Local Equilibrium Controllability of Multibody Systems Controlled via Shape Change

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***Abstract*—We study local equilibrium controllability of shape controlled multibody systems. The multibody systems are defined on a trivial principal fiber bundle by a Lagrangian that characterizes the base body motion and shape dynamics. A potential dependent on an advected parameter, e.g., uniform gravitational potential, is considered. This potential breaks base body symmetries, but a symmetry subgroup is assumed to exist. Symmetric product formulas are derived and important properties are obtained for symmetric products of horizontal shape control vector fields and a potential vector field that is dependent on an advected parameter. Based on these properties, sufficient conditions for local equilibrium controllability and local fiber equilibrium controllability are developed. These results are applied to two classes of shape controlled multibody systems in a uniform gravitational field: multibody attitude systems and neutrally buoyant multibody underwater vehicles.**

***Index Terms*—Nonlinear controllability, symmetry, underactuated mechanical systems.**

I. Introduction

A MULTIBODY system is a mechanical system consisting of several rigid bodies, in which one of the bodies is viewed as a base body that can translate and/or rotate. Additional bodies are connected to the base body and undergo constrained relative motion with respect to the base body. We refer to the relative configuration of the additional bodies with respect to the base body as the shape of the multibody system. Using shape change, one can achieve net changes in the multibody configuration, e.g., position and attitude of the base body, via interesting but complex coupling mechanisms that arise from a conservation law or interaction with the environment [28]. This idea has motivated research in attitude control of multibody aerospace systems [37], motion planning of robotic systems [35], locomotion of biological systems [18], and steering of land and underwater vehicles [19], [44]. In this paper, we focus on multibody systems controlled via shape change only. It is clear that such systems are underactuated.

Controllability is a fundamental issue in control system analysis. This is particularly important for underactuated mechanical control systems. Although linear controllability analysis is simple and elegant, many underactuated systems are not linearly controllable at an equilibrium. Hence, nonlinear controllability analysis is essential.

A variety of nonlinear controllability notions have been proposed, such as local accessibility, small-time local controllability [42], [43], global controllability in the sense of Bonnard [25], large-time local controllability [16], and controllability along a trajectory [1]. See [2], [33], and [41] for details. An important class of mechanical systems is that of drift free systems. For these systems, the Lie algebra rank condition implies both local accessibility and small time local controllability [32]. Another useful controllability concept for this class of systems is local fiber controllability [18]. If a nonlinear system has a drift vector field and cannot be transformed to drift free form, controllability analysis becomes much more complicated; the most widely used results are perhaps Sussmann's sufficient conditions [43]. Using the structure of mechanical systems, simplified controllability conditions can be obtained; see [34] and [36] for such results. Other controllability results for mechanical systems include global controllability of a spacecraft attitude control system [13] and its extension to mechanical systems with symmetries [27], controllability analysis for a planar body with unilateral thrusters [26] and for kinematic control systems on stratified configuration spaces [15]. Also, see [3], [31], [2], or [4] for controllability of systems with fiber bundle structure or nonholonomic constraints.

In [24], Lewis and Murray introduce the important notion of local configuration controllability for simple mechanical systems starting from zero velocity. It is shown that Lie brackets of simple mechanical systems have a special form when evaluated at zero velocity. This special form leads to the notion of a symmetric product, which was first noticed by Crouch [12]. Sufficient conditions for local configuration controllability are given in terms of Lie brackets and symmetric products of control vector fields and a potential vector field. An extension of local configuration controllability is equilibrium controllability [24]. Sufficient conditions for local configuration controllability imply equilibrium controllability if the initial configuration is an equilibrium.

A series of results on local configuration controllability have been obtained following [24]. Controllability of a class of underactuated Lagrangian systems on Lie groups is analyzed in [6]. Lewis investigates controllability for systems with nonholonomic constraints in [23]. Necessary conditions for single-input mechanical systems are given in [21]. Symmetric product properties are studied for multibody systems with full symmetry and constraints in [11], where local fiber configuration controlla-

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bility is defined and sufficient conditions are given. Also, see [7] for the related treatment. The case of dissipative forces is studied in [10].

In this paper, we study local equilibrium controllability for a class of shape controlled multibody systems whose potential is dependent on an advected parameter. This class of multibody systems has the property that base body symmetries are broken by the advected parameter but a symmetry subgroup does exist. This symmetry property results in a conserved quantity, which implies that the complete phase space state is not reachable, even locally. However, motivated by rest-to-rest maneuvers of multibody systems, we study whether the system can be transferred from one equilibrium configuration to a nearby equilibrium configuration, that is, local equilibrium controllability. To show this controllability property, we develop symmetric product properties for this class of multibody systems based on multibody structure and shape actuation assumptions. Sufficient local (fiber) equilibrium controllability conditions are obtained for this class of systems and are applied to multibody attitude systems and neutrally buoyant multibody underwater vehicles in a uniform gravitational field. The symmetric product results developed in this paper can also be used for open-loop motion planning of rest-to-rest maneuvers; see [39] for preliminary results in this direction.

The paper is organized as follows. In Section II, mathematical preliminaries are introduced. Section III studies the dynamics of a class of multibody systems dependent on an advected parameter; equilibrium manifolds and conservation properties are obtained and several nontrivial examples are given. Symmetric product properties are investigated in Section IV, where symmetric product formulas are derived and are used to demonstrate symmetric product properties. In Section V, we apply these properties to obtain sufficient conditions for local equilibrium controllability and for local fiber equilibrium controllability. In Section VI, two classes of shape controlled multibody systems in a uniform gravitational field are studied: multibody attitude systems and neutrally buoyant multibody underwater vehicles. Specific examples illustrate the general controllability results. Finally, conclusions are made in Section VII.

II. Mathematical Preliminaries

Consider a simple mechanical system on a trivial principal fiber bundle $Q = G \times Q_s$, where G is a Lie group and Q_s is a manifold we shall call shape space [2]; see examples of such systems in Section III-D. For every $q = (g, r) \in Q$ where $g \in G$ and $r \in Q_s$, the left action of G on Q is a smooth map $\Phi: G \times Q \to Q$ given by $\Phi_h q = (hg, r)$ for an arbitrary $h \in G$, where Φ is assumed to be free and proper.

Let $\hat{M}(q)$ be a G-invariant Riemannian metric on Q, that is, $\hat{M}(q)(u, v) = \hat{M}(\Phi_g q)(T_q\Phi_g u, T_q\Phi_g v)$ for all $g \in G$ and all tangent vectors $u, v \in T_qQ$ at all $q \in Q$. For notational convenience, we also use $\langle\langle\cdot,\cdot\rangle\rangle$ to denote the metric $\hat{M}(q)$. Let the Lagrangian be given by

$$\tilde{L}(v_q) = \frac{1}{2}\hat{M}(q)(v_q, v_q) - V(q) \tag{1}$$

where $v_q \in TQ$. In the bundle coordinates (g, r), the Lagrangian can be written as

$$\tilde{L}(g, \dot{g}, r, \dot{r}) = \frac{1}{2}\begin{pmatrix}\dot{g} & \dot{r}\end{pmatrix}\hat{M}(g, r)\begin{pmatrix}\dot{g}\\ \dot{r}\end{pmatrix} - V(g, r).$$

Let L_g be the left action of G, thus $\xi = T_gL_{g^{-1}}\dot{g} \in \mathfrak{g}$, the Lie algebra of G. By the invariance of the metric, we obtain the reduced kinetic energy

$$T(\xi, r, \dot{r}) = \frac{1}{2}\begin{pmatrix}\xi^T & \dot{r}^T\end{pmatrix}M(r)\begin{pmatrix}\xi\\ \dot{r}\end{pmatrix} \tag{2}$$

where ξ and $\dot{r}$ are treated as column vectors, and

$$M(r) = \begin{bmatrix} M_{11}(r) & M_{12}(r)\\ M_{21}(r) & M_{22}(r)\end{bmatrix} := \hat{M}(e, r)$$

is the reduced inertia tensor. Here, $e \in G$ is the identity of G. Let X be a smooth vector field on Q of the form $X(q) = (g\xi_X(q), v_X(q))$. We use $\tilde{X}$ or $X^{\sim}$ to denote the left translation of X to the identity of G, that is, $\tilde{X}(q) = X^{\sim}(q) := T_q\Phi_{g^{-1}}X(q) = (\,\xi_X(q), v_X(q)\,)$, where both ξ_X and v_X are treated as vector functions. Let $Y = (g\xi_Y(q), v_Y(q))$ be another vector field on Q. Using this notation and the definition of the reduced metric defined by $M(r)$, we have $\langle\langle X, Y\rangle\rangle = \tilde{X}^T M(r)\tilde{Y}$.

The mechanical connection $\mathcal{A}$, a $\mathfrak{g}$-valued one-form on Q, can be defined with respect to the reduced metric [2]. In coordinates, the connection $\mathcal{A}(g, r)\cdot(\dot{g}, \dot{r}) = \mathrm{Ad}_g\big(\xi + A(r)\dot{r}\big)$, where $A(r) = M_{11}^{-1}(r)M_{12}(r)$ is treated as a matrix function on Q_s. Using this connection, the tangent space T_qQ can be expressed as a direct sum of the vertical subpace $V_qQ = \{v_q \in T_qQ : v_q = \xi_Q(q) \text{ for some } \xi \in \mathfrak{g}\}$, where ξ_Q denotes the infinitesimal generator of Φ corresponding to ξ, and the horizontal subspace $H_qQ = \{v_q \in T_qQ : \mathcal{A}(v_q) = 0\}$ at each $q \in Q$. Let $\pi: Q \to Q_s$ be a projection. At each $q \in Q$, $T_q\pi: T_qQ \to T_{\pi(q)}Q_s$ maps H_qQ isomorphically onto $T_{\pi(q)}Q_s$ and $V_qQ = \mathrm{Ker}(T_q\pi)$. Moreover, a vertical vector field X on Q has the form $X = \big(g\xi(q), 0\big)$, and a horizontal vector field Y on Q is a horizontal lift of a vector field Y_s on Q_s, that is $Y = \big(-gA(r)\cdot Y_s(r), Y_s(r)\big)$, where $Y_s(r)$ is viewed as a vector function and $A(r)\cdot Y_s(r)$, a $\mathfrak{g}$-valued function, is also identified with a vector function. It is clear that a horizontal vector field Y is G-invariant and satisfies $\langle\langle X, Y\rangle\rangle = 0$ for an arbitrary vertical vector field X.

We now consider the mechanical control system defined by the Lagrangian (1). Let $\Lambda \subset T^*Q$ be a set of one-forms that represent the control forces on the system, and let $\mathcal{Y} = \hat{M}^{-1}(q)\cdot\Lambda$ denote the family of control vector fields. Without confusion, Λ and $\mathcal{Y}$ also denote their corresponding distributions respectively. A symmetric product of two vector fields X and Y is defined as $\langle X : Y\rangle := \nabla_X Y + \nabla_Y X$[24], where ∇ is the Levi–Civita connection defined by the G-invariant Riemannian metric $\hat{M}$. Let HQ denote the distribution of all horizontal vector fields; let $\overline{\mathrm{Lie}}$ and $\overline{\mathrm{Sym}}$ denote the involutive closure and symmetric product closure of a distribution, respectively.

Proposition 1: Consider the mechanical control system defined by the Lagrangian (1). Let H be a subgroup of G, and let

$\operatorname{grad} V$ be an H-invariant potential vector field. Suppose the control distribution $\mathcal{Y}$ is horizontal, that is, $\mathcal{Y} \subseteq HQ$. Then, the following hold.

1) Iterative symmetric products of $\mathcal{Y}$ are G-invariant and are horizontal.
2) Iterative symmetric products of $\{\mathcal{Y} \cup \operatorname{grad} V\}$ are H-invariant. Moreover, symmetric products involving $\operatorname{grad} V$ only are zero when evaluated at an equilibrium where $\operatorname{grad} V = 0$.

Proof: We prove the first claim now. The control vector fields are G-invariant since they are horizontal. Moreover, the G-invariant metric $\hat{M}$ implies that if vector fields X and Y are G-invariant, then $\nabla_X Y$ is also G-invariant. Hence, iterative symmetric products of the control vector fields are G-invariant.

To prove the horizontal property, we show that a symmetric product of two horizontal vector fields X and Y is also horizontal, i.e., $\langle\langle \langle X : Y\rangle, Z\rangle\rangle = 0$ for an arbitrary vertical vector field Z. Let $[\cdot,\cdot]$ represent the Lie bracket of two vector fields. Using the formalism of the connection [14]

$$\langle\langle \nabla_X Y, Z\rangle\rangle = \frac{1}{2}\left(L_X\langle\langle Y,Z\rangle\rangle + L_Y\langle\langle X,Z\rangle\rangle - L_Z\langle\langle X,Y\rangle\rangle\right.$$
$$\left. +\langle\langle [X,Y],Z\rangle\rangle - \langle\langle [X,Z],Y\rangle\rangle - \langle\langle [Y,Z],X\rangle\rangle\right)$$

we have

$$\langle\langle \langle X:Y\rangle, Z\rangle\rangle = L_X\langle\langle Y,Z\rangle\rangle + L_Y\langle\langle X,Z\rangle\rangle - L_Z\langle\langle X,Y\rangle\rangle - \langle\langle [X,Z],Y\rangle\rangle - \langle\langle [Y,Z],X\rangle\rangle. \quad (3)$$

It is clear that $L_X\langle\langle Y,Z\rangle\rangle = L_Y\langle\langle X,Z\rangle\rangle = 0$ since $\langle\langle Y,Z\rangle\rangle \equiv 0$ and $\langle\langle X,Z\rangle\rangle \equiv 0$. Moreover, the G-invariance of X, Y and the metric implies that $\langle\langle X,Y\rangle\rangle$ is a smooth function only on Q_s, thus $L_Z\langle\langle X,Y\rangle\rangle = 0$. Next, we show $[X,Z]$ is vertical. Consider X and Z in local coordinates (g^i, r^j)

$$X = X_g^i(g,r)\frac{\partial}{\partial g^i} + X_s^j(r)\frac{\partial}{\partial r^j} \quad Z = Z_g^i(g,r)\frac{\partial}{\partial g^i}.$$

Hence

$$[X,Z] = \left(X_g^k(g,r)\frac{\partial Z_g^i(g,r)}{\partial g^k} + X_s^l(r)\frac{\partial Z_g^i(g,r)}{\partial r^l} - Z_g^j(g,r)\frac{\partial X_g^i(g,r)}{\partial g^j}\right)\frac{\partial}{\partial g^i}.$$

Therefore, $[X,Z]$ is vertical. Similarly, $[Y,Z]$ is vertical. Thus $\langle\langle [X,Z],Y\rangle\rangle = \langle\langle [Y,Z],X\rangle\rangle = 0$. This completes the proof of the first claim.

We show the second claim. Note that $\mathcal{Y}$ and the Riemannian metric $\hat{M}$ are H-invariant because they are G-invariant. Using the fact that $\nabla_X Y$ is H-invariant if vector fields X and Y are H-invariant, we see that iterative symmetric products of $\{\mathcal{Y} \cup \operatorname{grad} V\}$ are H-invariant. Furthermore, using local coordinate expressions, it is easy to show that for two vector fields X and Y, if $X(q) = 0, Y(q) = 0$, then $\langle X : Y\rangle(q) = 0$. By this result and the fact that $\operatorname{grad} V$ is zero at equilibrium, symmetric products involving $\operatorname{grad} V$ only are zero when evaluated at equilibrium. □

Corollary 1: Consider the mechanical control system defined by the Lagrangian (1). Suppose the control distribution $\mathcal{Y} = HQ$. Then $\mathcal{Y} = \overline{\mathrm{Sym}}(\mathcal{Y})$.

Proof: It is clear that $\mathcal{Y} \subseteq \overline{\mathrm{Sym}}(\mathcal{Y})$. Furthermore, according to Proposition 1, we obtain $\overline{\mathrm{Sym}}(\mathcal{Y}) \subseteq HQ = \mathcal{Y}$. Therefore $\mathcal{Y} = \overline{\mathrm{Sym}}(\mathcal{Y})$. □

III. Dynamics of Shape Controlled Multibody Systems Dependent on an Advected Parameter

Consider a multibody system on $Q = G \times Q_s$ as described in Section II. To simplify the development, we restrict our attention to the case where G is a matrix Lie group that describes base body motion, and the shape space Q_s is an n-dimensional Abelian Lie group with local coordinates $r = (r_1, \ldots, r_n)$. Thus Q_s is diffeomorphic to $\mathbb{S}^k \times \mathbb{R}^{n-k}$ for some $0 \le k \le n$.

In addition to the assumption that the kinetic energy (or equivalently the metric $\hat{M}$) is G-invariant, we make two more assumptions that hold throughout this paper.

1) The potential energy depends on an advected parameter, see its definition in Section III-A.
2) Controls act on the shape space such that the shape is fully actuated, i.e., $\Lambda = T^*Q_s$.

Based on the metric assumption, we see that the full shape actuation assumption implies $\mathcal{Y} = HQ$.

A. Reduced Lagrangian and Equations of Motion

Let W be a vector space, and W^* be its dual space. Suppose there is an appropriate left action of G on W^*, denoted by ga, $\forall\, g \in G$ and $\forall\, a \in W^*$. The induced action of $\mathfrak{g}$ on $a \in W^*$ is denoted by ξa for $\xi = g^{-1}\dot{g} \in \mathfrak{g}$.

Let the Lagrangian $\tilde{L}: TG \times W^* \times TQ_s \to \mathbb{R}$ be $\tilde{L}(g,\dot{g},a_0,r,\dot{r}) = \tilde{T}(g,\dot{g},r,\dot{r}) - \tilde{V}(g,a_0,r)$, where $\tilde{T}: TG \times TQ_s \to \mathbb{R}$ is the kinetic energy, $\tilde{V}: G \times W^* \times Q_s \to \mathbb{R}$ is the potential energy, and $a_0 \in W^*$ is fixed. Note that $\tilde{V}(g,a_0,r)$ is generally not equal to $\tilde{V}(g^{-1}g,a_0,r)$; this shows that the base body symmetries are broken by the potential. We assume that $\tilde{L}$ is invariant with respect to the G-action if we include the G-action on a_0, that is, for arbitrary $g,h \in G$, $\tilde{L}(g,\dot{g},a_0,r,\dot{r}) = \tilde{L}(hg,h\dot{g},ha_0,r,\dot{r})$. This invariance property allows us to define a reduced Lagrangian $L: \mathfrak{g} \times W^* \times TQ_s \to \mathbb{R}$ as $L(\xi,a,r,\dot{r}) = \tilde{L}(g^{-1}g, g^{-1}\dot{g}, a, r, \dot{r})$, where $a = g^{-1}a_0 \in W^*$ is referred to as the advected parameter [8] and summarizes the dependence of the potential energy on the group action. The general treatment involves the semidirect product theory; see [17], [30], and [38].

The reduced Lagrangian can be written as $L(\xi,a,r,\dot{r}) = T(\xi,r,\dot{r}) - V(a,r)$, where the reduced kinetic energy $T(\xi,r,\dot{r})$ is given in (2) and $V(a,r)$ is the reduced potential energy. The equations of motion are obtained from $L(\xi,a,r,\dot{r})$ using the variational principle of Hamilton; see [17] and [39] for a derivation. The base body dynamics are described by the Euler–Poincaré equations [8], [38]

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \xi}\right) = \mathrm{ad}_\xi^*\left(\frac{\partial L}{\partial \xi}\right) + \frac{\partial L}{\partial a} \diamond a \quad (4)$$

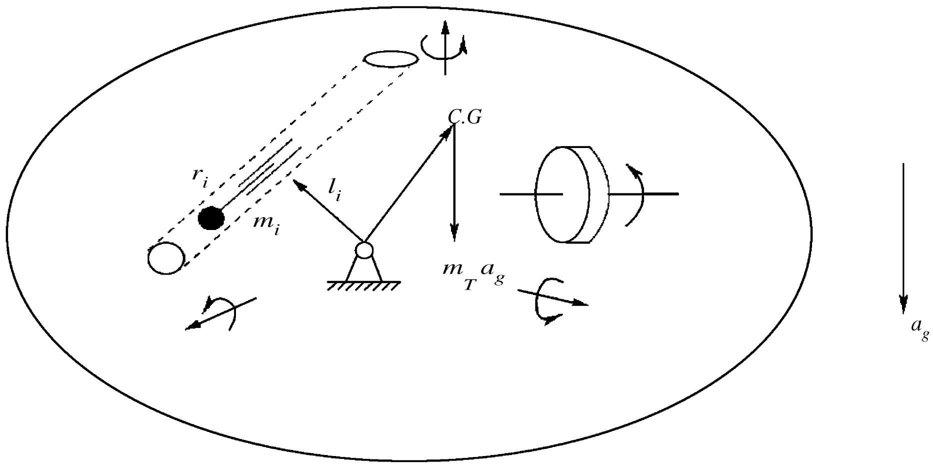

Fig. 1. Schematic configuration of a multibody attitude system in a uniform gravitational field with a reaction wheel and a proof mass actuator.

with $\dot{a} = -\xi a$, where the operation $\diamond$: $W \times W^* \to \mathfrak{g}^*$, denoted by $v \diamond a$ for $v \in W, a \in W^*$, is defined by $\langle v \diamond a, \xi \rangle = -\langle \xi a, v \rangle$, $\forall\, \xi \in \mathfrak{g}$. Here, $\langle \cdot, \cdot \rangle$ denotes the natural pairing operation. Subsequently, we also use conjugate momentum $p = \partial L / \partial \xi = M_{11}(r)\xi + M_{12}(r)\dot{r}$. The shape dynamics are described by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = u_s \tag{5}$$

where u_s denotes the shape control inputs.

B. Equilibrium Manifold

The conditions for a controlled equilibrium $q_e = (g_e, r_e)$ at which $(\xi, \dot{r}) \equiv 0$ are

$$\frac{\partial V(a_e, r_e)}{\partial a} \diamond a_e = 0 \quad \frac{\partial V(a_e, r_e)}{\partial r} = u_{se}$$

where $a_e = g_e^{-1} a_0$, and u_{se} denotes a constant shape control input. Note that u_{se} is generally not zero. Moreover, $\{h \in G | a_e = h a_e\}$ is a symmetry (or isotropy) group of a_e. The symmetry (or isotropy) algebra of a_e is $\{\xi \in \mathfrak{g} | \xi a_e = 0\}$ [29]. It is easy to verify that $S_e = \{(g_e h^{-1}, r_e) \in Q | a_e = h a_e\}$ is a set of equilibrium configurations corresponding to r_e. We call S_e the equilibrium configuration set associated with r_e.

C. Conserved Quantity

We consider the Kelvin–Noether theorem for the Euler–Poincaré equations (4). Using a result in [17], we obtain the following.

Proposition 2 [30], [39]: Let $\xi(t)$, $a(t)$, $r(t)$ satisfy (4) and (5) and let $g(t)$ be the solution of $\dot{g}(t) = g(t)\xi(t)$, $t \geq 0$. Let $\mathcal{C}$ be a manifold on which G acts through a left action, and let $c(t) = g^{-1}(t)g(0)c_0$, where $c_0 \in \mathcal{C}$ is fixed. Let $\mathcal{K}$: $\mathcal{C} \times W^* \to \mathfrak{g}$ be an equivariant map, satisfying $\mathcal{K}(c(t), a(t))a(t) = 0$, $\forall\, t \geq 0$. Then, the Kelvin–Noether quantity $I(c, \xi, a, r, \dot{r}) = \langle \mathcal{K}(c, a), (\partial L / \partial \xi)(\xi, a, r, \dot{r}) \rangle$ is a conserved quantity.

This conserved quantity is important in the subsequent controllability analysis.

D. Examples of Shape Controlled Multibody Systems

We apply the results developed above to three classes of shape controlled multibody systems: multibody systems with full base body symmetry, multibody attitude systems, and neutrally buoyant multibody underwater vehicles.

1) Multibody Systems With G-Invariant Potential: A special case is where the reduced potential is only dependent on the shape coordinates, so that the Lagrangian is G-invariant. The Euler–Poincaré equations for the base body dynamics are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \xi}\right) = \mathrm{ad}^*_\xi \left(\frac{\partial L}{\partial \xi}\right)$$

and the equations for the shape dynamics are the same as (5). Note that any $(g, r) \in Q$ is a controlled equilibrium if we choose $u_s = \partial V(r) / \partial r$.

According to Noether's theorem and the assumption that the system is initially in equilibrium, we have $p(t) = 0$, $\forall\, t \geq 0$. This implies $\xi = -A(r)\dot{r}$. Thus, we obtain the simplified control model: $g^{-1}\dot{g} = -A(r)\dot{r}$, $\ddot{r} = v_s$, where v_s is the shape acceleration control. The previous equations, with $\dot{r}$ viewed as an input, have been studied for numerous examples of multibody systems with G-invariant potential [37].

2) Multibody Attitude Systems: Consider the following class of multibody attitude systems: the base body rotates about a fixed pivot point in a uniform gravitational field; see Fig. 1 for a schematic configuration illustrating a base body, a reaction wheel and a proof mass actuator. A base body fixed coordinate frame is chosen with its origin located at the pivot point. The configuration manifold is $Q = \mathrm{SO}(3) \times Q_s$, where $R \in \mathrm{SO}(3)$ represents the base body attitude, and $\widehat{\omega} \in \mathfrak{so}(3)$. Here, $\omega \in \mathbb{R}^3$ represents the base body angular velocity expressed in the base body frame and $\widehat{\cdot}$: $\mathbb{R}^3 \to \mathfrak{so}(3)$ is the usual isomorphism.

The reduced kinetic energy is given by

$$T(\omega, r, \dot{r}) = \frac{1}{2} \begin{pmatrix} \omega^T & \dot{r}^T \end{pmatrix} \underbrace{\begin{bmatrix} M_{11}(r) & M_{12}(r) \\ M_{21}(r) & M_{22}(r) \end{bmatrix}}_{M(r)} \begin{pmatrix} \omega \\ \dot{r} \end{pmatrix}$$

where $M(r)$ is symmetric and positive definite for all $r \in Q_s$.

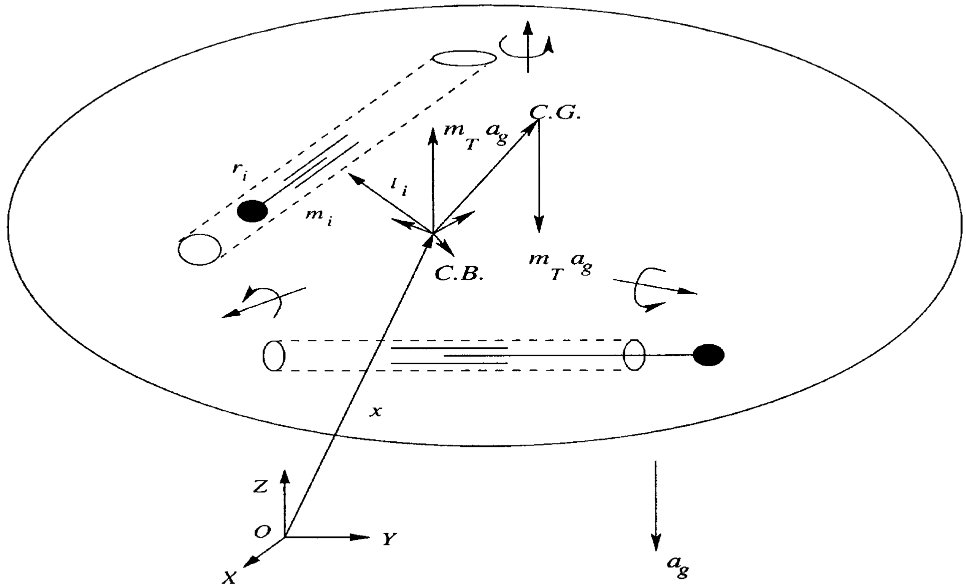

Fig. 2. Schematic configuration of a neutrally buoyant multibody underwater vehicle with two proof mass actuators.

The inertial coordinate frame is chosen with the origin located at the pivot point. The gravitational potential energy is $V_G(R,r) = m_T a_g e_3^T R \rho_c(r)$, where $e_3 = (0,0,1)^T$ denotes the opposite direction of the gravity vector in the inertial frame, a_g is the gravity acceleration constant, m_T is the total mass, and $\rho_c(r)$ denotes the position vector of the center of mass of the system with respect to the base body coordinate frame. Let $V_s(r)$ denote a shape-dependent potential energy, e.g., elastic potential. Thus the total potential energy is $V(R,r) = V_G(R,r) + V_s(r)$. Let $W^* = \mathbb{R}^3$, and $a_0 = e_3 \in W^*$. The action of SO(3) on W^* is given by matrix multiplication Ra for all $R \in \mathrm{SO}(3)$ and $a \in W^*$. Hence, $V_G(R, a_0, r) = m_T a_g \left(R^T e_3\right)^T \rho_c(r)$ is SO(3)-invariant; see [39] and [30]. As a result, the reduced potential energy is $V(\Gamma, r) = m_T a_g \Gamma^T \rho_c(r) + V_s(r)$, where $\Gamma = R^T e_3 \in W^*$. Let $\xi = \widehat{\eta} \in \mathfrak{so}(3)$, where $\eta \in \mathbb{R}^3$. It can be verified that the induced action of ξ on $a \in W^* = \mathbb{R}^3$ is identified with the usual vector cross product, i.e., $\xi a = \eta \times a$, and that the operator $\diamond$ is also identified with the cross product $\times$. Thus $\dot{\Gamma} = \Gamma \times \omega$.

The equations of motion are

$$M(r)\begin{bmatrix}\dot{\omega}\\ \ddot{r}\end{bmatrix} = -\frac{dM(r)}{dt}\begin{bmatrix}\omega\\ \dot{r}\end{bmatrix} + \begin{bmatrix}\Pi\times\omega\\ \frac{\partial[T(\omega,r,\dot{r})-V_s(r)]}{\partial r}\end{bmatrix} + m_T a_g \begin{bmatrix}\Gamma\times\rho_c(r)\\ -\left[\frac{\partial\rho_c(r)}{\partial r}\right]^T\Gamma\end{bmatrix} + \begin{bmatrix}0\\ u_s\end{bmatrix} \qquad (6)$$

where $\Pi = M_{11}(r)\omega + M_{12}(r)\dot{r}$ is the conjugate angular momentum. These equations are similar to the classical heavy top equations [29].

The conditions for a controlled equilibrium (R_e, r_e) are

$$\Gamma_e \times \rho_c(r_e) = 0 \quad \frac{\partial V_s(r_e)}{\partial r} + m_T a_g \left[\frac{\partial \rho_c(r_e)}{\partial r}\right]^T \Gamma_e = u_{se}$$

where $\Gamma_e = R_e^T e_3$, and u_{se} denotes a constant shape control input. We distinguish two classes of equilibria: balanced equilibria where $\rho_c(r_e) = 0$ and unbalanced equilibria where $\rho_c(r_e) \neq 0$. For the balanced case, any base body attitude is an equilibrium. For the unbalanced case, it is shown in [39] that the equilibrium configuration set associated with a shape equilibrium r_e is given by $S_e = \left\{(R, r_e) | R = R_e e^{\psi \widehat{\Gamma}_e},\ \psi \in [0, 2\pi)\right\}$, where R_e is an equilibrium attitude at r_e.

Applying Proposition 2, it can be shown that $\Gamma \cdot \Pi$ is a conserved quantity [9], [39]. This is consistent with the fact that the vertical component of the inertial angular momentum is conserved.

3) Neutrally Buoyant Multibody Underwater Vehicles: In this section, we consider a class of multibody underwater vehicles in an ideal fluid; a uniform gravitational field is assumed [20]. See Fig. 2 for a schematic configuration illustrating a base body and two proof masses. The base body of the vehicle can translate and rotate freely in three dimensional space according to Kirchoff's equations. The vehicle is assumed to be neutrally buoyant, but the center of gravity need not be coincident with the center of buoyancy. A base body coordinate frame is chosen with its origin at the center of buoyancy of the system. The configuration manifold is $Q = \mathrm{SE}(3) \times Q_s$, where $(x, R) \in \mathrm{SE}(3)$, $R \in \mathrm{SO}(3)$ represents the base body attitude, and $x \in \mathbb{R}^3$ represents the position of the center of buoyancy of the vehicle in the inertial frame. The twist of $\mathfrak{se}(3)$ is (v, ω), where $v \in \mathbb{R}^3$, $\omega \in \mathbb{R}^3$ represent the base body linear and angular velocities in the base body frame, respectively.

The reduced kinetic energy is

$$T(v,\omega,r,\dot{r}) = \frac{1}{2}\begin{pmatrix}v\\ \omega\\ \dot{r}\end{pmatrix}^T \underbrace{\begin{bmatrix}M_T & K(r) & B_t(r)\\ K^T(r) & J(r) & B_r(r)\\ B_t^T(r) & B_r^T(r) & m(r)\end{bmatrix}}_{M(r)} \begin{pmatrix}v\\ \omega\\ \dot{r}\end{pmatrix}$$

where $M(r)$ is symmetric and positive definite for all $r \in Q_s$, and M_T and $J(r)$ are the mass and inertia matrices of the body-fluid system, respectively; these matrices include the "added" masses and inertia due to the fluid [20]. We also have $K(r) = -m_T\widehat{\rho}_c(r)$, $B_t(r) = m_T(\partial\rho_c(r)/\partial r)$, where $\rho_c(r)$ represents the relative position of the center of mass (or center of gravity) of the system expressed in the base body coordinate frame, and m_T is the total mass of the multibody vehicle.

Since the system is assumed to be neutrally buoyant, the gravitational potential energy $V_G(R,r) = m_T a_g\left(R^T e_3\right)^T \rho_c(r)$, where $e_3 = (0,0,1)^T$ denotes the opposite direction of gravity as before. Let $V_s(r)$ denote a shape-dependent potential energy. The total potential energy is $V(R,r) = V_G(R,r) + V_s(r)$. In this case, $W^* = \mathbb{R}^3$, and $a_0 = e_3 \in W^*$. The action of SE(3) on W^* is given by Ra for all $R \in$ SO(3) and $a \in W^*$ as before. Hence, $V(R,a_0,r)$ is SE(3)-invariant. Let $\Gamma = R^T e_3 \in W^*$; the reduced potential energy is given by $V(\Gamma,r) = m_T a_g \Gamma^T \rho_c(r) + V_s(r)$. It can be verified that the induced action of $\xi \in \mathfrak{se}(3)$ with twist (v,ω) on $a \in W^* = \mathbb{R}^3$ is identified with $\omega \times a$, and that $\forall\, \Sigma \in W$, $\Sigma \diamond \Gamma = (0, \Sigma \times \Gamma) \in \mathfrak{se}(3)$.

The equations of motion, expressed in terms of $(v,\omega,r,\dot r)$, are given by

$$M(r)\begin{bmatrix}\dot v\\ \dot\omega\\ \ddot r\end{bmatrix} = -\frac{dM(r)}{dt}\begin{bmatrix}v\\ \omega\\ \dot r\end{bmatrix} + \begin{bmatrix}P\times\omega\\ \Pi\times\omega + P\times v\\ \frac{\partial[T(v,\omega,r,\dot r)-V_s(r)]}{\partial r}\end{bmatrix} + m_T a_g\begin{bmatrix}0\\ \Gamma\times\rho_c(r)\\ -\left[\frac{\partial\rho_c(r)}{\partial r}\right]^T\Gamma\end{bmatrix} + \begin{bmatrix}0\\0\\u_s\end{bmatrix} \tag{7}$$

where $P = M_T v + K(r)\omega + B_t(r)\dot r$, and $\Pi = K^T(r)v + J(r)\omega + B_r(r)\dot r$ are the conjugate linear and angular momenta, respectively.

The conditions for a controlled equilibrium (x_e, R_e, r_e) are

$$\Gamma_e \times \rho_c(r_e) = 0 \quad \frac{\partial V_s(r_e)}{\partial r} + m_T a_g\left[\frac{\partial\rho_c(r_e)}{\partial r}\right]^T \Gamma_e = u_{se}$$

where $\Gamma_e = R_e^T e_3$, and u_{se} denotes a constant shape control. For the unbalanced case where $\rho_c(r_e) \neq 0$, the equilibrium configuration set associated with a shape r_e is given by $S_e = \left\{(x,R,r_e) | x\in\mathbb{R}^3,\ R = R_e e^{\psi\widehat{\Gamma}_e},\ \psi\in[0,2\pi)\right\}$, where (x,R_e,r_e) is a controlled equilibrium configuration. For the balanced case where $\rho_c(r_e) = 0$, any base body position and attitude (x,R) is an equilibrium configuration.

As in the previous example, $\Gamma\cdot\Pi$ is conserved. Furthermore, the Lagrangian is translation invariant. Thus the total inertial linear momentum RP is conserved. If the system is initially in equilibrium, then $P(t) = 0, \forall\, t \geq 0$. Note that the inertial linear momentum RP is integrable: its integral is $M_T x + m_T R\rho_c(r)$ and remains constant along all trajectories.

IV. Symmetric Products for Shape Controlled Multibody Systems on $G \times Q_s$

As we will see in Section IV-A, sufficient conditions for local configuration controllability and local equilibrium controllability are expressed in terms of symmetric products. Moreover, a series expansion for simple mechanical systems starting from zero velocity, a useful tool for motion planning, can be written in terms of symmetric products [5], [6]. Therefore, symmetric product properties are not only crucial for controllability analysis but they are also important in motion planning. Prior results on symmetric products of mechanical control systems can be found in [7], [11], and [45]. A key assumption in those results is that there is no potential or the potential does not break G-symmetry. In this section, we study symmetric product properties, emphasizing the case where the potential depends on an advected parameter. Note that these properties do not require the assumption that the shape is fully actuated made in Section III.

A. Symmetric Products for Multibody Systems on $G \times Q_s$

We introduce some notation and a Lie bracket formula for vector fields on $Q = G \times Q_s$[39]. Consider two vector fields $X = (g\xi_X, v_X)$ and $Y = (g\xi_Y, v_Y)$ on $G\times Q_s$. The Lie bracket of X and Y is given by

$$[X,Y]^{\sim}(q) = \Big(\mathrm{ad}_X Y\Big)^{\sim}(q) + \frac{d}{dt}\Big|_{t=0}\Big(\tilde Y\circ\phi_t^X(q) - \tilde X\circ\phi_t^Y(q)\Big)$$

where $\phi_t^X(q)$, $\phi_t^Y(q)$ denote the flows of the vector fields X and Y starting from q, respectively, and $\Big(\mathrm{ad}_X Y\Big)^{\sim}(q) = (\mathrm{ad}_{\xi_X}\xi_Y(q), 0)^T$ denotes the adjoint operation for X and Y, where $\mathrm{ad}_\gamma\eta = [\gamma,\eta]_{\mathfrak{g}}$ is the adjoint operation of $\mathfrak{g}$ for $\gamma,\eta \in \mathfrak{g}$. Let $Y^* = (g\xi_Y^*, u_Y^*)$ be a co-vector field on $G\times Q_s$. The co-adjoint operator $\mathrm{ad}_X^* Y^*(q)$ is $\Big(\mathrm{ad}_X^* Y^*\Big)^{\sim}(q) = \big(\mathrm{ad}_{\xi_X}^*\xi_Y^*(q),\ 0\big)^T$. For notational convenience, we use $\mathrm{ad}_{\tilde X}\tilde Y$ and $\mathrm{ad}_{\tilde X}^*\tilde Y^*$ to denote $\Big(\mathrm{ad}_X Y\Big)^{\sim}$ and $\Big(\mathrm{ad}_X^* Y^*\Big)^{\sim}$, respectively. In the following proposition, we give formulas for symmetric products on $G\times Q_s$; the proofs are given in the Appendix. These formulas allow us to compute symmetric products even when G-symmetry is broken.

Proposition 3: Let $q = (g,r) \in G\times Q_s$, and let $X(q) = (g\xi_X(q), v_X(q))$ and $Y(q) = (g\xi_Y(q), v_Y(q))$ be two smooth vector fields on $G\times Q_s$. Their symmetric product is given by

$$\begin{aligned}\langle X:Y\rangle^{\sim}(q) = {} & \frac{d}{dt}\Big|_{t=0}\Big(\tilde X\circ\phi_t^Y(q) + \tilde Y\circ\phi_t^X(q)\Big)\\ & + M^{-1}(r)\\ & \times\Bigg\{\left(\frac{d}{dt}\Big|_{t=0} M\circ\phi_t^{v_Y}(q)\right)\tilde X(q)\\ & \quad + \left(\frac{d}{dt}\Big|_{t=0} M\circ\phi_t^{v_X}(q)\right)\tilde Y(q)\\ & \quad - DM(\tilde X(q),\tilde Y(q))\\ & \quad - \mathrm{ad}_{\tilde X(q)}^* M(r)\tilde Y(q)\\ & \quad - \mathrm{ad}_{\tilde Y(q)}^* M(r)\tilde X(q)\Bigg\}\end{aligned} \tag{8}$$

where $\phi_t^X(q)$, $\phi_t^Y(q)$ denote the flows of the vector fields X and Y, respectively, $\phi_t^{v_X}(q)$, $\phi_t^{v_Y}(q)$ denote the flows of the vector fields $\left(0, v_X(q)\right)$ and $\left(0, v_Y(q)\right)$, respectively, and DM is a $(\dim G+n)$-dimensional column vector given by

$$DM(\tilde{X}(q),\tilde{Y}(q)) = \left(\underbrace{0,\ldots,0}_{\dim G}, DM_s(\tilde{X}(q),\tilde{Y}(q))^T\right)^T$$

where

$$DM_s\left(\tilde{X}(q),\tilde{Y}(q)\right) = \left(\tilde{X}^T(q)\frac{\partial M(r)}{\partial r_1}\tilde{Y}(q),\ldots,\tilde{X}^T(q)\frac{\partial M(r)}{\partial r_n}\tilde{Y}(q)\right)^T.$$

Here, $(\partial M(r)/\partial r_i)$ denotes the derivative of the matrix valued function $M(r)$ with respect to r_i. Furthermore, let $X(q)$ and $Y(q)$ be of the form

$$\tilde{X}(q) = \begin{bmatrix}\xi_X(q)\\ v_X(q)\end{bmatrix} = M^{-1}(r)\begin{bmatrix}X_g(q)\\ X_s(q)\end{bmatrix} \quad (9)$$

$$\tilde{Y}(q) = \begin{bmatrix}\xi_Y(q)\\ v_Y(q)\end{bmatrix} = M^{-1}(r)\begin{bmatrix}Y_g(q)\\ Y_s(q)\end{bmatrix} \quad (10)$$

where X_g, Y_g are $\mathfrak{g}^*$-valued functions, and X_s, Y_s are T^*Q_s-valued functions, all on Q. Then, their symmetric product is

$$\langle X:Y\rangle^{\sim}(q) = M^{-1}(r)\begin{bmatrix}\langle X:Y\rangle_g(q)\\ \langle X:Y\rangle_s(q)\end{bmatrix} \quad (11)$$

where

$$\langle X:Y\rangle_g(q) = \left.\frac{d}{dt}\right|_{t=0}\left(X_g\circ\phi_t^Y(q)+Y_g\circ\phi_t^X(q)\right) - \mathrm{ad}^*_{\xi_X(q)}Y_g(q) - \mathrm{ad}^*_{\xi_Y(q)}X_g(q) \quad (12)$$

$$\langle X:Y\rangle_s(q) = \left.\frac{d}{dt}\right|_{t=0}\left(X_s\circ\phi_t^Y(q)+Y_s\circ\phi_t^X(q)\right) - DM_s\left(\tilde{X}(q),\tilde{Y}(q)\right). \quad (13)$$

B. Symmetric Products Only Involving Shape Control Vector Fields

In this section, we study symmetric products of shape control vector fields that are horizontal.

Proposition 4: Consider the multibody system described by (4) and (5) satisfying $\mathcal{Y}\subseteq HQ$. Then, iterative symmetric products of vector fields in $\mathcal{Y}$ are G-invariant and are in HQ. Moreover, let two vector fields $X, Y\in HQ$ be

$$\tilde{X}(q) = M^{-1}(r)\begin{bmatrix}0\\ X_s(r)\end{bmatrix}$$

$$\tilde{Y}(q) = M^{-1}(r)\begin{bmatrix}0\\ Y_s(r)\end{bmatrix} \quad (14)$$

then

$$\langle X:Y\rangle^{\sim}(q) = M^{-1}(r)\begin{bmatrix}0\\ \langle X:Y\rangle_s(r)\end{bmatrix} \quad (15)$$

where in coordinates

$$\langle X:Y\rangle_s(r) = \frac{\partial X_s(r)}{\partial r}\bar{M}_{22}(r)Y_s(r) + \frac{\partial Y_s(r)}{\partial r}\bar{M}_{22}(r)X_s(r) + DM_{22}(X_s(r),Y_s(r)) \quad (16)$$

and

$$DM_{22}(X_s(r),Y_s(r)) = \left(X_s^T(r)\frac{\partial\bar{M}_{22}(r)}{\partial r_1}Y_s(r),\ldots,X_s^T(r)\frac{\partial\bar{M}_{22}(r)}{\partial r_n}Y_s(r)\right)^T.$$

Here, $\bar{M}_{22} = \left(M_{22}-M_{21}M_{11}^{-1}M_{12}\right)^{-1}$.

Proof: The proof follows from Proposition 1 and straightforward computations using the formulas (12) and (13) in Proposition 3. □

Let Y^s denote an iterative symmetric product generated from a family of horizontal control vector fields. Thus, Y^s is horizontal, that is

$$M(r)\tilde{Y}^s(r) = (0, Y_s^s(r))^T \quad (17)$$

where $Y_s^s(r)\in T_r^*Q_s$. This agrees with the fact that all the controls influence the shape dynamics only. For the multibody system whose potential is G-invariant studied in Section III-D1, any symmetric product is horizontal. This property is consistent to conservation of the conjugate momentum p in this case.

C. Symmetric Products Involving a Potential Vector Field Dependent on an Advected Parameter

Consider the multibody system in (4) and (5). Since the shape is fully actuated, a control transformation can be introduced such that the transformed potential vector field is given by

$$\mathrm{grad}\, V^{\sim} = M^{-1}(r)\begin{bmatrix}-\frac{\partial V(a,r)}{\partial a}\diamond a\\ 0\end{bmatrix} \quad (18)$$

where $a=g^{-1}a_0$ is the advected parameter. This transformation does not change the spanning relations for symmetric products or controllability results, but it simplifies the computations. Thus, we consider this form for $\mathrm{grad}\, V$ in the subsequent development.

Note that at $q=(g,r)$, $\mathrm{grad}\, V$ is invariant under the subgroup $H=\{h\in G|a=ha\}$, the symmetry group of a. By Proposition 1, we see that symmetric products of $\{\mathcal{Y}\cup\mathrm{grad}\, V\}$ involving $\mathrm{grad}\, V$ are H-invariant at q. We show additional properties for such symmetric products in the following proposition; the proof is given in the Appendix.

We introduce some notation. An induced operation $\star\colon\mathfrak{g}\times W\to W$, denoted by $\xi\star v$ for $\xi\in\mathfrak{g}$, $v\in W$, is defined by $\langle a,\xi\star v\rangle = -\langle\xi a, v\rangle$, $\forall\, a\in W^*$.

Proposition 5: Let $q=(g,r)\in G\times Q_s$, and let X and Y be two vector fields dependent on the advected parameter of the form

$$\tilde{X}(q) = \begin{bmatrix}\xi_X(a,r)\\ v_X(a,r)\end{bmatrix} = M^{-1}(r)\begin{bmatrix}G_X(a,r)\diamond a\\ X_s(a,r)\end{bmatrix} \quad (19)$$

$$\tilde{Y}(q) = \begin{bmatrix}\xi_Y(a,r)\\ v_Y(a,r)\end{bmatrix} = M^{-1}(r)\begin{bmatrix}G_Y(a,r)\diamond a\\ Y_s(a,r)\end{bmatrix}. \quad (20)$$

Then, their symmetric product is

$$\langle X:Y\rangle^{\sim}(q) = M^{-1}(r)\begin{bmatrix}G(a,r)\diamond a\\ \langle X:Y\rangle_s(a,r)\end{bmatrix} \quad (21)$$

where in coordinates, $G(a,r) \in W$ is given by

$$G(a,r) = -\frac{\partial G_X(a,r)}{\partial a}(\xi_Y a) + \frac{\partial G_X(a,r)}{\partial r}v_Y + \xi_Y \star G_X(a,r) - \frac{\partial G_Y(a,r)}{\partial a}(\xi_X a) + \frac{\partial G_Y(a,r)}{\partial r}v_X + \xi_X \star G_Y(a,r) \quad (22)$$

and

$$\langle X : Y\rangle_s(a,r) = -\frac{\partial X_s}{\partial a}(\xi_Y a) - \frac{\partial Y_s}{\partial a}(\xi_X a) + \frac{\partial X_s}{\partial r}v_Y + \frac{\partial Y_s}{\partial r}v_X - DM_s(\tilde{X},\tilde{Y}). \quad (23)$$

With Proposition 5, we obtain the following result.

Proposition 6: Consider the multibody system described by (4) and (5) satisfying $\mathcal{Y} \subseteq HQ$. Assume that $\mathrm{grad}\, V$ depends on the advected parameter and is of the form in (18). Then iterative symmetric products of $\{\mathcal{Y} \cup \mathrm{grad}\, V\}$ involving $\mathrm{grad}\, V$ have the form in (21). Furthermore, symmetric products involving $\mathrm{grad}\, V$ only are zero when evaluated at any equilibrium q_e where $\mathrm{grad}\, V(q_e) = 0$.

Proof: Note that the control vector fields satisfy the form in (21) with $G(a,r) \equiv 0$. Hence, based on Proposition 5, each symmetric product generated from $\{\mathcal{Y} \cup \mathrm{grad}\, V\}$ involving $\mathrm{grad}\, V$ has the form in (21). The last claim follows from Proposition 1. □

Remark: Let $Z = (g\xi_Z, 0)$ be vertical, where ξ_Z is in the symmetry algebra of a corresponding to r. Hence, $\langle G(a,r) \diamond a, \xi_Z\rangle = -\langle \xi_Z a, G(a,r)\rangle = 0$. Therefore by Proposition 6, we conclude that $\langle\langle X, Z\rangle\rangle(a,r) = 0$ for any symmetric product X generated from $\{\mathcal{Y} \cup \mathrm{grad}\, V\}$. Using this observation and the series expansion in [5], one can obtain a conserved quantity.

We introduce some terminology for the subsequent development. We call $Y_s^s(r)$ in (17) for Y^s the *shape momentum function* associated with Y^s, and we call $\langle X : Y\rangle_s$ and $G(a,r)$ in (21) for $\langle X : Y\rangle$ the *shape momentum function* and the *advected momentum function* associated with $\langle X : Y\rangle$, respectively. This terminology will be used to describe local equilibrium controllability results in Section V.

V. Local Equilibrium Controllability of Shape Controlled Multibody Systems

Recall that if the conditions of Proposition 2 hold, then a conserved quantity exists. This quantity imposes a constraint on the system state and thus implies the phase space state is not completely reachable, even locally. Controllability analysis in another sense is needed. Moreover, it is shown in [39] that a multibody system described by (4) and (5) may not be linearly controllable at an equilibrium q_e if a symmetry subgroup exists, i.e., $\{h \in G | a_e = ha_e\}$ is not empty. These observations motivate our study of local equilibrium controllability, a nonlinear controllability concept, for such multibody systems.

A. Controllability Definitions and Sufficient Conditions

A simple mechanical system on Q is *locally equilibrium controllable* at an equilibrium $q_e \in Q$ if for each sufficiently small neighborhood of q_e in Q, and for every existing equilibrium $\bar{q}_e$ in such a neighborhood, there exist $T > 0$ and a solution and control pair (c,u), where $c: [0,T] \to Q$ such that $c(0) = q_e$, $c(T) = \bar{q}_e$ and $\dot{c}(0) = 0$, $\dot{c}(T) = 0$ using an admissible control.

Note that if there is no potential or the potential is only dependent on the shape, then any configuration is an equilibrium when the shape is fully actuated; but if there exists a potential that breaks base body symmetries, equilibrium configurations may be isolated or form an equilibrium manifold. Only in the latter case are there equilibria in any neighborhood of a given equilibrium so that local equilibrium controllability analysis is meaningful. The symmetry properties considered in this paper imply that such an equilibrium manifold always exists.

Sufficient conditions for local equilibrium controllability follow from conditions for small time local configuration controllability [24]; we refer to [24] for definitions and sufficient conditions for related controllability notions.[1]

Theorem 1 [24]: Let $q_e \in Q$ be an equilibrium configuration. Assume that the multibody system (4) and (5) is locally configuration accessible at q_e. If every bad symmetric product of $\{\mathcal{Y} \cup \mathrm{grad}\, V\}$ is a linear combination of lower degree good symmetric products when evaluated at q_e, then the multibody system is locally equilibrium controllable at q_e.

We also define local fiber equilibrium controllability. The multibody system in (4) and (5) is *locally fiber equilibrium controllable* at an equilibrium $q_e = (g_e, r_e) \in Q$ if for each sufficiently small neighborhood $U \subset G$ of g_e, and for every existing equilibrium $\bar{q}_e = (\bar{g}_e, \bar{r}_e)$ such that $\bar{g}_e \in U$, there exist $T > 0$ and (c,u), where $c: [0,T] \to Q$ such that $c(0) = q_e$, $c(T) = \bar{q}_e$ and $\dot{c}(0) = 0$, $\dot{c}(T) = 0$.

Let $\tau: G \times Q_s \to G$ denote a projection, and let $\tau_*: T(G \times Q_s) \to TG$ be its differential map. Sufficient conditions for local fiber equilibrium controllability follow from conditions for small time local fiber configuration controllability [11]; we refer to [11] for definitions and sufficient conditions for related controllability notions.

Theorem 2 [11]: Let $q_e \in Q$ be an equilibrium configuration. Assume that the multibody system (4) and (5) is locally fiber configuration accessible at q_e. If the projection through τ of every bad symmetric product of $\{\mathcal{Y} \cup \mathrm{grad}\, V\}$ is a linear combination of projections through τ of lower degree good symmetric products when evaluated at q_e, then the multibody system is locally fiber equilibrium controllable at q_e.

B. Local Configuration Accessibility and Local Fiber Configuration Accessibility

Let $C_{\mathrm{hor}}(\mathcal{Y}, V)$ denote the distribution that characterizes local configuration accessibility [24]. The general condition for local configuration accessibility is as follows [24]: if $\mathrm{rank}\{C_{\mathrm{hor}}(\mathcal{Y}, V)\} = \mathrm{dim} Q$ at q_e, then the system is locally configuration accessible at q_e. However, computation of $C_{\mathrm{hor}}(\mathcal{Y}, V)$ is complicated when a potential exists; see [24]

[1]We gratefully acknowledge Dr. Andrew D. Lewis for helpful discussions on these issues.

for the algorithm. In this section, we provide less general but computationally tractable sufficient conditions using the fact that $\overline{\text{Lie}}(\overline{\text{Sym}}(\mathcal{Y})) \subseteq C_{\text{hor}}(\mathcal{Y}, V)$. The proof of this fact follows from the algorithm in [24]; see [39] for details.

Proposition 7 [39]: If $\text{rank}\{\overline{\text{Lie}}(\overline{\text{Sym}}(\mathcal{Y}))\} = \dim G + \dim Q_s$ at an equilibrium (g_e, r_e), then the multibody system described by (4) and (5) is locally configuration accessible at (g, r_e) for all $g \in G$.

The condition in Proposition 7 only makes use of coupling of the horizontal control vector fields that are characterized by the mechanical connection; the interaction between the control vector fields and $\text{grad}\, V$ is ignored. This condition can be further simplified by utilizing the assumption that the shape is fully actuated, which implies $\mathcal{Y} = HQ$. According to Corollary 1 $\mathcal{Y} = \overline{\text{Sym}}(\mathcal{Y})$. Therefore, $\overline{\text{Lie}}(\overline{\text{Sym}}(\mathcal{Y})) = \overline{\text{Lie}}(\mathcal{Y})$. Using this observation and G-invariance of $\mathcal{Y}$, we have:

Corollary 2: If $\text{rank}\{\overline{\text{Lie}}(\mathcal{Y})\} = \dim G + \dim Q_s$ at an equilibrium (g_e, r_e), then the multibody system described by (4) and (5) is locally configuration accessible at (g, r_e) for any $g \in G$. Furthermore, if $\text{rank}\{\tau_* \overline{\text{Lie}}(\mathcal{Y})\} = \dim G$ at (g_e, r_e), then the system is locally fiber configuration accessible at (g, r_e) for any $g \in G$.

C. Local Equilibrium Controllability and Local Fiber Equilibrium Controllability

We first consider the G-invariant multibody systems studied in Section III-D1, where each configuration in Q can be an equilibrium.

Proposition 8 [39]: Consider the multibody system defined by (4) and (5) with a G-invariant potential. If $\text{rank}\{\overline{\text{Lie}}(\mathcal{Y})\} = \dim G + \dim Q_s$ holds at an equilibrium (g_e, r_e), then the multibody system is locally equilibrium controllable at (g, r_e) for all $g \in G$. Moreover, if $\text{rank}\{\tau_* \overline{\text{Lie}}(\mathcal{Y})\} = \dim G$ at (g_e, r_e), then the multibody system is locally fiber equilibrium controllable at (g, r_e) for all $g \in G$.

It is interesting to compare this controllability result with those for the simplified control model in Section III-D1. It has been shown that if the system is initially in equilibrium, then we have the simplified kinematic model: $g^{-1}\dot{g} = -A(r)v_s$, $\dot{r} = v_s$, where v_s denotes the shape velocity control. This is related to the equivalence between kinematic models and dynamic models discussed in [22].

We now consider multibody systems dependent on an advected parameter.

Propositition 9: Consider the multibody system given in (4) and (5). Let $q_e = (g_e, r_e)$ be an equilibrium configuration. Assume the following.

1) The system is locally configuration accessible at q_e.
2) For every bad symmetric product involving $\text{grad}\, V$, its advected momentum function is a linear combination of $(\partial V(a, r)/\partial a)$ and the advected momentum functions associated with lower degree good symmetric products involving $\text{grad}\, V$, all evaluated at q_e.

Then the multibody system is locally equilibrium controllable at q_e and at each equilibrium configuration in the equilibrium configuration set associated with r_e.

Furthermore, if the multibody system is locally fiber configuration accessible at q_e and Condition 2 holds, then the multibody system is locally fiber equilibrium controllable at q_e and at each equilibrium configuration in the equilibrium configuration set associated with r_e.

Proof: The proof follows from Theorem 1 and the symmetric product properties. Obviously, we only need to show the good and bad symmetric product relations. It is clear that iterative symmetric products of the control vector fields satisfy these relations. Consider bad symmetric products involving $\text{grad}\, V$. First, note that the symmetric products involving $\text{grad}\, V$ only are bad, but they trivially satisfy the relations since they vanish when evaluated at an equilibrium. Let $\tilde{Y}^g = M^{-1}(r)\left(G(a, r) \diamond a,\, Y_s^g(a, r)\right)^T$ be a nontrivial bad symmetric product involving $\text{grad}\, V$. According to the second assumption, its advected momentum function satisfies $G(a_e, r_e) \diamond a_e = \sum_{i=1}^{l} \alpha_i \left[G_i(a_e, r_e) \diamond a_e\right]$, where α_i, $i = 1, \ldots, l$, are scalars, and $G_i(a, r)$ are the advected momentum functions associated with lower degree good symmetric products Z_i, $i = 1, \ldots, l$, that involve $\text{grad}\, V$ and are given by $\tilde{Z}_i = M^{-1}(r)\left(G_i(a, r) \diamond a,\, Z_{is}(a, r)\right)^T$. Let Y_{is} denote the shape momentum function associated with the ith control vector field Y_i. Since $\{Y_{is}(r),\, i = 1, \ldots, n\}$ span $T_r^* Q_s$ at each r, there must be n scalars β_j, $j = 1, \ldots, n$, so that $\tilde{Y}_s^g(a_e, r_e) = \sum_{i=1}^{l} \alpha_i \tilde{Z}_{is}(a_e, r_e) + \sum_{j=1}^{n} \beta_j \tilde{Y}_{is}(r_e)$, where Z_{is} are the shape momentum function associated with $Z_i, i = 1, \ldots, l$. Hence, using the symmetric product properties given in Propositions 4 and 6, the bad symmetric product $Y^g(q_e)$ can be written as $Y^g(q_e) = \sum_{i=1}^{l} \alpha_i Z_i(q_e) + \sum_{j=1}^{n} \beta_j Y_j(q_e)$. This means that each bad symmetric product is a linear combination of lower degree good symmetric products at q_e. Consequently, the system is locally equilibrium controllable at q_e. By the invariance property, the system is also locally equilibrium controllable at each equilibrium configuration in the equilibrium configuration set associated with r_e.

The proof for local fiber equilibrium controllability follows a similar argument using projections through τ. □

Define

$$b_i = -\frac{\partial^2 V(a_e, r_e)}{\partial a^2}\Big(A_i(r_e) a_e\Big) - \frac{\partial^2 V(a_e, r_e)}{\partial a \partial r_i} + A_i(r_e) \star \frac{\partial V(a_e, r_e)}{\partial a}, \qquad i = 1, \ldots, n$$

where $A_i(r_e)$, the ith column of $A(r_e)$, is identified with an element in $\mathfrak{g}$, and $\Big(A_i(r_e) a_e\Big)$ denotes the induced action of ξ on a_e when ξ is identified with $A_i(r_e)$. The following corollary provides sufficient conditions that only make use of the first-order symmetric products for local equilibrium controllability.

Corollary 3 [39]: Consider the multibody system described by (4) and (5). Let $q_e = (g_e, r_e)$ be an equilibrium configuration. Assume the multibody system is locally configuration accessible at q_e and the following condition holds at q_e:

$$\text{rank}\Big\{b_1, \ldots, b_n, \frac{\partial V(a_e, r_e)}{\partial a}\Big\} = \dim \mathfrak{g}. \tag{24}$$

Then, the multibody system is locally equilibrium controllable at q_e and at each equilibrium configuration in the equilibrium configuration set associated with shape r_e. Furthermore, if the system is locally fiber configuration accessible at q_e and (24) holds, then the multibody system is locally fiber equilibrium controllable at q_e and at each equilibrium in the equilibrium configuration set associated with r_e.

VI. Local Equilibrium Controllability: Examples of Shape Controlled Multibody Systems

A. Local Equilibrium Controllability of Multibody Attitude Systems

Consider the multibody attitude system in a uniform gravitational field studied in Section III-D2. We first summarize local equilibrium controllability conditions, then we analyze local equilibrium controllability for three examples: a system controlled by two reaction wheels, a system controlled by two proof mass actuators, and a system controlled by one reaction wheel; the detailed proofs are given in [39].

1) Summary of Local Equilibrium Controllability Conditions: As shown in Section III-D2, the multibody attitude system is SO(3)-invariant with an advected parameter in $\mathbb{R}^3$. Moreover, the induced action of $\xi \in \mathfrak{so}(3)$ on $a \in W^* = \mathbb{R}^3$ is identified with the usual vector cross product, and the induced operations $\diamond$ and $\star$ are also identified with the cross product operation $\times$. Consider the following control vector fields Y_i, $i = 1, \ldots, n$, and the potential vector field:

$$\tilde{Y}_i = M^{-1}(r)\begin{bmatrix} 0 \\ Y_{is}(r) \end{bmatrix} \quad \mathrm{grad}\, V^{\sim} = M^{-1}(r)\begin{bmatrix} \Gamma \times \rho_c(r) \\ 0 \end{bmatrix}$$

where, without loss of generality, we scale the gravity constant as $a_g = 1/m_T$ such that $m_T a_g = 1$. Let Y^s be an iterative symmetric product involving the control vector fields only, and let Y^g be an iterative symmetric product involving $\mathrm{grad}\, V$. Then, Y^s and Y^g have the following forms:

$$\tilde{Y}^s = M^{-1}(r)\begin{bmatrix} 0 \\ Y_s^s(r) \end{bmatrix} \quad \tilde{Y}^g = M^{-1}(r)\begin{bmatrix} \Gamma \times G(\Gamma, r) \\ Y_s^g(\Gamma, r) \end{bmatrix}.$$

We now summarize local equilibrium controllability conditions using the equilibrium condition and Proposition 9 and Corollary 3.

Corollary 4: Let (R_e, r_e) be an equilibrium configuration. Assume that the multibody attitude system described by (6) is locally configuration accessible at (R_e, r_e) and that for every bad symmetric product involving $\mathrm{grad}\, V$, its advected momentum function is a linear combination of Γ_e and advected momentum functions associated with lower degree good symmetric products involving $\mathrm{grad}\, V$, evaluated at (R_e, r_e). Then, the multibody system is locally equilibrium controllable at (R_e, r_e) and at each equilibrium in the equilibrium configuration set associated with r_e.

Corollary 5: Let (R_e, r_e) be an equilibrium configuration. Assume that the multibody attitude system described by (6) is locally configuration accessible at (R_e, r_e) and the following condition holds:

$$\mathrm{rank}\Big\{\rho_c(r_e) \times A_1(r_e) + \frac{\partial \rho_c(r_e)}{\partial r_1}, \ldots, \rho_c(r_e) \times A_n(r) + \frac{\partial \rho_c(r_e)}{\partial r_n}, \Gamma_e\Big\} = 3. \quad (25)$$

Then, the multibody system is locally equilibrium controllable at (R_e, r_e) and at each equilibrium configuration in the equilibrium configuration set associated with r_e.

2) Local Equilibrium Controllability of a Multibody Attitude System Controlled by Two Reaction Wheels: In this example, the shape space $Q_s = \mathbb{S}^1 \times \mathbb{S}^1$. An important feature for this case is that the system is also Q_s-invariant, e.g., the inertia tensor M and the position vector ρ_c are constant; we assume $\rho_c \neq 0$. Let $A_i \in \mathbb{R}^3$ denote the ith column of $A = M_{11}^{-1} M_{12}$, $i = 1$, 2. The equilibrium manifold is given by $S_e = \{(R, r) | R = R_e e^{\psi \widehat{\Gamma'}}, \forall\, r \in \mathbb{S}^1 \times \mathbb{S}^1\}$, where $\psi \in [0, 2\pi)$ and $\Gamma' = \rho_c / \|\rho_c\|$ is a unit vector. This manifold corresponds to rotations about the gravity direction and characterizes all equilibrium configurations.

Proposition 10: If $A_1 \times A_2 \neq 0$ and $\{A_1, A_2, \rho_c\}$ are linearly independent, then the multibody attitude system described by (6) controlled by two reaction wheels is locally equilibrium controllable at any equilibrium configuration in S_e.

Suppose $A_1 \times A_2 \neq 0$ holds; that is, A_1 and A_2 are linearly independent. If $\{A_1, A_2, \rho_c\}$ are linearly dependent, or equivalently ρ_c is a linear combination of A_1 and A_2, then the sufficient conditions for local equilibrium controllability may not be satisfied [39]. An exception is given in the following proposition.

Proposition 11: Suppose $A_1^T A_2 = 0$ and $\rho_c \times A_1 = 0$. Then the multibody attitude system (6) controlled by two reaction wheels is locally equilibrium controllable at any equilibrium in S_e.

3) Local Equilibrium Controllability of a Multibody Attitude System Controlled by Two Proof Mass Actuators: In this example, the shape space $Q_s = \mathbb{R} \times \mathbb{R}$, ignoring stroke limits for the proof masses. In this case, shape changes influence the system mass and inertia distribution.

For simplicity, we assume that the center of mass of the base body is at the pivot point. The base body inertia with respect to the pivot point is given by $J_B = \mathrm{diag}(J_1, J_2, J_3)$. The two proof mass actuators have identical mass m_p. In the base body frame, the positions of the proof mass actuators are given by $\rho_1(r) = (r_1, 0, l)^T$ and $\rho_2(r) = (0, r_2, -l)^T$, where l denotes offset of each track to the pivot point. Therefore, the position vector of the system center of mass is $\rho_c(r) = (m_p/m_T)\big[\rho_1(r) + \rho_2(r)\big] = (m_p/m_T)(r_1, r_2, 0)^T$, where m_T is the total mass of the system. The expressions for $M_{11}(r)$, M_{12} are

$$M_{11}(r) = \begin{bmatrix} M_{11}^{11} & 0 & M_{11}^{13} \\ 0 & M_{11}^{22} & M_{11}^{23} \\ M_{11}^{31} & M_{11}^{32} & M_{11}^{33} \end{bmatrix} \quad M_{12} = \begin{bmatrix} 0 & m_p l \\ m_p l & 0 \\ 0 & 0 \end{bmatrix}$$

where $M_{11}^{11} = J_1 + 2m_p l^2 + m_p r_2^2$, $M_{11}^{22} = J_2 + 2m_p l^2 + m_p r_1^2$, $M_{11}^{33} = J_3 + m_p r_1^2 + m_p r_2^2$, $M_{11}^{31} = M_{11}^{13} = -m_p l r_1$, $M_{11}^{32} = M_{11}^{23} = m_p l r_2$, and $A(r) = M_{11}^{-1}(r) M_{12}$.

Consider a shape equilibrium given by $r_e = (r_1, r_2) = (0,0)$. Thus the system is balanced at this equilibrium since $\rho_c(0) = 0$; the base body equilibrium attitude can be arbitrary. Define $J_{01} = J_1 + 2m_p l^2$, $J_{02} = J_2 + 2m_p l^2$, and $J_{03} = J_3$. Straightforward computations show that if J_{03} satisfies the following condition:

$$\begin{aligned} J_{03} &\neq \frac{(J_2 - J_1)(J_{01} + 2J_{02})}{J_{01}}, \text{ when } J_1 < J_2 \\ \text{or } J_{03} &\neq \frac{(J_1 - J_2)(2J_{01} + J_{02})}{J_{02}}, \text{ when } J_1 > J_2 \end{aligned} \tag{26}$$

then the multibody attitude system is locally configuration accessible at any equilibrium with zero shape. Following Corollary 5, we obtain the following proposition.

Proposition 12: If J_{03} satisfies the condition (26), then the multibody attitude system given by (6) with two proof mass actuators is locally equilibrium controllable at any equilibrium configuration satisfying $\left\{(R, r_e) \middle| R^T e_3 \notin \operatorname{span}\{e_1, e_2\},\ r_e = 0\right\}$.

4) Local Equilibrium Controllability of a Multibody Attitude System Controlled by One Reaction Wheel: In the above examples, we have used the coupling of the horizontal control vector fields to guarantee local configuration accessibility for the multibody systems with two shape change actuators. In each setup, we have ignored the coupling between shape change and gravitational effects. We now consider a more interesting case: a multibody attitude system controlled by a single reaction wheel. This is the case where the number of shape change actuators is minimal. In this case, we must take into account the coupling between shape change and gravity effects.

The shape space $Q_s = \mathbb{S}^1$, and the position vector ρ_c is a nonzero constant vector. The equilibrium manifold is the same as defined in Section VI-A2.

Proposition 13: Let R_e be a base body equilibrium attitude, i.e., $\Gamma_e \times \rho_c = 0$ where $\Gamma_e = R_e^T e_3$. Suppose $A \in \mathbb{R}^3$ and ρ_c satisfy $\rho_c^T A = 0$, $\rho_c^T (A \times M_{11}^{-1} A) \neq 0$, and $(M_{11}\rho_c \times A)^T (A \times M_{11}^{-1} A) \neq 0$. Then, the multibody attitude system given by (6) controlled by one reaction wheel is locally equilibrium controllable at any base body equilibrium attitude satisfying $R^T e_3 = \Gamma_e$.

B. Local Fiber Equilibrium Controllability of Neutrally Buoyant Multibody Underwater Vehicles

Consider the neutrally buoyant multibody underwater vehicle in an ideal fluid acted on by uniform gravity studied in Section III-D3. As shown in Section VI-B1, this system *cannot* be locally equilibrium controllable because of the conserved inertial linear momentum; we study local fiber equilibrium controllability. We give conditions for this type of controllability, and we apply them to the case where the multibody vehicle is controlled by three independent proof masses.

1) Summary of Local Fiber Equilibrium Controllability Conditions: As shown in Section III-D3, the multibody underwater vehicle is SE(3)-invariant with an advected parameter in $W^* = \mathbb{R}^3$ and the advected parameter is $\Gamma = R^T e_3$. See Section III-D3 for the definitions of the operation $\diamond$. Moreover, the induced action of $\xi \in \mathfrak{se}(3)$ with twist (v, ω) on $a \in W^* = \mathbb{R}^3$ is identified with $\omega \times a$, and the operation $\star$ is defined as $\xi \star \Sigma = \omega \times \Sigma$, $\forall\, \xi \in \mathfrak{se}(3)$, $\forall\, \Sigma \in W$.

Consider the following control vector fields Y_i, $i = 1, \ldots, n$, and the potential vector field $\operatorname{grad} V$

$$\tilde{Y}_i = M^{-1}(r) \begin{bmatrix} 0 \\ 0 \\ Y_{is}(r) \end{bmatrix}$$

$$\operatorname{grad} V^{\sim} = M^{-1}(r) \begin{bmatrix} 0 \\ \Gamma \times \rho_c(r) \\ 0 \end{bmatrix}.$$

As before, we scale a_g as $a_g = 1/m_T$ such that $m_T a_g = 1$. Let Y^s be an iterative symmetric product involving the control vector fields only, and let Y^g be an iterative symmetric product involving $\operatorname{grad} V$. Then Y^s and Y^g have the following forms:

$$\tilde{Y}^s = M^{-1}(r) \begin{bmatrix} 0 \\ 0 \\ Y_s^s(r) \end{bmatrix} \quad \tilde{Y}^g = M^{-1}(r) \begin{bmatrix} 0 \\ \Gamma \times G(\Gamma, r) \\ Y_s^g(\Gamma, r) \end{bmatrix}.$$

It has been pointed out in Section III-D3 that the inertial linear momentum is integrable; its integral $M_T x + m_T R \rho_c(r)$ is conserved. This integral implies that the complete configuration $q = (x, R, r)$ is not reachable. Hence, we turn our attention to local fiber equilibrium controllability. Note that the aforementioned integral leads to a necessary condition for fiber controllability: A necessary condition for locally fiber equilibrium controllability is $\operatorname{rank}\left(\partial \rho_c(r_e)/\partial r\right) = 3$; see [39] for the proof. Physically, this condition means that to achieve local fiber controllability, at least three independent shape actuators are required such that $\rho_c(r)$ can (locally) arbitrarily change.

Local fiber equilibrium controllability conditions for the underwater multibody vehicle given in (7) are similar to those given in Corollaries 4 and 5, except that only local fiber configuration accessibility for SE(3) is required.

2) Local Fiber Equilibrium Controllability of an Underwater Vehicle Controlled by Three Proof Mass Actuators: In this example, the shape space $Q_s = \mathbb{R}^3$, ignoring stroke limits. The actuators, modeled as ideal mass particles, can slide along straight slots fixed in the base body. For simplicity, we assume that the base body center of mass is at the base body center of buoyancy, and the position of the ith proof mass actuator relative to the base body is given by $\rho_i = R_i \left(r_i, l_i, 0\right)^T$, $i = 1, 2, 3$, where R_i is a constant rotation matrix that describes the orientation of the ith slot frame relative to the base body frame, and l_i is a constant offset of ith slot. Note that $R_i(1)$, the first column of R_i, characterizes the ith slot direction in the base body frame. Let m_i be the mass of the ith proof mass, $\rho_c(r) = 1/m_T \sum_{i=1}^{3} m_i \rho_i(r)$. Thus the necessary condition $\operatorname{rank}\left(\partial \rho_c(r)/\partial r\right) = 3$ holds for all shapes if three actuators are independent, i.e., the three slot directions are linearly independent.

We now consider a special case where the three independent proof mass actuators have zero offsets, i.e., $l_1 = l_2 = l_3 = 0$. Furthermore, we assume that the initial equilibrium shape is zero, i.e., $r_e = 0$. This implies that $\rho_c(r_e) = 0$ and any $(x, R) \in \mathrm{SE}(3)$ is an equilibrium at r_e. We first show local fiber configuration accessibility using Corollary 2. The mechanical connection in matrix form is given by

$$A(r) = \begin{bmatrix} A_t(r) \\ A_r(r) \end{bmatrix} = \begin{bmatrix} M_T & K(r) \\ K^T(r) & J(r) \end{bmatrix}^{-1} \begin{bmatrix} B_t(r) \\ B_r(r) \end{bmatrix}.$$

It can be shown that $A_t(r) = m_T M_T^{-1}\Big[(\partial\rho_c(r)/\partial r) + \rho_c(r) \times A_r(r)\Big]$. Consider $\mathcal{Y} = \{Y_i^A,\ i = 1,2,3\} \subseteq HQ$, where $\tilde{Y}_i^A = (\,-A_{ti}(r),\ \ -A_{ri}(r),\ \ e_i\,)^T$, and A_{ti} and A_{ri} denote the ith column of A_t and A_r, respectively. Following similar computations in [39] and [40], it can be shown that $\mathrm{rank}\{\tau_* \overline{\mathrm{Lie}}(\mathcal{Y})\} = \dim \mathrm{SE}(3)$ at (x, R, r_e) for arbitrary $(x, R) \in \mathrm{SE}(3)$, where $\tau\colon \mathrm{SE}(3) \times Q_s \to \mathrm{SE}(3)$. Hence, the multibody vehicle described by (7) is locally fiber configuration accessible at (x, R, r_e) for arbitrary $(x, R) \in \mathrm{SE}(3)$. Moreover, since $\rho_c(r_e) \times A_{ri}(r_e) + (\partial\rho_c(r_e)/\partial r_i) = (m_i/m_T)R_i(1)$, $i = 1,2,3$, we see that (25) in Corollary 5 holds. Consequently, we have the following result.

Proposition 14: Consider the neutrally buoyant multibody underwater vehicle described by (7) controlled by the three independent proof mass actuators with zero offsets. The multibody vehicle is locally fiber equilibrium controllable at each equilibrium configuration associated with the zero shape, i.e., $r_e = 0$.

VII. Conclusion

In this paper, we have studied local equilibrium controllability of shape controlled multibody systems with certain assumptions on symmetries and shape actuation. We make use of symmetric product tools for controllability analysis. Important symmetric product properties involving an advected parameter dependent potential are investigated. Based on these properties, we obtain sufficient conditions for local (fiber) equilibrium controllability. The controllability results are illustrated by application to two classes of multibody systems in a uniform gravitational field: multibody attitude systems, and neutrally buoyant multibody underwater vehicles. We intend to use the theoretical results proved here as a basis for constructive control approaches; see [39] for preliminary results.

Appendix

Let $q = (g, r) \in G \times Q_s$, and let $X(q) = (g\xi_X(q), v_X(q))$ and $Y(q) = (g\xi_Y(q), v_Y(q))$ be two smooth vector fields on $G \times Q_s$. Recall that the Lie bracket of X and Y is given in Section IV-A as

$$[X,Y]^{\sim}(q) = \Big(\mathrm{ad}_X Y\Big)^{\sim}(q) + \frac{d}{dt}\Big|_{t=0}\Big(\tilde{Y}\circ\phi_t^X(q) - \tilde{X}\circ\phi_t^Y(q)\Big). \tag{27}$$

For notational convenience, we use $\langle\!\langle \cdot,\cdot \rangle\!\rangle_M$ to denote the reduced metric defined by M in the subsequent development. That is, for the two vector fields X and Y given previously, $\langle\!\langle X, Y\rangle\!\rangle = \tilde{X}^T M(r)\tilde{Y} = \langle\!\langle \tilde{X}, \tilde{Y}\rangle\!\rangle_M$.

A. Proof [Proof of Proposition 3]

It has been shown in the proof of Proposition 1 that the symmetric product of X and Y is defined as

$$\begin{aligned}\langle\!\langle \langle X : Y\rangle, Z\rangle\!\rangle = L_X\langle\!\langle Y, Z\rangle\!\rangle + L_Y\langle\!\langle X, Z\rangle\!\rangle - L_Z\langle\!\langle X, Y\rangle\!\rangle \\ -\langle\!\langle [X,Z], Y\rangle\!\rangle - \langle\!\langle [Y,Z], X\rangle\!\rangle \end{aligned} \tag{28}$$

for an arbitrary vector field Z. We compute the terms on the right-hand side of (28) as follows.

The first term can be calculated as

$$\begin{aligned} L_X\langle\!\langle Y, Z\rangle\!\rangle(q) &= \frac{d}{dt}\Big|_{t=0} \langle\!\langle \tilde{Y}, \tilde{Z}\rangle\!\rangle_M \circ \phi_t^X(q) \\ &= +\left\langle\!\!\left\langle \frac{d}{dt}\Big|_{t=0} \tilde{Y}\circ\phi_t^X(q), \tilde{Z}(q)\right\rangle\!\!\right\rangle_M \\ &\quad + \left\langle\!\!\left\langle \tilde{Y}(q), \frac{d}{dt}\Big|_{t=0} \tilde{Z}\circ\phi_t^X(q)\right\rangle\!\!\right\rangle_M \\ &\quad + \tilde{Y}^T(q)\left(\frac{d}{dt}\Big|_{t=0} M\circ\phi_t^X(q)\right)\tilde{Z}(q). \end{aligned}$$

Similarly, the second and third terms can be computed as

$$\begin{aligned} L_Y\langle\!\langle X, Z\rangle\!\rangle(q) &= \left\langle\!\!\left\langle \frac{d}{dt}\Big|_{t=0} \tilde{X}\circ\phi_t^Y(q), \tilde{Z}(q)\right\rangle\!\!\right\rangle_M \\ &\quad + \left\langle\!\!\left\langle \tilde{X}(q), \frac{d}{dt}\Big|_{t=0} \tilde{Z}\circ\phi_t^Y(q)\right\rangle\!\!\right\rangle_M \\ &\quad + \tilde{X}^T(q)\left(\frac{d}{dt}\Big|_{t=0} M\circ\phi_t^Y(q)\right)\tilde{Z}(q) \end{aligned}$$

and

$$\begin{aligned} L_Z\langle\!\langle X, Y\rangle\!\rangle(q) &= \left\langle\!\!\left\langle \frac{d}{dt}\Big|_{t=0} \tilde{X}\circ\phi_t^Z(q), \tilde{Y}(q)\right\rangle\!\!\right\rangle_M \\ &\quad + \left\langle\!\!\left\langle \tilde{X}(q), \frac{d}{dt}\Big|_{t=0} \tilde{Y}\circ\phi_t^Z(q)\right\rangle\!\!\right\rangle_M \\ &\quad + \tilde{X}^T(q)\left(\frac{d}{dt}\Big|_{t=0} M\circ\phi_t^Z(q)\right)\tilde{Y}(q). \end{aligned}$$

Since M is only dependent on Q_s, we obtain

$$\tilde{X}^T(q)\left(\frac{d}{dt}\Big|_{t=0} M\circ\phi_t^Z(q)\right)\tilde{Y}(q) = \tilde{Z}^T(q) DM(\tilde{X}(q), \tilde{Y}(q))$$

where

$$DM(\tilde{X}(q), \tilde{Y}(q)) = \Big(\underbrace{0,\ldots,0}_{\dim G}, DM_s(\tilde{X}(q), \tilde{Y}(q))^T\Big)^T$$

and

$$DM_s(\tilde{X}(q), \tilde{Y}(q)) = \left(\tilde{X}^T(q)\frac{\partial M(r)}{\partial r_1}\tilde{Y}(q), \ldots, \tilde{X}^T(q)\frac{\partial M(r)}{\partial r_n}\tilde{Y}(q)\right)^T.$$

Now, we compute the last two terms. Using the Lie bracket formula (27), we have

$$\begin{aligned}\langle\!\langle [X,Z],Y\rangle\!\rangle(q) = &\left\langle\!\left\langle \mathrm{ad}_{\tilde{X}(q)}\tilde{Z}(q),\tilde{Y}(q)\right\rangle\!\right\rangle_M\\ &+\left\langle\!\left\langle \frac{d}{dt}\bigg|_{t=0}\tilde{Z}\circ\phi_t^X(q),\tilde{Y}(q)\right\rangle\!\right\rangle_M\\ &-\left\langle\!\left\langle \frac{d}{dt}\bigg|_{t=0}\tilde{X}\circ\phi_t^Z(q),\tilde{Y}(q)\right\rangle\!\right\rangle_M\end{aligned}$$

and

$$\begin{aligned}\langle\!\langle [Y,Z],X\rangle\!\rangle(q) = &\left\langle\!\left\langle \mathrm{ad}_{\tilde{Y}(q)}\tilde{Z}(q),\tilde{X}(q)\right\rangle\!\right\rangle_M\\ &+\left\langle\!\left\langle \frac{d}{dt}\bigg|_{t=0}\tilde{Z}\circ\phi_t^Y(q),\tilde{X}(q)\right\rangle\!\right\rangle_M\\ &-\left\langle\!\left\langle \frac{d}{dt}\bigg|_{t=0}\tilde{Y}\circ\phi_t^Z(q),\tilde{X}(q)\right\rangle\!\right\rangle_M.\end{aligned}$$

Substituting these results into (28), we have

$$\begin{aligned}&\langle\!\langle \langle X:Y\rangle,Z\rangle\!\rangle(q)\\ &=\left\langle\!\left\langle \frac{d}{dt}\bigg|_{t=0}\tilde{Y}\circ\phi_t^X(q),\tilde{Z}(q)\right\rangle\!\right\rangle_M\\ &\quad+\tilde{Y}^T(q)\left(\frac{d}{dt}\bigg|_{t=0}M\circ\phi_t^X(q)\right)\tilde{Z}(q)\\ &\quad+\left\langle\!\left\langle \frac{d}{dt}|_{t=0}\tilde{X}\circ\phi_t^Y(q),\tilde{Z}(q)\right\rangle\!\right\rangle_M\\ &\quad+\tilde{X}^T(q)\left(\frac{d}{dt}\bigg|_{t=0}M\circ\phi_t^Y(q)\right)\tilde{Z}(q)\\ &\quad-\tilde{Z}^T(q)DM(\tilde{X}(q),\tilde{Y}(q))-\left\langle\!\left\langle \mathrm{ad}_{\tilde{X}(q)}\tilde{Z}(q),\tilde{Y}(q)\right\rangle\!\right\rangle_M\\ &\quad-\left\langle\!\left\langle \mathrm{ad}_{\tilde{Y}(q)}\tilde{Z}(q),\tilde{X}(q)\right\rangle\!\right\rangle_M\\ &=\left\langle\!\left\langle \frac{d}{dt}\bigg|_{t=0}(\tilde{Y}\circ\phi_t^X(q)+\tilde{X}\circ\phi_t^Y(q)),\tilde{Z}(q)\right\rangle\!\right\rangle_M\\ &\quad+\left[\left(\frac{d}{dt}\bigg|_{t=0}M\circ\phi_t^X(q)\right)\tilde{Y}(q)\right.\\ &\qquad+\left(\frac{d}{dt}\bigg|_{t=0}M\circ\phi_t^Y(q)\right)\tilde{X}(q)\\ &\qquad\left.-DM(\tilde{X}(q),\tilde{Y}(q))\right]^T\tilde{Z}(q)\\ &\quad-\left[\mathrm{ad}^*_{\tilde{X}(q)}M(r)\tilde{Y}(q)+\mathrm{ad}^*_{\tilde{Y}(q)}M(r)\tilde{X}(q)\right]^T\tilde{Z}(q).\end{aligned}$$

As a result, we obtain

$$\begin{aligned}\langle X:Y\rangle^{\sim}(q)=&\frac{d}{dt}\bigg|_{t=0}(\tilde{X}\circ\phi_t^Y(q)+\tilde{Y}\circ\phi_t^X(q))\\ &+M^{-1}(r)\\ &\times\left\{\left(\frac{d}{dt}\bigg|_{t=0}M\circ\phi_t^Y(q)\right)\tilde{X}(q)\right.\\ &\qquad+\left(\frac{d}{dt}\bigg|_{t=0}M\circ\phi_t^X(q)\right)\tilde{Y}(q)\\ &\qquad-DM(\tilde{X}(q),\tilde{Y}(q))\\ &\qquad-\mathrm{ad}^*_{\tilde{X}(q)}M(r)\tilde{Y}(q)\\ &\qquad\left.-\mathrm{ad}^*_{\tilde{Y}(q)}M(r)\tilde{X}(q)\right\}.\end{aligned}$$

Since M is only dependent on Q_s, $M\circ\phi_t^X(q)=M\circ\phi_t^{v_X}(q)$, $M\circ\phi_t^Y(q)=M\circ\phi_t^{v_Y}(q)$. This yields the desired formula (8).

Now, we show (11) for the vector fields of the form (9) and (10). Let $\bar{M}(r)=M^{-1}(r)$. Since

$$\frac{d}{dt}\bigg|_{t=0}\bar{M}\circ\phi_t^{v_Y}(q)+M^{-1}(r)\left(\frac{d}{dt}\bigg|_{t=0}M\circ\phi_t^{v_Y}(q)\right)\bar{M}(r)=0$$

the following identity holds:

$$\begin{aligned}&\left(\frac{d}{dt}\bigg|_{t=0}\bar{M}\circ\phi_t^{v_Y}(q)\right)\begin{bmatrix}X_g\\ X_s\end{bmatrix}\\ &\qquad+M^{-1}(r)\left(\frac{d}{dt}\bigg|_{t=0}M\circ\phi_t^{v_Y}(q)\right)\tilde{X}=0.\end{aligned}$$

Using this identity and the given expression for X, we have

$$\begin{aligned}&\frac{d}{dt}\bigg|_{t=0}\tilde{X}\circ\phi_t^Y(q)+M^{-1}(r)\left(\frac{d}{dt}\bigg|_{t=0}M\circ\phi_t^{v_Y}(q)\right)\tilde{X}(q)\\ &\quad=\frac{d}{dt}\bigg|_{t=0}\left(\bar{M}\circ\phi_t^{v_Y}(q)\right)\begin{bmatrix}X_g(q)\\ X_s(q)\end{bmatrix}\\ &\qquad+\bar{M}(r)\frac{d}{dt}\bigg|_{t=0}\begin{bmatrix}X_g\circ\phi_t^Y(q)\\ X_s\circ\phi_t^Y(q)\end{bmatrix}\\ &\qquad+M^{-1}(r)\left(\frac{d}{dt}\bigg|_{t=0}M\circ\phi_t^{v_Y}(q)\right)\tilde{X}(q)\\ &\quad=\bar{M}(r)\frac{d}{dt}\begin{bmatrix}X_g\circ\phi_t^Y(q)\\ X_s\circ\phi_t^Y(q)\end{bmatrix}_{t=0}.\end{aligned}$$

Similarly

$$\begin{aligned}&\frac{d}{dt}\bigg|_{t=0}\tilde{Y}\circ\phi_t^X(q)+M^{-1}(r)\left(\frac{d}{dt}\bigg|_{t=0}M\circ\phi_t^{v_X}(q)\right)\tilde{Y}(q)\\ &\qquad\qquad=\bar{M}(r)\frac{d}{dt}\begin{bmatrix}Y_g\circ\phi_t^X(q)\\ Y_s\circ\phi_t^X(q)\end{bmatrix}_{t=0}.\end{aligned}$$

Consequently, we simplify (8) and obtain

$$\begin{aligned}\langle X:Y\rangle^{\sim}=\bar{M}(r)&\left\{\frac{d}{dt}\begin{bmatrix}X_g\circ\phi_t^Y(q)\\ X_s\circ\phi_t^Y(q)\end{bmatrix}_{t=0}\right.\\ &+\frac{d}{dt}\begin{bmatrix}Y_g\circ\phi_t^X(q)\\ Y_s\circ\phi_t^X(q)\end{bmatrix}_{t=0}\\ &-DM(\tilde{X}(q),\tilde{Y}(q))\\ &-\mathrm{ad}^*_{\tilde{X}(q)}M(r)\tilde{Y}(q)\\ &\left.-\mathrm{ad}^*_{\tilde{Y}(q)}M(r)\tilde{X}(q)\right\}.\end{aligned}$$

The final result follows from the expression for $DM(\tilde{X}(q),\tilde{Y}(q))$, and

$$\begin{aligned}\mathrm{ad}^*_{\tilde{X}(q)}M(r)\tilde{Y}(q)&=\begin{bmatrix}\mathrm{ad}^*_{\xi_X(q)}Y_g(q)\\ 0\end{bmatrix}\\ \mathrm{ad}^*_{\tilde{Y}(q)}M(r)\tilde{X}(q)&=\begin{bmatrix}\mathrm{ad}^*_{\xi_Y(q)}X_g(q)\\ 0\end{bmatrix}.\end{aligned}$$

□

B. *Proof [Proof of Proposition 5]*

Let $F(a,r)$ be a smooth function on $W^*\times Q_s$, and let $X(q)=(g\xi_X(q),v_X(q))$ be a vector field on Q, whose flow is denoted by $\phi_t^X(q)$. Then it is easy to verify the following identity:

$$\frac{d}{dt}\bigg|_{t=0}F\circ\phi_t^X=\frac{\partial F}{\partial a}\Big(-\xi_X a\Big)+\frac{\partial F}{\partial r}v_X.$$

Using this result, we obtain the symmetric product of two vector fields X and Y of the form in (19) and (20) as

$$\langle X : Y\rangle^{\sim}(q) = M^{-1}(r)\begin{bmatrix}\langle X : Y\rangle_g(a,r)\\ \langle X : Y\rangle_s(a,r)\end{bmatrix}$$

where

$$\langle X : Y\rangle_g(a,r) = -\frac{\partial X_g}{\partial a}(\xi_Y a) - \frac{\partial Y_g}{\partial a}(\xi_X a) + \frac{\partial X_g}{\partial r}v_Y + \frac{\partial Y_g}{\partial r}v_X - \mathrm{ad}^*_{\xi_X} Y_g' - \mathrm{ad}^*_{\xi_Y} X_g \quad (29)$$

and $\left\langle X : Y\right\rangle_s(a,r)$ is given in (23).

We compute $\left\langle X : Y\right\rangle_g$ from (29). Since $X_g(a,r) = G_X(a,r)\diamond a$, we obtain

$$-\frac{\partial X_g}{\partial a}(\xi_Y a) = -\frac{\partial G_X(a,r)}{\partial a}(\xi_Y a)\diamond a - G_X(a,r)\diamond(\xi_Y a)$$

and

$$\frac{\partial X_g}{\partial r}v_Y = \left(\frac{\partial G_X(a,r)}{\partial r}v_Y\right)\diamond a.$$

We now simplify

$$-G_X(a,r)\diamond(\xi_Y a) - \mathrm{ad}^*_{\xi_Y}X_g = -G_X(a,r)\diamond(\xi_Y a) - \mathrm{ad}^*_{\xi_Y}(G_X(a,r)\diamond a).$$

Let σ be an arbitrary element in $\mathfrak{g}$. Since G is a matrix Lie group, it is easy to verify that $(\mathrm{ad}_{\xi_Y}\sigma)a = \xi_Y(\sigma a) - \sigma(\xi_Y a)$. Using this result and the definitions of the operations $\diamond$, $\star$, and ad^*, we have

$$\begin{aligned}&\left\langle -G_X(a,r)\diamond(\xi_Y a) - \mathrm{ad}^*_{\xi_Y}(G_X(a,r)\diamond a),\sigma\right\rangle\\ &= \langle\sigma(\xi_Y a), G_X(a,r)\rangle - \langle \mathrm{ad}_{\xi_Y}\sigma, G_X(a,r)\diamond a\rangle\\ &= \langle\sigma(\xi_Y a), G_X(a,r)\rangle + \langle(\mathrm{ad}_{\xi_Y}\sigma)a, G_X(a,r)\rangle\\ &= \langle\sigma(\xi_Y a) + \xi_Y(\sigma a) - \sigma(\xi_Y a), G_X(a,r)\rangle\\ &= \langle\xi_Y(\sigma a), G_X(a,r)\rangle = -\langle\sigma a, \xi_Y\star G_X(a,r)\rangle\\ &= \langle[\xi_Y\star G_X(a,r)]\diamond a,\sigma\rangle.\end{aligned}$$

Therefore

$$-G_X(a,r)\diamond(\xi_Y a) - \mathrm{ad}^*_{\xi_Y}(G_X(a,r)\diamond a) = [\xi_Y\star G_X(a,r)]\diamond a.$$

Hence, we obtain

$$\begin{aligned}&-\frac{\partial X_g}{\partial a}(\xi_Y a) + \frac{\partial X_g}{\partial r}v_Y - \mathrm{ad}^*_{\xi_Y}X_g\\ &= \left(-\frac{\partial G_X(a,r)}{\partial a}(\xi_Y a) + \frac{\partial G_X(a,r)}{\partial r}v_Y + \xi_Y\star G_X(a,r)\right)\diamond a.\end{aligned}$$

Similarly

$$\begin{aligned}&-\frac{\partial Y_g}{\partial a}(\xi_X a) + \frac{\partial Y_g}{\partial r}v_X - \mathrm{ad}^*_{\xi_X}Y_g\\ &= \left(-\frac{\partial G_Y(a,r)}{\partial a}(\xi_X a) + \frac{\partial G_Y(a,r)}{\partial r}v_X + \xi_X\star G_Y(a,r)\right)\diamond a.\end{aligned}$$

Finally, we have $\left\langle X : Y\right\rangle_g = G(a,r)\diamond a$, where $G(a,r)$ is given by (22). □

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