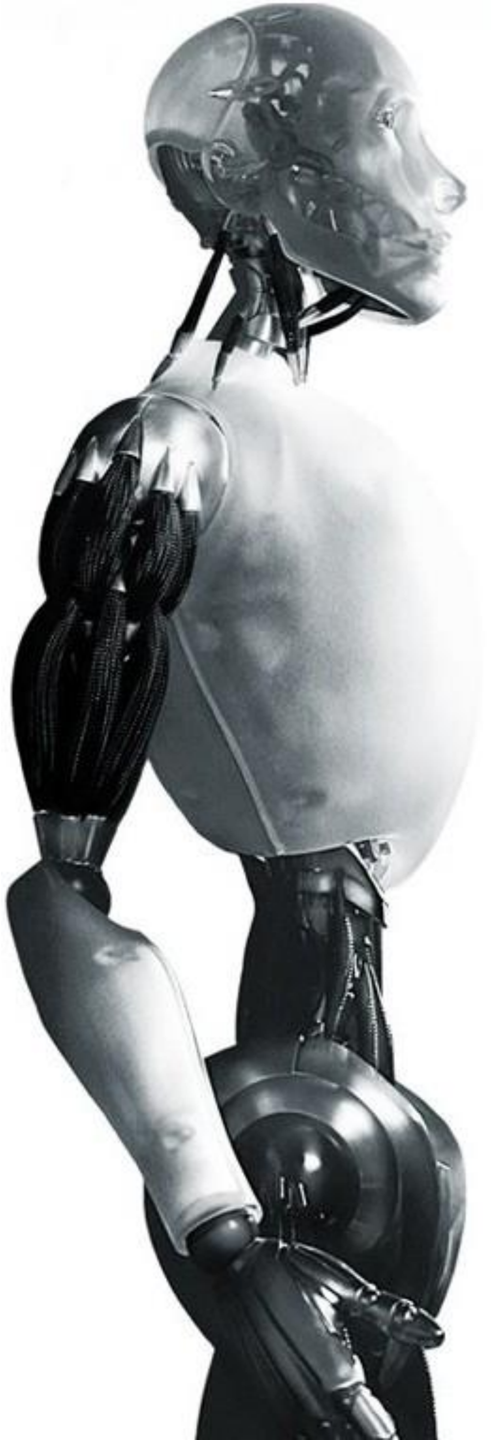
Disclaimer

These slides are intended as presentation aids for the lecture. They contain information that would otherwise be to difficult or time-consuming to reproduce on the board. But they are incomplete, not self-explanatory, and are not always used in the order they appear in this presentation. As a result, these slides should not be used as a script for this course. I recommend you take notes during class, maybe on the slides themselves. It has been shown that taking notes improves learning success.

Reading for this set of slides

- Probabilistic Robotics
  - Chapters 1-4, 7, 8-10 (please match the level of detail from the lectures, not all the material in these chapters is required)

Please note that this set of slides is intended as support for the lecture, not as a stand-alone script. If you want to study for this course, please use these slides in conjunction with the indicated chapters in the text books. The textbooks are available online or in the TUB library (many copies that can be checked out for the entire semester. There are also some aspects of the lectures that will not be covered in the text books but can still be part of the homework or exam. For those It is important that you attend class or ask somebody about what was covered in class.

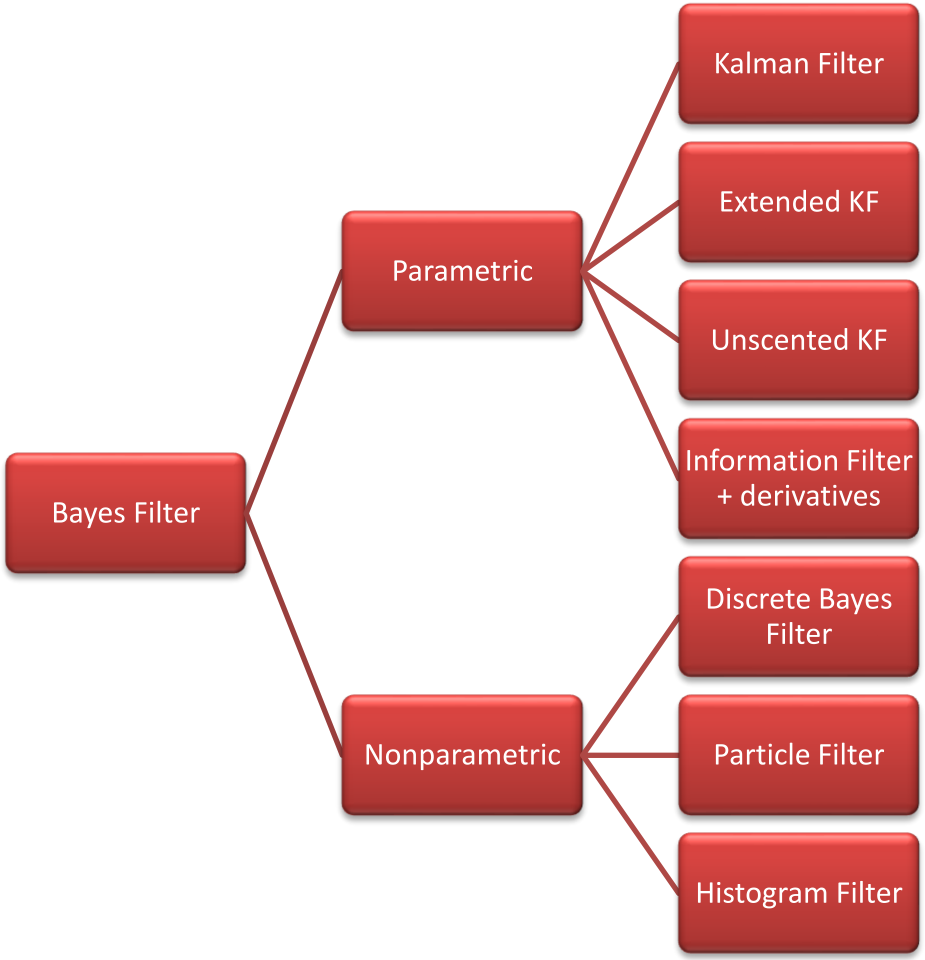


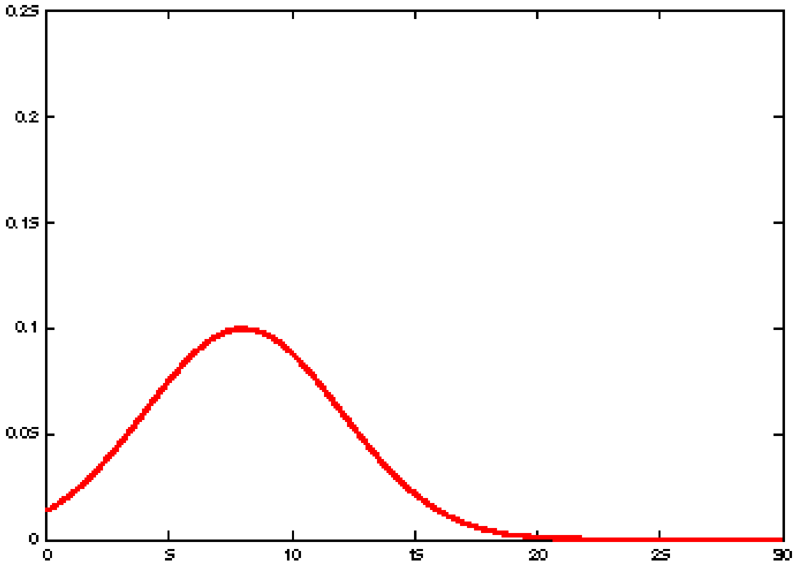
Robotics

Recursive State Estimation
Bayes Filters

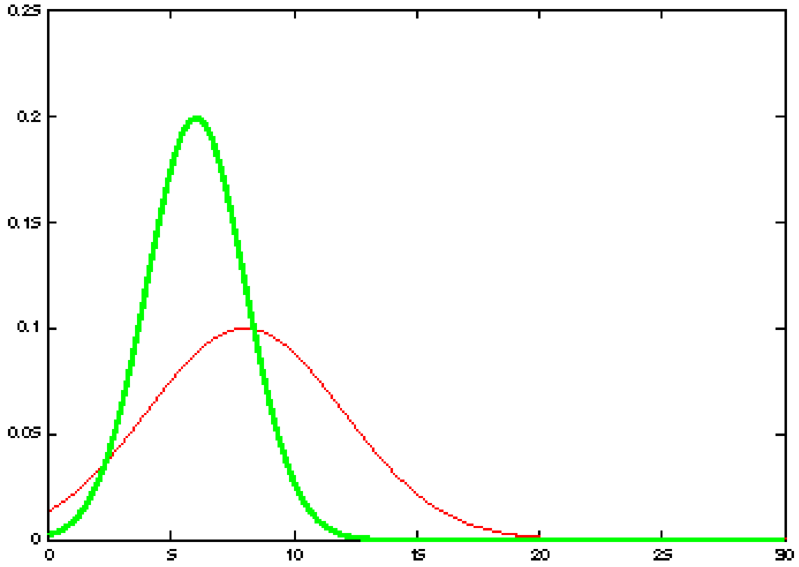
TU Berlin
Oliver Brock

The Family of Bayes Filters

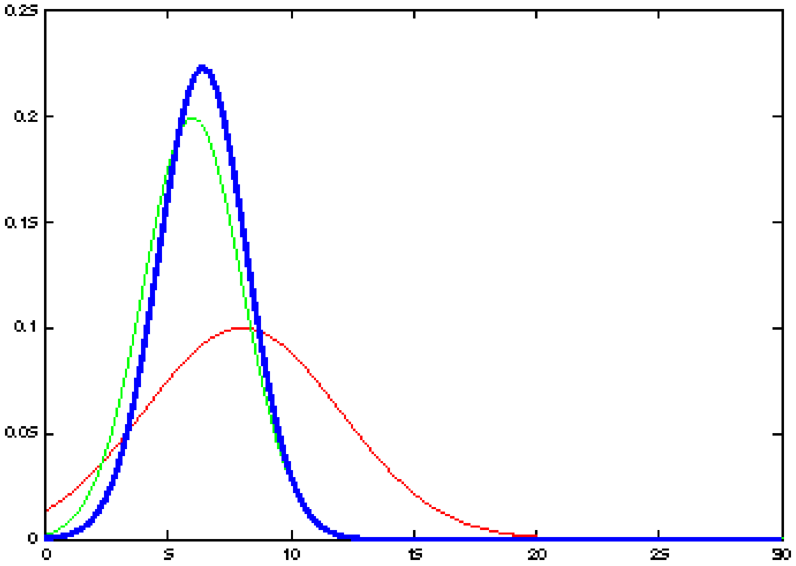




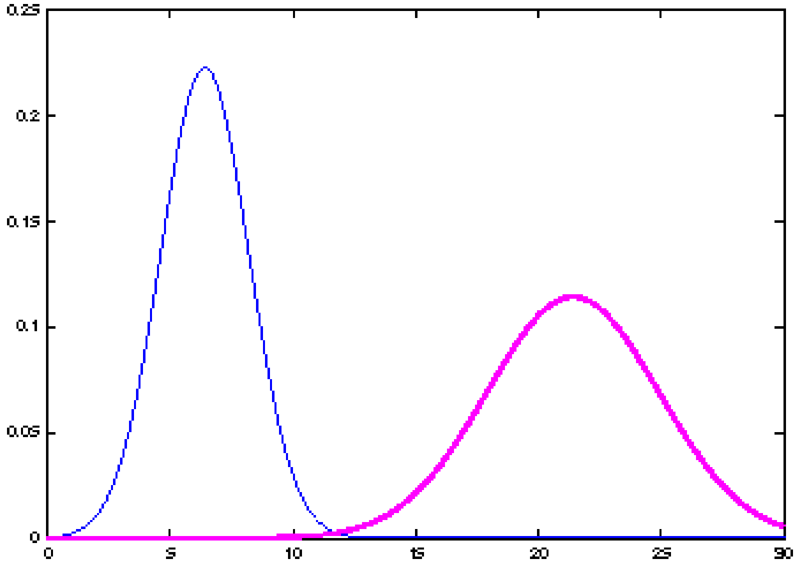
Initial estimate



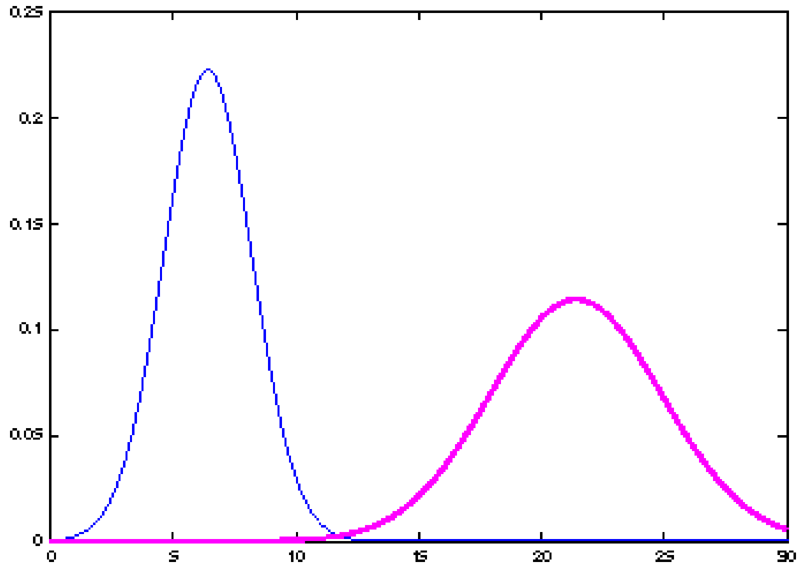
Sensing



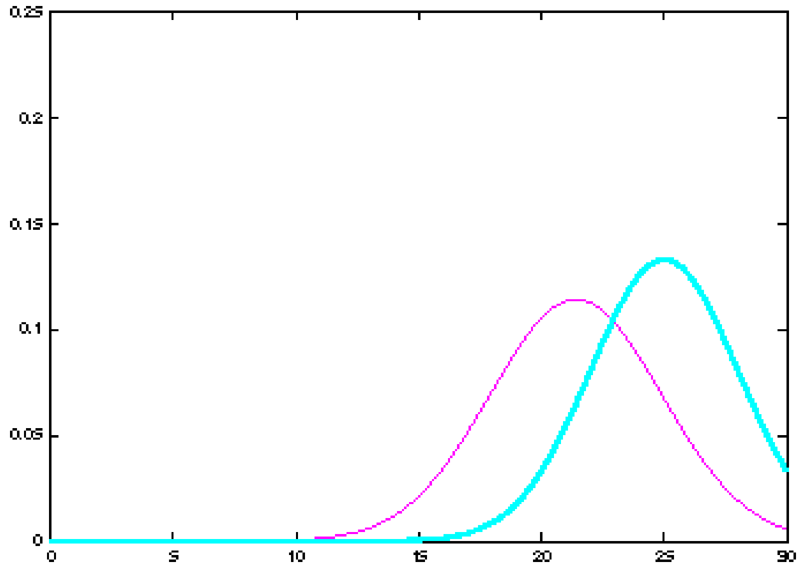
Correction



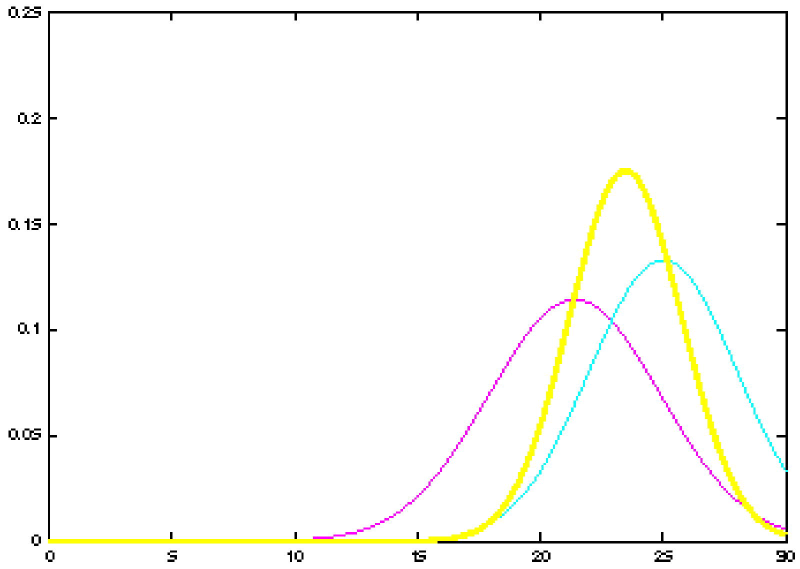
Prediction



Prediction



Sensing



Correction

Bayes Filter

Bayes Filter $(b(s_{t-1}), a_{t-1}, o_t)$

for all s_t do

prediction $\boxed{b'(s_t) = \int p(s_t \mid a_{t-1}, s_{t-1}) b(s_{t-1}) ds_{t-1}}$

measurement update $\boxed{b(s_t) = \mu \, p(o_t \mid s_t) \cdot b'(s_t)}$

end for

return $b(s_t)$

Gaussian Bayesian Filters

- Kalman Filter
  - linear update of states based on action

$$p(s_t|a_{t-1}, s_{t-1})$$

  - linear sensor model

$$p(o_t|s_t)$$

  - belief can be described by a normal distribution
  - computationally efficient and elegant

Nonparametric: Discrete Bayes

Discrete Bayes Filter $(\{p_{k,t-1}\}, a_{t-1}, o_t)$

for all k do

$$p'_{k,t} = \sum_i p(X_t = x_k \,|\, a_{t-1}, X_{t-1} = x_i) p_{i,t-1}$$

$$p_{k,t} = \mu \, p(o_t \,|\, X_t = x_k) \cdot p'_{k_t-1}$$

end for

return $\{p_{k,t}\}$

Kalman Filter

- [Swerling 1958] [Kalman 1960]
- Assumptions:
  - State transition probability $p(x_t|u_t,x_{t-1})$ is a linear function with Gaussian noise

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t$$

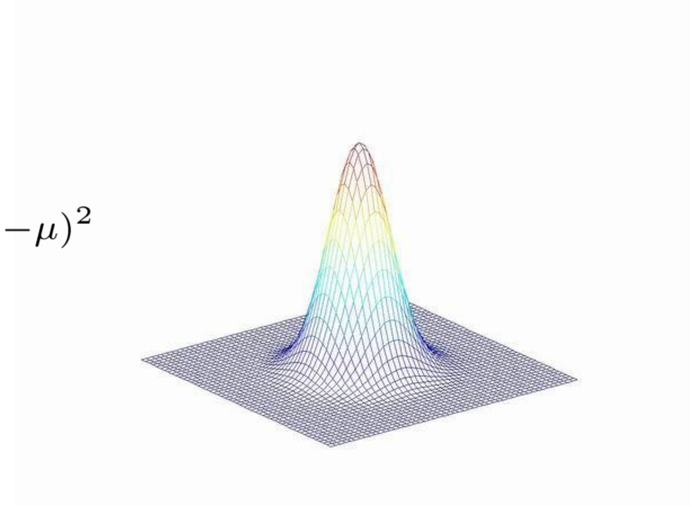
  - Measurement probability $p(z_t|x_t)$ is linear with Gaussian noise

$$z_t = C_t x_t + \delta_t$$

  - Initial belief is normally distributed

Multivariate Gaussian

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$



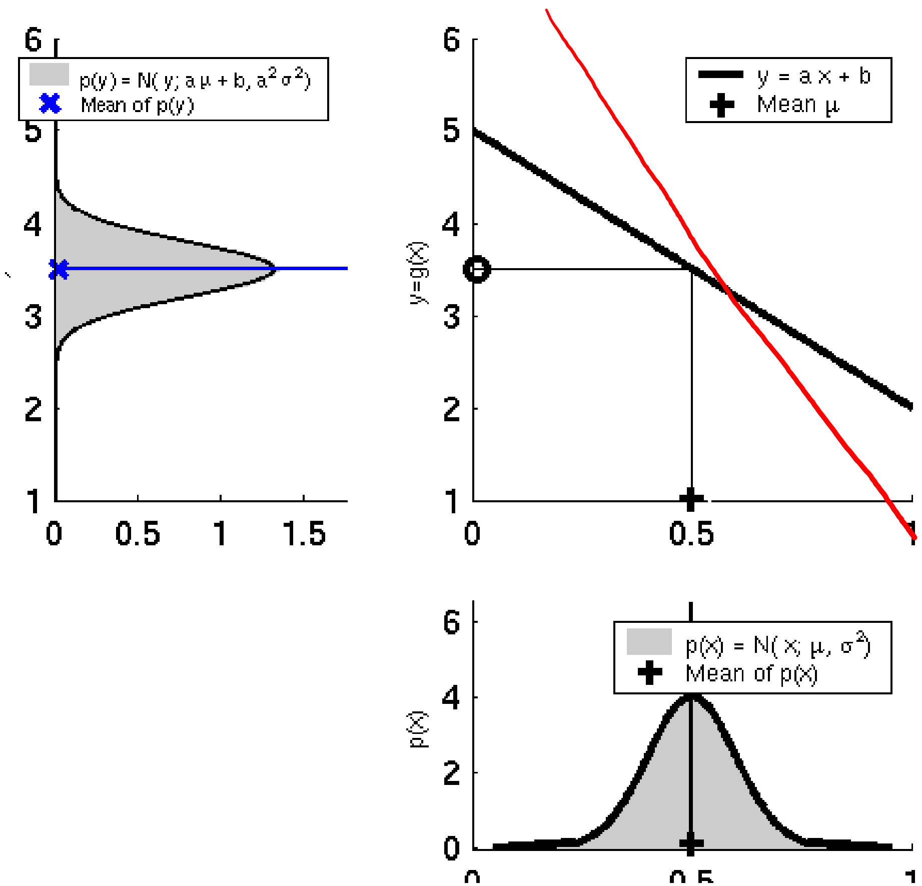
$$p(x) = \frac{1}{(2\pi)^{n/2}\det(\Sigma)^{1/2}}\ e^{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)}$$

Covariance Matrix Σ

$$\Sigma_{ij} = \mathrm{cov}(X_i, X_j) = \mathrm{E}\big[(X_i - \mu_i)(X_j - \mu_j)\big]$$

$$\Sigma = \begin{bmatrix} \mathrm{E}[(X_1-\mu_1)(X_1-\mu_1)] & \mathrm{E}[(X_1-\mu_1)(X_2-\mu_2)] & \cdots & \mathrm{E}[(X_1-\mu_1)(X_n-\mu_n)] \\ \mathrm{E}[(X_2-\mu_2)(X_1-\mu_1)] & \mathrm{E}[(X_2-\mu_2)(X_2-\mu_2)] & \cdots & \mathrm{E}[(X_2-\mu_2)(X_n-\mu_n)] \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{E}[(X_n-\mu_n)(X_1-\mu_1)] & \mathrm{E}[(X_n-\mu_n)(X_2-\mu_2)] & \cdots & \mathrm{E}[(X_n-\mu_n)(X_n-\mu_n)] \end{bmatrix}.$$

Linear Transformation of a Gaussian



Falling Body Example (state equation)

$$y_t = y_0 + \dot{y}_0 t - \frac{g}{2} t^2$$

$$y_t = y_{t-1} + \dot{y}_{t-1} \Delta t - \frac{g}{2} \Delta t^2$$

$$x_t = \begin{bmatrix} y_t \\ \dot{y}_t \end{bmatrix}$$

$$\begin{aligned} x_t &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_{t-1} + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} (-g) \\ &= Ax + Bu + \varepsilon_t \end{aligned}$$

Falling Body Example (measurement)

$$
\begin{aligned}
z_t &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_t + \delta_t \\
&= C_t x_t + \delta_t
\end{aligned}
$$

Components of a Kalman Filter

A_t Matrix (nxn) that describes how the state evolves from t to $t\text{-}1$ without controls or noise.

B_t Matrix (nxl) that describes how the control u_t changes the state from t to $t\text{-}1$.

C_t Matrix (kxn) that describes how to map the state x_t to an observation z_t.

ε_t

δ_t

Random variables representing the process and measurement noise that are assumed to be independent and normally distributed with covariance R_t and Q_t , respectively.

Kalman Filter (KF)

Kalman Filter $(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$

$$\bar{\mu}_t = A_t \, \mu_{t-1} + B_t \, u_t$$

$$\bar{\Sigma}_t = A_t \, \Sigma_{t-1} \, A_t^T + R_t$$

Prediction

$$K_t = \bar{\Sigma}_t \, C_t^T \, (C_t \, \bar{\Sigma}_t \, C_t^T + Q_t)^{-1}$$

Kalman gain

$$\mu_t = \bar{\mu}_t + K_t \, (z_t - C_t \, \bar{\mu}_t)$$

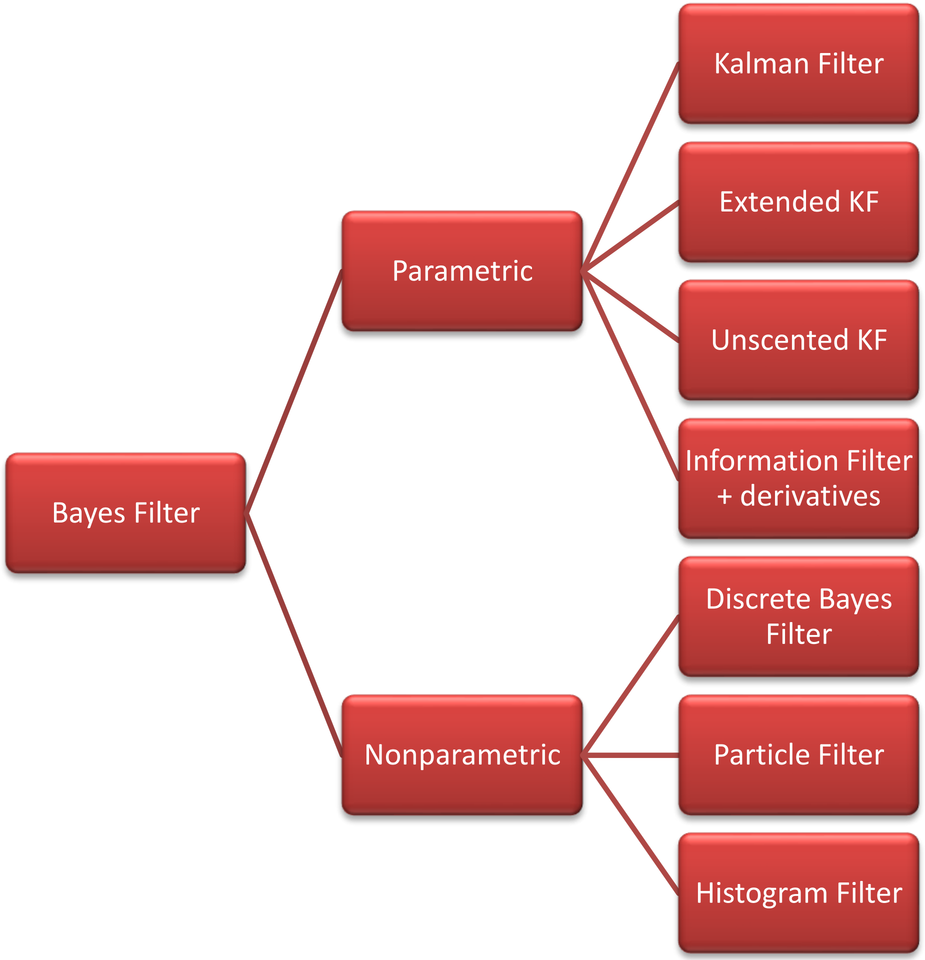
Innovation

$$\Sigma_t = (I - K_t \, C_t) \, \bar{\Sigma}_t$$

Correction

return μ_t, Σ_t

The Family of Bayes Filters



Extended Kalman Filter

- Assumptions relaxed:
  - State transition probability $p(x_t|u_t,x_{t-1})$ is a **nonlinear** function with Gaussian noise

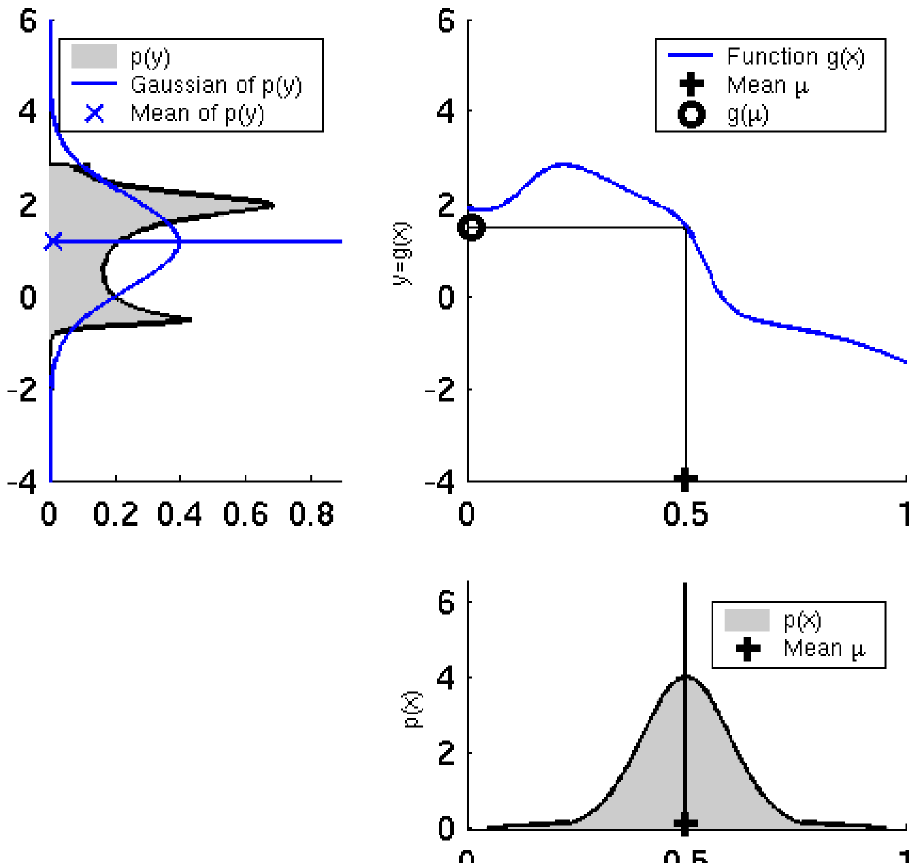
    $$x_t = g(u_t, x_{t-1}) + \varepsilon_t$$

  - Measurement probability $p(z_t|x_t)$ is a **nonlinear** with Gaussian noise

    $$z_t = h(x_t) + \delta_t$$

  - Initial belief is normally distributed

Effect of Nonlinearities in the Motion Model



Linearization with Taylor Expansion

$$\begin{aligned} g(u_t, x_{t-1}) &\approx g(u_t, \mu_{t-1}) + g'(u_t, \mu_{t-1})(x_{t-1} - \mu_{t-1}) \\ &= g(u_t, \mu_{t-1}) + G_t(x_{t-1} - \mu_{t-1}) \end{aligned}$$

G_t is the Jacobian!

Extended Kalman Filter (EKF)

Extended Kalman Filter $(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$

$$\bar{\mu}_t = g(u_t, \mu_{t-1})$$

$$\bar{\Sigma}_t = G_t \, \Sigma_{t-1} \, F_t^T + R_t$$

Prediction

$$K_t = \bar{\Sigma}_t \, C_t^T \, (C_t \, \bar{\Sigma}_t \, C_t^T + Q_t)^{-1}$$

Kalman gain

$$\mu_t = \bar{\mu}_t + K_t (z_t - h(\bar{\mu}_t))$$

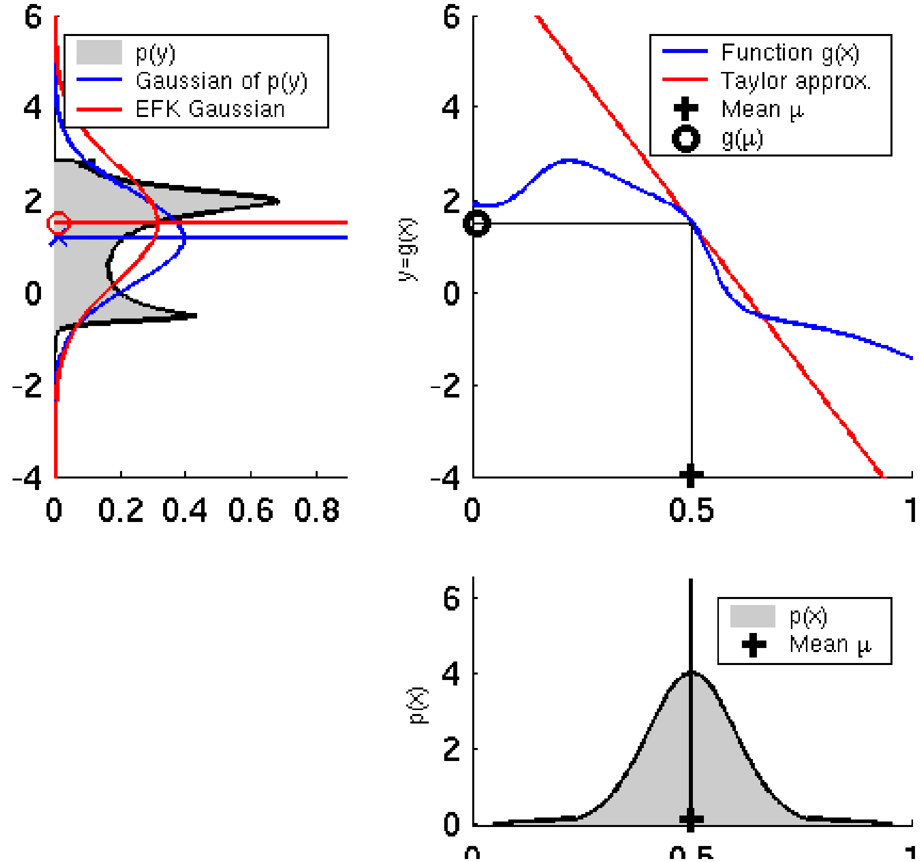
Innovation

$$\Sigma_t = (I - K_t \, H_t) \bar{\Sigma}_t$$

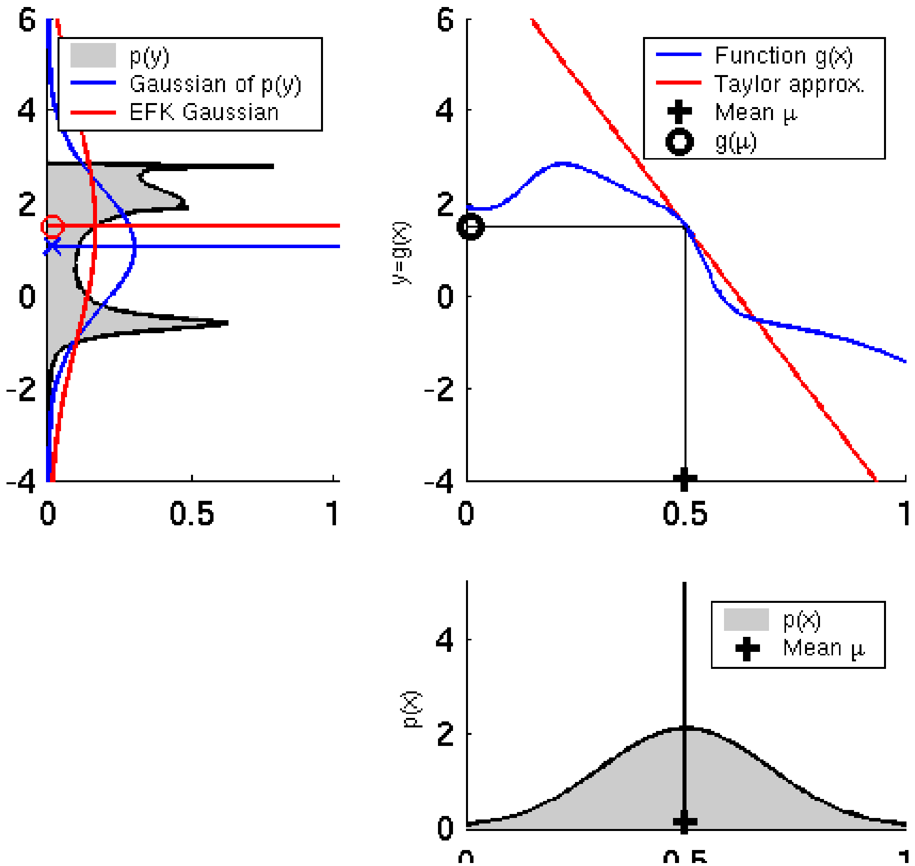
Correction

return μ_t, Σ_t

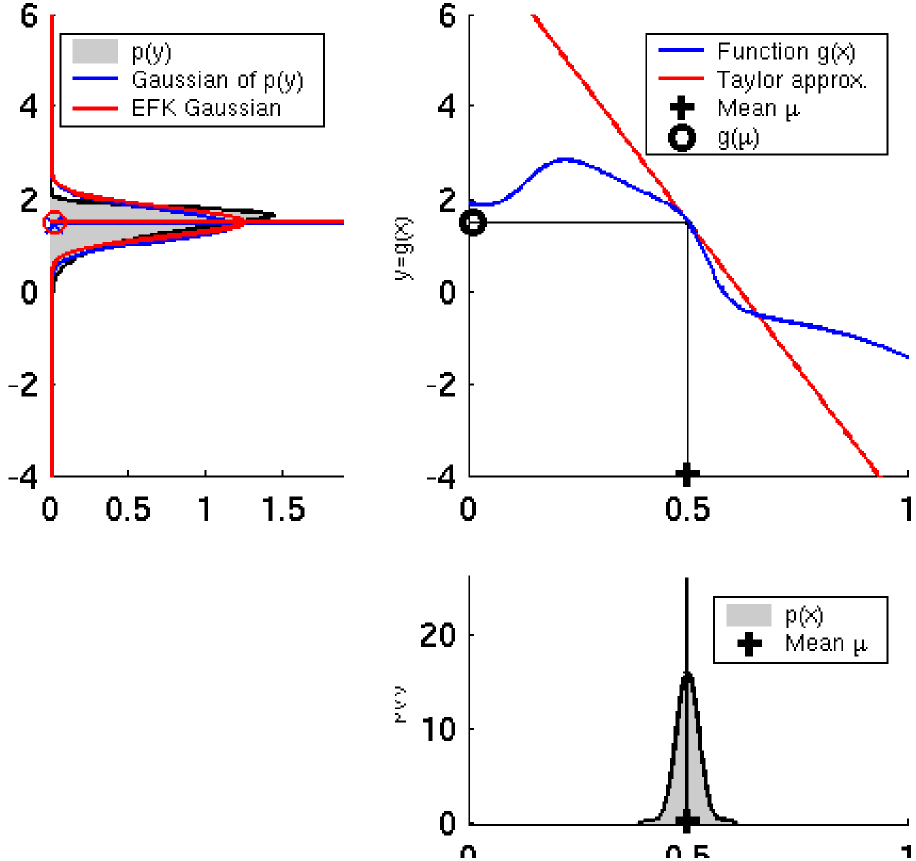
EKF depends on uncertainty (1)



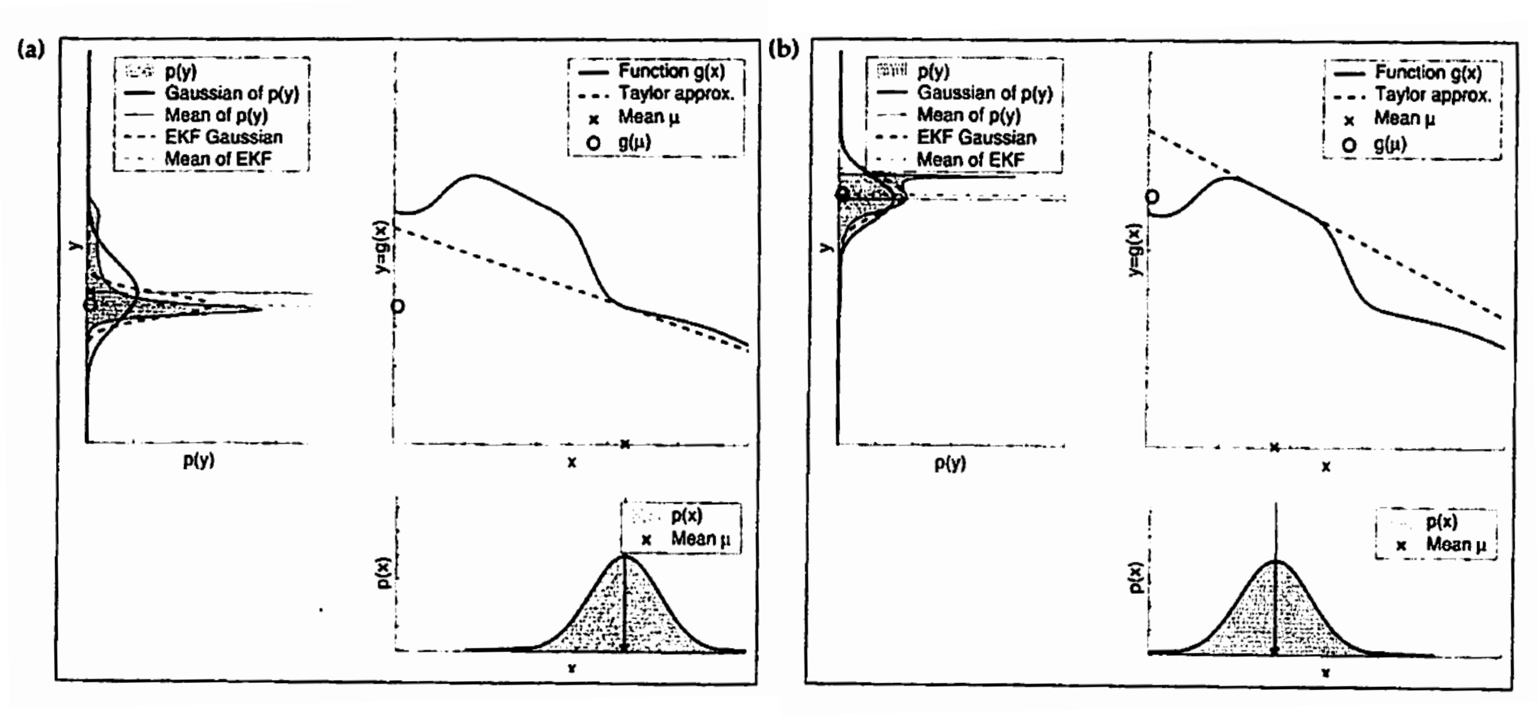
EKF depends on uncertainty (2)



EKF depends on uncertainty (3)



EKF depends on quality of approximation



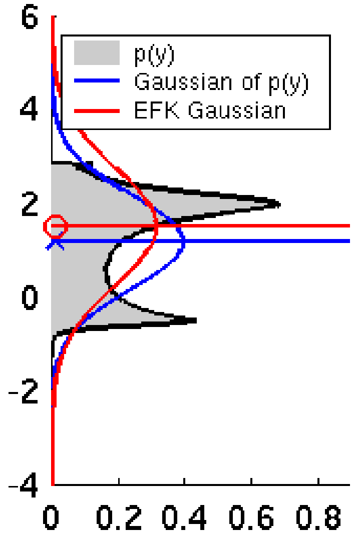
EKF Summary

- Highly efficient: Polynomial in measurement dimensionality *k* and state dimensionality *n*:

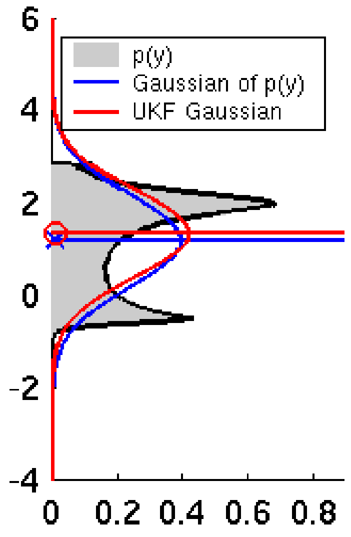
$$O(k^{2.376} + n^2)$$

- Not optimal!
- Can diverge if nonlinearities are large!
- Works surprisingly well even when all assumptions are violated!

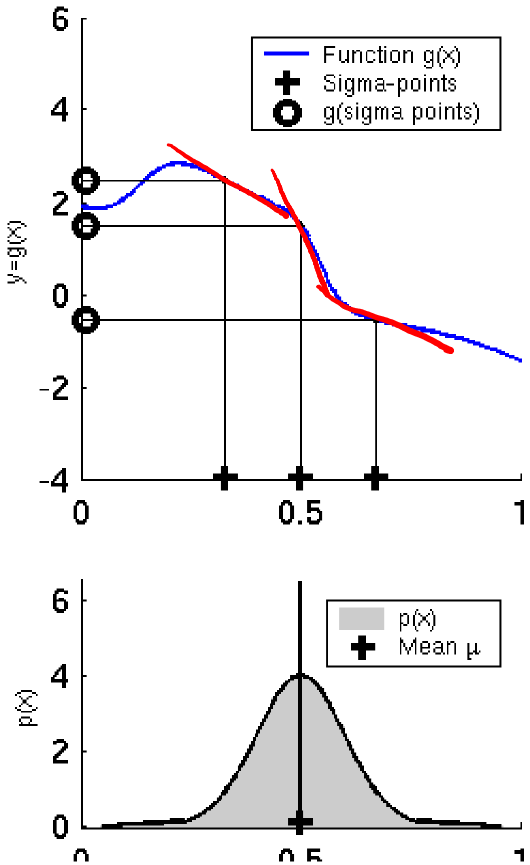
Unscented Kalman Filter (UKF)



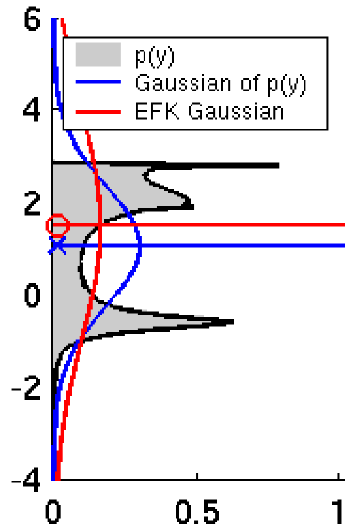
EKF



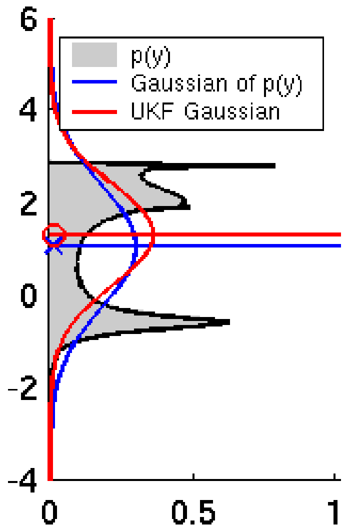
UKF



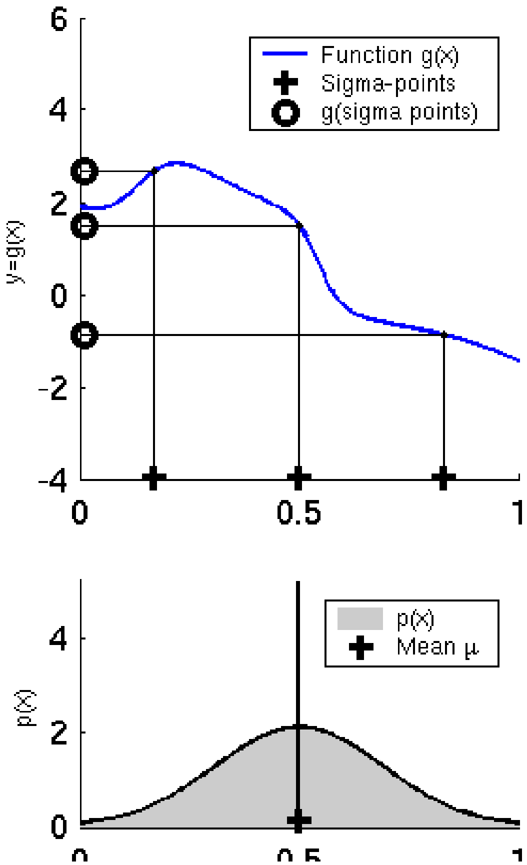
UKF Sigma-Point Estimate (2)



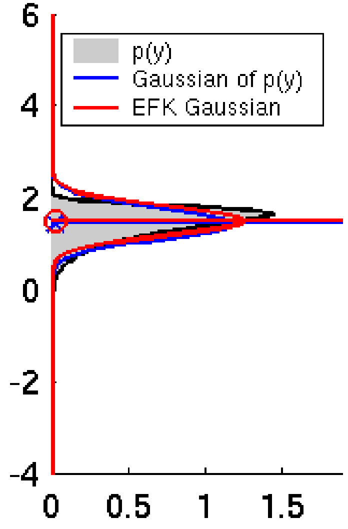
EKF



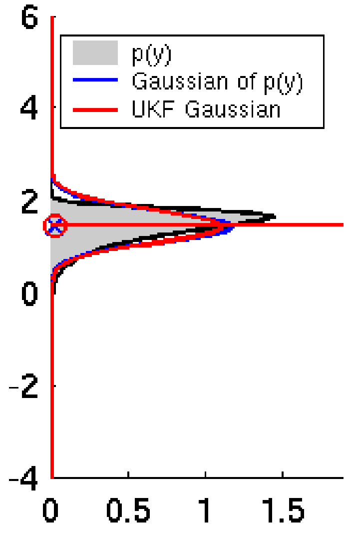
UKF



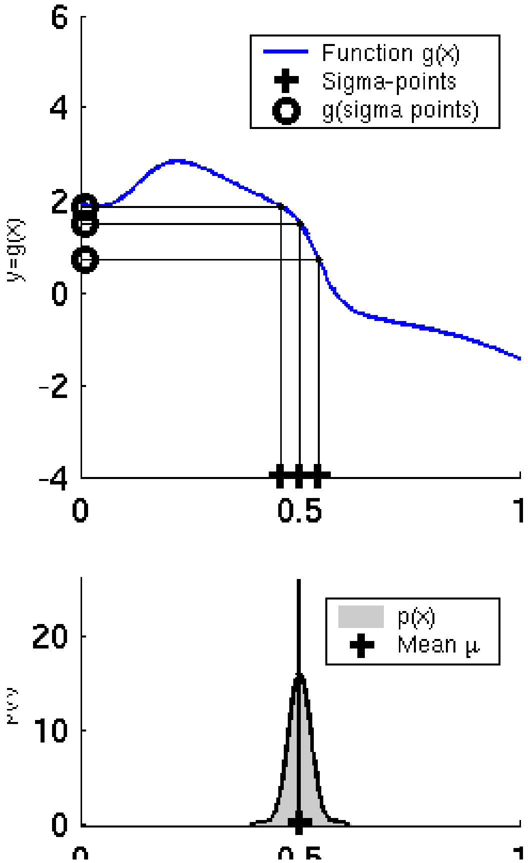
UKF Sigma-Point Estimate (3)



EKF



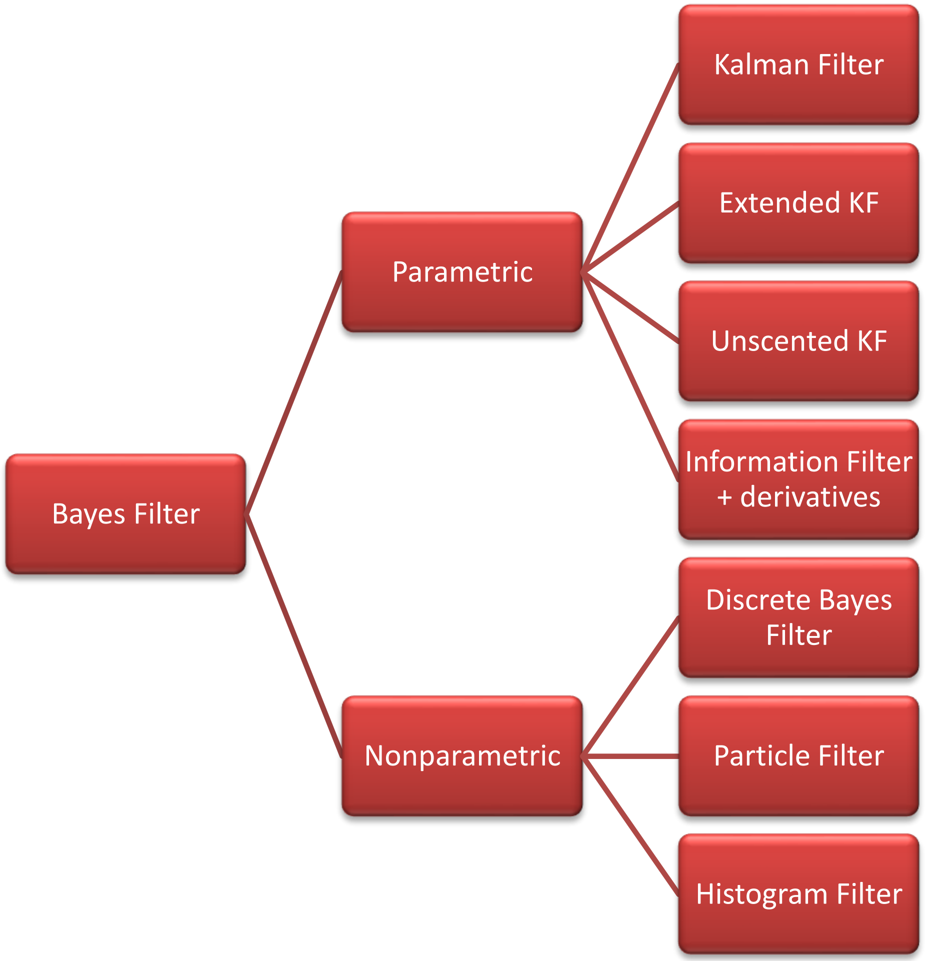
UKF



UKF Summary

- Highly efficient: Same complexity as EKF, with a constant factor slower in typical practical applications
- Better linearization than EKF: Accurate in first two terms of Taylor expansion (EKF only first term)
- Derivative-free: No Jacobians needed
- Still not optimal!

The Family of Bayes Filters



LINEARITY
GAUSSIAN STATE + NOISE

LOCAL LINEARITY (SIMPLE)

LOCAL LINEARITY (COMPLEX)

SIMPLE

COMPLEX