Math 650 Fall 2016 Homework #3

Due Oct. 11, Tue in class

1. Consider the equality constrained optimization problem on $\mathbb{R}^n$:

$$\min \ -x^T x, \qquad \text{subject to } \ x^T x \ = \ c^T x,$$

where $c \in \mathbb{R}^n$ is a nonzero vector.

   (1) Show that the constraint set $D = \{x \in \mathbb{R}^n \,|\, x^T x = c^T x\}$ is compact, and the optimization problem has a global minimizer.

   (2) Show that any Fritz John point is a KKT point.

   (3) Find the minimizer(s) using the 1st and 2nd order optimality conditions.

2. Consider the equality constrained quadratic program on $\mathbb{R}^n$:

$$\min \ \frac{1}{2} x^T Q x + c^T x, \qquad \text{subject to } Ax \ = \ b,$$

where $Q \in \mathbb{R}^{n\times n}$ is symmetric, $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$ has full row rank, and $b \in \mathbb{R}^m$ is in the range of A. Assume that a (local) minimizer exists.

   (1) Show that a local minimizer x^* satisfies the KKT conditions: $Qx^* + c$ is in the range of A^T and $Ax^* = b$.

   (2) Show that a local minimizer also satisfies the 2nd order necessary condition: Q is positive semidefinite on $N(A)$, i.e., the null space of A.

   (3) Show that a KKT point x^* satisfying the 2nd order necessary condition in (2) is a global minimizer of the quadratic program. (*Hint*: Let $f(x)$ be the objective function, and show that for any $v \in N(A)$, $f(x^* + v) = f(x^*) + (Qx^* + c)^T v + \frac{1}{2} v^T Q v$.)

3. Consider the equality constrained optimization problem on $\mathbb{R}^n$:

$$\min \ -\sum_{i=1}^{n} x_i^3, \qquad \text{subject to } \ \sum_{i=1}^{n} x_i^2 = 1,$$

where $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$.

   (1) Show that the optimization problem has a global minimizer.

   (2) Show that each local minimizer must be a KKT point, and find all (but finitely many) KKT points.

   (3) Determine which of the KKT points is a global minimizer (using the 1st and 2nd order optimality conditions, and the values of the objective function at the KKT points).

   (4) Use (3) to show the following inequality:

$$\sum_{i=1}^{n} |x_i|^3 \ \le \ \Big(\sum_{i=1}^{n} x_i^2 \Big)^{3/2} \ = \ \|x\|_2^3, \quad \forall \ x \in \mathbb{R}^n.$$

   (*Hint*: Note that for any $x \neq 0$, $\frac{\sum_{i=1}^n |x_i|^3}{\|x\|_2^3} = \sum_{i=1}^n \left(\frac{|x_i|}{\|x\|_2} \right)^3$.)

More practice problems: *Do not submit*

1. Consider the equality constrained optimization problem on $\mathbb{R}^2$:

$$\min\ x^2 + y^2, \qquad \text{subject to}\ \ x^2 - (y-1)^3 = 0$$

   (1) Plot the constraint set $D := \{(x, y) \in \mathbb{R}^2 \mid x^2 - (y-1)^3 = 0\}$ and find the minimizer(s). Does the KKT condition hold at the minimizer(s)? Explain why.

   (2) One may attempt to solve this problem by substituting $x^2 = (y-1)^3$ into the objective function and reducing the original problem to the *unconstrained* problem: $\min\ y^2 + (y-1)^3$. Is this correct? Justify your answer.