Disclaimer

These slides are intended as presentation aids for the lecture. They contain information that would otherwise be to difficult or time-consuming to reproduce on the board. But they are incomplete, not self-explanatory, and are not always used in the order they appear in this presentation. As a result, these slides should not be used as a script for this course. I recommend you take notes during class, maybe on the slides themselves. It has been shown that taking notes improves learning success.

Reading for this set of slides

- Craig – Intro to Robotics (3rd Edition)
  - Chapter 5 (5.1 – 5.10)

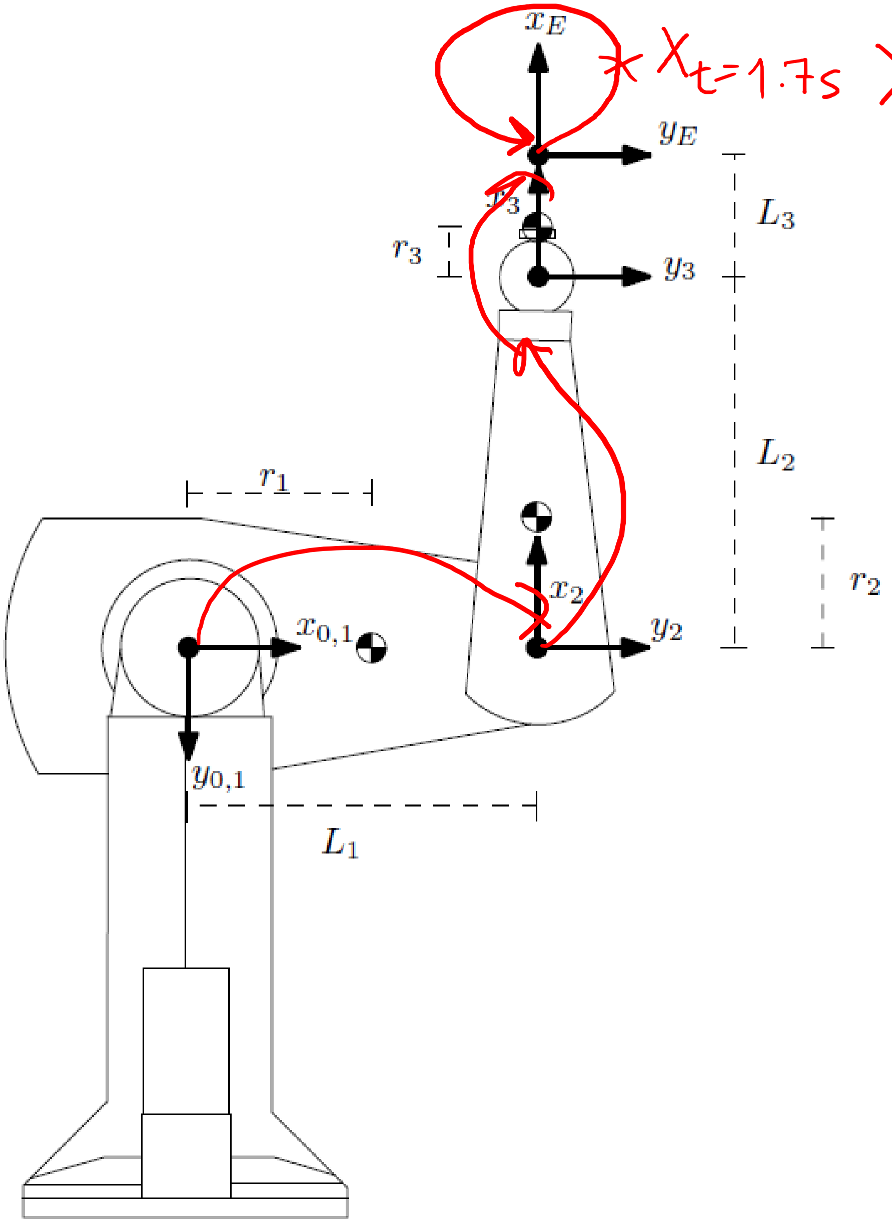
Please note that this set of slides is intended as support for the lecture, not as a stand-alone script. If you want to study for this course, please use these slides in conjunction with the indicated chapters in the text books. The textbooks are available online or in the TUB library (many copies that can be checked out for the entire semester. There are also some aspects of the lectures that will not be covered in the text books but can still be part of the homework or exam. For those It is important that you attend class or ask somebody about what was covered in class.

Robotics

The Jacobian Matrix

TU Berlin

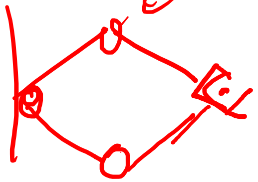
Oliver Brock



$* X_{t=1.7s}$ $X = f(q) \longrightarrow q = f^{-1}(x)$

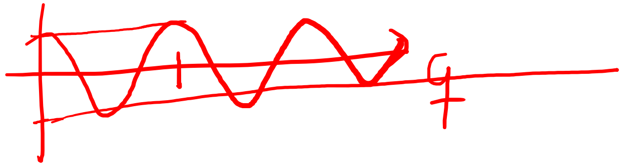
DIFFICULT

UP



(x, y) (x', y')

DOWN



$\hat{f}(x) \rightsquigarrow \{q_i\}$

COMPUTABLE

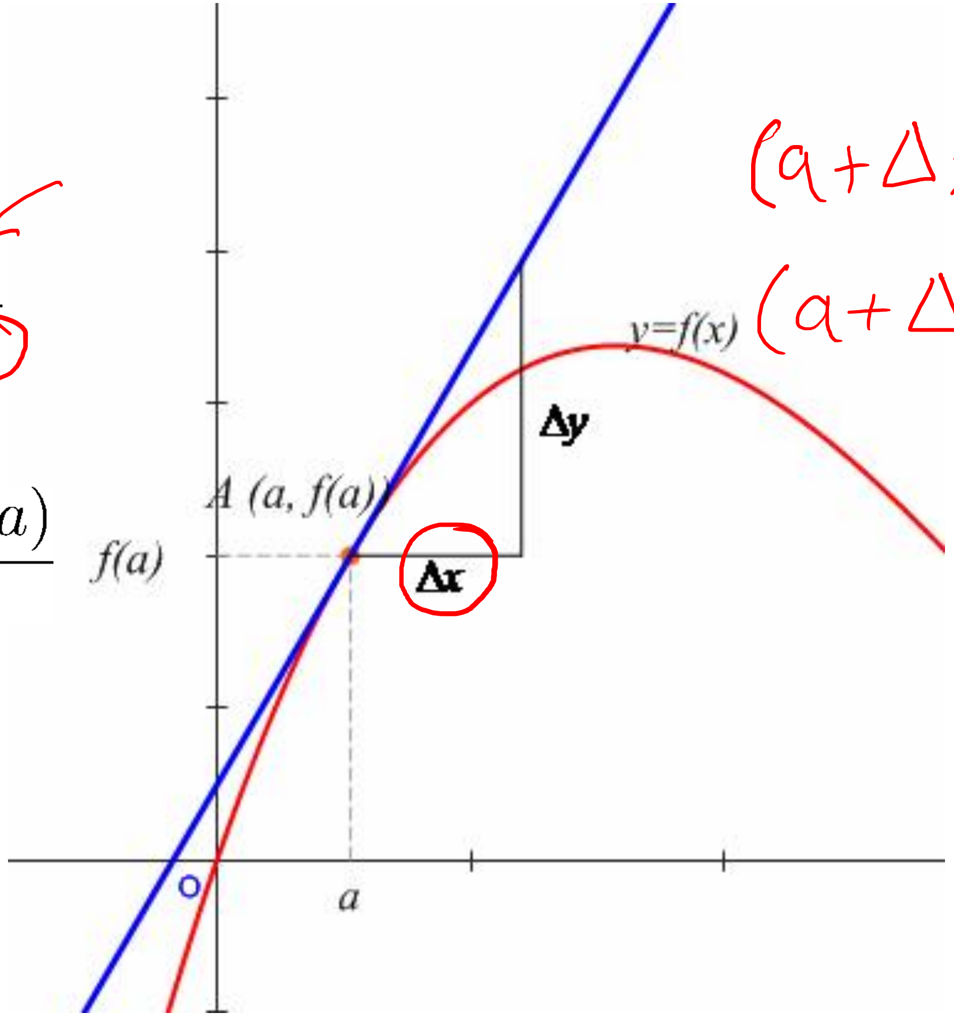
Notation

- To make equations more readable we will adopt the following:
  - Vector: $\mathbf{x}_{(n\times 1)} = (x_1, x_2, \cdots, x_n)^T$
  - Matrix: $M_{(n\times m)} = [\mathbf{x}_1\mathbf{x}_2 \ldots \mathbf{x}_m]$
  - Derivative of vectors and matrices: $\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}$
  - We'll introduce more as we go along…

Warm-Up: Derivative

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}$$

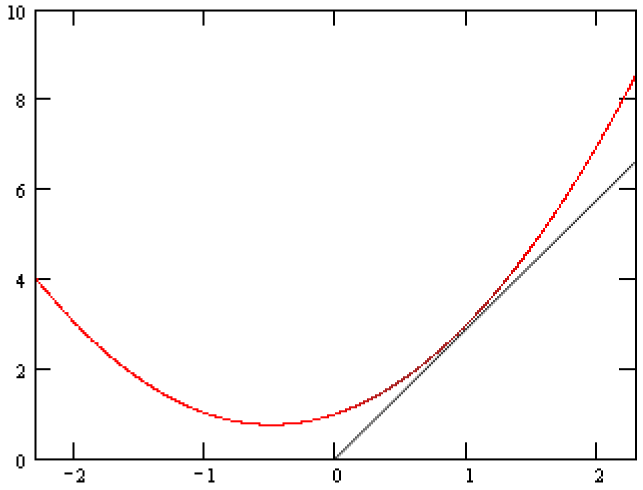
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



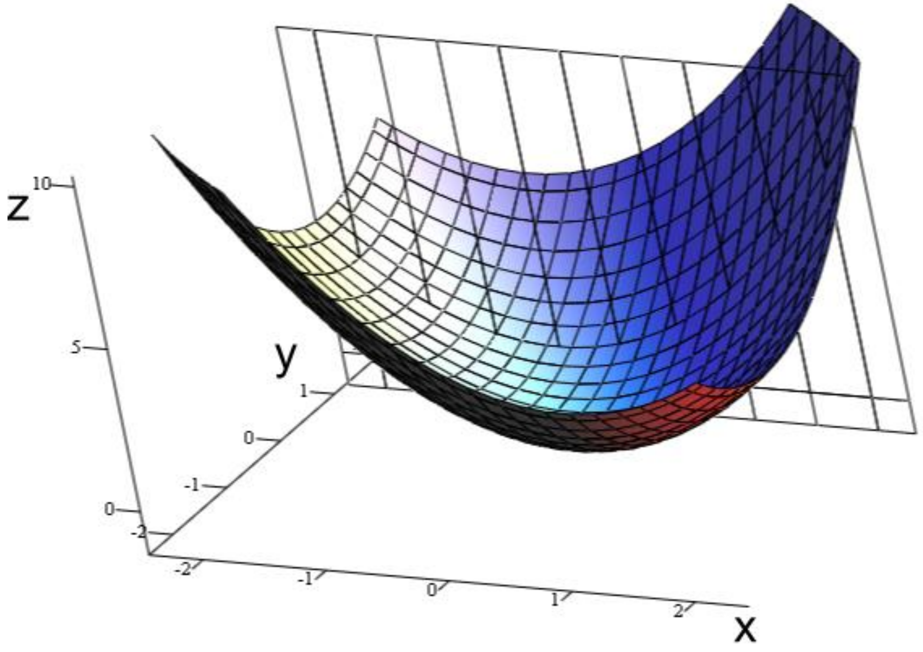
$(a+\Delta x, f(a+\Delta x)) \approx$

$(a+\Delta x, f(a) + \Delta y)$

Warm-Up: Partial Derivative



$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



$$\frac{\partial f}{\partial x_i}(a_1, \ldots, a_n) = \lim_{h \to 0} \frac{f(a_1, \ldots, a_i + h, \ldots, a_n) - f(a_1, \ldots, a_i, \ldots, a_n)}{h}$$

Infinitesimal Change and Velocities

$$\mathbf{x}_{(6\times1)} = \begin{pmatrix} \mathbf{x}_{p(3\times1)} \\ \mathbf{x}_{r(3\times1)} \end{pmatrix} \qquad \mathbf{q}_{(n\times1)} = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$$

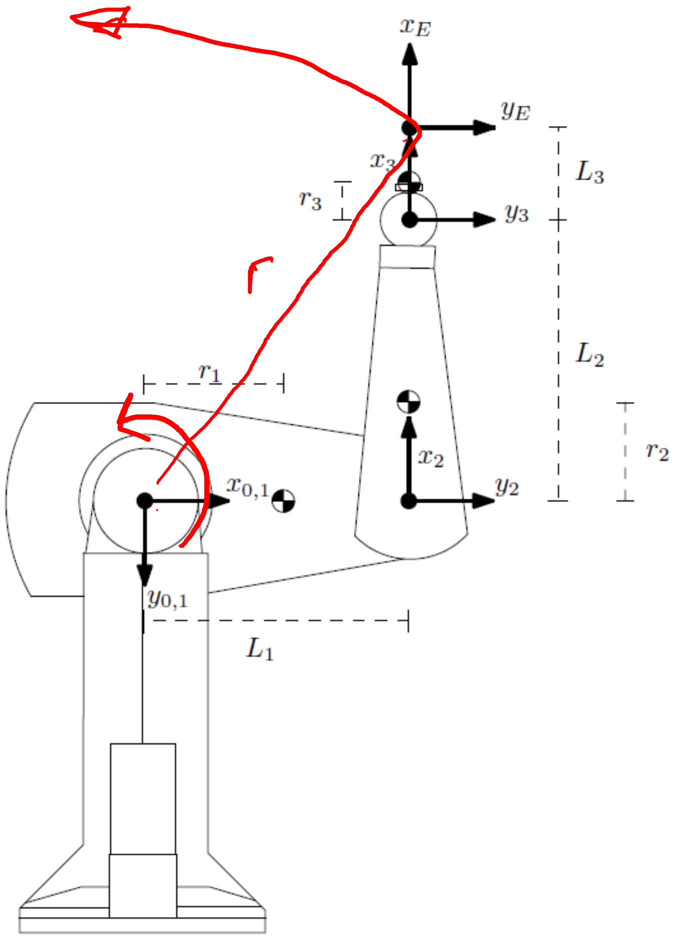
$$\begin{aligned} \mathbf{x} &= f(\mathbf{q}) \\ \mathbf{q} &= \cancel{f^{-1}}(\mathbf{x}) \end{aligned}$$

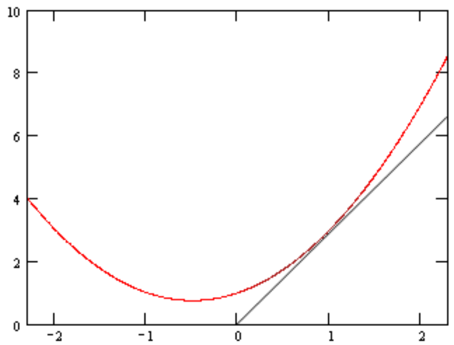
$$\delta\mathbf{x} = g(\delta\mathbf{q})$$

$$\delta\mathbf{q} = g^{-1}(\delta\mathbf{x})$$

$$g = ?$$

What is the relationship between velocities?





$$\delta(x, y, \theta) \rightsquigarrow \delta q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$q_1 \overset{?}{\rightsquigarrow} x, y, \theta$$

$$q_2 \overset{?}{\rightsquigarrow} x, \cancel{y}, \theta$$

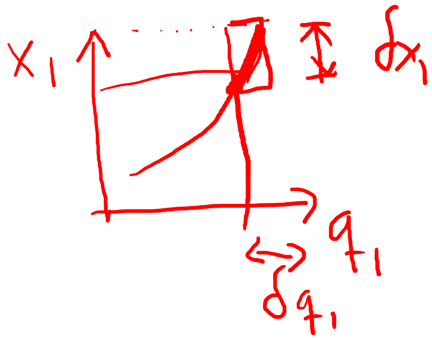
$$q_3 \overset{?}{\rightsquigarrow} x, \cancel{y}, \theta$$

Rewriting Forward Kinematics

$$\mathbf{x} = f(\mathbf{q}) = {}^0_nT(\mathbf{q}) = {}^0_1T(q_1)\ {}^1_2T(q_2)\ {}^2_3T(q_3)\ \cdots\ {}^{n-1}_{\ \ n}T(q_n)$$

$$\begin{aligned}
x_1 &= f_1((q_1, q_2, \cdots, q_n)^T) \\
x_2 &= f_2((q_1, q_2, \cdots, q_n)^T) \\
x_3 &= f_3((q_1, q_2, \cdots, q_n)^T) \\
&\vdots \\
x_m = 6 &= f_m((q_1, q_2, \cdots, q_n)^T)
\end{aligned}$$

$$x_i = f_i((q_1, q_2, \cdots, q_n)^T)$$



How does the end-effector move?

The Jacobian Matrix

$$
\begin{aligned}
\delta x_1 &= \frac{\partial f_1}{\partial q_1}\delta q_1 + \frac{\partial f_1}{\partial q_2}\delta q_2 + \cdots + \frac{\partial f_1}{\partial q_n}\delta q_n \\
\delta x_2 &= \frac{\partial f_2}{\partial q_1}\delta q_1 + \frac{\partial f_2}{\partial q_2}\delta q_2 + \cdots + \frac{\partial f_2}{\partial q_n}\delta q_n \\
&\vdots \\
\delta x_m &= \frac{\partial f_m}{\partial q_1}\delta q_1 + \frac{\partial f_m}{\partial q_2}\delta q_2 + \cdots + \frac{\partial f_m}{\partial q_n}\delta q_n
\end{aligned}
$$

$$
J = \frac{\partial f(\mathbf{q})}{\partial \mathbf{q}}
$$

$$
\delta \mathbf{x} = \begin{bmatrix}
\frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} & \cdots & \frac{\partial f_1}{\partial q_n} \\
\frac{\partial f_2}{\partial q_1} & \frac{\partial f_2}{\partial q_2} & \cdots & \frac{\partial f_2}{\partial q_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_m}{\partial q_1} & \frac{\partial f_m}{\partial q_2} & \cdots & \frac{\partial f_m}{\partial q_n}
\end{bmatrix} \delta \mathbf{q} = J_{(m \times n)}(\mathbf{q}) \delta \mathbf{q}
$$

Jacobian and Velocities

$$\lim_{\delta t \to 0} \left(\frac{\delta \mathbf{x}}{\delta t} = J(\mathbf{q}) \, \frac{\delta \mathbf{q}}{\delta t} \right)$$

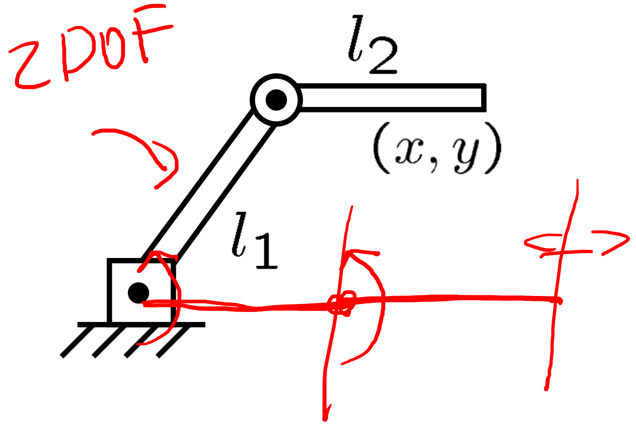
$$\dot{\mathrm{x}} = J(\mathrm{q}) \, \dot{\mathrm{q}}$$

$$\delta \mathbf{x}_{(m \times 1)} = J_{(m \times n)}(\mathbf{q}) \; \delta \mathbf{q}_{(n \times 1)}$$

$$\dot{\mathbf{x}}_{(m \times 1)} = J_{(m \times n)}(\mathbf{q}) \; \dot{\mathbf{q}}_{(n \times 1)}$$

Reminder: $\sin'(\theta) = \cos(\theta)$ $\cos'(\theta) = -\sin(\theta)$ $\mathbf{x} = f(\mathbf{q})$ $\mathbf{x} = (x, y)^T$ $\mathbf{q} = (\theta_1, \theta_2)^T$

Jacobian: Example



$$
\begin{aligned}
x &= f_1(\mathbf{q}) = l_1 c_1 + l_2 c_{12} \\
y &= f_2(\mathbf{q}) = l_1 s_1 + l_2 s_{12}
\end{aligned}
$$

$$
J_{(2\times 2)}(\mathbf{q}) = \begin{bmatrix} -(l_1 s_1 + l_2 s_{12}) & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}
$$

$$
\begin{aligned}
\frac{\partial f_1}{\partial \theta_1} &= -(l_1 s_1 + l_2 s_{12}) \\
\frac{\partial f_1}{\partial \theta_2} &= -l_2 s_{12} \\
\frac{\partial f_2}{\partial \theta_1} &= l_1 c_1 + l_2 c_{12} \\
\frac{\partial f_2}{\partial \theta_2} &= l_2 c_{12}
\end{aligned}
$$

$$
\begin{aligned}
\delta\mathbf{x} &= J\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) \begin{pmatrix} \delta\theta_1 \\ \delta\theta_2 \end{pmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ l_1 + l_2 & l_2 \end{bmatrix} \begin{pmatrix} \delta\theta_1 \\ \delta\theta_2 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ (l_1 + l_2)\delta\theta_1 + l_2\delta\theta_2 \end{pmatrix}
\end{aligned}
$$

1DOF

What the Jacobian can do…

$$\dot{\mathbf{x}} = J(\mathbf{q})\,\dot{\mathbf{q}}$$

$$\delta\mathbf{x} = J(\mathbf{q})\,\delta\mathbf{q}$$

$$\boxed{\delta\mathbf{q} = J(\mathbf{q})^{-1}\,\delta\mathbf{x}}$$