Math 650 Fall 2011 Homework #4

Due Oct. 26, Wed. in class

P.1 Let $\mathcal{A}$ be a subset in $\mathbb{R}^n$. Show that $\mathcal{A}$ is affine if and only if there exist a matrix $A \in \mathbb{R}^{m\times n}$ and a vector $b \in \mathbb{R}^m$ such that $\mathcal{A} = \{ x \in \mathbb{R}^n \mid Ax = b \}$. (*Hint*: use the fact in linear algebra: any subspace in $\mathbb{R}^n$ is a null space of some matrix.)

P.2 Let K_1 and K_2 be two convex cones in a vector space, both containing the origin.

(1) Show that $K_1 + K_2$ is a convex cone.

(2) Show that $K_1 + K_2 = \operatorname{conv}(K_1 \cup K_2)$.

P.3 Let C, D be nonempty sets in $\mathbb{R}^n$.

(1) Show that if C is open, so is $C + D$.

(2) Use (1) to show that if C is open, then its convex hull $\operatorname{conv}(C)$ is open.

(3) Construct an example of a closed set in $\mathbb{R}^2$ whose convex hull is not closed.

P.4 For a nonnegative vector $x \in \mathbb{R}^n$ (i.e. $x \in \mathbb{R}^n_+$), the index set $\operatorname{supp}(x) := \{ i \mid x_i > 0\}$ is called the *support* of x. Let $A \in \mathbb{R}^{m\times n}$ which has no zero column, and a *nonzero* vector $b \in \mathbb{R}^m$ be such that the equation $Ax = b,\ x \geq 0$ has a solution.

(1) Show that the solution set of the equation is closed and convex. Is the solution set affine? Is the solution set a cone? Explain why.

(2) Show that the equation attains a nonzero solution $x^* \geq 0$ with $\alpha = \operatorname{supp}(x^*)$ such that the columns of $A_{\bullet\alpha}$ are linearly independent. (*Hint*: refer to the proof of Caratheodory Theorem.)

P.5 An ordered real k-tuple $a \equiv \big(a_1, \ldots, a_k\big)$ is called *lexicographically nonnegative* if either $a = 0$ or its first nonzero element (from the left) is positive. Let $\mathcal{S}$ be the set of lexicographically nonnegative k-tuples.

(1) Show that $\mathcal{S}$ is a convex cone in $\mathbb{R}^k$.

(2) Is $\mathcal{S}$ closed? If so, give a proof; otherwise, construct a counterexample.

P.6 Consider the Lorentz cone $\mathcal{L} := \{x \in \mathbb{R}^n \mid x_n \geq \sqrt{\sum_{i=1}^{n-1} x_i^2}\}$. Show that the Lorentz cone is closed, convex, and self-dual (i.e. $\mathcal{L} = \mathcal{L}^*$). (*Hint*: for convexity, treat $\mathcal{L}$ as an epigraph of a *nice* function. For self-duality, use contradiction for $\mathcal{L}^* \subseteq \mathcal{L}$ and use Cauchy-Scharwz inequality for $\mathcal{L} \subseteq \mathcal{L}^*$.)