Positive Invariance of Constrained Affine Dynamics and Its Applications to Hybrid Systems and Safety Verification

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Abstract

Motivated by long-time dynamic analysis of hybrid systems and safety verification problems, this paper addresses fundamental positive invariance issues of an affine dynamical system on a general polyhedron and their applications. Necessary and sufficient algebraic conditions are established for the existence of a positively invariant set of an affine system on a polyhedron using the tools of lexicographic relation and long-time oscillatory dynamic analysis. A linear program based algorithm is proposed to verify these conditions and its computational complexity is analyzed. The positive invariance results are applied to obtain an explicit characterization of global switching behaviors of piecewise affine systems. In particular, the PASs with isolated equilibria and infinite mode switchings are characterized via the positive invariance conditions. The positive invariance techniques developed in this paper are also exploited to show decidability of safety verification of a class of affine dynamics on semialgebraic sets.

I. INTRODUCTION

Modeling, control, and computation of hybrid dynamical systems has received a fast growing interest, driven by various important applications such as air traffic control, nonsmooth mechanical/electrical systems, embedded systems, dynamical optimization, and systems biology. A hybrid ODE system consists of a family of ODEs with mode transitions occurring along state trajectories following switching rules. As a distinct and intrinsic feature of hybrid dynamical systems, mode switching complicates various fundamental issues, e.g. well-posedness and Zeno behavior [23]. Other challenging issues pertain to long-time dynamic properties, e.g., stability, reachability/safety, and long-time system properties such as observability, which have found a wide range of applications [16].

Inspired by the study of global and long-time dynamics of hybrid systems, the present paper investigates positive invariance of an ODE system on a constraint set. The concept of positive invariance is essential in asymptotic analysis of unconstrained smooth dynamical systems and Lyapunov stability theory. For example, if a trajectory is contained in a compact set for all positive times, then its positive limit set is

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nonempty, compact, connected, and positively invariant [32, Proposition 1.1.14]. A prominent application of this result is in the proof of LaSalle's invariance principle [17]. Also see the excellent survey paper [6] for a variety of applications in systems and control. In the realm of hybrid systems, it has been recently recognized that positive invariance of each mode of a hybrid system plays a crucial role in addressing important long-time dynamic issues [27]. Positive invariance results also shed light on reachability analysis and safety verification with interesting applications in engineering [5] and systems biology [2].

Given an ODE system Σ and a nonempty constraint set $\mathcal{S}$, two related but different positive invariance problems naturally arise: (1) find conditions on Σ and $\mathcal{S}$ such that $\mathcal{S}$ is a positively invariant set of Σ; (2) characterize the (largest) positively invariant set of Σ contained in $\mathcal{S}$, where specific issues include the existence and algebraic/geometric properties of such a set. The first problem bears a larger body of the literature, partly because it is more tractable and more relevant to control synthesis. Loosely speaking, this problem can be solved by imposing inward-pointing conditions on the vector field of Σ on the boundary of $\mathcal{S}$. Such conditions usually lead to verifiable algebraic results together with algebraic structure of $\mathcal{S}$; see [6], [10] for details and [2], [5], [29] for applications. Extensions to differential inclusions with set-valued right-hand sides can be found in [4].

Being more analytic in nature, the second positive invariance problem has received relatively less attention so far, in spite of equally important applications in hybrid systems. One exception is [16, Therorem 3.1], which provides a neat necessary and sufficient existence condition for affine dynamics on a polytope; its proof is based on an elegant use of the topological fixed point theorem and the convex structure. However, if the boundedness assumption is dropped, then characterization of the existence of a positively invariant set becomes nontrivial, even for relatively simpler dynamics and constraint sets. In this paper, we focus on affine dynamics on a (possibly unbounded) polyhedron. While this problem can be formulated as an infinite dimensional linear programming and tackled from a topological perspective [13, Theorems 12-13] and [3], no finitely verifiable conditions are given. Combining results of long-time linear dynamic analysis and lexicographical relation, we develop verifiable necessary and sufficient existence conditions and an algorithm for numerical verification. These results are applied for explicit characterization of global switching properties of piecewise affine hybrid systems with each mode defined on a possibly unbounded polyhedral set. Moreover, we exploit positive invariance techniques to show decidability of exact safety verification problems for a class of affine dynamics on semi-algebraic sets.

The rest of the paper is organized as follows. In Section II, we define the positive invariance and safety verification problems and present preliminary technical results. In Section III, a detailed development of the necessary and sufficient algebraic conditions for the existence of a positively invariant set on a polyhedron is given; computational issues are discussed. Section IV focuses on global switching properties

of piecewise affine hybrid systems, for which verifiable algebraic conditions are established using the positive invariance results. In section V, decidability analysis is performed for safety verification of affine systems with the aid of the long-time analysis techniques developed in Sections II and III. The paper concludes with final remarks and discussions on future research directions in Section VI.

II. PROBLEM FORMULATION AND PRELIMINARIES

A. Positive Invariance and Safety Verification

Consider an ODE system on $\mathbb{R}^n$: $\dot{x} = f(x)$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be globally Lipschitz continuous. Let $x(t, x^0)$ denote the unique C^1 trajectory corresponding to an initial state $x^0 \in \mathbb{R}^n$.

Definition 2.1: [32] A set $\mathcal{N} \subseteq \mathbb{R}^n$ is called *positively invariant* if the following implication holds: $x^0 \in \mathcal{N} \implies x(t, x^0) \in \mathcal{N}, \, \forall \, t \geq 0$. The set $\mathcal{N}$ is called *invariant* if $x^0 \in \mathcal{N} \implies x(t, x^0) \in \mathcal{N}, \forall \, t \in \mathbb{R}$.

Two problems emerge from many applied dynamical systems:

- Positive invariance analysis (of the second kind): given a set $\mathcal{S}$ in $\mathbb{R}^n$, characterize the subset $\mathcal{A} \subseteq \mathcal{S}$ that consists of all the x^0 satisfying $x^0 \in \mathcal{A} \implies x(t, x^0) \in \mathcal{S}, \, \forall \, t \geq 0$.

- Safety verification on a time interval $\Delta \subseteq \mathbb{R}$: given two sets $\mathcal{S}_0$ and $\mathcal{S}_f$ in $\mathbb{R}^n$ such that $\mathcal{S}_0 \subseteq \mathcal{S}_f$, determine if the following implication holds: $x^0 \in \mathcal{S}_0 \implies x(t, x^0) \in \mathcal{S}_f, \, \forall \, t \in \Delta$. Typical time intervals include $\mathbb{R}$ and $\mathbb{R}_+ \equiv [0, \infty)$.

In the second problem, $\mathcal{S}_f$ represents the safe region and the implication means that any trajectory starting from the initial set $\mathcal{S}_0$ will not leave $\mathcal{S}_f$ at any time in Δ, and thus remains *safe* from $\mathcal{S}_0$ on Δ. The safety verification problem can be treated as a positive invariance problem. For example, the dynamics is safe on $\Delta = \mathbb{R}$ if and only if $\mathcal{S}_0$ is contained in $\mathcal{A} \cap \mathcal{A}'$, where $\mathcal{A}$ and $\mathcal{A}'$ are the largest positively invariant sets of $\mathcal{S}_f$ with respect to $\dot{x} = f(x)$ and $\dot{x} = -f(x)$, respectively.

We mention some basic facts. If $\mathcal{S}$ is closed and its positively invariant set $\mathcal{A}$ is nonempty, then $\mathcal{A}$ is closed. Moreover, if f is affine and $\mathcal{S}$ is convex, then $\mathcal{A}$ is convex. Hence, if f is affine and $\mathcal{S}$ is a closed convex polyhedron, then $\mathcal{A}$ is closed and convex, albeit not necessarily polyhedral, upon its existence. If $\mathcal{S}$ is additionally bounded (i.e., $\mathcal{S}$ is a polytope), then $\mathcal{A}$ is compact and convex.

Remark 2.1: An explicit characterization of a general positively invariant set is usually unavailable or does not yield efficient or even finite computation, except for very simple dynamics on a lower dimensional space; see planar examples in Figure 1. On the other hand, many analysis and design problems only require certain algebraic/geometric properties of a positively invariant set. The characterization of these properties may be reduced to easily verifiable algebraic conditions and yield efficient computation; see, for instance, Theorem 3.1 and Section III-D. More examples are given in [26]. A primary goal of this paper is to establish equivalent positive invariance conditions that attain efficient numerical resolution.

B. Preliminary Technical Results

Two key techniques are exploited for the positive invariance analysis through this paper: long-time dynamic analysis of oscillatory dynamics and lexicographic relation.

1) Long-time dynamic analysis of oscillatory modes: The following results provide major tools to deal with oscillatory modes (corresponding to complex eigenvalues) in positive invariance analysis.

Lemma 2.1: [27, Lemma 13] Given (finitely many) continuous periodic functions $g_i : \mathbb{R} \to [a_i, b_i]$ with frequency $\omega_i > 0$, where $[a_i, b_i] \subseteq \mathbb{R}$ and $i = 1, \cdots, m$. Assume that each g_i is onto $[a_i, b_i]$ and the frequency ratio ω_i / ω_j is irrational for $i \neq j$. Then for any given $\widetilde{y} \in [a_1, b_1] \times \cdots \times [a_m, b_m]$ and any scalar $\varepsilon > 0$, there is a $\widetilde{t} \geq 0$ such that $\| \widetilde{y} - \big(g_1(\widetilde{t}), \cdots, g_m(\widetilde{t}) \big) \| \leq \varepsilon$.

The next result states that a nontrivial linear combination of sinusoidal functions has persistent sign alternating and its positive/negative variations are not diminishing as time evolves.

Lemma 2.2: [27, Corollary 15] Let $f : \mathbb{R} \to \mathbb{R}$ be $f(t) \equiv \sum_{i=1}^{m} \big[\alpha_i \cos(\omega_i t) + \beta_i \sin(\omega_i t) \big]$, where $\omega_i > 0$, $\omega_i \neq \omega_j$ whenever $i \neq j$, and $|\alpha_i| + |\beta_i| \neq 0$ for all i. Then there exist two scalars $\gamma_1 > 0$ and $\gamma_2 < 0$ such that for any t_*, two time instants $t_1, t_2 \in [t_*, \infty)$ exist satisfying $f(t_1) \geq \gamma_1$ and $f(t_2) \leq \gamma_2$.

Corollary 2.1: Let $v^i \in \mathbb{R}^m$ and $\omega_i > 0$, θ_i be given, where $i = 1, \cdots, \ell$ and $\omega_i \neq \omega_j$ whenever $i \neq j$. Then $\sum_{i=1}^{\ell} v^i \sin(\omega_i t + \theta_i) \geq 0$ for all $t \geq 0$ sufficiently large if and only if $v^i = 0$ for all $i = 1, \cdots, \ell$.

We discuss more properties of $f(t) \equiv \sum_{i=1}^{m} \big[\alpha_i \cos(\omega_i t) + \beta_i \sin(\omega_i t) \big]$. For notational simplicity, let $d_i(t) \equiv \alpha_i \cos(\omega_i t) + \beta_i \sin(\omega_i t)$. We obtain the collection of (disjoint and distinct) equivalent classes $E_{\omega_j} = \{ d_i(t) \mid \omega_i / \omega_j \text{ is rational} \}$ as shown in [27, Lemma 14]. Note that each equivalent class E_{ω_j} attains a basis frequency $\widetilde{\omega}_s > 0$, namely, $\omega_i / \widetilde{\omega}_s$ is a positive integer for any frequency ω_i associated with the function $d_i(t) \in E_{\omega_j}$. Let $E_{\widetilde{\omega}_s}$ denote the equivalent class and let $q_{\widetilde{\omega}_s}(t) \equiv \sum_{d_i \in E_{\widetilde{\omega}_s}} d_i(t)$. Then the following hold: (1) $q_{\widetilde{\omega}_s}(\cdot)$ is a real-valued smooth and periodic function with the frequency $\widetilde{\omega}_s$; (2) if $q_{\widetilde{\omega}_s}(\cdot)$ is not identically zero, then it attains the maximal and minimal values $\sigma_{\widetilde{\omega}_s} > 0$ and $\nu_{\widetilde{\omega}_s} < 0$ on $(-\infty, \infty)$ respectively; (3) $q_{\widetilde{\omega}_s}(\cdot)$ is onto $[\nu_{\widetilde{\omega}_s}, \ \sigma_{\widetilde{\omega}_s}]$; and (4) the ratio of two basis frequencies associated with two distinct equivalent classes is irrational. Suppose there are k equivalent classes $E_{\widetilde{\omega}_s}$ and thus $f(t) = \sum_{s=1}^{k} q_{\widetilde{\omega}_s}(t)$. While each $q_{\widetilde{\omega}_s}$ is periodic, f is generally not and hence may not attain its maximum and minimum on $\mathbb{R}$. In spite of this, the following lemma shows that its supremum (resp. infimum) is the sum of the maxima (resp. minima) of $q_{\widetilde{\omega}_s}$'s. This observation is essential to decidability of safety verification treated in Section V.

Lemma 2.3: Let $\sigma_{\widetilde{\omega}_s}$ and $\nu_{\widetilde{\omega}_s}$ be defined above for the function f. Then $\sup_{[t_*, \infty)} f(t) = \sum_{s=1}^{k} \sigma_{\widetilde{\omega}_s}$ and $\inf_{[t_*, \infty)} f(t) = \sum_{s=1}^{k} \nu_{\widetilde{\omega}_s}$ for any $t_* \in \mathbb{R}$.

Proof. It is clear that $\sum_{s=1}^{k} \sigma_{\widetilde{\omega}_s}$ and $\sum_{s=1}^{k} \nu_{\widetilde{\omega}_s}$ are upper and lower bounds of f, respectively. To show $\sum_{s=1}^{k} \sigma_{\widetilde{\omega}_s}$ is the least upper bound, we assume that each $q_{\widetilde{\omega}_s}$ is not identically zero without loss of generality. Hence, by Lemma 2.1 and the properties (1)-(4) stated above, we see that for any $\varepsilon > 0$ sufficiently small, there is a $\widetilde{t} \geq t_*$ such that $q_{\widetilde{\omega}_s}(\widetilde{t}) \in (\sigma_{\widetilde{\omega}_s} - \varepsilon/k, \sigma_{\widetilde{\omega}_s}]$ for all $s = 1, \cdots, k$. Therefore, $(\sum_{s=1}^{k} \sigma_{\widetilde{\omega}_s} - \varepsilon)$ is not an upper bound of f. Hence, $\sum_{s=1}^{k} \sigma_{\widetilde{\omega}_s}$ is the least upper bound. Similarly, $\sum_{s=1}^{k} \nu_{\widetilde{\omega}_s}$ is the greatest lower bound of f for any t_*. $\square$

It is worth noticing that Lemma 2.3 further implies: (1) $\sum_{s=1}^{k} \nu_{\widetilde{\omega}_s} = \inf_{(-\infty,\infty)} f(t)$, and (2) $f(t) \geq \rho, \forall\, t$ for a scalar ρ if and only if $\sum_{s=1}^{k} \nu_{\widetilde{\omega}_s} \geq \rho$.

2) Lexicographic relation: An ordered real ℓ-tuple $a = (a_1, \cdots, a_\ell)$ is called lexicographically nonnegative if either $a = 0$ or its first nonzero element (from the left) is positive and we write $a \succcurlyeq 0$. In the latter case, we call the first positive element the positive leading term/entry of the tuple. If a is not only lexicographically nonnegative but also nonzero, then a is called lexicographically positive and we write $a \succ 0$. For two tuples a and b, we write $a \succcurlyeq b$ if $(a - b) \succcurlyeq 0$. Hence, the lexicographical relation defines a total order on the vector space of ℓ-tuples. An n-dimensional vector tuple $(x^1, \cdots, x^\ell)$ is called lexicographically nonnegative (resp. positive) if each real tuple $(x_i^1, \cdots, x_i^\ell)$ is lexicographically nonnegative (resp. positive) for all $i = 1, \cdots, n$ and we write $(x^1, \cdots, x^\ell) \succcurlyeq (\succ)0$. In what follows, we let $\mathbb{R}_+^n$ (resp. $\mathbb{R}_{++}^n$) be the nonnegative (resp. positive) orthant of $\mathbb{R}^n$.

Lemma 2.4: Let $E : \mathbb{R}^n \to \mathbb{R}^{m_1}$ and $F : \mathbb{R}^n \to \mathbb{R}^{m_2}$ be two generalized positively homogeneous functions, i.e., $E_i(\lambda x) = p_i(\lambda) E_i(x)$, $F_j(\lambda x) = p_j(\lambda) F_j(x)$, $\forall\, x \in \mathbb{R}^n$, $\forall\, \alpha \in \mathbb{R}_+$, $\forall\, i, j$, where $p_i, p_j : \mathbb{R}_+ \to \mathbb{R}_+$ are bijective and strictly increasing. Suppose there exist $x^1, \cdots, x^k \in \mathbb{R}^n$ such that $(E(x^1), \cdots, E(x^k)) \succ 0$ and $(F(x^1), \cdots, F(x^k)) \succcurlyeq 0$. Then for any $a \in \mathbb{R}_{++}^{m_1}$, there exist nonnegative scalars $\mu_1, \cdots, \mu_k$ such that the positive leading terms $E_i(\mu_\ell\, x^\ell)$ and $F_s(\mu_p\, x^p)$ of any row of $(E(x^1), \cdots, E(x^k))$ and $(F(x^1), \cdots, F(x^k))$ satisfy $E_i(\mu_\ell\, x^\ell) \geq \left(\sum_{j=\ell+1}^{k} |E_i(\mu_j\, x^j)| \right) + a_i$ and

$F_s(\mu_p\, x^p) \geq \left(\sum_{j=p+1}^{k} |F_s(\mu_j\, x^j)| \right)$, respectively.

Proof. We assume, via suitable row switchings, that both $(E(x^1), \cdots, E(x^k))$ and $(F(x^1), \cdots, F(x^k))$ are of echelon form, namely, the nonzero row(s) are above all the zero rows and the positive leading term of a row is either at the same column or to the right of any leading term above it. We also assume that each column $z^i \equiv \begin{pmatrix} E(x^i) \\ F(x^i) \end{pmatrix} \in \mathbb{R}^{m_1+m_2}$ contains at least one positive leading term of some row, since otherwise we can simply drop such a column (or equivalently by choosing $\mu_i = 0$) without affecting the conclusion of the lemma. To find the desired μ_i's, define $\theta_i \subseteq \{1, \cdots, m_1 + m_2\}$ to be the set of the indices of the positive leading terms in z^i for each $i = 1, \cdots, k$. Clearly, θ_i is nonempty and

$z_{\theta_i}^i > 0$ for each i. Let $\bar{a} = (a^T, 0)^T$. We now start from the rightmost column, i.e., z^k, and choose $\mu_k := \max\limits_{j \in \theta_k} p_j^{-1}\left(\frac{\bar{a}_j}{z_j^k}\right)$, where p_j^{-1} is the inverse function of p_j. It is clear that $p_j^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$ is also bijective and strictly increasing. Hence, for any $j \in \theta_k$ corresponding to $E_j(x^k)$, we have

$$E_j(\mu_k x^k) = p_j(\mu_k)E_j(x^k) \geq \left[p_j \circ p_j^{-1}\left(\frac{\bar{a}_j}{z_j^k}\right)\right]E_j(x^k) \geq \bar{a}_j,$$

where we use $z_j^k \equiv E_j(x^k) > 0$ and the construction of μ_k. This also holds for $j \in \theta_k$ corresponding to $F_j(x^k)$. For $i = 1, \cdots, k-1$, we then recursively compute μ_i (from $i = k-1$ to $i = 1$) as

$$\mu_i := \max_{j \in \theta_i} p_j^{-1}\left(\frac{\sum\limits_{\ell=i+1}^{k} p_j(\mu_\ell)|z_j^\ell| + \bar{a}_j}{z_j^i}\right) \tag{1}$$

Notice that $\sum\limits_{\ell=i+1}^{k} p_j(\mu_\ell)|z_j^\ell|$ equals to $\sum\limits_{\ell=i+1}^{k} |E_j(\mu_\ell x^\ell)|$ or $\sum\limits_{\ell=i+1}^{k} |F_j(\mu_\ell x^\ell)|$ for any $j \in \theta_i$. It follows from the echelon structure and an induction argument that these μ_i's are the desired ones. $\square$

Corollary 2.2: Let E be the generalized positively homogeneous function defined above and suppose $\left(E(z^\ell), \cdots, E(z^0)\right) \succ 0$ for $z^0, \cdots, z^\ell \in \mathbb{R}^n$. Then there exist nonnegative scalars $\mu_1, \cdots, \mu_\ell$ such that $E(z^0) + \sum\limits_{i=1}^{\ell} E(\mu_i z^i) > 0$ and $E(z^j) + \sum\limits_{i=j+1}^{\ell} E(\mu_{i-j} z^i) \geq 0$ for all $j = 1, \cdots, \ell-1$. In particular, if $\left(z^\ell, \cdots, z^0\right) \succ 0$ holds, then there exist nonnegative scalars $\mu_1, \cdots, \mu_\ell$ such that $z^0 + \sum\limits_{i=1}^{\ell} \mu_i z^i > 0$ and $z^j + \sum\limits_{i=j+1}^{\ell} \mu_{i-j} z^i \geq 0$ for all $j = 1, \cdots, \ell-1$.

Proof. Let $y^i \equiv E(z^i)$. Since $\left(y^\ell, \cdots, y^0\right) \succ 0$, $\left(y^\ell, \cdots, y^i\right) \succcurlyeq 0$ for all $i = 1, \cdots, \ell$. Hence,

$$\begin{pmatrix} y^\ell & y^{\ell-1} & \cdots & y^1 & y^0 \end{pmatrix} \succ 0$$

$$\begin{pmatrix} & y^\ell & \cdots & y^2 & y^1 \\ & & \ddots & \vdots & \vdots \\ & & & y^\ell & y^{\ell-1} \\ & & & & y^\ell \end{pmatrix} \succcurlyeq 0$$

We also assume, via suitable row switchings, that the above tuple is of the echelon form as in the previous lemma. Letting $\mu_0 \equiv 1$, which corresponds to the rightmost column, and applying Lemma 2.4, we obtain nonnegative reals $\mu_1, \cdots, \mu_\ell$ such that the positive leading term y_j^s in the jth row of $\left(y^\ell, \cdots, y^0\right)$ satisfies $E_j(\mu_s z^s) > \sum\limits_{i=0}^{s-1} |E_j(\mu_i z^i)|$ and that the positive leading term y_k^s in the kth row of $\left(y^\ell, \cdots, y^p\right)$ satisfies $E_k(\mu_{s-p} z^s) \geq \sum\limits_{i=p}^{s-1} |E_k(\mu_{i-p} z^i)|$, where $p \in \{1, \cdots, \ell-1\}$. Noting that $\mu_0 \equiv 1$ and $E_j(z^r) \equiv 0$ for all $r > s$, we have $E_j(z^0) + \sum\limits_{i=1}^{\ell} E_j(\mu_i z^i) = E_j(\mu_s z^s) + \sum\limits_{i=0}^{s-1} E_j(\mu_i z^i) > \sum\limits_{i=0}^{s-1} |E_j(\mu_i z^i)| + \sum\limits_{i=0}^{s-1} E_j(\mu_i z^i) \geq 0$. Hence, $E(z^0) + \sum\limits_{i=1}^{\ell} E(\mu_i z^i) > 0$. Similarly, we can show $E(z^j) + \sum\limits_{i=j+1}^{\ell} E(\mu_{i-j} z^i) \geq 0$ for all $j = 1, \cdots, \ell-1$. Letting E be the identify mapping, we obtain the special case stated in the corollary. $\square$

III. Positive Invariance of Affine Dynamics on a Polyhedron

In this section, we are concerned with fundamental positive invariance issues of affine dynamics on a general polyhedron. Specifically, we address the existence of a positively invariant set and its numerical verification. Throughout this section, let the affine dynamics be $\dot{x} = Ax + d$ and the nonempty polyhedron be $\mathcal{P} = \{x \in \mathbb{R}^n \mid Cx \geq b\}$, where $A \in \mathbb{R}^{n \times n}$, $d \in \mathbb{R}^n$, $C \in \mathbb{R}^{m \times n}$, and $b \in \mathbb{R}^m$.

A. Positive Invariance of Linear Dynamics on a Polyhedron

To ease the presentation, we focus on linear dynamics first, i.e., $d = 0$. Without loss of generality, we assume throughout this section that the matrix A is of the real Jordan canonical form. Let J_{ij} be the jth real Jordan block associated with a possibly complex eigenvalue λ_i of A. Then A is the direct sum of J_{ij}'s and we write $A = \bigoplus_{i,j} J_{ij}$. For each J_{ij}, let C^{ij} denote the corresponding block in C. Let n_{ij} be the order of J_{ij} and $\bar{n}_i = \max_j n_{ij}$. For each real eigenvalue λ_i of A, suppose there are ℓ_i real Jordan blocks J_{ij}. For the jth Jordan block $J_{ij} \in \mathbb{R}^{n_{ij} \times n_{ij}}$, let the corresponding n-vector be of the form $v = \left(0, \cdots, 0, (v^{ij})^T, 0, \cdots, 0\right)^T$, where $v^{ij} \in \mathbb{R}^{n_{ij}}$. The collection of such the vectors, denoted by V_{ij}, is a subspace of $\mathbb{R}^n$ isomorphic to $\mathbb{R}^{n_{ij}}$. The direct sum of all the subspaces V_{ij}'s is a subspace of $\mathbb{R}^n$ given by $\left\{ \left(0, \cdots, 0, (v^{i1})^T, \cdots, (v^{i\ell_i})^T, 0, \cdots, 0\right)^T \in \mathbb{R}^n \mid v^{ij} \in \mathbb{R}^{n_{ij}}, \ j = 1, \cdots, \ell_i \right\}$ and is isomorphic to $\mathbb{R}^{n_{i1} + \cdots + n_{i\ell_i}}$, denoted by $V(\lambda_i) = \bigoplus_{j=1}^{\ell_i} \mathbb{R}^{n_{ij}}$. Each vector $v^i \in V(\lambda_i)$ is formed by vector stacking, i.e., $v^i = \left((v^{i1})^T, \cdots, (v^{i\ell_i})^T\right)^T$, where $v^{ij} \in \mathbb{R}^{n_{ij}}$. For notational convenience, we write $v^i = \bigoplus_j v^{ij}$. For a vector $u = (u_1, \cdots, u_\ell)^T$ in $\mathbb{R}^\ell$, we define the lifting operator $\mathcal{L}(u) := \left(u_2, \cdots, u_\ell, 0\right)^T$. Let $\mathcal{L}^k$ be the composition of k-copies of $\mathcal{L}$. By convention, we let $\mathcal{L}^0$ be the identity mapping, i.e., $\mathcal{L}^0(u) \equiv u$, and $\mathcal{L}^{-1}(u) \equiv 0, \forall u$. It is easy to verify that $\mathcal{L}$ is a linear operator and for any $u \in \mathbb{R}^\ell$, $\mathcal{L}^k(u) \equiv 0$ if $k \geq \ell$. The lifting operator $\mathcal{L}$ provides a compact way to express the solution of a linear dynamics defined by a Jordan canonical form. Indeed, for any Jordan block J_{ij} associated with a real eigenvalue λ_i and any vector $v^{ij} \in \mathbb{R}^{n_{ij}}$, we have $e^{J_{ij}t} v^{ij} = e^{\lambda_i t} \sum_{k=0}^{n_{ij}-1} \frac{t^k}{k!} \mathcal{L}^k(v^{ij})$. Let $C^i = \begin{bmatrix} C^{i1}, \cdots, C^{i\ell_i} \end{bmatrix}$ and $J_i = \bigoplus_j J_{ij}$. Similarly, for any $v^i \equiv \bigoplus_j v^{ij} \in V(\lambda_i)$, we have

$$C^i e^{J_i t} v^i = \sum_j C^{ij} e^{J_{ij} t} v^{ij} = e^{\lambda_i t} \sum_{k=0}^{\bar{n}_i - 1} \frac{t^k}{k!} \left(\sum_j C^{ij} \mathcal{L}^k(v^{ij}) \right), \tag{2}$$

where we use $\bar{n}_i = \max_j n_{ij}$ and $\mathcal{L}^k(v^{ij}) = 0$ once $k \geq n_{ij}$. For such a $v^i \in V(\lambda_i)$ and $\theta \subseteq \{1, \cdots, m\}$ whose cardinality is denoted by $|\theta|$, we define the following $|\theta|$-dimensional vector tuple

$$P(C_\theta^i, v^i) \equiv \left(\sum_j C_\theta^{ij} \mathcal{L}^{\bar{n}_i - 1}(v^{ij}), \sum_j C_\theta^{ij} \mathcal{L}^{\bar{n}_i - 2}(v^{ij}), \cdots, \sum_j C_\theta^{ij} \mathcal{L}^0(v^{ij}) \right) \tag{3}$$

Proposition 3.1: For a pair (C^i, J_i) associated with a nonnegative real eigenvalue λ_i of A, if $v^i \equiv \bigoplus_j v^{ij} \in V(\lambda_i)$ and the index subsets $\theta, \theta' \subseteq \{1, \cdots, m\}$ exist such that $P(C_\theta^i, v^i) \succ 0$ and $P(C_{\theta'}^i, v^i) \equiv$

0, then for any $a \in \mathbb{R}_{++}^{|\theta|}$, there exists $u^i \in V(\lambda_i)$ such that $C_\theta^i e^{J_i t} u^i \geq e^{\lambda_i t} a \geq a$ and $C_{\theta'}^i e^{J_i t} u^i \equiv 0, \ \forall \ t \geq 0$.

Proof. For notational convenience, let $z^k \equiv \sum_j C_\theta^{ij} \mathcal{L}^k(v^{ij})$ where $k = 0, \cdots, \bar{n}_i - 1$. Therefore, we have $P(C_\theta^i, v^i) = \left(z^{\bar{n}_i - 1}, \cdots, z^0 \right)$ such that $z^{\bar{n}_i - 1} \geq 0$ and $\left(z^{\bar{n}_i - 1}, \cdots, z^0 \right) \succ 0$. For the given $v^i = \bigoplus_j v^{ij}$, define $u^{ij} \equiv \sum_{s=0}^{\bar{n}_i - 1} \mu_s \mathcal{L}^s(v^{ij})$, where $\mu_0 \equiv 1$ and the nonnegative real numbers $\mu_1, \cdots, \mu_{\bar{n}_i - 1}$ are to be determined. Let $u^i \equiv \bigoplus_j u^{ij} \in V(\lambda_i)$. Hence,

$$
\begin{aligned}
C_\theta^i e^{J_i t} u^i &= e^{\lambda_i t} \sum_{k=0}^{\bar{n}_i - 1} \frac{t^k}{k!} \left(\sum_j C_\theta^{ij} \mathcal{L}^k(u^{ij}) \right) = e^{\lambda_i t} \sum_{k=0}^{\bar{n}_i - 1} \frac{t^k}{k!} \left[\sum_j C_\theta^{ij} \left(\sum_{s=0}^{\bar{n}_i - 1} \mu_s \mathcal{L}^{k+s}(v^{ij}) \right) \right] \\
&= e^{\lambda_i t} \sum_{k=0}^{\bar{n}_i - 1} \frac{t^k}{k!} \left[\sum_j C_\theta^{ij} \left(\sum_{s=0}^{\bar{n}_i - 1 - k} \mu_s \mathcal{L}^{k+s}(v^{ij}) \right) \right] \\
&= e^{\lambda_i t} \sum_{k=0}^{\bar{n}_i - 1} \frac{t^k}{k!} \left[\sum_{s=0}^{\bar{n}_i - 1 - k} \mu_s \sum_j C_\theta^{ij} \mathcal{L}^{k+s}(v^{ij}) \right] = e^{\lambda_i t} \sum_{k=0}^{\bar{n}_i - 1} \frac{t^k}{k!} \left[\sum_{p=k}^{\bar{n}_i - 1} \mu_{p-k} z^p \right]
\end{aligned}
$$

where $p \equiv k + s$, and we use the facts that the operator $\mathcal{L}$ is linear and that $\mathcal{L}^{k+s}(u^{ij}) = 0$ if $k + s \geq \bar{n}_i - 1$ for all j. Furthermore, by using Corollary 2.2 and recalling $\mu_0 \equiv 1$ and $z^{\bar{n}_i - 1} \geq 0$, we deduce that there exist nonnegative reals μ_p, $p = 1, \cdots, \bar{n}_i - 1$ such that $\sum_{p=0}^{\bar{n}_i - 1} \mu_p z^p > 0$ and $\sum_{p=k}^{\bar{n}_i - 1} \mu_{p-k} z^p \geq 0$ for all $k = 1, \cdots, \bar{n}_i - 1$. In particular, we can further positively scale μ_p's for each $p \geq 0$ such that $\sum_{p=0}^{\bar{n}_i - 1} \mu_p z^p \geq a$. These results, together with $\lambda_i \geq 0$, yield $C_\theta^i e^{J_i t} u^i = e^{\lambda_i t} \sum_{k=0}^{\bar{n}_i - 1} \frac{t^k}{k!} \left[\sum_{p=k}^{\bar{n}_i - 1} \mu_{p-k} z^p \right] \geq e^{\lambda_i t} \sum_{p=0}^{\bar{n}_i - 1} \mu_p z^p \geq e^{\lambda_i t} a \geq a$ for all $t \geq 0$ as desired. Similarly, letting $\widetilde{z}^p \equiv \sum_j C_{\theta'}^{ij} \mathcal{L}^p(v^{ij})$, we have $C_{\theta'}^i e^{J_i t} u^i = e^{\lambda_i t} \sum_{k=0}^{\bar{n}_i - 1} \frac{t^k}{k!} \left[\sum_{p=k}^{\bar{n}_i - 1} \mu_{p-k} \widetilde{z}^p \right]$. Since $P(C_{\theta'}^i, v^i) = \left(\widetilde{z}^{\bar{n}_i - 1}, \cdots, \widetilde{z}^0 \right)$ and $P(C_{\theta'}^i, v^i) \equiv 0$, $C_{\theta'}^i e^{J_i t} u^i \equiv 0$ for all t. $\square$

Remark 3.1: For the vector tuple $\left(z^{\bar{n}_i - 1}, \cdots, z^0 \right)$ in the above proof, let the index set ϕ consist of the indices corresponding to the positive leading terms of the tuple in the last column z^0, i.e., $\phi \equiv \{ j \mid z_j^0$ is the positive leading term in the tuple $\}$. If ϕ is nonempty, then the lexicographical relation of z^p's shows that $z_\phi^p = 0$ for all $p \geq 1$ and thus $C_\phi^i u^i = \sum_{p=0}^{\bar{n}_i - 1} \mu_p z_\phi^p = \mu_0 z_\phi^0$. Moreover, if $z_\phi^0 \geq a_\phi > 0$, then it is easy to see that $\mu_0 \equiv 1$ needs not be positively scaled in order for $\sum_{p=0}^{\bar{n}_i - 1} \mu_p z^p \geq a$ (but other μ_p's may need). That is to say, μ_0 can always be chosen as one in this case. These observations will be used in the proof of Theorem 3.1. By convention, we assume that $P(C_\theta^i, v^i) \succcurlyeq 0$ (resp. $\succ 0$) vacuously holds if the index set θ is empty.

Example 3.1: Consider $J_i = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}$ with $\lambda_1 \geq 0$, $C_\theta^i = \begin{bmatrix} 0 & 5 \\ 2 & -1 \end{bmatrix}$, and $v^i = (-1, 1)^T$. Hence,

$P(C_\theta^i, v^i) = (C_\theta^i \mathcal{L}(v^i), \; C_\theta^i v^i) = \begin{pmatrix} 0 & 5 \\ 2 & -3 \end{pmatrix} \succ 0$. Define $z^0 = (5, -3)^T$ and $z^1 = (0, 2)^T$. It follows

from Corollary 2.2 that $\mu_1 = 2$ renders $z^0 + \mu_1 z^1 > 0$. Let $\mu_0 = 1$ and $u = \mu_1 \mathcal{L}(v^i) + \mu_0 v^i =$

$(1, 1)^T$. Proposition 3.1 asserts that $C_\theta^i e^{J_i t} u = e^{\lambda_1 t} [\mu_0 z^1 t + (\mu_1 z^1 + \mu_0 z^0)] > 0, \forall \; t \geq 0$. Indeed,

a straightforward computation shows that $C_\theta^i e^{J_i t} u = e^{\lambda_1 t}(5, \; 2t + 1)^T$. Moreover, since $e^{\lambda_1 t} [\mu_0 z^1 t +$

$(\mu_1 z^1 + \mu_0 z^0)]$ is linear in (μ_0, μ_1), one can positively scale (μ_0, μ_1) such that $e^{\lambda_1 t} [\mu_0 z^1 t + (\mu_1 z^1 + \mu_0 z^0)]$

is greater than any positive vector for all $t \geq 0$.

We introduce more notation. For the given vector $b = (b_1, \cdots, b_m)^T \in \mathbb{R}^m$ characterizing the

polyhedron $\mathcal{P}$, define three index sets: $\alpha \equiv \{i \,|\, b_i > 0\}$, $\beta \equiv \{i \,|\, b_i = 0\}$, and $\gamma \equiv \{i \,|\, b_i < 0\}$.

For notational convenience, we let $b^+ = b_\alpha$, $b^0 = b_\beta = 0$, and $b^- = b_\gamma$. Likewise, we use C^+, C^0, C^-

for the corresponding blocks C_α, C_β, C_γ in C respectively. Let $F, G : \mathbb{R} \to \mathbb{R}^\ell$ be two functions. We say

that $F(t)$ tends to $G(t)$ as $t \to +\infty$ if for any $\varepsilon > 0$, there is $t_\varepsilon \geq 0$ such that $\|F(t) - G(t)\| \leq \varepsilon, \forall \, t \geq t_\varepsilon$.

With this notion, we present the following proposition for the necessity proof of the main result.

Proposition 3.2: Let $f : \mathbb{R} \to \mathbb{R}$ and $b_\ell \in \mathbb{R}$ be given such that $f(t) \geq b_\ell, \forall \, t \geq 0$. Suppose that

$f(t)$ tends to $e^{\lambda t} \dfrac{t^p}{p!} \Big[\rho_0 + \sum_{s=1}^k \rho_s \sin(\omega_s t + \theta_s) \Big]$ as $t \to +\infty$, where the real tuple $(\rho_0, \rho_1, \cdots, \rho_k) \neq 0$,

p is a nonnegative integer, $\lambda \in \mathbb{R}_+$, $\omega_s \in \mathbb{R}_{++}$ with $\omega_i \neq \omega_j$ whenever $i \neq j$, and $\theta_s \in \mathbb{R}$. Then the

following hold: (a) if $(\lambda, p) \succ (0, 0)$ or $b_\ell \geq 0$, then $\rho_0 > 0$; (b) if $\rho_0 \leq 0$, then $(\lambda, p) = (0, 0)$; and (c)

if $(\lambda, p) = (0, 0)$, then $\rho_0 \geq b_\ell$.

Proof. We prove (a) as follows. Let $h(t) \equiv \rho_0 + \sum_{s=1}^k \rho_s \sin(\omega_s t + \theta_s)$ and assume $\rho_0 \leq 0$. Since

$(\rho_0, \rho_1, \cdots, \rho_k) \neq 0$, either $\rho_0 < 0$ with $(\rho_1, \cdots, \rho_k) = 0$ or $\rho_0 = 0$ with $(\rho_1, \cdots, \rho_k) \neq 0$. For both the

cases, we obtain a scalar $\eta < 0$, particularly via Lemma 2.2 for the latter case, such that for any $t_* \geq 0$,

there exists $\tilde{t} \in [t_*, \infty)$ with $h(\tilde{t}) \leq \eta$. Consider the following two cases:

(1) $(\lambda, p) \succ (0, 0)$. Since, as $t \to \infty$, $e^{\lambda t} t^p$ is arbitrarily large and $f(t)$ is sufficiently close to $e^{\lambda t} \frac{t^p}{p!} h(t)$,

   we deduce that for any $\zeta < 0$ and any small $\varepsilon > 0$, there exists $t' \geq 0$ such that $e^{\lambda t'} \frac{(t')^p}{p!} h(t') \leq \zeta$

   and $|f(t') - e^{\lambda t'} \frac{(t')^p}{p!} h(t')| \leq \varepsilon$. This shows that for any $\zeta < 0$, there exists $t'' \geq 0$ with $f(t'') \leq \zeta/2$,

   which contradicts $f(t) \geq b_\ell, \forall \, t \geq 0$ for any given b_ℓ.

(2) $b_\ell \geq 0$. We only need to look at the case $(\lambda, p) = (0, 0)$ since $(\lambda, p) \succcurlyeq (0, 0)$ and the case

   $(\lambda, p) \succ (0, 0)$ has been treated in (1). Notice that $f(t)$ tends to $h(t)$ as $t \to +\infty$. Hence, via the

   property of $h(t)$ with $\eta < 0$ stated above, we see that for any $t_* \geq 0$, there exists $\tilde{t} \in [t_*, \infty)$ with

   $f(\tilde{t}) \leq \eta/2 < 0$, a contradiction.

Hence, we must have $\rho_0 > 0$ in (a). Statement (b) is a direct consequence of $(\lambda, p) \succcurlyeq (0, 0)$ and (a). To

show (c), notice that $(\lambda, p) = (0, 0)$ implies that $f(t)$ tends to $h(t)$ as $t \to +\infty$. If $(\rho_1, \cdots, \rho_k) = 0$,

then $\rho_0 \geq b_\ell$ holds true obviously. Otherwise, we apply Lemmas 2.2 and 2.3 to obtain $\rho_0 \geq b_\ell$. $\square$

The following result provides main necessary and sufficient existence conditions of a positively invariant set of a linear dynamics on a polyhedron; its extension to affine dynamics is given in Theorem 3.3. The proof for sufficiency relies on the lexicographic relation results developed before. The necessity proof shares the similar spirit in the recent paper [27]. However, a major difference is that [27] imposes observability-like conditions on the pairs (C^{ij}, J_{ij}), while the present paper does not. The latter allows us to handle a general polyhedron on one hand, but considerably complicates the analysis on the other hand. This difficulty is overcome by introducing the lifting operator and making full advantage of the echelon structure of the lexicographic relation and the long-time dynamic analysis results as shown below. The proof is given in the Appendix in order to maintain a smooth paper flow.

Theorem 3.1: The positively invariant set $\mathcal{A}$ of the linear dynamics on the polyhedron $\mathcal{P}$ is nonempty if and only if the index set α is empty or there exist real eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_k \geq 0$ of A and $v^i \in V(\lambda_i)$, $i = 1, \cdots, k$ such that

(a) $P^+ \equiv \big(P(C_\alpha^1, v^1), \cdots, P(C_\alpha^k, v^k) \big) \succ 0$, $P^0 \equiv \big(P(C_\beta^1, v^1), \cdots, P(C_\beta^k, v^k) \big) \succcurlyeq 0$, and

(b) the following implication holds:

$$P^- \equiv \big(P(C_\gamma^1, v^1), \cdots, P(C_\gamma^k, v^k) \big) \not\succeq 0 \implies \tag{4}$$
$$\Big[\lambda_k = 0, \quad \sum_j C_\phi^{kj} v^{kj} \geq b_\phi, \quad \sum_j C_\psi^{kj} v^{kj} \geq b_\psi, \quad \big(P(C_{\gamma\backslash\psi}^1, v^1), \cdots, P(C_{\gamma\backslash\psi}^k, v^k) \big) \succcurlyeq 0 \Big],$$

where the index sets $\phi \subseteq \alpha$ and $\psi \subseteq \gamma$ consist of the indices corresponding to the rows whose positive (resp. negative) leading terms appear in the last columns of P^+ and P^- respectively.

We say a few more words about the implication (4). This condition should be read as: if P^- is not lexicographically nonnegative, then all the following four conditions must hold: (i) the last real eigenvalue λ_k is zero; (ii) each negative leading term of P^- only appears in the last column of P^- (corresponding to the constant mode), and all the rest of the rows of P^- are lexicographically nonnegative; (iii) letting $\psi \subseteq \gamma$ denote the index set corresponding to the rows of P^- with negative leading terms (in the last column of P^-), then $\sum_j C_\psi^{kj} v^{kj} \geq b_\psi$; and (iv) letting $\phi \subseteq \alpha$ denote the index set corresponding to the rows of P^+ whose positive leading terms appear in the last column of P^+, then $\sum_j C_\phi^{kj} v^{kj} \geq b_\phi$.

The conditions of Theorem 3.1 can be greatly simplified if A is diagonal. To see this, we assume, without loss of generality, that $A = \text{diag}(J_1, J_2, \cdots, J_p)$, where the diagonal matrix block $J_i = \text{diag}(\lambda_i, \cdots, \lambda_i)$, and the eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_p$. Accordingly, the matrix C and an n-vector v can be partitioned as $C = \begin{bmatrix} C^1 & C^2 & \cdots C^p \end{bmatrix}$ and $v = \big((v^1)^T, (v^2)^T, \cdots, (v^p)^T \big)^T$ respectively, where C^i and v^i correspond to J_i. Note that for any J_i, each of its Jordan block is the scalar λ_i and $\bar{n}_i = 1$. Therefore, for any index set $\theta \subseteq \{1, \cdots, m\}$, $P(C_\theta^i, v^i) = C_\theta^i v^i$ and P^+ becomes $(C_\alpha^1 v^1, \cdots, C_\alpha^k v^k)$; the similar simplification can be made for P^0 and P^-. This is illustrated in the following example.

Example 3.2: Let $A \in \mathbb{R}^{4 \times 4}$, $C \in \mathbb{R}^{5 \times 4}$, and $b \in \mathbb{R}^5$ be $A = \text{diag}(5, 4, 0, -6)$, $C = \begin{bmatrix} -1 & \star & \kappa & \star \\ 0 & 1 & 2 & \star \\ 0 & -1 & 1 & \star \\ 0 & -2 & 1 & \star \\ 0 & 0 & -0.5 & \star \end{bmatrix}$,

and $b = (1, 3, 0.2, 0, -1)^T$, where $\star$ denotes the uninteresting terms and κ is a real parameter we shall discuss below. Note that the polyhedron defined by C and b is unbounded, and the index sets $\alpha = \{1, 2, 3\}$, $\beta = \{4\}$, and $\gamma = \{5\}$. Let $v = (v^1, v^2, v^3, v^4)^T$, where each scalar v^i corresponds to the ith eigenvalue of A. Furthermore, C^i is the ith column of C in this case. Choose $v^1 = -1$ and $v^3 = 2$. We obtain $P^+ = (C_\alpha^1 v^1, C_\alpha^3 v^3) = \begin{pmatrix} 1 & 2\kappa \\ 0 & 4 \\ 0 & 2 \end{pmatrix} \succ 0$, $P^0 = (C_\beta^1 v^1, C_\beta^3 v^3) = (0, 2) \succcurlyeq 0$, and

$P^- = (C_\gamma^1 v^1, C_\gamma^3 v^3) = (0, -1) \not\succeq 0$. To verify the implication (4), notice that v^3 corresponds to the zero eigenvalue, the negative leading term of P^- is in the last column of P^-, and the index sets $\phi = \{2, 3\}$ and $\psi = \{5\}$. Moreover, $C_\phi^3 v^3 = (4, 2)^T \geq b_\phi = (3, 0.2)^T$ and $C_\psi^3 v^3 = -1 \geq b_\psi = -1$. Hence, the positively invariant set $\mathcal{A}$ exists. In fact, for any κ, $x = (-1 - 2|\kappa|, 0, 2, 0)^T \in \mathcal{A}$. Finally, we point out that a nonzero v^2 will not make $(C_\alpha^1 v^1, C_\alpha^2 v^2, C_\alpha^3 v^3) \succ 0$ since the leading terms of the second and third rows of C have opposite signs.

Note that the conditions (a) and (b) of Theorem 3.1 do not involve the complex eigenvalues of A and can be cast as a constrained eigenvector problem. We elaborate more on this via the following example.

Example 3.3: Let $A = \text{diag}(J_1, J_2) \in \mathbb{R}^{4 \times 4}$, $C \in \mathbb{R}^{3 \times 4}$, and $b \in \mathbb{R}^3$ be $J_1 = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}$, $J_2 = \begin{bmatrix} \lambda_2 & \omega \\ -\omega & \lambda_2 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 5 & \star & \star \\ 2 & -1 & \star & \star \\ 0 & -1 & \star & \star \end{bmatrix}$, $b = (3, 0, -4)^T$ with $\lambda_1 \geq 0, \omega > 0$. Corresponding to J_1 and J_2, we partition C and $v \in \mathbb{R}^4$ as $C = \begin{bmatrix} C^1 & C^2 \end{bmatrix}$, and $v = ((v^1)^T, (v^2)^T)^T$, where $C^1 = (C_{\bullet 1}^1, C_{\bullet 2}^1)$ contains the first two columns of C and $v^1 = (v_1^1, v_2^1)^T \in \mathbb{R}^2$. Since J_1 has order 2, we obtain from (3) that, for any $\theta \subseteq \{1, 2, 3\}$, $P(C_\theta^1, v^1) = (C_{\theta 1}^1 v_2^1, \ C_{\theta 1}^1 v_1^1 + C_{\theta 2}^1 v_2^1)$. Moveover, $\alpha = \{1\}$, $\beta = \{2\}$, and $\gamma = \{3\}$. Since $P^+ = (0, 5 v_2^1)$, $P^0 = (2 v_2^1, 2 v_1^1 - v_2^1)$, and $P^- = (0, -v_2^1)$, we must have $v_2^1 > 0$ for $P^+ \succ 0$, but this implies $P^- \not\succeq 0$. Therefore, if $\lambda_1 > 0$, then the condition (4) fails, which rules out the existence of the positively invariant set. On the other hand, if $\lambda_1 = 0$, then, as partially shown in Example 3.1, $v_1^1 = -1$ and $v_2^1 = 1$ satisfy the conditions of Theorem 3.1 and thus the positively invariant set exists.

Finally we explain why the oscillatory mode associated with the complex eigenvalues plays no role in the positive invariance conditions. Consider a positively invariant trajectory $x(t, x^0)$. If $\lambda_2 > \lambda_1 \geq 0$, then the oscillatory mode would dominate as t sufficiently large. However, due to $\lambda_2 > 0$ and the persistent

sign alternating property of the oscillatory mode (cf. Lemma 2.2), $Cx(t, x^0) \geq b$ cannot hold for large t, which contradicts the positive invariance of $x(t, x^0)$. Hence, such the oscillatory mode must vanish. If $\lambda_2 < \lambda_1$, then for t sufficiently large, the mode corresponding to J_1 will dominate. The case where $\lambda_2 = \lambda_1$ is subtle, but it can also be shown via the sign alternating property that the mode corresponding to J_1 plays a major role in $x(t, x^0)$. In all the cases, we obtain the (necessary) positive invariance conditions that only depend on the mode associated with λ_1 for sufficiently large t. Conversely, under the positive invariance conditions, one can always use the lexicographical relation and perform positive scaling to obtain a positively invariant trajectory without considering the oscillatory mode.

B. Non-trivial Positive Invariance of Linear Dynamics on a Polyhedron

We discuss a special case in the positive invariance analysis of the linear dynamics. Recall that if the index set α is empty, then $0 \in \mathcal{A}$. We shall derive necessary and sufficient conditions for this case such that $\mathcal{A}$ is nontrivial, i.e., $\mathcal{A} \neq \{0\}$, in this section. This result is useful to characterize global switching behaviors of piecewise affine systems in Section IV. Notice that when $\mathcal{P}$ is a polyhedral cone, i.e., $b = 0$, a similar result is obtained in [22] using the fixed point theorem. The following results deal with the case where $\mathcal{P}$ is other than a cone, i.e., γ is nonempty.

Lemma 3.1: Let (C^i, J_i) be a pair corresponding to a real eigenvalue λ_i of A. If there exists a nonzero $v^i \in V(\lambda_i)$ such that $P(C_\theta^i, v^i) \succcurlyeq 0$ for a nonempty index set θ, then there exists an eigenvector $u \in V(\lambda_i)$ of J_i (associated with λ_i) such that $C_\theta^i u \geq 0$.

Proof. Given the nonzero $v^i \equiv \bigoplus_j v^{ij} \in V(\lambda_i)$, let ℓ be the largest integer such that $\mathcal{L}^\ell(v^{ij}) \neq 0$ for some j, namely, $\mathcal{L}^p(v^{ij}) = 0$, $\forall j$ if $p > \ell$. Since $v^i \neq 0$, clearly ℓ exists and $0 \leq \ell \leq \bar{n}_i - 1$. For this ℓ, define the nonempty index set $\vartheta \equiv \{j \mid \mathcal{L}^\ell(v^{ij}) \neq 0\}$. By the property of the lifting operator $\mathcal{L}$ discussed previously, we have, for any $j \in \vartheta$, $\mathcal{L}^\ell(v^{ij}) = \left(\rho_j, 0, \cdots, 0 \right)^T$, where $\rho_j \neq 0$. Therefore $\sum_j C_\theta^{ij} \mathcal{L}^\ell(v^{ij}) = \sum_{j \in \vartheta} \rho_j C_{\theta 1}^{ij}$, where $C_{\theta 1}^{ij}$ denotes the first column of C_θ^{ij}. This thus shows

$$P(C_\theta^i, v^i) = \left(0, \cdots, 0, \sum_j C_\theta^{ij} \mathcal{L}^\ell(v^{ij}), \star, \cdots, \star \right) = \left(0, \cdots, 0, \sum_{j \in \vartheta} \rho_j C_{\theta 1}^{ij}, \star, \cdots, \star \right) \succcurlyeq 0$$

where $\star$ denotes the terms we are not interested in. Consequently, $\sum_{j \in \vartheta} \rho_j C_{\theta 1}^{ij} \geq 0$. It is also noted that for each $j \in \vartheta$, $\mathcal{L}^\ell(v^{ij})$ is an eigenvector of J_{ij} associated with λ_i. Hence, $u = \bigoplus_j \mathcal{L}^\ell(v^{ij})$ is an eigenvector of J_i associated with λ_i. The lemma thus holds as $C_\theta^i u = \sum_{j \in \vartheta} \rho_j C_{\theta 1}^{ij} \geq 0$. $\square$

Theorem 3.2: Consider the linear dynamics and the polyhedron $\mathcal{P}$ with $\alpha = \emptyset$. Then $\mathcal{A}$ is nontrivial if and only if any one of the following conditions holds:

(a) (C, A) is an unobservable pair, i.e., there exists a nonzero vector $v \in \overline{O}(C, A)$;

(b) there exists an eigenvector v associated with a real eigenvalue λ of A such that $C^0 v \geq 0$ and

$\lambda > 0 \Rightarrow C^- v \geq 0$;

(c) there exist a complex eigenvalue $(\lambda + \imath \omega)$ of A (with $\omega \neq 0$) and an associated complex eigenvector $(u + \imath v)$ such that $\lambda \leq 0$ and $C^0 u = C^0 v = 0$.

Proof. "Sufficiency". Case (a) is trivial. We consider (b) and (c) as follow:

Case (b). Let λ be a real eigenvalue of A and $v \neq 0$ be an associated eigenvector satisfying the conditions stated in (b). First, it is easy to see $C^0 e^{At} v = e^{\lambda t} C^0 v \geq 0, \forall\, t \geq 0$. Consider two subcases: (i) $\lambda > 0$; and (ii) $\lambda \leq 0$. For the first subcase, we have $C^- e^{At} v = e^{\lambda t} C^- v \geq 0 \geq b^-,\ \forall\, t \geq 0$. Thus $v \in \mathcal{A}$. For subcase (ii), we have a scalar $\varepsilon > 0$ such that $C^-(\varepsilon\, v) \geq b^-$. Notice that $C^0 e^{At}(\varepsilon\, v) \geq 0, \forall\, t \geq 0$. Define the index set $\phi = \{ i \mid (C^- v)_i \geq 0 \}$. Clearly, $[C^-(\varepsilon v)]_\phi \geq 0$ and $0 > [C^-(\varepsilon v)]_{\overline{\phi}} \geq (b^-)_{\overline{\phi}}$. Therefore, $[C^- e^{At}(\varepsilon v)]_\phi = e^{\lambda t}[C^-(\varepsilon\, v)]_\phi \geq 0 \geq (b^-)_\phi$ and $[C^- e^{At}(\varepsilon v)]_{\overline{\phi}} = e^{\lambda t}[C^-(\varepsilon\, v)]_{\overline{\phi}} \geq [C^-(\varepsilon\, v)]_{\overline{\phi}} \geq (b^-)_{\overline{\phi}}$ for all $t \geq 0$ (as $\lambda \leq 0$). In conclusion, $C^- e^{At}(\varepsilon\, v) \geq b^-, \forall\, t \geq 0$. Thus $0 \neq \varepsilon\, v \in \mathcal{A}$.

Case (c). Notice that $v \neq 0$ and the condition $C^0 u = C^0 v = 0$ implies $(u + \imath v) \in \overline{O}(C^0, A)$, i.e., $C^0 e^{At}(u + \imath v) = 0, \forall\, t$. Therefore, $C^0 e^{At} v = 0, \forall\, t \geq 0$. Moreover, observing that $C^- e^{At} v = e^{\lambda t}[\cos(\omega t) C^- v + \sin(\omega t) C^- u]$ and $\lambda \leq 0$, we deduce that $\|C^- e^{At} v\|_2 \leq \delta, \forall\, t \geq 0$ for some scalar $\delta > 0$. Hence, there exists a scalar $\varepsilon > 0$ such that $\|C^- e^{At}(\varepsilon\, v)\|_2 \leq \min_{i \in \gamma}(|b_i^-|), \forall\, t \geq 0$. Therefore $C^- e^{At}(\varepsilon\, v) \geq b^-$ and $C^0 e^{At}(\varepsilon\, v) \equiv 0$ for all $t \geq 0$. This implies $\varepsilon\, v \in \mathcal{A}$.

"Necessity". Let $0 \neq x^* \in \mathcal{A}$, i.e., $C e^{At} x^* \geq b, \forall\, t \geq 0$. To avoid triviality, we assume $x^* \notin \overline{O}(C, A)$. Since $C e^{At} x^*$ is not identically zero, there is an eigenvalue λ_i associated with the largest non-vanishing mode in $C e^{At} x^*$ as shown in the proof of Theorem 3.1. Consider two cases:

(N1) $\lambda_i > 0$. In this case, by the similar argument as in the necessity proof of Theorem 3.1, we deduce that a nonzero $v^i \in V(\lambda_i)$ exists such that $P(C^i, v^i) \succcurlyeq 0$. This yields, via Lemma 3.1, an eigenvector u of J_i with $C^i u \geq 0$. Appropriately expanding u to a vector $v \in \mathbb{R}^n$ by adding the zero subvectors, we obtain an eigenvector v associated with λ_i such that $C^0 v \geq 0$ and $C^- v \geq 0$. This leads to (b).

(N2) $\lambda_i \leq 0$. We consider two subcases: (i) $x^* \notin \overline{O}(C^0, A)$; or (ii) $x^* \in \overline{O}(C^0, A)$ but $x^* \notin \overline{O}(C^-, A)$. For the first subcase, $C^0 e^{At} x^* = C_\beta\, e^{At} x^* \geq 0, \forall\, t \geq 0$ but is not identically zero. Hence, $P(C_\beta^i, v^i) \succcurlyeq 0$ for some $0 \neq v^i \in V(\lambda_i)$. Following the argument in (N1), we obtain an eigenvector u associated with λ_i such that $C^0\, u \geq 0$. This results in (b) by observing that the implication in (b) vacuously holds as $\lambda_i \leq 0$. For the second subcase, we have the non-vanishing terms in $C_\gamma e^{At} x^*$ corresponding to λ_i as $e^{\lambda_i t} \sum_{k=0}^{\bar{n}_i - 1} \frac{t^k}{k!} \left(\sum_j C_\gamma^{ij} \mathcal{L}^k(v^{ij}) + \sum_s (u_s^{ik})_\gamma \sin(\omega_s^{ik} t + \theta_s^{ik}) \right)$ where $v^{ij}, u_s^{ik}, \omega_s^{ik}, \theta_s^{ik}$ depend on C, A, x^* only. If all $(u_s^{ik})_\gamma = 0$, then $v^i = \bigoplus_j v^{ij} \neq 0$. But since $C_\beta^i e^{J_i t} v^i \equiv 0, \forall\, t$, we have $P(C_\beta^i, v^i) \equiv 0$ which leads to an eigenvector v of A associated with λ_i such that $C_\beta v \equiv C^0 v \geq 0$. This gives rise to (b). If, however, $(u_s^{ik})_\gamma \neq 0$ for some s and k, then $u_s^{ik} \neq 0$ and we must have a complex eigenvector

$(u + \imath\, v)$ associated with the complex eigenvalue $\lambda_i + \imath\, \omega_s^{ik}$ such that $C^0(u + \imath\, v) = 0$, or equivalently $C^0 u = C^0 v = 0$. Since $\lambda_i \leq 0$, we obtain (c) as desired. $\square$

C. Positive Invariance of Affine Dynamics on a Polyhedron

We now return to positive invariance analysis of the affine dynamics $\dot{x} = Ax + d$ on the polyhedron $\mathcal{P} = \{ x \in \mathbb{R}^n \mid Cx \geq b \}$, where $A \in \mathbb{R}^{n \times n}$ and $d \in \mathbb{R}^n$. For the given A and d, we can uniquely decompose d into $d = d_c + d_n$, where d_c is the orthogonal projection of d onto the column space of A and d_n is the projection onto the null space of A^T such that d_c is orthogonal to d_n. Since $d_c = Au_c$ for some vector u_c, the state transformation $\widetilde{x} = x + u_c$ converts the affine dynamics into $\dot{\widetilde{x}} = A\widetilde{x} + d_n$ and the polyhedron into $\widetilde{\mathcal{P}} = \{ \widetilde{x} \mid C\widetilde{x} \geq \widetilde{b} \}$, where $\widetilde{b} \equiv b + Cu_c$. Hence, if $d_n = 0$, i.e. d is in the range of A, then the affine dynamics can be transformed into the linear dynamics. Consequently, we assume, without loss of generality, that $d \neq 0$ is in the null space of A^T. Equivalently, d is assumed to be an eigenvector of A^T associated with the zero eigenvalue and thus A has the zero eigenvalue.

We present a technical lemma extended from Proposition 3.1. Let $J_i = \bigoplus_j J_{ij}$ be the Jordan block associated with the zero eigenvalue λ_i and $d^i = \bigoplus_j d^{ij}$ be an eigenvector of J_i^T. Hence, $d^{ij} = \left(0, \cdots, 0, d^{ij}_{n_{ij}}\right)^T$, where $d^{ij}_{n_{ij}} \neq 0$ is the last element of d^{ij}. Recall that $\bar{n}_i = \max_j n_{ij}$ where n_{ij} is the order of J_{ij}. For a given index set θ and $v^i \in V(0)$, define the vector tuple:

$$P(C_\theta^i, v^i, d^i) \equiv \Big(\sum_j C_\theta^{ij} [\mathcal{L}^{\bar{n}_i}(v^{ij}) + \mathcal{L}^{\bar{n}_i - 1}(d^{ij})], \; \cdots, \; \sum_j C_\theta^{ij} [\mathcal{L}(v^{ij}) + \mathcal{L}^0(d^{ij})], \; \sum_j C_\theta^{ij} \mathcal{L}^0(v^{ij}) \Big) \tag{5}$$

Lemma 3.2: Let d^i be an eigenvector of J_i^T associated with the eigenvalue $\lambda_i = 0$. For a given vector $b \in \mathbb{R}^m$, if there exist $v^i \in V(\lambda_i)$ and the index subsets $\theta, \theta' \subseteq \{1, \cdots, m\}$ such that $P(C_\theta^i, v^i, d^i) \succ 0$, $P(C_{\theta'}^i, v^i, d^i) \equiv 0$, and $\sum_j C_\phi^{ij} \mathcal{L}^0(v^{ij}) \geq b_\phi$, where $\phi \subseteq \theta$ consists of the indices corresponding to the positive leading term of the tuple in the last column of $P(C_\theta^i, v^i, d^i)$, then there exists $u^i \in V(\lambda_i)$ such that $C_\theta^i [e^{J_i t} u^i + \int_0^t e^{J_i(t-\tau)} d\tau\, d^i] \geq b_\theta$, $C_{\theta'}^i [e^{J_i t} u^i + \int_0^t e^{J_i(t-\tau)} d\tau\, d^i] \equiv 0$, $\forall\, t \geq 0$.

Proof. Let $d^i = \bigoplus_j d^{ij}$, where d^{ij} is in the null space of the Jordan block J_{ij}^T associated with the zero eigenvalue λ_i and at least one of d^{ij}'s is nonzero. Since $d^{ij} = \left(0, \cdots, 0, d^{ij}_{n_{ij}}\right)^T$ with $d^{ij}_{n_{ij}} \neq 0$, we have $\int_0^t e^{J_{ij}(t-\tau)} d\tau\, d^{ij} = \sum_{k=0}^{n_{ij}-1} \frac{t^{k+1}}{(k+1)!} (J_{ij})^k d^{ij} = \sum_{k=0}^{n_{ij}-1} \frac{t^{k+1}}{(k+1)!} \mathcal{L}^k(d^{ij})$. Let $z^0 \equiv \sum_j C_\theta^{ij} \mathcal{L}^0(v^{ij})$ and $z^k \equiv \sum_j C_\theta^{ij} [\mathcal{L}^k(v^{ij}) + \mathcal{L}^{k-1}(d^{ij})]$ for $k = 1, \cdots, \bar{n}_i$. Likewise, let $y^0 \equiv \sum_j C_{\theta'}^{ij} \mathcal{L}^0(v^{ij})$ and $y^k \equiv \sum_j C_{\theta'}^{ij} [\mathcal{L}^k(v^{ij}) + \mathcal{L}^{k-1}(d^{ij})]$ for $k = 1, \cdots, \bar{n}_i$. Hence, $\left(z^{\bar{n}_i}, \cdots, z^0\right) \succ 0$ with $z^0_\phi \geq b_\phi$ and $\left(y^{\bar{n}_i}, \cdots, y^0\right) \equiv 0$. By Corollary 2.2 and Remark 3.1, we obtain nonnegative scalars $\mu_1, \cdots, \mu_{\bar{n}_i}$ with $\mu_0 \equiv 1$ such that $\sum_{s=0}^{\bar{n}_i} \mu_s z^s \geq b_\theta$ and $\sum_{s=k}^{\bar{n}_i} \mu_{s-k} z^s \geq 0$ for all $k = 1, \cdots, \bar{n}_i$, and $\sum_{s=k}^{\bar{n}_i} \mu_{s-k} y^s \equiv 0$ for all

$k \geq 0$. Let $u^{ij} \equiv \mu_0 \mathcal{L}^0(v^{ij}) + \sum_{s=1}^{\bar{n}_i} \mu_s [\mathcal{L}^s(v^{ij}) + \mathcal{L}^{s-1}(d^{ij})]$, and $u^i \equiv \bigoplus_j u^{ij} \in V(\lambda_i)$. Hence,

$$C_\theta^i \left[e^{J_i t} u^i + \int_0^t e^{J_i(t-\tau)} d\tau \, d^i \right] = \sum_{k=0}^{\bar{n}_i} \frac{t^k}{k!} \left(\sum_j C_\theta^{ij} [\mathcal{L}^k(u^{ij}) + \mathcal{L}^{k-1}(d^{ij})] \right)$$

$$= \sum_{k=0}^{\bar{n}_i} \frac{t^k}{k!} \left(\sum_j C_\theta^{ij} \left[\sum_{s=1}^{\bar{n}_i-k} \mu_s [\mathcal{L}^{k+s}(v^{ij}) + \mathcal{L}^{k+s-1}(d^{ij})] + \mu_0 \mathcal{L}^k(v^{ij}) + \mu_0 \mathcal{L}^{k-1}(d^{ij}) \right] \right)$$

$$= \sum_{k=0}^{\bar{n}_i} \frac{t^k}{k!} \left(\sum_j C_\theta^{ij} \sum_{s=0}^{\bar{n}_i-k} \mu_s [\mathcal{L}^{k+s}(v^{ij}) + \mathcal{L}^{k+s-1}(d^{ij})] \right)$$

$$= \sum_{k=0}^{\bar{n}_i} \frac{t^k}{k!} \left(\sum_{p=k}^{\bar{n}_i} \mu_{p-k} \sum_j C_\theta^{ij} [\mathcal{L}^p(v^{ij}) + \mathcal{L}^{p-1}(d^{ij})] \right) = \sum_{k=0}^{\bar{n}_i} \frac{t^k}{k!} \left[\sum_{p=k}^{\bar{n}_i} \mu_{p-k} z^p \right],$$

where $\mathcal{L}^{-1}(w) = 0$ for any w is used. Therefore, we have $C_\theta^i [e^{J_i t} u^i + \int_0^t e^{J_i(t-\tau)} d\tau \, d^i] \geq \sum_{p=0}^{\bar{n}_i} \mu_p z^p \geq b_\theta$, $\forall\, t \geq 0$. Similarly, we obtain

$$C_{\theta'}^i [e^{J_i t} u^i + \int_0^t e^{J_i(t-\tau)} d\tau \, d^i] = \sum_{k=0}^{\bar{n}_i} \frac{t^k}{k!} \left[\sum_{p=k}^{\bar{n}_i} \mu_{p-k} y^p \right] = 0, \quad \forall\, t \geq 0. \quad \square$$

Theorem 3.3: Consider the affine dynamics $\dot{x} = Ax + d$, where $d \neq 0$ is in the null space of A^T. The positively invariant set $\mathcal{A}$ of the affine dynamics on $\mathcal{P}$ is nonempty if and only if there exist real eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_{k-1} > 0 = \lambda_k$ of A and $v^i \in V(\lambda_i)$, $i = 1, \cdots, k$ such that $\left(P(C_\alpha^1, v^1), \cdots, P(C_\alpha^{k-1}, v^{k-1}), P(C_\alpha^k, v^k, d^k) \right) \succ 0$ with $\sum_j C_\phi^{kj} v^{kj} \geq b_\phi$, $\left(P(C_\beta^1, v^1), \cdots, P(C_\beta^{k-1}, v^{k-1}), P(C_\beta^k, v^k, d^k) \right) \succcurlyeq 0$, and the following implication holds:

$$\left(P(C_\gamma^1, v^1), \cdots, P(C_\gamma^{k-1}, v^{k-1}), P(C_\gamma^k, v^k, d^k) \right) \nsucceq 0 \implies$$

$$\left[\sum_j C_\psi^{kj} v^{kj} \geq b_\psi, \quad \left(P(C_{\gamma\setminus\psi}^1, v^1), \cdots, P(C_{\gamma\setminus\psi}^k, v^{k-1}), P(C_{\gamma\setminus\psi}^k, v^k, d^k) \right) \succcurlyeq 0 \right],$$

where the index sets $\phi \subseteq \alpha$ and $\psi \subseteq \gamma$ consist of the indices corresponding to the rows whose positive (resp. negative) leading terms appear in the last columns of the respective tuples.

Proof. Since $d \neq 0$ is in the null space of A^T, A has the zero eigenvalue and thus A always has a nonnegative eigenvalue. The sufficiency of the theorem can be proved in the similar argument as in Theorem 3.1 by making use of Lemma 3.2 and $\lambda_k = 0$ for the zero eigenvalue mode. To prove the necessity, let $x^* \in \mathcal{A}$. It is clear that

$$C \left[e^{At} x^* + \int_0^t e^{A(t-\tau)} d\tau \, d \right] = \sum_{\lambda_i \neq 0} \left\{ e^{\lambda_i t} \sum_{k=0}^{\hat{n}_i - 1} \frac{t^k}{k!} \left(\sum_j C^{ij} \mathcal{L}^k(v^{ij}) + \sum_s u_s^{ik} \sin(\omega_s^{ik} t + \theta_s^{ik}) \right) \right\}$$

$$+ \sum_{k=0}^{\hat{n}_0} \frac{t^k}{k!} \left(\sum_j C^{0j} [\mathcal{L}^k(v^{0j}) + \mathcal{L}^{k-1}(d^{0j})] + \sum_s u_s^{0k} \sin(\omega_s^{0k} t + \theta_s^{0k}) \right),$$

where λ_i and $\widehat{n}_i$ are defined in the same way as in Theorem 3.1, v^{0j} and u_s^{0k} correspond to the zero eigenvalue and (possibly existing) imaginary eigenvalues of A respectively, which depend on C, A, x^* only. Let the real parts of the eigenvalues of A be labeled in a descending order, i.e., $\lambda_1 > \cdots > \lambda_{k-1} > \lambda_k = 0 > \lambda_{k+1} > \cdots > \lambda_q$. Define the vector tuples $P(C^i, v^i)$ corresponding to positive $\lambda_i, i = 1, \cdots, k-1$ in the same fashion as in Theorem 3.1. In particular, if λ_i corresponds to a complex eigenvalue of A, then λ_i must be positive and $P(C^i, v^i) \equiv 0$. We also define $P(C^k, v^k, d^k)$ for the zero eigenvalue λ_k with $v^k = \bigoplus_j v^{0j} \in V(\lambda_k)$ and $d^k = \bigoplus_j d^{0j}$, which is an eigenvector of J_0^T. The remaining proof thus follows from the necessity arguments in (N1)-(N3) of Theorem 3.1 by considering the large-time dominating mode of $C_\ell [e^{At} x^* + d \, t]$ for each $\ell \in \alpha$, $\ell \in \beta$ and $\ell \in \gamma$ respectively. Recall that if any index set is empty, then the associated lexicographical relation is assumed to hold vacuously (cf. Remark 3.1). Lastly, since A has the zero eigenvalue, by removing the positive λ_i that corresponds to a complex eigenvalue of A and its corresponding zero block in the obtained tuples, we attain the desired nonnegative real eigenvalues $\widetilde{\lambda}_i$ and associated subvectors $v^i \in V(\widetilde{\lambda}_i)$ as shown in Theorem 3.1. $\square$

Example 3.4: Consider Example 3.2, where A is diagonal and $d = (0, 0, \rho, 0)^T$ with $\rho \neq 0$ such that d is in the null space of A^T. Hence, $d^i = 0, i = 1, 2, 4$ and $d^3 = \rho$. Recall that the polyhedron is unbounded and $\alpha = \{1, 2, 3\}$, $\beta = \{4\}$, and $\gamma = \{5\}$. Since d^3 corresponds to the zero eigenvalue, it follows from (5) that $P(C_\theta^3, v^3, d^3) = (C_\theta^3 \, d^3, C_\theta^3 \, v^3)$ for any index set $\theta \subseteq \{1, \cdots, 5\}$. Let $\rho > 0$. Note that no matter what v^1, v^2 and v^3 are chosen, $\big(P(C_\gamma^1 \, v^1), \, P(C_\gamma^2 \, v^2), \, P(C_\gamma^3 \, v^3, d^3) \big) = (0, 0, -0.5 \, \rho, -0.5 v^3) \not\succeq 0$ and the negative leading term $-0.5 \, \rho$ does not appear in the last column. Hence, the conditions of Theorem 3.3 fail such that the positively invariant set does not exist. This result can be directly verified as the last entry of $C[e^{At} x + \int_0^t e^{A(t-\tau)} d\tau \, d]$ takes the form $-0.5(\rho \, t + x_3 + x_4 \, e^{-6t})$, which cannot be greater than -1 for all $t \geq 0$. Now consider $\rho < 0$. Since

$$\big(P(C_\alpha^1, v^1), \, P(C_\alpha^2, v^2), \, P(C_\alpha^3, v^3, d^3) \big) = \begin{pmatrix} -v^1 & \star & \kappa \rho & \kappa v^3 \\ 0 & v^2 & 2\rho & 2v^3 \\ 0 & -v^2 & \rho & v^3 \end{pmatrix},$$

no v^1, v^2, v^3 will make the above vector tuple lexicographically positive. Hence, the positively invariant set does not exist either. Finally, we comment that if $\rho > 0$ and the $(5, 3)$-entry of C changes from -0.5 to a nonnegative number, then the conditions of Theorem 3.3 are satisfied for any κ, which ascertains the existence of the positively invariant set.

Remark 3.2: A special case is when $\mathcal{P}$ is bounded, i.e., $\mathcal{P}$ is a polytope. For a general affine dynamics $\dot{x} = Ax + d$ with (possibly zero) d in the null space of A^T, it has been shown in [16, Therorem 3.1] via the fixed point argument that the positively invariant set $\mathcal{A}$ exists if and only if the affine system has an equilibrium in $\mathcal{P}$, i.e., $Av + d = 0$ for some $v \in \mathcal{P}$. The long-time dynamic analysis techniques
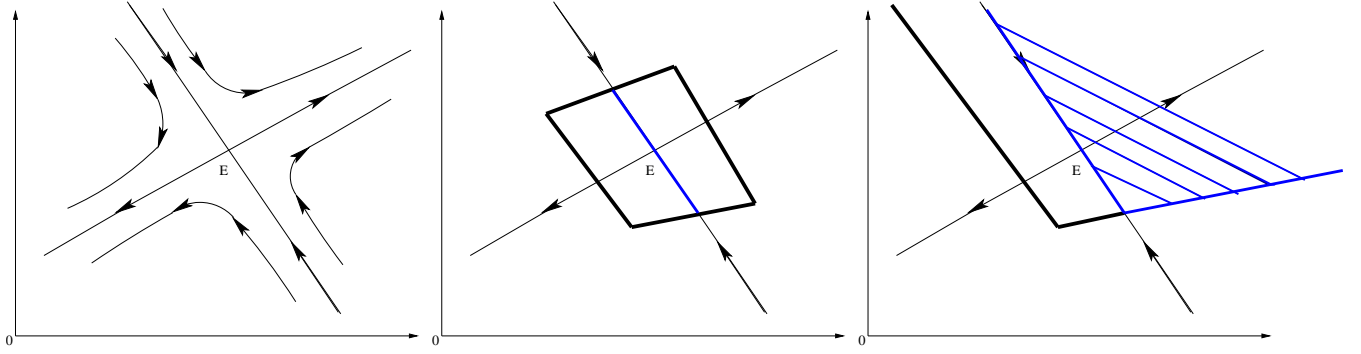
Fig. 1. Left: phase diagram of the affine dynamics: E denotes the equilibrium; Middle: positively invariant set in a bounded polytope; Right: positively invariant set in an unbounded polyhedron

in Theorems 3.1 and 3.3 provide an alternative proof for the necessity of this result (the sufficiency is trivial). Indeed, letting $x^* \in \mathcal{A}$ and noticing $e^{At}x^* + \int_0^t e^{A(t-\tau)}d\tau\, d$ is bounded on $[0, \infty)$, we have

$$e^{At}x^* + \int_0^t e^{A(t-\tau)}d\tau\, d \;=\; \bigoplus_j \left[v^{0j} + t\left(\mathcal{L}(v^{0j}) + d^{0j}\right) + h_j(t)\right],$$ where v^{0j} corresponds to the zero eigenvalue, $d = \bigoplus_j d^{0j}$, and $h_j(t)$ contains all the terms associated with the eigenvalues with non-positive real parts. Hence $\mathcal{L}(v^{0j}) + d^{0j} = 0$ for all j due to the boundedness. By (c) of Proposition 3.2, we deduce $\sum_j C^{0j} v^{0j} \geq b$. Moreover, $\mathcal{L}(v^{0j}) = J_{0j}\, v^{0j}$ holds. Hence, by appropriately expanding v^0, we obtain v satisfying $Cv = \sum_j C^{0j} v^{0j} \geq b$ and $Av = \bigoplus_j J_{0j}\, v^{0j} = -\bigoplus_j d^{0j} = -d$. This shows that $v \in \mathcal{P}$ and $Av + d = 0$. It can also be obtained by directly checking the conditions of Theorem 3.3 and observing $v^i = 0$ for $\lambda_i > 0$ and $\phi = \alpha$ if $\alpha \neq \emptyset$ due to the boundedness of $\mathcal{P}$. To further illustrate positively invariant sets, consider the planar affine dynamics $\dot{x} = Ax + d$, where $x, d \in \mathbb{R}^2$, and $A \in \mathbb{R}^{2\times 2}$ has two nonzero real eigenvalues of opposite signs. Hence the unique equilibrium $-A^{-1}d$ is a saddle point and the phase diagram is shown in Figure 1. The positively invariant sets contained in a polytope and in an unbounded polyhedron are the blue regions in Figures 1, where the boundary of each constraint polyhedron is highlighted in the black color. Note that positively invariant sets are not necessarily polyhedral in general; see [26, Example 10].

D. Computation of Positive Invariance Conditions

In this section, we discuss computational aspects of the positive invariance conditions obtained in Theorems 3.1 and 3.3. For the ease of discussion, we focus on the necessary and sufficient conditions of Theorem 3.1 pertaining to a linear system on a polyhedron; the resulting algorithm (cf. Algorithm 1) for verifying these conditions can be easily extended to Theorem 3.3. Further, to avoid triviality, we assume (i) the polyhedron $\mathcal{P}$ is unbounded (otherwise, one only needs to check if $0 \in \mathcal{P}$); (ii) the index set α is nonempty; and (iii) A has at least one nonnegative real eigenvalue. Under these assumptions, it is shown as follows that verification of the positive invariance conditions of Theorem 3.1 can be accomplished via

(finitely many) combinatorial linear programs, and in turn, this leads to computational complexity. Let $\lambda_1 > \lambda_2 > \cdots > \lambda_r \geq 0$ be all the nonnegative real eigenvalues of A (as before, A is assumed to be in the Jordan canonical form). It is easy to see that there exist vectors satisfying (a)-(b) of Theorem 3.1 if and only if there exist $v^i \in V(\lambda^i), i = 1, \cdots, r$ satisfying one of the following conditions:

(a′) $P^+ \equiv \big(P(C_\alpha^1, v^1), \cdots, P(C_\alpha^r, v^r) \big) \succ 0$, $P^0 \equiv \big(P(C_\beta^1, v^1), \cdots, P(C_\beta^r, v^r) \big) \succcurlyeq 0$, and $P^- \equiv$ $\big(P(C_\gamma^1, v^1), \cdots, P(C_\gamma^r, v^r) \big) \succcurlyeq 0$;

(b′) $\lambda_r = 0$, $P^+ \succ 0$, $P^0 \succcurlyeq 0$, $P^- \not\succcurlyeq 0$ but the right-hand side of the implication (4) holds (with k replaced by r).

Consider the condition (a′) first. Let $\mathbf{v} = \bigoplus_i v^i \in \mathbb{R}^{\widetilde{n}}$ with $\widetilde{n} \equiv \sum_{i=1}^r \sum_j n_{ij}$, and $\widehat{n} \equiv \sum_{i=1}^r \bar{n}_i$; see the definitions of n_{ij} and $\bar{n}_i$ in Section III-A. Note that the lifting operator $\mathcal{L}$ can be represented by a matrix operation. Therefore, the lexicographical conditions in (a′) can be formulated as $\big(E_1 \mathbf{v}, \cdots, E_{\widehat{n}} \mathbf{v} \big) \succ 0$ and $\big(F_1 \mathbf{v}, \cdots, F_{\widehat{n}} \mathbf{v} \big) \succcurlyeq 0$, where $E_i \in \mathbb{R}^{|\alpha| \times \widetilde{n}}$, $F_i \in \mathbb{R}^{(|\beta|+|\gamma|) \times \widetilde{n}}$ are suitable matrices determined via (3) from associated C^i and a matrix corresponding to $\mathcal{L}$. Let $(E_i)_{\ell \bullet}$ (resp. $(F_i)_{\ell \bullet}$) denote the ℓth row of E_i (resp. F_i). Hence, a vector $\mathbf{v}$ satisfies $\big(E_1 \mathbf{v}, \cdots, E_{\widehat{n}} \mathbf{v} \big) \succ 0$ and $\big(F_1 \mathbf{v}, \cdots, F_{\widehat{n}} \mathbf{v} \big) \succcurlyeq 0$ if and only if (i) for each $\ell \in \{1, \cdots, |\alpha|\}$, there exists $k \in \{1, \cdots, \widehat{n}\}$ such that $(E_j)_{\ell \bullet} \mathbf{v} = 0$ for all $j = 1, \cdots, k-1$ and $(E_k)_{\ell \bullet} \mathbf{v} > 0$; and (ii) for each $\ell \in \{1, \cdots, |\beta| + |\gamma|\}$, either $(F_j)_{\ell \bullet} \mathbf{v} = 0$ for all j or there exists $k \in \{1, \cdots, \widehat{n}\}$ such that $(F_j)_{\ell \bullet} \mathbf{v} = 0$ for all $j = 1, \cdots, k-1$ and $(F_k)_{\ell \bullet} \mathbf{v} > 0$. Recall $m = |\alpha| + |\beta| + |\gamma|$. This yields $\widehat{n}^{|\alpha|} (\widehat{n}+1)^{m-|\alpha|}$ sets of linear inequalities of the form $\big[G_i \mathbf{v} > 0, H_i \mathbf{v} = 0 \big]$, where each indexed matrix pair (G_i, H_i) is established from the combinatorial procedure descried above. Hence, the condition (a′) holds for some $\mathbf{v}$ if and only if one of $\{\mathbf{v} \,|\, G_i \mathbf{v} > 0, H_i \mathbf{v} = 0\}$, or equivalently $\mathcal{V}_i \equiv \{\mathbf{v} \,|\, G_i \mathbf{v} \geq \mathbf{1}, H_i \mathbf{v} = 0\}$, is feasible, due to the homogeneity of the former set. The feasibility of each $\mathcal{V}_i$ can be checked via linear programming techniques, e.g. interior-point methods, whose computational complexity is given by $O(N^\mu)$, where N is the sum of the numbers of variables and constraints; a typical μ can be taken as 3.5. In view of this and via careful study of the number of constraints of each linear inequality established from the combinatorial procedure (whose details are omitted here), we obtain an upper bound of the worst-case computational complexity for checking (a′):

$$\sum_{p_i \in \{1, \cdots, \widehat{n}+1\}} (p_1 + \cdots + p_m + \widetilde{n})^\mu \leq \Big[\sum_{p_i \in \{1, \cdots, \widehat{n}+1\}} (p_1 + \cdots + p_m + \widetilde{n}) \Big]^\mu$$

$$\leq \Big[\frac{(\widehat{n}+1)(\widehat{n}+2)}{2} \sum_{i=1}^m i \binom{m}{i} + (\widehat{n}+1)^m \widetilde{n} \Big]^\mu = \Big[(\widehat{n}+1)(\widehat{n}+2) m 2^{m-2} + (\widehat{n}+1)^m \widetilde{n} \Big]^\mu$$

Next we consider the condition (b′), where $\lambda_r = 0$ and $P^- \not\succcurlyeq 0$ but its negative elements appear in the last column of P^- only. This condition leads to a family of mixed lexicographical and linear inequalities. It can be shown via a similar but more involved combinatorial argument that these inequalities can be converted into the total number of $\big[\sum_{i=0}^{|\alpha|} \binom{|\alpha|}{i} (\widehat{n}-1)^{|\alpha|-i} \big] (\widehat{n}+1)^{|\beta|} \big[\sum_{i=1}^{|\gamma|} \binom{|\gamma|}{i} (\widehat{n}+1)^{|\gamma|-i} \big]$ linear

inequalities of the form $[L_i\mathbf{v} > 0, M_i\mathbf{v} = 0, N_i\mathbf{v} > p_i, S_i\mathbf{v} \geq q_i]$, where the indexed matrix-vector tuple $(L_i, M_i, N_i, S_i, p_i, q_i)$ is determined via the combinatorial procedure. It follows from [15, Section 2] that by introducing a slack variable $\widetilde{\mathbf{v}}_{\widetilde{n}+1}$ and $\widetilde{\mathbf{v}} \equiv (\mathbf{v}, \widetilde{\mathbf{v}}_{\widetilde{n}+1})$, the feasibility of each $[L_i\mathbf{v} > 0, M_i\mathbf{v} \geq 0, N_i\mathbf{v} > p_i, S_i\mathbf{v} \geq q_i]$ is equivalent to that of $[P_i\widetilde{\mathbf{v}} \geq \mathbf{1}, U_i\widetilde{\mathbf{v}} = 0, W_i\widetilde{\mathbf{v}} \geq 0]$ for suitable matrices P_i, U_i, W_i. Let $\mathcal{U}_i \equiv \{\widetilde{\mathbf{v}} \mid P_i\widetilde{\mathbf{v}} \geq \mathbf{1}, U_i\widetilde{\mathbf{v}} = 0, W_i\widetilde{\mathbf{v}} \geq 0\}$. Thus the existence of $\mathbf{v}$ satisfying the condition (b') is equivalent to the feasibility of at least one of $\mathcal{U}_i$ that can be tested via linear programs. An upper bound of the worst-case computational complexity for verifying the condition (b') is obtained as

$$\sum_{i=1}^{|\gamma|}\sum_{j=0}^{|\alpha|}\left[i(\widehat{n}+1) + j\,\widehat{n} + \sum_{p_i\in\{1,\cdots,\widehat{n}+1\}}(p_1 + \cdots + p_{m-i-j} + \widetilde{n} + 2)\right]^{\mu} \leq \left[m^2(\widehat{n}+1)\right.$$
$$\left. + \frac{(\widehat{n}+1)(\widehat{n}+2)}{2}m(2^m - 2^{m-1-|\alpha|})(1 - 2^{-|\gamma|}) + (1+\widehat{n}^{-1})(\widetilde{n}+2)[(1+\widehat{n})^{m-1} - (1+\widehat{n})^{m-|\gamma|-1}]\right]^{\mu}$$

The numerical procedure discussed above is summarized in Algorithm 1 for checking the positive invariance conditions of Theorem 3.1.

The total computational complexity is the sum of the above two upper bounds, which are tight for large $\widehat{n}, \widetilde{n}, m$. For a fixed $m \geq 3$, the computational complexity is given by $O((\widehat{n}^m\widetilde{n})^{\mu})$ for large $\widehat{n}$ and $\widetilde{n}$, namely, the complexity is polynomial in the number of state variables corresponding to nonnegative real eigenvalues. On the other hand, for a large number of (non-redundant) polyhedral constraints, i.e. m is large, the complexity grows exponentially with respect to m. This is not surprising due to the combinatorial complexity of testing lexicographical inequalities.

Example 3.5: Algorithm 1 is implemented on MATLAB, where the function LINPROG is called for feasibility test of linear inequalities. Extensive numerical experiments are carried out to evaluate computational complexity. In particular, two scenarios are considered: (i) m is fixed but the number of state variables n' (associated with the nonnegative real eigenvalues) varies; (ii) n' is fixed but m varies. For the first scenario, two fixed m's are considered: $m = 3$ and $m = 4$, and n' ranges from 2 to 25 and from 2 to 15 respectively; for the second, $n' = 5$ and $n' = 7$ are chosen, and m ranges from 3 to 7 and from 3 to 6 respectively. The first scenario corresponds to a moderate size of constraints but with a large of number of state variables, and the second to a moderate number of variables but with many constraints. For each pair (m, n'), various numerical tests are run for different system data that require the worst-case computation (corresponding to the non-existence of a positively invariant set) and the average of computation times of these tests is obtained. Figure 2 displays the plots of the logarithm of averaging computation times (in seconds) versus the number of variables or constraints. It can be seen that the slopes of the top logarithm curves tend to be decreasing, whereas the bottom logarithm curves are almost linear. This confirms that the computational complexity of Algorithm 1 is polynomial in the number of variables but exponential in the number of constraints. Finally, it is worth mentioning that [10], [6]

Algorithm 1 Checking the Existence of a Positively Invariant Set of Linear Dynamics with Nonnegative Real Eigenvalues and Nonempty α on an Unbounded Polyhedron

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Given the matrix A for linear dynamics and the polyhedron $\mathcal{P} := \{x \,|\, Cx \geq b\}$

Find an invertible T such that $T^{-1}AT$ is in the Jordan canonic form and let $C \leftarrow CT^{-1}$

Let $\lambda_1 > \cdots > \lambda_r \geq 0$ be all the nonnegative real eigenvalues of A

Find the indexed matrix pairs (G_i, H_i)

for each pair (G_i, H_i) **do**

    Check the feasibility of $\mathcal{V}_i \equiv \{\mathbf{v} \,|\, G_i\mathbf{v} \geq \mathbf{1}, H_i\mathbf{v} = 0\}$ via linear program

    if $\mathcal{V}_i$ is feasible **then**

        return [a positively invariant set exists]

    end if

end for

if $\lambda_r = 0$ **then**

    Find the indexed matrix tuples (P_i, U_i, W_i)

    for each tuple (P_i, U_i, W_i) **do**

        Check the feasibility of $\mathcal{U}_i \equiv \{\widetilde{\mathbf{v}} \,|\, P_i\widetilde{\mathbf{v}} \geq \mathbf{1}, U_i\widetilde{\mathbf{v}} = 0, W_i\widetilde{\mathbf{v}} \geq 0\}$ via linear program

        if $\mathcal{U}_i$ is feasible **then**

            return [a positively invariant set exists]

        end if

    end for

end if

return [a positively invariant set does not exist]

---

establish conditions for a (parameterized) polyhedron to be a positively invariant set of a linear system. Their computation costs are polynomial in the number of state variables and constraints. These results offer numerically efficiently (but possibly conservative) sufficient conditions for the existence problem treated in Theorem 3.1.

IV. APPLICATION TO GLOBAL SWITCHING CHARACTERIZATION OF PIECEWISE AFFINE SYSTEMS

In this section, the positive invariance results are applied to a class of affine hybrid systems. Specifically, necessary and sufficient conditions are derived to characterize global long-time switching behaviors of piecewise affine systems with isolated equilibria and infinite mode switchings.

A. Piecewise Affine Systems

A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is called piecewise affine (PA) if there exists a finite family of affine functions $\{f_i\}_{i=1}^{\ell}$ such that $f(x) \in \{f_i(x)\}_{i=1}^{\ell}$ for each $x \in \mathbb{R}^n$ [12]. Consider the ODE system: $\dot{x} = f(x)$, where
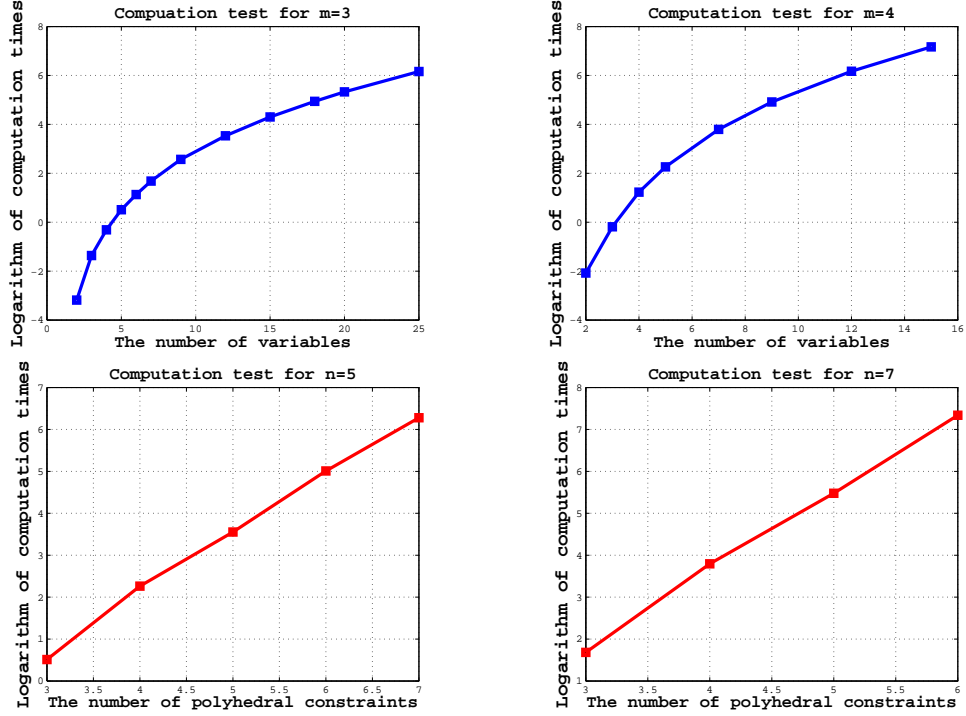
Fig. 2. Top: the number of state variables (corresponding to nonnegative real eigenvalues) versus the logarithm of computation times for a fixed m. Top-left: $m = 3$; Top-right: $m = 4$. Bottom: the number of constraints versus the logarithm of computation times for a fixed n'. Bottom-left: $n' = 5$; Bottom-right: $n' = 7$.

$f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and piecewise affine. We call such a system the *piecewise affine system* (PAS) [9]. A continuous and PA function possesses an appealing geometric structure for its domain, which provides an alternative representation for the PAS. To elaborate on this, let Ξ be a finite collection of polyhedra $\mathcal{P}_i = \{ x \in \mathbb{R}^n \mid C_i x \geq b_i \}$ for $C_i \in \mathbb{R}^{m_i \times n}$ and $b_i \in \mathbb{R}^{m_i}$. We call Ξ a *polyhedral subdivision* of $\mathbb{R}^n$ [12] if (a) $\bigcup_{i=1}^m \mathcal{P}_i = \mathbb{R}^n$; (b) each $\mathcal{P}_i$ has a nonempty interior; and (c) the intersection of any two polyhedra in Ξ is either empty or a common proper face of both polyhedra.

For a continuous and PA function f, one can always find a polyhedral subdivision of $\mathbb{R}^n$ and finitely many affine functions $g_i \equiv A_i x + d_i$ such that f coincides with one of g_i's on each polyhedron in Ξ [12, Proposition 4.2.1]. With these notions, we can rewrite the PAS in the equivalent form $\dot{x} = A_i x + d_i, \forall\, x \in \mathcal{P}_i$ where $[\, x \in \mathcal{P}_i \cap \mathcal{P}_j \,] \Rightarrow [\, A_i x + d_i = A_j x + d_j \,]$ holds true due to the continuity of f. In what follows, we call each affine dynamics $\dot{x} = A_i x + d_i$ and its associated polyhedron $\mathcal{P}_i$ a *mode* of the PAS. Since a continuous and PA function is globally Lipschitz continuous [12], the PAS has a unique solution for any initial state. The PASs form a class of affine hybrid systems, for which the time-invariant vector fields are affine, the invariant sets are the polyhedrons $\mathcal{P}_i$ of dimension n, the guard sets are the boundaries of these polyhedra, and the reset maps are all identities. An important class of the PASs is the conewise linear systems (CLSs) [9], [27], [28], for which $b_i = 0$ and $d_i = 0$ for each i such that the PAS becomes $\dot{x} = A_i x$ if $x \in \mathcal{C}_i \equiv \{x \mid C_i x \geq 0\}$.

B. Mode Switchings and Global Long-time Switching Behaviors

A PAS is subject to state-dependent mode switchings along its trajectories with implicit transition times and implicit mode selection at switching times. This complicates dynamical and control analysis of the hybrid dynamics. As a special case of the PASs, state-dependent switching behaviors of the CLSs have been extensively studied in the recent papers [9], [27]. It can be proved in a similar fashion as in [9], [27] that the PAS enjoys the same local switching features as the CLS, i.e. it is Zeno free and possesses simple switching property. However, long-time switching properties of the PAS turns out to be rather nontrivial since these properties are closely related to positively invariant sets of affine modes of the PASs that have not been fully treated before. The positive invariance results in Section III provide essential tools that lead to verifiable characterization conditions as shown below.

Let $\mathcal{A}_i$ and $\mathcal{E}_i \equiv \{x \mid A_i x + d_i = 0, \ x \in \mathcal{P}_i\}$ denote the positively invariant set and equilibrium set of the ith mode of the PAS, respectively. An equilibrium x^e of the PAS is called *isolated* if a neighborhood of x^e exists such that it does not contain any other equilibrium. We call a PAS *with isolated equilibria* if each of its equilibria is isolated. Notice that an equilibrium is asymptotically stable only if it is isolated. Hence a PAS with isolated equilibria is of special interest in asymptotic stability analysis. It is easy to show via the convex structure of the PAS that each equilibrium of the PAS is isolated if and only if the equilibrium set of each mode contains at most one element.

In the sequel, we concentrate on a specific class of PASs, namely, the PASs whose trajectories have infinitely many mode transitions (in the positive time direction) whenever they start from non-equilibria. These PASs are referred to as the PASs with infinite mode switchings. It can be shown using the similar argument as in [27, Proposition 4.1] that the PAS has infinite mode switchings if and only if $\mathcal{A}_i = \mathcal{E}_i$ for each i. In view of this, we focus on each affine mode of the PAS. For the ith mode and its equilibrium set $\mathcal{E}_i$, we assume the (possibly zero) d_i to be in the null space of A_i^T as before (cf. Section III-C). Obviously if $\mathcal{E}_i$ is nonempty, then d_i must be zero. Consider the following cases: (1) $\mathcal{E}_i = \emptyset$. In this case, $\mathcal{A}_i = \mathcal{E}_i$ if and only if exactly one of the following holds: (1.1) the existence conditions in Theorem 3.1 fail when $d_i = 0$; (1.2) the existence conditions in Theorem 3.3 do not hold when $d_i \neq 0$. (2) $\mathcal{E}_i$ is a singleton set. In this case, since $d_i = 0$, the sole equilibrium in $\mathcal{E}_i$ can be taken as $x^e = 0$ (by a suitable state translation). Therefore, $b_i \leq C_i x^e = 0$, which implies that b_i has no positive element or equivalently the index set α associated with b_i is empty. Hence, $\mathcal{A}_i = \mathcal{E}_i$ if and only if $\mathcal{A}_i$ is trivial, i.e., $\mathcal{A}_i = \{0\}$, or equivalently the nontrivial positive invariance conditions in Theorem 3.2 fail. By virtue of the above discussions, we immediately have the following result:

Corollary 4.1: A PAS with isolated equilibria has infinite mode switchings if and only if the conditions in (1) or (2) discussed above hold for each of its modes.

Remark 4.1: We make two comments here. First, for a CLS, all of its equilibria are isolated if and only if it has a unique equilibrium at the origin. Hence, we deduce from Theorem 3.2 that a CLS with isolated equilibria has infinite mode switchings if and only if $\mathcal{A}_i$ is trivial for each i, which is further equivalent to the conditions that (C_i, A_i) is an observable pair and that A_i has no eigenvector in the polyhedral cone $\{x \mid C_i x \geq 0\}$. This recovers the result of [27, Proposition 17]. Second, if $\mathcal{E}_i$ is non-degenerate, i.e., $\mathcal{E}_i \cap \operatorname{int} \mathcal{P}_i$ is nonempty, then the similar characterization conditions for $\mathcal{A}_i = \mathcal{E}_i$ can be obtained as those in [27, Proposition 19] via the positive invariant results and the observation that $d_i = 0$.

V. APPLICATION TO EXACT SAFETY VERIFICATION OF AFFINE DYNAMICS

Safety verification and reachability analysis of complex systems has received tremendous interests in computer science and control engineering, motivated by applications such as air traffic control and embedded systems. Various analytic and numerical techniques have been proposed [1], [30], and two technical paths have been widely followed. One is based on approximation (e.g., over-/under-approximation and asymptotic approximation) for general nonlinear dynamics or fast computation, including the approximate bisimulation technique [14], the barrier certificate method [25], the abstraction predicate approach [30], and geometric programming [33]. The other path focuses on exact safety verification, particularly decidable verification schemes [7], but only for simpler dynamics such as linear or affine systems; typical approaches include model checking techniques via bisimulations and abstractions [18], and semi-algebraic methods [11], [19], [20], [34], [18]. Also see [31] for additional numerical techniques.

In this section, we consider exact safety verification of a class of affine systems whose defining matrix contains complex eigenvalues only. Examples of such systems include linear Hamiltonian dynamics with complex eigenvalues only. Furthermore, we assume that $\mathcal{S}_0$ and $\mathcal{S}_f$ are closed semi-algebraic sets, i.e., they are described by multi-variate polynomial equations or inequalities which are neither convex nor polyhedral in general. It is shown that the safety verification problem is decidable even if the ratios of mode frequencies of the dynamics are irrational. Two key tools are exploited: (i) Lemmas 2.2 and 2.3 that express the infimum of a linear combination of periodic functions in term of the sum of the minima of the periodic functions, and (ii) an algebraic technique that formulates finding the minima of the periodic functions as a semi-algebraic problem. Compared with the existing decidability and safety verification techniques, the proposed method makes a novel use of the positive invariance techniques developed in Section II.

Proposition 5.1: Let $\mathcal{S}_0 = \{x \in \mathbb{R}^n \mid p(x) \geq 0, w(x) = 0\}$ and $\mathcal{S}_f = \{x \in \mathbb{R}^n \mid f(x) \geq 0\}$ be closed semi-algebraic sets, where p, w, and f are vector-valued multivariate polynomials. Suppose that all the eigenvalues of A are known and have nonzero imaginary parts. Then checking safety of the affine dynamics $\dot{x} = Ax + d$ on the time interval $\Delta = \mathbb{R}$ is decidable.

Proof. Since A has no real eigenvalue, it is invertible and d is in the range of A. Therefore via a suitable state transformation, we may assume $d = 0$ and consider the linear dynamics $\dot{x} = Ax$. The sets $\mathcal{S}_0$ and $\mathcal{S}_f$ remain semialgebraic after the transformation. Since all the eigenvalues of A are complex, each entry of $e^{At}x$ can be written as $e^{\lambda t} t^p l(x) \cos(\omega t)$ or $e^{\lambda t} t^p l(x) \sin(\omega t)$ for some known λ, p, ω, where $p \geq 0$ is an integer, $\omega > 0$, $\lambda \pm \imath \omega$ is the corresponding complex eigenvalue of A, and $l(x)$ is a linear function of x. Therefore, via basic trigonometric relations and straightforward computations, we have $f(e^{At}x) = c(x) + \sum_{(\widetilde{\lambda}, \widetilde{p}, \widetilde{\omega})} e^{\widetilde{\lambda} t} t^{\widetilde{p}} \big[g_{(\widetilde{\lambda}, \widetilde{p}, \widetilde{\omega})}(x) \cos(\widetilde{\omega} t) + h_{(\widetilde{\lambda}, \widetilde{p}, \widetilde{\omega})}(x) \sin(\widetilde{\omega} t) \big]$, where $c(x)$ is a (vector-valued) multivariate polynomial, $g_{(\widetilde{\lambda}, \widetilde{p}, \widetilde{\omega})}(x)$, $h_{(\widetilde{\lambda}, \widetilde{p}, \widetilde{\omega})}(x)$ are (vector-valued) multivariate polynomials corresponding to the tuple $(\widetilde{\lambda}, \widetilde{p}, \widetilde{\omega})$, and $\widetilde{\omega}$'s associated with the same $(\widetilde{\lambda}, \widetilde{p})$ are all distinct. Notice that there are finitely many such tuples. We have the following claim:

Claim C1: the linear system is safe on $\Delta = \mathbb{R}$ if and only if for all $x \in \mathcal{S}_0$, $g_{(\widetilde{\lambda}, \widetilde{p}, \widetilde{\omega})}(x) = h_{(\widetilde{\lambda}, \widetilde{p}, \widetilde{\omega})}(x) = 0$ for $(\widetilde{\lambda}, \widetilde{p}) \neq 0$ and $c(x) + \sum_{\omega_j} \big[g_{(0,0,\omega_j)}(x) \cos(\omega_j t) + h_{(0,0,\omega_j)}(x) \sin(\omega_j t) \big] \geq 0, \forall \, t \in \mathbb{R}$, where each ω_j in the latter condition corresponds to the pure imaginary eigenvalues of A.

The proof for (C1) is as follows. The sufficiency is obvious. To see necessity, we first show that for each $x \in \mathcal{S}_0$, $g_{(\widetilde{\lambda}, \widetilde{p}, \widetilde{\omega})}(x) = h_{(\widetilde{\lambda}, \widetilde{p}, \widetilde{\omega})}(x) = 0$ whenever $(\widetilde{\lambda}, \widetilde{p}) \succ (0,0)$. Suppose not. Then there exist $x^* \in \mathcal{S}_0$ and $(\widetilde{\lambda}, \widetilde{p}) \succ (0,0)$ with $\big(g_{(\widetilde{\lambda}, \widetilde{p}, \widetilde{\omega})}(x^*), h_{(\widetilde{\lambda}, \widetilde{p}, \widetilde{\omega})}(x^*)\big) \neq 0$ for some $\widetilde{\omega} > 0$. Let (λ^*, p^*) be the largest such a pair in the lexicographical sense, i.e., $(\lambda^*, p^*) \succ (\widetilde{\lambda}, \widetilde{p})$ for any pair $(\widetilde{\lambda}, \widetilde{p})$ with nonzero $\big(g_{(\widetilde{\lambda}, \widetilde{p}, \widetilde{\omega})}(x^*), h_{(\widetilde{\lambda}, \widetilde{p}, \widetilde{\omega})}(x^*)\big)$. Therefore, $f(e^{At}x^*)$ tends to $e^{\lambda^* t} t^{p^*} \sum_{\widetilde{\omega}} \big[g_{(\lambda^*, p^*, \widetilde{\omega})}(x^*) \cos(\widetilde{\omega} t) + h_{(\lambda^*, p^*, \widetilde{\omega})}(x^*) \sin(\widetilde{\omega} t) \big]$ as $t \to +\infty$. However, by Corollary 2.1 (or Proposition 3.2) and $f(e^{At}x^*) \geq 0, \forall \, t \geq 0$, we have $g_{(\lambda^*, p^*, \widetilde{\omega})}(x^*) = h_{(\lambda^*, p^*, \widetilde{\omega})}(x^*) = 0$ for all $\widetilde{\omega}$, a contradiction. Similarly, by the reverse-time argument, we conclude that if $(\widetilde{\lambda}, \widetilde{p}) \prec (0,0)$ (or equivalently $\widetilde{\lambda} < 0$), $g_{(\widetilde{\lambda}, \widetilde{p}, \widetilde{\omega})}(x) = h_{(\widetilde{\lambda}, \widetilde{p}, \widetilde{\omega})}(x) = 0$ for all $x \in \mathcal{S}_0$. Hence, the claim holds.

Next we show that the necessary and sufficient conditions established by Claim C1 can be finitely verified. It is easy to see that checking the first condition, i.e., $g_{(\widetilde{\lambda}, \widetilde{p}, \widetilde{\omega})}(x) = h_{(\widetilde{\lambda}, \widetilde{p}, \widetilde{\omega})}(x) = 0$ with $(\widetilde{\lambda}, \widetilde{p}) \neq 0$ for all $x \in \mathcal{S}_0$, can be cast as a semialgebraic decision problem and thus is decidable. Hence, we only focus on the second condition which possesses a universal quantifier t. The case where the ratio of any two frequencies associated with pure imaginary eigenvalues of A is rational has been treated in [34, Section IV]. Hence, we assume that among all the frequencies ω_i of the pure imaginary eigenvalues of A, some of $\omega_i/\omega_j, i \neq j$, are rational and the others are irrational as follows.

Define the function $s_{\omega_j}(t, x) \equiv g_{(0,0,\omega_j)}(x) \cos(\omega_j t) + h_{(0,0,\omega_j)}(x) \sin(\omega_j t)$ for each ω_j. Following the similar treatment as in Section II-B, we obtain the collection of (disjoint) equivalent classes $E_{\omega_j} = \{ s_{\omega_i}(t, x) \,|\, \omega_i/\omega_j \text{ is rational} \}$ (which is independent of x). Let $\widetilde{\omega}_\ell > 0$ be a basis frequency associated with each E_{ω_j} and denote such the equivalent class by $E_{\widetilde{\omega}_\ell}$. Let $q_{\widetilde{\omega}_\ell}(t, x) \equiv \sum_{s_{\omega_i} \in E_{\widetilde{\omega}_\ell}} s_{\omega_i}(t, x)$. For any fixed

$x \in \mathbb{R}^n$, we see that each component of $q_{\widetilde{\omega}_\ell}(t, x)$ enjoys the same four properties of $q_{\widetilde{\omega}_s}$ stated in Section II-B. In particular, if the ith component $(q_{\widetilde{\omega}_\ell})_i(\cdot, x)$ is not identically zero, then it attains the maximal and minimal values $\mu_{i,\widetilde{\omega}_\ell}(x) > 0$ and $\nu_{i,\widetilde{\omega}_\ell}(x) < 0$ on $(-\infty, \infty)$ respectively. Furthermore, recall that for each $s_{\omega_j} \in E_{\widetilde{\omega}_\ell}$, $\omega_j/\widetilde{\omega}_\ell$ is a positive integer. Therefore, by the basic trigonometric results, we see that each $q_{\widetilde{\omega}_\ell}(t, x)$ can be expressed as a (vector-valued) multivariate polynomial function in terms of x, $\sin(\widetilde{\omega}_\ell t)$, and $\cos(\widetilde{\omega}_\ell t)$. In other words, letting $u_{\widetilde{\omega}_\ell} \equiv \sin(\widetilde{\omega}_\ell t)$ and $v_{\widetilde{\omega}_\ell} \equiv \cos(\widetilde{\omega}_\ell t)$ for each $E_{\widetilde{\omega}_\ell}$, we obtain a (vector-valued) polynomial function $\widetilde{q}_{\widetilde{\omega}_\ell}(x, u_{\widetilde{\omega}_\ell}, v_{\widetilde{\omega}_\ell})$ that is equivalent to $q_{\widetilde{\omega}_\ell}(t, x)$, where $u_{\widetilde{\omega}_\ell}^2 + v_{\widetilde{\omega}_\ell}^2 = 1$. Likewise, we write the time derivative $\frac{\partial q_{\widetilde{\omega}_\ell}(t,x)}{\partial t}$ as an equivalent (vector-valued) polynomial function $\widetilde{d}_{\widetilde{\omega}_\ell}(x, u_{\widetilde{\omega}_\ell}, v_{\widetilde{\omega}_\ell})$. Suppose there are k equivalent classes $E_{\widetilde{\omega}_\ell}$. Since $c(x) + \sum_{\omega_j} \left[g_{(0,0,\omega_j)}(x) \cos(\omega_j t) + h_{(0,0,\omega_j)}(x) \sin(\omega_j t) \right] = c(x) + \sum_{\ell=1}^k q_{\widetilde{\omega}_\ell}(t, x)$, we have:

Claim C2: for any fixed x and each i, $c_i(x) + \sum_{\ell=1}^k (q_{\widetilde{\omega}_\ell})_i(t, x) \geq 0$, $\forall\ t \in \mathbb{R}$ if and only if the following implication holds

$$\left[(\widetilde{d}_{\widetilde{\omega}_\ell})_i(x, u_{\widetilde{\omega}_\ell}, v_{\widetilde{\omega}_\ell}) = 0,\ u_{\widetilde{\omega}_\ell}^2 + v_{\widetilde{\omega}_\ell}^2 = 1,\ \forall\ \ell = 1, \cdots, k \right] \Rightarrow c_i(x) + \sum_{\ell=1}^k (\widetilde{q}_{\widetilde{\omega}_\ell})_i(x, u_{\widetilde{\omega}_\ell}, v_{\widetilde{\omega}_\ell}) \geq 0 \quad (6)$$

The proof for Claim C2 is as follows:

"Sufficiency": Without loss of generality, we assume that $(q_{\widetilde{\omega}_\ell})_i(\cdot, x)$ is not identically zero for each ℓ. Observe that for every ℓ, the real pairs $\left(u_{\widetilde{\omega}_\ell}, v_{\widetilde{\omega}_\ell} \right)$ that satisfy $(\widetilde{d}_{\widetilde{\omega}_\ell})_i(x, u_{\widetilde{\omega}_\ell}, v_{\widetilde{\omega}_\ell}) = 0$ and $u_{\widetilde{\omega}_\ell}^2 + v_{\widetilde{\omega}_\ell}^2 = 1$ correspond to critical points of the real-valued function $(q_{\widetilde{\omega}_\ell})_i(\cdot, x)$ (by the definition of $\widetilde{d}_{\widetilde{\omega}_\ell}$). Therefore, one of such the $(u_{\widetilde{\omega}_\ell}, v_{\widetilde{\omega}_\ell})$'s, say $\left(u_{\widetilde{\omega}_\ell}^*, v_{\widetilde{\omega}_\ell}^* \right)$, is a minimizer of the bounded periodic function $(q_{\widetilde{\omega}_\ell})_i(\cdot, x)$. This shows that $(\widetilde{q}_{\widetilde{\omega}_\ell})_i(x, u_{\widetilde{\omega}_\ell}^*, v_{\widetilde{\omega}_\ell}^*) = \nu_{i,\widetilde{\omega}_\ell}(x) < 0$ (see the definition of $\nu_{i,\widetilde{\omega}_\ell}$ above). We thus deduce from (6) that $c_i(x) + \sum_{\ell=1}^k \nu_{i,\widetilde{\omega}_\ell}(x) \geq 0$. Since $(q_{\widetilde{\omega}_\ell})_i(t, x) \geq \nu_{i,\widetilde{\omega}_\ell}(x)$ for all t, we have $c_i(x) + \sum_{\ell=1}^k (q_{\widetilde{\omega}_\ell})_i(t, x) \geq c_i(x) + \sum_{\ell=1}^k \nu_{i,\widetilde{\omega}_\ell}(x) \geq 0$ for all t as desired.

"Necessity": We show this by contradiction. Suppose $c_i(x) + \sum_{\ell=1}^k q_{\widetilde{\omega}_i}(t, x) \geq 0$, $\forall\ t \in \mathbb{R}$ but there exist pairs $(u_{\widetilde{\omega}_\ell}^*, v_{\widetilde{\omega}_\ell}^*)$ with $(u_{\widetilde{\omega}_\ell}^*)^2 + (v_{\widetilde{\omega}_\ell}^*)^2 = 1$ and $(\widetilde{d}_{\widetilde{\omega}_\ell})_i(x, u_{\widetilde{\omega}_\ell}^*, v_{\widetilde{\omega}_\ell}^*) = 0$, $\ell = 1, \cdots, k$, such that $c_i(x) + \sum_{\ell=1}^k (\widetilde{q}_{\widetilde{\omega}_\ell})_i(x, u_{\widetilde{\omega}_\ell}^*, v_{\widetilde{\omega}_\ell}^*) < 0$. Notice that for each ℓ, $(\widetilde{q}_{\widetilde{\omega}_\ell})_i(x, u_{\widetilde{\omega}_\ell}^*, v_{\widetilde{\omega}_\ell}^*) \in \left[\nu_{i,\widetilde{\omega}_\ell}(x), \mu_{i,\omega_\ell}(x) \right]$ and that $(q_{\widetilde{\omega}_\ell})_i(t, x)$ is onto $\left[\nu_{i,\widetilde{\omega}_\ell}(x), \mu_{i,\widetilde{\omega}_\ell}(x) \right]$. Consequently, by the properties of $(q_{\widetilde{\omega}_\ell})_i(\cdot, x)$ and Lemmas 2.2 and 2.3 (which rely on irrational ratio of any two basis frequenecies), we deduce the existence of $t_* \in \mathbb{R}$ such that $(q_{\widetilde{\omega}_\ell})_i(t_*, x)$ is arbitrarily close to $(\widetilde{q}_{\widetilde{\omega}_\ell})_i(x, u_{\widetilde{\omega}_\ell}^*, v_{\widetilde{\omega}_\ell}^*)$ for each $\ell = 1, \cdots, k$. This, together with the continuity of $(q_{\widetilde{\omega}_\ell})_i(\cdot, x)$, shows that $c_i(x) + \sum_{\ell=1}^k (q_{\widetilde{\omega}_\ell})_i(t_*, x) < 0$, a contradiction.

In view of Claim C2, we now complete the proof for the second case. Recalling $x \in \mathcal{S}_0 \Leftrightarrow \left[p(x) \geq 0, w(x) = 0 \right]$ and defining $\widehat{u}^i, \widehat{v}^i \in \mathbb{R}^k, i = 1, \cdots, m$, we obtain the following implication that is

equivalent to the second condition of Claim C1: for each i,

$$\left[\, p(x) \geq 0,\ w(x) = 0,\ (\widetilde{d}_{\widetilde{\omega}_\ell})_i(x, \widehat{u}_j^i, \widehat{v}_j^i) = 0,\ (\widehat{u}_j^i)^2 + (\widehat{v}_j^i)^2 - 1 = 0 \,\right] \implies c_i(x) + \sum_{\ell=1}^{k}(\widetilde{q}_{\widetilde{\omega}_\ell})_i(x, \widehat{u}_j^i, \widehat{v}_j^i) \geq 0$$

Denoting $\widetilde{x} = \left(x, \widehat{u}^1, \cdots, \widehat{u}^m, \widehat{v}^1, \cdots, \widehat{v}^m \right) \in \mathbb{R}^{n+k \times 2m}$, we can rewrite the above implication as $\left[\, \widetilde{p}(\widetilde{x}) \geq 0,\ \widetilde{w}(\widetilde{x}) = 0 \,\right] \implies \left[\, \widetilde{g}(\widetilde{x}) = 0,\ \widetilde{h}(\widetilde{x}) \geq 0 \,\right]$, where $\widetilde{p}$, $\widetilde{w}$, $\widetilde{g}$ and $\widetilde{h}$ are vector-valued polynomials. Since checking this implication can be made in finite steps via the Tarski-Seidenberg decision procedure, the safety verification problem is decidable. $\square$

Example 5.1: Consider the linear system on $\mathbb{R}^8$ whose defining matrix $A = \mathrm{diag}(A_1, A_2, A_3, A_4)$. Here the matrix blocks A_i are $A_1 = \begin{bmatrix} \sigma_1 & \omega_1 \\ -\omega_1 & \sigma_1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & \pi \\ -\pi & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 2\pi \\ -2\pi & 0 \end{bmatrix}$, $A_4 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, where $\sigma_1 \neq 0$ and $\omega_1 > 0$. Let the initial set $\mathcal{S}_0 = \{x \in \mathbb{R}^8 \,|\, \|x - x^*\|_2^2 \leq 1\}$ for a given x^* and the safe region $\mathcal{S}_f = \{x \in \mathbb{R}^8 \,|\, c^T x \geq b\}$ for some $c \in \mathbb{R}^8$ and $b \in \mathbb{R}$. To simplify notation, let $c^T = (c_1^T, \cdots, c_4^T)$ and $x = ((x^1)^T, \cdots, (x^4)^T)^T$, where $c_i, x^i \in \mathbb{R}^2$ correspond to the matrix block A_i, and let the symplectic matrix $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Therefore, $c^T e^{At} x = e^{\sigma_1 t}[c_1^T x^1 \cos(\omega_1 t) + (Sc_1)^T x^1 \sin(\omega_1 t)] + q_1(x,t) + q_2(x,t)$, where $q_1(t,x) = c_2^T x^2 \cos(\pi t) + (Sc_2)^T x^2 \sin(\pi t) + c_3^T x^3 \cos(2\pi t) + (Sc_3)^T x^3 \sin(2\pi t)$ and $q_2(t,x) = c_4^T x^4 \cos(t) + (Sc_4)^T x^4 \sin(t)$. Notice that $q_1(t,x)$ and $q_2(t,x)$ are periodic in t but their frequency ratio is irrational. Moreover, for a fixed x, even though q_1 is the sum of two sinusoidal functions with frequencies π and 2π respectively, the maximal (resp. minimal) values of q_1 cannot be simply written as the sum of the maximal (resp. minimal) values of the two sinusoidal functions. This is why we introduce the time derivative of q_1 to characterize the extremal value of q_1. Now define $u_1 \equiv \cos(\pi t), v_1 \equiv \sin(\pi t)$ and $u_2 \equiv \cos(t), v_2 \equiv \sin(t)$. Rewriting $q_i(t,x)$ and $\frac{\partial q_i(t,x)}{\partial t}, i = 1, 2$ in terms of x, u_1, v_1, u_2, v_2 and applying the argument in Proposition 5.1, we transform the safety verification problem on the time domain $\Delta = \mathbb{R}$ into the following decidable semi-algebraic decision problem $\left[\, P \Rightarrow Q \,\right]$ on $\mathbb{R}^{12}$, where $P \equiv \left[\pi(-c_2^T x^2 v_1 + (Sc_2)^T x^2 u_1 - 4c_3^T x^3 u_1 v_1 + 2(Sc_3)^T x^3 (u_1^2 - v_1^2)) = 0\right] \wedge [u_1^2 + v_1^2 - 1 = 0] \wedge \left[-c_4^T x^4 v_2 + (Sc_4)^T x^4 u_2 = 0\right] \wedge \left[u_2^2 + v_2^2 - 1 = 0\right] \wedge \left[(x - x^*)^T(x - x^*) - 1 \leq 0\right]$ and $Q \equiv \left[c_2^T x^2 u_1 + (Sc_2)^T x^2 v_1 + c_3^T x^3 (u_1^2 - v_1^2) + 2(Sc_3)^T x^3 u_1 v_1 + c_4^T x^4 u_2 + (Sc_4)^T x^4 v_2 - b \geq 0\right] \wedge \left[c_1^T x^1 = 0\right] \wedge \left[(Sc_1)^T x^1 = 0\right]$.

Remark 5.1: Admittedly, the assumption on A's eigenvalues poses a limitation on its applicability of Proposition 5.1. For linear dynamics with purely real eigenvalues, decidability and numerical issues have been studied [11], [19], [20], [34]. The case where A has both real and complex eigenvalues is much harder to deal with as shown in model theory [21]. Nevertheless Proposition 5.1 reveals an interesting perspective for decidability analysis via dynamic system techniques.

Remark 5.2: While Proposition 5.1 satisfactorily addressees the decidability issue, more practically important issues include numerical verification and computational complexity. For a small scale problem, real algebraic geometry provides instrumental arithmetic tools, e.g. the quantifier elimination technique

[8]. However, for a large size problem, global verification of semialgebraic conditions is NP-hard. This difficulty can be overcome by employing recently developed polynomial optimization techniques that offer a numerical alternative with arbitrary precision, e.g. the sum-of-squares relaxation approach. This approach, armed with semidefinite programming methods, offers an efficient numerical solutions for semialgebraic verification problems [24]; see [25], [34] for its applications in safety verification.

Remark 5.3: Compared with approximation methods such as the abstraction predicate technique [30] and the barrier certificate approach [25], the above proposition establishes necessary and sufficient safety conditions without construction of Lyapunov-like functions, which are known to be difficult for general dynamics and constraint sets. Future work will focus on extensions of the positive invariance techniques to nonlinear dynamics and switching systems, particularly those with structural properties.

VI. CONCLUSIONS

In this paper, we have addressed the existence of a positively invariant set of an affine dynamics on a polyhedron and its numerical verification. Necessary and sufficient existence conditions are established via algebraic properties of the lexicographical relation and long-time dynamic analysis techniques. These positive invariance results form a cornerstone for long-time switching characterization of piecewise affine systems and safety verification on semi-algebraic sets. Deep understanding of switching behaviors of affine hybrid dynamics and safety analysis problems warrants further investigation of the positive invariance issues and their implications in stability analysis and other long-time dynamic analysis.

VII. APPENDIX: PROOF OF THEOREM 3.1

Proof. The case where the index set α is empty is trivial since $0 \in \mathcal{A}$. Note that the matrix A needs not to have a real eigenvalue in this case. In the subsequent development we assume that α is nonempty.

"Sufficiency". We consider two cases as follows:

(S1) $\alpha \neq \emptyset, \gamma = \emptyset$. In this case, there exist real eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_k \geq 0$ and $v^i \in V(\lambda_i)$ such that $\left(P(C_\alpha^1, v^1), \cdots, P(C_\alpha^k, v^k)\right) \succ 0$ and $\left(P(C_\beta^1, v^1), \cdots, P(C_\beta^k, v^k)\right) \succcurlyeq 0$. Let $\alpha \subseteq \theta \subseteq \alpha \cup \beta$ be such that $\widetilde{P} \equiv \left(P(C_\theta^1, v^1), \cdots, P(C_\theta^k, v^k)\right) \succ 0$ and $\left(P(C_{\bar\theta}^1, v^1), \cdots, P(C_{\bar\theta}^k, v^k)\right) \equiv 0$, where $\bar\theta$ is the complement of θ. Hence, each row of $\widetilde{P}$ contains a positive leading entry. Without loss of generality, we further assume that $\widetilde{P}$ is of the echelon structure via suitable row switchings and that each $P(C_\theta^i, v^i)$ contains a positive leading entry in $\widetilde{P}$. For each $P(C_\theta^i, v^i)$, define the index set

$$\theta_i \equiv \left\{ j \in \theta \mid \text{the } j\text{th row of } P(C_\theta^i, v^i) \text{ contains a positive leading entry of } \widetilde{P} \right\}.$$

It is clear that θ_i is nonempty, $\theta = \cup_{i=1}^k \theta_i$, $\theta_i \cap \theta_j = \emptyset$ for any $i \neq j$. Besides, due to the lexicographical relation and echelon structure of $\widetilde{P}$, we have, for each i, $P(C_{\theta_i}^i, v^i) \succ 0$ and $P(C_{\theta_i}^j, v^j) \equiv 0$ for $1 \leq j < i$.

The following diagram illustrates this structure which is critical to the rest of the proof:

$$\widetilde{P} = \begin{pmatrix} P(C_{\theta_1}^1, v^1) & \star & \cdots & \cdots & & \star \\ & P(C_{\theta_2}^2, v^2) & \star & & \cdots & & \star \\ & & \ddots & & & & \vdots \\ & & & P(C_{\theta_{k-1}}^{k-1}, v^{k-1}) & & \star \\ & & & & P(C_{\theta_k}^k, v^k) \end{pmatrix}$$

where $\star$ denotes the blocks without a leading entry in $\widetilde{P}$. For each i, define the index set $\widetilde{\theta}_i \equiv \cup_{j=i}^k \theta_j$. We claim that for every $i = k, \cdots, 1$ and any $h \in \mathbb{R}_{++}^{|\widetilde{\theta}_i|}$, there exists a vector $u \equiv \bigoplus_{j=i}^k u^j \in \bigoplus_{j=i}^k V(\lambda_j)$ such that $\sum_{j=i}^k C_{\widetilde{\theta}_i}^j e^{J_j t} u^j \geq h$ for all $t \geq 0$. We prove the claim by induction on i as follows. First, consider $i = k$. In this case, the claim follows immediately from Proposition 3.1. Suppose that the claim holds for all $i = k, \cdots, \ell+1$, where $1 \leq \ell \leq k-1$. Now consider $i = \ell$. By the induction hypothesis, for a given $h = (h_1, h_2) \in \mathbb{R}_{++}^{|\widetilde{\theta}_\ell|}$ with $h_1 \in \mathbb{R}_{++}^{|\theta_\ell|}$ and $h_2 \in \mathbb{R}_{++}^{|\widetilde{\theta}_{\ell+1}|}$, there exists $\widetilde{u} \equiv \bigoplus_{j=\ell+1}^k \widetilde{u}^j$ with $\widetilde{u}^j \in V(\lambda_j)$ such that $\sum_{j=\ell}^k C_{\widetilde{\theta}_{\ell+1}}^j e^{J_j t} \widetilde{u}^j \geq h_2$, $\forall\, t \geq 0$. Notice that $\lambda_\ell > \lambda_j$ for all $j = \ell+1, \cdots, k$. Hence, $e^{-\lambda_\ell t} \| \sum_{j=\ell}^k C_{\theta_\ell}^j e^{J_j t} \widetilde{u}^j \|_2$ is bounded on $[0, \infty)$ as $\widetilde{u}^j$'s are fixed, namely, there exists a positive scalar ρ satisfying $e^{-\lambda_\ell t} \| \sum_{j=\ell}^k C_{\theta_\ell}^j e^{J_j t} \widetilde{u}^j \|_2 \leq \rho$, $\forall\, t \geq 0$. Moreover, by Proposition 3.1 and $P(C_{\theta_\ell}^\ell, v^\ell) \succ 0$, we deduce the existence of $\widetilde{u}^\ell \in V(\lambda_\ell)$ such that $C_{\theta_\ell}^\ell e^{J_\ell t} \widetilde{u}^\ell \geq e^{\lambda_\ell t}(h_1 + \rho \mathbf{1})$, where $\mathbf{1} = (1, \cdots, 1)^T$. Consequently, $C_{\theta_\ell}^\ell e^{J_\ell t} \widetilde{u}^\ell + \sum_{j=\ell+1}^k C_{\theta_\ell}^j e^{J_j t} \widetilde{u}^j \geq e^{\lambda_\ell t} h_1 \geq h_1$, $\forall\, t \geq 0$. Furthermore, it is noted, by the definition of θ_j and the echelon structure of $\widetilde{P}$, that $P(C_{\theta_j}^\ell, v^\ell) \equiv 0$ for all $j = \ell+1, \cdots, k$. Hence, we obtain, via Proposition 3.1, that $C_{\theta_j}^\ell e^{J_\ell t} \widetilde{u}^\ell \equiv 0$ for all $j = \ell+1, \cdots, k$. As a result, for any $i = \ell+1, \cdots, k$, we have $C_{\theta_i}^\ell e^{J_\ell t} \widetilde{u}^\ell + \sum_{j=\ell+1}^k C_{\theta_i}^j e^{J_j t} \widetilde{u}^j = \sum_{j=\ell+1}^k C_{\theta_i}^j e^{J_j t} \widetilde{u}^j$, $\forall\, t \geq 0$. Letting $u = \bigoplus_{j=\ell}^k \widetilde{u}_i \in \bigoplus_{j=\ell}^l V(\lambda_j)$, it is easy to see from the above development that

$$\sum_{j=\ell}^k C_{\widetilde{\theta}_\ell}^j e^{J_j t} \widetilde{u}^j = \begin{pmatrix} C_{\theta_\ell} e^{J_j t} \widetilde{u}^j \\ ----- \\ C_{\widetilde{\theta}_{\ell+1}} e^{J_j t} \widetilde{u}^j \end{pmatrix} = \begin{pmatrix} \sum_{j=\ell}^k C_{\theta_\ell}^j e^{J_j t} \widetilde{u}^j \\ ------ \\ \sum_{j=\ell}^k C_{\theta_{\ell+1}}^j e^{J_j t} \widetilde{u}^j \\ \vdots \\ \sum_{j=\ell}^k C_{\theta_k}^j e^{J_j t} \widetilde{u}^j \end{pmatrix} = \begin{pmatrix} \sum_{j=\ell}^k C_{\theta_\ell}^j e^{J_j t} \widetilde{u}^j \\ ------ \\ \sum_{j=\ell+1}^k C_{\theta_{\ell+1}}^j e^{J_j t} \widetilde{u}^j \\ \vdots \\ \sum_{j=\ell+1}^k C_{\theta_k}^j e^{J_j t} \widetilde{u}^j \end{pmatrix} \geq \begin{pmatrix} h_1 \\ -- \\ h_2 \end{pmatrix} = h$$

for all $t \geq 0$. This shows that u is the desired vector and thus completes the proof of the claim. Letting $i = 1$, then for $h \in \mathbb{R}_{++}^{|\theta|}$ with $h_\alpha = b^+ > 0$, we obtain $u \equiv \bigoplus_{j=1}^k u^j$ with $u^j \in V(\lambda_j)$ satisfying the condition stated in the claim. Note that $C_{\widetilde{\theta}}^j e^{J_j t} u^j \equiv 0, \forall\, t$ for all $j = 1, \cdots, k$, where $\bar{\theta} \subseteq \beta$.

Therefore, appropriately expanding u to $u^* = (u^T, 0)^T \in \mathbb{R}^n$ by adding the zero subvectors, we have $C_\alpha e^{At} u^* = \sum_{j=1}^{k} C_\alpha^j e^{J_j t} u^j \geq b^+$ and $C_\beta e^{At} u^* = \sum_{j=1}^{k} C_\beta^j e^{J_j t} u^j \geq 0$ for all $t \geq 0$. This shows $u^* \in \mathcal{A}$.

(S2) $\alpha \neq \emptyset, \gamma \neq \emptyset$. If, in addition to $P^+ \succ 0$ and $P^0 \succcurlyeq 0$, $P^- \succcurlyeq 0$ holds true, then we have $\widetilde{P} \succ 0$ and $\left(P(C_\theta^1, v^1), \cdots, P(C_\theta^k, v^k) \right) \equiv 0$, where $\widetilde{P}$ is defined as before with $\theta \subseteq \beta \cup \gamma$. Hence we can apply the similar argument as in (S1) to obtain $u^* \in \mathbb{R}^n$ such that $C_\alpha e^{At} u^* \geq b^+$, $C_\beta e^{At} u^* \geq 0$, and $C_\gamma e^{At} u^* \geq 0 \geq b^-$ for all $t \geq 0$. Thus $u^* \in \mathcal{A}$. Now suppose $P^- \nsucceq 0$ but the condition (4) holds. Let $\bar{\psi} \equiv \{1, \cdots, m\} \setminus \psi$, where ψ is defined in (4). Therefore $\bar{\psi} = \alpha \cup \beta \cup (\gamma \setminus \psi)$. Hence, $P_{\bar{\psi}} \equiv \left(P(C_{\bar{\psi}}^1, v^1), \cdots, P(C_{\bar{\psi}}^k, v^k) \right) \succcurlyeq 0$. By (S1), there exists $u = \bigoplus_{j=1}^{k} u^j$ with $u^j \in V(\lambda_j)$ such that

$$\sum_{j=1}^{k} C_\alpha^j e^{J_i t} u^j \geq b^+, \quad \sum_{j=1}^{k} C_\beta^j e^{J_i t} u^j \geq 0, \quad \sum_{j=1}^{k} C_{\gamma \setminus \psi}^j e^{J_i t} u^j \geq 0 \geq b_{\gamma \setminus \psi}, \quad \forall\, t \geq 0$$

Particularly, since $\sum_j C_\phi^{kj} \mathcal{L}^0(v^{kj}) = \sum_j C_\phi^{kj} v^{kj} \geq b_\phi$, we deduce, via the proof of Proposition 3.1 and Remark 3.1, that the positive coefficient μ_0 in u^k can be taken as one. Therefore, using the lexicographical relation and $\lambda_k = 0$, we have

$$\sum_{i=1}^{k} C_\psi^i e^{J_i t} u^i = C_\psi^k e^{J_k t} u^k = e^{\lambda_k t} \mu_0 \sum_j C_\psi^{kj} \mathcal{L}^0(v^{kj}) = \sum_j C_\psi^{kj} v^{kj} \geq b_\psi, \ \forall\, t \geq 0.$$

Consequently, appropriately expanding u to $u^* \in \mathbb{R}^n$ by adding the zero subvectors, we obtain $u^* \in \mathcal{A}$.

"Necessity". Let $x^* \in \mathcal{A}$, i.e., $C e^{At} x^* \geq b, \forall\, t \geq 0$. Since $\alpha \neq \emptyset$, $C e^{At} x^*$ is not identically zero, i.e., $x^* \notin \overline{O}(C, A)$, where $\overline{O}(C, A)$ denotes the unobservable subspace of (C, A). We deduce from (2) that

$$C e^{At} x^* = \sum_i \left\{ e^{\lambda_i t} \sum_{k=0}^{\widehat{n}_i - 1} \frac{t^k}{k!} \left(\sum_j C^{ij} \mathcal{L}^k(v^{ij}) + \sum_s u_s^{ik} \sin(\omega_s^{ik} t + \theta_s^{ik}) \right) \right\}, \tag{7}$$

where λ_i's are the real parts of the eigenvalues of A, $\widehat{n}_i$ is the largest order of the Jordan blocks associated with (possibly complex) eigenvalues with the real part λ_i, and (possibly zero) $u_s^{ik} \in \mathbb{R}^m$, $\omega_s^{ik} \in \mathbb{R}_{++}$, $\theta_s^{ik} \in \mathbb{R}$ depends on C, A, x^* only. It should be noted that if λ_i is such that the matrix A has no Jordan block associated with a real eigenvalue λ_i (namely, λ_i corresponds to a complex eigenvalue of A), then $\sum_j C^{ij} \mathcal{L}^k(v^{ij}) = 0$ for all k in (7). Let the real parts of the eigenvalues of A be labeled in a descending order, i.e., $\lambda_1 > \lambda_2 > \cdots > \lambda_k \geq 0 > \lambda_{k+1} > \cdots > \lambda_q$. For each nonnegative real part λ_i, define the vector tuple $P(C^i, v^i)$ as follows: if λ_i corresponds a complex eigenvalue of A, then we set $P(C^i, v^i) \equiv 0$; otherwise, i.e., λ_i is a real eigenvalue of A,

$$P(C^i, v^i) \equiv \left(\sum_j C^{ij} \mathcal{L}^{\widehat{n}_i - 1}(v^{ij}), \sum_j C^{ij} \mathcal{L}^{\widehat{n}_i - 2}(v^{ij}), \cdots, \sum_j C^{ij} \mathcal{L}^0(v^{ij}) \right), \quad i = 1, \cdots, k$$

where $v^i = \bigoplus_j v^{ij} \in V(\lambda_i)$ is the subvector in x^* corresponding to λ_i. We show below that the tuples $P^+ \equiv \left(P(C_\alpha^1, v^1), \cdots, P(C_\alpha^k, v^k) \right), P^0 \equiv \left(P(C_\beta^1, v^1), \cdots, P(C_\beta^k, v^k) \right)$, and $P^- \equiv \left(P(C_\gamma^1, v^1), \cdots, P(C_\gamma^k, v^k) \right)$ satisfy the conditions (a) and (b) in the theorem. For notational simplicity, let P_i^+, P_i^0, P_i^- denote the ith

row of P^+, P^0, and P^- respectively.

(N1) To prove $P^+ \succ 0$, it is sufficient to show that for each $\ell \in \alpha$, $P_\ell^+ \succ 0$, i.e., $P_\ell^+ \neq 0$ and the leading entry in P_ℓ^+ is positive. Let the real pair (λ, p) represent the "mode" of $e^{\lambda t} t^p$ in $C_\ell e^{At} x^*$, where p is a nonnegative integer, and let $(\widetilde{\lambda}, \widetilde{p})$ represent the largest non-vanishing mode in $C_\ell e^{At} x^*$ in the sense that $(\widetilde{\lambda}, \widetilde{p}) \succ (\lambda_i, p_i)$ for any other non-vanishing mode defined by (λ_i, p_i). Since $C_\ell e^{At} x^* \geq b_\ell > 0, \forall\, t \geq 0$, we must have $(\widetilde{\lambda}, \widetilde{p}) \succcurlyeq (0,0)$ because otherwise, $C_\ell e^{At} x^* \to 0$ as $t \to \infty$, a contradiction. Since $\widetilde{\lambda} \geq 0$, $\widetilde{\lambda} = \lambda_r$ for some $r \in \{1, \cdots, k\}$. Furthermore, we deduce from (7) that $C_\ell e^{At} x^*$ tends to $e^{\lambda_r t} \dfrac{t^{\widetilde{p}}}{\widetilde{p}!} \Big[\rho_0 + \sum_{s=1}^{k_\ell} \rho_s \sin(\omega_s t + \theta_s) \Big]$ as $t \to +\infty$, where $(\rho_0, \rho_1, \cdots, \rho_{k_\ell}) \neq 0$ depends on C_ℓ, A, x^* only and $\rho_0 = \sum_j C_\ell^{rj} \mathcal{L}^{\widetilde{p}}(v^{rj})$. Since $(\lambda_r, \widetilde{p})$ denotes the largest non-vanishing mode in $C_\ell e^{At} x^*$, all the terms before ρ_0 in P_ℓ^+ must be zero. Therefore, ρ_0 is the first nonzero entry in P_ℓ^+. Since $b_\ell > 0$, we deduce $\rho_0 > 0$ in view of (a) of Proposition 3.2.

(N2) To prove $P^0 \succcurlyeq 0$, we show that $P_{i_\ell}^0 \succcurlyeq 0$ for each $\ell \in \beta$, where i_ℓ is the row index of P^0 corresponding to $\ell \in \beta$. Suppose, in contrast, that there exists $\ell \in \beta$ such that the first nonzero element in $P_{i_\ell}^0$ is negative which corresponds to the mode in $C_\ell e^{At} x^*$ defined by $(\lambda_r, \widetilde{p})$. Hence, $(\lambda_r, \widetilde{p}) \succcurlyeq (0,0)$, $\sum_j C_\ell^{rj} \mathcal{L}^{\widetilde{p}}(v^{rj}) < 0$, $\sum_j C_\ell^{rj} \mathcal{L}^p(v^{rj}) = 0$ for $\widetilde{p} < p \leq \bar{n}_r - 1$, and $P(C_\ell^i, v^i) = 0$ for all $i = 1, \cdots, r+1$. Let $(\widehat{\lambda}, \widehat{p})$ represent the largest non-vanishing mode in $C_\ell e^{At} x^*$. Therefore, $(\widehat{\lambda}, \widehat{p})$ exists and satisfies $(\widehat{\lambda}, \widehat{p}) \succcurlyeq (\lambda_r, \widetilde{p}) \succcurlyeq (0,0)$. Moreover, $C_\ell e^{At} x^*$ tends to $e^{\widehat{\lambda} t} \dfrac{t^{\widehat{p}}}{\widehat{p}!} \Big[\rho_0 + \sum_{s=1}^{k_\ell} \rho_s \sin(\omega_s t + \theta_s) \Big]$ as $t \to +\infty$, where $(\rho_0, \rho_1, \cdots, \rho_{k_\ell}) \neq 0$. Notice that either $\rho_0 = 0$ when $(\widehat{\lambda}, \widehat{p}) \succ (\lambda_r, \widetilde{p})$ or $\rho_0 < 0$ when $(\widehat{\lambda}, \widehat{p}) = (\lambda_r, \widetilde{p})$. However, this contradicts (a) of Proposition 3.2 since $b_\ell = 0$.

(N3) To show the implication (4) holds for P^-, define the (nonempty) index set $\vartheta \equiv \{ j \in \gamma \mid$ the jth row of P^- has a negative leading entry $\}$. For each $\ell \in \vartheta$, let the pair $(\lambda_r, \widetilde{p}) \succcurlyeq (0,0)$ correspond to the mode in $C_\ell e^{At} x^*$ associated with the negative leading entry in $P_{i_\ell}^-$ (again i_ℓ corresponds to the $\ell \in \vartheta$), and let $(\widehat{\lambda}, \widehat{p}) \succcurlyeq (\lambda_r, \widetilde{p})$ correspond to the largest non-vanishing mode in $C_\ell e^{At} x^*$. We claim that $(\widehat{\lambda}, \widehat{p}) = (\lambda_r, \widetilde{p})$. Suppose not. Then $(\widehat{\lambda}, \widehat{p}) \succ (0,0)$ and $C_\ell e^{At} x^*$ tends to $e^{\widehat{\lambda} t} \dfrac{t^{\widehat{p}}}{\widehat{p}!} \Big[\rho_0 + \sum_{s=1}^{k_\ell} \rho_s \sin(\omega_s t + \theta_s) \Big]$ as $t \to +\infty$ with $\rho_0 = 0$ and $(\rho_1, \cdots, \rho_{k_\ell}) \neq 0$, a contradiction to (a) of Proposition 3.2. Hence $(\widehat{\lambda}, \widehat{p}) = (\lambda_r, \widetilde{p})$ and $C_\ell e^{At} x^*$ is tends to $e^{\lambda_r t} \dfrac{t^{\widetilde{p}}}{\widetilde{p}!} \Big[\widetilde{\rho}_0 + \sum_{s=1}^{k_\ell} \widetilde{\rho}_s \sin(\omega_s t + \theta_s) \Big]$ for $t \to +\infty$ with $\widetilde{\rho}_0 < 0$. We thus deduce $(\lambda_r, \widetilde{p}) = (0,0)$ and $\lambda_r = \lambda_k = 0$ by (b) of Proposition 3.2, namely, the negative leading entry in $P_{i_\ell}^-$ appears in the last column of P^-. This shows that ϑ equals to ψ defined in (4) and thus $\big(P(C_{\gamma \setminus \psi}^1, v^1), \cdots, P(C_{\gamma \setminus \psi}^k, v^k) \big) \succcurlyeq 0$. Moreover, we obtain $\widetilde{\rho}_0 \equiv \sum_j C_\ell^{kj} v^{kj} \geq b_\ell$ for each $\ell \in \psi$ from (c) of Proposition 3.2. Therefore, $\sum_j C_\psi^{kj} v^{kj} \geq b_\psi$ holds. Following the similar argument, we also have, for each $\ell \in \phi \subseteq \alpha$, $C_\ell e^{At} x^*$ tends to $\sum_j C_\ell^{kj} v^{kj} + \sum_{s=1}^{k_\ell} \widetilde{\rho}_s \sin(\omega_s t + \theta_s)$ as $t \to +\infty$. This yields, via (c) of Proposition 3.2, that $\sum_j C_\ell^{kj} v^{kj} \geq b_\ell > 0$. As a result, $\sum_j C_\phi^{kj} v^{kj} \geq b_\phi$.

Finally we remove each nonnegative λ_i corresponding to a complex eigenvalue of A from $(\lambda_1, \cdots, \lambda_k)$ and its associated zero block from P^+, P^0, P^-. Since the index set α is nonempty, there exists $\ell \in \alpha$ such that the long-time dominating mode of $C_\ell\, e^{At} x^*$ has a positive ρ_0 as shown in (N1). This implies that at least one of $\lambda_i \geq 0, i = 1, \cdots, k$ is a real eigenvalue of A. For each remaining real eigenvalue $\lambda_i \geq 0$, let $\bar{n}_i$ be the largest order of the Jordan blocks associated with the real λ_i. If the integer $\widehat{n}_i$, the largest order of the Jordan blocks associated with (possibly complex) eigenvalues with the real part λ_i, is greater than $\bar{n}_i$, then we can replace $P(C^i, v^i)$ by

$$P(C^i, v^i) \equiv \Big(\sum_j C^{ij}\, \mathcal{L}^{\bar{n}_i - 1}(v^{ij}), \sum_j C^{ij}\, \mathcal{L}^{\bar{n}_i - 2}(v^{ij}), \cdots, \sum_j C^{ij}\, \mathcal{L}^0(v^{ij}) \Big),$$

since $\sum_j C^{ij}\, \mathcal{L}^p(v^{ij}) = 0$ for $p \geq \bar{n}_i$. We thus obtain the nonnegative real eigenvalues $\widetilde{\lambda}_1 > \cdots > \widetilde{\lambda}_k \geq 0$ and the tuples $\widetilde{P}^+, \widetilde{P}^0, \widetilde{P}^-$. It is easy to verify that removing the zero blocks from P^+, P^0, P^- does not change the lexicographical relations or the implications proved in (N1–N3) for $\widetilde{P}^+, \widetilde{P}^0, \widetilde{P}^-$, i.e., $\widetilde{P}^+ \succ 0$, $\widetilde{P}^- \succeq 0$ and the implication (4) holds for $\widetilde{P}^-$. This shows that nonnegative eigenvalues $\widetilde{\lambda}_i$ and $v^i \in V(\widetilde{\lambda}_i)$ satisfy the conditions (a)–(b). $\square$

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