INCOMPATIBILITIES AND PARADOXICAL CASES IN VOTING THEORY

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INTRODUCTION

The work of Daugherty has shown that different voting methods yield different winners. Her work has inspired us to explore the cases in which two voting methods, positional tally and pairwise comparison, generate conflicting voting results. We define this difference as an incompatibility. In this project, we present our results of calculations of the number of incompatibilities. We applied methods such as permutation matrices, combinatorics, and linear inequalities to show how incompatibilities can be constructed. Calculations are performed in Python. We also explored the Condorcet paradox, i.e. that social preferences can be cyclic even when the individual choices are not cyclic. By applying combinatorial methods, we have proved various patterns exhibited by Condorcet paradoxes when there are three candidates.

DEFINITIONS

Positional Tally - *Borda Count*: a weighting vector w, whose entries correspond to the number of points given to a voter's ranking of each candidate.

Pairwise Comparison - *Condorcet Method* : We could compare candidates pairwise, and give points according to how many times one candidate beats each other candidate. According to Daugherty, "Candidate A wins over candidate B if A is ranked higher than B more times than B is ranked higher than A."

A **Condorcet paradox** occurs when the outcome of a head-to-head comparison is cyclic. For example, the result might have candidate A over candidate B, candidate B over candidate C, and candidate C over candidate A.

An **incompatibility** occurs when the two voting methods, Pairwise Comparison and Positional Tally, do not agree, including the possibility of a tie. If both methods declare candidate A as the winner but have different runner-ups, it is considered as an incompatibility.

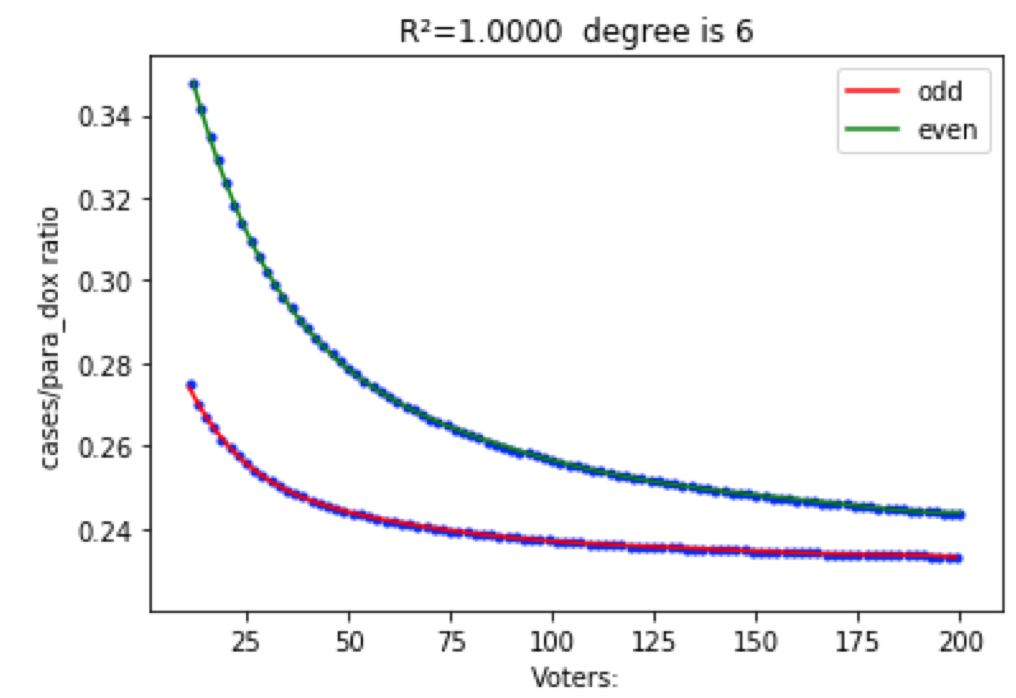
A **profile** is a vector $p = (p_1, p_2, p_3, p_4, p_5, p_6)$ where p_i represents the number of voters voting for each type of full rankings as listed below:

Type	Ranking	Type	Ranking
1	$A \succ B \succ C$	4	$B \succ C \succ A$
2	$A \succ C \succ B$	5	$C \succ A \succ B$
3	$B \succ A \succ C$	6	$C \succ B \succ A$

GRAPHIC RESULT

Voters	Total Profiles	Incom.	Percentage
10	3003	1062	26.25%
20	17190	53130	21.07%
71	18474840	4433754	24.00%
72	19757815	5249826	26.57%
73	21111090	5060640	23.97%
74	22537515	5969106	26.49%
...	...	...	...
200	2872408791	699509982	24.35%

The figure below shows the percentage of incompatible profiles among all possible profiles. In the graph below, **odd/even** represents the percentage of incompatibility when we have odd/even number of voters voting for 3 candidates.



The figure above and the table both show that, for even number of voters and odd number of voters, the cases of incompatibility both increase. The percentage of incompatibility decreases at different rates. However, they both converge to approximately 25% as the number of voters n get larger.

KEY IDEA

How we count the number of incompatible profiles.
Generator: The idea is that if we know one incompatible profiles, we can find 5 more incompatible profiles without calculating its positional tally score and pairwise comparison result by swapping the roles of the candidates.

Profile	Creation Method	Groupings	Resulting Conflict
$P = (p_1, p_2, p_3, p_4, p_5, p_6)$	The original profile	e	Between A and B
$P_1 = (p_2, p_1, p_5, p_6, p_3, p_4)$	Switching the roles of B and C	(23)	Between A and C
$P_2 = (p_3, p_4, p_1, p_2, p_6, p_5)$	Switching the roles of A and B	(12)	Between A and B
$P_3 = (p_6, p_5, p_4, p_3, p_2, p_1)$	Switching the roles of A and C	(13)	Between B and C
$P_4 = (p_4, p_3, p_6, p_5, p_1, p_2)$	Switching A to B, B to C, C to A	(123)	Between B and C
$P_5 = (p_5, p_6, p_2, p_1, p_4, p_3)$	Switching A to C, C to B, B to A	(132)	Between A and C

OBSERVATIONS

If we observe the profiles that yield Condorcet paradox where the outcome of the positional tally method has candidate A over B, B over C, and C over A, we can see that there is a one-to-one correspondence between the profiles. (Notice that for profiles that yield Condorcet paradox, the outcome of the positional tally method can also have candidate A over C, C over B and B over A.)

n	Profiles	$n+3$	Profiles
3	$(1,0,0,1,1,0)$	6	$(2,0,0,2,2,0)$
5	$(1,0,0,2,2,0)$	8	$(2,0,0,3,3,0)$
	$(1,0,1,1,2,0)$		$(2,0,1,2,3,0)$
	$(1,1,0,2,1,0)$		$(2,1,0,3,2,0)$
	$(2,0,0,1,1,1)$		$(3,0,0,2,2,1)$
	$(2,0,0,1,2,0)$		$(3,0,0,2,3,0)$
	$(2,0,0,2,1,0)$		$(3,0,0,3,2,0)$

Table 1: Corresponding paradox-generating profiles given n voters and $n+3$ voters.

If we add 1 to each of the values of p_1, p_4, and p_5 of the paradox-generating profiles when there are n candidates, we get a paradox-generating profile when there are $n+3$ candidates. Similarly, we can subtract 1 from each of the values of p_1, p_4, and p_5 of the $(n+3)$-candidate profiles to get the n-candidate profiles.

CONDORCET PARADOX THEOREMS

Theorem 1 *When the number of voters n is odd, the number of profiles that generate Condorcet paradox is equal to the number when there are $n+3$ voters.*

Voters	Paradoxes	Voters	Paradoxes
3	2	6	2
5	12	8	12
7	42	10	42
9	112	12	112
11	252	14	252
13	504	16	504
15	924	18	924
17	1584	20	1584

Table 2: Number of paradox-generating profile given the number of voters.

Theorem 2 *Let n be a non-negative integer. When there are $2n+1$ or $2n+4$ voters voting for 3 candidates, the number of profiles that generate Condorcet paradox is*

$$2\binom{n+5}{5} = \frac{(n+5)(n+4)(n+3)(n+2)(n+1)}{60}.$$

Theorem 3 *The fraction of profiles that generate Condorcet paradox given n voters and 3 candidates converges to $\frac{1}{16}$.*

REFERENCE

[1] Daugherty, Zajj, "An Algebraic Approach to Voting Theory" (2005). HMC Senior Theses. 169. https://scholarship.claremont.edu/hmc_theses/169

ONGOING RESEARCH

In future work, we plan to tackle the problem of Condorcet paradoxes and incompatibility in the n-candidate model. Furthermore, we hope to find combinatorially meaningful expressions for the number of paradox-generating profiles. Last but not least, we wish to find a way to decrease the percentage of incompatibility to make the voting result less prone to manipulation.