Delaunay Triangulations

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Outlines

- Introduction
  - Convex hull
  - Triangulation
  - Delaunay triangulation
  - The Lawson Flip algorithm
- Properties
  - Empty Circle
  - Maximize the minimum angle
- Optimal Delaunay triangulation

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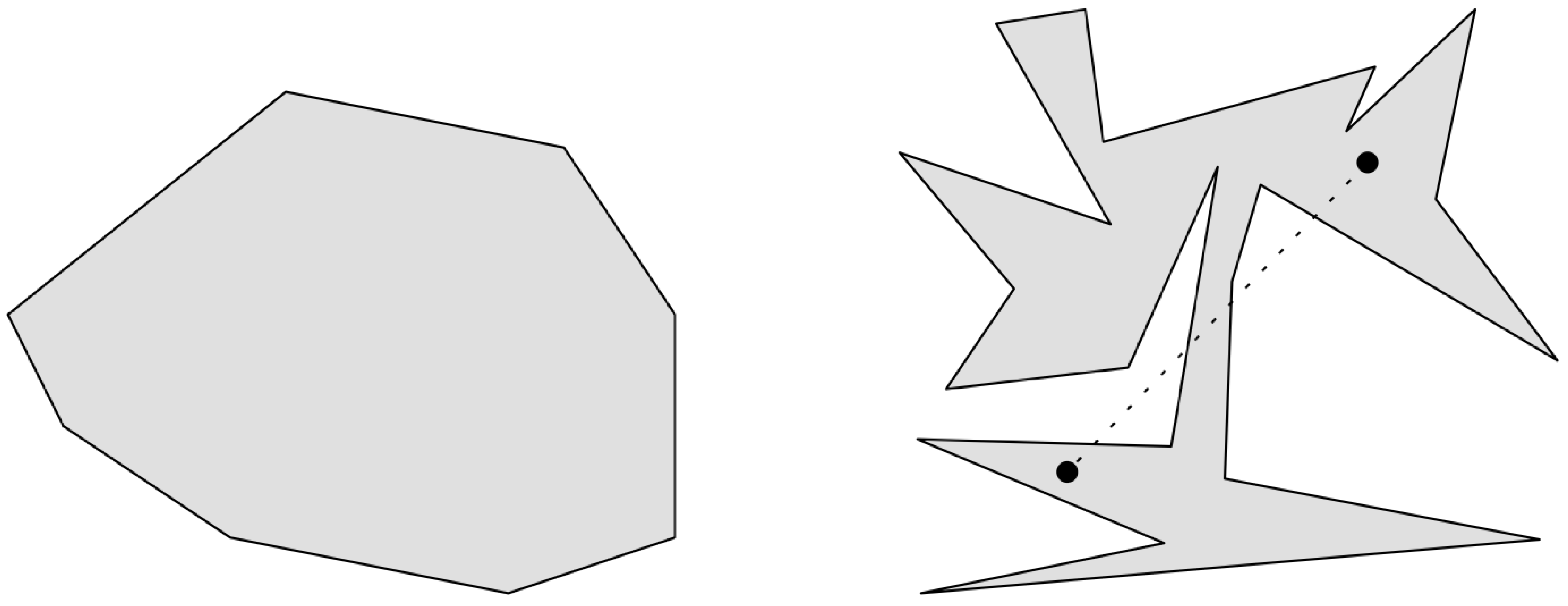
Convex polygon

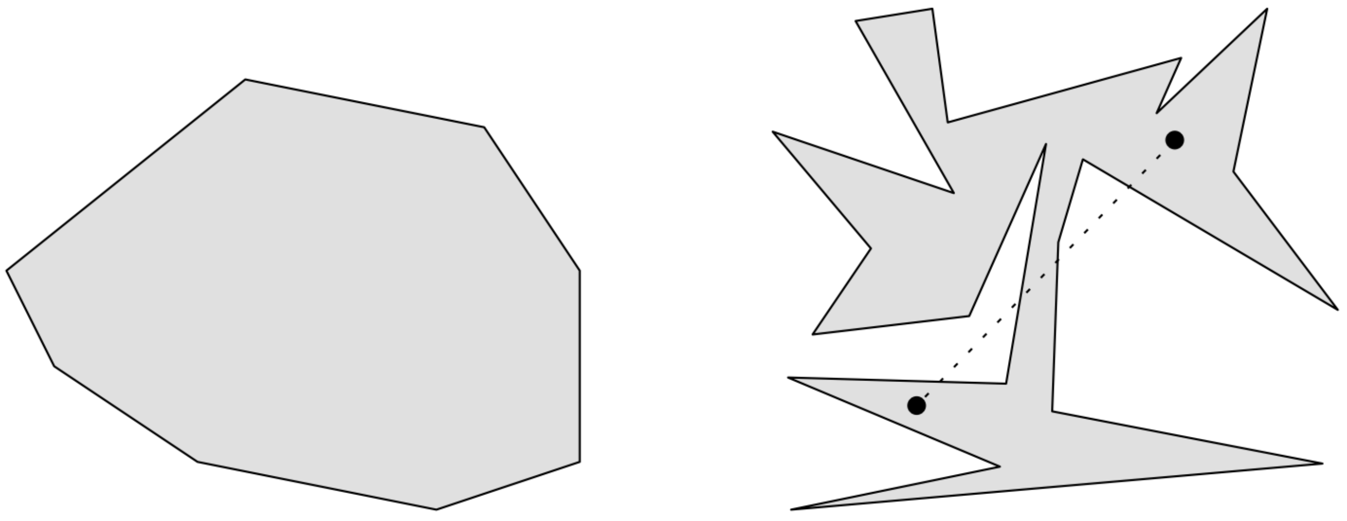
- One can walk between any two vertices along a straight line without ever leaving the polygon.



(a) A convex polygon. (b) A non-convex polygon

Convex polygon

- A set $P \in R^d$ is *convex* if $\overline{pq} \in P, \forall p, q \in P$.
- An alternatively equivalent way to phrase convexity:
  - For every line $l \in R^d$, the intersection $l \cap P$ is connected



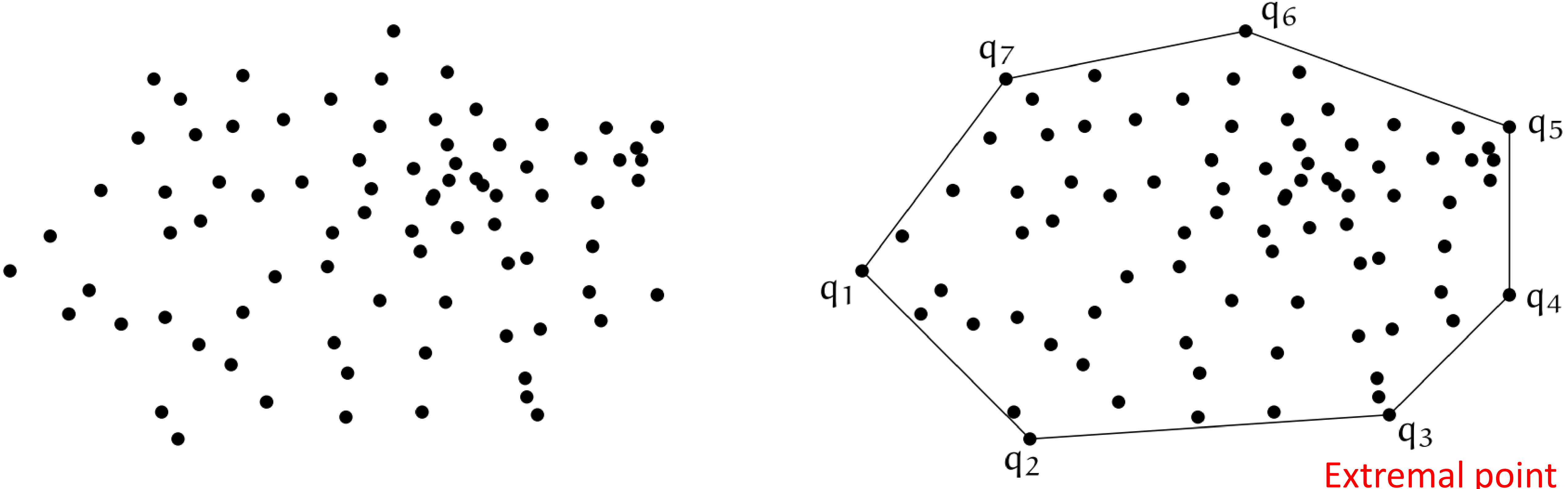
(a) A convex polygon. **(b)** A non-convex polygon

- For any family $\{P_i\}$ of convex sets, the intersection $\cap_i P_i$ is convex.

Convex hull

- The convex hull of a finite point set $P \in R^d$ forms a convex polytope, denoted as $conv(P)$.
- Each $p \in P$ for which $p \notin conv(P \setminus \{p\})$ is called a vertex of $conv(P)$.
- A vertex of $conv(P)$ is also called an *extremal point* of P.
- A convex polytope in R^2 is called a convex polygon.

An example of $conv(P)$



(a) Input. (b) Output.

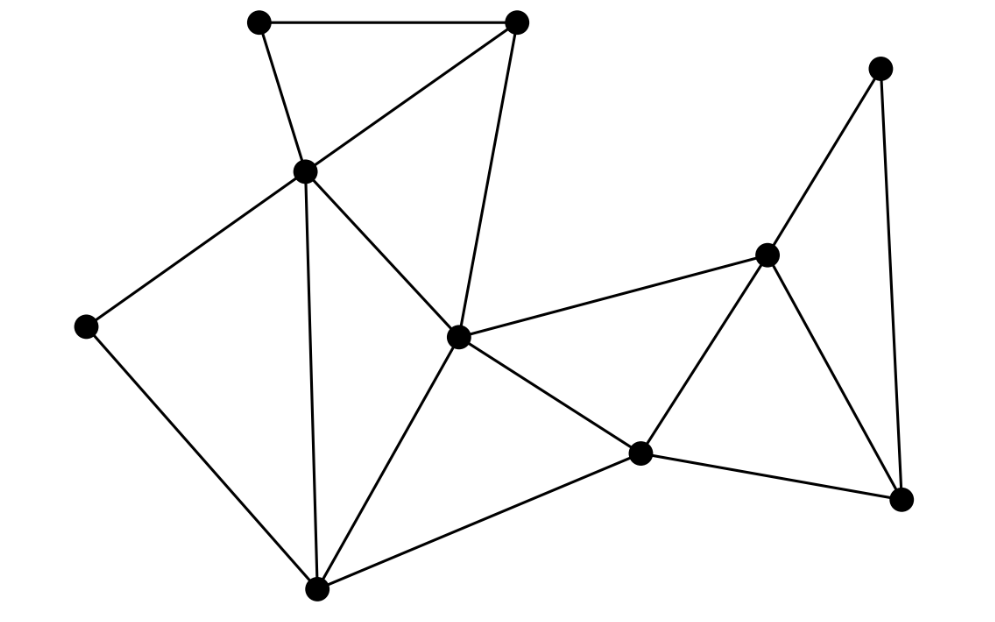
Trivial algorithms of Convex hull

- Carathéodory's Theorem
  - Test for every point $p \in P$ whether there are $q, r, s \in P \setminus \{p\}$ such that p is inside the triangle with vertices $q, r,$ and s.
  - Runtime $O(n^4)$.

- The Separation Theorem:
  - Test for every pair $(p, q) \in P^2$ whether all points from $P \setminus \{p, q\}$ are to the left of the directed line through p and q (or on the line segment $\overline{pq}$).
  - Runtime $O(n^3)$.

Triangulation of polygon

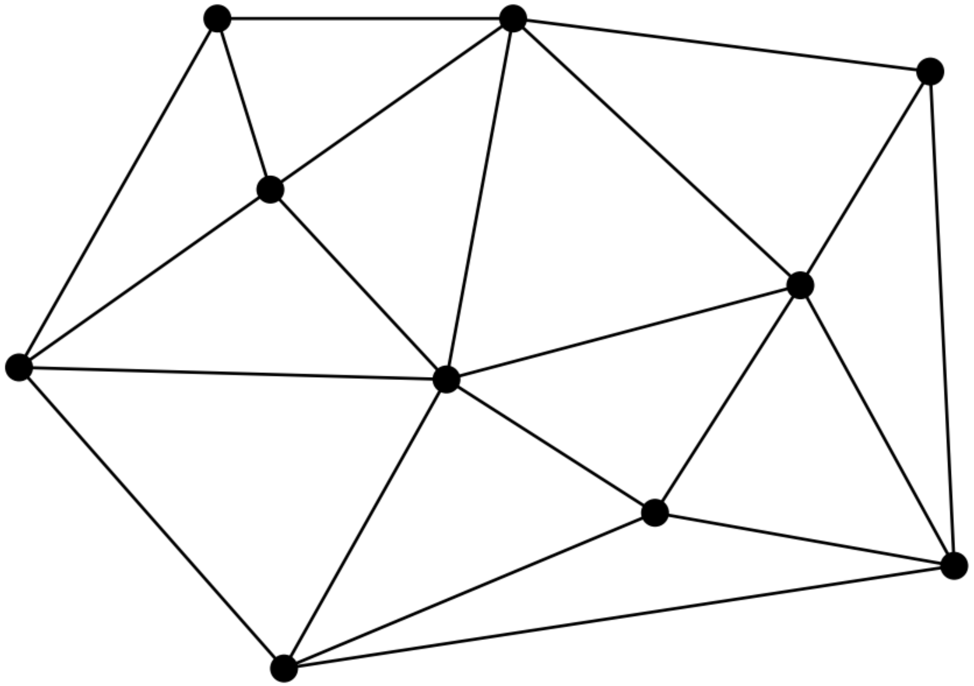
- A triangulation nicely partitions a polygon into triangles, which allows, for instance, to easily compute the area or a guarding of the polygon.
- Another typical application scenario is to use a triangulation T for interpolation.



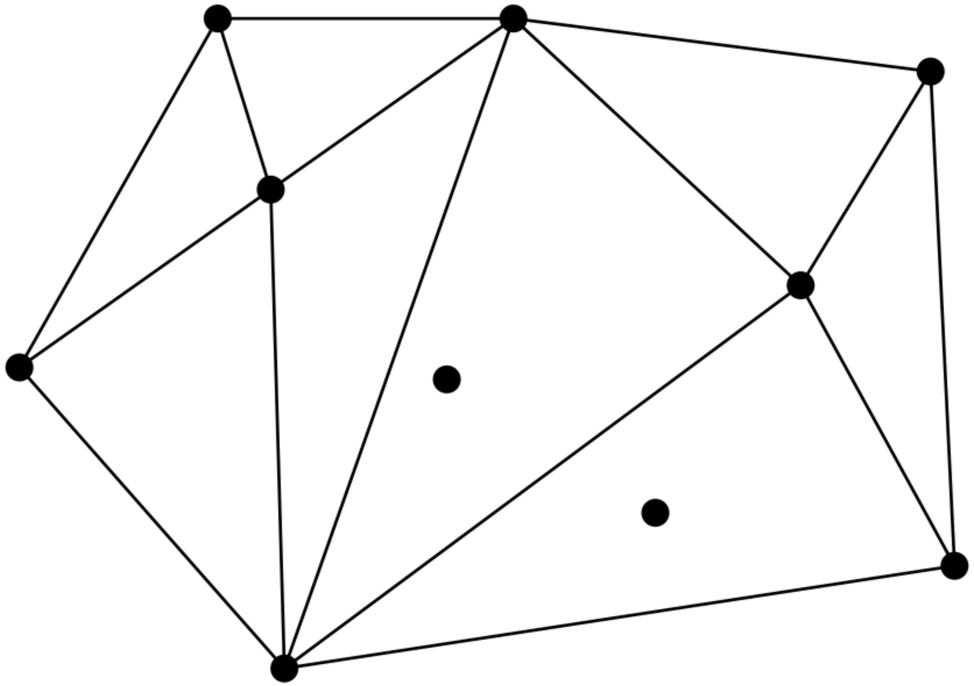
(a) Simple polygon triangulation.

Triangulation of a point set

- A triangulation should then partition the convex hull while respecting the points in the interior.



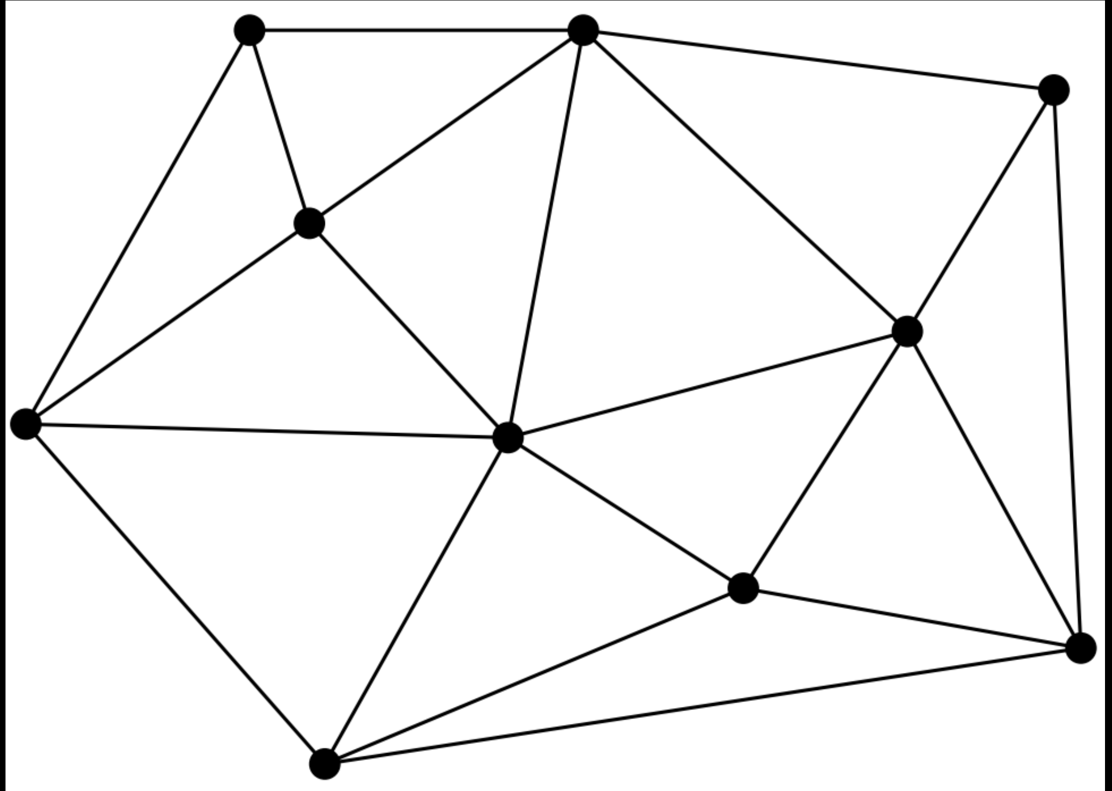
(b) Point set triangulation.



(c) Not a triangulation.

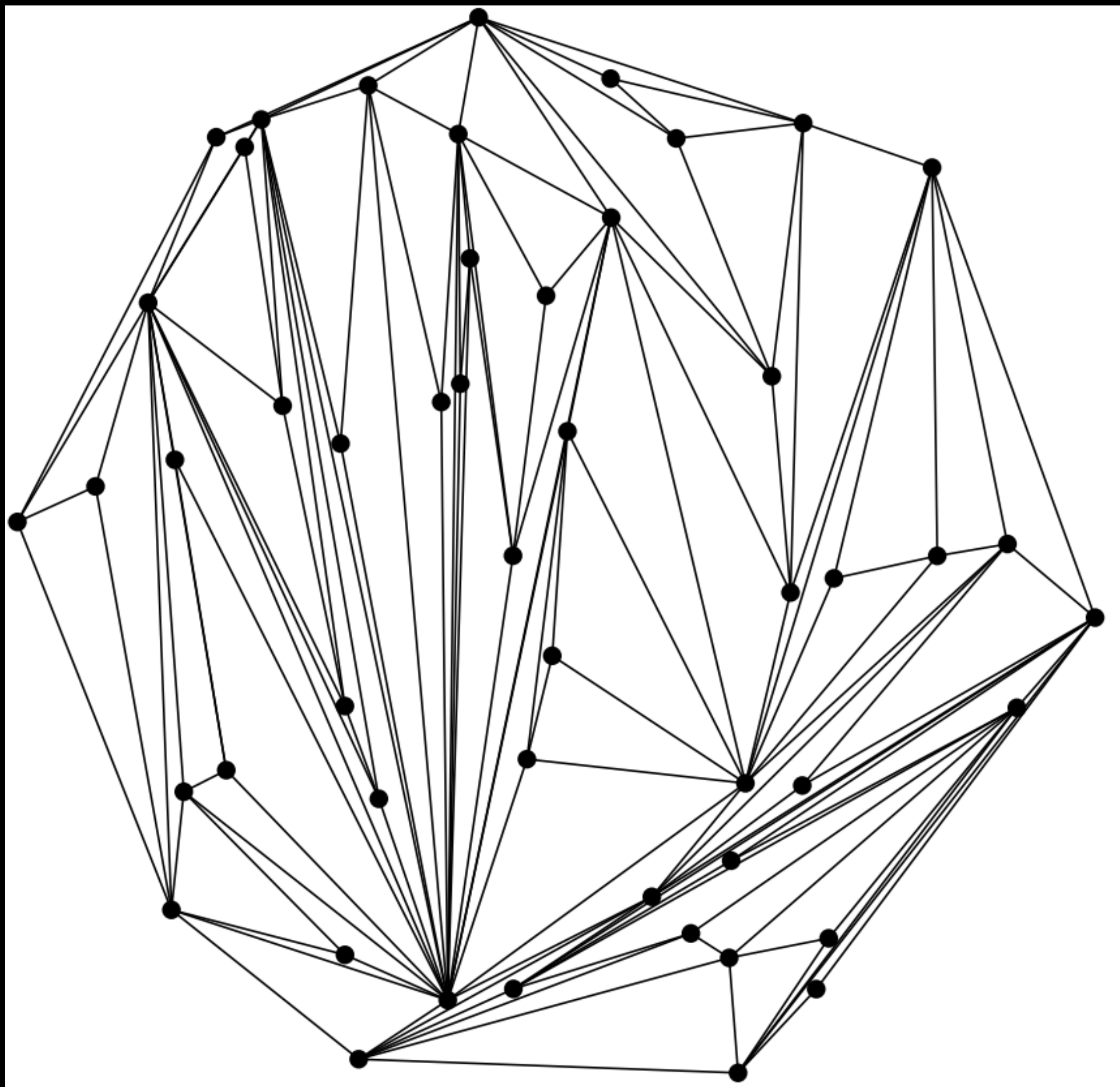
Definition

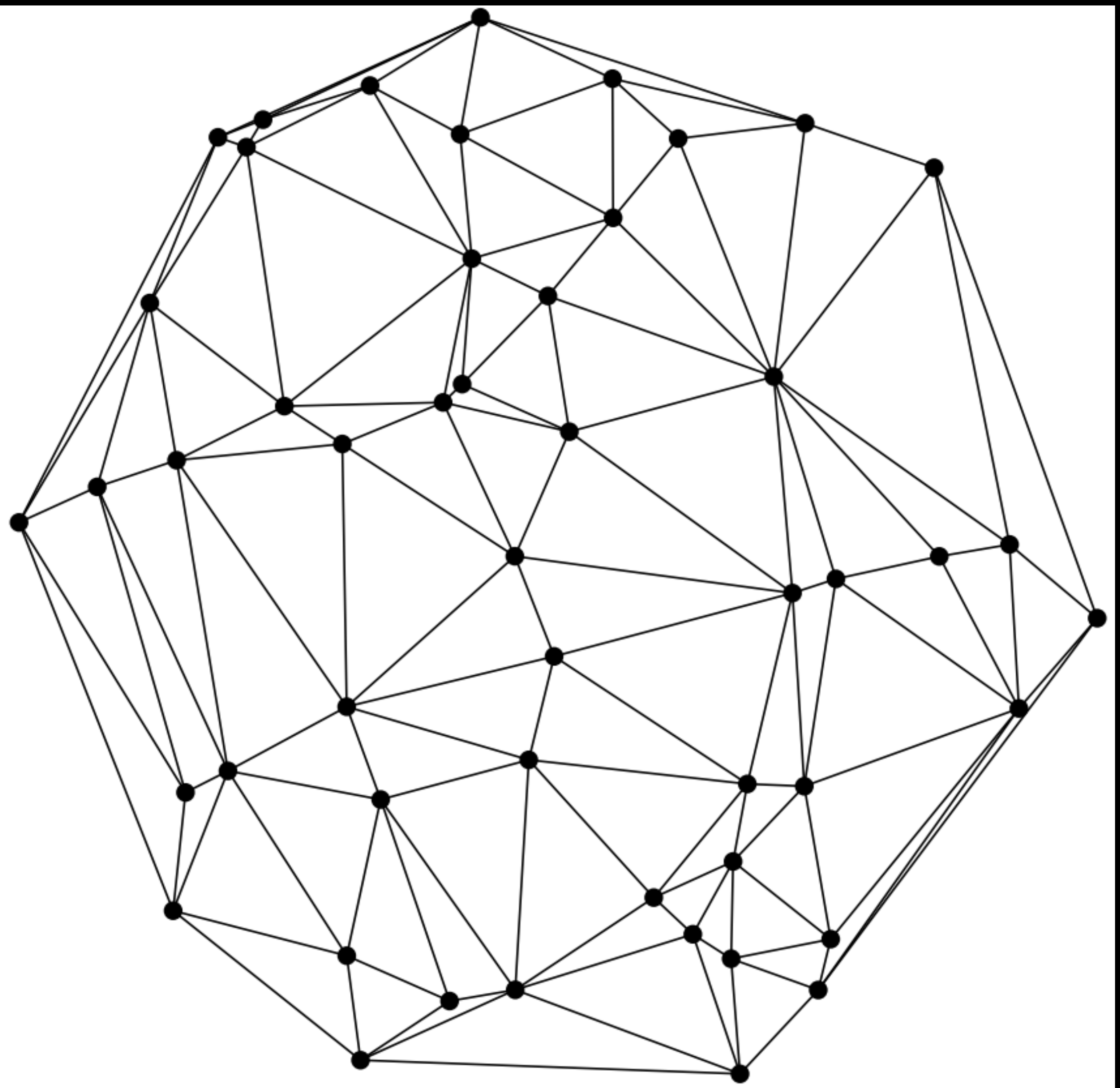
- A triangulation of a finite point set $P \subset R^2$ is a collection $\mathcal{T}$ of triangles, such that:
  - (1) $conv(P) = \cup_{T \in \mathcal{T}} T$
  - (2) $P = \cup_{T \in \mathcal{T}} V(T)$
  - (3) For every distinct pair $T, U \in \mathcal{T}$, the intersection $T \cap U$ is either a common vertex, or a common edge, or empty.



(b) Point set triangulation.

Various triangulations

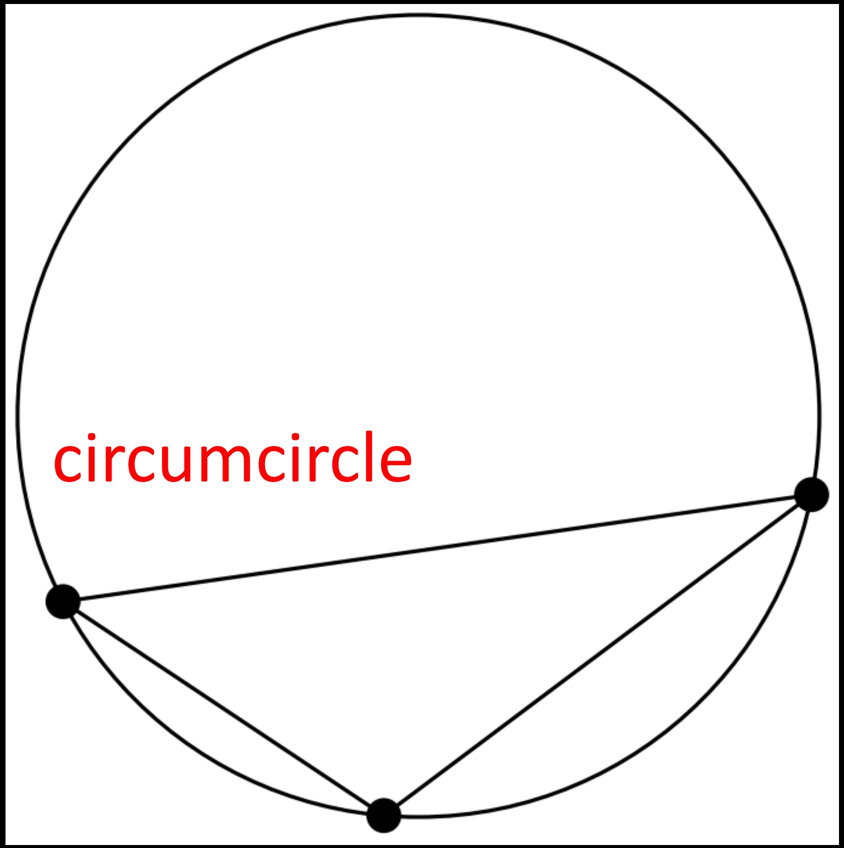




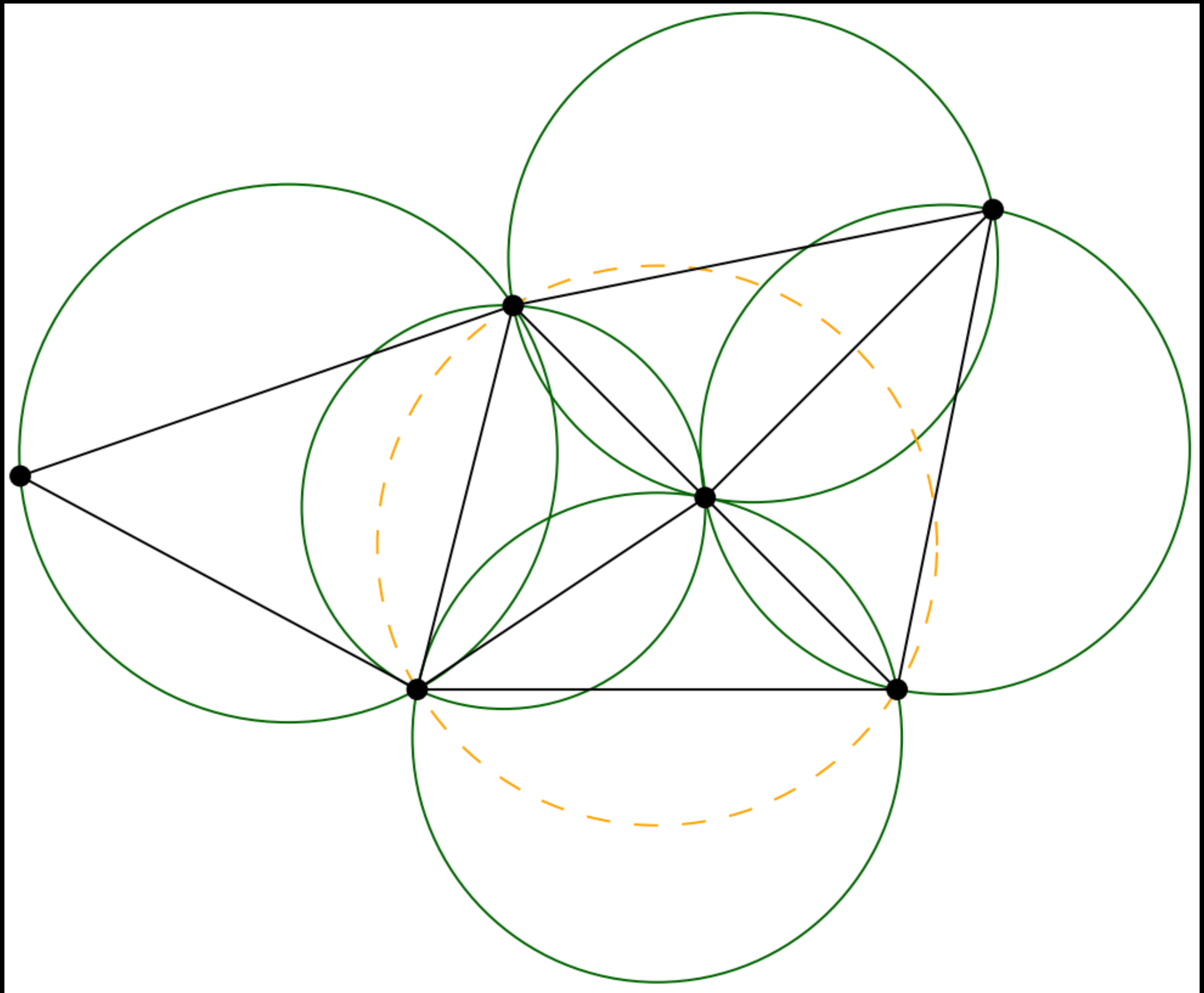
Delaunay triangulation

- Definition: For a given set P of discrete points in a plane is a triangulation $DT(P)$ such that no point in P is inside the circumcircle of any triangle in $DT(P)$.

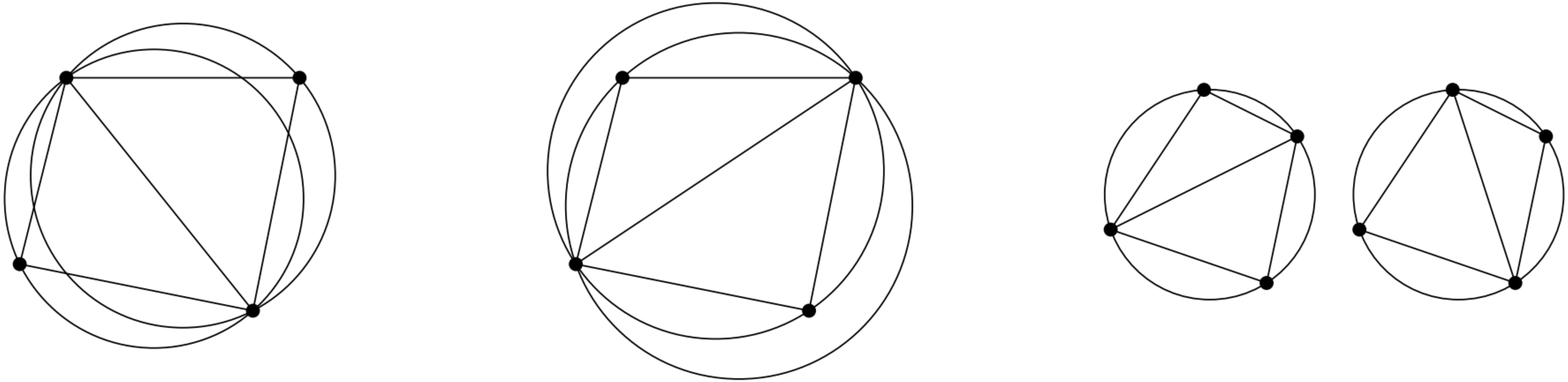
- Empty Circle property



Empty Circle



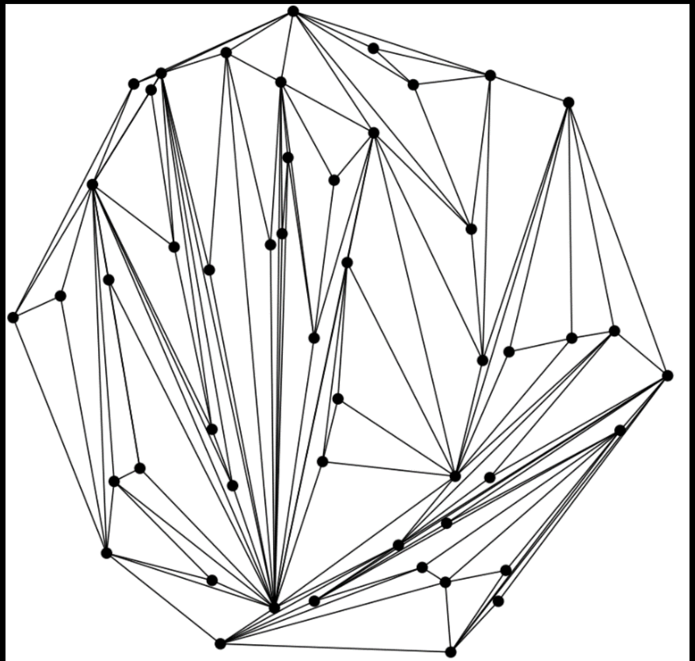
Four points in convex position

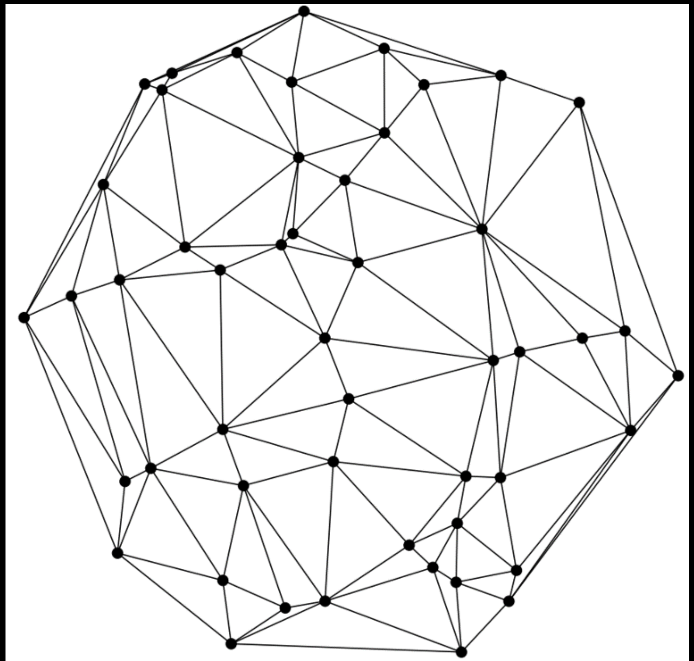


(a) Delaunay triangulation. (b) Non-Delaunay triangulation. (c) Two Delaunay triangulations.

The Lawson Flip algorithm

- (1) Compute some triangulation of P
- (2) While there exists a subtriangulation of four points in convex position that is not Delaunay, replace this subtriangulation by the other triangulation of the four points.





Theorem

Let $P \subseteq R^2$ *be a set of* n *points, equipped with some triangulation* $\mathcal{T}$*.*
The Lawson flip algorithm terminates after at most $\binom{n}{2} = O(n^2)$ *flips,*
and the resulting triangulation D *is a Delaunay triangulation of* P*.*

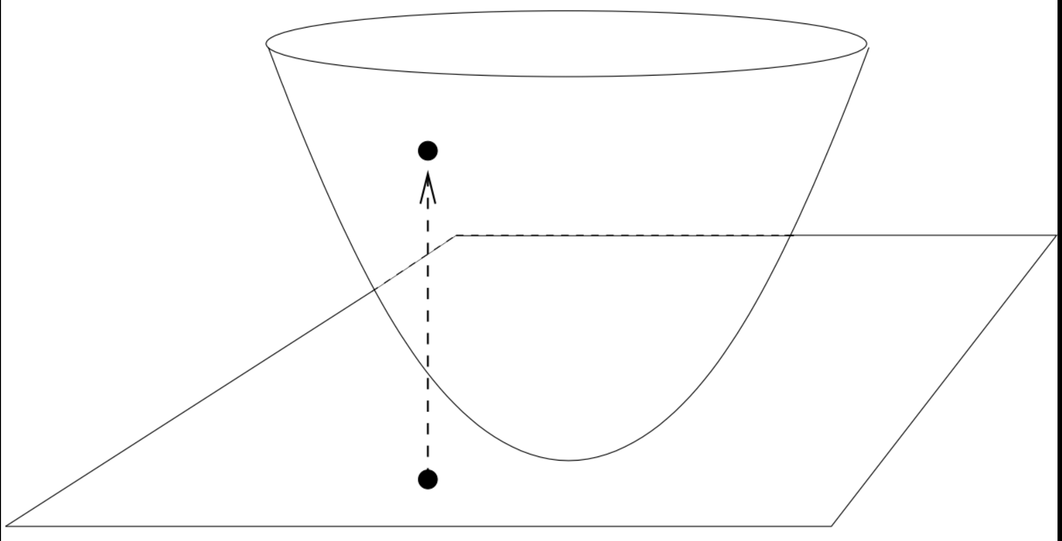
Two-step proof:

1. The program described above always terminates.
2. The algorithm does what it claims to do, namely the result is a Delaunay triangulation.

The Lifting Map

- Given a point $p = (x, y) \in R^2$, its lifting $l(p)$ is the point
$$l(p) = (x, y, x^2 + y^2) \in R^3$$

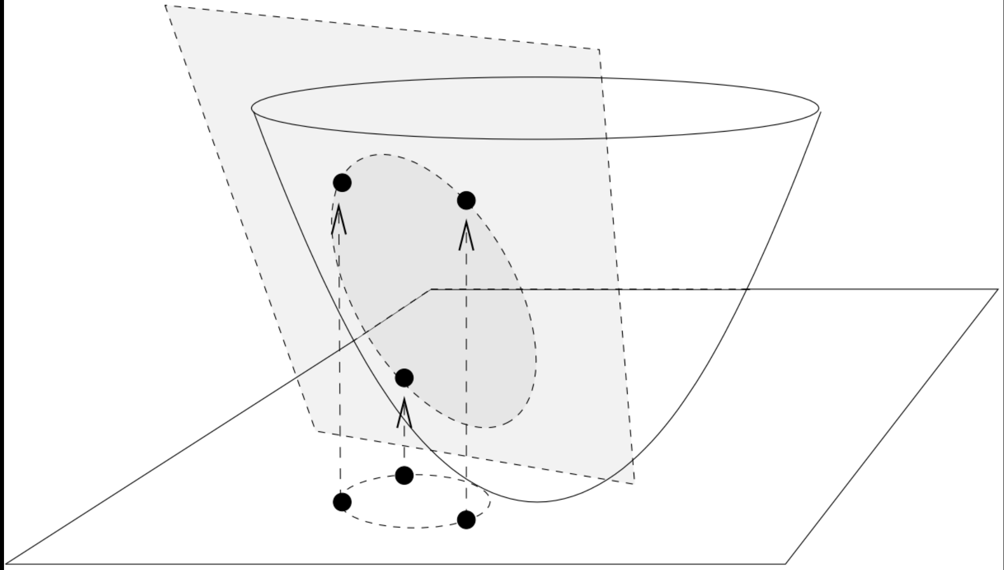
Geometrically, l “lifts” the point vertically up until it lies on the unit *paraboloid:*



(a) The lifting map.

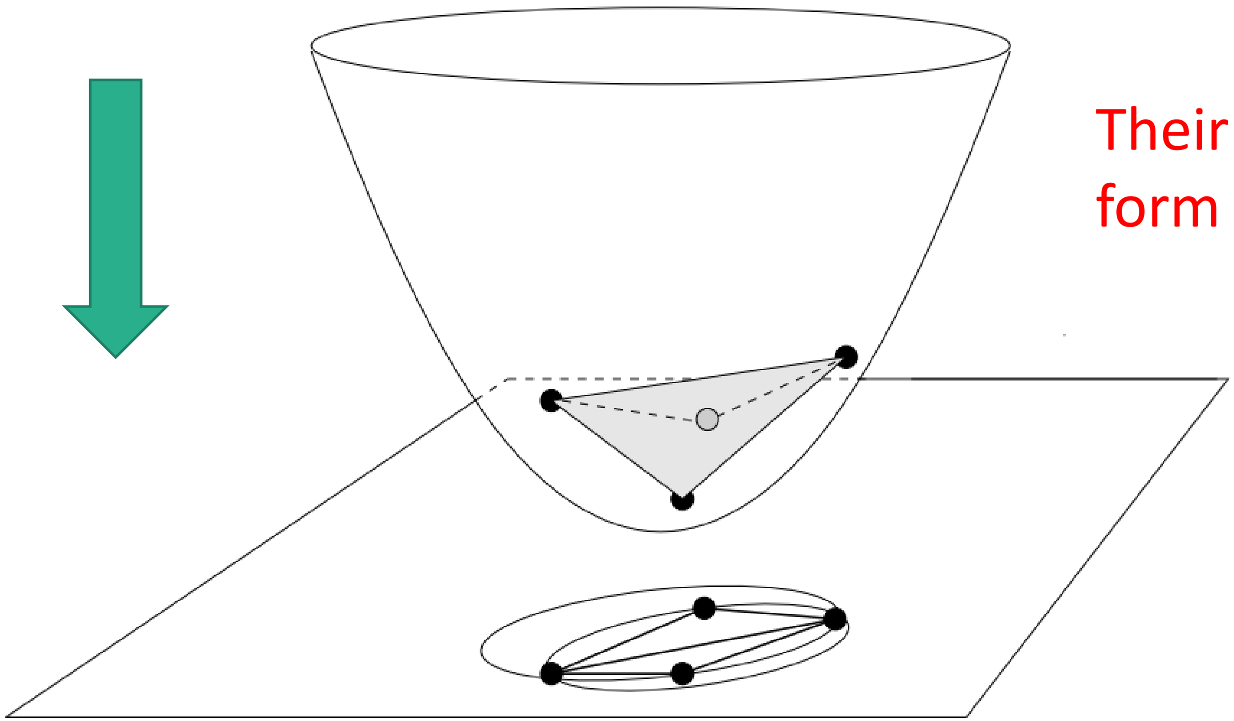
Important property of the lifting map

- Lemma: Let $C \subseteq R^2$ be a circle of positive radius. The "lifted circle" $l(C) = \{l(p) | p \in C\}$ is contained in a unique plane $h(C) \subseteq R^3$.
- Moreover, a point $p \in R^2$ is strictly inside (outside, respectively) of C if and only if the lifted point $l(p)$ is strictly below (above, respectively) $h(C)$.

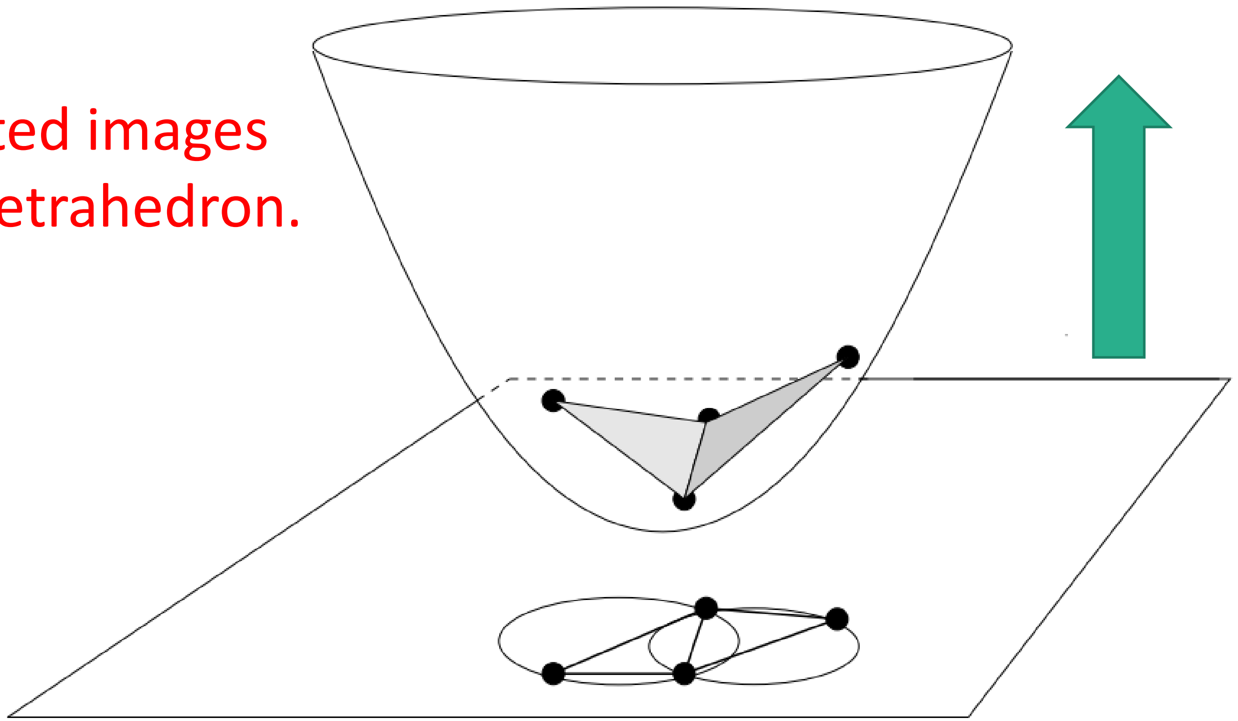


(b) Points on/inside/outside a circle are lifted to points on/below/above a plane.

(1) Termination



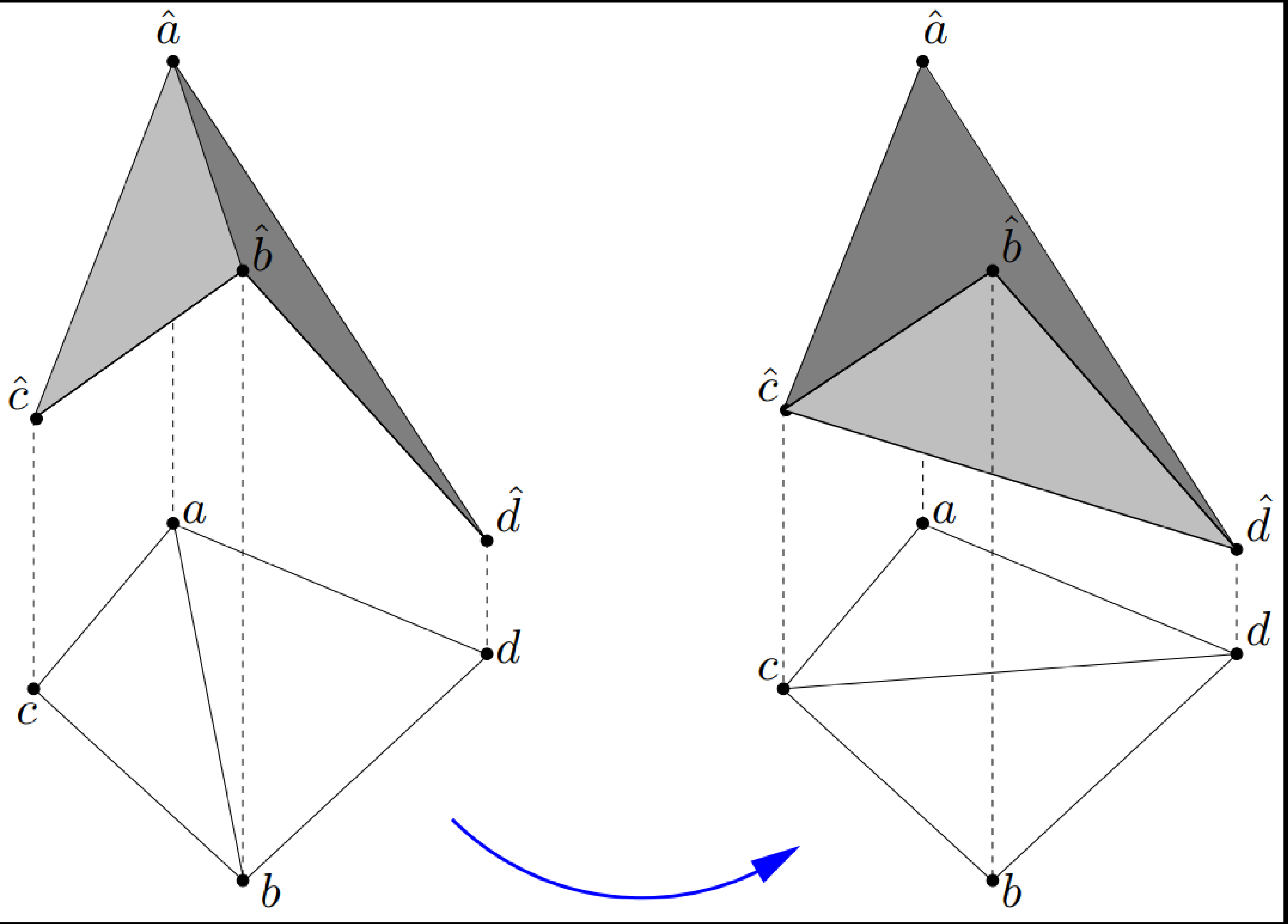
Their lifted images form a tetrahedron.



(a) Before the flip: the top two triangles of the tetrahedron and the corresponding non-Delaunay triangulation in the plane.

(b) After the flip: the bottom two triangles of the tetrahedron and the corresponding Delaunay triangulation in the plane.

(1) Termination



(1) Termination

- A Lawson flip can therefore be interpreted as an operation that replaces the top two triangles of a tetrahedron by the bottom two ones.
- If we consider the lifted image of the current triangulation, we therefore have a surface in R^3 whose pointwise height can only decrease through Lawson flips.
- In particular, once an edge has been flipped, this edge will be strictly above the resulting surface and can therefore never be flipped a second time. Since n points can span at most $\binom{n}{2}$ edges, the bound on the number of flips follows.

(2) Correctness

- Locally Delaunay: Let Δ, Δ' be two adjacent triangles in the triangulation D that results from the Lawson flip algorithm. Then the circumcircle of Δ does not have any vertex of Δ' in its interior, and vice versa.

- Locally Delaunay $\Longleftrightarrow$ Globally Delaunay:
  - contradiction

Locally Delaunay ⟺ Globally Delaunay

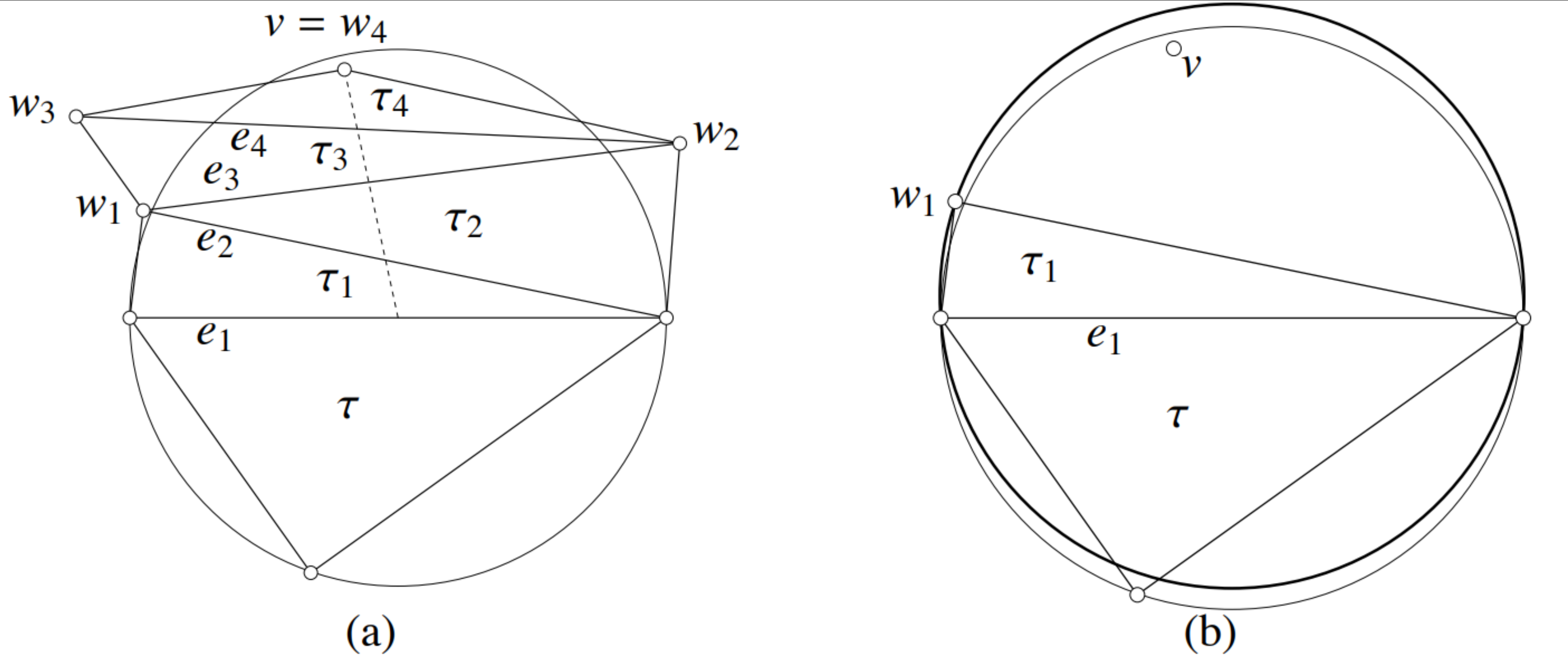


Figure 2.7: (a) Because τ's open circumdisk contains v, some edge between v and τ is not locally Delaunay. (b) Because v lies above e_1 and in τ's open circumdisk, and because w_1 lies outside τ's open circumdisk, v must lie in τ_1's open circumdisk.

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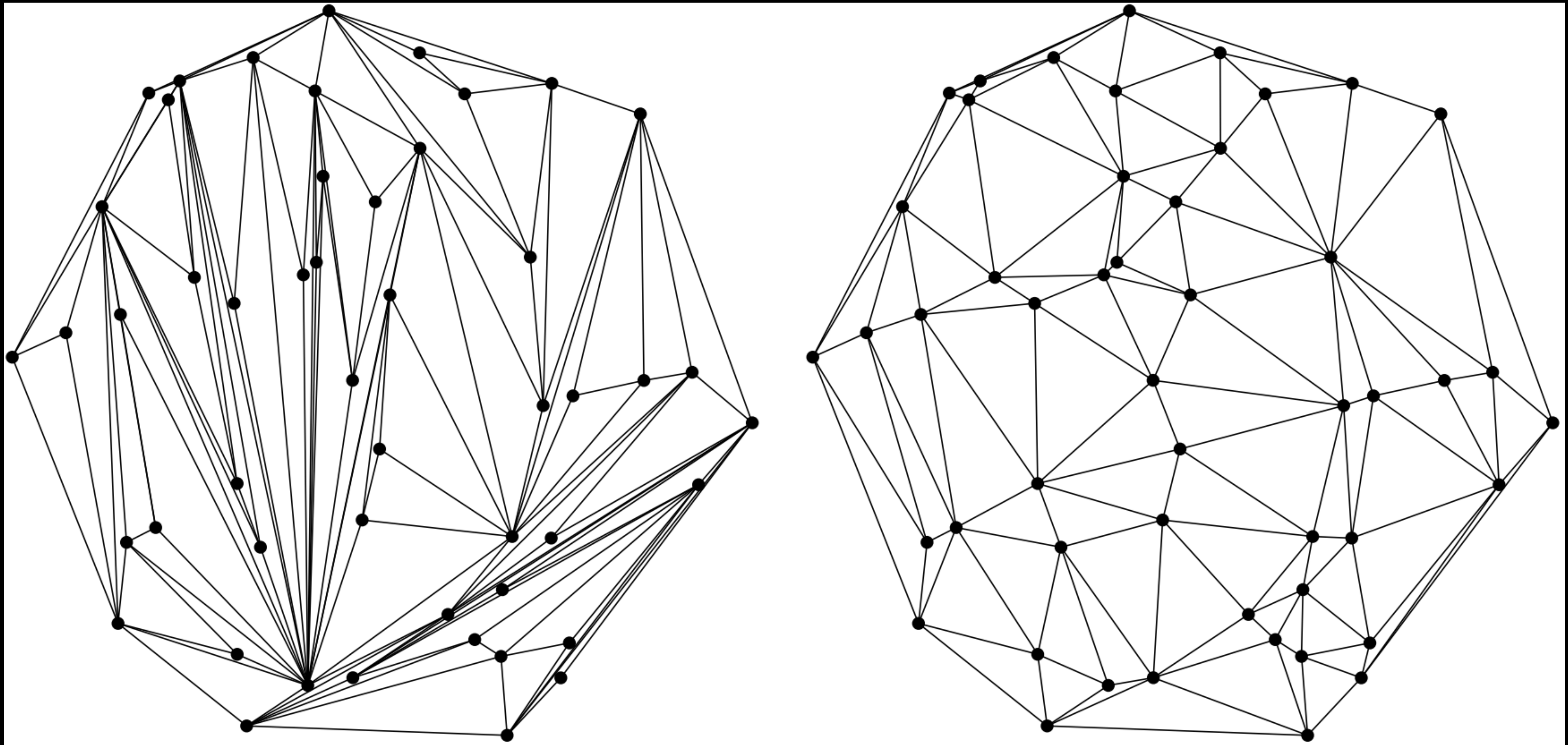
Maximize the minimum angle



Long and skinny triangles

Much closer to an equilateral triangle

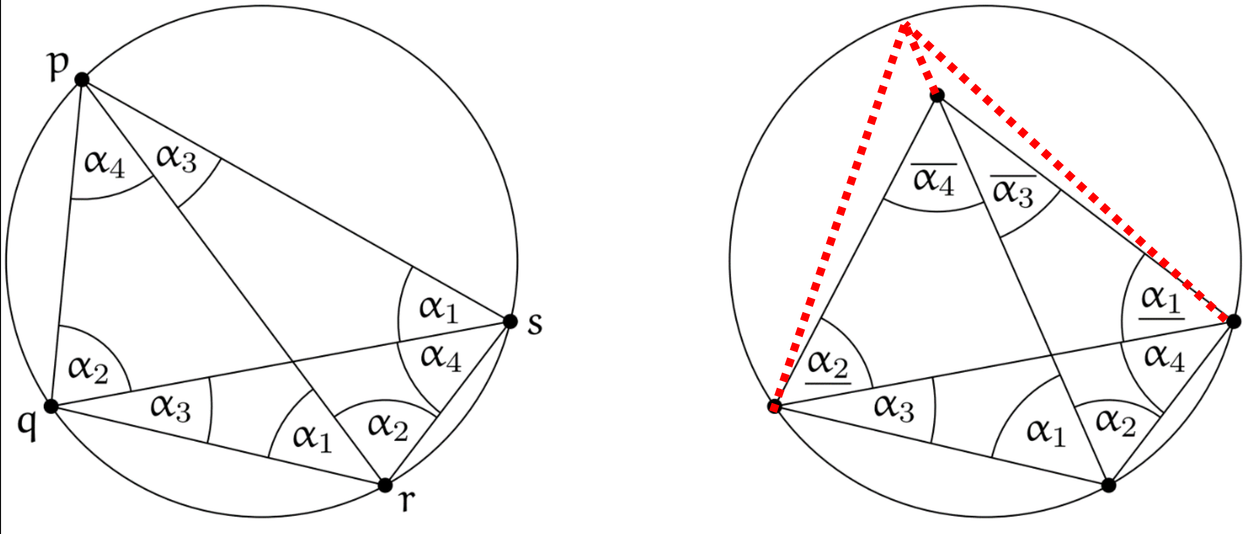
Maximize the minimum angle

- Indeed, we will show that Delaunay triangulations maximize the smallest angle among all triangulations of a given point set.
- Note that this does not imply that there are no long and skinny triangles in a Delaunay triangulation.
- But if there is a long and skinny triangle in a Delaunay triangulation, then there is an at least as long and skinny triangle in ***every*** triangulation of the point set.

Maximize the minimum angle

- A flip replaces six interior angles by six other interior angles, and we will actually show that the smallest of the six angles strictly increases under the flip.
  - Before the flip:
    - $\alpha_1+\alpha_2, \alpha_3, \alpha_4, \underline{\alpha_1}, \underline{\alpha_2}, \overline{\alpha_3}+\overline{\alpha_4}$
  - After the flip:
    - $\alpha_1, \alpha_2, \overline{\alpha_3}, \overline{\alpha_4}, \underline{\alpha_1}+\alpha_4, \underline{\alpha_2}+\alpha_3$
  - $\alpha_1 > \underline{\alpha_1}, \alpha_2 > \underline{\alpha_2}, \overline{\alpha_3} > \alpha_3, \overline{\alpha_4} > \alpha_4$

  $\underline{\alpha_1}+\alpha_4 > \alpha_4, \underline{\alpha_2}+\alpha_3 > \alpha_3$



(a) Four cocircular points and the induced eight angles.

(b) The situation before a flip.

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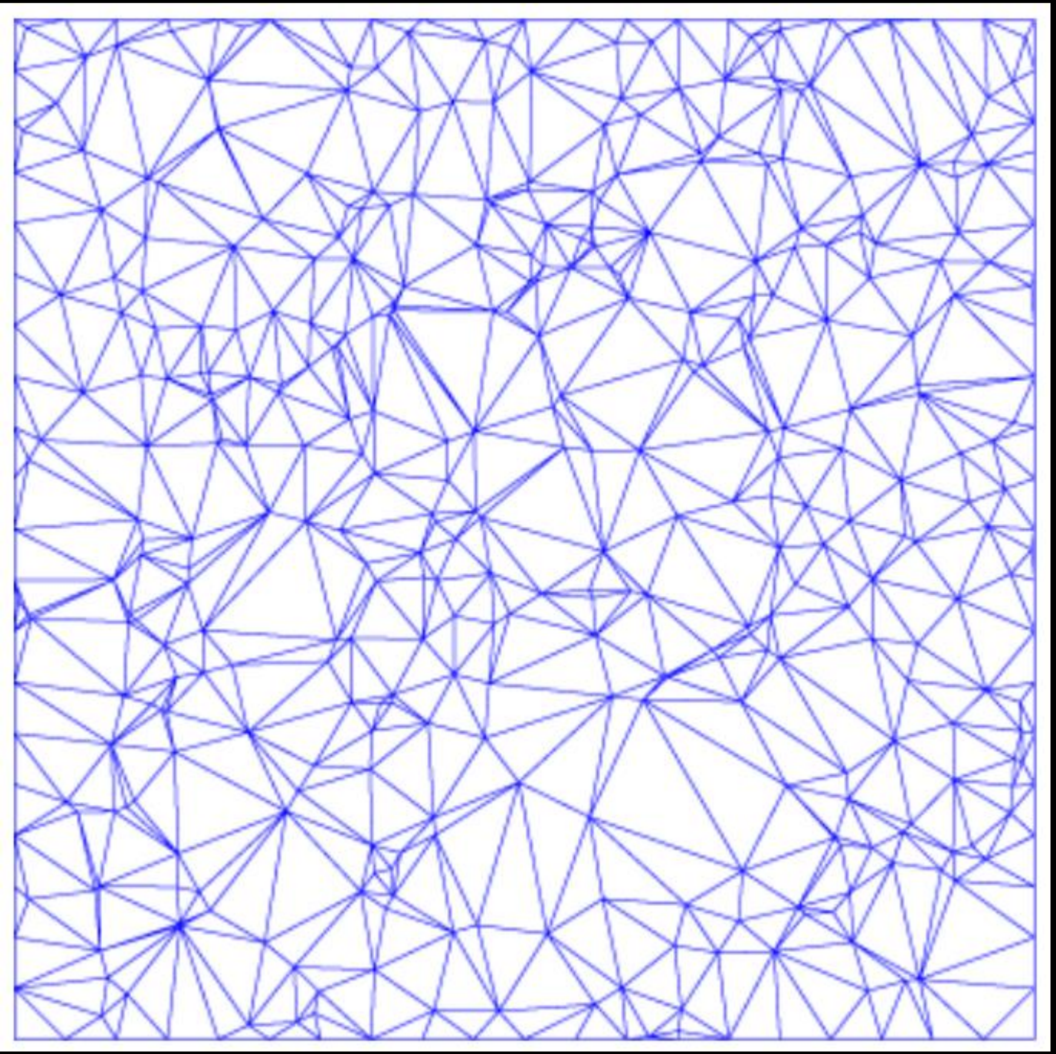
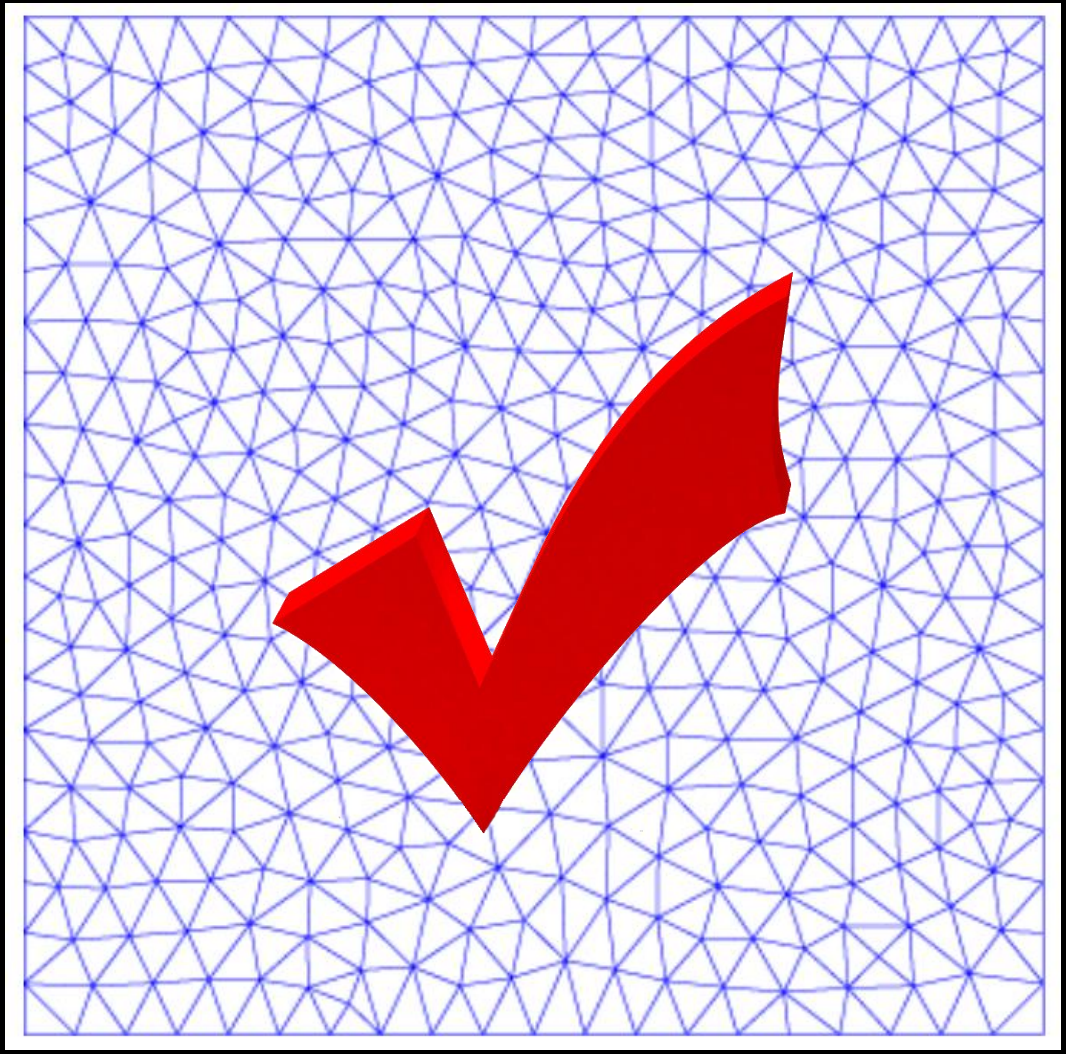
  - Euclidean Minimum Spanning Tree
- Optimal Delaunay triangulation

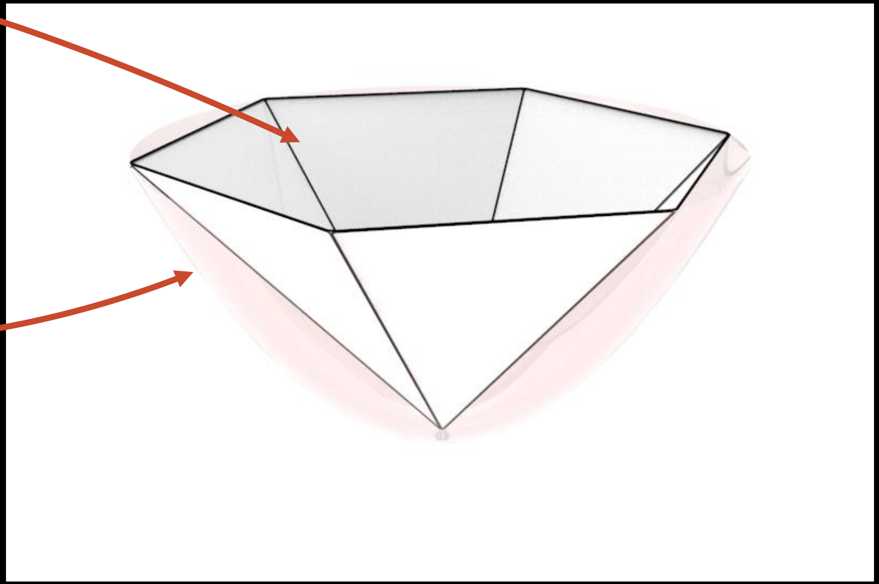
Optimal Delaunay triangulation





Thinking from surface approximation

$$E = \sum_{T \in \mathcal{T}} \int_T |\hat{u}(x) - u(x)| \, d\boldsymbol{x}$$



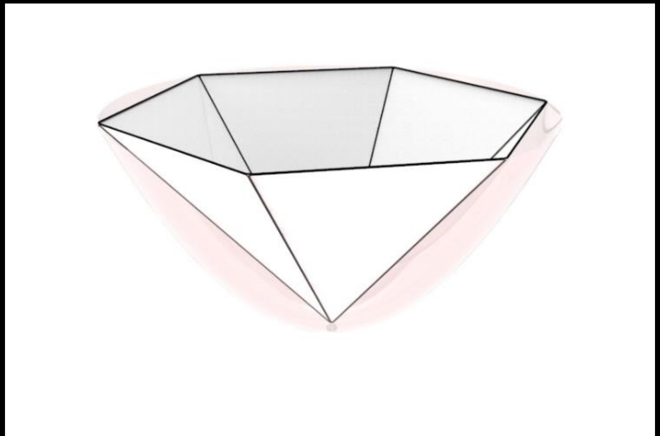
$u(x)$: $z = x^2 + y^2$

$\hat{u}(x)$: piecewise linear interpolation of u

$\mathcal{T}$: a triangulation

Fix positions of vertices, Delaunay triangulation is optimal.

Update of vertices' positions



- Fix the triangulation, update the vertices.

$$E = \sum_{T\in\mathcal{T}} \int_T |\hat{u}(x) - u(x)|\, d\boldsymbol{x} = \sum_{T\in\mathcal{T}} \int_T \hat{u}(x)\, d\boldsymbol{x} + C$$

$$= \sum_{T\in\mathcal{T}} \frac{|T|}{3}\left(u(p_i) + u\left(p_j\right) + u(p_k)\right) + C$$

$$\nabla E_{p_i} = \sum_{T\in\Omega(i)} \frac{\nabla|T|}{3}\left(u(p_i) + u\left(p_j\right) + u(p_k)\right) + \frac{|\Omega|}{3}\nabla u(p_i) = 0$$

Because $\sum_{T\in\Omega(i)} \frac{\nabla|T|}{3} u(p_i) = 0$

$$\nabla u(p_i) = -\frac{1}{|\Omega|} \sum_{T\in\Omega(i)} \frac{\nabla|T|}{3}\left(u\left(p_j\right) + u(p_k)\right)$$

Optimal Delaunay triangulation

- Alternately iterate:
  - Update triangulation
  - Update vertices
- Extension to any convex function $u(x)$:
  - Delaunay triangulation → regular triangulation

$$u(x,y) = \mathrm{e}^{\frac{(x^2+y^2)}{10}}$$

$$\Omega = [-5,5]^2$$