Math 302/401/600 Fall 2010 Homework #5

Due Oct. 13, Wed. in class

1. Let A be a sequentially compact set in the metric space (M, d). Show that any closed subset of A is also sequentially compact.

2. In $(\mathbb{R}^n, \|\cdot\|)$, where $\|\cdot\|$ is an arbitrary norm on $\mathbb{R}^n$, show that any sequence in a bounded set of $\mathbb{R}^n$ has a subsequence that converges in $\mathbb{R}^n$. And illustrate by an example that the statement does not hold in a general metric space.

3. Use the definition of compactness (i.e. the open cover definition) to show that the following two sets are *not* compact:

   (1) the open ball $B(x, 1)$ with the radius 1 contained in the metric space (M, d), where $x \in M$;

   (2) the set $A = \{(x, y) \in \mathbb{R}^2 : -1 < x \leq 1,\ 0 \leq y \leq 2\}$ (using the standard metric on $\mathbb{R}^2$).

4. Use the definition of compactness to show that the union of two (nonempty) compact sets is compact.

5. What are compact and connected sets in $\mathbb{R}$?

6. Which of the following sets are connected in $\mathbb{R}$:

$$\{\sqrt{2}\}, \quad \{3, -10\}, \quad [0, 1), \quad (-7, \infty), \quad \mathbb{Z}, \quad \mathbb{Q}, \quad \mathbb{R},$$

   where $\mathbb{Z}$ and $\mathbb{Q}$ are the sets of integers and rational numbers, respectively.