Math 650 Fall 2011 Homework #2

Due Sept. 26, Mon. in class

P.1 Let $f : \mathbb{R} \to \mathbb{R}$ be a *univariate*, real-valued function with a continuous derivative. Show that if f has a local minimizer that is not a global minimizer, then f must have another critical point. (*Remark*: the extension to *multivariable* functions is not true in general; see Ex. 8 on p. 56 of the Text.)

P.2 Consider the function $f(x, y) = x^3 - 3\alpha xy + y^3$, where $\alpha \in \mathbb{R}$ is a parameter.

(1) Show that f has no global minimizer or global maximizer for any α.

(2) For each value of α, find all the critical point(s) of f and determine whether a critical point is a local minimizer, local maximizer or saddle point.

P.3 Let $f(x) = \frac{1}{2}x^T Ax + c^T x + \alpha$ be a quadratic function, where $A \in \mathbb{R}^{n\times n}$, $c \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$. We shall prove the following claim in two steps:
Claim: if f is bounded from below on $\mathbb{R}^n$, i.e. $f(x) \geq \gamma, \forall\, x \in \mathbb{R}^n$ for some $\gamma \in \mathbb{R}$, then A is positive semidefinite and f achieves its minimum on $\mathbb{R}^n$.

(1) Show that the claim holds when A is symmetric using diagonalization of A.

(2) Show that the claim holds for an arbitrary A (without the symmetry assumption).

P.4 Consider the equality constrained optimization problem on $\mathbb{R}^2$:

$$\min\ x^2 + y^2, \qquad \text{subject to}\ \ x^2 - (y-1)^3 = 0$$

(1) Plot the constraint set $D := \{(x, y) \in \mathbb{R}^2 \mid x^2 - (y-1)^3 = 0\}$ and find the minimizer(s). Does the first-order necessary condition hold at the minimizer(s)? Explain why.

(2) One may attempt to solve this problem by substituting $x^2 = (y-1)^3$ into the objective function and reducing the original problem to the *unconstrained* problem: min $y^2+(y-1)^3$. Is this correct? Justify your answer.

P.5 Consider the equality constrained quadratic program on $\mathbb{R}^n$:

$$\min\ \frac{1}{2}x^T Qx + c^T x, \qquad \text{subject to}\ Ax = b,$$

where $Q \in \mathbb{R}^{n\times n}$ is symmetric, $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$ has full row rank, and $b \in \mathbb{R}^m$.

(1) Show that a local minimizer x^* satisfies the KKT conditions: $Qx^* + c$ is in the range of A^T and $Ax^* = b$.

(2) Show that a local minimizer also satisfies the 2nd order necessary condition: Q is positive semidefinite on the null space of A.

(3) Show that a KKT point satisfying the 2nd order necessary condition in (2) is a (global) minimizer of the quadratic program.

P.6 Consider the equality constrained optimization problem on $\mathbb{R}^n$:

$$\min\ -x^T x, \qquad \text{subject to}\ \ x^T x = c^T x,$$

where c is a nonzero vector in $\mathbb{R}^n$.

(1) Show that any Fritz John point is a KKT point.

(2) Find the minimizer(s) using the 1st and 2nd order optimality conditions.