Math 302/600 Spring 2015 Homework #11

Due May 7, Thu. in class

1. Let $f_n : [1,2] \to \mathbb{R}$ be defined by $f_n(x) = \frac{x}{(x+1)^n}$.
   (1) Determine if $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent on $A = [1,2]$.
   (2) Determine if $\int_1^2 \big(\sum_{n=1}^{\infty} f_n(x)\big)dx = \sum_{n=1}^{\infty} \int_1^2 f_n(x)dx$.

2. Let $A = [-a, a] \subset \mathbb{R}$ with $a > 0$, and let
$$f_n(x) = \frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!}, \quad x \in \mathbb{R}.$$
   (1) Use the Weierstrass M-test to show uniform convergence of the series $\sum_{n=1}^{\infty} f_n$ on A.
   (2) Let f_* be the limit function of the series on A, i.e., $f_*(x) = \sum_{n=1}^{\infty} f_n(x)$. Is f_* differentiable on $(-a, a)$? If so, is $f_*'(x) = \sum_{n=1}^{\infty} f_n'(x)$ on $(-a, a)$? Prove your answers.

3. Find the largest possible constant $r \in (0,1)$ such that the function $f : [0,r] \to [0,r]$ defined by $f(x) = x^2$ is a contraction.

4. Let $(V, \|\cdot\|)$ be a complete normed vector space and its induced metric $d(x,y) = \|x-y\|$ for $x, y \in V$. Let $f : V \to V$ be a *linear mapping/function*, i.e., $f(x+y) = f(x) + f(y), \forall\, x, y \in V$ and $f(\alpha x) = \alpha f(x)$ for all $x \in V$ and $\alpha \in \mathbb{R}$. You may assume the following facts without proof: $f(0) = 0$ and $f(x-y) = f(x) - f(y), \forall\, x, y \in V$.
   (1) Show that f is a contraction if and only if there exists a constant C with $0 < C < 1$ such that $\|f(x)\| \le C\|x\|$ for all $x \in V$.
   (2) Let $x_0 \in V$ be arbitrary, and define the sequence (x_n) recursively by $x_n = f(x_{n-1}), \ n \in \mathbb{N}$. Show that (x_n) converges to the zero vector in V.

5. Let the constant K satisfy $0 < K < 1$. Consider the *linear* function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by
$$f(x) = \frac{K}{\sqrt{2}}\big(x_1 + x_2,\ x_2 - x_1\big), \quad \forall\, x = (x_1, x_2) \in \mathbb{R}^2.$$
   In the following, you may use the results of Problem 4.
   (1) Show that when the 2-norm (i.e., $\|\cdot\|_2$) is used, f is a contraction.
   (2) Show that when the 1-norm (i.e., $\|\cdot\|_1$) is used, f is *not* a contraction if $\frac{1}{\sqrt{2}} < K < 1$.
   (3) Let $x^0 = (x_1^0, x_2^0) \in \mathbb{R}^2$ be arbitrary. Define the sequence (x^k) as $x^k = f(x^{k-1}), \ k \in \mathbb{N}$. Explain why the sequence (x^k) is convergent when the 2-norm is used. (*Note:* recall that $(\mathbb{R}^2, \|\cdot\|_2)$ is complete.)
   (4) Show that the sequence defined in (3) is convergent when the 1-norm is used. (*Hint:* use the equivalence of norms on a Euclidean space shown in Problem 2 of Homework #8.)

   $\star$ This example shows that the contractive property is a *sufficient* condition for convergence but not a necessary one.

The following extra problems are for **Math 600** *students only:*

6. Let $f_n : \mathbb{R} \to \mathbb{R}$ be
$$f_n(x) = \frac{(-1)^{n+1}x}{n}.$$
Let A be a bounded set in $\mathbb{R}$. Show that the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on A. (*Hint*: use the Cauchy criterion.)

7. Suppose that each $f_n : \mathbb{R} \to \mathbb{R}$ is continuous on the set A, and (f_n) converges to f_* uniformly on A. Let (x_n) in A converge to $x_* \in A$. Show that $(f_n(x_n))$ converges to $f_*(x_*)$.