Math 302/600 Spring 2017 Homework #10

Due April 25, Tue in class

Note: For any Euclidean space $\mathbb{R}^n$, consider the usual metric induced by the Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^n$, unless otherwise stated.

1. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous on $\mathbb{R}^n$. Define the function $g : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$ as:

$$g(x) := (x, f(x)), \qquad \forall\, x \in \mathbb{R}^n.$$

   (1) Show that g is continuous on $\mathbb{R}^n$. (*Hint*: use the sequential criterion.)

   (2) For the function f given above, define the following set (called the *graph* of f):

$$\mathcal{S} := \{(x, f(x)) \,|\, x \in \mathbb{R}^n\} \subseteq \mathbb{R}^n \times \mathbb{R}^m.$$

   Show that $\mathcal{S}$ is closed and connected.

2. Show that the following sets in $\mathbb{R}^2$ are path connected by demonstrating specific continuous paths joining two points in the sets:

   (1) $\{(x, \sin(x)) \,|\, x \geq 0\}$, where you may assume that $\sin(\cdot)$ is continuous on $\mathbb{R}$;

   (2) $\{(x_1, x_2) \,|\, 0 < x_2 < 1\}$.

3. Let A be a path connected set in a metric space (M, d), and f be a continuous function on M. Show that $f(A)$ is path connected.

4. Solve the following problems.

   (1) Let f be a continuous function on a metric space (M, d) and A be a nonempty set in M. Show that if f is uniformly continuous on the closure of A, so is on A.

   (2) Let $g : \mathbb{R}^2 \to \mathbb{R}$ be continuous on $\mathbb{R}^2$. Let $(a, b]$ and (c, d) be two intervals in $\mathbb{R}$. Use (1) to show that g is uniformly continuous on $(a, b] \times (c, d)$.

The following extra problem(s) are for **Math 600** *students only:*

5. Show that the closure of a connected set is connected.

6. (Optional) Show that the Cartesian product of two path connected sets is path connected.