Sizing the bets in a focused portfolio

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1 Summary

This paper consist of two parts, one without the mathematics and one with the mathematics. For a non–mathematical reader, it is best to skip the two sections related to mathematics and numerics, namely section 4 and section 5. The reader who is interested in mathematics should still read the preceding sections since they provide context within which the mathematics is built on.

2 Motivation

Before going on to explain the motivation about this work, a bit of context might come useful. In silico is an employee–owned engineering consultancy company that invests its excess cash into publicly traded stocks. The excess cash comes sporadically, and In silico has been a net buyer over the last four years: A trend that will likely continue. When the excess cash comes in, the following question arises: How much money to put in which stocks? That question had been answered by looking at our fundamental analysis for each company, and performing some vague hand calculations. That of course works well because of all the other uncertainties in the investment process, but we were still motivated to:

1. Improve the odds of us behaving rationally (i.e. minimize psychological misjudgments such as anchoring bias [1], consistency and commitment tendency [2], and others with their combined effects [3]),

2. Save time (i.e. avoid doing hand calculations).

Since mathematics is usually good at keeping people rational, and software is great for saving time, a decision has been made to write a software that answers the following question: For a set of candidate companies and their current market capitalization, each having a set of scenarios defined by a probability and intrinsic value estimate, how much of our capital to invest in each?

3 Introduction

The answer to this question has been given by Kelly with his widely known Kelly formula (sometimes called Kelly criterion) [4]. The Kelly's approach starts by maximizing long–term growth of capital when one is presented with an infinite amount of opportunities to bet on. This work extends that idea by considering:

1. Multiple companies (stocks) in parallel,
2. An arbitrary number of scenarios for each company,
3. A fundamentals–based analysis (after all, a stock is an ownership share of business [5]).

The generalization leads to a non–linear system of equations that when solved, yields a fraction of capital to invest in each of the candidate companies. The mathematical derivation is presented in section 4, while in the next two subsections (subsection 3.1 and subsection 3.2), a discussion on assumptions in a non–mathematical way, and the margin of safety, are presented.

3.1 Disclaimers and Assumptions

The underlying philosophy is that one should spend the majority of his time analyzing investments and thinking about intrinsic values under different scenarios that might play out, independently of outside thoughts and events. However, calculating intrinsic value of a company is more of an art than a science, especially for a high-quality, growing businesses within one's circle of competence. And according to Charlie Munger, Warren Buffett, Mohnish Pabrai and the like-minded others from whom the author got the inspiration for this work, one should focus precisely on getting such great businesses for a fair price. That means that one shouldn't take what this approach says at face value, and one should probably use its guidance infrequently.

During the mathematical derivation presented in section 4, an assumption is made that the number of bets is very high (tends to infinity). This work does not try to justify this assumption in a strong mathematical form [1]. Here's a soft, non-mathematical reasoning on why the author thinks this is fine: The assumption is made in order to write the equations in terms of probabilities instead of the number of outcomes divided by the number of bets. Therefore, as long as the input probabilities are *conservatively* estimated, the framework should still be valid. This essentially represents the most important margin of safety [5].

3.2 Margin of Safety

In addition to the most important margin of safety mentioned above, there are however a couple of more margins of safety embedded in the current framework. These are:

[1] It is on the author's TODO list to try and prove this.

- No shorting allowed. From a purely mathematical point of view, shorting would be allowed. Without detailed math, it's easy to see how a company with a negative expected return would result in a short position. However, due to the asymmetry of the potential losses compared to gains, coupled with the usual time–frame limit that comes with shorting, the author decided to avoid it.

- No use of leverage allowed. Again, from a purely mathematical point of view, use of leverage would sometimes be useful. The thinking in avoiding to use leverage is that no–one should be in a hurry to get rich, and should avoid risking good night's sleep based on short–term market fluctuations, which are fairly hard to predict consistently (unless you work at Renaissance Technologies, in which case you can scratch this).

- No company without at least one downside scenario is allowed. By disallowing inputs without downside, the framework forces you to focus and think about what can go wrong, as opposed to dreaming about what can go right. However, if one is absolutely (100%) sure that a company does not have a downside, then the solution is to lever up as much as possible, and put all the money in that one company. It is our feeling that the best way to handle such cases is outside of this framework. Alternatively, one can always model an unknown downside scenario with a small probability (say 5–20%) and intrinsic value of zero.

These assumptions and margins of safety are embedded into the framework in order to try and tie the rational mathematics with common–sense from other disciplines.

4 Mathematics

The following derivation mostly follows the first part of the work by Byrnes and Barnett [6]. The problem statement is repeated here for convenience: Given a set of candidate companies, each having a set of scenarios described by the probability p and estimate of the intrinsic value $\mathcal{V}$, calculate the optimal allocation fraction f for each candidate company by maximizing the long–term growth rate of assets. After a single outcome (realization), the change in value of assets can be written as follows:

$$\mathcal{A}_{after} = \mathcal{A}_{before} \left(1 + \sum_{j}^{N_c} f_j k_j \right) \tag{1}$$

where $\mathcal{A}$ is the value of assets (capital), N_c is the number of candidate companies to consider, f_j is the allocated fraction to jth company, and k_j is a return for a company j defined as the relative difference between the estimated intrinsic value under a given scenario ($\mathcal{V}$) and the market capitalization at the time of investing ($\mathcal{M}$):

$$k_j = \frac{\mathcal{V}_j - \mathcal{M}_j}{\mathcal{M}_j} \tag{2}$$

If a significant number of (re)allocations (N_a) is performed in succession, the equation (1) can be written as follows:

$$\mathcal{A}_{N_a} = \mathcal{A}_0 \prod_{i_1, i_2, \ldots, i_{N_o}} \left(1 + \sum_j^{N_c} f_j k_{ij}\right)^{n_i} \tag{3}$$

where $n_i \in n_{i1}, n_{i2}, \ldots, n_{N_o}$ is the number of times the ith outcome has occurred. Note that k_{ij} represents the return of the jth company for the ith outcome. Following original Kelly's approach, a logarithmic growth function $\mathcal{G}$ is introduced:

$$\mathcal{G} = \lim_{N_a \to \infty} \frac{1}{N_a} \ln \frac{\mathcal{A}_{N_a}}{\mathcal{A}_0} \tag{4}$$

and the goal is to find its maximum with respect to allocation fractions f_j:

$$\frac{\partial \mathcal{G}}{\partial f_j} = 0 \tag{5}$$

Substituting equation (1) into equation (5) results in the following equation, after some calculus and algebra:

$$\lim_{N_a \to \infty} \frac{1}{N_a} \sum_i^{N_o} \frac{n_i k_{ij}}{1 + \sum_j^{N_c} f_j k_{ij}} = 0 \tag{6}$$

If one assumes an infinite number of (re)allocations ${N_a}^2$, the following relation holds:

$$\lim_{N_a \to \infty} \frac{n_i}{N} = p_i \tag{7}$$

Where p_i is the probability of the ith outcome. For example, if there are two companies, each with two 50–50 scenarios, there will be four outcomes in total with the probability of each outcome equal to 25%. Finally, substituting equation (7) into (6) results in a system of equations written in terms of probabilities p_i, expected returns for each company in each of the outcomes k_{ij}, and allocation fractions for each company f_j:

$$\sum_i^{N_o} \frac{p_i k_{ij}}{1 + \sum_j^{N_c} f_j k_{ij}} = 0 \tag{8}$$

The equation (8) represents a non–linear system of equations in the unknown fractions f_j, which when solved, should yield optimal allocation strategy for maximizing long–term growth of capital.

[2] The assumption regarding the infinite number of allocations is something that the author is slightly uncomfortable with, but *feels* it is fine because of the margins of safety embedded into the thinking that goes into assessing each investment opportunity.

5 Numerics

The equation (8) can be written succinctly as:

$$\mathcal{F}_i(f_j) = 0 \tag{9}$$

Because $\mathcal{F}_i$ is a non–linear equation in f_j, the Newton–Raphson method is used to find a numerical solution. The method is iterative and starts by linearizing the equation around the previous solution from the previous iteration:

$$\mathcal{F}_i^o + \sum_j^{N_c} \mathcal{J}_{ij}^o (f_j^n - f_j^o) = 0 \tag{10}$$

where $\mathcal{J}_{ij}$ is the Jacobian of $\mathcal{F}_\rangle$, and superscripts ^{n} and ^{o} denote the new value and the old value, respectively. The Jacobian is:

$$\mathcal{J}_{ij} = -\sum_o^{N_o} \frac{p_o k_{oi} k_{oj}}{\left(1 + \sum_j^{N_c} f_j k_{ij}\right)^2} \tag{11}$$

where subscript for outcome i has been changed to o in order to be able to write the Jacobian in the standard ij notation.

The numerical procedure starts by assuming the uniform allocation across all companies, i.e. $f_j = f = \frac{1}{N_c}$. Based on the f_j^o in the current iteration, the linear system in equation (10) is solved to find the new solution f_j^n. The process is repeated until sufficient level of accuracy is reached [3].

A practical approach is used to prevent shorting and use of leverage. Shorting would occur if any of the fractions is negative, and is prevented by filtering out the candidate companies with negative expected return before solving the equations. The use of leverage is avoided by normalizing the fractions after obtaining a numerical solution, if the sum of all fractions exceeds 1. The author is well–aware that this is somewhat a brute–force approach, and it will be a topic of future work to avoid shorting and use of leverage by using inequality constraints.

6 Example

As an example, consider seven candidate companies, each with two to four scenarios. Each scenario is represented by an intrinsic value and the probability of the scenario happening (or intrinsic value being reached at some point in the future). Note that how these numbers are obtained is outside of the scope of this work, although it is important to stress that the validity and conservative assumptions behind these numbers are probably the most important part of an investor's job. The example inputs are presented in Table 1 to Table 6.

Table 1: Company A with current market cap of 214B USD.

Scenario	Intrinsic value	Probability
Total loss	0 USD	5%
Bear thesis	170B USD	30%
Base thesis	270B USD	50%
Bull thesis	360B USD	15%

Table 2: Company B with current market cap of 306M USD.

Scenario	Intrinsic value	Probability
Total loss	0 USD	5%
Bear thesis	300M USD	50%
Base thesis	900M USD	45%

Table 3: Company C with current market cap of 34M GBP.

Scenario	Intrinsic value	Probability
Total loss	0 GBP	10%
Bear thesis	33.5M GBP	50%
Base thesis	135M GBP	40%

Table 4: Company D with current market cap of 806M SGD.

Scenario	Intrinsic value	Probability
Bear thesis	330M SGD	40%
Base thesis	1B SGD	50%
Bull thesis	4B SGD	10%

Table 5: Company E with current market cap of 17.6M CAD.

Scenario	Intrinsic value	Probability
Total loss	0 CAD	5%
Bear thesis	10M CAD	25%
Base thesis	25M CAD	70%

Table 6: Company F with current market cap of 581M USD.

Scenario	Intrinsic value	Probability
Total loss	0 USD	20%
Base thesis	1B USD	80%

Table 7: Portfolio that maximizes long–term growth–rate of capital.

Company	A	B	C	D	E	F
Allocation fractions	13%	27%	26%	12%	12%	10%

Based on the inputs, the portfolio allocation that maximizes the long–term growth–rate of capital is presented in Table 7.

With the obtained fractions, it is easy to obtain some useful statistics on the portfolio, namely:

- Expected gain of 61 cents for every dollar invested,
- Cumulative probability of loss of capital of 13%,
- Permanent loss of 95% of capital with probability of 0.001%.

The last item is particularly interesting to the author. According to Actuarial Life Tables in [7], the probability of the (currently 33 year–old) author dying within the next year is approximately 0.25%. That is two orders of magnitude higher than the probability of the permanent loss of capital for this portfolio. Considering that the portfolio has six stocks, that is a very strong argument against excessive diversification, especially if:

- One thinks of stocks as ownership shares of businesses, which implies long–term thinking and not being bothered by market fluctuations,
- One embeds a margin of safety in different scenarios for different companies by e.g. recognizing that both unknown and known bad things may happen.

The observation about excessive diversification is inline with the thoughts from the Poor Charlie's Almanack [3].

7 Future Work

The future work will focus on adding a constraint for the maximum allowable risk of the permanent loss of capital. The idea is that an investor could input the probability of the permanent loss of capital and the fraction of capital one is willing to risk, and the allocation will change accordingly. For example, an investor might be fine with losing 50% of capital with the probability of 0.1% as a worst–case outcome, which should be an additional input to the system.

[3] Usually two significant digits is enough considering the uncertainty and the judgment required for the inputs.

References

[1] D. Kahneman. *Thinking, Fast and Slow.* Farrar, Straus and Giroux, 2011.

[2] R.B. Cialdini. *Influence: The Psychology of Persuasion.* Harper Business, 2006.

[3] P.D. Kaufman, E. Wexler, W.E. Buffett, and C.T. Munger. *Poor Charlie's Almanack: The Wit and Wisdom of Charles T. Munger.* Walsworth Publishing Company, 2005.

[4] J.L. Kelly. A New Interpretation of Information Rate. *Bell System Technical Journal*, 35(4):917–926, 1956.

[5] B. Graham. *The Intelligent Investor.* Harper Business, 2003.

[6] T. Byrnes and T Barnett. Generalized framework for applying the Kelly criterion to stock markets. *International Journal of Theoretical and Applied Finance*, 21(5), 2018.

[7] United States Social Security Administration. Actuarial Life Table. `https://www.ssa.gov/oact/STATS/table4c6.html`, 2023. [Online; accessed 15 May 2023].