Math 302/600 Spring 2015 Homework #4

Due March 5, Thu. in class

Note: For any Euclidean space $\mathbb{R}^n$, consider the usual metric induced by the Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^n$, unless otherwise stated.

1. Let (M, d) be a metric space, and A, B be two subsets of M. Show the following:

   (1) $\mathrm{cl}(A \cap B) \subseteq (\mathrm{cl}A) \cap (\mathrm{cl}B)$;

   (2) $M \setminus \mathrm{int}A = \mathrm{cl}(M \setminus A)$.

2. Let $\mathbb{Q}$ denote the set of rational numbers, and $\mathbb{I}$ denote the set of irrational numbers. Let the set $\mathbb{Q} \times \mathbb{I} := \{(x_1, x_2) : x_1 \in \mathbb{Q},\ x_2 \in \mathbb{I}\} \subseteq \mathbb{R}^2$. Determine the interior, closure, and boundary of $\mathbb{Q} \times \mathbb{I}$ (without proof).

3. Let (M, d) be a metric space, and S be a nonempty subset of M such that (S, d) is complete. Show that S is closed in (M, d).

4. Let (M, d) be a metric space with the discrete metric d. A sequence (x_n) in M is said to have a *constant tail* if there exist $K \in \mathbb{N}$ and $c \in M$ such that $x_n = c$, $\forall n \geq K$. Prove the following:

   (1) A sequence in (M, d) is convergent if and only if it has a constant tail;

   (2) A Cauchy sequence in (M, d) has a constant tail;

   (3) Show that (M, d) is complete using (1)-(2).

The following extra problem is for **Math 600** *students only:*

5. Let (M, d) be a metric space, and a point $c \in M$ is a *cluster point* of a sequence (x_n) in (M, d) if for any $\varepsilon > 0$, there are infinitely many x_n's in the sequence such that $d(c, x_n) < \varepsilon$.

   (1) Show that c is a cluster of a sequence (x_n) if and only if there is a subsequence of (x_n) that converges to c.

   (2) Suppose that any bounded sequence in (M, d) has at least one cluster point in M. Show that (M, d) is complete. (*Hint*: show that a Cauchy sequence has at most one cluster point.)