Math 302/600 Spring 2017 Homework #2

Due Feb. 14, Tue. in class

Note: For any Euclidean space $\mathbb{R}^n$, consider the usual metric induced by the Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^n$, unless otherwise stated.

1. Prove that the discrete metric d satisfies the triangle inequality, i.e., $d(x,y) \leq d(x,z) + d(y,z), \forall\, x,y,z \in M$.

2. Let d_1 and d_2 be two metrics on a set M. Define the sum $d_1 + d_2$ on M as

$$(d_1 + d_2)(x,y) := d_1(x,y) + d_2(x,y), \quad \forall\, x,y \in M.$$

   Show that $d_1 + d_2$ is a metric on M.

3. Let (M,d) be a metric space. Show the following inequality:

$$|d(x,y) - d(z,y)| \leq d(x,z), \quad \forall\, x,y,z \in M.$$

4. Let A be an open set in $\mathbb{R}$, and define the set $B := \{(x_1,x_2) \in \mathbb{R}^2 \,|\, x_1 \in A\} \subseteq \mathbb{R}^2$. Show that B is open in $\mathbb{R}^2$.

5. Let (M,d) be a metric space. Given a nonempty set $A \subseteq M$, let $B := \{x \in M \,|\, d(x,y) < 1 \text{ for some } y \in A\}$. Show that B is open. (*Hint*: write B as the union of open sets.)

6. Consider the metric induced by the 1-norm on $\mathbb{R}^2$: $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$, $\forall\, x,y \in \mathbb{R}^2$. Let $A = \{x = (x_1,x_2) \in \mathbb{R}^2 \,|\, x_1 \geq 1 \text{ and } x_2 \leq 1\}$. Find the interior of A using the given metric, and prove your answer.

The following extra problems are for **Math 600** *students only:*

7. Given a normed space $(V, \|\cdot\|)$, let A be a nonempty open set in V, and B be a nonempty set in V. Define $A + B := \{a + b \,|\, a \in A,\ b \in B\}$. Show that $A + B$ is open.

8. Let M be a set endowed with two metrics d_1 and d_2, namely, both (M,d_1) and (M,d_2) are metric spaces. Suppose that there exist two positive real numbers α and β such that

$$\beta d_1(x,y) \leq d_2(x,y) \leq \alpha d_1(x,y), \quad \forall\, x,y \in M.$$

   Show that a set $A \subseteq M$ is open with respect to d_1 if and only if A is open with respect to d_2.