Math 302/600 Spring 2015 Homework #2

Due Feb. 17, Tue. in class

Note: For any Euclidean space $\mathbb{R}^n$, consider the usual metric induced by the Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^n$, unless otherwise stated.

1. Let (M, d) be a metric space. Show the following inequality:

$$|d(x,y) - d(z,y)| \leq d(x,z), \quad \forall\, x, y, z \in M.$$

2. Let A be an open set in $\mathbb{R}$, and define the set $B := \{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_1 \in A\} \subseteq \mathbb{R}^2$. Show that B is open in $\mathbb{R}^2$.

3. Let (M, d) be a metric space. Given a nonempty set $A \subseteq M$, let $B := \{x \in M \,|\, d(x,y) < 1 \text{ for some } y \in A\}$. Show that B is open. (*Hint*: write B as the union of open sets.)

4. Consider $\mathbb{R}^2$ and the metric induced by the 1-norm: $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$, $\forall\, x, y \in \mathbb{R}^2$. Let $A = \{x = (x_1, x_2) \in \mathbb{R}^2 \,|\, x_1 \geq 1 \text{ and } x_2 \leq 1\}$. Find the interior of A (using the given metric), and prove your answer.

5. Let (M, d) be a metric space and A, B be two subsets of M. Show the following:

   (1) if $A \subseteq B$, then $\text{int}A \subseteq \text{int}B$;

   (2) $\text{int}(A \cap B) = (\text{int}A) \cap (\text{int}B)$.

The following extra problems are for **Math 600** *students only:*

6. Given a normed space $(V, \|\cdot\|)$, let A be a nonempty open set in V, and B be a nonempty set in V. Define $A + B := \{a + b \in V \,|\, a \in A,\ b \in B\}$. Show that $A + B$ is open.

7. Let M be a set endowed with two metrics d_1 and d_2, namely, both (M, d_1) and (M, d_2) are metric spaces. Suppose that there exist two positive real numbers α and β such that

$$\beta d_1(x,y) \leq d_2(x,y) \leq \alpha d_1(x,y), \quad \forall\, x, y \in M.$$

   Show that a set $A \subseteq M$ is open with respect to d_1 if and only if A is open with respect to d_2.