Spherical Parametrizations

Xiao-Ming Fu

Outlines

- Definition & Applications
- Hierarchical method
  - Paper: Advanced Hierarchical Spherical Parameterizations
- Two hemispheres
- Curvature flow

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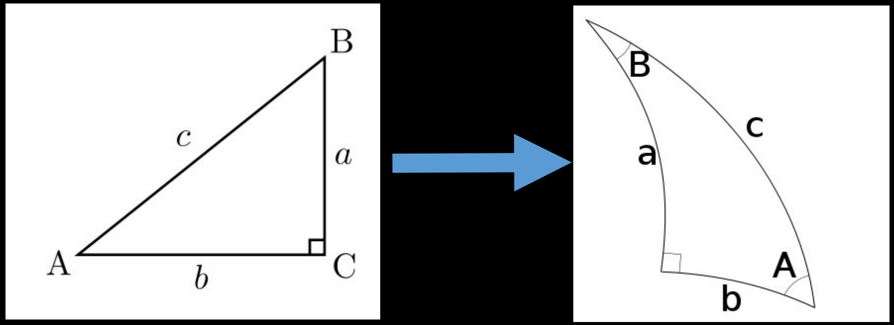
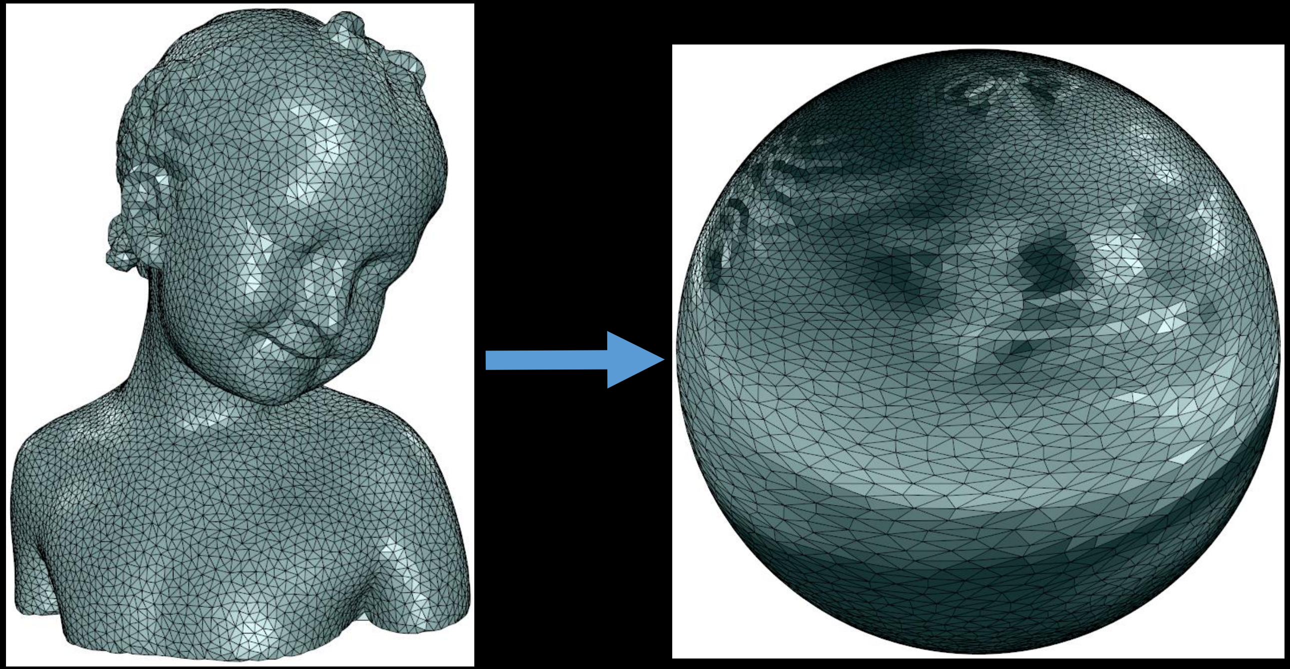
Spherical Parametrizations

- Homeomorphic mapping between a genus-0 closed surface to a sphere
- If the surface is represented by a triangle mesh, each triangle is projected to a spherical triangle



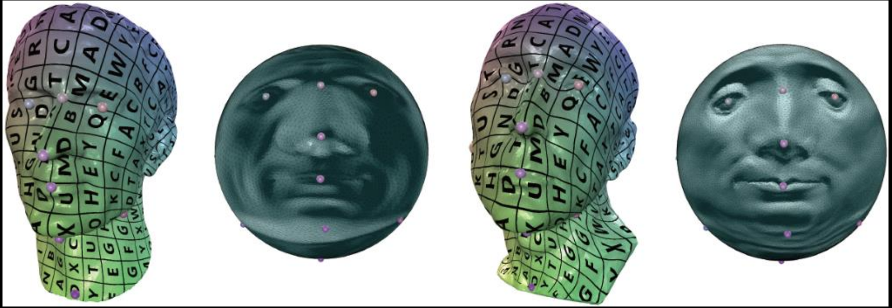


Genus-0 surface

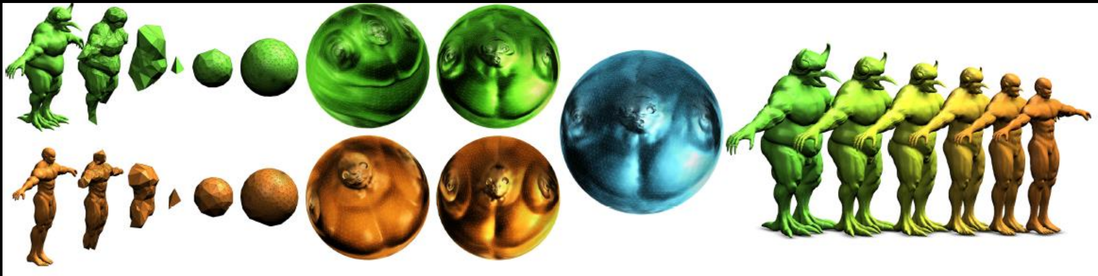
Sphere

Applications

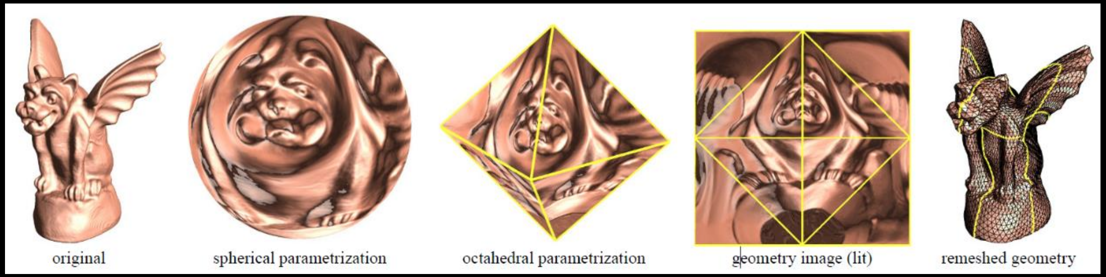
- Correspondence



- Morphing



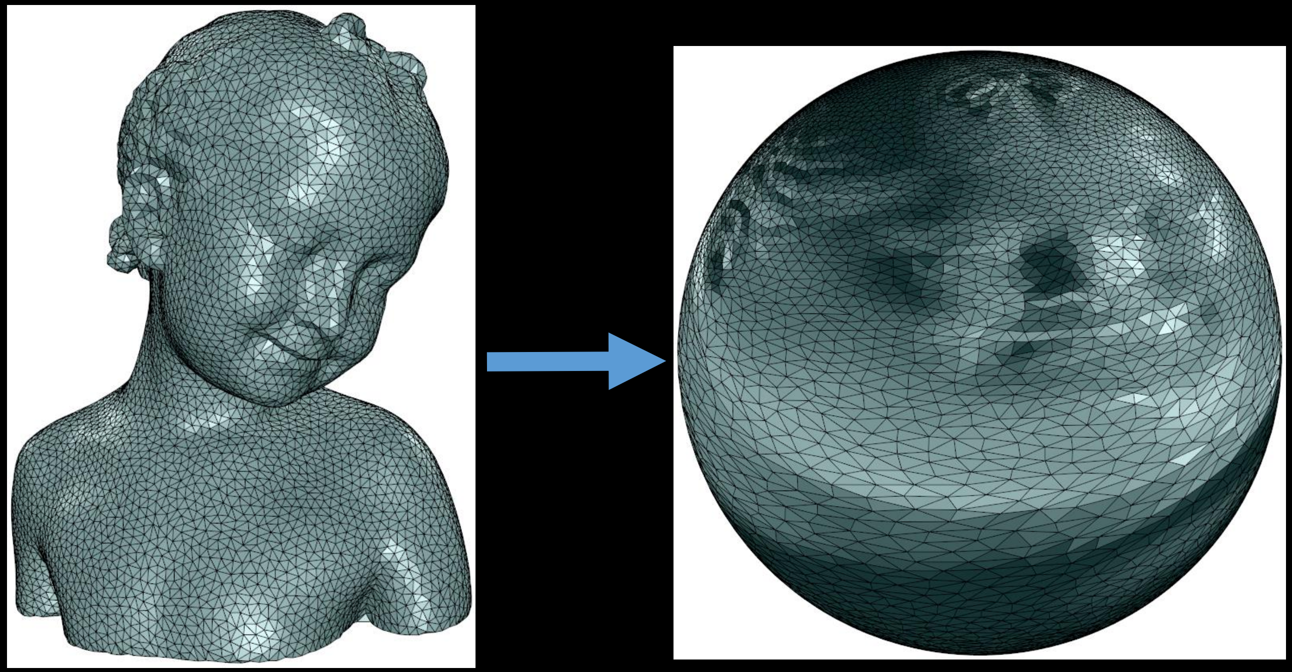
- Remeshing



- ……

Constraints

- Spherical constraints
- Bijective constraints
  - No foldover
- Low distortion



$$x^2 + y^2 + z^2 = r^2$$

Non-linear, non-convex

Challenge

- No Tutte’s embedding method
- Non-linear, non-convex optimization problem.

Very challenging!!!

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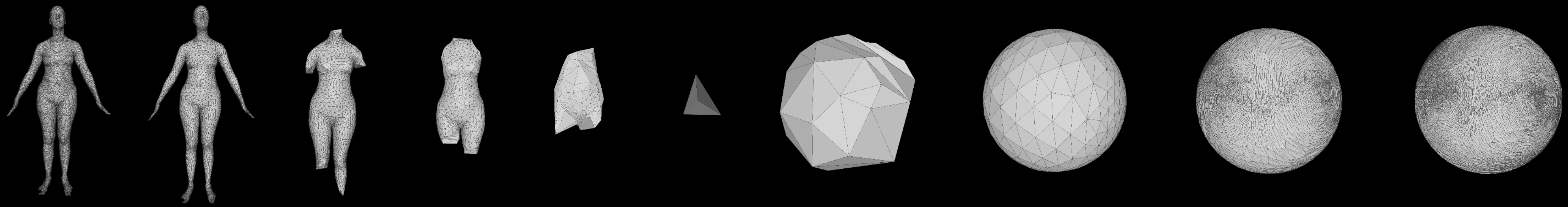
Hierarchical scheme pipeline

Input:
A triangle mesh → Decimation → Refinement → Output:
A triangular sphere



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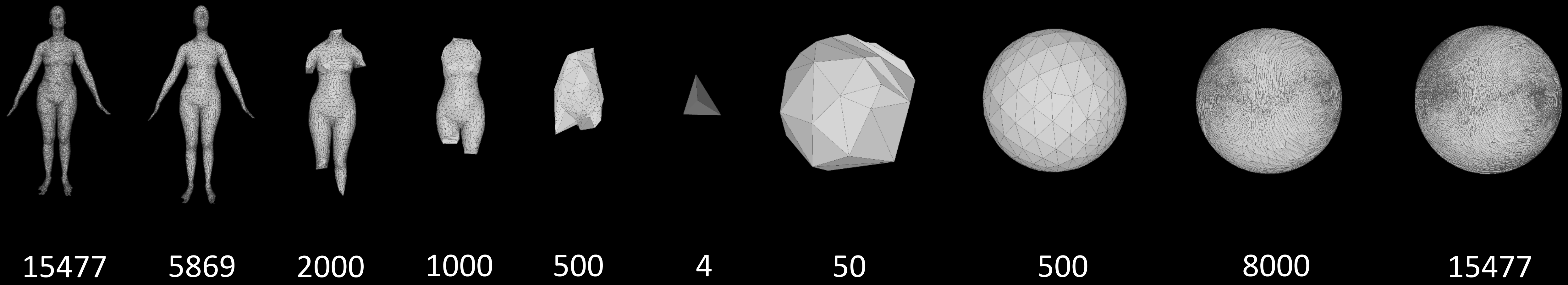
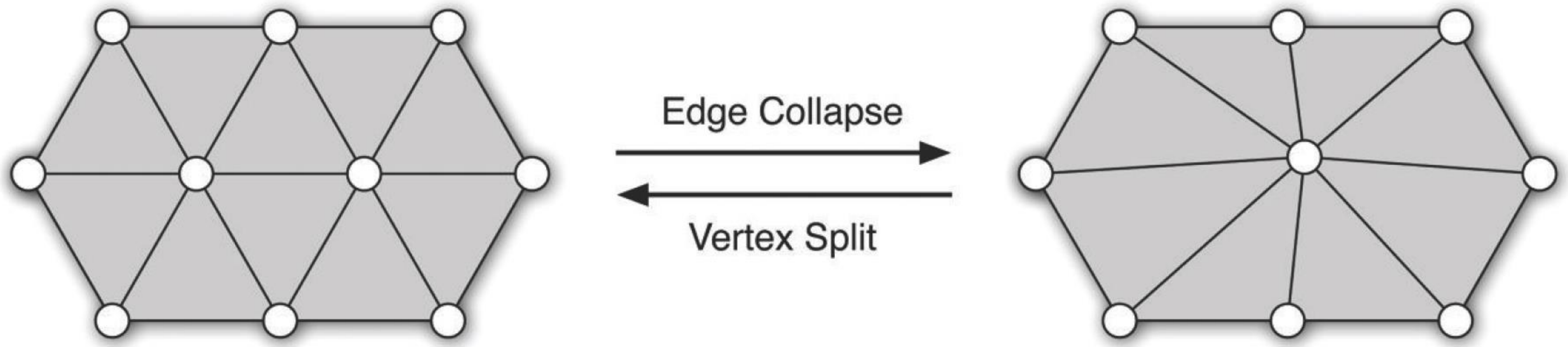
Hierarchical scheme pipeline

Input: A triangle mesh → **Decimation** → **Refinement** → Output: A triangular sphere

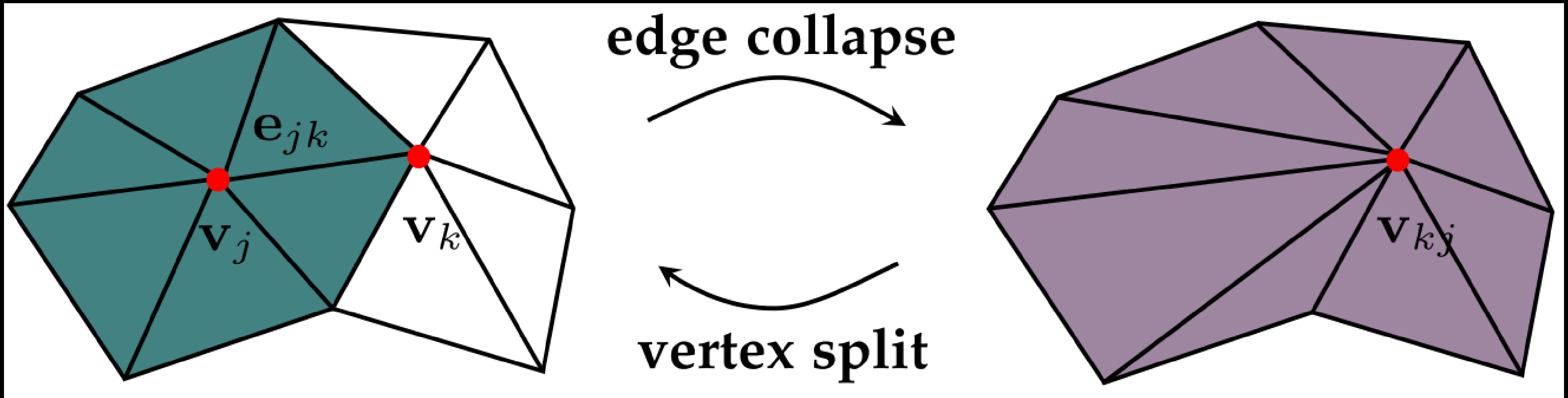


Decimation



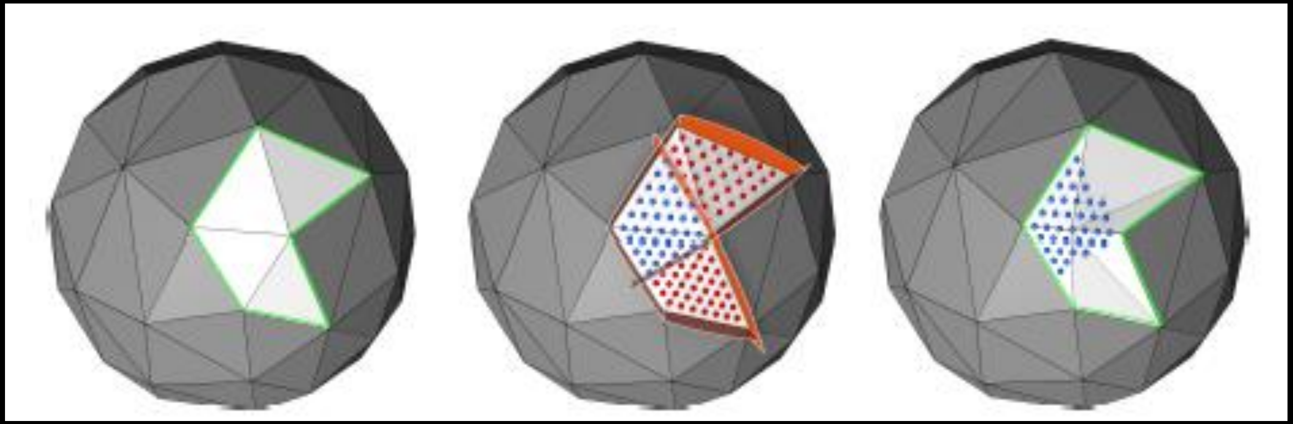
Curvature error metric (CEM)

- $\mathrm{E}_{\mathrm{jk}}^{\mathrm{C}} = \dfrac{g(V_j)}{d_e(V_{kj}) \cdot \rho(V_{kj})}$
  - $g(V_j)$ is the Gaussian curvature at V_j
  - $d_e(V_{kj}) = e^{(d(V_{kj})-6)^2}$, $d(V_{kj})$ is the valence of V_{kj}
  - $\rho(V_{kj}) = \sum_{f \in \Omega(V_{kj})} \frac{c_r(f)}{2 \times ir(f)}$



Refinement

- Insert a new vertex on sphere



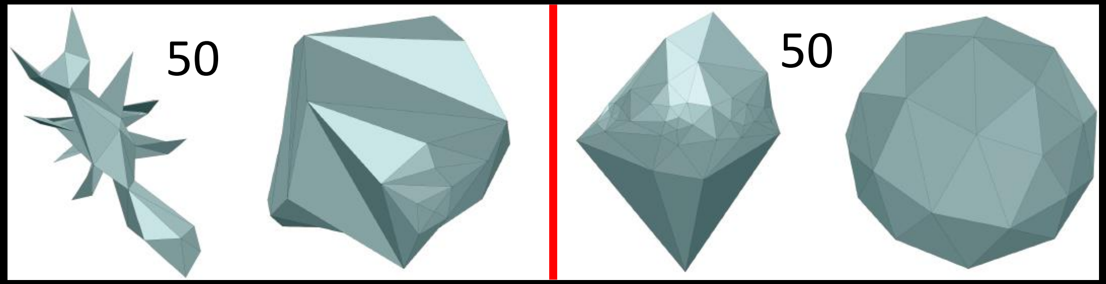
Such sphere kernel is non-empty !

Paper: Advanced Hierarchical Spherical Parameterizations

Flat-to-extrusive decimation strategy

Once an approximate surface has been spherically parameterized, the details of the input surface can be refined easily.

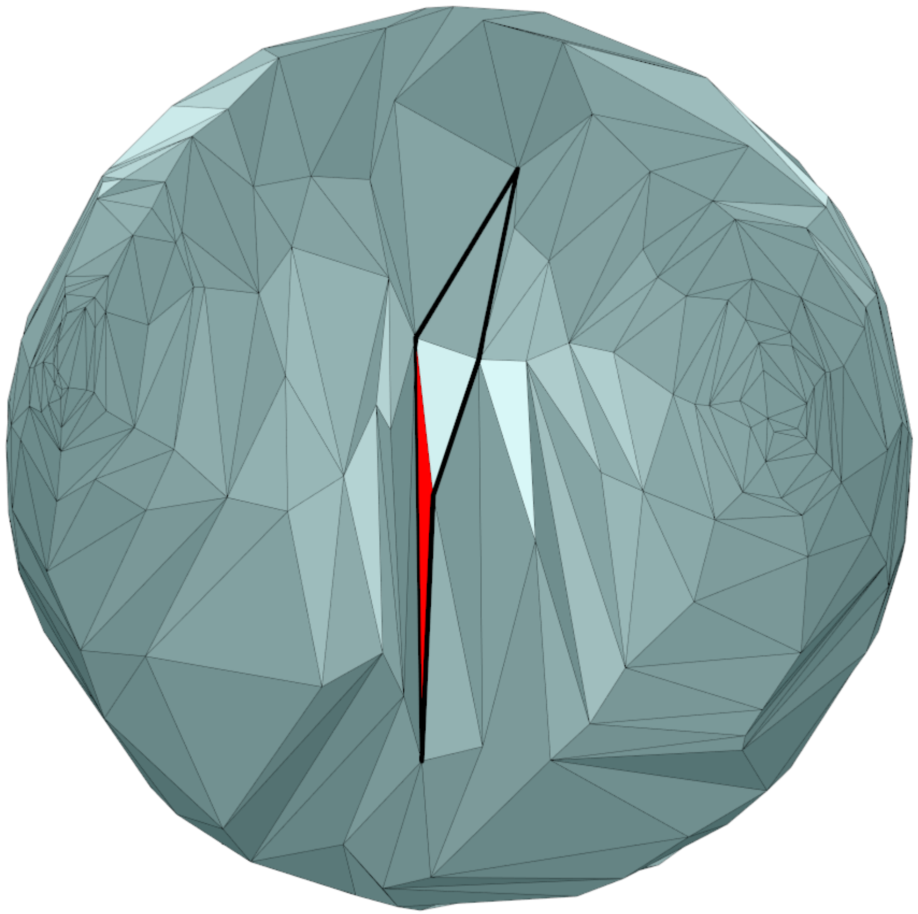
The shape contains highly curved areas at the beginning of refinement, it is hard to make the vertices evenly.



A flat-to-extrusive mesh decimation scheme, first simplifies the **flat** regions by QEM and then the **extrusive** regions by CEM.

Depression illustration

- Long and thin triangles (red) may block the next vertex insertion.



Make the vertices evenly distribute over the sphere as much as possible.

It is easier to insert the later vertices.

Flexible group mesh refinement

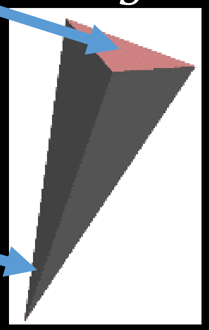
- Group insertion
  - Distortion control: iteratively insert vertices until the maximal distortion exceeds a threshold.
  - Global optimization: reduce distortion.

- Former methods optimize the vertices after inserting fixed number of vertices.
  - [Praun and Hoppe 2003], [Peng et al.2016], [Wan et al. 2012]

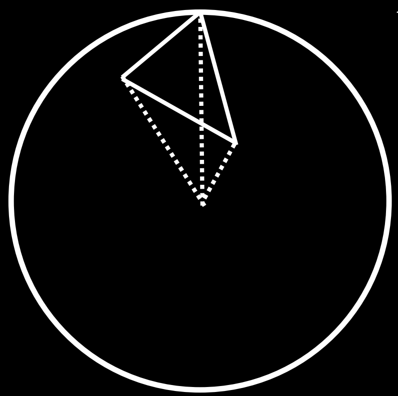
- This group scheme is much more robust and efficient.

Global optimization

- We want to make the vertices evenly distribute on the sphere
  - The ideal tetrahedron T
    - S is a equilateral triangle
    - The area of spherical triangle S^t is decided by the current face number
    - Use $spherical\ excess$ to compute dihedral angle
    - Use $Cosing\ Law$ to compute the angle θ



  - Optimizing the tetrahedral mesh
    - Tetrahedron: formed by mesh triangles and coordinate origin
    - Inexact block coordinate descent



- Only the topology information of origin mesh is used during the global optimization.

Global optimization energy

- 3D AMIPS energy:

$$E_i^{con} = \frac{1}{8}(\|J_i\|_F^2 \cdot \left\|J_i^{-1}\right\|_F^2 - 1)$$

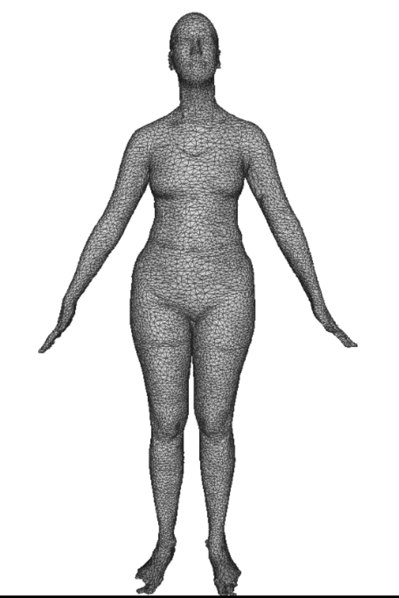
$$E_i^{\mathrm{vol}} = \frac{1}{2}(\det J_i + (\det J_i)^{-1})$$

$$E_i^{iso} = \frac{1}{2}(E_i^{con} + E_i^{\mathrm{vol}})$$

$$E_i^* = \exp\left(\left(E_i^{iso}\right)^s\right)$$

- $\mathrm{s} = \frac{\ln \tau}{\ln E_{\max}^m}, \quad \left(E_i^{iso}\right)^s < \tau$
- Volume-based energy significantly improve the robustness of our refinement process.

Triangle-based failure cases



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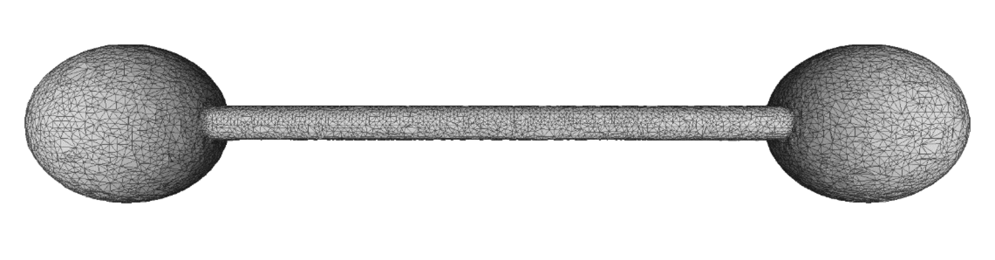
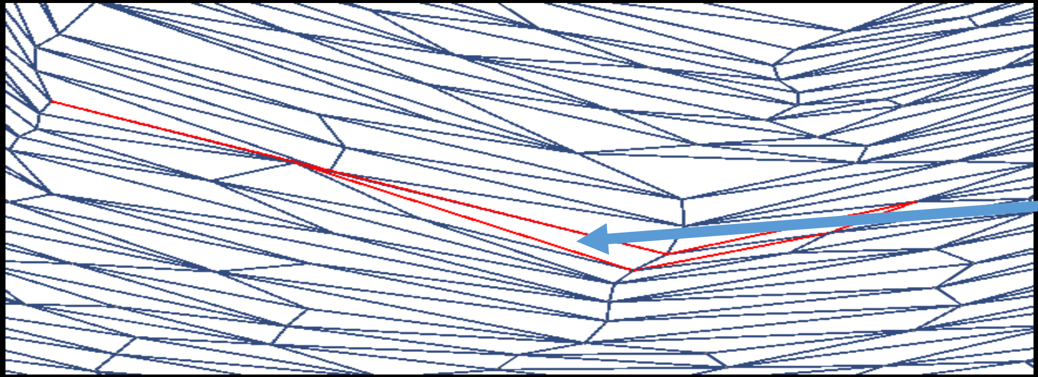
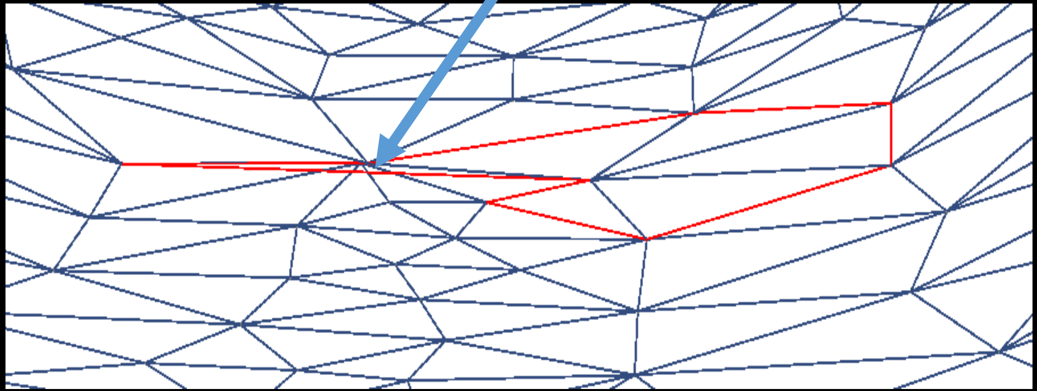




Narrow and long triangle make it hard to insert new vertex

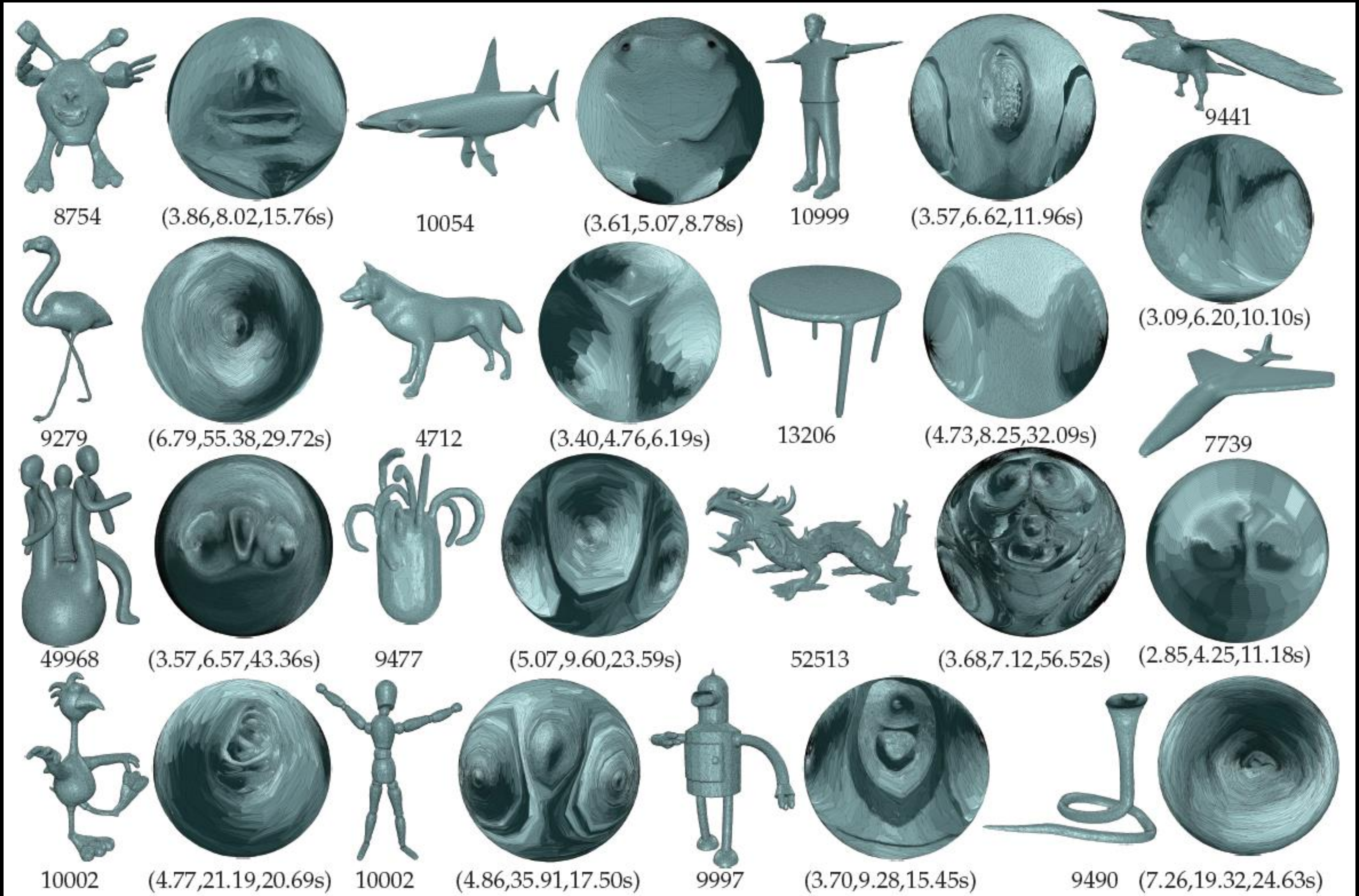


Post optimization

- Rigidly transfer each triangle t_i of original mesh to $\widetilde{t_i}$ on the sphere
- Form tetrahedron $\widetilde{T_i}$ by the origin and three vertices of $\widetilde{t_i}$
- Use AMIPS energy to optimize T_i toward $\widetilde{T_i}$

Before optimization After optimization

Results



8754
(3.86,8.02,15.76s)
10054
(3.61,5.07,8.78s)
10999
(3.57,6.62,11.96s)
9441
(3.09,6.20,10.10s)
9279
(6.79,55.38,29.72s)
4712
(3.40,4.76,6.19s)
13206
(4.73,8.25,32.09s)
7739
49968
(3.57,6.57,43.36s)
9477
(5.07,9.60,23.59s)
52513
(3.68,7.12,56.52s)
(2.85,4.25,11.18s)
10002
(4.77,21.19,20.69s)
10002
(4.86,35.91,17.50s)
9997
(3.70,9.28,15.45s)
9490
(7.26,19.32,24.63s)

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Pipeline

1. Partition M into two balanced sub-meshes.
2. Embed each sub-mesh in a planar disk using the barycentric method with weights w.
3. Combine the embeddings of the two submeshes into one planar embedding using Moebius inversion.
4. Use inverse stereo projection to obtain a spherical embedding.

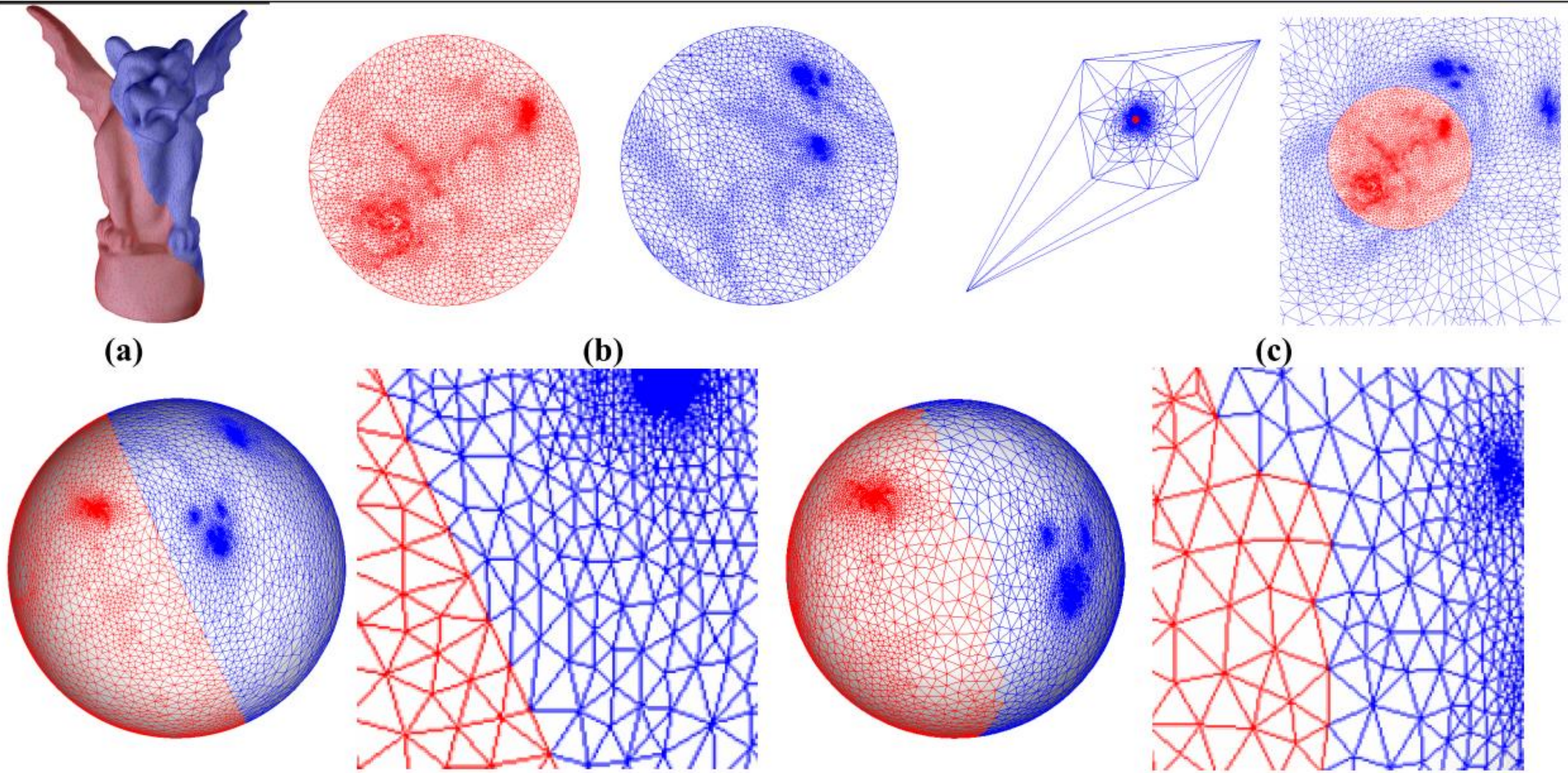

Figure 1. Stages in generating a spherical embedding for the gargoyle model (using uniform weights): (a) Partition into two sub-meshes using MeTiS. (b) Planar parameterization of the sub-meshes. (c) Combined planar embedding (with zoom); (d) Initial spherical parameterization generated by inverse stereo projection (with zoom) (f) final result after projected Gauss-Seidel and local Newton iterations (with zoom).

Improvements

- How to cut?
  - MeTiS graph partitioning package: obtain a balanced minimal vertex separator of the mesh graph.
- How to map the planar parameterizations onto the sphere?
  - Moebius inversion $f(z) = 1/conj(z)$
    - This maps the interior of the unit disk to its exterior.
  - Mapped to the unit sphere using the inverse stereo projection.

$$P(u,v) = \frac{1}{1+u^2+v^2}\left(2u, 2v, 1-u^2-v^2\right)$$

More papers

- Connectivity Shapes
- Spherical Parameterization Balancing Angle and Area Distortions, TVCG 2017

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Calabi flow

- Calabi energy: it is squared difference between current curvature vector and target curvature.

Mean curvature flow