Math 600 Fall 2015 Homework #2

Due Sept. 24, Thu. in class

Note: For the Euclidean space $\mathbb{R}^n$, consider the usual metric induced by the Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^n$, unless otherwise stated.

1. Let A be a set in the metric space (M, d). Show the following:
   (1) The interior of A is equal to the union of all open sets contained in A (*Hint*: is $\text{int}A$ an open set contained in A?);
   (2) The closure of A is equal to the intersection of all closed sets containing A. (*Hint*: let S be the above-mentioned intersection. To show $\text{cl}A \subseteq S$, it suffices to show $S^c \subseteq (\text{cl}A)^c$.)

2. Let (M, d) be a metric space, and A, B be two subsets of M. Show the following:
   (1) $\text{cl}(A \cap B) \subseteq (\text{cl}A) \cap (\text{cl}B)$;
   (2) $\big(\text{int}A\big)^c = \text{cl}(A^c)$.

3. Let (M, d) be a metric space. Prove any two of the following statements (also think about the other but do not submit):
   (1) a convergent sequence in (M, d) is bounded and has a unique limit;
   (2) a convergent sequence in (M, d) is Cauchy;
   (3) a Cauchy sequence in (M, d) is bounded.

4. Let (M, d) be a metric space with the discrete metric d. A sequence (x_n) in M is said to have a *constant tail* if there exist $K \in \mathbb{N}$ and $c \in M$ such that $x_n = c$, $\forall n \geq K$. Prove the following:
   (1) A sequence in (M, d) is convergent if and only if it has a constant tail;
   (2) A Cauchy sequence in (M, d) has a constant tail;
   (3) Show that (M, d) is complete using (1)-(2).

Miscellaneous practice problems: *Do not submit*

1. Find all the limit points of each of the following sets (without proof):
   (1) $A = \{x = (x_1, x_2) \in \mathbb{R}^2 \,:\, 1 < x_1 \leq 2, \text{ and } x_2 < -2\}$ in $\mathbb{R}^2$;
   (2) $B = \{x = (\frac{1}{n}, 1 - \frac{1}{n}) \in \mathbb{R}^2 \,:\, n \in \mathbb{N}\} \cup \{(1, 2)\}$ in $\mathbb{R}^2$;
   (3) $C = \{x = (x_1, x_2) \in \mathbb{R}^2 \,:\, x_1 \text{ is rational, and } x_2 = \frac{1}{n}, n \in \mathbb{N}\}$ in $\mathbb{R}^2$.

2. Let $\mathbb{Q}$ denote the set of rational numbers, and $\mathbb{I}$ denote the set of irrational numbers. Let the set $\mathbb{Q} \times \mathbb{I} := \{(x_1, x_2) \,:\, x_1 \in \mathbb{Q}, \ x_2 \in \mathbb{I}\} \subseteq \mathbb{R}^2$. Determine the interior, closure, and boundary of $\mathbb{Q} \times \mathbb{I}$ (without proof).

4. Consider the metric induced by the 1-norm on $\mathbb{R}^2$: $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$, $\forall\, x, y \in \mathbb{R}^2$. Let the set $A = \{x = (x_1, x_2) \in \mathbb{R}^2 \,:\, x_1 \geq 1 \text{ and } x_2 \leq 1\}$.
   (1) Prove that the set A is closed via the definition, namely, the complement of A is open;
   (2) Prove that the set A is closed using the sequential criterion.