Voronoi Diagram

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Outlines

- Introduction
  - Post Office Problem
  - Voronoi Diagram

- Duality: Delaunay triangulation

- Centroidal Voronoi tessellations (CVT)
  - Definition
  - Applications
  - Algorithms

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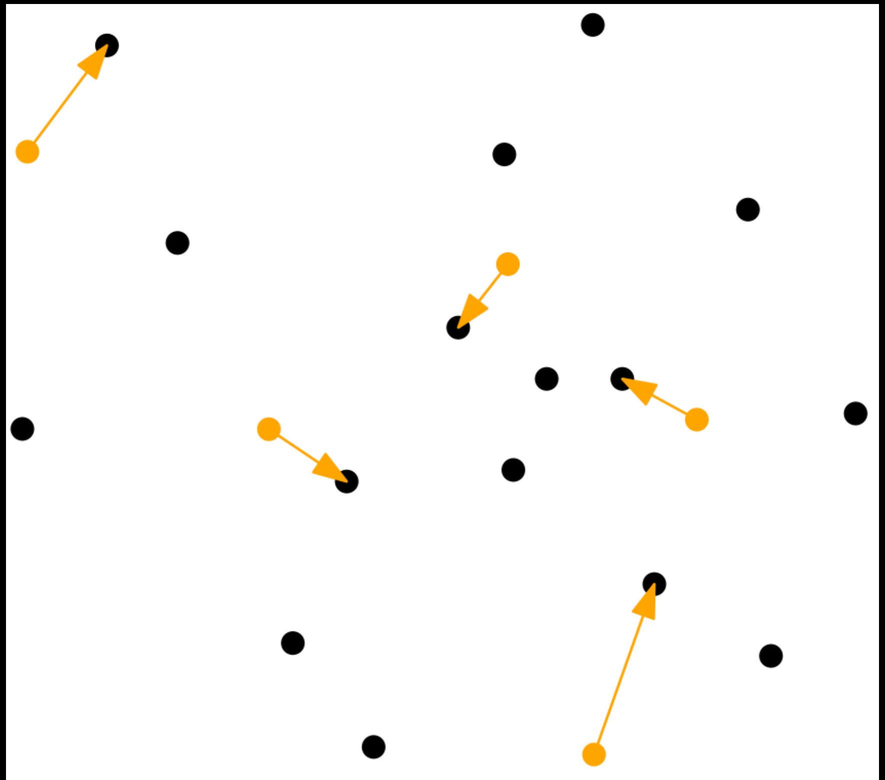
Post Office Problem

- Suppose there are n post offices $p_1, \ldots, p_n$ in a city.
- Someone who is located at a position q within the city would like to know which post office is closest to him.



Post Office Problem

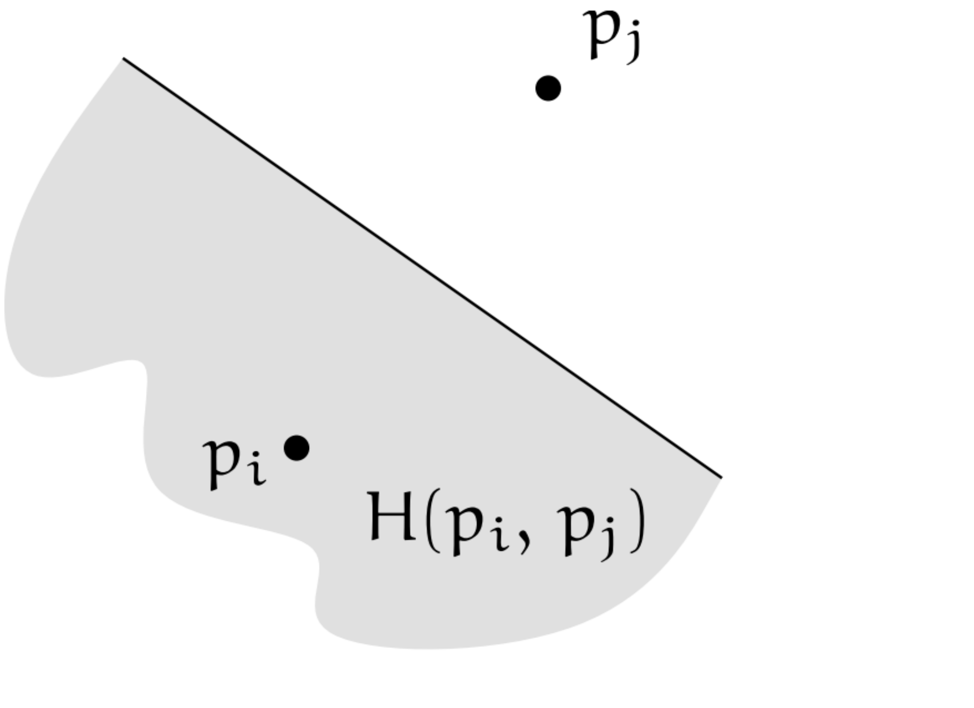
- Do not think from the queries.

- Our long term goal is to come up with a data structure on top of P that allows to answer any possible query efficiently.

- Basic idea:
  - Partition the query space into regions on which is the answer is the same.
  - In our case, this amounts to partition the plane into regions such that for all points within a region the same point from P is closest.

Two post offices

- Proposition
  - For any two distinct points in R^d, the bisector is a hyperplane, that is, in R^2 it is a line.



The bisector of two points.

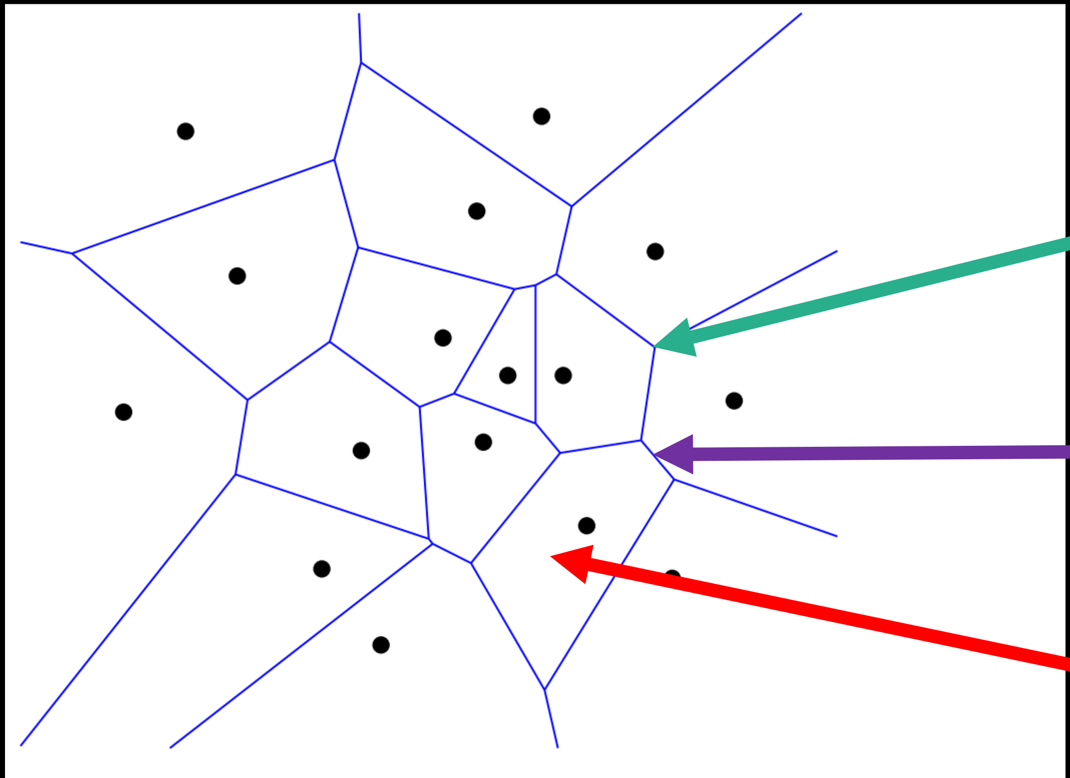
Voronoi cell

- Given a set $P = \{p_1, \ldots, p_n\}$ of points in R^2, for $p_i \in P$ denote the Voronoi cell $VP(i)$ of p_i by
$$VP(i) \coloneqq \{q \in R^2 \mid \|q - p_i\| \leq \|q - p\|, \forall p \in P\}$$

1. $VP(i) = \cap_{j \neq i} H(p_i, p_j)$
2. $VP(i)$ is non-empty and convex.
3. Observe that every point of the plane lies in some Voronoi cell but no point lies in the interior of two Voronoi cells. Therefore these cells form a subdivision of the plane.

Voronoi Diagram

- The Voronoi Diagram $VD(P)$ of a set $P = \{p_1, \ldots, p_n\}$ of points in R^2 is the subdivision of the plane induced by the Voronoi cells $VP(i)$, for $i = 1, \ldots, n$.



Example: The Voronoi diagram of a point set.

$VV(P)$: the set of vertices

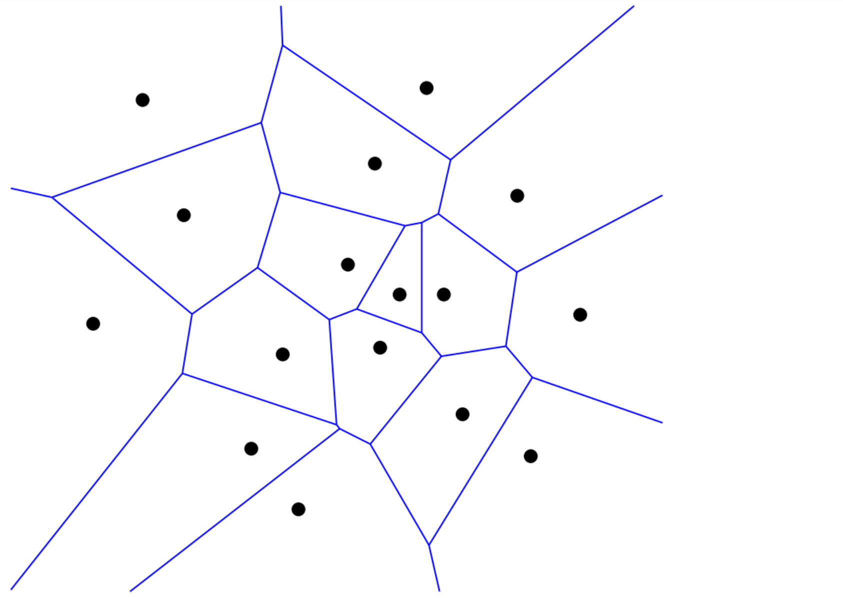
$VE(P)$: the set of edges

$VR(P)$: the set of regions

Lemma 1

- For every vertex $v \in VV(P)$ the following statements hold.
  - 1) v is the common intersection of at least three edges from $VE(P)$;
  - 2) v is incident to at least three regions from $VR(P)$;

Proof: As all Voronoi cells are convex, each interior angle is less than π, thus $k \geq 3$ of them must be incident to v.



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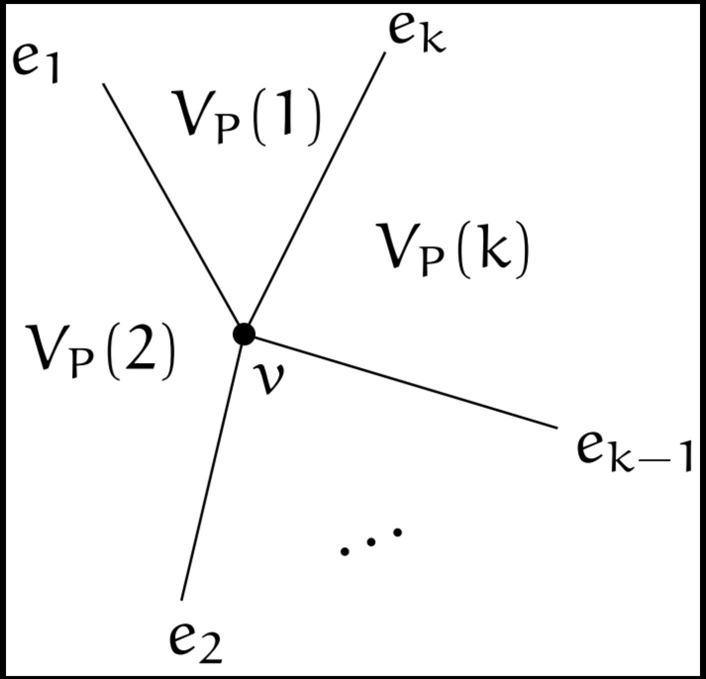
  - 3) v is the center of a circle $C(v)$ through at least three points from P;



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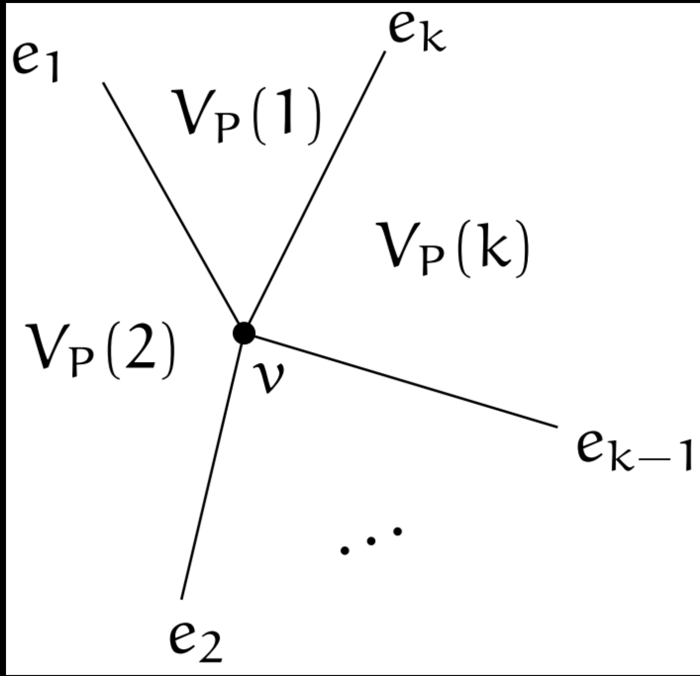
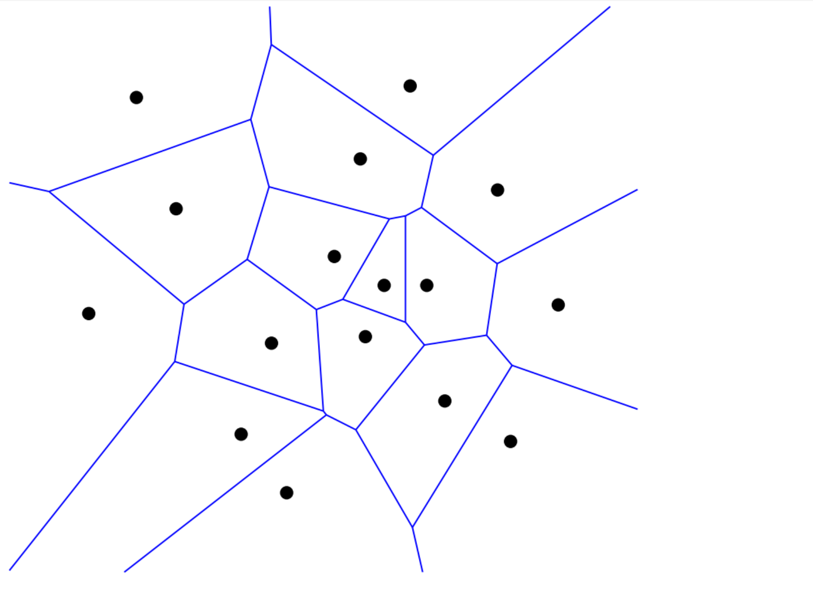
  - 1) v is the common intersection of at least three edges from $VE(P)$;
  - 2) v is incident to at least three regions from $VR(P)$;
  - 3) v is the center of a circle $C(v)$ through at least three points from P;
  - 4) $C(v)^{\circ} \cap P = \emptyset$. $C(v)^{\circ}$: The interior of $C(v)$.



Suppose there exists a point $p_l \in C(v)^{\circ}$. Then the vertex v is closer to p_l than it is to any of $p_1, \ldots, p_k$, in contradiction to the fact that v is contained in all of $VP(1), \ldots, VP(k)$.

Lemma 2

- There is an unbounded Voronoi edge bounding $VP(i)$ and $VP\ (j) \Leftrightarrow \overline{p_i p_j} \cap P = \{p_i, p_j\}$ and $\overline{p_i p_j} \in \partial conv(P)$, where the latter denotes the boundary of the convex hull of P.
- Proof: There is an unbounded Voronoi edge bounding $VP(i)$ and $VP(j) \Leftrightarrow$ there is a ray $\rho \subset b_{i,j}$ such that $\|r - p_k\| > \|r - p_i\|(= \|r - p_j\|), \forall r \in \rho\ and\ p_k \in P \backslash \{p_i, p_j\}$. Equivalently, there is a ray $\rho \subset b_{i,j}$ such that for every point $r \in \rho$ the circle $C \in D$ centered at r does not contain any point from P in its interior.



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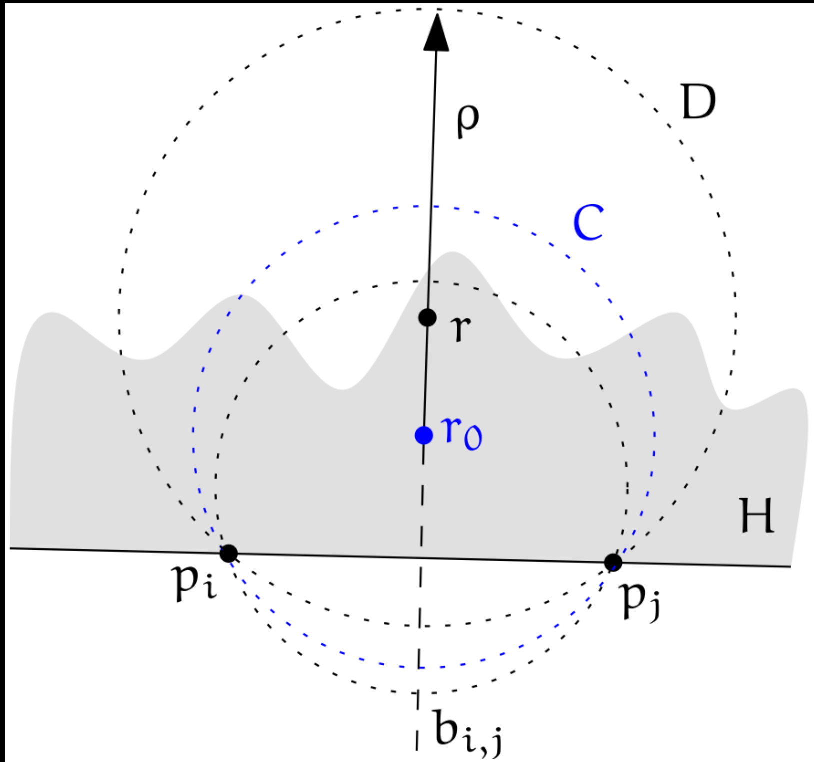

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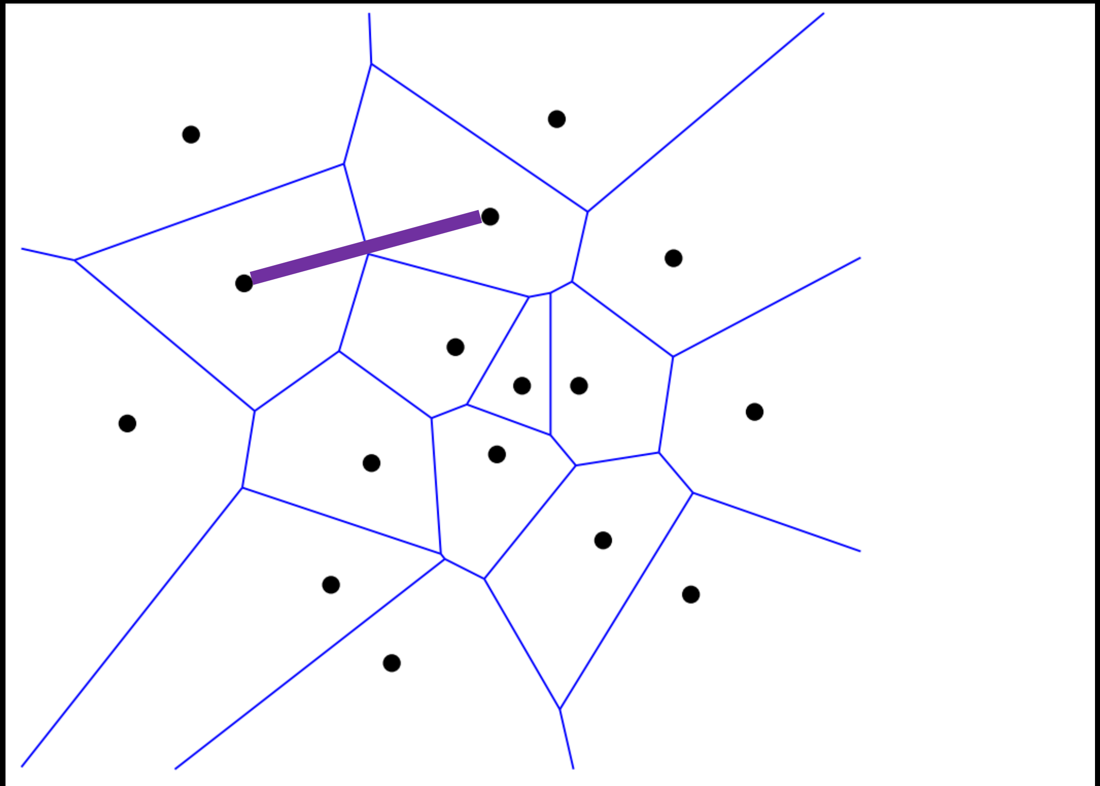
Duality

- A straight-line dual of a plane graph G is a graph G' defined as follows: choose a point for each face of G and connect any two such points by a straight edge, if the corresponding faces share an edge of G.

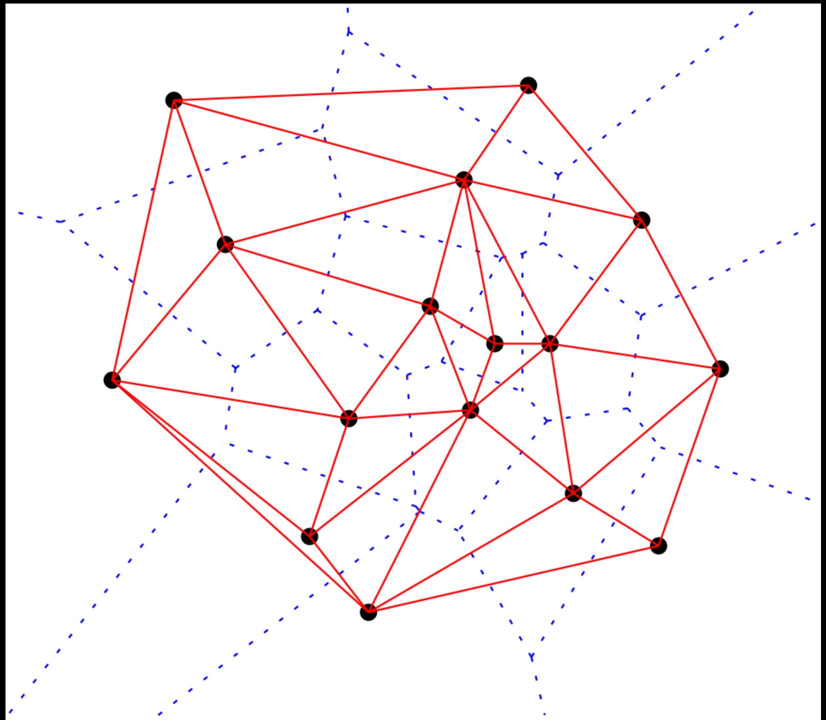


Delaunay triangulation

- Theorem: The straight-line dual of $VD(P)$ for a set $P \subset R^2$ of $n > 3$ points in general position (no three points from P are collinear and no four points from P are cocircular) is a triangulation: the unique Delaunay triangulation of P.

Proof: $\Longrightarrow$

1. convex hull
2. Triangles
3. Empty circle property



Proof: $\Longleftarrow$

1. Circumcenter is selected for each face.
2. Empty circle property.

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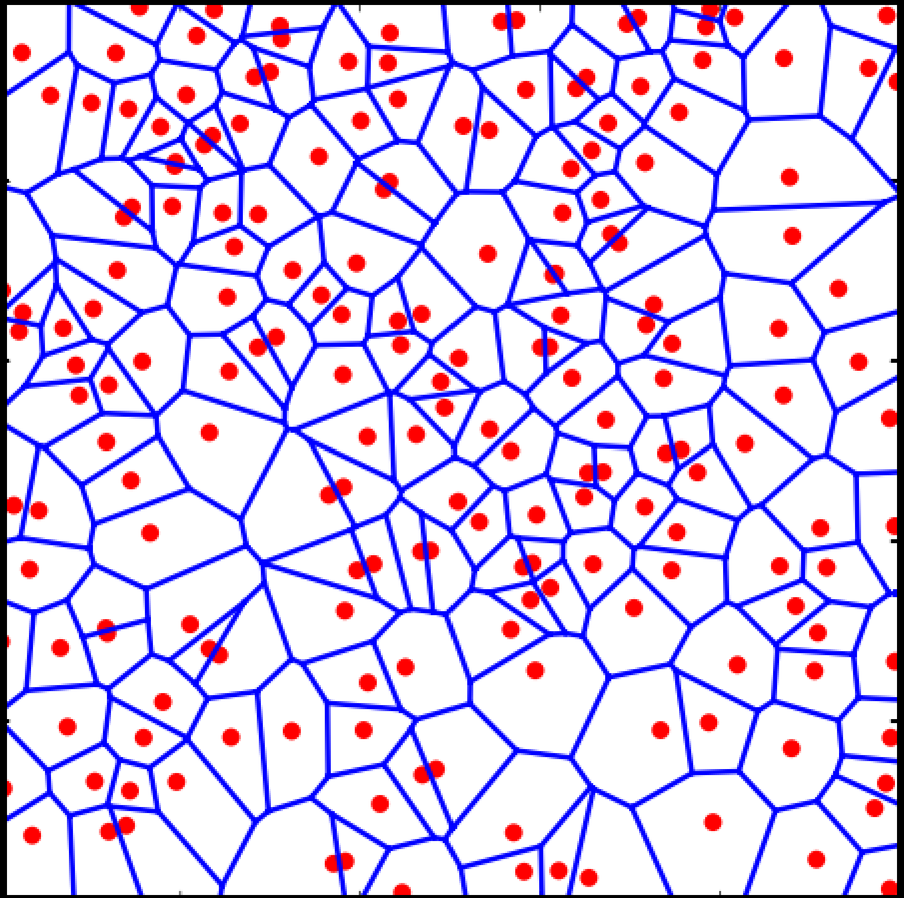
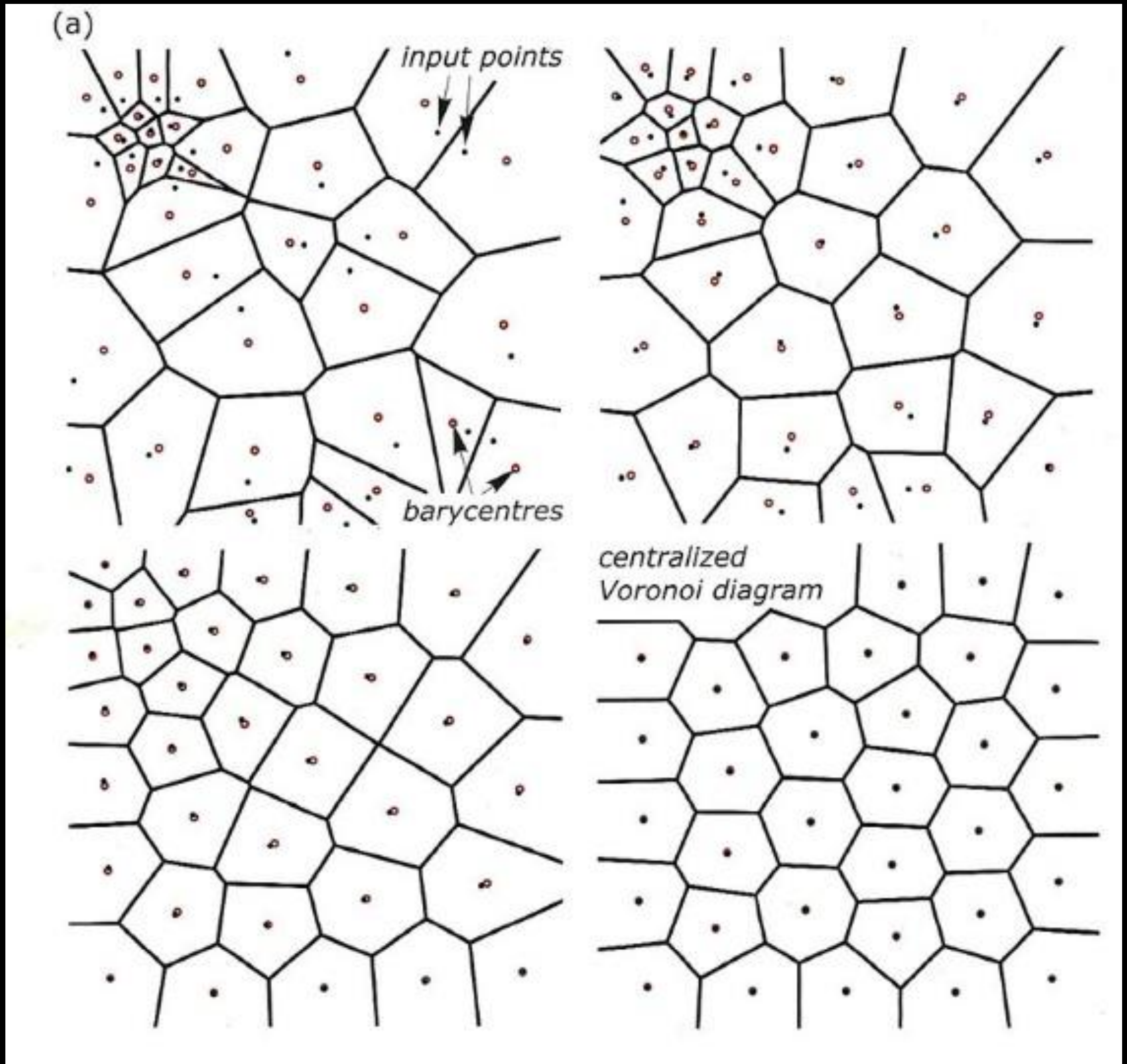
Problem



Update vertices

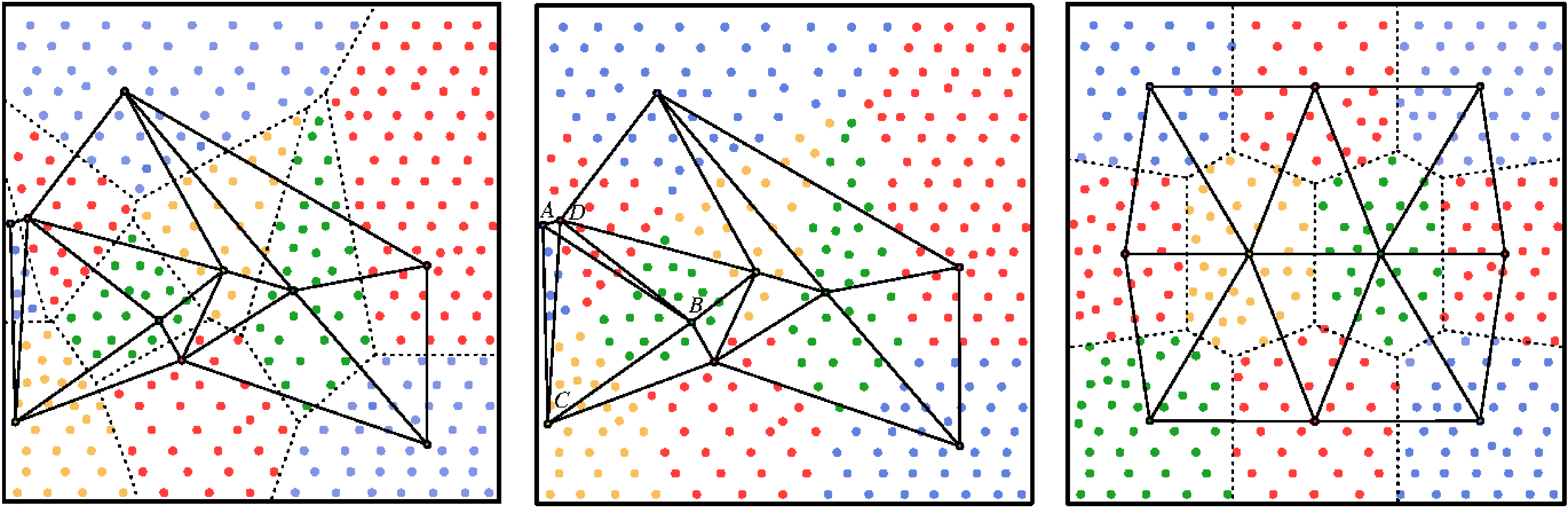
Definition – CVT



A class of Voronoi tessellations where each site coincides with the centroid (i.e., center of mass) of its Voronoi region.

$$c_i = \frac{\int_{V_i} x\rho(x)\,dx}{\int_{V_i} \rho(x)dx}$$

Applications – Remeshing



(a) (b) (c)

Energy function

$$E(p_1, \dots, p_n, V_1, \dots, V_n) = \sum_{i=1}^{n} \int_{V_i} \|x - p_i\|^2 \, dx$$

1. For a fixed set of sites $P = \{p_1, \dots, p_n\}$, the energy function is minimized if $\{V_1, \dots, V_n\}$ is a Voronoi tessellation.
2. For the fixed regions, the p_i are the mass centroids c_i of their corresponding regions V_i.

Lloyd iteration

- 1. Construct the Voronoi tessellation corresponding to the sites p_i.
- 2. Compute the centroids c_i of the Voronoi regions V_i and move the sites p_i to their respective centroids c_i.
- 3. Repeat steps 1 and 2 until satisfactory convergence is achieved.