Math 650 Fall 2011 Homework #6

Due Nov. 21, Mon. in class

P.1 Let $C \subseteq \mathbb{R}^n$ be convex. Show that the closure of C is convex.

P.2 Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable convex function, $C \subseteq \mathbb{R}^n$ be a closed convex set, and $\alpha \geq 0$ be a fixed real number. Show that $x^* \in C$ is a global min of the convex optimization problem $\min_{x \in C} f(x)$ if and only if $x^* = \Pi_C(x^* - \alpha \nabla f(x^*))$, where $\Pi_C(\cdot)$ denotes the Euclidean projection onto C. (*Hint*: use the optimality condition (i.e. VI) of convex optimization and the property of Euclidean projection.)

P.3 Let $C \subseteq \mathbb{R}^n$ be a closed convex set, and $x^* \in \mathbb{R}^n \setminus C$. Define the (Euclidean) distance from x^* to C as $d_C(x^*) := \|x^* - \Pi_C(x^*)\|_2$.

(1) Let D be a closed convex set containing C. Show that $d_D(x^*) \leq d_C(x^*)$.

(2) Let H be a closed half-space containing C. Show that $d_H(x^*) \leq d_C(x^*)$.

(3) Consider the closed half-space $H := \{x \in \mathbb{R}^n : \langle a, x \rangle \leq \alpha\}$, where $a = x^* - \Pi_C(x^*)$ and $\alpha = \langle a, \Pi_C(x^*) \rangle$. Show that (i) $a \neq 0$; (ii) $C \subseteq H$; and (iii) $d_C(x^*) = d_H(x^*)$.

(4) Use the above results to show that the minimum distance from x^* to C (with $x^* \notin C$) is the maximum among the distances from x^* to closed half-spaces containing C, i.e., $d_C(x^*) = \max_{C \subseteq H} d_H(x^*)$, where H's are closed half-spaces.

P.4 For a given nonempty set $S \subseteq \mathbb{R}^n$, recall that its support function $\sigma_S(x) := \sup\{\langle x, z \rangle \mid z \in S\}$. We have shown that the function σ_S is convex and sublinear.

(1) For any nonempty sets $S_1, S_2 \in \mathbb{R}^n$, show that $\sigma_{S_1+S_2}(x) = \sigma_{S_1}(x) + \sigma_{S_2}(x), \ \forall\, x \in \mathbb{R}^n$.

(2) Show that if S is compact, then $\sigma_S(x) = \max\{\langle x, z \rangle \mid z \in S\}$.

(3) Let F and G be two compact convex sets in $\mathbb{R}^n$ such that $\sigma_F(x) = \sigma_G(x), \forall\, x \in \mathbb{R}^n$. Show that $F = G$. (*Hint*: use an appropriate separation theorem.)

(4) Use the above results to show that if A, B, C are compact and convex sets in $\mathbb{R}^n$ such that $A + C = B + C$, then $A = B$. Discuss what compactness/convexity assumptions on A, B, C can be relaxed without invalidating the statement.

P.5 Let $P := \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a nonempty polyhedron for a matrix A and a vector b.

(1) Show that a nonempty intersection of P and an affine set in $\mathbb{R}^n$ is a polyhedron.

(2) Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be an affine transformation defined by $F(x) := Tx + c$, where $T \in \mathbb{R}^{m \times n}$ and $c \in \mathbb{R}^m$. Show that the set $F(P) \subseteq \mathbb{R}^m$ is polyhedral (namely, any affine transformation of P is polyhedral). (*Hint*: use the Minkowski-Weyl Theorem.)