Math 302 Fall 2011 Homework #5

Due Oct. 17, Mon. in class

1. Consider the real line $\mathbb{R}$ endowed with the discrete metric d. A real sequence (x_n) is said to have a *constant tail* if there exist $K \in \mathbb{N}$ and $c \in \mathbb{R}$ such that $x_n = c$ for all $n \geq K$. Prove the following statements:

   (1) A sequence in $\mathbb{R}$ with a constant tail is convergent in $\mathbb{R}$;

   (2) A Cauchy sequence in $\mathbb{R}$ has a constant tail;

   (3) Show that $(\mathbb{R}, d)$ is complete using (1)-(2).

2. Textbook, page 147, Ex. 32, 38.

3. Textbook, page 125, Ex. 1.

4. Determine which of the following sets is sequentially compact (using the standard metric on $\mathbb{R}^n$):

   (1) In $\mathbb{R}$: $\{2/n : n \in \mathbb{N}\}$, $\mathbb{Q} \cap [0, 1]$ ($\mathbb{Q}$ is the set of rational numbers);

   (2) In $\mathbb{R}^2$: $\mathbb{Q} \times \mathbb{Q}$, $\{(x, y) \in \mathbb{R}^2 : x^2 + 2y^2 = 4\}$.

5. Show that the union and intersection of two sequentially compact sets remain sequentially compact.