Math 302/600 Spring 2015 Homework #3

Due Feb. 26, Thursday in class

Note: For any Euclidean space $\mathbb{R}^n$, consider the usual metric induced by the Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^n$, unless otherwise stated.

1. Consider the metric induced by the 1-norm on $\mathbb{R}^2$: $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$, $\forall\, x, y \in \mathbb{R}^2$. Let the set $A = \{x = (x_1, x_2) \in \mathbb{R}^2 \,:\, x_1 \geq 1 \text{ and } x_2 \leq 1\}$.
   (1) Prove that the set A is closed via the definition, namely, show that the complement of A is open;
   (2) Prove that the set A is closed using sequential criterion.

2. Find all the accumulation points of each of the following sets (without proof):
   (1) $A = \{x = (x_1, x_2) \in \mathbb{R}^2 \,:\, 1 < x_1 \leq 2, \text{ and } x_2 < -2\}$ in $\mathbb{R}^2$;
   (2) $B = \{x = (\frac{1}{n}, 1 - \frac{1}{n}) \in \mathbb{R}^2 \,:\, n \in \mathbb{N}\} \cup \{(1,2)\}$ in $\mathbb{R}^2$;
   (3) $C = \{x = (x_1, x_2) \in \mathbb{R}^2 \,:\, x_1 \text{ is rational, and } x_2 = \frac{1}{n},\, n \in \mathbb{N}\}$ in $\mathbb{R}^2$.

3. Given a metric space (M, d). Prove the following statements:
   (1) any convergent sequence in (M, d) has a unique limit;
   (2) any convergent sequence in (M, d) is Cauchy;
   (3) any Cauchy sequence in (M, d) is bounded.

The following extra problem is for **Math 600** *students only:*

4. Let A be a nonempty set in $\mathbb{R}^n$, and d be the metric induced by the Euclidean norm on $\mathbb{R}^n$. Let $z \in \mathbb{R}^n$ be given.
   (1) Show that the infimum of the *real* set $\{d(z,x) : x \in A\}$ exists. In the following, define $d(z, A) := \inf\{d(z,x) : x \in A\}$.
   (2) Show that there exists a sequence (x_k) in A such that the *real* sequence $(d(z, x_k))$ converges to $d(z, A)$. Furthermore, show that (x_k) has a convergent subsequence in $\mathbb{R}^n$. (*Hint*: for the latter statement, consider the Bolzano-Weierstrass Theorem, which says that a bounded sequence in $\mathbb{R}^n$ has ...)
   (3) Use (2) to show that if the set A is closed, then there exists $x^* \in A$ such that $d(z, A) = d(z, x^*)$.