Math 302/600 Spring 2015 Homework #5

Due March 24, Tue. in class

Note: For any Euclidean space $\mathbb{R}^n$, consider the usual metric induced by the Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^n$, unless otherwise stated.

1. Determine which of the following sets are sequentially compact:

   (1) On $\mathbb{R}$: $A = \{2/n : n \in \mathbb{N}\}$, $A = \mathbb{Q} \cap [0, 1]$ (where $\mathbb{Q}$ is the set of rational numbers);

   (2) On $\mathbb{R}^2$: $A = \mathbb{Q} \times \mathbb{Q}$, $A = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| + x_2^2 \leq 4\}$.

   If a set is *not* sequentially compact, briefly explain why; otherwise, give a proof.

2. Show that any (nonempty) closed subset of a sequentially compact set in a metric space is also sequentially compact.

3. Show that the intersection and union of two sequentially compact sets in a metric space (M, d) remain sequentially compact.

4. Let (M, d) be a metric space. Show that $A \subseteq M$ is sequentially compact if and only if every infinite subset of A (i.e., a subset that contains infinitely many elements) has an accumulation point in A.

The following extra problems are for **Math 600** *students only:*

5. Let (x_n) be a sequence in a metric space that converges to x_*, and the set $A := \{x_1, x_2, \ldots, x_n, \ldots\} \cup \{x_*\}$. Show that A is sequentially compact.

6. Let (M, d) be a metric space such that M is sequentially compact. Show that (M, d) is complete.