0, 0) u^*(t) \\ &= \left\{ \int_0^1 \left[J_u F(x^*(t), \tau u^*(t)) - J_u F(x^0, 0) \right] d\tau \right\} u^*(t)\end{aligned}$$

satisfies $\lim_{\substack{t \downarrow 0 \\ u^*(t) \neq 0}} (e_x(t))/(\|u^*(t)\|) = 0$, by Lebesgue's dominated convergence theorem, which allows us to exchange the limit with the integral. In the rest of the proof, we define 0/0 to be zero. Notice that $u^*(t)$ satisfies

$$0 \leq u^*(t) \perp H(x^*(t), u^*(t)) \geq 0. \qquad (23)$$

We have

$$\begin{aligned}&H(x^*(t), u^*(t)) \\ &\quad = H(x^*(t), u^*(t)) - H(x^*(t), 0) + H(x^*(t), 0) \\ &\quad = C(x^*(t)) + e_u(t) + D u^*(t)\end{aligned}$$

where, similar to $e_x(t)$,

$$\begin{aligned}e_u(t) &\equiv H(x^*(t), u^*(t)) - H(x^*(t), 0) - J_u H(x^0, 0) u^*(t) \\ &= \left\{ \int_0^1 \left[J_u H(x^*(t), \tau u^*(t)) - J_u H(x^0, 0) \right] d\tau \right\} u^*(t)\end{aligned}$$

satisfies

$$\lim_{\substack{t \downarrow 0 \\ u^*(t) \neq 0}} \frac{e_u(t)}{\|u^*(t)\|} = 0. \qquad (24)$$

Comparing (23) with the LCP

$$0 \leq v \perp L_f^k C(x^0) + D v \geq 0 \qquad (25)$$

we deduce, by (20)

$$\|u^*(t) - t^k v^{k*}/k!\| \leq \rho_D \|e(t)\| \qquad (26)$$

where v^{k*} is the solution of the LCP (25) and

$$\begin{aligned}e(t) &\equiv C(x^*(t)) + e_u(t) - \frac{t^k}{k!} L_f^k C(x^0) \\ &= e_u(t) + [C(x^*(t)) - C(\hat{x}^k(t))] \\ &\quad + \left[C(\hat{x}^k(t)) - \frac{t^k}{k!} L_f^k C(x^0) \right].\end{aligned}$$

By the analyticity, and thus Lipschitz continuity, of $f(x)$ and $C(x)$ and Lemma 5(a), it follows that for some constants $\rho_f > 0, \rho_c > 0$ and $p_k \geq 0$, we have, for all $t > 0$ sufficiently small

$$\begin{aligned}\|f(x^*(t)) - f(\hat{x}^{k-1}(t))\| &\leq \rho_f \|x^*(t) - \hat{x}^{k-1}(t)\| \\ &\leq \rho_f \left[\|x^*(t) - \hat{x}^k(t)\| + \|\hat{x}^k(t) - \hat{x}^{k-1}(t)\| \right] \\ &\leq \rho_f \left[\|z(t)\| + p_k t^k \right]\end{aligned}$$

and $\|C(x^*(t)) - C(\hat{x}^k(t))\| \leq \rho_c \|x^*(t) - \hat{x}^k(t)\| = \rho_c \|z(t)\|$. Moreover, by the analyticity of $C(\hat{x}^k(t))$, we have, for all $t > 0$ sufficiently small

$$C(\hat{x}^k(t)) = \sum_{j=0}^{\infty} \left. \frac{d^j C(\hat{x}^k(t))}{d\, t^j} \right|_{t=0} \frac{t^j}{j!} = L_f^k C(x^0) \frac{t^k}{k!} + O(t^{k+1}).$$

Therefore, (26) yields

$$\begin{aligned}&\left\| u^*(t) - t^k v^{k*}/k! \right\| \\ &\quad \leq \rho_D \left[\rho_c \|z(t)\| + \frac{\|e_u(t)\|}{\|u^*(t)\|} \|u^*(t)\| + O(t^{k+1}) \right].\end{aligned}$$

Hence

$$\begin{aligned}&\left(1 - \rho_D \frac{\|e_u(t)\|}{\|u^*(t)\|} \right) \left\| u^*(t) - \frac{t^k}{k!} v^{k*} \right\| \\ &\quad \leq \rho_D \left[\rho_c \|z(t)\| + \frac{\|e_u(t)\|}{\|u^*(t)\|} \frac{t^k}{k!} \|v^{k*}\| + O(t^{k+1}) \right]\end{aligned}$$

which yields, in view of (24),

$$\left\| u^*(t) - t^k v^{k*}/k! \right\| \leq \rho'_D \left[\rho_c \|z(t)\| + o(t^k) \right] \qquad (27)$$

for some constant $\rho'_D > 0$. It follows that

$$\|e_x(t)\| \leq \frac{\|e_x(t)\|}{\|u^*(t)\|} \left\| u^*(t) - \frac{t^k}{k!} v^{k*} \right\| + \frac{\|e_x(t)\|}{\|u^*(t)\|} \frac{t^k}{k!} \|v^{k*}\|.$$

Rewriting

$$\begin{aligned}\dot{z}(t) &= f(x^*(t)) - f(\hat{x}^{k-1}(t)) + e_x(t) \\ &\quad + J_u F(x^0, 0) \left(u^*(t) - \frac{t^k}{k!} v^{k*} \right) + \frac{t^k}{k!} J_u F(x^0, 0) v^{k*}\end{aligned}$$

we obtain

$$\begin{aligned}\|z(t)\| &\leq \int_0^t \|\dot{z}(s)\| ds \leq \rho_f \left\{ \int_0^t \left[\|z(s)\| + p_k s^k \right] \right. \\ &\quad + \left(\|J_u F(x^0, 0)\| + \frac{\|e_x(s)\|}{\|u^*(s)\|} \right) \left\| u^*(s) - \frac{s^k}{k!} v^{k*} \right\| \\ &\quad \left. + \frac{s^k}{k!} \|J_u F(x^0, 0) v^{k*}\| + o(s^k) \right\} ds \\ &\leq \rho_z \int_0^t \left[\|z(s)\| + s^k + o(s^k) \right] ds.\end{aligned}$$

for some constant $\rho_z > 0$. By Gronwall's lemma, we deduce

$$\|z(t)\| \leq \rho_z e^{\rho_z t} \int_0^t \left[s^k + o(s^k) \right] ds \leq \rho'_z \left[t^{k+1} + o(t^{k+1}) \right]$$

for some constant $\rho'_z > 0$ and for all $t > 0$ sufficiently small. Substituting this into the inequality (27) readily gives the expression for $u^*(t)$ in (21). To prove the expression for $x^*(t)$, note that

$$x^*(t) - \hat{x}^{k+1}(t) = \int_0^t [F(x^*(s), u^*(s)) - F(\hat{x}^k(s), 0)] ds$$

$$= \int_0^t \{[F(x^*(s), u^*(s)) - F(x^*(s), 0)] + [F(x^*(s), 0) - F(\hat{x}^k(s), 0)]\} ds.$$

As before, we write

$$\begin{aligned} F(x^*(s), u^*(s)) &- F(x^*(s), 0) \\ &= e_F(s) + J_u F(x^0, 0) u^*(s) \\ &= e_F(s) + J_u F(x^0, 0) \\ &\quad \times \left(u^*(s) - \frac{s^k}{k!} v^{k*} \right) + \frac{s^k}{k!} J_u F(x^0, 0) v^{k*} \end{aligned}$$

where $e_F(s) \equiv F(x^*(s), u^*(s)) - F(x^*(s), 0) - J_u F(x^0, 0) u^*(s)$ satisfies $\lim_{t \downarrow 0} (\|e_F(t)\|)/(\|u^*(t)\|) = 0$. Hence

$$\begin{aligned} \|e_F(s)\| &\le \frac{\|e_F(s)\|}{\|u^*(s)\|} \left[\left\| u^*(s) - \frac{s^k}{k!} v^{k*} \right\| + \frac{s^k}{k!} \|v^{k*}\| \right] \\ &\le \frac{\|e_F(s)\|}{\|u^*(s)\|} \left[\frac{s^k}{k!} \|v^{k*}\| + o(s^k) \right]. \end{aligned}$$

Moreover, for some constant $\rho_f' > 0$

$$\begin{aligned} &\|F(x^*(s), 0) - F(\hat{x}^k(s), 0)\| \\ &\quad \le \rho_f \|x^*(s) - \hat{x}^k(s)\| \le \rho_f' \left[s^{k+1} + o(s^{k+1}) \right]. \end{aligned}$$

Consequently

$$\begin{aligned} x^*(t) - \hat{x}^{k+1}(t) &= J_u F(x^0, u^0) v^{k*} \int_0^t \frac{s^k}{k!} ds + o(t^{k+1}) \\ &= \frac{t^{k+1}}{(k+1)!} J_u F(x^0, u^0) v^{k*} + o(t^{k+1}) \end{aligned}$$

as desired. To complete the proof of the lemma, we have, by the definition of $\hat{x}^{k+1}(t)$

$$\begin{aligned} \hat{x}^{k+1}(t) &= x^0 + \int_0^t f(\hat{x}^k(s)) ds \\ &= x^0 + \int_0^t \left[\sum_{j=0}^k L_f^j f(x^0) \frac{s^j}{j!} + O(s^{k+1}) \right] ds \\ &= x^0 + \sum_{j=0}^k L_f^j f(x^0) \frac{t^{j+1}}{(j+1)!} + O(t^{k+2}) \end{aligned}$$

where the middle equality follows from (14) extended to the vector function f. Thus (21) follows from (22). □

The expansion (21) expresses $x^*(t)$ as a polynomial function of t up to order $o(t^{k+1})$. It is interesting to note that in the last formula, all the terms up to t^{k+1} depend only on the function $f \equiv F(\cdot, 0)$. As an illustration of the formula, consider the case where $F(x, u) = Ax + Bu$ is a linear function defined by the constant matrices $A \in \Re^{n \times n}$ and $B \in \Re^{n \times m}$. We have $f(x) = Ax$. Furthermore, by induction, we can show that $L_f^i f(x) = A^{i+1} x$ for all nonnegative integers i. Hence, it follows that in this case, for all $t > 0$ sufficiently small

$$x^*(t) = \sum_{j=0}^{k+1} \frac{t^j}{j!} A^j x^0 + \frac{t^{k+1}}{(k+1)!} B v^{k*} + o(t^{k+1})$$

which recovers the first formula in [26, Lemma 14] for the LCS, up to the term t^{k+1}. The expansion in the reference extends to the next term t^{k+2}; this is possible because of the linearity of the functions F and H. Further expansion in the nonlinear case is beyond the scope of this paper.

IV. Proof of Theorem 2

We apply induction on the dimension m of the algebraic variable u. The theorem is obviously true when $m = 0$. Assume that the theorem is true when there are $m - 1$ or fewer algebraic variables and consider an NCS with $u \in \Re^m$. Without loss of generality, suppose $u^0 = H(x^0, u^0) = 0$. By Lemma 7, we may further assume that there is a first nonnegative integer k such that $L_f^k C(x^0) \neq 0$. It follows from the proof of Lemma 8 that

$$\begin{aligned} H(x^*(t), u^*(t)) &= e(t) + D \left(u^*(t) - \frac{t^k}{k!} v^{k*} \right) \\ &\quad + \frac{t^k}{k!} \left[L_f^k C(x^0) + D v^{k*} \right] \\ &= \frac{t^k}{k!} \left[L_f^k C(x^0) + D v^{k*} \right] + o(t^k). \end{aligned}$$

Since $L_f^k C(x^0) \neq 0$, there must be at least one index i such that either $v_i^{k*} > 0$ or $(L_f^k C(x^0) + D v^{k*})_i > 0$. For such an index i, it follows that either $[u_i^*(t) > 0 = H_i(x^*(t), u^*(t))$ for all $t > 0$ sufficiently small] or $[H_i(x^*(t), u^*(t)) > 0 = u_i^*(t)$ for all $t > 0$ sufficiently small]. Thus letting α_1 and γ_1 denote the set of indices satisfying the former and latter condition, respectively, we must have $\alpha_1 \cup \gamma_1 \neq \emptyset$. If $\alpha_1 \neq \emptyset$, then $J_{u_{\alpha_1}} H_{\alpha_1}(x^0, u^0)$, being a principal submatrix of the P-matrix $J_u H(x^0, u^0)$, is nonsingular. Moreover, the Schur complement of $J_{u_{\alpha_1}} H_{\alpha_1}(x^0, u^0)$ in $J_u H(x^0, u^0)$ remains a P-matrix. Consequently, by the dimension reduction described in Subsection III-B, the NCS (1) is locally (near the time 0) equivalent to one where the algebraic variable is of dimension at least one less than m. Moreover, the strong regularity condition is preserved in the reduced NCS whose defining functions remain analytic. The induction hypothesis now completes the proof. The argument for the case where $\gamma_1 \neq \emptyset$ is similar and is omitted. □

V. Extension to DVIs Via KKT Conditions

Theorem 2 can be extended to a differential variational inequality where the NCP is replaced by a VI. Specifically, consider the DVI

$$\begin{aligned} \dot{x}(t) &= F(x(t), u(t)) \\ u(t) &\in \mathrm{SOL}(K(x(t)), H(x(t), \cdot)) \end{aligned}$$

where, for each $x \in \Re^n$,

$$K(x) \equiv \{u \in \Re^m : g(x, u) \le 0\}$$

is such that $g : \Re^{n+m} \to \Re^\ell$ is analytic, and for each $i = 1, \ldots, \ell$, $g_i(x, \cdot)$ is convex. The notation $u \in \mathrm{SOL}(K(x), H(x, \cdot))$ means that u is a solution of the VI defined by the pair $(K(x), H(x, \cdot))$, i.e., $u \in K(x)$ and

$$(u' - u)^T H(x, u) \ge 0 \qquad \forall u' \in K(x).$$

Under some standard constraint qualifications for the function g at all such pairs (x, u), it follows that $u \in \mathrm{SOL}(K(x), H(x, \cdot))$ if and only if there exists $\lambda \in \Re^\ell$ such that

$$\begin{aligned} H(x, u) + \sum_{j=1}^\ell \lambda_j \nabla_u g_j(x, u) = 0 \\ 0 \le \lambda \perp g(x, u) \le 0 \end{aligned}$$

which are the KKT conditions of the VI $(K(x), H(x, \cdot))$. We can similarly speak about the strong regularity of a KKT pair (u, λ) corresponding to a given x [23], [10]. There is also an extended matrix-theoretic characterization of this condition; see the cited references.

Let $(x^*(t), u^*(t), \lambda^*(t))$ be a solution trajectory passing through (x^0, u^0, λ^0) at time t_0 and satisfying

$$\begin{aligned}\dot{x}(t) &= F(x(t), u(t))\\ 0 &= H(x(t), u(t)) + \sum_{j=1}^{\ell} \lambda_j(t) \nabla_u g_j(x(t), u(t)) \\ 0 \le \lambda(t) &\perp g(x(t), u(t)) \le 0.\end{aligned} \tag{28}$$

For notational convenience, we let

$$\mathbf{L}(x, u, \lambda) \equiv H(x, u) + \sum_{j=1}^{\ell} \lambda_j \nabla_u g_j(x, u)$$

be the (vector-valued) VI Lagrangian function. Assuming the pair (u^0, λ^0) is a strongly regular solution of the KKT conditions corresponding to x^0, Theorem 9 extends Theorem 2 to the DVI. The extended theorem is a strong non-Zenoness result for the DVI phrased in terms of the constancy of the active constraints and their multipliers. There is a similar extension of Corollary 3 which we omit.

Theorem 9: In the above setting, there exist a scalar $\varepsilon > 0$ and two triples of index sets, $(\alpha_+, \beta_+, \gamma_+)$ and $(\alpha_-, \beta_-, \gamma_-)$ such that for all $t \in (t_0, t_0 + \varepsilon]$

$$\begin{aligned}\alpha_+ &= \{j : \lambda_j^*(t) > 0 = g_j(x^*(t), u^*(t))\}\\ \beta_+ &= \{j : \lambda_j^*(t) = 0 = g_j(x^*(t), u^*(t))\}\\ \gamma_+ &= \{j : \lambda_j^*(t) = 0 > g_j(x^*(t), u^*(t))\}\end{aligned}$$

and for all $t \in [t_0 - \varepsilon, t_0)$,

$$\begin{aligned}\alpha_- &= \{j : \lambda_j^*(t) > 0 = g_j(x^*(t), u^*(t))\}\\ \beta_- &= \{j : \lambda_j^*(t) = 0 = g_j(x^*(t), u^*(t))\}\\ \gamma_- &= \{j : \lambda_j^*(t) = 0 > g_j(x^*(t), u^*(t))\}\end{aligned}$$

Proof: The strong regularity of the pair (u^0, λ^0) implies that the matrix

$$\begin{bmatrix} J_u\mathbf{L}(x^0, u^0, \lambda^0) & J_u g_{\alpha_0}(x^0, u^0)^T \\ -J_u g_{\alpha_0}(x^0, u^0) & 0 \end{bmatrix}$$

is nonsingular; moreover, the Schur complement of the previous matrix in

$$\begin{bmatrix} J_u\mathbf{L}(x^0, u^0, \lambda^0) & J_u g_{\alpha_0}(x^0, u^0)^T & J_u g_{\beta_0}(x^0, u^0)^T \\ -J_u g_{\alpha_0}(x^0, u^0) & 0 & 0 \\ -J_u g_{\beta_0}(x^0, u^0) & 0 & 0 \end{bmatrix}$$

which is equal to

$$\begin{aligned}&[\, J_u g_{\beta_0}(x^0, u^0) \quad 0 \,]\\ &\times \begin{bmatrix} J_u\mathbf{L}(x^0, u^0, \lambda^0) & J_u g_{\alpha_0}(x^0, u^0)^T \\ -J_u g_{\alpha_0}(x^0, u^0)^T & 0 \end{bmatrix}^{-1}\\ &\times \begin{bmatrix} J_u g_{\beta_0}(x^0, u^0)^T \\ 0 \end{bmatrix}\end{aligned}$$

is a P-matrix. These two properties imply that locally near time t_0, we can apply the implicit function theorem for smooth equations to solve for the variables (u, λ_{α_0}) in terms of (x, λ_{β_0}), thereby reducing the system (28) to an equivalent NCS:

$$\dot{x} = \hat{f}(x, \lambda_{\beta_0}), \quad 0 \le \lambda_{\beta_0} \perp \hat{H}_{\beta_0}(x, \lambda_{\beta_0}) \ge 0$$

for some functions $\hat{f}$ and $\hat{H}_{\beta_0}$ that are analytic in a neighborhood of the pair $(x^0, \lambda^0_{\beta_0})$. Moreover, by a proof similar to that of Lemma 6, it can be shown that $\lambda^0_{\beta_0} = 0$ is a strongly regular solution of the NCP: $0 \le \lambda_{\beta_0} \perp \hat{H}_{\beta_0}(x^0, \lambda_{\beta_0}) \ge 0$. The theorem now follows from Theorem 2. □

VI. Local Observability of the NCS

We address the second topic of this paper, namely the local observability of a given state of a differential variational system. Consider the DVI

$$\begin{aligned}&\dot{x}(t) = F(x(t), u(t)), \quad x(0) = \xi\\ &u(t) \in \mathrm{SOL}(K, H(x(t), \cdot))\end{aligned} \tag{29}$$

where the set $K \subseteq \Re^\ell$ is closed convex and is independent of the state variable. For simplicity, we assume that F and H are continuously differentiable on $\Re^{n+m}$. (The analyticity of these functions is not needed until we apply Theorem 2.) Of particular importance is the case where K is a polyhedron. This case is already quite broad and includes the mixed NCP, where $K \equiv \Re^{\ell_1} \times \Re^{\ell_2}_+$ for some positive integers ℓ_1 and ℓ_2 whose sum is ℓ; in turn, the differential KKT system (28) is a special mixed NCP. Let $(x(t,\xi), u(t,\xi))$ denote a solution trajectory of (29) defined on a time interval $[0, T]$ for some $T > 0$. Note that $x(0, \xi) = \xi$ for every ξ. Moreover, it is worth mentioning that $u(t, \xi)$ is a function of $x(t, \xi)$ (see the discussion at the beginning of Section II), thus the observability of NCS is different from that of nonlinear systems with u being treated as a control input. Consider a given output function $\Phi : \Re^n \to \Re^p$; we are interested in the uniqueness of an initial condition ξ^0 based on the "observed trajectory" $y(t, \xi) \equiv \Phi(x(t, \xi))$. This is made precise in the following.

Definition 10: For a given $T > 0$, the state $\xi^0 \in \Re^n$ is said to be T- *time locally observable* for the DVI (29) if there exists a neighborhood $\tilde{\mathcal{N}_0}$ of ξ^0 such that

$$\left.\begin{array}{rcll} \Phi(x(t,\xi)) & = & \Phi(x(t,\xi^0)) & \forall t \in [0, T] \\ \xi & \in & \tilde{\mathcal{N}_0} & \end{array}\right\} \Rightarrow \xi = \xi^0.$$

If there exists a scalar $\varepsilon_0 > 0$ such that ξ^0 is ε-time locally observable for the DVI (29) for all $\varepsilon \in (0, \varepsilon_0]$, then we say that ξ^0 is *short-time locally observable* for the DVI (29). □

Clearly, for any two positive scalars $T < T'$, T-time local observability implies T'-time local observability. In particular, short-time local observability implies T-time local observability for any $T > 0$, but the converse is not necessarily true. In fact, the treatment of T-time observability for a given $T > 0$ is not an easy task; for details, see [7] where other notions of observability are also defined and analyzed. Here, we only treat short-time local observability for forward-time trajectories, although a similar analysis can be made for backward-time trajectories.

Implicit in Definition 10 is the existence of the solution trajectory $x(t, \xi)$ in the time interval $[0, T]$ for all initial conditions ξ sufficiently near ξ^0. For the analysis in this section, we assume that the VI $(K, H(\xi^0, \cdot))$ has a strongly regular solution u^0. Under this assumption, there exist neighborhoods $\mathcal{V}_0$ of ξ^0 and $\mathcal{U}_0$ of u^0 and a Lipschitz continuous function $u : \mathcal{V}_0 \to \mathcal{U}_0$ such that for every $x \in \mathcal{V}_0$, $u(x)$ is the unique solution of the VI $(K, H(x, \cdot))$ that lies in $\mathcal{U}_0$. Consequently, locally near ξ^0, the DVI (29) is equivalent to the ODE: $\dot{x} = F(x, u(x))$, whose right-hand side is Lipschitz continuous in x. Hence, there exist a neighborhood $\mathcal{N}_0$ of ξ^0 and an appropriate time $T_0 > 0$ such that the solution trajectory $(x(t,\xi), u(t,\xi))$ is well-defined for all $(t,\xi) \in [0,T_0] \times \mathcal{N}_0$; moreover, for every $\xi \in \mathcal{N}_0, x(\cdot,\xi)$ is continuously differentiable, $u(t,\xi) = u(x(t,\xi))$, and for any ξ and ξ' in $\mathcal{N}_0$

$$\|x(t,\xi) - x(t,\xi')\| \le e^{tL_F}\|\xi - \xi'\| \qquad \forall t \in [0,T_0]$$

where L_F is a Lipschitz constant of the composite function $x \mapsto F(x,u(x))$ near ξ^0. Furthermore, if K is polyhedral, then for every $t \in [0,T_0], x(t,\cdot)$ is directionally differentiable at ξ^0 for every fixed $t \in [0,T_0]$; this was first proved in [28] in the general context of differential inclusions and reproved in [22] by an elementary argument. The directional derivative of the solution map $x(t,\cdot)$ at ξ^0 along the direction η, denoted by and defined as

$$x'_\xi(t,\xi^0;\eta) \equiv \lim_{\tau\downarrow 0} \frac{x(t,\xi^0+\tau\eta) - x(t,\xi^0)}{\tau}$$

is the unique function $y(t)$, which, together with $(x(t,\xi^0), u(t,\xi^0))$ and a suitable $v(t)$, satisfies the following "first variational system":

$$\begin{aligned}
\dot{x}(t) &= F(x(t),u(t))\\
\dot{y}(t) &= J_xF(x(t),u(t))y(t) + J_uF(x(t),u(t))v(t)\\
u(t) &\in \mathrm{SOL}(K, H(x(t),\cdot))\\
\mathcal{C}(u(t)) \ni v(t) &\perp J_xH(x(t),u(t))y(t)\\
&\quad + J_uH(x(t),u(t))v(t) \in \mathcal{C}(u(t))^*\\
x(0) &= \xi^0, \quad y(0) = \eta
\end{aligned}$$

where $\mathcal{C}(u)$ is the "critical cone" [10] of the pair $(K, H(x,\cdot))$ at the solution $u \in \mathrm{SOL}(K, H(x,\cdot))$; see [22]. In turn, the latter cone is equal to the intersection of the tangent cone $\mathcal{T}(u;K)$ of K at $u \in K$ and the orthogonal complement $H(x,u)^\perp$ of the vector $H(x,u)$, and $\mathcal{C}(u)^*$ is the dual cone of $\mathcal{C}(u)$. Apart from the previous characterization, $x'_\xi(t,\xi^0;\cdot)$ is positively homogeneous in the third argument, i.e., $x'_\xi(t,\xi^0;\tau\eta) = \tau x'_\xi(t,\xi^0;\eta)$ for all scalars $\tau \ge 0$; moreover the following limit holds:

$$\lim_{\xi^0 \ne \xi \to \xi^0} \frac{x(t,\xi) - x(t,\xi^0) - x'_\xi(t,\xi^0;\xi-\xi^0)}{\|\xi - \xi^0\|} = 0. \tag{30}$$

The latter is due to the Lipschitz continuity of $x(t,\cdot)$ near ξ^0; see [25]. These properties are sufficient for the applicability of the following general result pertaining to the locally observability of a given state of an ODE with a B(ouligand)-differentiable right-hand side. (Formally, a function $\Psi : \Re^n \to \Re^n$ is *B-differentiable* at a vector $x \in \Re^n$ if Ψ is Lipschitz continuous in a neighborhood of x and directionally differentiable at x.) In the following, the directional derivative of Φ at x along the direction η is denoted by $\Phi'(x;\eta)$, i.e.,

$$\Phi'(x;\eta) \equiv \lim_{\tau\downarrow 0} \frac{\Phi(x+\tau\eta) - \Phi(x)}{\tau}.$$

Theorem 11: Let $\Psi, \Phi : \Re^n \to \Re^n$ be B-differentiable at $\xi^0 \in \Re^n$. Let $x(t,\xi)$, for $(t,\xi) \in [0,T] \times \mathcal{N}$, be a solution trajectory of the ODE: $\dot{x} = \Psi(x), x(0) = \xi$, where $T > 0$ is a suitable scalar and $\mathcal{N}$ is a suitable neighborhood of ξ^0. Then, $x(t,\cdot)$ is B-differentiable at ξ^0 for all $t \in [0,T]$. Moreover, if $\varepsilon_0 \in (0,T]$ exists such that for all $\varepsilon \in (0,\varepsilon_0]$

$$\left\{\Phi'(x(t,\xi^0); x'_\xi(t,\xi^0;\eta)) = 0 \ \forall t \in [0,\varepsilon]\right\} \Rightarrow \eta = 0 \tag{31}$$

then ξ^0 is T'-time locally observable for the ODE: $\dot{x} = \Psi(x)$ with respect to Φ for any $T' \in (0,\varepsilon_0]$.

Proof: Assume for contradiction that ξ^0 is not locally observable as stated. Then, there exists a sequence of vectors $\{\xi^k\}$ converging to ξ^0 such that $\xi^k \ne \xi^0$ for every $k \ne 0$ and $\Phi(x(t,\xi^k)) = \Phi(x(t,\xi^0))$ for all $t \in [0,T']$, for some $T' \in (0,\varepsilon_0]$. For every fixed but arbitrary $t \in [0,T']$, we can write

$$\begin{aligned}
\Phi(x(t,\xi^k)) &= \Phi(x(t,\xi^0)) + \Phi'(x(t,\xi^0);\, x(t,\xi^k) - x(t,\xi^0))\\
&\quad + \Phi(x(t,\xi^k)) - \Phi(x(t,\xi^0))\\
&\quad - \Phi'(x(t,\xi^0); x(t,\xi^k) - x(t,\xi^0))\\
&= \Phi(x(t,\xi^0)) + \Phi'(x(t,\xi^0); x'_\xi(t,\xi^0;\xi^k - \xi^0))\\
&\quad + [\Phi'(x(t,\xi^0); x(t,\xi^k) - x(t,\xi^0))\\
&\quad - \Phi'(x(t,\xi^0); x'_\xi(t,\xi^0;\xi^k - \xi^0))]\\
&\quad + \Phi(x(t,\xi^k)) - \Phi(x(t,\xi^0))\\
&\quad - \Phi'(x(t,\xi^0); x(t,\xi^k) - x(t,\xi^0)).
\end{aligned}$$

Therefore, for every $t \in [0,T']$

$$\begin{aligned}
0 &= \Phi'(x(t,\xi^0); x'_\xi(t,\xi^0;\xi^k - \xi^0))\\
&\quad + \Phi'(x(t,\xi^0); x(t,\xi^k) - x(t,\xi^0))\\
&\quad - \Phi'(x(t,\xi^0); x'_\xi(t,\xi^0;\xi^k - \xi^0))\\
&\quad + \Phi(x(t,\xi^k)) - \Phi(x(t,\xi^0))\\
&\quad - \Phi'(x(t,\xi^0); x(t,\xi^k) - x(t,\xi^0)).
\end{aligned}$$

Let η be any accumulation point of the normalized sequence $\{(\xi^k - \xi^0)/\|\xi^k - \xi^0\|\}$; such a point must exist and be nonzero. By the B-differentiability of Φ, it follows that there exist scalars $T_0 \in (0,T']$ and $L_\Phi > 0$ such that for all $t \in [0,T_0]$

$$\|\Phi'(x(t,\xi^0);v) - \Phi'(x(t,\xi^0);v')\| \le L_\Phi\|v - v'\|$$

for all $v, v' \in \Re^n$. Hence, we have

$$\begin{aligned}
&\|\Phi'(x(t,\xi^0); x(t,\xi^k) - x(t,\xi^0))\\
&\quad - \Phi'(x(t,\xi^0); x'_\xi(t,\xi^0;\xi^k - \xi^0))\|\\
&\qquad \le L_\Phi\|x(t,\xi^k) - x(t,\xi^0)) - x'_\xi(t,\xi^0;\xi^k - \xi^0))\|.
\end{aligned}$$

Consequently, using (30) and taking limits in the above expressions, we deduce $\Phi'(x(t,\xi^0); x'_\xi(t,\xi^0;\eta)) = 0$ for all $t \in [0,T_0]$. This contradicts (31) because $\eta \ne 0$. $\square$

Theorem 11 can be applied to the DVI (29) at an initial condition ξ^0 via the equivalent formulation $\dot{x} = F(x,u(x))$, provided that the VI $(K, H(\xi^0,\cdot))$ has a strongly regular solution u^0 and that the implicit function $u(x)$ is B-differentiable (recall that the latter is valid if K is polyhedral). Instead of discussing the result in its generality, we consider the special case of the NCS treated in Theorem 2. For this theorem to be applicable, we assume throughout the rest of this section that F and H are analytic functions near the pair (ξ^0, u^0).

We begin by identifying the critical cone of an NCP (which has $K \equiv \Re^m_+$) at a solution. Specifically, for any pair (x,u) that satisfies $0 \le u \perp H(x,u) \ge 0$, the critical cone of this NCP at u is given by

$$\mathcal{C}(x,u) = \Re^{|\alpha(x,u)|} \times \Re_+^{|\beta(x,u)|} \times \{0\}^{|\gamma(x,u)|}$$

where

$$\begin{aligned}
\alpha(x,u) &\equiv \{i : u_i > 0 = H_i(x,u)\}\\
\beta(x,u) &\equiv \{i : u_i = 0 = H_i(x,u)\}\\
\gamma(x,u) &\equiv \{i : u_i = 0 < H_i(x,u)\}.
\end{aligned}$$

By Theorem 2, it follows that a scalar $\varepsilon_{\rm n} > 0$ and a triple of index sets $(\alpha_{\rm n}, \beta_{\rm n}, \gamma_{\rm n})$ exist such that, for all $t \in (0, \varepsilon_{\rm n}]$

$$\begin{aligned}\alpha(x(t,\xi^0), u(t,\xi^0)) &= \alpha_{\rm n}\\ \beta(x(t,\xi^0), u(t,\xi^0)) &= \beta_{\rm n}\\ \gamma(x(t,\xi^0), u(t,\xi^0)) &= \gamma_{\rm n}.\end{aligned}$$

We call $(\alpha_{\rm n}, \beta_{\rm n}, \gamma_{\rm n})$ the *nominal triple* because it pertains to the *nominal trajectory* $(x(t,\xi^0), u(t,\xi^0))$. Note that

$$\alpha_0 \subseteq \alpha_{\rm n} \subseteq \alpha_0 \cup \beta_0, \quad \beta_0 \supseteq \beta_{\rm n},\ \gamma_0 \subseteq \gamma_{\rm n} \subseteq \gamma_0 \cup \beta_0$$

where $(\alpha_0, \beta_0, \gamma_0) \equiv (\alpha(\xi^0, u^0), \beta(\xi^0, u^0), \gamma(\xi^0, u^0))$ is the triple of index sets corresponding to the initial pair (ξ^0, u^0). For all $t \in (0, \varepsilon_{\rm n}]$, the condition

$$\begin{aligned}&\mathcal{C}(u(t,\xi^0)) \ni v(t) \perp J_x H(x(t,\xi^0), u(t,\xi^0)) y(t)\\ &\quad + J_u H(x(t,\xi^0), u(t,\xi^0)) v(t) \in \mathcal{C}(u(t,\xi^0))^*\end{aligned}$$

becomes the mixed LCP

$$\begin{aligned}&J_x H_{\alpha_{\rm n}}(x(t,\xi^0), u(t,\xi^0)) y(t)\\ &\qquad + J_u H_{\alpha_{\rm n}}(x(t,\xi^0), u(t,\xi^0)) v(t) = 0\\ &0 \le v_{\beta_{\rm n}}(t) \perp J_x H_{\beta_{\rm n}}(x(t,\xi^0), u(t,\xi^0)) y(t)\\ &\qquad + J_u H_{\beta_{\rm n}}(x(t,\xi^0), u(t,\xi^0)) v(t) \ge 0\\ &0 = v_{\gamma_{\rm n}}(t).\end{aligned}$$

Moreover, provided that $\varepsilon_{\rm n} > 0$ is properly restricted, the principal submatrix $J_{u_{\alpha_{\rm n}}} H_{\alpha_{\rm n}}(x(t,\xi^0), u(t,\xi^0))$ is nonsingular. Hence we can solve for the variable $v_{\alpha_{\rm n}}(t)$ from the first equation, yielding

$$\begin{aligned}v_{\alpha_{\rm n}}(t) = &-(J_{u_{\alpha_{\rm n}}} H_{\alpha_{\rm n}}(x(t,\xi^0), u(t,\xi^0)))^{-1}\\ &[J_x H_{\alpha_{\rm n}}(x(t,\xi^0), u(t,\xi^0)) y(t)\\ &+ J_{u_{\beta_{\rm n}}} H_{\alpha_{\rm n}}(x(t,\xi^0), u(t,\xi^0)) v_{\beta_{\rm n}}(t)]\end{aligned}$$

which we can substitute into the middle complementarity condition, thereby reducing the aforementioned mixed LCP into an equivalent standard LCP in the variable $v_{\beta_{\rm n}}(t)$ only

$$0 \le v_{\beta_{\rm n}}(t) \perp C^{\rm n}_{\beta_{\rm n}\bullet}(t) y(t) + D^{\rm n}_{\beta_{\rm n}\beta_{\rm n}}(t) v_{\beta_{\rm n}}(t) \ge 0$$

where $C^{\rm n}_{\beta_{\rm n}\bullet}(t)$ is equal to

$$\begin{aligned}&J_x H_{\beta_{\rm n}}(x(t,\xi^0), u(t,\xi^0)) - J_{u_{\alpha_{\rm n}}} H_{\beta_{\rm n}}(x(t,\xi^0)\\ &u(t,\xi^0)) \left[J_{u_{\alpha_{\rm n}}} H_{\alpha_{\rm n}}(x(t,\xi^0), u(t,\xi^0))\right]^{-1}\\ &J_x H_{\alpha_{\rm n}}(x(t,\xi^0), u(t,\xi^0))\end{aligned}$$

and $D^{\rm n}_{\beta_{\rm n}\beta_{\rm n}}(t)$, which is equal to

$$\begin{aligned}&J_{u_{\beta_{\rm n}}} H_{\beta_{\rm n}}(x(t,\xi^0), u(t,\xi^0)) - J_{u_{\alpha_{\rm n}}} H_{\beta_{\rm n}}(x(t,\xi^0)\\ &u(t,\xi^0)) \left[J_{u_{\alpha_{\rm n}}} H_{\alpha_{\rm n}}(x(t,\xi^0), u(t,\xi^0))\right]^{-1}\\ &J_{u_{\beta_{\rm n}}} H_{\alpha_{\rm n}}(x(t,\xi^0), u(t,\xi^0))\end{aligned}$$

is a P-matrix. Thus, the directional derivative $x'_\xi(t,\xi^0;\eta)$ is the unique function $y(t)$, which together with an appropriate $v_{\beta_{\rm n}}(t)$, satisfies the following time-dependent LCS:

$$\begin{aligned}&\dot y(t) = A^{\rm n}(t) y(t) + B^{\rm n}_{\bullet\beta_{\rm n}}(t) v_{\beta_{\rm n}}(t), \quad y(0) = \eta\\ &0 \le v_{\beta_{\rm n}}(t) \perp C^{\rm n}_{\beta_{\rm n}\bullet}(t) y(t) + D^{\rm n}_{\beta_{\rm n}\beta_{\rm n}}(t) v_{\beta_{\rm n}}(t) \ge 0\end{aligned} \tag{32}$$

where

$$\begin{aligned}A^{\rm n}(t) \equiv\ &J_x F(x(t,\xi^0), u(t,\xi^0)) - J_{u_{\alpha_{\rm n}}} F(x(t,\xi^0), u(t,\xi^0))\\ &\times \left[J_{u_{\alpha_{\rm n}}} H_{\alpha_{\rm n}}(x(t,\xi^0), u(t,\xi^0))\right]^{-1}\\ &\times J_x H_{\alpha_{\rm n}}(x(t,\xi^0), u(t,\xi^0))\end{aligned}$$

$$\begin{aligned}B^{\rm n}_{\bullet\beta_{\rm n}}(t) \equiv\ &J_{u_{\beta_{\rm n}}} F(x(t,\xi^0), u(t,\xi^0)) - J_{u_{\alpha_{\rm n}}} F(x(t,\xi^0), u(t,\xi^0))\\ &\times \left[J_{u_{\alpha_{\rm n}}} H_{\alpha_{\rm n}}(x(t,\xi^0), u(t,\xi^0))\right]^{-1}\\ &\times J_{u_{\beta_{\rm n}}} H_{\alpha_{\rm n}}(x(t,\xi^0), u(t,\xi^0)).\end{aligned}$$

We let $(y^{\rm n}(t,\eta), v^{\rm n}_{\beta_{\rm n}}(t,\eta))$ denote the pair of solution trajectories to (32), which we call the *directional* LCS of the nominal trajectory $(x(t,\xi^0), u(t,\xi^0))$. Specializing Theorem 11, we obtain the following observability result for the NCS (1) with a differentiable output function. No proof is needed.

Corollary 12: Suppose that F and H are analytic functions in a neighborhood of (ξ^0, u^0), where u^0 is a strongly regular solution of the NCP: $0 \le u \perp H(\xi^0, u) \ge 0$. Let the output function Φ be continuously differentiable in a neighborhood of ξ^0. If $\varepsilon_0 > 0$ exists such that for all $\varepsilon \in (0, \varepsilon_0]$

$$\{J\Phi(x(t,\xi^0)) y^{\rm n}(t;\eta) = 0\ \forall t \in [0,\varepsilon]\} \Rightarrow \eta = 0 \tag{33}$$

then ξ^0 is short-time locally observable for the NCS (1). □

Informally speaking, the implication (33) postulates that the zero state is short-time locally observable for the directional LCS (32) with respect to the time-dependent matrix $J\Phi(x(t,\xi^0))$. In the case where the output function Φ is linear, say $\Phi(x) = Hx$ for some matrix $H \in \Re^{p\times n}$, Corollary 12 reduces the short-time local observability of a given state of the NCS (1) to that of the zero state of the directional LCS (32) with respect to the linear output. In general, with the matrices $A^{\rm n}(t), B^{\rm n}_{\bullet\beta_{\rm n}}(t), C^{\rm n}_{\beta_{\rm n}\bullet}(t)$, and $D^{\rm n}_{\beta_{\rm n}\beta_{\rm n}}(t)$ being analytic functions of t, and with $D^{\rm n}_{\beta_{\rm n}\beta_{\rm n}}(t)$ being a P-matrix, by Corollary 4, it follows that for every $\eta \in \Re^n$, the solution pair $(y^{\rm n}(t,\eta), v^{\rm n}(t,\eta))$ is analytic at every time $t > 0$ sufficiently small. Since the nominal trajectory $x(t,\xi^0)$ is also analytic at such times, one could use a series expansion for $J\Phi(x(t,\xi^0)) y^{\rm n}(t;\eta)$ and obtain some sufficient conditions in terms of the matrices $A^{\rm n}(t), B^{\rm n}_{\bullet\beta_{\rm n}}(t), C^{\rm n}_{\beta_{\rm n}\bullet}(t), D^{\rm n}_{\beta_{\rm n}\beta_{\rm n}}(t)$, and the derivatives of the output function Φ to ensure the validity of the implication (33), thereby establishing the short-time local observability of ξ^0. These details are fairly technical in general. In the next section, we apply this idea to the LCS where improved results can be obtained more succinctly in terms of some constant matrices.

VII. Local Observability of the LCS

Throughout this section, we specialize Corollary 12 to the LCS

$$\begin{aligned}&\dot x = Ax + Bu, \quad x(0) = \xi\\ &0 \le u \perp Cx + Du \ge 0\end{aligned} \tag{34}$$

with a linear output $\Phi(x) \equiv Hx$ for some given matrix $H \in \Re^{p\times n}$. For simplicity, D is assumed to be a P-matrix. Thus, the pair of solution trajectories $(x(t,\xi), u(t,\xi))$ exists and is unique for all $(t,\xi) \in [0,\infty) \times \Re^n$. Due to the constancy of the matrices A, B, C, and D, the analytic functions mentioned at the end of the last paragraph can be written down explicitly. More importantly, as we see later (cf. Definition 13 and Theorem 16) the sufficient condition (33) is related to the nonnegativity of certain analytic functions near $t = 0$, which in turn is equivalent to the "lexicographical nonnegativity" of the sequences of coefficient vectors in the infinite series representations of these functions.

It should be pointed out that an LCS is *not* a linear system, though it has certain linear structure in the dynamics and the associated complementarity problem. In fact, it is a piecewise linear, thus nonsmooth, system. Thus, unlike time-invariant linear systems (possibly with control inputs) for which all the observability notions are equivalent, one cannot assert the same equivalence for the LCS.

For any subset β_a of $\beta_{\rm n}$, let $\bar{\beta}_a \equiv \beta_{\rm n} \backslash \beta_a$ and define the $m \times n$ matrix $\overline{C}^{\rm n}(\beta_a)$ to be as shown in

$$\begin{bmatrix} -\begin{bmatrix} D_{\alpha_{\rm n}\alpha_{\rm n}} & D_{\alpha_{\rm n}\beta_a} \\ D_{\beta_a\alpha_{\rm n}} & D_{\beta_a\beta_a} \end{bmatrix}^{-1} \begin{bmatrix} C_{\alpha_{\rm n}\bullet} \\ C_{\beta_a\bullet} \end{bmatrix} \\ \begin{bmatrix} C_{\bar{\beta}_a\bullet} \\ C_{\gamma_{\rm n}\bullet} \end{bmatrix} - \begin{bmatrix} D_{\bar{\beta}_a\alpha_{\rm n}} & D_{\bar{\beta}_a\beta_a} \\ D_{\gamma_{\rm n}\alpha_{\rm n}} & D_{\gamma_{\rm n}\beta_a} \end{bmatrix} \begin{bmatrix} D_{\alpha_{\rm n}\alpha_{\rm n}} & D_{\alpha_{\rm n}\beta_a} \\ D_{\beta_a\alpha_{\rm n}} & D_{\beta_a\beta_a} \end{bmatrix}^{-1} \begin{bmatrix} C_{\alpha_{\rm n}\bullet} \\ C_{\beta_a\bullet} \end{bmatrix} \end{bmatrix}$$

and the $n \times n$ matrix $A^{\rm n}(\beta_a)$ to be

$$A - [\,B_{\bullet\alpha_{\rm n}} \quad B_{\bullet\beta_a}\,] \begin{bmatrix} D_{\alpha_{\rm n}\alpha_{\rm n}} & D_{\alpha_{\rm n}\beta_a} \\ D_{\beta_a\alpha_{\rm n}} & D_{\beta_a\beta_a} \end{bmatrix}^{-1} \begin{bmatrix} C_{\alpha_{\rm n}\bullet} \\ C_{\beta_a\bullet} \end{bmatrix}.$$

Observe that there are only $2^{|\beta_{\rm n}|}$ matrices $\overline{C}^{\rm n}(\beta_a)$, each corresponding to one subset β_a of $\beta_{\rm n}$. In the important special case where the initial u^0 is a *nondegenerate solution* of the LCP: $0 \leq u \perp C\xi^0 + Du \geq 0$, then $\beta_0 = \beta_{\rm n} = \emptyset$ and all the matrices $\overline{C}^{\rm n}(\beta_a)$ collapse to one single matrix, which is equal to

$$\overline{C}^{\rm n}(\emptyset) = \begin{bmatrix} -(D_{\alpha_0\alpha_0})^{-1}C_{\alpha_0\bullet} \\ C_{\gamma_0\bullet} - D_{\gamma_0\alpha_0}(D_{\alpha_0\alpha_0})^{-1}C_{\alpha_0\bullet} \end{bmatrix}. \tag{35}$$

The same is true for the matrices $A^{\rm n}(\beta_0)$, which collapses to

$$A^{\rm n}(\emptyset) = A^{\rm n} = A - B_{\bullet\alpha_0}(D_{\alpha_0\alpha_0})^{-1}C_{\alpha_0\bullet}. \tag{36}$$

The matrices $\overline{C}^{\rm n}(\beta_a)$ and $A^{\rm n}(\beta_a)$ arise when we consider the following restricted differential algebraic equation:

$$\begin{aligned} \dot{y} &= Ay + B_{\bullet\alpha_{\rm n}}v_{\alpha_{\rm n}} + B_{\bullet\beta_a}v_{\beta_a} \\ 0 &= C_{\alpha_{\rm n}\bullet}y + D_{\alpha_{\rm n}\alpha_{\rm n}}v_{\alpha_{\rm n}} + D_{\alpha_{\rm n}\beta_a}v_{\beta_a} \\ 0 &= C_{\beta_a\bullet}y + D_{\beta_a\alpha_{\rm n}}v_{\alpha_{\rm n}} + D_{\beta_a\beta_a}v_{\beta_a} \end{aligned} \tag{37}$$

which is derived from fixing some of the complementarity conditions in (34) at equalities. The unique solution of (37) satisfying the initial condition $y(0) = \eta$ is given by

$$y(t,\eta) = e^{A^{\rm n}(\beta_a)t}\eta = \sum_{j=0}^{\infty} \frac{t^j}{j!} A^{\rm n}(\beta_a)^j \eta$$

$$\begin{pmatrix} v^{\rm n}_{\alpha_{\rm n}}(t,\eta) \\ v^{\rm n}_{\beta_a}(t,\eta) \end{pmatrix} = -\begin{bmatrix} D_{\alpha_{\rm n}\alpha_{\rm n}} & D_{\alpha_{\rm n}\beta_a} \\ D_{\beta_a\alpha_{\rm n}} & D_{\beta_a\beta_a} \end{bmatrix}^{-1} \begin{bmatrix} C_{\alpha_{\rm n}\bullet} \\ C_{\beta_a\bullet} \end{bmatrix} y(t,\eta).$$

For our purpose, the subvector

$$\begin{aligned} \begin{pmatrix} v^{\rm n}_{\beta_a}(t,\eta) \\ w^{\rm n}_{\bar{\beta}_a}(t,\eta) \end{pmatrix} &= \overline{C}^{\rm n}_{\beta_{\rm n}\bullet}(\beta_a)y(t,\eta) \\ &= \sum_{j=0}^{\infty} \frac{t^j}{j} \overline{C}^{\rm n}_{\beta_{\rm n}\bullet}(\beta_a)A^{\rm n}(\beta_a)^j\eta \end{aligned} \tag{38}$$

plays a particularly important role in the observability of the state ξ^0. Note that the aforementioned vector is a polynomial function of the form $\sum_{j=0}^{\infty}(t^j)/(j!)v^j$, where $v^j \equiv \overline{C}^{\rm n}_{\beta_{\rm n}\bullet}(\beta_a)A^{\rm n}(\beta_a)^j\eta$. We introduce a lexicographic notion that will ensure $(v^{\rm n}_{\beta_a}(t,\eta), w^{\rm n}_{\bar{\beta}_a}(t,\eta)) \geq 0$ for all $t \geq 0$ sufficiently small.

Definition 13: We say that a sequence of vectors $\{v^j\}_{j=0}^{\infty} \subset \Re^k$ is *lexicographically nonnegative* if for every $i = 1,\ldots,k$, either the sequence of scalars $\{v_i^j\}_{j=0}^{\infty}$ is identically zero, or the first nonzero member of the sequence is positive. □

Recall that for a given pair of matrices $M \in \Re^{n\times n}$ and $N \in \Re^{m\times n}$, the unobservable space of (N,M), denoted $\overline{O}(N,M)$, is the set of vectors $\xi \in \Re^n$ such that $NM^j\xi = 0$ for all $j = 0,1,2,\ldots$. By the well-known Cayley-Hamilton theorem, it follows that $\xi \in \overline{O}(N,M)$ if and only if $NM^j\xi = 0$ for all $j = 0,1,2,\ldots,n-1$. Elements of the space $\overline{O}(N,M)$ are said to be *unobservable* with respect to the pair (N,M). If $\overline{O}(N,M)$ consists only of the zero vector, then (N,M) is called an *observable pair*. If $N' \in \Re^{m\times n}$ is another matrix of the same dimension as N, we call the elements in $\overline{O}(N,M)\cap\overline{O}(N',M)$ *jointly unobservable* with respect to the two pairs (N,M) and (N',M). We call these two pairs of matrices *jointly observable* if $\overline{O}(N,M)\cap\overline{O}(N',M) = \{0\}$. Note that it is possible for the two pairs (N,M) and (N',M) to be jointly observable without each individual pair being observable by itself. We give in the next result a necessary condition for the LCS to have at least one short-time locally observable state in terms of two jointly observable pairs of matrices.

Proposition 14: A necessary condition for the LCS (34) to have a short-time locally observable state is that the two pairs (H,A) and (C,A) are jointly observable.

Proof: Let η be any nonzero jointly unobservable vector with respect to the two pairs (H,A) and (C,A). It is easy to show that, for any $\xi^0 \in \Re^n$, $x(t,\xi^0+\tau\eta) = x(t,\xi^0) + \tau e^{tA}\eta$ and $u(t,\xi^0+\tau\eta) = u(t,\xi^0)$ for all $t \geq 0$ and all τ. Hence $Hx(t,\xi^0+\tau\eta) = Hx(t,\xi^0)$ for all $t \geq 0$. Since $\eta \neq 0$, the state ξ^0 cannot be short-time locally observable. □

Joint observability is in general not sufficient for a state of the LCS to be short-time locally observable. We need to extend this classical observability concept. Specifically, we define the *semiunobservable cone* of (N,M), which we denote $\overline{SO}(N,M)$, to be the set of vectors $\xi \in \Re^n$ such that the sequence of vectors $\{NM^j\xi\}_{j=0}^{\infty}$ is lexicographically nonnegative. Elements of $\overline{SO}(N,M)$ are said to be *semiunobservable* with respect to the pair (N,M). It can be seen that $\overline{SO}(N,M)$ is a convex cone, containing the closed convex cone of vectors $\xi \in \Re^n$ for which $NM^j\xi \geq 0$ for all $j \geq 0$. Unlike $\overline{O}(N,M)$, which is a linear subspace, the cone $\overline{SO}(N,M)$ is in general not even closed as demonstrated by simple examples. Note that $\overline{SO}(N,M) = \bigcap_{i=1}^m \overline{SO}(N_{i\bullet},M)$. Interestingly, $\overline{SO}(N,M)\cap(-\overline{SO}(N,M)) = \overline{O}(N,M)$. Hence, $\overline{SO}(N,M)$ is pointed, i.e., $\overline{SO}(N,M)\cap(-\overline{SO}(N,M)) = \{0\}$, if and only if the pair (N,M) is observable. The following lemma shows that checking whether a vector belongs to the cone $\overline{SO}(N,M)$ can be done by a finite procedure.

Lemma 15: A vector ξ belongs to $\overline{SO}(N,M)$ if and only if the finite sequence of vectors $\{NM^j\xi\}_{j=0}^{n-1}$ is lexicographically nonnegative.

Proof: Clearly, if the infinite sequence $\{NM^j\xi\}_{j=0}^{\infty}$ is lexicographically nonnegative, then so is the finite sequence $\{NM^j\xi\}_{j=0}^{n-1}$. Conversely, suppose that for some vector $\xi \in \Re^n$, the latter finite sequence is lexicographically nonnegative, but the former infinite sequence is not. Then there exist an index i and a nonnegative integer ℓ such that $(NM^j\xi)_i = 0$ for all $j = 0,1,\ldots,\ell-1$ and $(NM^\ell\xi)_i < 0$. We must have $\ell \geq n$. But by the Cayley-Hamilton theorem, we have coefficients $\{\lambda_j\}_{j=0}^{n-1}$ such that $M^\ell = \sum_{j=0}^{n-1}\lambda_j M^j$, which implies $(NM^\ell\xi)_i = \sum_{j=0}^{n-1}\lambda_j(NM^j\xi)_i = 0$. This is a contradiction. □

It follows from Lemma 15 that, for any cone P in $\Re^n$, $P \cap \overline{SO}(N, M) = \{0\}$ if and only if for every subset $\mathcal{I}$ of $\{1, \ldots, m\}$ and any tuple of integers $\{\ell_i\}_{i \in \mathcal{I}}$, where each $0 \leq \ell_i \leq n-1$, the linear inequality system as follows:

$$\begin{aligned} (NM^\ell \xi)_i &= 0 \qquad \forall \ell = 0, \ldots, \ell_i - 1 \qquad \forall i \in \mathcal{I} \\ (NM^{\ell_i} \xi)_i &\geq 1 \qquad \forall i \in \mathcal{I} \\ (NM^\ell \xi)_i &= 0 \qquad \forall \ell = 0, \ldots, n-1 \qquad \forall i \notin \mathcal{I} \end{aligned}$$

has no solution $\xi \in P$ if $\mathcal{I} \neq \emptyset$ and no nonzero solution $\xi \in P$ if $\mathcal{I} = \emptyset$. In turn, when P is polyhedral, checking the feasibility of such an inequality system can be accomplished by linear programming. Since there are only finitely many such index subsets $\mathcal{I}$ and tuples $\{\ell_i\}_{i \in \mathcal{I}}$, the problem of determining if $P \cap \overline{SO}(N, M) = \{0\}$ can therefore be decided by solving finitely many linear programs.

Recalling the expression (38), we deduce that the vectors $v^{\rm n}_{\beta_a}(t, \eta)$ and $w^{\rm n}_{\beta_a}(t, \eta)$ are nonnegative for all $t \geq 0$ sufficiently small if and only if $\eta \in \overline{SO}(C^{\rm n}_{\beta_{\rm n}\bullet}(\beta_a), A^{\rm n}(\beta_a))$. Thus a sufficient and a necessary condition for short-time local observability are obtained as follows.

Theorem 16: Consider the LCS (34). Let u^0 satisfy $0 \leq u^0 \perp C\xi^0 + Du^0 \geq 0$. The following statements hold.

a) If $\overline{O}(H, A^{\rm n}(\beta_a)) \cap \overline{SO}(\overline{C}^{\rm n}_{\beta_{\rm n}\bullet}(\beta_a), A^{\rm n}(\beta_a)) = \{0\}$, then ξ^0 is short-time locally observable.

b) A state ξ^0 is short-time locally observable only if $\overline{O}(H, A^{\rm n}(\beta_a)) \cap \overline{SO}(\overline{C}^{\rm n}_{\beta_0\bullet}(\beta_a), A^{\rm n}(\beta_a)) = \{0\}, \forall \beta_a \subseteq \beta_{\rm n}$.

Proof: "Sufficiency". It suffices to verify the implication (33). Note that $J\Phi(x(t, \xi^0))$ is the constant matrix H. Let η satisfy the left-hand side of (33) for some $\varepsilon > 0$. The time-dependent LCS (32) becomes a time-invariant LCS

$$\begin{aligned} \dot{y}^{\rm n}(t) &= A^{\rm n} y^{\rm n}(t) + B^{\rm n}_{\bullet\beta_{\rm n}} v^{\rm n}_{\beta_{\rm n}}(t), \quad y^{\rm n}(0) = \eta \\ 0 &\leq v^{\rm n}_{\beta_{\rm n}}(t) \perp C^{\rm n}_{\beta_{\rm n}\bullet} y^{\rm n}(t) + D^{\rm n}_{\beta_{\rm n}\beta_{\rm n}} v^{\rm n}_{\beta_{\rm n}}(t) \geq 0 \end{aligned}$$

where

$$\begin{aligned} A^{\rm n} &= A - B_{\bullet\alpha_{\rm n}} (D_{\alpha_{\rm n}\alpha_{\rm n}})^{-1} C_{\alpha_{\rm n}\bullet} \\ B^{\rm n}_{\bullet\beta_{\rm n}} &= B_{\bullet\beta_{\rm n}} - B_{\bullet\alpha_{\rm n}} (D_{\alpha_{\rm n}\alpha_{\rm n}})^{-1} D_{\alpha_{\rm n}\beta_{\rm n}} \\ C^{\rm n}_{\beta_{\rm n}\bullet} &= C_{\beta_{\rm n}\bullet} - D_{\beta_{\rm n}\alpha_{\rm n}} (D_{\alpha_{\rm n}\alpha_{\rm n}})^{-1} C_{\alpha_{\rm n}\bullet} \\ D^{\rm n}_{\beta_{\rm n}\beta_{\rm n}} &= D_{\beta_{\rm n}\beta_{\rm n}} - D_{\beta_{\rm n}\alpha_{\rm n}} (D_{\alpha_{\rm n}\alpha_{\rm n}})^{-1} D_{\alpha_{\rm n}\beta_{\rm n}}. \end{aligned}$$

Since $D^{\rm n}_{\beta_{\rm n}\beta_{\rm n}}$ is a P-matrix, it follows that a scalar $\varepsilon_\eta > 0$ such that the triple of index sets

$$\begin{aligned} &\{i \in \beta_{\rm n} : v^{\rm n}_i(t, \eta) > 0 = [C^{\rm n}_{\beta_{\rm n}\bullet} y^{\rm n}(t, \eta) + D^{\rm n}_{\beta_{\rm n}\beta_{\rm n}} v^{\rm n}_{\beta_{\rm n}}(t, \eta)]_i\} \\ &\{i \in \beta_{\rm n} : v^{\rm n}_i(t, \eta) = 0 = [C^{\rm n}_{\beta_{\rm n}\bullet} y^{\rm n}(t, \eta) + D^{\rm n}_{\beta_{\rm n}\beta_{\rm n}} v^{\rm n}_{\beta_{\rm n}}(t, \eta)]_i\} \\ &\{i \in \beta_{\rm n} : v^{\rm n}_i(t, \eta) = 0 < [C^{\rm n}_{\beta_{\rm n}\bullet} y^{\rm n}(t, \eta) + D^{\rm n}_{\beta_{\rm n}\beta_{\rm n}} v^{\rm n}_{\beta_{\rm n}}(t, \eta)]_i\} \end{aligned}$$

are equal to a constant triple of index sets $(\beta_a, \beta_b, \beta_c)$ for all $t \in (0, \varepsilon_\eta]$. Like ε_η, the triple $(\beta_a, \beta_b, \beta_c)$ depends on η. Recalling the derivation of the vectors $v^{\rm n}_{\beta_a}(t, \eta)$ and $w^{\rm n}_{\beta_a}(t, \eta)$, we deduce that, with $\varepsilon' \equiv \min(\varepsilon, \varepsilon_\eta)$,

$$\left.\begin{aligned} 0 &= H y^{\rm n}(t, \eta) = \textstyle\sum_{j=0}^\infty \frac{t^j}{j!} H A^{\rm n}(\beta_a)^j \eta \\ 0 &\leq \textstyle\sum_{j=0}^\infty \frac{t^j}{j!} \overline{C}^{\rm n}_{\beta_{\rm n}\bullet}(\beta_a) A^{\rm n}(\beta_a)^j \eta \end{aligned}\right\} \qquad \forall t \in [0, \varepsilon'].$$

For the above to hold, η must necessarily belong to $\overline{O}(H, A^{\rm n}(\beta_a)) \cap \overline{SO}(\overline{C}^{\rm n}_{\beta_{\rm n}\bullet}(\beta_a), A^{\rm n}(\beta_a) = \{0\}$.

"Necessity". Suppose that η is a nonzero vector belonging to $\overline{O}(H, A^{\rm n}(\beta_a)) \cap \overline{SO}(\overline{C}^{\rm n}_{\beta_0\bullet}(\beta_a), A^{\rm n}(\beta_a))$ for some subset β_a of $\beta_{\rm n}$. Define $y^{\rm n}(t, \eta) \equiv e^{A^{\rm n}(\beta_a)t}\eta$ and write $\overline{C}^{\rm n}(\beta_a) y(t, \eta) \equiv (v^{\rm n}_{\alpha_{\rm n}}(t, \eta), v^{\rm n}_{\beta_a}(t, \eta), w^{\rm n}_{\beta_a}(t, \eta), w^{\rm n}_{\gamma_{\rm n}}(t, \eta))$. Let $v^{\rm n}_i(t, \eta) = 0$ for all $i \notin \alpha_{\rm n} \cup \beta_a$. We then have $[Cy^{\rm n}(t, \eta) + Dv^{\rm n}(t, \eta)]_{\alpha_{\rm n}} = 0$ and $v^{\rm n}_{\gamma_{\rm n}}(t, \eta) = 0$. By the definition of the two index sets $\alpha_{\rm n}$ and $\gamma_{\rm n}$, we have $[Cx(t, \xi^0) + Du(t, \xi^0)]_{\alpha_{\rm n}} = 0 \leq u_{\alpha_{\rm n}}(t, \xi^0)$ and $[Cx(t, \xi^0) + Du(t, \xi^0)]_{\gamma_{\rm n}} \geq 0 = u_{\gamma_{\rm n}}(t, \xi^0)$. Since the sequence of vectors $\{C^{\rm n}_{\beta_0\bullet}(\beta_a) A^{\rm n}(\beta_a)^j \eta\}_{j=0}^\infty$ is lexicographically nonnegative, it follows that, for all $t \geq 0$ sufficiently small,

$$0 \leq [Cy^{\rm n}(t, \eta) + Dv^{\rm n}(t, \eta)]_{\beta_0} \perp v^{\rm n}_{\beta_0}(t, \eta) \geq 0 \tag{39}$$

which can be seen as follows. If $i \in \beta_0 \cap \alpha_{\rm n}$, then $[Cy^{\rm n}(t, \eta) + Dv^{\rm n}(t, \eta)]_i = 0$ for all $t \geq 0$ and $v^{\rm n}_i(t, \eta) \geq 0$ for all $t \geq 0$ is sufficiently small. The same is true for all $i \in \beta_a \subseteq \beta_0$. If $i \in \beta_0 \backslash (\alpha_{\rm n} \cup \beta_a)$, then $[Cy^{\rm n}(t, \eta) + Dv^{\rm n}(t, \eta)]_i = w^{\rm n}_i(t, \eta) \geq 0$ for all $t \geq 0$ sufficiently small and $v^{\rm n}_i(t, \eta) = 0$ for all $t \geq 0$. We also have (the dot is time derivative)

$$\begin{aligned} \dot{x}(t, \xi^0) &= Ax(t, \xi^0) + Bu(t, \xi^0) \\ \dot{y}^{\rm n}(t, \eta) &= Ay^{\rm n}(t, \eta) + Bv^{\rm n}(t, \eta). \end{aligned}$$

Since $u^0_{\alpha_0} > 0$ and $[C\xi^0 + Du^0]_{\gamma_0} > 0$, it follows that for some $\tilde{t}_0 > 0$, we have

$$\inf_{t \in [0, \tilde{t}_0]} \min \left\{ \min_{i \in \alpha_0} u_i(t, \xi^0), \min_{i \in \gamma_0} [Cx(t, \xi^0) + Du(t, \xi^0)]_i \right\} > 0.$$

This implies that for some $\tilde{\tau}_0 > 0$

$$\begin{aligned} \inf_{(t,\tau) \in [0, \tilde{t}_0] \times [0, \tilde{\tau}_0]} \min \Big\{ &\min_{i \in \alpha_0} \left(u_i(t, \xi^0) + \tau v^{\rm n}_i(t, \eta) \right), \\ &\min_{i \in \gamma_0} \left([Cx(t, \xi^0) + Du(t, \xi^0)]_i + \tau w^{\rm n}_i(t, \eta) \right) \Big\} > 0. \end{aligned}$$

Hence, for all $(t, \tau) \in [0, \tilde{t}_0] \times [0, \tilde{\tau}_0]$, $u_i(t, \eta) + \tau v^{\rm n}_i(t, \eta) \geq 0$ for all $i \in \alpha_{\rm n}$. This is true for all $i \in \alpha_0$ by the choice of $\tilde{t}_0$ and $\tilde{\tau}_0$; for $i \in \alpha_{\rm n} \setminus \alpha_0$, we must have $i \in \beta_0$; hence $u_i(t, \eta)$ and $v^{\rm n}_i(t, \eta)$ are both nonnegative, the latter being due to (39). Similarly, we have, for the same pairs (t, τ),

$$[C(x(t, \xi^0) + \tau y^{\rm n}(t, \eta)) + D(u(t, \xi^0) + \tau v^{\rm n}(t, \eta)]_i \geq 0$$

for all $i \in \gamma_{\rm n}$. Consequently, for all $(t, \tau) \in [0, \tilde{t}_0] \times [0, \tilde{\tau}_0]$, the pair $(x(t, \xi^0) + \tau y^{\rm n}(t, \eta), u(t, \xi^0) + \tau v^{\rm n}(t, \eta))$ satisfy the LCS (34) with the initial condition $x(0) = \xi^0 + \tau\eta$. Indeed, we have

$$\begin{aligned} 0 &= [C(x(t, \xi^0) + \tau y^{\rm n}(t, \eta)) + D(u(t, \xi^0) + \tau v^{\rm n}(t, \eta))]_{\alpha_{\rm n}} \\ &\quad \perp u_{\alpha_{\rm n}}(t, \xi^0) + \tau v^{\rm n}_{\alpha_{\rm n}}(t, \eta) \geq 0 \\ 0 &\leq [C(x(t, \xi^0) + \tau y^{\rm n}(t, \eta)) + D(u(t, \xi^0) + \tau v^{\rm n}(t, \eta))]_{\beta_{\rm n}} \\ &\quad \perp u_{\beta_{\rm n}}(t, \xi^0) + \tau v^{\rm n}_{\beta_{\rm n}}(t, \eta) \geq 0 \\ 0 &\leq [C(x(t, \xi^0) + \tau y^{\rm n}(t, \eta)) + D(u(t, \xi^0) + \tau v^{\rm n}(t, \eta))]_{\gamma_{\rm n}} \\ &\quad \perp u_{\gamma_{\rm n}}(t, \xi^0) + \tau v^{\rm n}_{\gamma_{\rm n}}(t, \eta) = 0. \end{aligned}$$

Hence, by the uniqueness of the solution to the LCS corresponding to a given initial condition, we have $x(t, \xi^0 + \tau\eta) = x(t, \xi^0) + \tau y^{\rm n}(t, \eta)$. Since $Hy^{\rm n}(t, \eta) = 0$ because $\eta \in \overline{O}(H, A^{\rm n}(\beta_a))$, we deduce $Hx(t, \xi^0 + \tau\eta) = Hx(t, \xi^0)$ for all $(t, \tau) \geq 0$ sufficiently small. Since $\eta \neq 0$, this is a contradiction to the short-time local observability at ξ^0. $\square$

Both the sufficient condition and the necessary condition in Theorem 16 are geometric in nature. By the discussion following Lemma 15, both conditions can be checked by a finite algebraic procedure via linear programming. Note that, since $\beta_{\rm n} \subseteq \beta_0$, we have

$$\begin{aligned}&\overline{O}(H, A^{\rm n}(\beta_a)) \cap \overline{O}(\overline{C}^{\rm n}_{\beta_0\bullet}(\beta_a), A^{\rm n}(\beta_a))\\ &\subseteq \overline{O}(H, A^{\rm n}(\beta_a)) \cap \overline{SO}(\overline{C}^{\rm n}_{\beta_0\bullet}(\beta_a), A^{\rm n}(\beta_a))\\ &\subseteq \overline{O}(H, A^{\rm n}(\beta_a)) \cap \overline{SO}(\overline{C}^{\rm n}_{\beta_{\rm n}\bullet}(\beta_a), A^{\rm n}(\beta_a)) \subseteq \overline{O}(H, A^{\rm n}(\beta_a))\end{aligned}$$

for all subsets β_a of $\beta_{\rm n}$. Based on the previous inclusions, we give two corollaries of Theorem 16. Corollary 17 provides a sufficient condition and a necessary condition for local observability via the classical concept of observability pairs in system theory (as opposed to the cone-based condition of semi-unobservability in the former theorem). Corollary 18 identifies situations under which conditions that are both necessary and sufficient for local observability can be obtained.

Corollary 17: In the setting of Theorem 16, the following two statements hold.

a) If $(H, A^{\rm n}(\beta_a))$ is an observable pair for every subset β_a of $\beta_{\rm n}$, then ξ^0 is short-time locally observable.

b) A state ξ^0 is short-time locally observable only if the two pairs of matrices $(H, A^{\rm n}(\beta_a))$ and $(\overline{C}^{\rm n}_{\beta_0\bullet}(\beta_a), A^{\rm n}(\beta_a))$ are jointly observable for every subset β_a of $\beta_{\rm n}$.

Proof: The assertions are clear because the respective observability conditions amount to saying that for every subset β_a of $\beta_{\rm n}$, $\overline{O}(H, A^{\rm n}(\beta_a)) = \{0\}$ and $\overline{O}(H, A^{\rm n}(\beta_a)) \cap \overline{O}(\overline{C}^{\rm n}_{\beta_0\bullet}(\beta_a), A^{\rm n}(\beta_a)) = \{0\}$. □

Corollary 18: In the setting of Theorem 16, suppose for every subset β_a of $\beta_{\rm n}$,

$$\begin{aligned}&\overline{O}(H, A^{\rm n}(\beta_a)) \cap \overline{SO}(\overline{C}^{\rm n}_{\beta_0\bullet}(\beta_a), A^{\rm n}(\beta_a))\\ &\quad = \overline{O}(H, A^{\rm n}(\beta_a)) \cap \overline{SO}(\overline{C}^{\rm n}_{\beta_{\rm n}\bullet}(\beta_a), A^{\rm n}(\beta_a)) \qquad (40)\end{aligned}$$

then ξ^0 is short-time locally observable if and only if $\overline{O}(H, A^{\rm n}(\beta_a)) \cap \overline{SO}(\overline{C}^{\rm n}_{\beta_0\bullet}(\beta_a), A^{\rm n}(\beta_a)) = \{0\}$ for all subsets β_a of $\beta_{\rm n}$.

Proof: This follows easily by combining the two parts in Theorem 16. □

A sufficient condition for (40) to hold is that every semi-unobservable state with respect to the pair $(\overline{C}^{\rm n}_{\beta_{\rm n}\bullet}(\beta_a), A^{\rm n}(\beta_a))$ must be semi-unobservable with respect to the pair $(\overline{C}^{\rm n}_{\beta_0\setminus\beta_{\rm n}\bullet}(\beta_a), A^{\rm n}(\beta_a))$. The latter condition, which is more restrictive than (40), pertains to the LCS (34) itself and does not involve the output matrix H. Among other things, this condition illustrates the significance of the indices in $\beta_0 \setminus \beta_{\rm n}$, whose presence is a source of difficulty for fully analyzing the locally observable states of the LCS. In particular, if $\beta_0 = \beta_{\rm n}$, which means that the nominal time $t_0 = 0$ is forward strongly nonswitching, then complete necessary and sufficient conditions for local observability can be obtained; we state this observation in part a) of Corollary 19. The other two parts of this next corollary pertain to two interesting special cases where the time $t_0 = 0$ is indeed forward strongly nonswitching. We define the family $\mathcal{F}$ of pairs of matrices $(\hat{A}, \hat{C})$ such that for some index subset $\alpha \subseteq \{1, 2, \ldots, m\}$ with complement $\bar{\alpha}$, $\hat{A} \equiv A - B_{\bullet\alpha}(D_{\alpha\alpha})^{-1}C_{\alpha\bullet} \in \Re^{n\times n}$ and

$$\hat{C} \equiv \begin{bmatrix} -(D_{\alpha\alpha})^{-1}C_{\alpha\bullet} \\ C_{\bar{\alpha}\bullet} - D_{\bar{\alpha}\alpha}(D_{\alpha\alpha})^{-1}C_{\alpha\bullet} \end{bmatrix} \in \Re^{m\times n}.$$

Note that the family $\mathcal{F}$ has 2^m elements.

Corollary 19: In the setting of Theorem 16, the following statements hold.

a) Suppose $t_0 = 0$ is forward strongly nonswitching, then ξ^0 is short-time locally observable if and only if $\overline{O}(H, A^{\rm n}(\beta_a)) \cap \overline{SO}(\overline{C}^{\rm n}_{\beta_0\bullet}(\beta_a), A^{\rm n}(\beta_a)) = \{0\}$ for all subsets β_a of β_0.

b) A state $\xi^0 \in \overline{O}(C, A)$ is short-time locally observable if and only if $\overline{O}(H, \hat{A}) \cap \overline{SO}(\hat{C}, \hat{A}) = \{0\}$ for any pair $(\hat{A}, \hat{C}) \in \mathcal{F}$.

c) Suppose u^0 is a nondegenerate solution of the LCP: $0 \leq u \perp C\xi^0 + Du \geq 0$, then ξ^0 is short-time locally observable if and only if $\overline{O}(H, A - B_{\bullet\alpha_0}(D_{\alpha_0\alpha_0})^{-1}C_{\alpha_0\bullet}) = \{0\}$.

Proof: As mentioned above, part a) follows from Corollary 18. For the state $\xi^0 \in \overline{O}(C, A)$, we have $Cx(t, \xi^0) = 0$ for all $t \geq 0$. Hence, $\beta_0 = \beta_{\rm n} = \{1, \ldots, m\}$, and so, noting that $\overline{C}^{\rm n}_{\beta_0\bullet}(\beta_a) = \overline{C}^{\rm n}(\beta_a)$ in this case and β_a ranges over all subsets of $\{1, \ldots, m\}$, we easily deduce part b) from part a). If u^0 is nondegenerate, then $\beta_0 = \beta_n = \emptyset$ and $\overline{C}^{\rm n}_{\beta_0\bullet}(\beta_a)$ is the vacuous matrix so that (c) also follows from part a). □

A. A SISO System

The single-input–single-output (SISO) system pertains to the LCS with a scalar u. Such a system belongs to the class of *bimodal system* whereby the system has only two modes. We can formulate the general strongly regular SISO system as

$$\dot{x} = Ax + bu \quad 0 \leq u \perp c^T x + u \geq 0 \qquad (41)$$

for some $n \times n$ matrix A and n-vectors b and c. An equivalent formulation of this LCS is: $\dot{x} = Ax + b\max(0, -c^T x)$.

Proposition 20: Consider the SISO LCS (41). Let $u^0 = \max(0, -c^T\xi^0)$. The following statements hold.

a) Suppose $c^T\xi^0 > 0$ or [$c^T\xi^0 = 0$ but $\alpha_{\rm n} = \{1\}$], then ξ^0 is short-time locally observable if and only if $\overline{O}(H, A) = \{0\}$.

b) Suppose $c^T\xi^0 < 0$ or [$c^T\xi^0 = 0$ but $\gamma_{\rm n} = \{1\}$], then ξ^0 is short-time locally observable if and only if $\overline{O}(H, A - bc^T) = \{0\}$.

c) Suppose $c^T\xi^0 = 0$ and $\beta_{\rm n} = \{1\}$, then ξ^0 is short-time locally observable if and only if $\overline{O}(H, A) = \{0\}$ and $\overline{O}(H, A - bc^T) = \{0\}$.

Proof: We prove the proposition by looking at the following three cases: i) $c^T\xi^0 < 0$, ii) $c^T\xi^0 = 0$, and iii) $c^T\xi^0 > 0$. In case i), we have $\alpha_0 = \{1\}$ and $\beta_0 = \gamma_0 = \emptyset$. In case ii), we have $\alpha_0 = \gamma_0 = \emptyset$ and $\beta_0 = \{1\}$. In case iii), we have $\gamma_0 = \{1\}$ and $\beta_0 = \alpha_0 = \emptyset$. The first and third case occur when u^0 is nondegenerate. In these two cases, it follows that x^0 is short-time locally observable if and only if either the pair $(H, A - bc^T)$ (first case) or the pair (H, A) is observable (third case). Consider case ii). There are two subcases: ii-a) $\beta_{\rm n} = \emptyset$ or ii-b) $\beta_{\rm n} = \{1\}$. Part a) of Corollary 19 is applicable to case (ii-b). In this case, ξ^0 is short-time locally observable if and only if $\overline{O}(H, A - bc^T) \cap \overline{SO}(-c^T, A - bc^T) = \{0\}$ (by taking $\beta_a = \{1\}$) and $\overline{O}(H, A) \cap \overline{SO}(c^T, A) = \{0\}$ (by taking $\beta_a = \emptyset$). We now prove assertion c) of the proposition by invoking Proposition 14. Suppose ξ^0 is short-time locally observable. By the cited proposition, it follows that $\overline{O}(H, A) \cap \overline{O}(c^T, A) = \{0\}$. We claim that $\overline{O}(H, A) \cap \overline{SO}(c^T, A) = \{0\}$ if and only if $\overline{O}(H, A) = \{0\}$ and that $\overline{O}(H, A - bc^T) \cap \overline{SO}(-c^T, A - bc^T) = \{0\}$ if and only if $\overline{O}(H, A - bc^T) = \{0\}$. Consider the former equivalence. The "if" part is trivial. Now suppose $\overline{O}(H,A)\cap\overline{SO}(c^T,A)=\{0\}$ but $O(H,A)\neq\{0\}$ so that there is a nonzero $\eta\in\overline{O}(H,A)$. Let k be the first integer such that $c^TA^k\eta\neq 0$. We must have $0\leq k\leq n-1$. If $c^TA^k\eta>0$, then $\eta\in\overline{SO}(c^T,A)$, which is a contradiction; if $c^TA^k\eta<0$, then $-\eta\in\overline{O}(H,A)$ and $c^TA^k(-\eta)>0$, hence $-\eta\in\overline{O}(H,A)\cap\overline{SO}(c^T,A)$, which is a contradiction as well. This shows that $\overline{O}(H,A)\cap\overline{SO}(c^T,A)=\{0\}$ is equivalent to $\overline{O}(H,A)=\{0\}$. Similarly, $\overline{O}(H,A-bc^T)\cap\overline{SO}(-c^T,A-bc^T)=\{0\}$ is equivalent to $\overline{O}(H,A-bc^T)=\{0\}$. This establishes the necessity in part c) of the proposition. The sufficiency of this part is trivial.

It remains to deal with case ii-a). For this, there are two further subcases: ii-a-1) $\alpha_{\rm n}=\{1\},\gamma_{\rm n}=\emptyset$; ii-a-2) $\alpha_{\rm n}=\emptyset,\gamma_{\rm n}=\{1\}$. We prove only the second one. In this subcase, a sufficient condition for ξ^0 to be short-time locally observable is that $\overline{O}(H,A)=\{0\}$ by Theorem 11. Conversely, if ξ^0 is short-time locally observable, then the same theorem implies that $\overline{O}(H,A)\cap\overline{SO}(c^T,A)=\{0\}$. By the equivalence proved above, the latter holds if and only if $\overline{O}(H,A)=\{0\}$. Hence the "or" assertion in part b) of the proposition is valid. □

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