Math 600 Fall 2017 Homework #8

Due Dec. 6, Wed. in class

Note: For the Euclidean space $\mathbb{R}^n$, consider the usual metric induced by the Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^n$, unless otherwise stated.

1. Let $C_b(\mathbb{R})$ be the space of real-valued continuous and bounded functions on $\mathbb{R}$ endowed with the sup-norm (or uniform norm) $\|\cdot\|_\infty$. Let $B \subset C_b(\mathbb{R})$ be

$$B = \Big\{ f \in C_b(\mathbb{R}) \,|\, 0 < f(x) < 2, \forall\, x \in \mathbb{R} \Big\}$$

   Is B bounded? Is B open? Is B closed? Justify your answers.

2. Consider the space $C([0,1])$ of real-valued continuous functions on $[0,1]$ endowed with the sup-norm (or uniform norm) $\|\cdot\|_\infty$. Let $B \subset C([0,1])$ be

$$B = \Big\{ f \in C([0,1]) \,|\, 0 \le f(x) \le 2, \forall\, x \in [0,1] \Big\}$$

   Show that B is closed and bounded but B is not compact.

3. Let $A \subset \mathbb{R}$ be a bounded set, and the set $B \subset C(A, \mathbb{R})$ be

$$B = \Big\{ \underbrace{\frac{x^2}{\alpha^2 + x^2}}_{f_\alpha} : A \to \mathbb{R} \;\Big|\; \alpha \ge 1 \Big\}.$$

   Show that B is equi-continuous.

4. Consider the space $C([0,1])$ of all real-valued continuous functions on $[0,1]$ endowed with the sup-norm (or uniform norm), i.e., $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$ for any $f \in C([0,1])$. Let $B \subset C([0,1])$ be

$$B = \Big\{ f \in C([0,1]) \,|\, f \text{ is differentiable on } [0,1],\ -1 \le f'(x) \le 2, \forall\, x \in [0,1],\ f(0) = 0 \Big\}$$

   (1) Show that $\mathrm{cl}(B)$ is bounded.

   (2) Show that $\mathrm{cl}(B)$ is equi-continuous and thus compact.

5. Let $\mathbb{R}_+ := \{x \in \mathbb{R} \,|\, x \ge 0\}$, and consider the sequence of functions $f_n : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} \frac{x}{n}, & \text{if } x \in [0, n] \\ 1, & \text{if } x \ge n \end{cases}$$

   Show that the sequence (f_n) is equi-continuous and bounded (with respect to the sup-norm defined before). Does the sequence (f_n) have a convergent subsequence (with respect to the sup-norm)? Is your answer contradictory to the Arzela-Ascoli Theorem? Explain why.

6. Let (f_n) be an equi-continuous sequence of functions $f_n : (M, d) \to \mathbb{R}$, where (M, d) is compact. Suppose that (f_n) converges pointwise to f_* on M. Show that (f_n) converges uniformly to f_* on M.

Other problems: *Do not submit*

1. Let A be an $n \times n$ real matrix, and the function $f : \mathbb{R}^n \to \mathbb{R}^n$ be $f(x) = (x^T Ax) \cdot x$, where x^T is the transpose of the vector $x \in \mathbb{R}^n$. Show that for any $x \in \mathbb{R}^n$ and any direction vector d, the directional derivative

$$f'(x; d) = (x^T Ad)x + (d^T Ax)x + (x^T Ax)d.$$

And determine the Frechet derivative of f at an arbitrary $x \in \mathbb{R}^n$.

2. Let A be a real $m \times n$ matrix and $b \in \mathbb{R}^m$ be a real vector, and define $f(x) := \|b - Ax\|_2^2 = (Ax - b)^T(Ax - b)$. Determine the Frechet derivative of f at an arbitrary $x \in \mathbb{R}^n$.

3. Consider the space $C([0, 1])$ of real-valued continuous functions on $[0, 1]$ endowed with the sup-norm $\| \cdot \|_\infty$. Let $h \in C([0, 1])$ be a fixed function. Define $F : C([0, 1]) \to C([0, 1])$ by

$$F(f)(x) = \int_0^x \Big[(f(t))^2 + h(t)f(t)\Big]\,dt, \qquad x \in [0, 1],$$

for all $f \in C[0, 1]$. Find the Frechet derivative $DF(f)$ at each $f \in C([0, 1])$, and prove your finding is indeed the Frechet derivative.