Math 650 Fall 2011 Homework #1

Due Sept. 14, Wed. in class

1. Let $f : \mathbb{R}^n \to \mathbb{R}$ and $x^*, d \in \mathbb{R}^n$ be given. Show that the (standard) D.D. $f'(x^*; d)$ exists if and only if the following limits exist and are identical:

$$\lim_{t \downarrow 0} \frac{f(x^* + td) - f(x^*)}{t}, \quad \text{and} \quad \lim_{t \uparrow 0} \frac{f(x^* + td) - f(x^*)}{t}.$$

2. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$: $f(x_1, x_2) = \max(x_1, x_2)$. Show that for any $d = (d_1, d_2)^T \in \mathbb{R}^2$, the one-sided D.D. $f'_+(x, d)$ is given by

$$f'_+(x; d) = \begin{cases} d_1 & \text{if } x_1 > x_2 \\ \max(d_1, d_2) & \text{if } x_1 = x_2 \\ d_2 & \text{if } x_1 < x_2 \end{cases}$$

Furthermore, show that f is (standard) directionally differentiable (D.D.) at $x = (x_1, x_2)^T$ along any d if and only if $x_1 \neq x_2$.

3. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$:

$$f(x, y) = \begin{cases} \frac{x^3 y}{x^4 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

   (1) Show that f is continuous and (standard) D.D. at $(0, 0)$.

   (2) Find $\nabla f(0, 0)$ and show that f is G(ateatux)-differentiable at $(0, 0)$.

   (3) Show that f is *not* F(rechet)-differentiable at $(0, 0)$. (*Hint*: consider two curves approaching $(0, 0)$: $y = x$ and $y = x^2$.)

4. Let the function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ be G-differentiable on the open set U. Let $x, y \in U$ be such that the line segment $[x, y] \subset U$. Define the function $g : [0, 1] \to \mathbb{R}$ by

$$g(t) := f(x + t(y - x)).$$

Show that for any $t_* \in (0, 1)$, $g'(t_*) = \langle \nabla f(x + t_*(y - x)), y - x \rangle$.

5. Let the function $F : \mathbb{R}^2 \to \mathbb{R}^2$ be $F(x_1, x_2) = (x_1^3, x_2^2)^T$. Let $x = (0, 0)$ and $y = (1, 1)$. Show that there is *no* vector z on the the line segment $[x, y]$ such that

$$F(y) - F(x) = JF(z)(y - x),$$

where $JF(z)$ denotes the Jacobian of F at z.