Math 650 Fall 2011 Homework #7

Due Nov. 30, Wed. in class

P.1 Let C_1, C_2 be two convex polyhedral cones in $\mathbb{R}^n$. Show that $C_1 + C_2$ ia also a convex polyhedral cone.

P.2 Let $P := \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$ be a nonempty convex polyhedron. Show that P is bounded (i.e., it is a polytope) if and only if the linear inequality $Ax = 0,\ x \geq 0$ has the trivial solution $x = 0$ only.

P.3 Let the polyhedral cone $C = \{x \in \mathbb{R}^n : Ax = 0,\ x \geq 0\}$ for some matrix $A \in \mathbb{R}^{m\times n}$. Show that its dual cone $C^* = \{A^T u + v : u \in \mathbb{R}^m,\ v \in \mathbb{R}^n_+\}$. (*Hint*: convert C into the standard form.)

P.4 Let C be a convex cone in $\mathbb{R}^n$, and let $\mathcal{V} := C \cap (-C)$.

(1) Show that $\mathcal{V}$ is a subspace of $\mathbb{R}^n$.

(2) A cone $\mathcal{C}$ is called *pointed* if the condition that $x_1 + \cdots + x_k = 0$ with $x_i \in \mathcal{C}$, $i = 1, \dots, k$ implies $x_i = 0$ for all i. Show that the convex cone C is pointed if and only if $\mathcal{V} = \{0\}$. (*Hint*: recall that if $x, y \in C$, then $x + y \in C$.)

(3) Let $\mathcal{K} := C \cap \mathcal{V}^{\perp}$. Show that $\mathcal{K}$ is a pointed convex cone and $C = \mathcal{K} + \mathcal{V}$ with $\mathcal{K} \perp \mathcal{V}$.

(4) Let C be a polyhedral cone given by $C = \{x : Ax \geq 0\}$. Show that C is pointed if and only if the null space of A is trivial.

(5) (Optional) Suppose the set S has nonempty interior. Show that its dual cone S^* is pointed.

P.5 Let

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 2 \\ -1 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}.$$

Show that the inequality system $A^T y \leq 0,\ b^T y > 0$ has a solution. (*Hint*: use Farkas' Lemma.)

P.6 Let $A \in \mathbb{R}^{m\times n}$. Show that exactly one of the following inequality systems has a solution:

$$\mathbf{I}: \ Ax \leq 0,\ x \geq 0,\ \sum_{i=1}^{n} x_i = 1; \qquad \text{and} \qquad \mathbf{II}: \ A^T y > 0,\ y \geq 0,\ \sum_{i=1}^{m} y_i = 1$$