Math 302/401/600 Fall 2010 Homework #7

Due Oct. 27, Wed. in class

1. Textbook, page 191, Section 4.4, 3. (Here use the standard Euclidean metric on $\mathbb{R}^n$.)

2. Let $f, g : (M, d) \to (V, \|\cdot\|)$ be two functions, where (M, d) is a metric space and $(V, \|\cdot\|)$ is a normed vector space.

   (1) Use the sequential argument to show that if f and g are continuous at $x_0 \in M$, so is $f + g$;

   (2) Let λ be a scalar. Use the $\varepsilon - \delta$ definition to show that if f is continuous at $x_0 \in M$, so is λf.

3. Let $\|\cdot\|_2$ be the standard Euclidean norm on $\mathbb{R}^n$ and $\|\cdot\|_\alpha$ be an arbitrary norm on $\mathbb{R}^n$. Recall that both $\|\cdot\|_2$ and $\|\cdot\|_\alpha$ are continuous on $\mathbb{R}^n$.

   (1) Let $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$. Show that $\mathbb{S}^{n-1}$ is closed. (*Hint*: convert $\mathbb{S}^{n-1}$ into the inverse image of a closed set in $\mathbb{R}$ under $\|\cdot\|_2$.)

   (2) Show that $\mathbb{S}^{n-1}$ is compact.

   (3) Show that there exist $x^* \in \mathbb{S}^{n-1}$ and $y^* \in \mathbb{S}^{n-1}$ such that

   $$\|x^*\|_\alpha = \max_{x \in \mathbb{S}^{n-1}} \|x\|_\alpha, \qquad \|y^*\|_\alpha = \min_{x \in \mathbb{S}^{n-1}} \|x\|_\alpha.$$

   Further show that $\|x^*\|_\alpha > 0$ and $\|y^*\|_\alpha > 0$.

   (4) Show that for any $0 \neq x \in \mathbb{R}^n$,

   $$\frac{x}{\|x\|_2} \in \mathbb{S}^{n-1}.$$

   (5) Let $c_1 := \|x^*\|_\alpha$ and $c_2 := \|y^*\|_\alpha$. Use (3)-(4) to show that

   $$c_2 \|x\|_2 \leq \|x\|_\alpha \leq c_1 \|x\|_2, \qquad \forall\, x \in \mathbb{R}^n.$$

   (6) Let $\|\cdot\|_\beta$ be another norm on $\mathbb{R}^n$. Use (5) to show that there exist $\kappa_1, \kappa_2 > 0$ such that

   $$\kappa_2 \|x\|_\beta \leq \|x\|_\alpha \leq \kappa_1 \|x\|_\beta, \qquad \forall\, x \in \mathbb{R}^n.$$

   $\star$ *This result shows that all norms on $\mathbb{R}^n$ are equivalent.*

4. Let $f : (M, d) \to (N, \rho)$ be continuous on M. Let $B \subseteq M$. Show that $f(\mathrm{cl}(B)) \subseteq \mathrm{cl}(f(B))$. (*Hint*: think of the sequential definition of closure and the sequential criteria of set closedness and continuity.)