Proceedings of the 2007 American Control Conference
Marriott Marquis Hotel at Times Square
New York City, USA, July 11-13, 2007

Linear Complementarity Systems with Singleton Properties: Non-Zenoness

Jinglai Shen and Jong-Shi Pang

Abstract— **Extending our previous work on linear complementarity systems (LCSs) with the P-property, this paper establishes that a certain class of LCSs of the positive semidefinite-plus type does not have Zeno states. An intrinsic feature of such an LCS is that it has a unique continuously differentiable state solution for any initial condition, albeit the associated algebraic linear complementarity problem has non-unique solutions. Applications of our results to constrained dynamic optimization, and more generally, to differential affine complementarity systems are discussed. The cornerstone of our proof of the main non-Zeno result is a recent theory for conewise linear systems.**

I. Introduction

Defined by a linear time-invariant ordinary differential equation (ODE) coupled with a linear complementarity problem (LCP), a linear complementarity system (LCS) is a special class of hybrid systems that possess multiple modes and exhibit state-dependent mode switchings. Motivated by the analysis and design of nonsmooth systems in engineering, robotics, and operations research, various analytic and system/control issues of LCSs have been studied, for example, solution existence and uniqueness [2], [10], [12], [13], (non-)Zeno behavior [7], [15], [19], stability [5], [11], controllability [3], and observability [6], [15]. The two survey papers [1], [18] and the references therein present a nice summary of basic aspects of the LCSs and their applications in operations research and nonsmooth physical systems.

The (non-)Zeno behavior of complementarity systems, which is referred to as (non-)existence of infinitely many mode transitions in finite time, is a critical issue in numerical and control analysis of LCSs. The article [4] establishes the non-Zeno behavior for a class of LCSs satisfying certain passivity conditions. The recent paper [19] introduces the concepts of strong and weak non-Zenoness and proves that the LCSs with the P-property is "globally strongly non-Zeno". The local strong non-Zenoness result is further extended to nonlinear complementarity systems and differential variational inequalities satisfying the strong regularity condition [15]. Moreover, it is proved in [6] that conewise linear systems (CLSs), which represent a large number of piecewise linear systems and LCSs with continuously differentiable state trajectories, do not exhibit the Zeno behavior. The connection and differences between these non-Zeno notions and implications will be discussed in Section IV-A.

This work is based on research supported by the National Science Foundation under a Focused Research Group grant DMS 0353216 and grant DMS 0508986.

Department of Mathematics and Statistics, University of Maryland Baltimore County, Baltimore, Maryland 21250, U.S.A. Email: shenj@umbc.edu.
Department of Mathematical Sciences, Rensselaer Polytechnic Institute, Troy, New York 12180-3590, U.S.A. Email:pangj@rpi.edu.

This paper continues our study of the Zeno behavior of LCSs. Specifically, we focus on a class of LCSs satisfying certain singleton properties. Such an LCS has a unique continuously differentiable state solution for any initial condition. This class of LCSs includes several important subclasses, such as those with the P-property and, more importantly, those with the positive-semidefinite-plus (PSD-plus) property, i.e., the defining matrix of the associated LCP is a "PSD-plus" matrix; see Section II for definition. The latter class of LCSs naturally arises from the multiplier formulations of certain dynamical equilibrium problems such as linear-quadratic differential Nash games and multibody contact problems subject to polygonal friction constraints and with state-invariant local compliance. The non-Zeno analysis in the present paper is based on a conewise linear formulation of the LCS. Unlike the non-Zeno argument in [19], [15] that depends on a reduction procedure, the conewise linear approach relies on the geometry of polyhedral subdivision of a piecewise linear function. By adopting this approach and exploiting the piecewise linear structure, not only are we able to recover the previous reduction-based non-Zeno results via simpler and more elegant arguments, but also extend them to more general LCSs for which the reduction approach fails.

The rest of the paper is organized as follows. In Section II, we present examples of a class of PSD-plus LCSs that motivate this work. Section III contains the essential background of LCPs and CLSs and discusses related non-Zeno results. Strong and uniformly weak non-Zeno concepts are formally defined and proved for the class of LCSs on hand in Section IV. Finally, finite verification of singleton conditions is discussed in Section V.

II. Motivation and Examples

The special class of LCSs to be studied in this section has a wide range of important applications in engineering and dynamic optimization. Recall that a real square matrix D is *positive semidefinite plus* (PSD-plus) [14] if it is positive semidefinite and satisfies $[\, v^T Dv = 0 \Rightarrow Dv = 0 \,]$. An equivalent definition is that D can be written as $D = FMF^T$ for some matrix F and a positive definite matrix M, where F can be further strengthened to be of full column rank if $D \neq 0$ (without loss of generality). The latter definition is the matrix-theoretic analog of the "monotonicity-plus" property of a nonlinear mapping [9], which has been employed in the study of differential variational inequalities [16]. In addition, letting $\mathrm{SOL}(q, D)$ denote the solution set of the LCP: $0 \leq u \perp q + Du \geq 0$ for a matrix D, we call a matrix E satisfying the *E-singleton property* (on a convex subset $\mathcal{S}$) if $\mathrm{SOL}(q, D)$ is nonempty and $E\mathrm{SOL}(q, D)$ is singleton for all $q \in \mathcal{S}$. It is known that $E\text{SOL}(q, D)$ is piecewise linear on $\mathcal{S}$ in this case.

Consider the LCS:

$$\dot{x} = Ax + BF^T u, \quad 0 \le u \perp FCx + FMF^T u \ge 0, \quad (1)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, F is nonzero, M is positive definite (though not necessarily symmetric), and (A, B, C) are of suitable dimensions. Clearly $D := FMF^T$ is PSD-plus and $F^T\text{SOL}(FCx, D)$ is a singleton for all $x \in \mathbb{R}^n$. Thus both $BF^T\text{SOL}(FCx, D)$ and $D\text{SOL}(FCx, D)$ are singletons for all $x \in \mathbb{R}^n$. In the following, we present several examples to illustrate various realizations of the PSD-plus LCS (1).

Example 1 Consider the linear cone complementarity system (LCCS):

$$\dot{x} = Ax + Bz, \qquad \mathcal{C} \ni z \perp Cx + Mz \in \mathcal{C}^*,$$

where $\mathcal{C} = \{v \,|\, v = F^T u,\, u \ge 0\}$ is a polyhedral cone represented by linear inequalities for a nonzero matrix F, and $\mathcal{C}^* = \{v \,|\, Fv \ge 0\}$ is the dual cone represented by generators. When F is the identity matrix, the LCCS reduces to the LCS. Conversely, the LCCS can be formulated as an LCS (1) using the multiplier u. Hence, if M is positive definite, the LCCS becomes a PSD-plus LCS.

In the above LCCS, if instead $\mathcal{C} = \{v \,|\, Ev \ge 0\}$ so that $\mathcal{C}^* = \{v \,|\, v = E^T u,\, u \ge 0\}$, then, with $\widehat{A} := A - BM^{-1}C$, $\widehat{B} := BM^{-1}$, $\widehat{C} := -M^{-1}C$, and $\widehat{z} := Cx + Mz$, the above LCCS is equivalent to:

$$\dot{x} = \widehat{A}x + \widehat{B}\widehat{z}, \qquad \mathcal{C}^* \ni \widehat{z} \perp \widehat{C}x + M^{-1}\widehat{z} \in \mathcal{C},$$

which is in the former form that is readily equivalent to a PSD-plus LCS.

An interesting special case is the following dynamic optimization problem where the algebraic variable u on the right-hand side of the differential equation is a solution of the state-dependent strictly convex quadratic program:

$$\dot{x} = Ax + Bu, \quad u \in \operatorname*{argmin}_{Eu \le 0,\, u \ge 0} \left[\frac{1}{2} u^T M u + (Cx)^T u \right]$$

Here $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the state and algebraic variables, respectively, and $E \in \mathbb{R}^{\ell_1 \times m}, M \in \mathbb{R}^{m \times m}$, and $C \in \mathbb{R}^{\ell_2 \times n}$ are constant matrices with M symmetric positive definite. The constraint set $\{u \,|\, Eu \le 0,\, u \ge 0\}$ is assumed to be nonempty. Let $\lambda \in \mathbb{R}^{\ell_1}$ be the Lagrange multiplier to the constraint $Eu \le 0$. The quadratic program can be characterized by the following linear complementarity conditions:

$$0 \le \begin{pmatrix} u \\ \lambda \end{pmatrix} \perp \begin{pmatrix} Cx \\ 0 \end{pmatrix} + \begin{bmatrix} M & E^T \\ -E & 0 \end{bmatrix} \begin{pmatrix} u \\ \lambda \end{pmatrix} \ge 0 \qquad (2)$$

Let $w := Cx + Mu + E^T\lambda$ so that $u = M^{-1}\big[w - Cx - E^T\lambda\big]$. The LCP (2) becomes

$$0 \le \widetilde{u} \perp \widetilde{F}\widetilde{C}x + \widetilde{F}M^{-1}\widetilde{F}^T\widetilde{u} \ge 0,$$

where $\widetilde{C} = -M^{-1}C$, $\widetilde{u} = (w, \lambda)$, $\widetilde{F} = \begin{bmatrix} I \\ -E \end{bmatrix}$. Letting $\widetilde{A} = A - BM^{-1}C$ and $\widetilde{B} = BM^{-1}$, we obtain the LCS of the form (1):

$$\dot{x} = \widetilde{A}x + \widetilde{B}\widetilde{F}^T\widetilde{u}, \quad 0 \le \widetilde{u} \perp \widetilde{F}\widetilde{C}x + \widetilde{F}M^{-1}\widetilde{F}^T\widetilde{u} \ge 0.$$

As shown in Example 9 of Section IV, the LCS (1) may not be strongly non-Zeno; nevertheless the bimodal case of (1) has been proved to be weakly non-Zeno [19, Theorem 23]. But the general case was not treated in the cited reference. In the present paper, we treat not only the PSD-plus LCS but also a larger class of LCSs satisfying a certain singleton property to be specified below.

III. Conewise Linear Systems

The LCSs considered in this paper are closely related to conewise linear systems (CLSs) recently introduced in [6], where non-Zenoness and observability are studied. In particular, the results established in the cited references can be borrowed for our purpose here. Hence, we first review some basic facts about the CLS, particularly, the key non-Zeno result.

Consider the system $\dot{x} = f(x)$ where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a (continuous) piecewise affine (PA) function, i.e. i) f is continuous, and ii) there exist an integer m and a family of affine functions $\{f_i\}_{i=1}^m$ such that $f(x) \in \{f_i(x)\}_{i=1}^m$ for any $x \in \mathbb{R}^n$. We call these systems *piecewise affine* systems. Another representation of PA functions is based upon the geometry of these functions. To elaborate on this, we recall that a finite collection of polyhedra in $\mathbb{R}^n$, denoted Ξ, is a *polyhedral subdivision* of $\mathbb{R}^n$ if (a) the union of all polyhedra in Ξ is equal to $\mathbb{R}^n$, (b) each polyhedron in Ξ is of dimension n, and (c) the intersection of any two polyhedra in Ξ is either empty or a common proper face of both polyhedra. For every PA function f, one can find a polyhedral subdivision of $\mathbb{R}^n$ and a finite family of affine functions $\{g_i\}$ such that f coincides with one of the functions $\{g_i\}$ on each polyhedron in Ξ [17], [9, Proposition 4.2.1]. In particular if the function f is piecewise linear, i.e. f is continuous and each of its component functions $\{f_i\}_{i=1}^m$ is linear, then one can find a *conic subdivision* of $\mathbb{R}^n$ for f, i.e. the finitely many polyhedra $\{\mathcal{X}_i\}_{i=1}^m$ defined above are replaced by a collection of cones. This leads to the *conewise linear systems* (CLSs):

$$\dot{x} = A_i x \quad \text{if} \quad x \in \mathcal{X}_i, \qquad (3)$$

where $\mathcal{X}_i = \{x \mid C_i x \ge 0\}$ for some matrix C_i and the continuity condition implies $x \in \mathcal{X}_i \cap \mathcal{X}_j \Rightarrow A_i x = A_j x$. Without loss of generality, we assume each C_i has no zero rows.

The CLSs represent a large number of important LCSs (A, B, C, D):

$$\dot{x} = Ax + Bu, \quad 0 \le u \perp Cx + Du \ge 0, \qquad (4)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and (A, B, C, D) are constant matrices of suitable dimensions, for instance, the LCSs with P-property [19]. Particularly, the LCSs with globally continuously differentiable state solution can be formulated as CLSs due to the following observation.

For the LCP in (4), it is clear that if the matrix B satisfies the singleton property for all $x \in \mathbb{R}^n$, then $B\mathrm{SOL}(Cx, D)$ is piecewise linear in Cx (thus in x). Moreover, it is shown in [5] that an LCS (A, B, C, D) has a continuously differentiable state solution for any initial state if and only if $B\mathrm{SOL}(Cx, D)$ is singleton for all $x \in \mathbb{R}^n$. Notice that such an LCS can be formulated as a CLS since $B\mathrm{SOL}(Cx, D)$ is piecewise linear.

An important feature of the CLS is that it does not have the Zeno behavior in the following sense [6, Theorem 3.5]: for any $x^0 \in \mathbb{R}^n$ and $t' \geq 0$, there exist two real numbers $\varepsilon_{\pm} > 0$ and two cones $\mathcal{X}_{i\pm} \in \Xi$ such that the state trajectory $x(t, x^0) \in \mathcal{X}_{i+}$ for all $t \in [t', t' + \varepsilon_+]$ (forward-time non-Zeno), and, for $t' > 0$, $x(t, x^0) \in \mathcal{X}_{i-}$ for all $t \in [t' - \varepsilon_-, t']$ (backward-time non-Zeno). Notice that the forward-time non-Zenoness implies that $x(t, x^0) = e^{A_{i+}(t-t')} x(t', x^0)$ for all $t \in [t', t' + \varepsilon_+]$.

Related to the proof of the above non-Zeno result, two technical lemmas in [6] are worth quoting here for the subsequent development:

Lemma 2 [6, Lemma 2.4] The following statements are equivalent for any vector $x^0 \in \mathbb{R}^n$:

- there exist a positive number ε such that $x(t, x^0) \in \mathcal{X}_i$ for all $t \in [0, \varepsilon]$;
- for some (equivalently, any) positive n-vector $c = (c_1, \cdots, c_n)$, there exists a number $\mu_0 > 0$ such that $\sum_{k=0}^{n-1} c_{k+1}\, \mu^k A_i^k x^0 \in \mathcal{X}_i$ for all $\mu \in [0, \mu_0]$.

Lemma 3 [6, Lemma 3.4] For any state $\xi \in \mathbb{R}^n$ and any polynomial $p : \mathbb{R} \to \mathbb{R}^n$ with $p(0) = \xi$, there exists an index i with $\xi \in \mathcal{X}_i$ such that $p(\mu) \in \mathcal{X}_i$ for all sufficiently small $\mu > 0$.

IV. Strong and Uniform Weak Non-Zenoness

A. Non-Zeno Concepts of LCSs

All the previous non-Zeno results in [2], [6], [15], [19] claim that for a given solution trajectory, the system under consideration remains in one of the "modes" in an open time interval. However, their respective implications are quite different, since the non-Zeno behavior is closely related to definitions of "modes" and "mode switching" in the systems. This leads to different non-Zenoness notions such as strong non-Zenoness and weak non-Zenoness. Roughly speaking, the "modes" in strong non-Zenoness are defined by three fundamental triple of index sets for an associated complementarity problem, while the "modes" in weak non-Zenoness are described by finitely many differential algebraic equations (DAEs) that partition the given complementarity systems. Such fine and delicate classifications of a seemingly intuitive concept turn out to be essential because of the mathematical subtleties associated with the complementarity condition; see [6, Example 3.12 and Proposition 3.13] for illustration.

To formally define strong non-Zenoness, consider the fundamental triple of index sets for a given solution pair $\big(x(t), u(t)\big)$ of the LCS (4):

$$
\begin{aligned}
\alpha(t) &\equiv \{\, j \mid u_j(t) > 0 = (\, Cx(t) + Du(t)\,)_j \,\}, \\
\beta(t) &\equiv \{\, j \mid u_j(t) = 0 = (\, Cx(t) + Du(t)\,)_j \,\}, \\
\gamma(t) &\equiv \{\, j \mid u_j(t) = 0 < (\, Cx(t) + Du(t)\,)_j \,\}.
\end{aligned}
$$

Definition 4 A given solution pair $\big(x(t), u(t)\big)$ of the LCS (4) is *strongly non-Zeno* at a $t_* \geq 0$ if there exist two real numbers $\varepsilon_{\pm} > 0$ and two triples of index sets $(\alpha_{\pm}, \beta_{\pm}, \gamma_{\pm})$ such that

$$
\begin{aligned}
\big(\alpha(t), \beta(t), \gamma(t)\big) &= \big(\alpha_+, \beta_+, \gamma_+\big), \quad \forall\, t \in (t_*, t_* + \varepsilon_+] \\
\big(\alpha(t), \beta(t), \gamma(t)\big) &= \big(\alpha_-, \beta_-, \gamma_-\big), \quad \forall\, t \in [t_* - \varepsilon_-, t_*)
\end{aligned}
$$

In contrast, to define the "modes" in weak non-Zenoness, consider a linear differential algebraic equation (LDAE) characterized by a pair of disjoint index sets $(\theta, \bar{\theta})$ whose union is $\{1, \cdots, m\}$:

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
0 &= (\, Cx(t) + Du(t)\,)_\theta \\
0 &= u_{\bar{\theta}}(t).
\end{aligned} \tag{5}
$$

It is clear that there are finitely many LDAEs (i.e. modes) and that every solution pair $\big(x(t), u(t)\big)$ of the LCS must satisfy one of such the LDAEs for some pair $\big(\theta, \bar{\theta}\big)$ at each time instant.

Definition 5 A solution pair $\big(x(t), u(t)\big)$ is *weakly non-Zeno* at a time t_* if a scalar $\varepsilon_* > 0$ and two pairs of index sets $(\theta_+, \bar{\theta}_+)$ and $(\theta_-, \bar{\theta}_-)$ partitioning $\{1, \cdots, m\}$ exist such that

C1. $\big(x(t), u(t)\big)$ satisfies the LDAE (5) corresponding to $(\theta_+, \bar{\theta}_+)$ for all $t \in (t_*, t_* + \varepsilon_*]$, and

C2. $\big(x(t), u(t)\big)$ satisfies the LDAE (5) corresponding to $(\theta_-, \bar{\theta}_-)$ for all $t \in [t_* - \varepsilon_*, t_*)$.

Moreover, for a given x-trajectory, if the above positive scalar ε_* and the index partition holds for any u-trajectory in both forward-time and backward-time directions, then we call the LCS *uniformly weakly non-Zeno* at a time t_*.

Clearly, strong non-Zenoness implies weak non-Zenoness, but not vice versa.

The following result is a direct consequence of [6, Corollary 3.8]: consider an LCS (A, B, C, D) with $B\mathrm{SOL}(Cx, D)$ being singleton for all $x \in \mathbb{R}^n$, then for every initial condition and any time t_*, there is a u-trajectory such that the solution pair $\big(x(t), u(t)\big)$ is weakly non-Zeno at t_*.

B. Main Non-Zeno Results

We first consider a class of LCSs satisfying the B-*singleton* and D-*singleton property*, i.e. both $B\mathrm{SOL}(Cx, D)$ and $D\mathrm{SOL}(Cx, D)$ are singletons for all $x \in \mathbb{R}^n$. Notice that the D-singleton property implies *global* w-*uniqueness*, i.e. for any $x \in \mathbb{R}^n$, $Cx + Du$ is unique for any solution $u \in \mathrm{SOL}(Cx, D)$.

Theorem 6 For an LCS (A, B, C, D) with $B\mathrm{SOL}(Cx, D)$ and $D\mathrm{SOL}(Cx, D)$ being both singleton for all $x \in \mathbb{R}^n$, the following two statements hold:

(a) the LCS is uniformly weakly non-Zeno at any time for any initial state;
(b) $Du(t)$ is continuous and piecewise analytic on any compact time interval for any u-trajectory.

Proof. Statement (a): we only need to prove the forward-time case when $t_* = 0$. Write $x(t) := x(t, x^0)$ and $u(t) := u(t, x^0)$. Since the mapping $x \mapsto B\mathrm{SOL}(Cx, D)$ is single-valued for all x, it is piecewise linear (and continuous). Thus it has the following conewise linear formulation [6]: there exist finitely many polyhedral cones $\{\mathcal{X}_i\}_{i=1}^k$ in $\mathbb{R}^n$ forming a polyhedral (more precisely, conic) subdivision of $\mathbb{R}^n$ such that for each cone $\mathcal{X}_i$, $B\mathrm{SOL}(Cx, D)$ is linear in x, i.e. a constant matrix F_i exists such that $B\mathrm{SOL}(Cx, D) = F_i x$ for all $x \in \mathcal{X}_i$. Therefore the LCS can be formulated as the (CLS): $\dot{x} = A_i x$ if $x \in \mathcal{X}_i$, where $A_i \equiv A + F_i$. By the non-Zeno result [6, Theorem 3.5] for the CLS, we conclude that there exist a scalar $\varepsilon > 0$ and an index $\ell \in \{1, \cdots, k\}$ such that the state trajectory $x(t) \in \mathcal{X}_\ell$ for all $t \in [0, \varepsilon]$, that is, $x(t) = e^{A_\ell t} x^0$ for all $t \in [0, \varepsilon]$. Furthermore, since $D\mathrm{SOL}(Cx, D)$ is a singleton for all x, the mapping admits a similar conewise linear formulation. Specifically, there exist finitely many polyhedral cones $\{\mathcal{C}_j\}_{j=1}^p$ in $\mathbb{R}^n$ forming a polyhedral subdivision of $\mathbb{R}^n$ such that for each cone $\mathcal{C}_j$, a constant matrix E_j exists such that $D\mathrm{SOL}(Cx, D) = E_j Cx$ for all $x \in \mathcal{C}_j$. Thus $Cx + D\mathrm{SOL}(Cx, D) = (I + E_j)Cx$ if $x \in \mathcal{C}_j$. Note that by Lemma 3, there exists an index $s \in \{1, \cdots, p\}$ such that the polynomial $\sum_{i=0}^{n-1} \frac{t^i}{i!} A_\ell^i x^0 \in \mathcal{C}_s$ for all $t \geq 0$ sufficiently small. Hence, we deduce from Lemma 2 that there exists a scalar $\varepsilon' > 0$ such that $x(t) \in \mathcal{C}_s$ for all $t \in [0, \varepsilon']$. Letting $\bar{\varepsilon} \equiv \min(\varepsilon, \varepsilon')$, we have $Cx(t) + D\mathrm{SOL}(Cx(t), D) = (I + E_s)Ce^{A_\ell t}x^0 \geq 0$ for all $t \in [0, \bar{\varepsilon}]$. Letting $G := I + E_s$, we thus deduce the existence of an index partition $(\theta, \bar{\theta})$ of $\{1, \cdots, m\}$, where $\bar{\theta}$ denotes the complement of θ, such that

$$\left(GCx^0,\, GCA_\ell x^0, \cdots, GCA_\ell^{n-1} x^0 \right)_\theta \;=\; 0,$$

i.e. $x^0 \in \overline{O}\big([GC]_{\theta\bullet}, A_\ell\big)$, and

$$\left(GCx^0,\, GCA_\ell x^0, \cdots, GCA_\ell^{n-1} x^0 \right)_{\bar{\theta}} \;\succ\; 0.$$

This further implies the existence of a scalar $\widetilde{\varepsilon} > 0$ such that for all $t \in (0, \widetilde{\varepsilon}]$,

$$\begin{array}{rcccl} \big(Cx(t) + Du(t)\big)_\theta & = & 0 & \leq & u_\theta(t), \\ \big(Cx(t) + Du(t)\big)_{\bar{\theta}} & > & 0 & = & u_{\bar{\theta}}(t) \end{array}$$

for any u-trajectory. Consequently, the uniform weak non-Zenoness holds.

To prove Statement (b), recall that we have shown above that $Cx(t) + Du(t)$ is continuous and piecewise analytic for any u-trajectory on any compact time interval. Noticing that $x(t)$ possesses the same property on the said interval, we thus obtain the desired result. □

If $\mathrm{SOL}(Cx, D)$ is a singleton for all $x \in \mathbb{R}^n$, i.e, the u-uniqueness or u-singleton property holds, then we show below that the LCS enjoys global strong non-Zenoness. Clearly, the LCS with the P-property treated in [19] is a special case of this class of the LCSs.

Theorem 7 Consider an LCS (A, B, C, D) with $\mathrm{SOL}(Cx, D)$ being a singleton for all $x \in \mathbb{R}^n$, then:

(a) the LCS is strongly non-Zeno at any time for any initial state;
(b) $u(t)$ is continuous and piecewise analytic on any compact time interval.

Proof. It is clear that the u-trajectory is unique, the mapping $\mathrm{SOL}(Cx, D)$ has a conewise linear formulation and the LCS can be formulated as a CLS. Hence, by the similar argument as that for Theorem 6, we deduce that there exist a scalar $\varepsilon > 0$ and two matrices A_ℓ and E_s such that $x(t) = e^{A_\ell t} x^0$ and $u(t) = E_s C e^{A_\ell t} x^0 \geq 0$ for all $t \in [0, \varepsilon]$. Hence, $Cx + Du(t) = (I + DE_s)Ce^{A_\ell t} x^0 \geq 0$ for all $t \in [0, \varepsilon]$. By checking $\left(E_s C x^0, E_s C A_\ell x^0, \cdots, E_s C A_\ell^{n-1} x^0 \right) \succeq 0$ for $u(t)$ and $\left(HCx^0, HCA_\ell x^0, \cdots, HCA_\ell^{n-1} x^0 \right) \succeq 0$ for $Cx(t) + Du(t)$, where $H := I + E_s D$, we see that there exist a scalar $\widetilde{\varepsilon} > 0$ and a fundamental index triple $(\alpha_+, \beta_+, \gamma_+)$ such that

$$\begin{array}{rcccl} u_{\alpha_+}(t) & > & 0 & = & \big(Cx(t) + Du(t)\big)_{\alpha_+} \\ u_{\beta_+}(t) & = & 0 & = & \big(Cx(t) + Du(t)\big)_{\beta_+} \\ u_{\gamma_+}(t) & = & 0 & < & \big(Cx(t) + Du(t)\big)_{\gamma_+} \end{array}$$

for all $t \in (0, \widetilde{\varepsilon}]$. Statement (b) follows readily. □

The singleton conditions assumed in the last two theorems are not restrictive for the non-Zeno notions to hold. Two examples below illustrate this: the first example pertains to uniform weak non-Zenoness and the second to strong non-Zenoness.

Example 8 Consider the LCS (4), where $A = \mu I_{3\times 3}$ for some scalar μ and

$$B = 0_{3\times 3}, \quad C = I_{3\times 3}, \quad D = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Note that by the pole-shifting technique [2], it is sufficient to look at the case where $\mu = 0$ such that the LCS becomes a static problem: $x(t) = x^0$ and LCP (Cx^0, D) or equivalently LCP (x^0, D). One can easily verify that the LCP (q, D) is globally feasible, i.e. for any $q \in \mathbb{R}^3$, there exists $z \geq 0$ such that $q + Dz \geq 0$. Since D is positive semidefinite, the global feasibility implies global solvability of the LCP [8, Theorem 3.1.2]. Hence, $B\mathrm{SOL}(Cx, D)$ is singleton for all $x \in \mathbb{R}^3$. However, the w-uniqueness does not hold for any vector $q = \begin{pmatrix} 0 & q_2 & 0 \end{pmatrix}$, where $q_2 > 0$. Indeed, in addition to the trivial solution pair

$$v^1 = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T, \quad w^1 = \begin{pmatrix} 0 & q_2 & 0 \end{pmatrix}^T,$$

the LCP (q, D) has multiple solution pairs

$$v^2 = \begin{pmatrix} 0 & 0 & \lambda q_2 \end{pmatrix}^T, \quad w^2 = \begin{pmatrix} 0 & (1-\lambda) q_2 & 0 \end{pmatrix}^T,$$

for $0 \le \lambda \le 1$. In particular, if $\lambda = 1$, then $v^2 = \begin{pmatrix} 0 & 0 & q_2 \end{pmatrix}$ and $w^2 = 0$. Since the index set $\theta = \{1, 3\}$ corresponding to the solution pair $(x(t), u(t)) = (x^0, v^1)$ but $\theta = \{1, 2, 3\}$ corresponding to $(x(t), u(t)) = (x^0, v^2)$ (for $\lambda = 1$), the LCS is *not* uniformly weakly non-Zeno at $x^0 = q$.

Example 9 Consider a special bimodal PSD-plus LCS treated in [19]

$$\dot{x} = Ax + bf^T u, \quad 0 \le u \perp \; fc^T x + ff^T u \ge 0,$$

where $A \in \mathbb{R}^{n \times n}$, $b, c \in \mathbb{R}^n$ with $n \ge 2$, and f is a non-zero m-vector. The LCS satisfies the B-singleton and D-singleton properties, thus has unique x-trajectory, but the LCP solution is clearly not unique. For an initial condition x^0 is such that $c^T x^0 < 0$ and $f_i \ne 0$ for all $i = 1, \cdots, n$, we have $f^T u(t) = -c^T x(t) > 0$ for all $t \ge 0$ sufficiently small. This yields a solution pair $\big(x(t), u(t)\big)$ such that the fundamental index triples vary in any small time interval containing $t = 0$. Hence, the strong non-Zenoness fails.

Finally we turn to the PSD-plus LCS (1) in Section II. Observing the singleton properties and applying Theorem 6, we immediately conclude:

Corollary 10 The PSD-plus LCS (1) is uniformly weakly non-Zeno at any state $x^* \in \mathbb{R}^n$.

V. Finite Verification

The characterization of uniform weak non-Zenoness in Theorem 6 begs the question of whether the singleton conditions for an arbitrary LCP (q, D) can be verified by a finite procedure. The answer turns out to be negative in general, since verifying the solvability of an LCP (q, D) for all q may not be fulfilled in finite steps, even under the feasibility assumption. This is related to the Q_0-matrices in linear complementarity theory [8]. However, if D is PSD-plus, then finite verification is possible as shown below.

For a PSD-plus matrix $D = FMF^T$ with a nonzero F and a positive definite M, we define the cone

$$\mathcal{C} = \{\, v \,|\, F^T v = 0, \; v \ge 0 \} = \{\, v \,|\, Dv = 0, \; v \ge 0 \}. \quad (6)$$

The following result is well known in LCP theory [8] e.g.

Proposition 11 Let $D = FMF^T$ be PSD-plus. For a given vector q, the LCP(q, D) is solvable if and only if a vector v exists such that $Fv + q \ge 0$ (or equivalently a vector v' exists such that $Dv' + q \ge 0$).

We introduce more notation for the subsequent development. Let $\mathcal{S}$ be a linear subspace of $\mathbb{R}^n$ and let $\mathcal{S}^\perp$ denote its orthogonal complement, which is another linear subspace such that $\mathcal{S} + \mathcal{S}^\perp = \mathbb{R}^n$. For a given $x \in \mathbb{R}^n$, we let $\pi_{\mathcal{S}}(x) \in \mathbb{R}^n$ denote its projection onto $\mathcal{S}$. For a set $\mathcal{U} \subseteq \mathbb{R}^n$, Aff$(\mathcal{U})$ denotes the affine hull of $\mathcal{U}$, i.e. the smallest affine set containing $\mathcal{U}$.

Proposition 12 Given a $C \in \mathbb{R}^{m \times n}$ and a PSD-plus $D \in \mathbb{R}^{m \times m}$ with $D = FMF^T$ for a positive definite matrix M and some matrix F, the following statements are equivalent:

1. SOL(Cx, D) is nonempty for all $x \in \mathbb{R}^n$;
2. $C\mathbb{R}^n \subseteq \mathbb{R}^m_+ - D\mathbb{R}^m$;
3. $\pi_{\text{Null}(F^T)} \,\text{Range}(C) \;\perp\; \mathcal{C}$, where $\pi_{\text{Null}(F^T)}\text{Range}(C)$ stands for the projection of Range(C) onto Null(F^T);
4. $\pi_{\text{Null}(F^T)} \,\text{Range}(C) \;\perp\; \text{Aff}(\mathcal{C})$.

Proof. (1) $\Leftrightarrow$ (2). Notice that Statement (2) is equivalent to: for any given $x \in \mathbb{R}^n$, there exist $w \in \mathbb{R}^m_+$ and $u \in \mathbb{R}^m$ such that $Cx = w - Du$ or equivalently $Cx + Du = w \ge 0$. Since the latter is equivalent to Statement (1) by Proposition 11, the equivalence of Statements (1) and (2) follows.

(1) $\Leftrightarrow$ (3). Notice that by Proposition 11 and the Theorem of Alternative, Statement (1) is equivalent to: for any $x \in \mathbb{R}^n$, the inequality system

$$(Cx)^T y < 0, \quad \forall\, y \in \mathcal{C} \quad (7)$$

has no solution y. Since Range$(F) + \big(\text{Range}(F)\big)^\perp = \mathbb{R}^m$ and $\big(\text{Range}(F)\big)^\perp = \text{Null}(F^T)$, we have, for an arbitrary $q \in \text{Range}(C)$, $q = u^1 + u^2$, where $u^1 \in \text{Range}(F)$ and $u^2 \in \text{Null}(F^T)$. Thus $q^T y = (u^1 + u^2)^T y = (u^2)^T y$. Note that if $y^* \in \mathcal{C}$ exists such that $q^T y^* = (u^2)^T y^* \ne 0$, then y^* is the solution to the inequality system (7) for either $q \in \text{Range}(C)$ (when $q^T y^* < 0$) or $-q \in \text{Range}(C)$ (when $q^T y^* > 0$). Consequently, Statement (1) holds if and only if $\pi_{\text{Null}(F^T)}(q) \perp \mathcal{C}$ for any $q \in \text{Range}(C)$. This proves the equivalence of Statements (1) and (3).

(1) $\Leftrightarrow$ (4). Note that $\mathcal{C}$ is a closed convex set containing the zero vector, hence Aff$(\mathcal{C})$ is a linear subspace. Since $\mathcal{C} \subseteq \text{Aff}(\mathcal{C})$ and Aff$(\mathcal{C})$ is an affine combination of all the elements of $\mathcal{C}$, a vector $u \perp \mathcal{C}$ if and only if $u \perp \text{Aff}(\mathcal{C})$, and the latter is equivalent to $\pi_{\text{Null}(F^T)} \,\text{Range}(C) \perp \text{Aff}(\mathcal{C})$. $\square$

The above proposition leads to a finite procedure to verify global solvability of LCP(Cx, D). Specifically, let $\widehat{b}^i := \pi_{\text{Null}(F^T)}(b^i) = \pi_{\text{Null}(D)}(b^i)$ for $i = 1, \cdots, k$. Thus checking Statement (3) can be achieved by solving two linear programs: $\min\limits_{F^T v = 0, \; v \ge 0} \big(\widehat{b}^i\big)^T v$, and $\max\limits_{F^T v = 0, \; v \ge 0} \big(\widehat{b}^i\big)^T v$.

In the sequel, we characterize a finite procedure to verify the global singleton property for BSOL(Cx, D), assuming global solvability of LCP(Cx, D). Letting $u^* \in \text{SOL}(q^*, D)$ for a given q^*, the following fundamental triple of index sets is uniquely determined:

$$\begin{aligned} \alpha_* &\equiv \{\, j \,|\, u^*_j > 0 = (\, q^* + Du^* \,)_j \,\}, \\ \beta_* &\equiv \{\, j \,|\, u^*_j = 0 = (\, q^* + Du^* \,)_j \,\}, \\ \gamma_* &\equiv \{\, j \,|\, u^*_j = 0 < (\, q^* + Du^* \,)_j \,\}. \end{aligned} \quad (8)$$

It is clear that γ_* is invariant for any $u \in \text{SOL}(q^*, D)$, due to the w-uniqueness. The lemma below provides a necessary and sufficient condition for the singleton property at q^*, which is valid for any D with the w-uniqueness property.

Lemma 13 In the above setting, $B\text{SOL}(q^*, D)$ is singleton if and only if the following condition holds: for all $v_{\alpha_*} \in \mathbb{R}^{|\alpha_*|}$, $v_{\beta_*} \in \mathbb{R}_+^{|\beta_*|}$,

$$\begin{aligned} D_{\bullet\alpha_*} v_{\alpha_*} + D_{\bullet\beta_*} v_{\beta_*} = 0 \\ \Rightarrow \; B_{\bullet\alpha_*} v_{\alpha_*} + B_{\bullet\beta_*} v_{\beta_*} = 0. \end{aligned} \tag{9}$$

Proof. The "if" part. Consider an arbitrary $u \in \text{SOL}(q^*, D)$. It is clear that $u_{\beta_*} \geq 0$ and $u_{\gamma_*} = 0$. Thanks to the w-uniqueness, we have $D(u - u^*) = 0$, or equivalently $D_{\bullet\alpha_*}(u - u^*)_{\alpha_*} + D_{\bullet\beta_*}(u - u^*)_{\beta_*} = 0$. Hence $B_{\bullet\alpha_*}(u - u^*)_{\alpha_*} + B_{\bullet\beta_*}(u - u^*)_{\beta_*} = 0$, by the implication (9), or equivalently $B(u - u^*) = 0$. Thus $Bu = Bu^*$.

The "only if" part. Suppose the implication does not hold, i.e. there exist $v'_{\alpha_*} \in \mathbb{R}^{|\alpha_*|}$ and $v'_{\beta_*} \in \mathbb{R}_+^{|\beta_*|}$ such that $D_{\bullet\alpha_*} v'_{\alpha_*} + D_{\bullet\beta_*} v'_{\beta_*} = 0$ but $B_{\bullet\alpha_*} v'_{\alpha_*} + B_{\bullet\beta_*} v'_{\beta_*} \neq 0$. (Obviously, $\big(v'_{\alpha_*}, v'_{\beta_*}\big) \neq 0$.) Note that for some $\lambda > 0$ sufficiently small, $u^*_{\alpha_*} + \lambda v'_{\alpha_*} > 0$, and $u^*_{\beta_*} + \lambda v'_{\beta_*} \geq 0$. Letting $\widehat{u} = u^* + \lambda\big(v'_{\alpha_*}, v'_{\beta_*}, 0\big)$, it is easy to verify that $\widehat{u} \geq 0$ with $\widehat{u}_{\alpha_*} > 0$, $\widehat{u}_{\beta_*} \geq 0$, $\widehat{u}_{\gamma_*} = 0$ and that $q^* + D\widehat{u} = q^* + Du^* \geq 0$. Consequently, $\widehat{u} \perp \big(q^* + D\widehat{u}\big)$. Thus $\widehat{u} \in \text{SOL}(q^*, D)$. However, $B(\widehat{u} - u^*) = \lambda\big[B_{\bullet\alpha_*} v'_{\alpha_*} + B_{\bullet\beta_*} v'_{\beta_*}\big] \neq 0$, which is a contradiction. $\square$

An important feature of the necessary and sufficient condition (9) is that it is independent of q^* but only dependent on the fundamental index sets. This leads to a finite verification procedure to check the global singleton condition for $B\text{SOL}(Cx, D)$. Specifically, for a given fundamental triple (α, β, γ), define the set

$$\begin{aligned} \mathcal{S}(\alpha,\beta,\gamma) \;=\; \big\{\, & x \in \mathbb{R}^n \mid \; \exists\, u \in \text{SOL}(Cx, D) \text{ such that} \\ & u_\alpha > 0 = (Cx + Du)_\alpha,\; u_\beta = 0 = (Cx + Du)_\beta, \\ & u_\gamma = 0 < (Cx + Du)_\gamma \,\big\}. \end{aligned}$$

(Note that these sets may overlap for different α, β with the same γ.) We thus see that if $\mathcal{S}(\alpha,\beta,\gamma)$ is nonempty, then $B\text{SOL}(Cx, D)$ is singleton for all $x \in \mathcal{S}(\alpha,\beta,\gamma)$ if and only if (9) holds for (α,β,γ). Hence we obtain the following result without further proof.

Proposition 14 Let $D \in \mathbb{R}^{m\times m}$ be PSD-plus such that $\text{SOL}(Cx, D) \neq \emptyset$ for all $x \in \mathbb{R}^n$. Then $B\text{SOL}(Cx, D)$ is singleton for all $x \in \mathbb{R}^n$ if and only if the implication (9) holds for any fundamental index triple (α,β,γ) such that $\mathcal{S}(\alpha,\beta,\gamma)$ is nonempty. $\square$

Given a matrix triple (B, C, D) with D PSD-plus and $\text{SOL}(Cx, D) \neq \emptyset$ for all $x \in \mathbb{R}^n$, a finite procedure can be developed to verify the global singleton property of $B\text{SOL}(Cx, D)$, based on Proposition 14 and the following observations: (1) there are finitely many fundamental index triples; (2) for any given fundamental index triple (α,β,γ), checking whether $\mathcal{S}(\alpha,\beta,\gamma)$ is nonempty can be solved by a linear program; (3) checking the implication (9) for a given triple (α,β,γ) can be solved by a linear program.

Acknowledgement We are grateful to the two referees for their constructive comments on the paper.

REFERENCES

[1] B. BROGLIATO. Some perspectives on analysis and control of complementarity systems. *IEEE Transactions on Automatic Control* 48 (2003) 918–935.

[2] M.K. ÇAMLIBEL. *Complementarity Methods in the Analysis of Piecewise Linear Dynamical Systems*. Ph.D. thesis, Center for Economic Research, Tilburg University, (2001).

[3] M.K. ÇAMLIBEL. Popov-Belevitch-Hautus type controllability tests for linear complementarity systems. *Systems and Control Letters*, to appear, 2006.

[4] M.K. ÇAMLIBEL, W.P.M.H. HEEMELS, AND J.M. SCHUMACHER. On linear passive complementarity systems. *European Journal of Control* 8 (2002) 220–237.

[5] M.K. ÇAMLIBEL, J.S. PANG, AND J. SHEN. Lyapunov stability of complementarity and extended systems. *SIAM Journal on Optimization*, Vol.17(4), p.1066–1101, 2006.

[6] M.K. ÇAMLIBEL, J.S. PANG, AND J. SHEN. Conewise linear systems: non-Zenoness and observability. *SIAM Journal on Control and Optimization*, Vol.45(6), p.1769–1800, 2006

[7] M.K. CAMLIBEL AND J.M. SCHUMACHER. On the Zeno behavior of linear complementarity systems. *Proceedings of the 40th IEEE Conference on Decision and Control, Orlando, Florida 2001*, pp. 346–351 (2001).

[8] R.W. COTTLE, J.S. PANG, AND R.E. STONE. *The Linear Complementarity Problem*, Academic Press Inc., (Cambridge 1992).

[9] F. FACCHINEI AND J.S. PANG. *Finite-dimensional Variational Inequalities and Complementarity Problems*. Spring-Verlag (New York 2003).

[10] W.P.M.H. HEEMELS. *Linear Complementarity Systems: A Study in Hybrid Dynamics*. Ph.D. thesis, Department of Electrical Engineering, Eindhoven University of Technology (November 1999).

[11] W.P.M.H. HEEMELS, M.K. ÇAMLIBEL, A.J. VAN DER SCHAFT, AND J.M. SCHUMACHER. Modelling, well-posedness, and stability of switched electrical networks. In: O. Maler and A. Pnueli, editors, *Hybrid Systems: Computation and Control*, (2003) 249–266.

[12] W.P.M.H. HEEMELS, J.M. SCHUMACHER, AND S. WEILAND. The rational linear complementarity systems. *Linear Algebra and its Applications* 294 (1999) 93–135.

[13] W.P.M.H. HEEMELS, J.M. SCHUMACHER, AND S. WEILAND. Linear complementarity systems. *SIAM Journal on Applied Mathematics* 60 (2000) 1234–1269.

[14] Z.Q. LUO AND P. TSENG. A descomposition property for a class of square matrices. *Applied Mathematics Letters* 4 (1991) 67-69.

[15] J.S. PANG AND J. SHEN. Strongly regular differential variational systems. *IEEE Transactions on Automatic Control*, Vol.52(2), p.242–255, 2007.

[16] J.S. PANG AND D.E. STEWART. Differential variational inequalities. *Mathematical Programming, Series A*. To appear (2006).

[17] S. SCHOLTES. Introduction to piecewise differentiable equations. Habilitation thesis, Institut für Statistik und Mathematische Wirtschaftstheorie, Universität Karlsruhe (1994).

[18] J.M. SCHUMACHER. Complementarity systems in optimization. *Mathematical Programming, Series B* 101 (2004) 263–296.

[19] J. SHEN AND J.S. PANG. Linear complementarity systems: Zeno states. *SIAM Journal on Control and Optimization* 44 (2005) 1040–1066.