Math 302/600 Spring 2017 Homework #13

Due May 16, Tue in class

Note: For any Euclidean space $\mathbb{R}^n$, consider the usual metric induced by the Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^n$, unless otherwise stated.

1. Let $A = [-a, a] \subset \mathbb{R}$ with $a > 0$, and let
$$f_n(x) = \frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!}, \quad x \in \mathbb{R}.$$
   (1) Use the Weierstrass M-test to show uniform convergence of the series $\sum_{n=1}^{\infty} f_n$ on A.

   (2) Let f_* be the limit function of the series on A, i.e., $f_*(x) = \sum_{n=1}^{\infty} f_n(x)$. Is f_* differentiable on $(-a, a)$? If so, is $f_*'(x) = \sum_{n=1}^{\infty} f_n'(x)$ on $(-a, a)$? Prove your answers.

2. Find the largest possible constant $r \in (0, 1)$ such that the function $f : [0, r] \to [0, r]$ defined by $f(x) = x^2$ is a contraction.

3. Let $(V, \|\cdot\|)$ be a complete normed vector space and its induced metric $d(x, y) = \|x - y\|$ for $x, y \in V$. Let $f : V \to V$ be a *linear mapping/function*, i.e., $f(x + y) = f(x) + f(y), \forall\, x, y \in V$ and $f(\alpha x) = \alpha f(x)$ for all $x \in V$ and $\alpha \in \mathbb{R}$. You may assume the following facts without proof: $f(0) = 0$ and $f(x - y) = f(x) - f(y), \forall\, x, y \in V$.

   (1) Show that f is a contraction if and only if there exists a constant C with $0 < C < 1$ such that $\|f(x)\| \le C\|x\|$ for all $x \in V$.

   (2) Suppose that f is a contraction. Let $x_0 \in V$ be arbitrary, and define the sequence (x_n) recursively by $x_n = f(x_{n-1}), \quad n \in \mathbb{N}$. Show that (x_n) converges to the zero vector in V.

4. Let the constant K satisfy $0 < K < 1$. Consider the *linear* function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by
$$f(x) = \frac{K}{\sqrt{2}}\big(x_1 + x_2,\ x_2 - x_1\big), \quad \forall\, x = (x_1, x_2) \in \mathbb{R}^2.$$
   In the following, you may use the results of Problem 3.

   (1) Show that when the 2-norm (i.e., $\|\cdot\|_2$) is used, f is a contraction.

   (2) Show that when the 1-norm (i.e., $\|\cdot\|_1$) is used, f is *not* a contraction if $\frac{1}{\sqrt{2}} < K < 1$.

   (3) Let $x^0 = (x_1^0, x_2^0) \in \mathbb{R}^2$ be arbitrary. Define the sequence (x^k) as $x^k = f(x^{k-1}), \quad k \in \mathbb{N}$. Explain why the sequence (x^k) is convergent when the 2-norm is used. (*Note:* recall that $(\mathbb{R}^2, \|\cdot\|_2)$ is complete.)

   (4) Show that the sequence defined in (3) is convergent when the 1-norm is used. (*Hint:* use the equivalence of norms on a Euclidean space shown in Problem 2 of Homework #9.)

   $\star$ This problem shows that the contractive property is a *sufficient* condition for convergence but not a necessary one.

5. Consider the space $C([0, 1])$ of all real-valued continuous functions on $[0, 1]$ endowed with the sup-norm (or uniform norm), i.e., $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$ for any $f \in C([0, 1])$. Let $B \subset C([0, 1])$ be
$$B = \Big\{ f \in C([0, 1]) \mid -1 \le f'(x) \le 2, \forall\, x \in [0, 1],\ f(0) = 0 \Big\}.$$
   Show that B is equi-continuous and compact.

6. Consider the space $C([0,1])$ of real-valued continuous functions on $[0,1]$ endowed with the sup-norm (or uniform norm) $\|\cdot\|_\infty$. Let $B \subset C([0,1])$ be

$$B = \Big\{ f \in C([0,1]) \,|\, 0 \leq f(x) \leq 2, \forall\, x \in [0,1] \Big\}.$$

Show that B is closed and bounded (with respect to the sup-norm) but B is not compact.

The following extra problem(s) are for **Math 600** *students only:*

7. Let $C_b(\mathbb{R})$ be the space of real-valued continuous and *bounded* functions on $\mathbb{R}$ endowed with the sup-norm (or uniform norm) $\|\cdot\|_\infty$. Let $B \subset C_b(\mathbb{R})$ be

$$B = \Big\{ f \in C_b(\mathbb{R}) \,|\, 0 < f(x) < 2, \forall\, x \in \mathbb{R} \Big\}.$$

Is B bounded (with respect to the sup-norm)? Is B open? Is B closed? If not, what is the closure of B? Justify your answers.

8. Let (f_n) be an equi-continuous sequence of functions $f_n : (M,d) \to \mathbb{R}$, where (M,d) is compact. Suppose that (f_n) converges pointwise to f_* on M. Show that (f_n) converges uniformly to f_* on M.