Math 600 Fall 2017 Homework #6

Due Nov. 13, Mon. in class

Note: For the Euclidean space $\mathbb{R}^n$, consider the usual metric induced by the Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^n$, unless otherwise stated.

1. Solve the following problems.

   (1) Let f be a continuous function on a metric space (M, d) and A be a nonempty set in M. Show that if f is uniformly continuous on the closure of A, then f is uniformly continuous on A.

   (2) Let $g : \mathbb{R}^2 \to \mathbb{R}$ be continuous on $\mathbb{R}^2$. Let $(a, b]$ and (c, d) be two intervals in $\mathbb{R}$. Use (1) to show that g is uniformly continuous on $(a, b] \times (c, d)$.

2. Let $f : (M, d) \to \mathbb{R}^n$ be Lipschitz continuous on M, and $f(A)$ be a closed and bounded set in $\mathbb{R}^n$ for some set $A \subset M$. Let $g : \mathbb{R}^n \to \mathbb{R}$ be continuous on $\mathbb{R}^n$. Show that the composition $g \circ f$ is uniformly continuous on A.

3. Let $f_n(x) = \sin(nx)/(1 + nx)$, and $A = [0, \infty)$.

   (1) Show that (f_n) converges pointwise on A, and find the limit function f_*;

   (2) Let $a > 0$. Show that (f_n) converges uniformly on $[a, \infty)$ to f_*;

   (3) Show that (f_n) does not converge uniformly on A to f_*.

4. Let $f_n(x) = x^n/(1 + x^n)$, and $A = [0, \infty)$.

   (1) Show that (f_n) converges pointwise on A, and find the limit function f_*;

   (2) Let $a \in (0, 1)$. Show that (f_n) converges uniformly on $[0, a]$ to f_*;

   (3) Show that (f_n) does not converge uniformly on $[0, 1]$ to f_*.

5. Suppose a sequence of *continuous* functions (f_n) converges pointwise to f_* on a *compact* set A. If f_* is *continuous* on A, does this imply that (f_n) always converge uniformly to f_* on A? If so, prove it; otherwise, give a counterexample.

6. Suppose that each f_n is continuous on the set A, and (f_n) converges to f_* uniformly on A. Suppose the sequence (x_n) in A converges to $x_* \in A$. Show that $(f_n(x_n))$ converges to $f_*(x_*)$.

7. Let $f_n : \mathbb{R} \to \mathbb{R}$ and $g_n : \mathbb{R} \to \mathbb{R}$ be two sequences of *bounded* functions that converge uniformly on the set A to f_* and g_*, respectively. Show that $(f_n \cdot g_n)$ converges uniformly to $f_* \cdot g_*$ on A.

Miscellaneous practice problems: *Do not submit*

1. Use the sequential criterion to show that the function $f(x) = x^2$ is *not* uniformly continuous on $A = [0, \infty)$.

2. Let $f_n : \mathbb{R} \to \mathbb{R}$ and $g_n : \mathbb{R} \to \mathbb{R}$ be two sequences of functions that converge uniformly on the set A to f_* and g_*, respectively. Show that $(f_n + g_n)$ converges uniformly on A to $f_* + g_*$.

3. Let $f_n(x) = x + 1/n,\ x \in \mathbb{R}$.

(1) Show that (f_n) converges pointwise on $\mathbb{R}$, and find the limit function f_*;

(2) Show that (f_n) converges uniformly to f_* on $\mathbb{R}$;

(3) Show that (f_n^2) does *not* converge uniformly to f_*^2 on $\mathbb{R}$.

$\star$ This problem shows that in general, the product of uniformly convergent sequences of functions need *not* be uniformly convergent on a set.