Math 650 Fall 2016 Homework #2

Due Sept. 29, Thu in class

1. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$: $f(x_1, x_2) = \max(x_1, x_2)$. Show that for any $d = (d_1, d_2)^T \in \mathbb{R}^2$, the one-sided D.D. $f'_+(x, d)$ is given by

$$f'_+(x; d) = \begin{cases} d_1 & \text{if } x_1 > x_2 \\ \max(d_1, d_2) & \text{if } x_1 = x_2 \\ d_2 & \text{if } x_1 < x_2 \end{cases}$$

2. Let the function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ be G-differentiable on the open set U. Let $x, y \in U$ be such that the line segment $[x, y] \subset U$. Define the function $g : [0, 1] \to \mathbb{R}$ as

$$g(t) := f(x + t(y - x)).$$

Show that for any $t_* \in [0, 1]$, $g'(t_*) = \langle \nabla f(x + t_*(y - x)), y - x\rangle$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a *univariate*, real-valued function with continuous derivative. Show that if f has a local minimizer that is not a global minimizer, then f must have another critical point. (*Remark*: the extension to *multivariable* functions is not true in general.)

4. Let the function f on $\mathbb{R}^2$ be $f(x, y) = x^3 - 3\alpha xy + y^3, \forall (x, y) \in \mathbb{R}^2$, where $\alpha \in \mathbb{R}$ is a parameter.

   (1) Show that f has no global minimizer or global maximizer for any α.

   (2) For each value of α, find all the critical point(s) of f and determine whether a critical point is a local minimizer, or a local maximizer, or a saddle point.

5. Let $f(x) = \frac{1}{2}x^T Ax + c^T x + \alpha$ be a quadratic function on $\mathbb{R}^n$, where $A \in \mathbb{R}^{n\times n}$ is symmetric, $c \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$. Prove the Frank-Wolfe Theorem: if f is bounded below on $\mathbb{R}^n$, i.e. $f(x) \geq \gamma, \forall\, x \in \mathbb{R}^n$ for some $\gamma \in \mathbb{R}$, then A is positive semidefinite, c is in the range of A, and f has a minimizer on $\mathbb{R}^n$.

More practice problems: *Do not submit*

1. Let the function $F : \mathbb{R}^2 \to \mathbb{R}^2$ be $F(x_1, x_2) = (x_1^3, x_2^2)^T$. Let $x = (0, 0)$ and $y = (1, 1)$. Show that there is *no* vector z on the the line segment $[x, y]$ such that

$$F(y) - F(x) = JF(z)(y - x),$$

where $JF(z)$ denotes the Jacobian of F at z. (*Hint*: This example shows that the Mean-value Theorem cannot be extended to vector-valued functions in general.)

2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be the 1-norm on $\mathbb{R}^n$, i.e., $f(x) := \|x\|_1$. Show that f is one-sided directionally differentiable on $\mathbb{R}^n$ but is not (standard) directionally differentiable at each $x \in \mathbb{R}^n$.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be $f(x) = \max(x_1, x_2)$ for $x = (x_1, x_2)^T \in \mathbb{R}^2$.

   (1) Show that f is (standard) directionally differentiable at $x = (x_1, x_2)^T$ along any d if and only if $x_1 \neq x_2$.

   (2) Fix arbitrary $x, d \in \mathbb{R}^n$, and define $g : \mathbb{R}_{++} \to \mathbb{R}$ as $g(t) := \frac{f(x+td)-f(x)}{t}$ for all $t > 0$. Show that g is monotonically increasing.