Part A

(1) Prove that the closure of S is the set of all limits of convergent sequences from S. That is, prove that

$$\overline{S} = \left\{ L \in X : \exists \{s_1, s_2, s_3, \cdots\} \text{ with each } s_i \in S \text{ and so that } L = \lim_{n\to\infty} \{s_n\}_{n=1}^{\infty} \right\}.$$

Solution. In a previous assignment we showed that $\bar{S} = \{x \in X : \forall \varepsilon > 0 \text{ we have } B(x, \varepsilon) \cap S \neq \emptyset\}$. This will be a useful result for us to use here.

First, suppose $L \in X$ is given so that there exists some $\{p_n\}_{n\in\mathbb{N}}$ from S with $L = \lim_{n\to\infty}\{p_n\}$. To show that $L \in \overline{S}$, we will prove that for any $\varepsilon > 0$ we have $B(L, \varepsilon) \cap S \neq \emptyset$. So let $\varepsilon > 0$ be given. By definition of limit, there is some $N \in \mathbb{N}$ so that $d(p_n, L) < \varepsilon$ for all $n \geq N$. In particular, this means that $p_N \in B(L, \varepsilon) \cap S$, so that the intersection $B(L, \varepsilon) \cap S$ is indeed nonempty.

Now suppose that $L \in \overline{S}$. By the result from the previous problem set, this means that for all $\varepsilon > 0$ on has $B(L, \varepsilon) \cap S \neq \emptyset$. With this in mind, for every $n \in \mathbb{N}$ we let $p_n \in B_E(L, \frac{1}{n}) \cap S$ be given. We claim that $\{p_n\}_{n\in\mathbb{N}}$ has limit L. To see this, let $\varepsilon > 0$ be given. By the archimedean property, there exists some $N \in \mathbb{N}$ so that $\frac{1}{N} < \varepsilon$. We also have for all $n > N$ that $\frac{1}{n} < \frac{1}{N}$. Hence for all such n we get $d(p_n, L) < \frac{1}{n} < \frac{1}{N} < \varepsilon$, as required. $\square$

(2) Suppose that $\{p_n\}_{n=1}^{\infty}$ is a sequence in a metric space (E, d) which converges to $L \in E$. Prove that $\{L, p_1, p_2, \cdots\}$ is a closed set in E. [Note: in this last expression, we consider $\{L, p_1, p_2, \cdots\}$ as a set in E, not as a sequence of points in E.] ¡p¿[Hint: you can do this directly by applying the definition of "closed".]

Solution. For convenience, let S be the set $\{L, p_1, p_2, \cdots\}$. We want to argue that S is closed, so we need to show that S^c is open. To do this, let $x \in S^c$ be given. We'll aim to find some $\varepsilon > 0$ so that $B(x, \varepsilon) \subseteq S^c$; said another way, this means we want $\varepsilon > 0$ so that $B(x, \varepsilon) \cap S = \emptyset$.

Let $\hat{\varepsilon} = d(L, x)$. Since $L \in S$ but $x \notin S$, we know that $x \neq L$, and by the axioms of a metric space this means that $\hat{\varepsilon} > 0$. Since $L = \lim_{n\to\infty}\{p_n\}$, we know there exists some $N \in \mathbb{N}$ so that for all $n \geq N$ we have $d(L, p_n) < \frac{\hat{\varepsilon}}{2}$. Notice in particular this this also forces $d(x, p_n) \geq \frac{\hat{\varepsilon}}{2}$, since otherwise the triangle inequality would give us that

$$d(L, x) \leq d(L, p_n) + d(p_n, x) < \frac{\hat{\varepsilon}}{2} + \frac{\hat{\varepsilon}}{2} = \hat{\varepsilon} = d(L, x),$$

a clear contradiction.

Now for each $1 \leq i \leq N - 1$, we know that $x \neq p_i$ (again, since $p_i \in S$ but $x \notin S$), and just as before this means that $\delta_i = d(x, p_i)$ is a positive number. With this in mind, we will choose $\varepsilon = \min\left\{\frac{\hat{\varepsilon}}{2}, \delta_1, \cdots, \delta_{N-1}\right\}$. Note that since ε is the minimum of a finite collection of positive number, we must have $\varepsilon > 0$ as well.

Furthermore, we claim that $B(x, \varepsilon) \cap S = \emptyset$. To see this is true, note first that $L \notin B(x, \varepsilon)$, since we know that

$$d(L, x) = \hat{\varepsilon} > \frac{\hat{\varepsilon}}{2} \geq \min\left\{\frac{\hat{\varepsilon}}{2}, \delta_1, \cdots, \delta_{N-1}\right\} = \varepsilon.$$

Now let $i \in \mathbb{N}$ be given. If $1 \leq i \leq N-1$, then we must have $p_i \notin B(x, \varepsilon)$ since

$$d(p_i, x) = \delta_i \geq \min\left\{\frac{\hat{\varepsilon}}{2}, \delta_1, \cdots, \delta_{N-1}\right\} = \varepsilon.$$

On the other hand, if $n \geq N$ then from above we've already seen that

$$d(x, p_n) \geq \frac{\hat{\varepsilon}}{2} \geq \min\left\{\frac{\hat{\varepsilon}}{2}, \delta_1, \cdots, \delta_{N-1}\right\} = \varepsilon.$$

□

Part B

(3) Suppose that $\{a_1, a_2, a_3, \cdots\}$ is a sequence in $\mathbb{R}$ with limit L, and construct a new sequence $\{b_n\}_{n\in\mathbb{N}}$ so that $b_n = \frac{1}{n}(a_1 + a_2 + \cdots + a_n)$. Prove that $\lim_{n\to\infty}\{b_n\} = L$.

Solution. Let $\varepsilon > 0$ be given, and let $N \in \mathbb{N}$ be chosen so that for all $m \geq N$ we have $|L - a_m| < \frac{\varepsilon}{2}$. Define $K = \sum_{i=1}^{N-1} |L - a_i|$, and use the archimedean property to choose $\hat{N}$ so that $\frac{1}{\hat{N}} < \frac{\varepsilon}{2K}$. We now claim that for all $n \geq \max\{N, \hat{N}\}$ we get $|b_n - L| < \varepsilon$.

To see this, first observe that the triangle inequality gives us

$$|b_n - L| = \left|\frac{a_1 + \cdots + a_n}{n} - L\right| = \left|\frac{(a_1 - L) + \cdots + (a_n - L)}{n}\right| \leq \left|\frac{a_1 - L}{n}\right| + \cdots + \left|\frac{a_n - L}{n}\right|.$$

Now we have

$$\left|\frac{a_1 - L}{n}\right| + \cdots + \left|\frac{a_{N-1} - L}{n}\right| = \frac{1}{n}\left(|a_1 - L| + \cdots + |a_{N-1} - L|\right) = \frac{K}{n} \leq \frac{K}{\hat{N}} < \frac{\varepsilon}{2},$$

with the second-to-last inequality following since $n \geq \max\{N, \hat{N}\} \geq \hat{N}$, and the last inequality following from our choice of $\hat{N}$ so that $\frac{1}{\hat{N}} < \frac{\varepsilon}{2K}$. On the other hand, since each $|a_k - L| < \frac{\varepsilon}{2}$ for all $k \geq N$, and we have $n - N$ terms from N to n, we get

$$\left|\frac{a_N - L}{n}\right| + \cdots + \left|\frac{a_n - L}{n}\right| < (n - N)\frac{\varepsilon}{2n} < \frac{\varepsilon}{2}.$$

Putting these two results together, then, we find that for all $n \geq \max\{N, \hat{N}\}$ we have

$$|b_n - L| \leq \left|\frac{a_1 - L}{n}\right| + \cdots + \left|\frac{a_N - L}{n}\right| + \left|\frac{a_{N+1} - L}{n}\right| + \cdots + \left|\frac{a_n - L}{n}\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

(4) A permutation of $\mathbb{N}$ is a bijection $\sigma : \mathbb{N} \to \mathbb{N}$. For a sequence $\{p_1, p_2, \cdots\}$ in a metric space (E, d), we say that $\{q_1, q_2, \cdots\}$ is a rearrangement of $\{p_1, p_2, \cdots\}$ if there exists some bijection $\sigma : \mathbb{N} \to \mathbb{N}$ so that for all $i \in \mathbb{N}$ we have $q_{\sigma(i)} = p_i$. As an example, if we let σ be the permutation given by

$$\sigma(n) = \begin{cases} n + 1 & \text{if } n \text{ is odd} \\ n - 1 & \text{if } n \text{ if even} \end{cases}$$

then the corresponding rearrangement of $\{p_1, p_2, \cdots\}$ is the sequence $\{p_2, p_1, p_4, p_3, p_6, p_5, \cdots,\}$.¡p¿Suppose that $\{p_1, p_2, \cdots\}$ is a sequence and there exists some $L \in E$ so that $\lim_{n\to\infty}\{p_n\} = L$. Prove that if $\{q_1, q_2, \cdots\}$ is a rearrangement of $\{p_1, p_2, \cdots\}$, then $\lim_{n\to\infty}\{q_n\} = L$ as well.

Solution. Let $\varepsilon > 0$ be given. Since $\{p_n\}_{n\in\mathbb{N}}$ converges to L, we know there is some $N \in \mathbb{N}$ so that for all $n \geq N$ we get $d(p_n, L) < \varepsilon$. Let $\hat{N} = \max\{\sigma(1), \cdots, \sigma(N)\} + 1$, and suppose that $m > \hat{N}$. Since σ is a surjection, there exists some $\ell \in \mathbb{N}$ so that $m = \sigma(\ell)$. Note furthermore that since σ is an injection and $m \neq \sigma(i)$ for any $1 \leq i \leq N$, we must have $\ell \notin \{1, \cdots, N\}$. Hence $\ell > N$, and so $d(p_\ell, L) < \varepsilon$. Hence we get

$$d(q_m, L) = d(q_{\sigma(\ell)}, L) = d(p_\ell, L) < \varepsilon,$$

as desired. □