Math 302/600 Spring 2017 Homework #3, Update

Due Feb. 21, Tue. in class

Note: For any Euclidean space $\mathbb{R}^n$, consider the usual metric induced by the Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^n$, unless otherwise stated.

1. Consider the metric induced by the 1-norm on $\mathbb{R}^2$: $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$, $\forall\ x, y \in \mathbb{R}^2$. Let $A = \{x = (x_1, x_2) \in \mathbb{R}^2 \,|\, x_1 \geq 1 \text{ and } x_2 \leq 1\}$.
   (1) Find the interior of A using the given metric, and prove your answer;
   (2) Prove that the set A is closed via the definition, namely, show that the complement of A is open.
2. Let (M, d) be a metric space and A, B be two subsets of M. Show the following:
   (1) if $A \subseteq B$, then $\text{int}A \subseteq \text{int}B$;
   (2) $\text{int}(A \cap B) = (\text{int}A) \cap (\text{int}B)$.
3. Find all the accumulation points of each of the following sets (without proof):
   (1) $A = \{x = (x_1, x_2) \in \mathbb{R}^2 \,:\, 1 < x_1 \leq 2, \text{ and } x_2 < -2\}$ in $\mathbb{R}^2$;
   (2) $B = \{x = (\frac{1}{n}, 1 - \frac{1}{n}) \in \mathbb{R}^2 \,:\, n \in \mathbb{N}\} \cup \{(1,2)\}$ in $\mathbb{R}^2$;
   (3) $C = \{x = (x_1, x_2) \in \mathbb{R}^2 \,:\, x_1 \text{ is rational, and } x_2 = \frac{1}{n}, n \in \mathbb{N}\}$ in $\mathbb{R}^2$.
4. Given a metric space (M, d). Prove the following statements:
   (1) any convergent sequence in (M, d) has a unique limit;
   (2) any convergent sequence in (M, d) is Cauchy;
   (3) any Cauchy sequence in (M, d) is bounded.

The following extra problem(s) are for **Math 600** *students only:*

5. Let (M, d) be a metric space, and a point $c \in M$ is a *cluster point* of a sequence (x_n) in (M, d) if for any $\varepsilon > 0$, there are infinitely many x_n's in the sequence such that $d(c, x_n) < \varepsilon$.
   (1) Show that c is a cluster of a sequence (x_n) if and only if there is a subsequence of (x_n) that converges to c.
   (2) Show that a Cauchy sequence has at most one cluster point.