Part A

(1) Suppose (E, d) is a metric space, and that $S \subseteq E$ is complete. Prove that S is closed (as a subset of E).

Solution. Suppose that S is a complete subset of a metric space (E, d). We will argue that S is closed by showing it's closed under limits. So suppose that $\{s_1, s_2, \cdots\}$ is some sequence whose elements are drawn from S, and suppose that $\lim\{s_n\} = L$ exists in E. Since $\{s_n\}_{n\in\mathbb{N}}$ is convergent, we know that the sequence is Cauchy (since convergent implies Cauchy, as proved in class). But since the sequence is Cauchy and S is complete, the definition of completeness tells us that $\{s_n\}_{n\in\mathbb{N}}$ converges in S, say $\lim\{s_n\} = \hat{L}$ for some $\hat{L} \in S$. Now by uniqueness of limits, we have $L = \hat{L}$. Hence we have the desired result. □

(2) Prove that compact sets are closed under finite union; that is, if $S_1, \cdots, S_n$ are each compact within a fixed metric space (E, d), prove that $\cup_{i=1}^n S_i$ is also compact. Then, give an example to show that compact sets are not closed under arbitrary union.

Solution. Suppose that $C_1, \cdots, C_n$ are compact sets, and let $\{V_i\}_{i\in\mathcal{I}}$ be an open cover of $C_1 \cup \cdots \cup C_n$:

$$C_1 \cup \cdots \cup C_n \subseteq \bigcup_{i\in\mathcal{I}} V_i.$$

By the definition of union, for each $1 \leq k \leq n$ we have

$$C_k \subseteq \bigcup_{j=1}^n C_j \subseteq \bigcup_{i\in\mathcal{I}} V_i,$$

so that $\{V_i\}_{i\in\mathcal{I}}$ is an open cover on C_k. But then by compactness of C_k we know there is some finite subcollection $\{V_j\}_{j\in\mathcal{J}_k}$ — where $\mathcal{J}_k \subseteq \mathcal{I}$ and $|J_k| < \infty$ — which covers C_k:

$$C_k \subseteq \bigcup_{j\in\mathcal{J}_k} V_j.$$

By the definition of union, we then have that

$$C_1 \cup \cdots \cup C_n \subseteq \left(\bigcup_{j\in\mathcal{J}_1} V_j\right) \cup \left(\bigcup_{j\in\mathcal{J}_2} V_j\right) \cup \cdots \cup \left(\bigcup_{j\in\mathcal{J}_n} V_j\right).$$

The latter cover has at most $|\mathcal{J}_1| + \cdots + |\mathcal{J}_n| < \infty$ many terms, and hence is a finite sub cover.

To show that an arbitrary union of compact sets is not compact, note that for any $x \in \mathbb{R}$ we have that $\{x\}$ is a compact set in $\mathbb{R}$ (since finite sets are always compact). However, we also have that $\mathbb{R} = \bigcup_{x\in\mathbb{R}}\{x\}$, and certainly $\mathbb{R}$ is not compact since $\mathbb{R}$ is not bounded. □

(3) Are compact sets closed under arbitrary intersection? Prove or give a counterexample.

Solution. We prove that compact sets are closed under arbitrary intersection. Let $\mathcal{I}$ be an indexing set, and suppose for each $i \in \mathcal{I}$ we have a compact set C_i. Now we know that all compact sets are closed from the first problem, and furthermore that closed sets are closed under arbitrary intersection by the topology of closed sets. Hence we have $\bigcap_{i\in\mathcal{I}} C_i$ is closed. Furthermore if we choose any $i_0 \in \mathcal{I}$

we have $\bigcap_{i\in\mathcal{I}} C_i \subseteq C_{i_0}$; from class we know that any closed subset of a compact set is again compact, and so we have the desired result. $\square$

Part B

(4) Prove that any sequence of $\mathbb{R}$ has a monotonic subsequence. [Hint: you might break this into two cases: first when there exists some subsequence which has no minimum element, and second when this condition does not hold.]

Solution. Let $\{p_1, p_2, \cdots\}$ be some sequence in $\mathbb{R}$. For the first case, suppose there exists some subsequence $\{p_{n_1}, p_{n_2}, \cdots\}$ which has no minimum element. It is useful to know that if this is the case, then we have for any $k \in \mathbb{N}$ some $j > k$ with $p_{n_k} > p_{n_j}$. For if this weren't the case, then we'd have $p_{n_k} \leq p_{n_j}$ for all $j > k$, and then $\min\{p_{n_1}, \cdots, p_{n_k}\}$ would be less than or equal to any element in the subsequence — certainly it's no larger than any element from $\{p_{n_1}, \cdots, p_{n_k}\}$ by construction, but since it's no larger than p_{n_k}, it's also no larger than p_{n_j} for $j > k$. Of course, this would contradict the assumption that $\{p_{n_1}, p_{n_2}, \cdots\}$ has no least element.

We build a new sequence $\{p_{m_i}\}_{i\in\mathbb{N}}$ inductively. First, we let $p_{m_1} = p_{n_1}$. Now if we assume that we have chosen $p_{m_1}, \cdots, p_{m_k}$ for some $k \geq 1$, and for concreteness suppose we have $p_{m_k} = p_{n_\ell}$. Then we choose $p_{m_{k+1}}$ to be the first term in the subsequence $\{p_{n_{\ell+1}}, p_{n_{\ell+2}}, \cdots\}$ which is smaller than p_{n_ℓ}; we know some such term must exist from our result in the previous paragraph.

We've build the sequence $\{p_{m_i}\}$ to be (strictly) decreasing and a subsequence of the original sequence, so we've completed this case.

Now we must consider the case when there is no subsequence which fails to have a minimum element. In other words, we must consider the case where every subsequence does have a minimum element. Again, we'll build our sequence inductively. We let p_{m_1} be the first term in the sequence $\{p_1, p_2, \cdots\}$ which equals $\min\{p_1, p_2, \cdots\}$; such a term exists because we're in the case where we assume every subsequence has a minimum element. For any $k > 1$, inductively construct p_{m_k} so that p_{m_k} is the first term in $\{p_{m_{k-1}+1}, p_{m_{k-1}+2}, \cdots\}$ which equals $\min\{p_{m_{k-1}+1}, p_{m_{k-1}+2}, \cdots\}$. Note that this certainly gives us $m_1 < m_2 < \cdots$, and furthermore it implies the sequence of terms $\{p_{m_1}, p_{m_2}, \cdots\}$ is increasing as well; to see this is true, note that since $m_1 < m_2 < \cdots$ we have $\{p_{m_{k-2}+1}, p_{m_{k-2}+2}, \cdots\} \supseteq \{p_{m_{k-1}+1}, p_{m_{k-1}+2}, \cdots\}$, and so

$$p_{m_{k-1}} = \min\{p_{m_{k-2}+1}, p_{m_{k-2}+2}, \cdots\} \leq \min\{p_{m_{k-1}+1}, p_{m_{k-1}+2}, \cdots\} = p_{m_k}.$$

$\square$

(5) Suppose that $\{a_n\}_{n\in\mathbb{N}}$ is a bounded sequence of real numbers. We define two new sequences from this original sequence. The first is $\{u_n\}_{n\in\mathbb{N}}$, where for each $n \in \mathbb{N}$ we define $u_n = l.u.b.\{a_n, a_{n+1}, a_{n+2}, \cdots\}$; and the second is $\{\ell_n\}_{n\in\mathbb{N}}$, where for each $n \in \mathbb{N}$ we define $\ell_n = g.l.b.\{a_n, a_{n+1}, a_{n+2}, \cdots\}$.

(a) Prove that $\{u_n\}_{n\in\mathbb{N}}$ is convergent. [N.B. $\lim_{n\to\infty}\{u_n\}$ is often called the limit supremum of the original sequence $\{a_n\}_{n\in\mathbb{N}}$, and is denoted $\limsup\limits_{n\to\infty}\{a_n\}$.]

Solution. Though the problem doesn't ask for this, it's worth pointing out that each set $\{a_n, a_{n+1}, a_{n+2}, \cdots\}$ actually has a least upper bound since $\{a_k\}_{k\in\mathbb{N}}$ is nonempty and bounded, and therefore so too is $\{a_n, a_{n+1}, a_{n+2}, \cdots\}$ for any given n. Hence u_n is well-defined.

Now for the problem itself. First, observe that if we have two nonempty bounded sets $A, B \subseteq \mathbb{R}$ such that $A \subseteq B$, then we must have $l.u.b.(A) \leq l.u.b.(B)$. To see this is true, let $\alpha = l.u.b.(A)$ and $\beta = l.u.b.(B)$. For any given $a \in A$, we have $a \in B$. Since β is an upper bound on B, it follows then that $a \leq \beta$. But since this is true for all $a \in A$, we must have β is an upper bound on A. By definition of least upper bound, we then have $\alpha \leq \beta$, as desired.

With this in mind, note that $\{a_{n+1}, a_{n+2}, a_{n+3}, \cdots\} \subseteq \{a_n, a_{n+1}, a_{n+2}, \cdots\}$. By the previous result, then, we have $u_{n+1} \leq u_n$. It therefore follows that $\{u_n\}_{n\in\mathbb{N}}$ is a decreasing sequence. If we can show that $\{u_n\}_{n\in\mathbb{N}}$ is bounded, then the Monotonic Convergence Theorem will then give us that $\lim_{n\to\infty}\{u_n\}$ exists, as desired.

So we now show that $\{u_n\}_{n\in\mathbb{N}}$ is bounded. Certainly we must have that u_1 is an upper bound on $\{u_n\}_{n\in\mathbb{N}}$ since the sequence is decreasing. Now we check that the sequence is bounded from below. By assumption we have $\{a_n\}_{n\in\mathbb{N}}$ is bounded; suppose that $\{a_n\}_{n\in\mathbb{N}} \subseteq B_E(0, M)$. We claim that $-M \leq u_n$ for all $n \in \mathbb{N}$. If this were not true, we'd have $-M > u_n$ for some n. But then since u_n is the least upper bound for $\{a_n, a_{n+1}, a_{n+2}, \cdots\}$, we'd have $-M > u_n \geq a_k$ for all $k \geq n$. This, of course, would contradict the assumption that $\{a_n\}_{n\in\mathbb{N}} \subseteq B_E(0, M) = (-M, M)$. □

(b) Prove that $\{\ell_n\}_{n\in\mathbb{N}}$ is convergent. [N.B. $\lim_{n\to\infty}\{\ell_n\}$ is often called the limit infimum of the original sequence $\{a_n\}_{n\in\mathbb{N}}$, and is denoted $\liminf\limits_{n\to\infty}\{a_n\}$.]

Solution. It should be no surprise that this proof is essentially identical to the first, with only some minor changes necessary. The statement at the start of the last proof is still relevant here: the ℓ_n are all well-defined, and so it makes sense to consider this sequence in the first place.

Again, as before, we start with a general fact about how greatest lower bounds behave under set containment. If we have two nonempty bounded sets $A, B \subseteq \mathbb{R}$ such that $A \subseteq B$, then we must have $g.l.b.(A) \geq g.l.b.(B)$. To see this is true, let $\alpha = g.l.b.(A)$ and $\beta = g.l.b.(B)$. For any given $a \in A$, we have $a \in B$. Since β is an lower bound on B, it follows then that $a \geq \beta$. But since this is true for all $a \in A$, we must have β is a lower bound on A. By definition of greatest lower bound, we then have $\alpha \geq \beta$, as desired.

Now observe that $\{a_{n+1}, a_{n+2}, a_{n+3}, \cdots\} \subseteq \{a_n, a_{n+1}, a_{n+2}, \cdots\}$. By the previous result, then, we have $\ell_{n+1} \leq \ell_n$. It therefore follows that $\{\ell_n\}_{n\in\mathbb{N}}$ is an increasing sequence. If we can show that $\{\ell_n\}_{n\in\mathbb{N}}$ is bounded, then the Monotonic Convergence Theorem will then give us that $\lim_{n\to\infty}\{\ell_n\}$ exists, as desired.

So we now show that $\{\ell_n\}_{n\in\mathbb{N}}$ is bounded. Certainly we must have that ℓ_1 is a lower bound on $\{\ell_n\}_{n\in\mathbb{N}}$ since the sequence is increasing. Now we check that the sequence is bounded from above. By assumption we have $\{a_n\}_{n\in\mathbb{N}}$ is bounded; suppose that $\{a_n\}_{n\in\mathbb{N}} \subseteq B_E(0, M)$. We claim that $M \geq \ell_n$ for all $n \in \mathbb{N}$. If this were not true, we'd have $M < \ell_n$ for some n. But then since ℓ_n is the greatest lower bound for $\{a_n, a_{n+1}, a_{n+2}, \cdots\}$, we'd have $M < \ell_n \leq a_k$ for all $k \geq n$. This, of course, would contradict the assumption that $\{a_n\}_{n\in\mathbb{N}} \subseteq B_E(0, M)$. □

(c) Suppose that $\lim_{n\to\infty}\{a_n\}$ exists. Prove that $\limsup_{n\to\infty}\{a_n\} = \lim_{n\to\infty}\{a_n\}$.

[NB. The same idea that you use in your proof can be used to prove that if $\lim_{n\to\infty}\{a_n\}$ exists, then $\liminf_{n\to\infty}\{a_n\} = \lim_{n\to\infty}\{a_n\}$, and so in fact we get $\liminf_{n\to\infty}\{a_n\} = \limsup_{n\to\infty}\{a_n\}$ in this case. As it turns out, if one has a sequence for which its limit inferior agrees with its limit superior, then one can use this to argue that the original sequence is convergent!]

Solution.

Suppose we know that $\lim\{a_n\}$ exists, and we hope to show that $\liminf\{a_n\} = \limsup\{a_n\}$. Let $\varepsilon > 0$ be given, and choose $N \in \mathbb{N}$ so that for all $n > N$ we get $|L - a_n| < \varepsilon$. This means that for all $n > N$ we have $-\varepsilon < a_n - L < \varepsilon$ (by "bounding absolute value"), and by subtracting L from both sides and using inequalities under arithmetic, we get

$$L - \varepsilon < a_n < L + \varepsilon.$$

As a consequence, we have $L + \varepsilon$ is an upper bound on $\{a_{N+1}, a_{N+2}, \cdots\}$, and since u_{N+1} is the least upper bound on this set, we have $u_{N+1} \leq L + \varepsilon$. Likewise we have $L - \varepsilon \leq \ell_n$. And since $\ell_{N+1} \leq u_{N+1}$ we therefore get

$$L - \varepsilon \leq \ell_{N+1} \leq u_{N+1} \leq L + \varepsilon.$$

But if k is any integer greater than $N + 1$ we have $\ell_{N+1} \leq \ell_k \leq u_k \leq u_{N+1}$, and so for any such k we have

$$L - \varepsilon \leq \ell_{N+1} \leq \ell_k \leq u_k \leq u_{N+1} \leq L + \varepsilon.$$

In summary: for all $k > n$ we have $|L - u_n| < \varepsilon$ and $|L - \ell_n| < \varepsilon$, whence $\lim\{u_n\} = \lim\{\ell_n\} = L$, as desired.

Remark: In proving that $\liminf_{n\to\infty}\{a_n\} = \limsup_{n\to\infty}\{a_n\}$ implies that $\lim_{n\to\infty}\{a_n\}$ exists, we'd like to use the squeeze theorem. The problem is that the squeeze theorem (as we stated it in class) relies on first knowing that the underlying sequences converge, and we don't know $\lim_{n\to\infty}\{a_n\}$ converges at this point. □

(6) Define a sequence in $\mathbb{R}$ by $a_1 = \sqrt{2}$ and $a_n = \sqrt{2 + a_{n-1}}$ for all $n \geq 2$. Prove that the sequence converges and determine its limit.¡p¿[Note: one way to view the content of this problem is to say that we're evaluating $\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}}$.]

Solution. First, we check that $a_n > 0$ for all n. For this, we of course have $a_1 > 0$ (since $\sqrt{2}$ denotes the (unique) positive real whose square is 2). Assume that we've shown that $a_n > 0$, and we argue that $a_{n+1} > 0$ as well. For this, note $a_n > 0$ implies $2 + a_n > 0$ (since positive numbers are closed under addition), whence $a_{n+1} = \sqrt{2 + a_n}$ is positive since every positive real has a unique positive real square root.

Now we prove that $a_n < 2$ for all n by induction. Certainly we have $a_1 < 2$ since $(\sqrt{2})^2 < 2^2$, and using the fact that the squaring function is strictly increasing on the set of positive reals. Now assume we've shown that $a_n < 2$, and we'll verify that $a_{n+1} < 2$. For this, observe that $2 + a_n < 2 + 2$ by arithmetic under inequalities; since the squaring function is strictly increasing on the set of positive reals, we then get

$$a_{n+1} = \sqrt{2 + a_n} < \sqrt{2 + 2} = 2.$$

The two facts we've shown so far prove that $\{a_n\}$ is bounded. Now we check it is monotonic by proving that $a_{n+1} > a_n$ for all n. We already know that $a_n < 2$ for all n, and so $a_n^2 < 2a_n = a_n + a_n < a_n + 2$. Since the squaring function is strictly increasing on the set of positive reals, we therefore have

$$a_n = \sqrt{a_n^2} < \sqrt{a_n + 2} = a_{n+1}.$$

Now that we know the sequence is monotonic and bounded, the monotonic convergence theorem tells us that the sequence has a limit; call this limit L. Note that we have

$$2 + L = 2 + \lim\{a_n\} = \lim\{2 + a_n\} = \lim\{\sqrt{2 + a_n}\sqrt{2 + a_n}\} = \lim\{a_{n+1}a_{n+1}\}.$$

Now we know that $\lim\{a_{n+1}\} = \lim\{a_n\}$ (since subsequences of convergent sequence are also converge, and converge to the same limit), hence arithmetic under limits tells us that the preceding calculation continues as

$$2 + L = \lim\{a_{n+1}a_{n+1}\} = \lim\{a_n\}\lim\{a_n\} = L^2.$$

Now we have $2 + L = L^2$, or $L^2 - L - 2 = (L - 2)(L - 1) = 0$. Since L is a real number and $\mathbb{R}$ is a field, this equation can only hold if $L = 2$ or $L = -1$. Certainly we can't have $L = -1$, since the sequence $\{a_n\}$ consists of non-negative reals, and the set of non-negative reals is closed (so that the limit of any convergent sequence of non-negative reals must be non-negative). Hence we get $L = 2$. $\square$