Visual Servoing and the Image Jacobian

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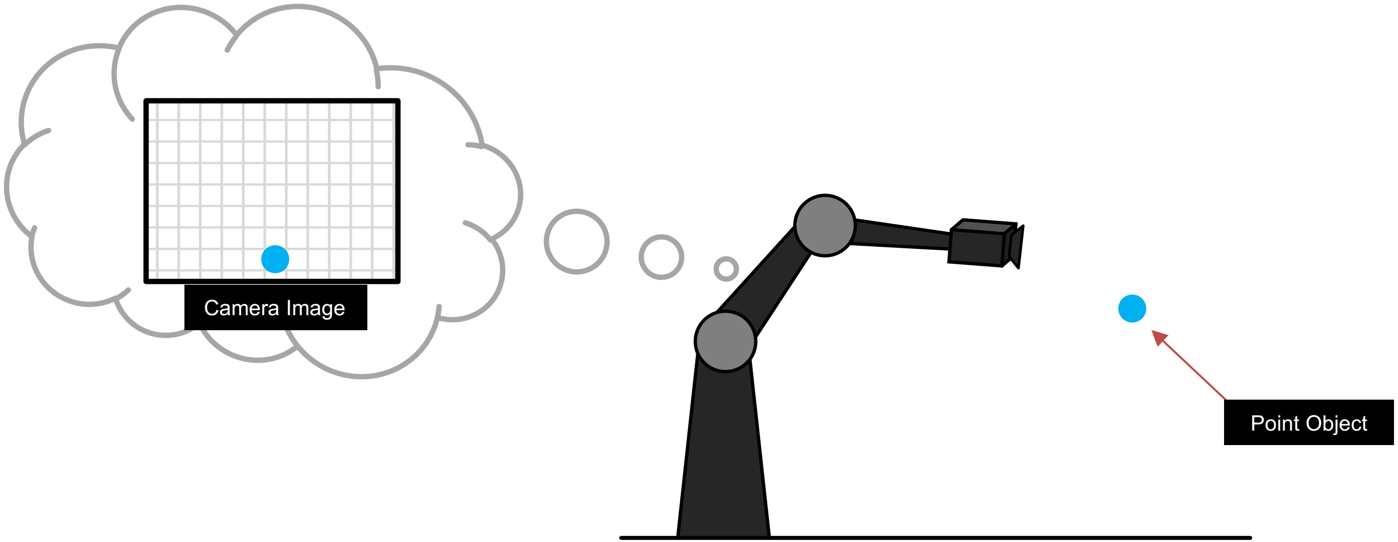
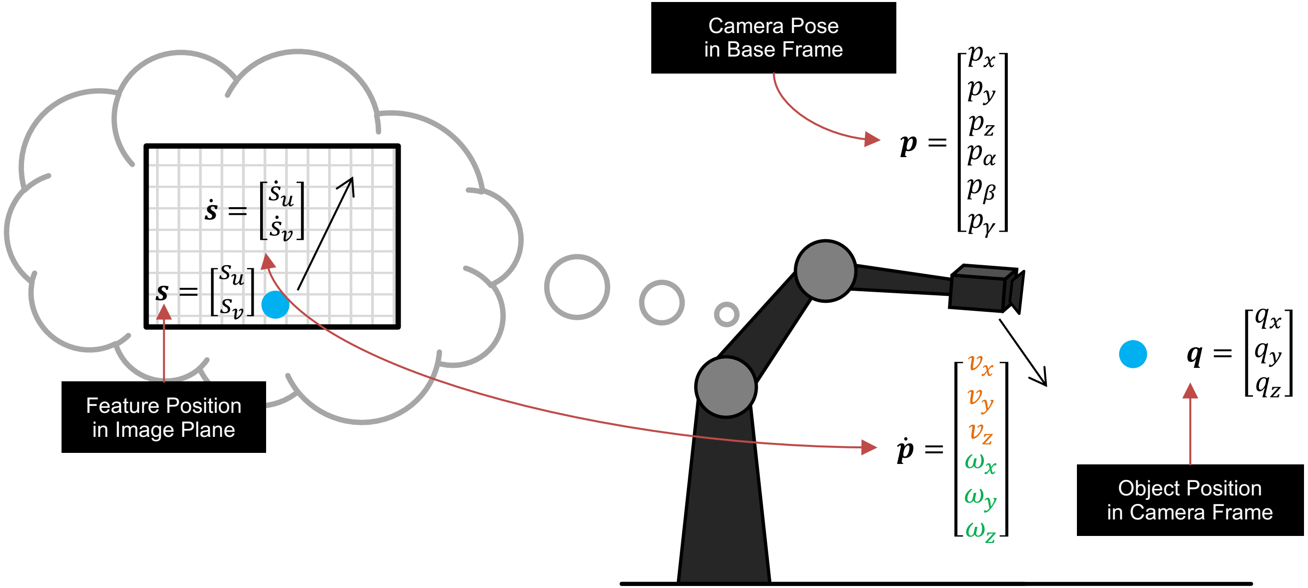
Eye-In-Hand Setup



Motivation



Camera Pose
in Base Frame
$\boldsymbol{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \\ p_\alpha \\ p_\beta \\ p_\gamma \end{bmatrix}$
$\dot{\boldsymbol{s}} = \begin{bmatrix} \dot{s}_u \\ \dot{s}_v \end{bmatrix}$
$\boldsymbol{s} = \begin{bmatrix} s_u \\ s_v \end{bmatrix}$
Feature Position
in Image Plane
$\dot{\boldsymbol{p}} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$
$\boldsymbol{q} = \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix}$
Object Position
in Camera Frame

The Image Jacobian

Velocity of Image Features

$$\dot{s} = J_I \dot{p}$$

Velocity of Camera

The Image Jacobian

$$\dot{\boldsymbol{s}} = \boldsymbol{J}_I \dot{\boldsymbol{p}}$$

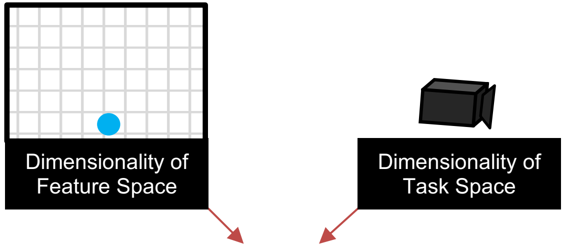
Dimensionality of Feature Space

Dimensionality of Task Space

$$\boldsymbol{J}_I(\boldsymbol{p}) = \begin{bmatrix} \frac{\partial s_1}{\partial p_1} & \cdots & \frac{\partial s_1}{\partial p_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial s_k}{\partial p_1} & \cdots & \frac{\partial s_k}{\partial p_m} \end{bmatrix}^{k \times m}$$

Camera Pose

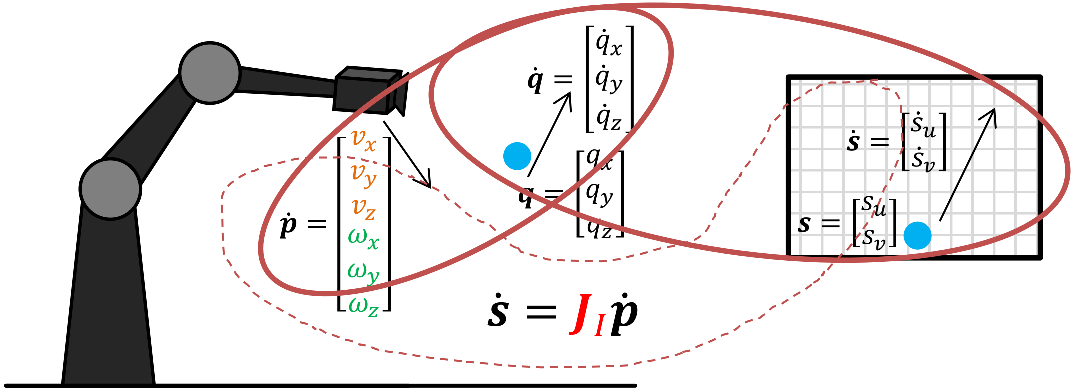
Our Example



$$J_I \in \mathbb{R}^{2\times 6}$$

$$\begin{bmatrix} \dot{s}_u \\ \dot{s}_v \end{bmatrix} = \begin{bmatrix} ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

Deriving the Jacobian: Three Steps

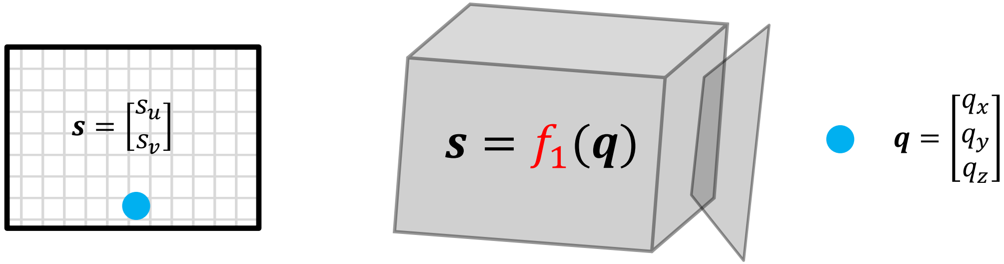


$$\boldsymbol{s} = f_1(\boldsymbol{q})$$

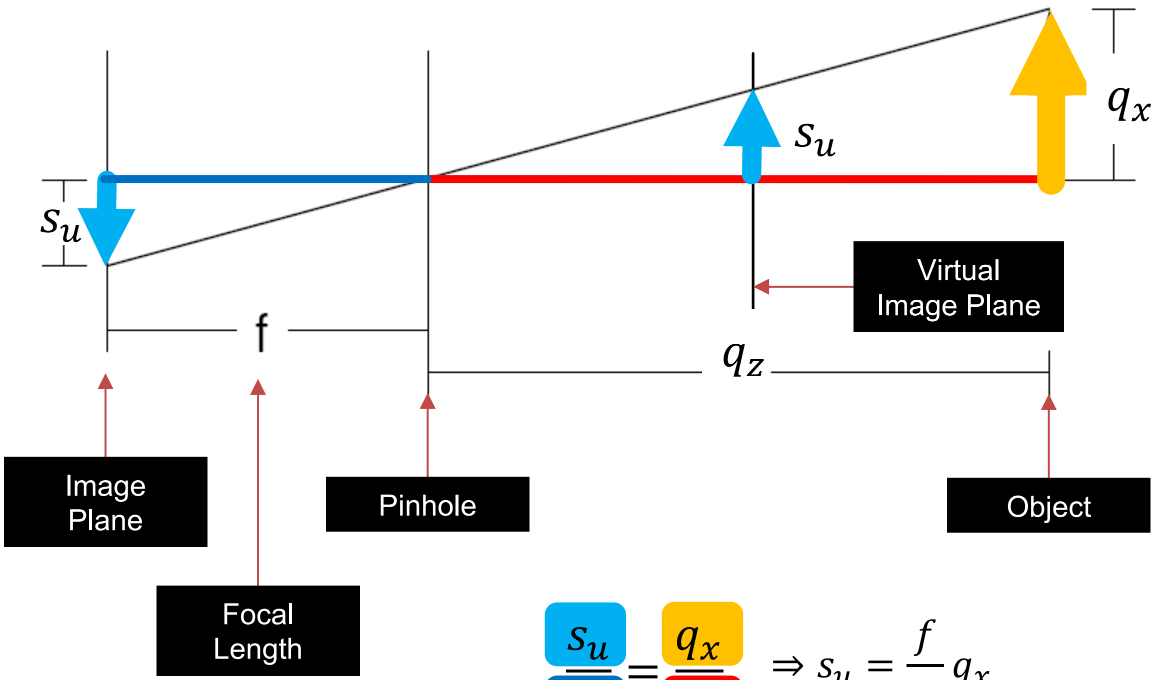
$$\dot{\boldsymbol{s}} = f_2(\dot{\boldsymbol{q}})$$

$$\dot{\boldsymbol{q}} = f_3(\dot{\boldsymbol{p}})$$

Step 1


$\boldsymbol{s} = \begin{bmatrix} s_u \\ s_v \end{bmatrix}$
$\boldsymbol{s} = f_1(\boldsymbol{q})$
$\boldsymbol{q} = \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix}$

Perspective Projection in 2D



$$\frac{s_u}{f} = \frac{q_x}{q_z} \Rightarrow s_u = \frac{f}{q_z} q_x$$

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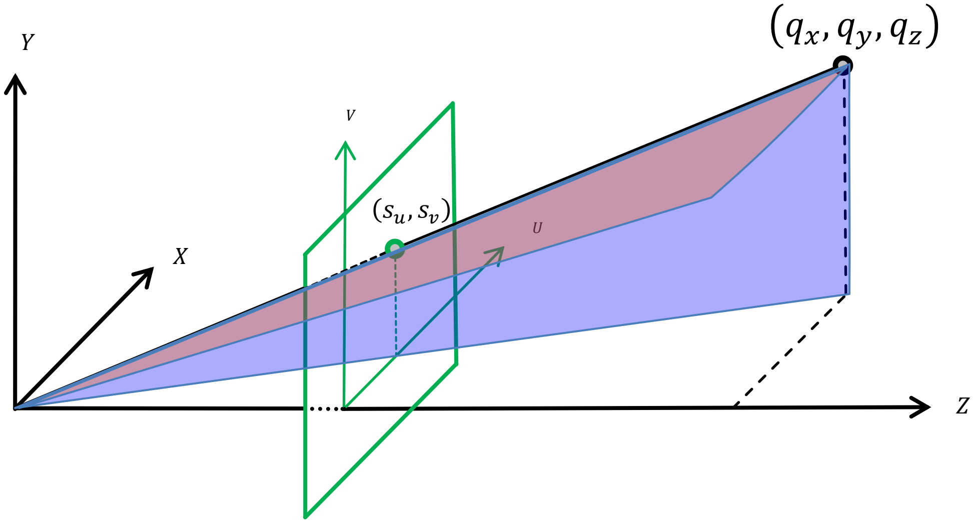
Perspective Projection in 3D



$$s_u = \frac{f}{q_z} q_x$$

$$s_v = \frac{f}{q_z} q_y$$

$$\begin{bmatrix} \dot{s}_u \\ \dot{s}_v \end{bmatrix} = \frac{f}{q_z} \begin{bmatrix} q_x \\ q_y \end{bmatrix}$$

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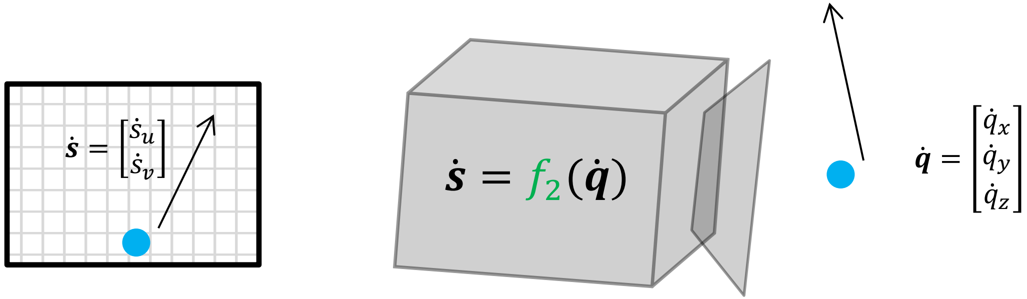
Step 2



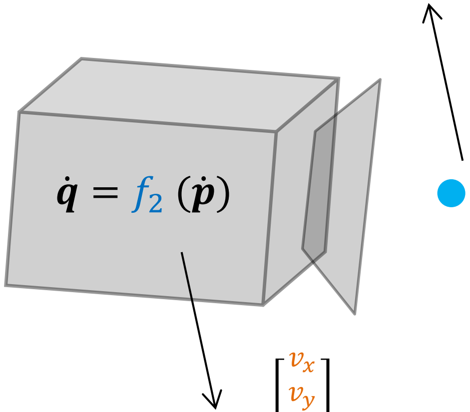
$\dot{\boldsymbol{s}} = \begin{bmatrix} \dot{s}_u \\ \dot{s}_v \end{bmatrix}$
$\dot{\boldsymbol{s}} = f_2(\dot{\boldsymbol{q}})$
$\dot{\boldsymbol{q}} = \begin{bmatrix} \dot{q}_x \\ \dot{q}_y \\ \dot{q}_z \end{bmatrix}$

Basic Differentiation

$$s_u = f\,\frac{q_x}{q_z} \qquad \xrightarrow{\frac{d}{dt}} \qquad \dot{s}_u = f\,\frac{q_z\dot{q}_x - q_x\dot{q}_z}{{q_z}^2}$$

$$s_v = f\,\frac{q_y}{q_z} \qquad\qquad\qquad \dot{s}_v = f\,\frac{q_z\dot{q}_y - q_y\dot{q}_z}{{q_z}^2}$$

Step 3

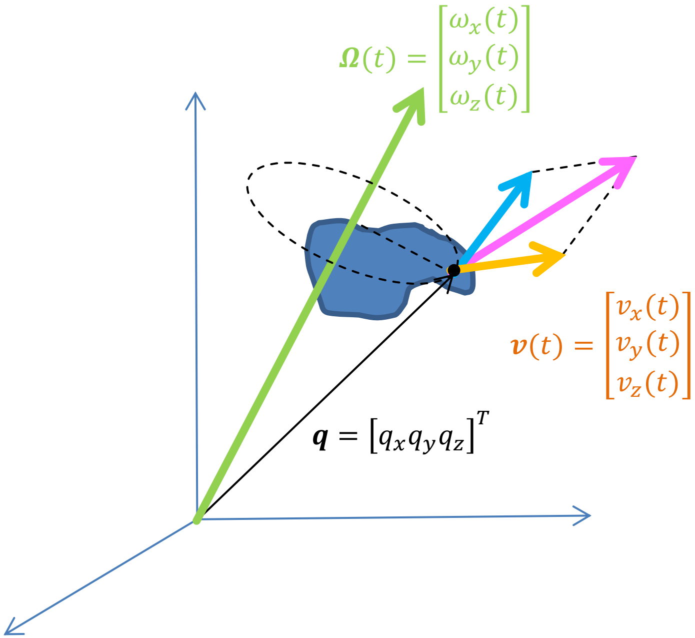


$$\dot{\boldsymbol{q}} = f_2\left(\dot{\boldsymbol{p}}\right)$$

$$\dot{\boldsymbol{q}} = \begin{bmatrix} \dot{q}_x \\ \dot{q}_y \\ \dot{q}_z \end{bmatrix}$$

$$\dot{\boldsymbol{p}} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\Omega} \end{bmatrix}$$

Reminder: Motion of a Point in 3D



$$\dot{\boldsymbol{q}} = \boldsymbol{\Omega} \times \boldsymbol{q} + \boldsymbol{v}$$

$$\begin{bmatrix} \dot{q}_x \\ \dot{q}_y \\ \dot{q}_z \end{bmatrix} = \begin{bmatrix} q_z\omega_y - q_y\omega_z + v_x \\ q_x\omega_z - q_z\omega_x + v_y \\ q_y\omega_x - q_x\omega_y + v_z \end{bmatrix}$$

Moving Reference Frame

If we move the point $\rightarrow$ $\dot{\boldsymbol{q}} = \boldsymbol{\Omega}\times\boldsymbol{q} + \boldsymbol{v}$

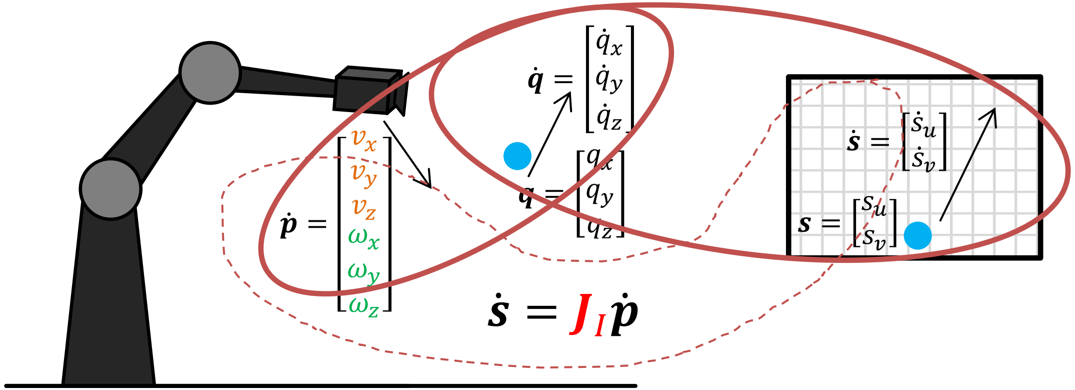
If we move the EE / camera $\rightarrow$ $\dot{\boldsymbol{q}} = -\boldsymbol{\Omega}\times\boldsymbol{q} - \boldsymbol{v}$

$$\begin{bmatrix} \dot{s}_u \\ \dot{s}_v \end{bmatrix} = \frac{f}{q_z}\begin{bmatrix} q_x \\ q_y \end{bmatrix}$$

From Step 1

$$\dot{\boldsymbol{q}} = \begin{bmatrix} \dot{q}_x \\ \dot{q}_y \\ \dot{q}_z \end{bmatrix} = \begin{bmatrix} -q_z\omega_y + q_y\omega_z - v_x \\ -q_x\omega_z + q_z\omega_x - v_y \\ -q_y\omega_x + q_x\omega_y - v_z \end{bmatrix} = \begin{bmatrix} -q_z\omega_y + \frac{s_v q_z}{f}\omega_z - v_x \\ -\frac{s_u q_z}{f}\omega_z + q_z\omega_x - v_y \\ -\frac{q_z}{f}\left(s_v\omega_x - s_u\omega_y\right) - v_z \end{bmatrix}$$

Putting it all together



$$\boldsymbol{s} = f_1(\boldsymbol{q})$$

$$\dot{\boldsymbol{s}} = f_2(\dot{\boldsymbol{q}})$$

$$\dot{\boldsymbol{q}} = f_3(\dot{\boldsymbol{p}})$$

Step 1

$$q_x = \frac{q_z s_u}{f}$$

Step 3

$$\dot{q}_x = -q_z\omega_y + \frac{s_v q_z}{f}\omega_z - v_x$$
$$\dot{q}_y = -\frac{s_u q_z}{f}\omega_z + q_z\omega_x - v_y$$
$$\dot{q}_z = -\frac{q_z}{f}\left(s_v\omega_x - s_u\omega_y\right) - v_z$$

Step 2

$$\dot{s}_u = f\frac{q_z\dot{q}_x - q_x\dot{q}_z}{{q_z}^2} = \frac{f}{{q_z}^2}\left[q_z\left(-q_z\omega_y + \frac{s_v q_z}{f}\omega_z - v_x\right) - \frac{s_u q_z}{f}\left(-\frac{q_z}{f}\left(s_v\omega_x - s_u\omega_y\right) - v_z\right)\right]$$

$$\dot{s}_u = -\frac{f}{q_z}v_x + \frac{s_u}{q_z}v_z + \frac{s_u s_v}{f}\omega_x - \frac{f^2 + {s_u}^2}{f}\omega_y + s_v\omega_z$$

$$\dot{s}_v = -\frac{f}{q_z}v_y + \frac{s_v}{q_z}v_z + \frac{f^2 + {s_v}^2}{f}\omega_x - \frac{s_u s_v}{f}\omega_y - s_u\omega_z$$

Image Jacobian for a Point Feature

$$\dot{s}_u = -\frac{f}{q_z} v_x + \frac{s_u}{q_z} v_z + \frac{s_u s_v}{f} \omega_x - \frac{f^2 + s_u{}^2}{f} \omega_y + s_v \omega_z$$

$$\dot{s}_v = -\frac{f}{q_z} v_y + \frac{s_v}{q_z} v_z + \frac{f^2 + s_v{}^2}{f} \omega_x - \frac{s_u s_v}{f} \omega_y - s_u \omega_z$$

$$\begin{bmatrix} \dot{s}_u \\ \dot{s}_v \end{bmatrix} = \begin{bmatrix} ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$\begin{bmatrix} \dot{s}_u \\ \dot{s}_v \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{f}{q_z} & 0 & \frac{s_u}{q_z} & \frac{s_u s_v}{f} & -\frac{f^2 + s_u{}^2}{f} & s_v \\ 0 & -\frac{f}{q_z} & \frac{s_v}{q_z} & \frac{f^2 + s_v{}^2}{f} & -\frac{s_u s_v}{f} & -s_u \end{bmatrix}}_{\boldsymbol{J_I}} \begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

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Another way

$$\dot{s}_u = f\frac{q_z\dot{\boldsymbol{q}}_{\boldsymbol{x}} - q_x\dot{\boldsymbol{q}}_{\boldsymbol{z}}}{{q_z}^2}$$

$$\dot{s}_v = f\frac{q_z\dot{\boldsymbol{q}}_{\boldsymbol{y}} - q_y\dot{\boldsymbol{q}}_{\boldsymbol{z}}}{{q_z}^2}$$

$$\dot{q}_x = -q_z\boldsymbol{\omega}_{\boldsymbol{y}} + \frac{s_v q_z}{f}\boldsymbol{\omega}_{\boldsymbol{z}} - \boldsymbol{v}_{\boldsymbol{x}}$$

$$\dot{q}_y = -\frac{s_u q_z}{f}\boldsymbol{\omega}_{\boldsymbol{z}} + q_z\boldsymbol{\omega}_{\boldsymbol{x}} - \boldsymbol{v}_{\boldsymbol{y}}$$

$$\dot{q}_z = -\frac{q_z}{f}\left(s_v\boldsymbol{\omega}_{\boldsymbol{x}} - s_u\boldsymbol{\omega}_{\boldsymbol{y}}\right) - \boldsymbol{v}_{\boldsymbol{z}}$$

$$\dot{\boldsymbol{s}} = \boldsymbol{J}_2\dot{\boldsymbol{q}} \qquad \mathbb{R}^{2\times 3}$$

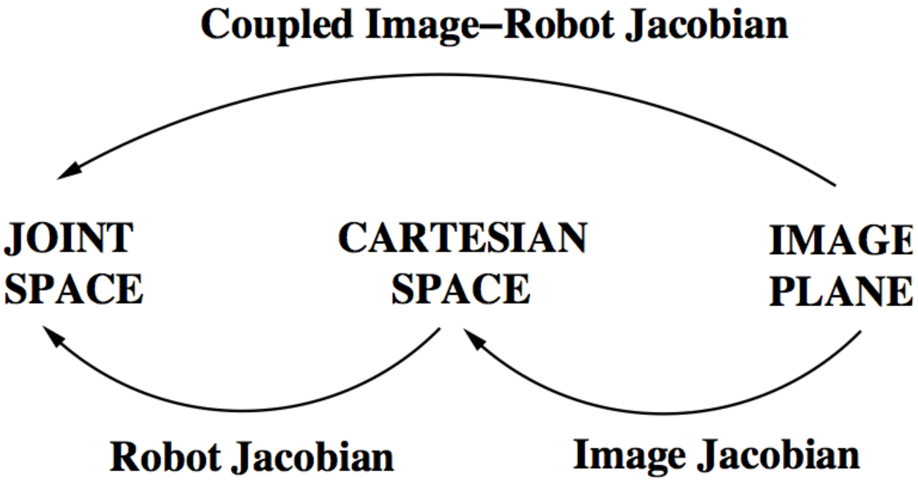
$$\dot{\boldsymbol{q}} = \boldsymbol{J}_3\dot{\boldsymbol{p}} \qquad \mathbb{R}^{3\times 6}$$

$$\dot{\boldsymbol{s}} = \boldsymbol{J}_2\boldsymbol{J}_3\dot{\boldsymbol{p}}$$

$$\downarrow$$

$$\dot{\boldsymbol{s}} = \boldsymbol{J}_I\dot{\boldsymbol{p}} \qquad \mathbb{R}^{2\times 6}$$

Jacobians all the way down



Mapping back to EE / Camera velocity

The Inverse $\boldsymbol{J_I}^{-1}$ only exists if $\boldsymbol{J_I}$ is square.

$$\boldsymbol{s} = \boldsymbol{J_I}\, \boldsymbol{p}$$

k=m

$$\boldsymbol{s} = \boldsymbol{J_I}\, \boldsymbol{p}$$

underconstrained

$$\boldsymbol{s} = \boldsymbol{J_I}\, \boldsymbol{p}$$

overconstrained

$$\dot{\boldsymbol{p}} = \boldsymbol{J_I}^{-1} \dot{\boldsymbol{s}}$$