Math 600 Fall 2017 Homework #5

Due Nov. 1, Wed. in class

Note: For the Euclidean space $\mathbb{R}^n$, consider the usual metric induced by the Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^n$, unless otherwise stated.

1. Let $f : M \to N$ be a continuous function on M. Use the sequential argument *only* to show that if A is a sequentially compact set in M, then $f(A)$ is sequentially compact.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$. Which of the following four sets are necessarily open, closed, compact, and connected (no proof is needed)?

   (1) $\{x \in \mathbb{R} \,|\, f(x) = 0\}$; (2) $\{x \in \mathbb{R} \,|\, f(x) > 1\}$; (3) $\{f(x) \in \mathbb{R} \,|\, x \geq 0\}$; (4) $\{f(x) \in \mathbb{R} \,|\, 0 \leq x \leq 1\}$.

3. Let $\|\cdot\|_2$ be the standard Euclidean norm on $\mathbb{R}^n$, and $\|\cdot\|_\alpha$ be an arbitrary norm on $\mathbb{R}^n$. Recall that both $\|\cdot\|_2$ and $\|\cdot\|_\alpha$ are continuous on $(\mathbb{R}^n, \|\cdot\|_2)$.

   (1) Let $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$. Show that $\mathbb{S}^{n-1}$ is closed in $(\mathbb{R}^n, \|\cdot\|_2)$. (*Hint*: write $\mathbb{S}^{n-1}$ as a suitable inverse image.)

   (2) Show that $\mathbb{S}^{n-1}$ is compact in $(\mathbb{R}^n, \|\cdot\|_2)$.

   (3) Show that there exist $x^* \in \mathbb{S}^{n-1}$ and $y^* \in \mathbb{S}^{n-1}$ such that

   $$\|x^*\|_\alpha = \max_{x \in \mathbb{S}^{n-1}} \|x\|_\alpha, \qquad \|y^*\|_\alpha = \min_{x \in \mathbb{S}^{n-1}} \|x\|_\alpha.$$

   Further show that $\|x^*\|_\alpha > 0$ and $\|y^*\|_\alpha > 0$.

   (4) Let $c_1 := \|x^*\|_\alpha$ and $c_2 := \|y^*\|_\alpha$. Use (3) to show that

   $$c_2\|x\|_2 \leq \|x\|_\alpha \leq c_1\|x\|_2, \qquad \forall\, x \in \mathbb{R}^n.$$

   (*Hint*: for any $0 \neq x \in \mathbb{R}^n$, $\frac{x}{\|x\|_2} \in \mathbb{S}^{n-1}$.)

   (5) Let $\|\cdot\|_\beta$ be another norm on $\mathbb{R}^n$. Use (4) to show that there exist $\kappa_1, \kappa_2 > 0$ such that

   $$\kappa_2\|x\|_\beta \leq \|x\|_\alpha \leq \kappa_1\|x\|_\beta, \qquad \forall\, x \in \mathbb{R}^n.$$

   $\star$ *This result shows that all norms on $\mathbb{R}^n$ are equivalent.*

4. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous on $\mathbb{R}^n$. Define the function $g : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$ as:

   $$g(x) := (x, f(x)), \qquad \forall\, x \in \mathbb{R}^n.$$

   (1) Show g is continuous on $\mathbb{R}^n$. (*Hint*: consider the sequential criterion.)

   (2) For the function f given above, define the following set (called the *graph* of f):

   $$\mathcal{S} := \{(x, f(x)) \,|\, x \in \mathbb{R}^n\} \subseteq \mathbb{R}^n \times \mathbb{R}^m.$$

   Show that $\mathcal{S}$ is closed and connected.

5. Show that a metric space (M, d) is connected if and only if the only open *and* closed sets in M are M and the empty set.

Miscellaneous practice problems: *Do not submit*

1. Let $f : [a, b] \to [a, b]$ be continuous on $[a, b]$, where the real numbers $a < b$. Show that f has a fixed point on $[a, b]$, namely, there exists $c \in [a, b]$ such that $f(c) = c$.

2. Let $f, g : (M, d) \to \mathbb{R}$ be continuous on M. Then (i) the set $\{x \in M : f(x) = g(x)\}$ is closed in (M, d); and (ii) the set $\{x \in M : f(x) > 2g(x)\}$ is open in (M, d).