# Math 650 Fall 2016 Homework #4

## Due Oct. 27, Thu in class

1. Construct an example of a closed set in $\mathbb{R}^2$ whose convex hull is not closed.

2. Let $C \subseteq \mathbb{R}^n$ be a convex set. Show that the closure of $C$ is convex. (*Hint*: use the sequential criterion for a point in the closure of a set.)

3. For a vector $x \in \mathbb{R}^n$, the index set $\text{supp}(x) := \{\, i \mid x_i \neq 0\}$ is called the *support* of $x$. Let a *nonzero* matrix $A \in \mathbb{R}^{m\times n}$ and a *nonzero* vector $b \in \mathbb{R}^m$ be such that the equation $Ax = b$, $x \geq 0$ has a solution, where $x \geq 0$ means that each component of $x$ is nonnegative.

   (1) Show that the solution set of the equation is closed and convex.

   (2) Show that the equation attains a nonzero solution $x^* \geq 0$ with $\alpha = \text{supp}(x^*)$ such that the columns of $A_{\bullet\alpha}$ are linearly independent. (*Hint*: refer to the proof of Caratheodory Theorem.)

4. Let $K_1$ and $K_2$ be two convex cones in a vector space. Show that $K_1 + K_2$ is a convex cone and $K_1 + K_2 = \text{conv}(K_1 \cup K_2)$.

5. An ordered real $k$-tuple $a \equiv \big(a_1, \ldots, a_k\big)$ is called *lexicographically nonnegative* if either $a = 0$ or its first nonzero element (from the left) is positive. Let $\mathcal{S}$ be the set of all lexicographically nonnegative $k$-tuples.

   (1) Show that $\mathcal{S}$ is a convex cone in $\mathbb{R}^k$.

   (2) Is $\mathcal{S}$ closed? If so, give a proof; otherwise, construct a counterexample.

**More practice problems:** *Do not submit*

1. Consider the Lorentz cone $\mathcal{L} := \{x \in \mathbb{R}^n \mid x_n \geq \sqrt{\sum_{i=1}^{n-1} x_i^2}\}$. Show that the Lorentz cone is closed, convex, and self-dual (i.e. $\mathcal{L} = \mathcal{L}^*$). (*Hint*: For the convexity, treat $\mathcal{L}$ as an epigraph of a *nice* function. For the self-duality, use contradiction for $\mathcal{L}^* \subseteq \mathcal{L}$ and use Cauchy-Schwarz inequality for $\mathcal{L} \subseteq \mathcal{L}^*$.)