Math 650 Fall 2016 Homework #6

Due Nov. 29, Tue in class

1. For a real number a, let $a_+ := \max(a, 0)$. For a vector $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$, define $x_+ := ((x_1)_+, \ldots, (x_n)_+)^T$. Show that $\Pi_{\mathbb{R}^n_+}(x) = x_+$.

2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a G-differentiable convex function, $C \subseteq \mathbb{R}^n$ be a closed convex set, and γ be a positive real number. Show that $x^* \in C$ is a global minimizer of the convex optimization problem $\min_{x \in C} f(x)$ if and only if $x^* = \Pi_C(x^* - \gamma \nabla f(x^*))$, where $\Pi_C(\cdot)$ denotes the Euclidean projection onto C.
(*Hint*: use the optimality condition (i.e., the VI: $\langle \nabla f(x^*), x - x^* \rangle \geq 0, \forall x \in C$) of convex optimization and the VI for Euclidean projection: $\langle x - \Pi_C(x), z - \Pi_C(x) \rangle \leq 0, \forall z \in C$.)

3. Let $C \subseteq \mathbb{R}^n$ be a closed convex set, and $x^* \in \mathbb{R}^n \setminus C$. Define the (Euclidean) distance from x^* to C as $d_C(x^*) := \|x^* - \Pi_C(x^*)\|_2 = \min_{z \in C} \|z - x^*\|_2$.

   (1) Let D be a closed convex set containing C. Show that $d_D(x^*) \leq d_C(x^*)$.

   (2) Let H be a closed half-space containing C. Show that $d_H(x^*) \leq d_C(x^*)$.

   (3) Consider the closed half-space $H := \{x \in \mathbb{R}^n : \langle a, x \rangle \leq \gamma\}$, where $a = x^* - \Pi_C(x^*)$ and $\gamma = \langle a, \Pi_C(x^*) \rangle$. Show that (i) $a \neq 0$; (ii) $C \subseteq H$; and (iii) $d_H(x^*) = d_C(x^*)$.

   (4) Use the above results to show that the minimum distance from x^* to C (with $x^* \notin C$) is the maximum among the distances from x^* to closed half-spaces H containing C, i.e., $d_C(x^*) = \max_{C \subseteq H} d_H(x^*)$, where H's are closed half-spaces.

4. For a given nonempty set $S \subseteq \mathbb{R}^n$, recall that its support function $\sigma_S(x) := \sup\{\langle x, z \rangle \mid z \in S\}$.

   (1) Let $\overline{S}$ denote the closure of S. Show that $\sigma_S(x) = \sigma_{\overline{S}}(x), \forall\, x \in \mathbb{R}^n$.

   (2) Let F and G be two compact convex sets in $\mathbb{R}^n$ such that $\sigma_F(x) = \sigma_G(x), \forall\, x \in \mathbb{R}^n$. Show that $F = G$. (*Hint*: use an appropriate separation theorem.)

More practice problems: *Do not submit*

1. Let σ_S denote the support function of a nonempty set S.

   (1) For any nonempty sets $S_1, S_2 \in \mathbb{R}^n$, show that $\sigma_{S_1+S_2}(x) = \sigma_{S_1}(x) + \sigma_{S_2}(x),\ \forall\, x \in \mathbb{R}^n$.

   (2) Use the above results to show that if A, B, C are compact and convex sets in $\mathbb{R}^n$ such that $A + C = B + C$, then $A = B$.