Math 302 Fall 2011 Homework #13

Due Dec. 12, Mon. in class

$\star$ *Use the standard Euclidean metric on $\mathbb{R}$, and all x below are in $\mathbb{R}$ unless otherwise indicated.*

1. Let
$$f_n(x) = \frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!}, \quad x \in \mathbb{R}.$$
It is shown in the last homework that the series $\sum_{n=1}^{\infty} f_n$ uniformly converges on $A = [-a, a]$ with $a > 0$. Let f_* be the limiting function of the series on A, i.e., $f_*(x) = \sum_{n=1}^{\infty} f_n(x)$.
   - (1.1) Is $\int_{-a}^{a} f_*(x)dx = \sum_{n=1}^{\infty} \int_{-a}^{a} f_n(x)dx$? Justify your answer.
   - (1.2) Is f_* differentiable on $(-a, a)$? If so, is $f'_*(x) = \sum_{n=1}^{\infty} f'_n(x)$ on $(-a, a)$? Prove your answers.

2. Textbook, page 282, Exercises 1, 3.

3. Let $(V, \|\cdot\|)$ be a complete normed vector space and its induced metric $d(x, y) = \|x - y\|$ for $x, y \in V$. Let $f : V \to V$ be a *linear* function, i.e., $f(x + y) = f(x) + f(y), \forall\, x, y \in V$ and $f(\alpha x) = \alpha f(x)$ for all $x \in V$ and $\alpha \in \mathbb{R}$.
   - (3.1) Show that $f(0) = 0$ and $f(x - y) = f(x) - f(y), \forall\, x, y \in V$. (You may assume $x - y = x + (-1)y$ for all $x, y \in V$.)
   - (3.2) Show that f is a contraction if and only if there exists a constant C with $0 < C < 1$ such that $\|f(x)\| \le C\|x\|$ for all $x \in V$.
   - (3.3) Suppose that f is a contraction. Let $x_0 \in V$ be arbitrary and define the sequence (x_n) recursively by
   $$x_n = f(x_{n-1}), \quad n \in \mathbb{N}.$$
   Show that (x_n) converges to the zero vector in V.
   - (3.4) Let the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ be
   $$f(x_1, x_2) = \frac{K}{\sqrt{2}}\big(x_1 + x_2,\ x_2 - x_1\big), \quad \forall\, x := (x_1, x_2) \in \mathbb{R}^2,$$
   where the constant K satisfies $0 < K < 1$. Show that f is a linear function and f is a contraction when the 2-norm (i.e., $\|\cdot\|_2$) is used.