Homework 3
ISyE 6420: Fall 2024

Question 1

Three devices are monitored until failure. The observed lifetimes are $1.1, 2.2$, and 0.4 years. If the lifetimes ate modeled as exponential distribution with rate λ,

$$T_i \sim \mathcal{E}xp(\lambda), \quad f(t \mid \lambda) = \lambda e^{-\lambda t}, t > 0, \lambda > 0$$

Assume an exponential prior on λ:

$$\lambda \sim \mathcal{E}xp(2), \quad \pi(\lambda) = 2e^{-2\lambda}, \lambda > 0$$

(a) Find the posterior distribution of λ.
(b) Find the Bayes estimator for λ.
(c) Find the MAP estimator for λ.
(d) Numerically find both the equi-tailed and highest posterior density credible sets for λ, at the 95% credibility level.
(e) Find the posterior probability of hypothesis $H_0 : \lambda \leq 1/2$.

Question 2

Let

$$\begin{aligned} y_i \mid \theta_i \quad &\sim^{ind.} \quad \text{Poisson}\,(\theta_i) \\ \theta_i \quad &\sim^{iid} \quad \text{Gamma}(2, b) \end{aligned}$$

for $i = 1, \ldots, n$, where b is unknown. Find the empirical Bayes estimator of $\theta_i, i = 1, \ldots, n$ (Note: If $X \sim \text{Gamma}(a, b)$, then its pdf is $p(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}$ for $x \geq 0, a, b > 0.$).

Question 3

Suppose $y \mid \beta \sim \text{Gamma}(\alpha, \beta)$, where α is known.

(a) Find the Jeffreys prior for β.

(b) Using the Jeffreys prior from Part 1, derive the posterior distribution $p(\beta \mid y_1, \ldots, y_n)$ for n i.i.d. observations $y_1, \ldots, y_n$.