Homework 4
ISyE 6420: Fall 2024

Question 1

Consider the following unnormalized posterior:

$$p(\boldsymbol{\theta} \mid \boldsymbol{y}) \propto e^{-\frac{1}{2}\left\{\theta_1^2\theta_2^2+\theta_1^2+\theta_2^2-2\theta_1\theta_2-4\theta_1-4\theta_2\right\}}$$

where $\boldsymbol{\theta} \in \mathbb{R}^2$. Plot a two-dimensional image of this distribution for $\boldsymbol{\theta} \in [-5, 10]^2$. Generate an MCMC sample of size 10,000 using the Metropolis algorithm with 1,000 additional burn-in iterations for a total of 11,000. This needs to be manually coded (without using a PPL) in Python, R, etc. The two-dimensional image can be created in Python using the Matplotlib function contourf or in R using the function image.

1. Choose the scale of the proposal distribution (bivariate normal distribution) so that the acceptance rate is around 0.40. Report the chosen scale and the actual acceptance rate.
2. Plot the sampled points over the two-dimensional image of the distribution.
3. Plot the marginal densities of the two parameters.
4. Obtain the 95% equi-tailed credible intervals for each of the two parameters.

Question 2

Consider the Bayesian model:

$$\begin{aligned} y \mid \theta_1, \theta_2 &\sim N\left(\theta_1+\theta_2, 1\right) \\ \theta_i &\sim^{\text{iid}} \quad N\left(0, \nu^2\right), \quad i=1,2. \\ \nu^2 &\sim \operatorname{Inv-Gamma}(10,10) \end{aligned}$$

Suppose $y = 1.2$ is observed. Then,

(a) Find the full conditional distributions of θ_1, θ_2, and ν^2 and use Gibbs sampling to sample from the posterior.

(b) Plot the marginal posterior densities of the three parameters and provide their mean and 95% credible intervals.

(c) Create trace plots for all three parameters. For the trace plots, the X-axis should be the iteration count, and the Y-axis should be the observed value of the chain at each iteration.