Math 302/600 Spring 2015 Homework #1

Due Feb. 10, Tue. in class

1. In an inner product space V with the induced norm $\|\cdot\|$, show that for any $x, y \in V$,
   (1) $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$;
   (2) $\|x+y\| \cdot \|x-y\| \leq \|x\|^2 + \|y\|^2$.

2. On the Euclidean space $\mathbb{R}^n$, define for each $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$,
$$\|x\|_\star := c_1|x_1| + c_2|x_2| + \cdots + c_n|x_n|,$$
where c_i is a positive real number for each $i = 1, \cdots, n$. Show that $\|\cdot\|_\star$ is a norm on $\mathbb{R}^n$.

3. Recall that $C([0,1])$ is the vector space of all continuous functions $f : [0,1] \to \mathbb{R}$ on the interval $[0,1]$. For any $f \in C([0,1])$, define
$$\|f\|_\infty := \max\{|f(t)| : t \in [0,1]\}.$$
   (1) Explain why $\|f\|_\infty$ exists for any $f \in C([0,1])$. (*Hint*: think of Theorem 5.3.4 of Bartle and Sherbert's book.)
   (2) Prove that $\|\cdot\|_\infty$ is a norm on $C([0,1])$.

4. Prove that the discrete metric d discussed in class satisfies the triangle inequality, i.e. $d(x,y) \leq d(x,z) + d(y,z), \forall\, x, y, z \in M$.

5. Let d_1 and d_2 be two metrics on a set M. Define the sum $d_1 + d_2$ on M as
$$(d_1 + d_2)(x,y) := d_1(x,y) + d_2(x,y), \quad \forall\, x, y \in M.$$
Show that $d_1 + d_2$ is a metric on M.

The following extra problems are for **Math 600** *students only:*

6. On the Euclidean space $\mathbb{R}^n$, define for each $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$,
$$\|x\|_\infty := \max(|x_1|, |x_2|, \cdots, |x_n|)$$
Show that $\|\cdot\|_\infty$ is a norm on $\mathbb{R}^n$.

7. Let (M, d) be a metric space. Define
$$\rho(x,y) := \frac{d(x,y)}{1 + d(x,y)}, \quad \forall\, x, y \in M.$$
Show that ρ is a metric on M. (*Hint*: observe that $t/(1+t)$ is an increasing function on $\mathbb{R}_+$.)