Math 302/401/600 Fall 2010 Homework #12

Due Dec. 8, Wed. in class

1. Textbook, Section 5.7, page 282, Exercises 1, 3.

2. Let $(V, \|\cdot\|)$ be a complete normed vector space and its induced metric $d(x, y) = \|x - y\|$ for $x, y \in V$. Let $f : V \to V$ be a *linear mapping/function*, i.e., $f(x + y) = f(x) + f(y), \forall\ x, y \in V$ and $f(\alpha x) = \alpha f(x)$ for all $x \in V$ and $\alpha \in \mathbb{R}$.

   (2.1) Show that $f(0) = 0$ and $f(x - y) = f(x) - f(y), \forall\ x, y \in V$. (You may assume $x - y = x + (-1)y$ for all $x, y \in V$.)

   (2.2) Show that f is a contraction if and only if there exists a constant C with $0 < C < 1$ such that $\|f(x)\| \leq C\|x\|$ for all $x \in V$.

   (2.3) Let $x_0 \in V$ be arbitrary and define the sequence (x_n) recursively by

   $$x_n = f(x_{n-1}), \quad n \in \mathbb{N}.$$

   Show that (x_n) converges to the zero vector in V.

3. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

   $$f(x, y) = \frac{K}{\sqrt{2}}\big(x + y,\ y - x\big), \quad \forall\ (x, y) \in \mathbb{R}^2,$$

   where the constant K satisfies $0 < K < 1$. In the following, you may use the results of Problem 2.

   (3.1) Show that f is a linear mapping.

   (3.2) Show that when the 2-norm (i.e., $\|\cdot\|_2$) is used, f is a contraction.

   (3.3) Show that when the 1-norm (i.e., $\|\cdot\|_1$) is used, f is *not* a contraction if $\frac{1}{\sqrt{2}} < K < 1$.

   (3.4) Let $(x_0, y_0) \in \mathbb{R}^2$ be arbitrary. Define the sequence (x_n, y_n) recursively by

   $$(x_n, y_n) = f(x_{n-1}, y_{n-1}), \quad n \in \mathbb{N}.$$

   Explain why the sequence (x_n, y_n) is convergent when the 2-norm is used. (*Note:* recall that $(\mathbb{R}^2, \|\cdot\|_2)$ is complete.)

   (3.5) Show that the sequence defined in (3.4) is convergent when the 1-norm is used. (*Hint:* use the equivalence of norms on a Euclidean space shown in Problem 3 of Homework 7.)

   $\star$ This example shows that the contractive property is a *sufficient* condition for convergence but not a necessary one.