Math 302/401/600 Fall 2010 Homework #4

Due Oct. 6, Wed. in class

1. Given a metric space (M, d). Prove the following statements:

   (1) In (M, d), any convergent sequence has a unique limit;

   (2) In (M, d), any convergent sequence is Cauchy;

   (3) In (M, d), any Cauchy sequence is bounded.

2. Consider the real line $\mathbb{R}$ endowed with the discrete metric d. A real sequence (x_n) is said to have a *constant tail* if there exist $K \in \mathbb{N}$ and $c \in \mathbb{R}$ such that $x_n = c$ for all $n \geq K$. Prove the following statements:

   (1) A sequence in $\mathbb{R}$ with a constant tail is convergent in $\mathbb{R}$;

   (2) A Cauchy sequence in $\mathbb{R}$ has a constant tail;

   (3) Show that $(\mathbb{R}, d)$ is complete using (2)-(3).

3. Textbook, page 125, Section 2.8, Exercise 1.

4. Determine which of the following sets are sequentially compact (using the standard metric on $\mathbb{R}^n$):

   (1) In $\mathbb{R}$: $\{2/n : n \in \mathbb{N}\}$, $\mathbb{Q} \cap [0, 1]$ ($\mathbb{Q}$ is the set of rational numbers);

   (2) In $\mathbb{R}^2$: $\mathbb{Q} \times \mathbb{Q}$, $\{(x, y) \in \mathbb{R}^2 : x^2 + 2y^2 = 4\}$.

5. Show that the union and intersection of two sequentially compact sets remain sequentially compact.

The following extra problem is for **Math 401/600** *students only:*

6. Let A be a nonempty set in $\mathbb{R}^n$ and d be the standard Euclidean norm on $\mathbb{R}^n$. Let $z \in \mathbb{R}^n$ be given.

   (1) Show that the infimum of the *real* set $\{d(z, x) : x \in A\}$ exists. In the following, define $d(z, A) := \inf\{d(z, x) : x \in A\}$.

   (2) Show that there exists a sequence $\{x_k\}$ in A such that the *real* sequence $\{d(z, x_k)\}$ converges to $d(z, A)$. Further, show that $\{x_k\}$ converges in $\mathbb{R}^n$.

   (3) Use (2) to show that if the set A is closed, then there exists $x^* \in A$ such that $d(z, A) = d(z, x^*)$.

$\star$ (Bonus problem that you don't have to turn in)

   (1) Textbook, page 125, Section 2.8, Exercise 2. (*hint*: refer to the similar proof of Theorem 3.5.5 of Bartle's book)

   (2) Use (1) to show that any sequentially compact metric space is complete.