Math 600 Fall 2015 Homework #3

Due Oct. 8, Thu. in class

Note: For the Euclidean space $\mathbb{R}^n$, consider the usual metric induced by the Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^n$, unless otherwise stated.

1. Let (x_k) be a Cauchy sequence in the metric space (M, d). Show that for any given $z \in M$, the real sequence $\big(d(x_k, z)\big)$ converges. (*Hint*: consider the reverse triangle inequality of d.)
2. Let (x_k) and (y_k) be two sequences in the metric space (M, d) that converge to $x \in M$ and $y \in M$ respectively. Show that the real sequence $\big(d(x_k, y_k)\big)$ converges to $d(x, y)$.
3. Let A be a nonempty set in the metric space (M, d). Show that A is sequentially compact if and only if any infinite subset of A has a limit point that belongs to A.
4. Let (M, d) be a metric space such that M is (sequentially) compact. Show that (M, d) is complete.
5. Show that the intersection and union of two compact sets in a metric space (M, d) remain compact. (*Note*: think of both the open cover definition and sequential argument, but only turn in one method.)

Miscellaneous practice problems: *Do not submit*

1. Use the definition of compactness (i.e. the open cover definition) to show that the following sets are *not* compact, by exhibiting an open cover with no finite sub-cover:
   (1) The open ball $B(x, 1)$ centered at a fixed $x \in \mathbb{R}^n$ with the radius 1 in the Euclidean space $\mathbb{R}^n$;
   (2) The set $A = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1,\, x_2 \geq 0\}$ in $\mathbb{R}^2$;
   (3) An infinite set in the metric space (M, d) with the discrete metric d.
2. Determine which of the following sets is (sequentially) compact:
   (1) On $\mathbb{R}$: $A = \{2/n : n \in \mathbb{N}\}$, $A = \mathbb{Q} \cap [0, 1]$ (where $\mathbb{Q}$ is the set of rational numbers);
   (2) On $\mathbb{R}^2$: $A = \mathbb{Q} \times \mathbb{Q}$, $A = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| + x_2^2 \leq 4\}$.

   If a set is *not* (sequentially) compact, briefly explain why; otherwise, give a proof.