Math 302 Fall 2011 Homework #4

Due Oct. 10, Mon. in class

1. Consider the metric induced by the 1-norm on $\mathbb{R}^2$: $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$, $\forall\ x, y \in \mathbb{R}^2$. Let the set $A = \{x = (x_1, x_2) \in \mathbb{R}^2 \ : \ x_1 \geq 1 \text{ and } x_2 \geq 1\}$. Prove that the set A is closed using the sequential criterion.

2. Given a metric space (M, d). Prove the following statements:

   (1) In (M, d), a convergent sequence has a unique limit (*Hint*: by contradiction);

   (2) In (M, d), a convergent sequence is Cauchy;

   (3) In (M, d), a Cauchy sequence is bounded.

3. Textbook, page 123, Ex. 2.

4. Let (x_n) be a *bounded* sequence in $\mathbb{R}$. Assume that $x_n - x_{n-1} \leq x_{n+1} - x_n$ for all $n \geq 2$. Consider the sequence (s_n) with $s_n := x_n - x_{n-1}$.

   (1) Show that (s_n) is monotone, bounded and convergent.

   (2) Show that $\lim(s_n) = 0$ (*Hint*: by contradiction and using the boundedness of (x_n)).