Math 302/600 Spring 2017 Homework #4

Due March 2, Thu. in class

Note: For any Euclidean space $\mathbb{R}^n$, consider the usual metric induced by the Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^n$, unless otherwise stated.

1. Consider the metric induced by the 1-norm on $\mathbb{R}^2$: $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$, $\forall\, x, y \in \mathbb{R}^2$. Let the set $A = \{x = (x_1, x_2) \in \mathbb{R}^2 \,:\, x_1 \geq 1 \text{ and } x_2 \leq 1\}$. Prove that the set A is closed using the sequential criterion.

2. Let the set A in $\mathbb{R}^2$ be
$$A = \Big\{(2,-1)\Big\} \cup \Big\{(x_1,x_2) \,:\, x_1 = x_2\Big\} \cup \Big\{(x_1,x_2) \,:\, x_1 \leq 0 \text{ and } x_2 > 0\Big\}.$$
Find the interior, the set of all accumulation points, and the closure of A (without proof).

3. Let $\mathbb{Q}$ denote the set of rational numbers, and $\mathbb{I}$ denote the set of irrational numbers. Let the set $\mathbb{Q} \times \mathbb{I} := \{(x_1,x_2) \,:\, x_1 \in \mathbb{Q},\ x_2 \in \mathbb{I}\} \subseteq \mathbb{R}^2$. Determine the interior, closure, and boundary of $\mathbb{Q} \times \mathbb{I}$ (without proof).

4. Let (M,d) be a metric space, and S be a nonempty subset of M such that (S,d) is complete. Show that S is closed in (M,d).

5. Let (M,d) be a metric space with the discrete metric d. A sequence (x_n) in M is said to have a *constant tail* if there exist $K \in \mathbb{N}$ and $c \in M$ such that $x_n = c$, $\forall n \geq K$. Prove the following:

   (1) A sequence in (M,d) is convergent if and only if it has a constant tail;

   (2) A Cauchy sequence in (M,d) has a constant tail;

   (3) Show that (M,d) is complete using (1)-(2).

The following extra problem(s) are for **Math 600** *students only:*

6. Let A be a nonempty set in $\mathbb{R}^n$, and d be the metric induced by the Euclidean norm on $\mathbb{R}^n$. Let $z \in \mathbb{R}^n$ be given.

   (1) Show that the infimum of the *real* set $\{d(z,x) \,:\, x \in A\}$ exists. In the following, define $d(z,A) := \inf\{d(z,x) : x \in A\}$.

   (2) Show that there exists a sequence (x_k) in A such that the *real* sequence $(d(z,x_k))$ converges to $d(z,A)$. Furthermore, show that (x_k) has a convergent subsequence in $\mathbb{R}^n$. (*Hint*: for the latter statement, consider the Bolzano-Weierstrass Theorem, which says that a bounded sequence in $\mathbb{R}^n$ has ...)

   (3) Use (2) to show that if the set A is closed, then there exists $x^* \in A$ such that $d(z,A) = d(z,x^*)$.