Math 302 Fall 2011 Homework #7

Due Oct. 31, Mon. in class

1. In $(\mathbb{R}^n, \|\cdot\|)$, where $\|\cdot\|$ is an arbitrary norm on $\mathbb{R}^n$, show that any sequence in a bounded set of $\mathbb{R}^n$ has a subsequence that converges in $\mathbb{R}^n$. And illustrate by an example that the statement does not hold in a general metric space.

2. Use the definition of compactness (i.e. the open cover definition) to show that the following two sets are *not* compact, by exhibiting an open cover with no finite subcover:

   (1) the open ball $B(x, 1)$ centered at a fixed $x \in M$ with the radius 1 in the metric space (M, d);

   (2) the set $A = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1,\ y \geq 0\}$ (using the standard metric on $\mathbb{R}^2$).

3. Use the definition of compactness (i.e. the open cover definition) to show that the union of two (nonempty) compact sets is compact.

4. Which of the following sets are connected in $\mathbb{R}$:

$$\{3, -10\}, \quad [0, 1), \quad (-7, \infty), \quad \mathbb{Z}, \quad \mathbb{Q}, \quad \mathbb{R},$$

   where $\mathbb{Z}$ and $\mathbb{Q}$ are the sets of integers and rational numbers, respectively.

5. What are compact and connected sets in $\mathbb{R}$?