Math 600 Fall 2017 Homework #7

Due Nov. 27, Mon. in class

Note: For the Euclidean space $\mathbb{R}^n$, consider the usual metric induced by the Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^n$, unless otherwise stated.

1. Let $f_n : [1,2] \to \mathbb{R}$ be $f_n(x) = \frac{x}{(x+1)^n}$.

   (1) Use the Weierstrass M-test to show that $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent on $A = [1,2]$;

   (2) Determine if $\int_1^2 \left(\sum_{n=1}^{\infty} f_n(x)\right)dx = \sum_{n=1}^{\infty} \int_1^2 f_n(x)dx$, and justify your answer. (*Note*: do NOT try to find the values of the integrals of f_n and the series over $[1,2]$.)

2. Let $A = [-a,a] \subset \mathbb{R}$ with $a > 0$, and let

$$f_n(x) = \frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!}, \quad x \in \mathbb{R}.$$

   (1) Use the Weierstrass M-test to show uniform convergence of the series $\sum_{n=1}^{\infty} f_n$ on A;

   (2) Let f_* be the limit function of the series on A, i.e., $f_*(x) = \sum_{n=1}^{\infty} f_n(x)$. Is f_* differentiable on $(-a,a)$? If so, is $f'_*(x) = \sum_{n=1}^{\infty} f'_n(x)$ on $(-a,a)$? Justify your answers.

3. Let A be a bounded set in $\mathbb{R}$, and $f_n : \mathbb{R} \to \mathbb{R}$ be

$$f_n(x) = \frac{(-1)^{n+1}x}{\sqrt{n}}.$$

   (1) Use the Cauchy criterion to show that the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on A.

   (2) Show that the Weierstrass M-test fails for this series.

   $\star$ *This problem shows that the Weierstrass M-test is a* sufficient *condition for uniform convergence but not a necessary one.*

4. Let $f_n : \mathbb{R} \to \mathbb{R}$ be $f_n(x) = \frac{x}{n^2+x^2}$.

   (1) Use the Weierstrass M-test to show that the series $s_* := \sum_{n=1}^{\infty} f_n$ converges uniformly on any bounded set $A \subset \mathbb{R}$. Furthermore, show that s_* is continuous at any point in $\mathbb{R}$.

   (2) Show that the series $\sum_{n=1}^{\infty} f_n$ does not converge uniformly on $\mathbb{R}$ via the Cauchy criterion.

5. Let $(V, \|\cdot\|)$ be a complete normed vector space and its induced metric $d(x,y) = \|x-y\|$ for $x, y \in V$. Let $f : V \to V$ be a *linear mapping/function*, i.e., $f(x+y) = f(x) + f(y), \forall\, x, y \in V$ and $f(\alpha x) = \alpha f(x)$ for all $x \in V$ and $\alpha \in \mathbb{R}$. You may assume the following facts without proof: $f(0) = 0$ and $f(x-y) = f(x) - f(y), \forall\, x, y \in V$.

   (1) Show that f is a contraction if and only if there exists a constant C with $0 < C < 1$ such that $\|f(x)\| \leq C\|x\|$ for all $x \in V$.

   (2) Suppose that f is a contraction. Let $x_0 \in V$ be arbitrary, and define the sequence (x_n) recursively by $x_n = f(x_{n-1}), \quad n \in \mathbb{N}$. Show that (x_n) converges to the zero vector in V.

6. Let the constant K satisfy $0 < K < 1$. Consider the linear function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x) = \frac{K}{\sqrt{2}}\big(x_1 + x_2,\ x_2 - x_1\big), \quad \forall\, x = (x_1, x_2) \in \mathbb{R}^2.$$

To solve the following problems, you may use the results of Problem 5.

(1) Show that when the 2-norm (i.e., $\|\cdot\|_2$) is used, f is a contraction.

(2) Show that when the 1-norm (i.e., $\|\cdot\|_1$) is used, f is *not* a contraction if $\frac{1}{\sqrt{2}} < K < 1$.

(3) Let $x^0 = (x_1^0, x_2^0) \in \mathbb{R}^2$ be arbitrary. Define the sequence (x^k) as $x^k = f(x^{k-1}),\ \ k \in \mathbb{N}$. Explain why the sequence (x^k) is convergent when the 2-norm is used. (*Note:* recall that $(\mathbb{R}^2, \|\cdot\|_2)$ is complete.)

(4) Show that the sequence defined in (3) is convergent when the 1-norm is used. (*Hint:* use the equivalence of norms on a Euclidean space shown in Problem 3 of Homework #5.)

⋆ *This problem shows that the contractive property is a* sufficient *condition for convergence but not a necessary one.*

Miscellaneous practice problems: *Do not submit*

1. Let $f_n : \mathbb{R} \to \mathbb{R}$ be such that the sequence (f_n) converges uniformly to f_* on the set A. Suppose that each f_n is bounded on A, i.e., for each f_n, there exists $M_n > 0$ (dependent on f_n) such that $|f_n(x)| \le M_n, \forall\, x \in A$. Show that f_* is bounded on A.

2. Find the largest possible constant $r \in (0, 1)$ such that the function $f : [0, r] \to [0, r]$ defined by $f(x) = x^2$ is a contraction.