Disclaimer

These slides are intended as presentation aids for the lecture. They contain information that would otherwise be to difficult or time-consuming to reproduce on the board. But they are incomplete, not self-explanatory, and are not always used in the order they appear in this presentation. As a result, these slides should not be used as a script for this course. I recommend you take notes during class, maybe on the slides themselves. It has been shown that taking notes improves learning success.

Reading for this set of slides

- Craig – Intro to Robotics (3rd Edition)
  - Chapter 5.6 (and Chapter 5 overall)

Please note that this set of slides is intended as support for the lecture, not as a stand-alone script. If you want to study for this course, please use these slides in conjunction with the indicated chapters in the text books. The textbooks are available online or in the TUB library (many copies that can be checked out for the entire semester. There are also some aspects of the lectures that will not be covered in the text books but can still be part of the homework or exam. For those It is important that you attend class or ask somebody about what was covered in class.

Robotics

Computing the Jacobian

TU Berlin

Oliver Brock

Different Jacobians

- So far we computed the Jacobian
  - wrt/ end-effector
  - for a given end-effector **representation x**
- Jacobian can be computed for
  - any point
  - any representation
- We want a uniquely defined Jacobian!

Basic Jacobian

$$\begin{pmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} \mathbf{v}_{(3\times1)} \\ \omega_{(3\times1)} \end{pmatrix} = J_{0(6\times n)}(\mathbf{q}) \;\; \dot{\mathbf{q}}_{(n\times1)}$$

For convenience, we will often drop the (**q**)

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{\mathbf{x}}_p \\ \dot{\mathbf{x}}_r \end{pmatrix} = \begin{bmatrix} E_p(\mathbf{x}_p) & 0 \\ 0 & E_r(\mathbf{x}_r) \end{bmatrix} \begin{pmatrix} \mathbf{v} \\ \omega \end{pmatrix}$$

$$E_p(\mathbf{x_p}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_r(\mathbf{x}_r) = \begin{bmatrix} -\frac{s\alpha c\beta}{s\beta} & \frac{c\alpha c\beta}{s\beta} & 1 \\ c\alpha & s\alpha & 0 \\ \frac{s\alpha}{s\beta} & -\frac{c\alpha}{s\beta} & 0 \end{bmatrix}$$

e.g., to convert from ω to α, β, γ Euler

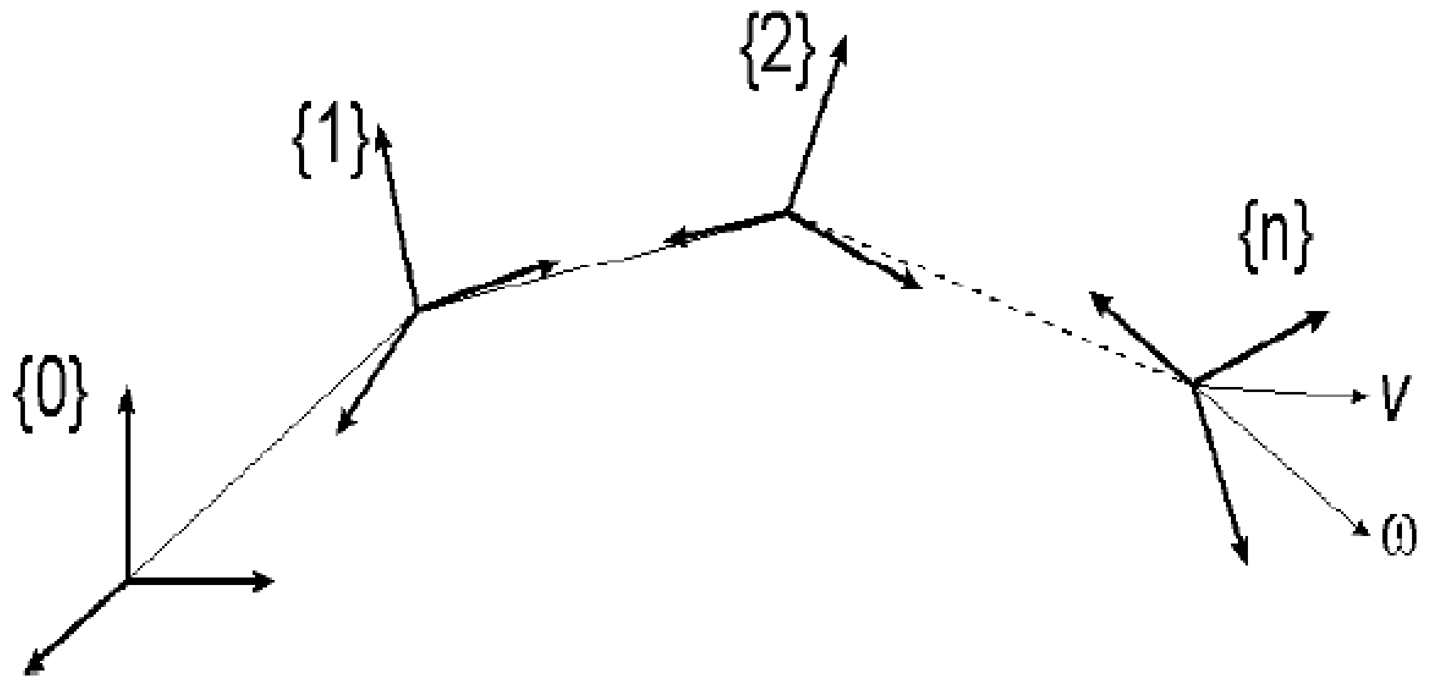
Relationship between $J_{\mathbf{x}}$ and J_0

$$J_{0(6\times n)} = \begin{bmatrix} J_{v(3\times n)} \\ J_{\omega(3\times n)} \end{bmatrix}$$

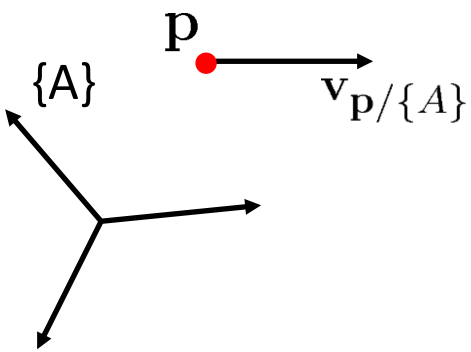
$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{\mathbf{x}}_p \\ \dot{\mathbf{x}}_r \end{pmatrix} = \begin{pmatrix} E_p(\mathbf{x}_p)\ \mathbf{v} \\ E_r(\mathbf{x}_r)\ \omega \end{pmatrix} = \begin{pmatrix} E_p(\mathbf{x}_p) J_{\mathbf{v}}\ \dot{\mathbf{q}} \\ E_r(\mathbf{x}_r) J_{\omega}\ \dot{\mathbf{q}} \end{pmatrix}$$

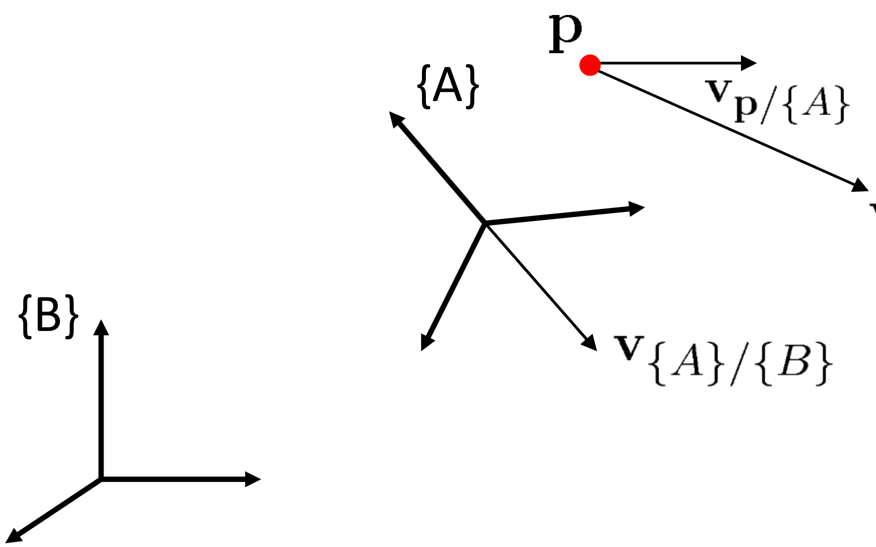
$$J_{\mathbf{x}} = \begin{bmatrix} J_p \\ J_r \end{bmatrix} = \begin{bmatrix} E_p & 0 \\ 0 & E_r \end{bmatrix} \begin{bmatrix} J_{\mathbf{v}} \\ J_{\omega} \end{bmatrix}$$

Velocity Propagation



Linear Motion



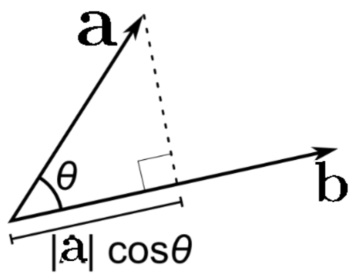


$$\mathbf{v}_{\mathbf{p}/\{B\}} = \mathbf{v}_{\mathbf{p}/\{A\}} + \mathbf{v}_{\{A\}/\{B\}}$$

Sidebar: dot / cross product

dot product

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \; \|\mathbf{b}\| \; \cos\phi$$



$$\|\mathbf{a}_\mathbf{b}\| = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|}$$

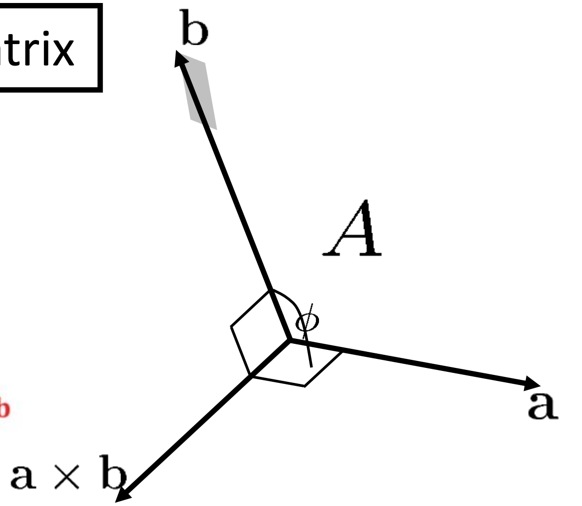
cross product

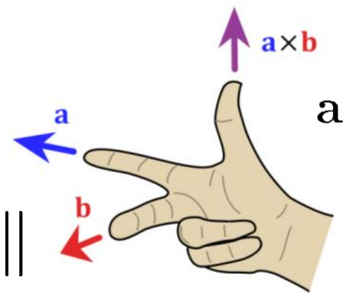
skew-symmetric matrix

$$\mathbf{a} \times \mathbf{b} = \hat{\mathbf{a}} \; \mathbf{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \mathbf{b}$$

cross product operator

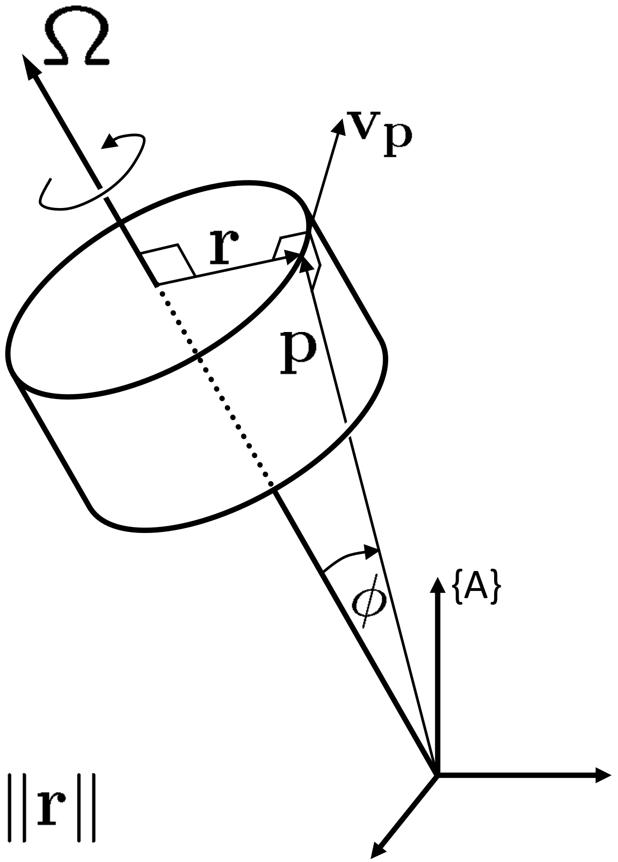
$$A = \|\mathbf{a}\| \; \|\mathbf{b}\| \; \sin\phi = \|\mathbf{a} \times \mathbf{b}\|$$





Rotational Motion

Angular Velocity



$$\mathbf{v_p} = \Omega \times \mathbf{p}$$

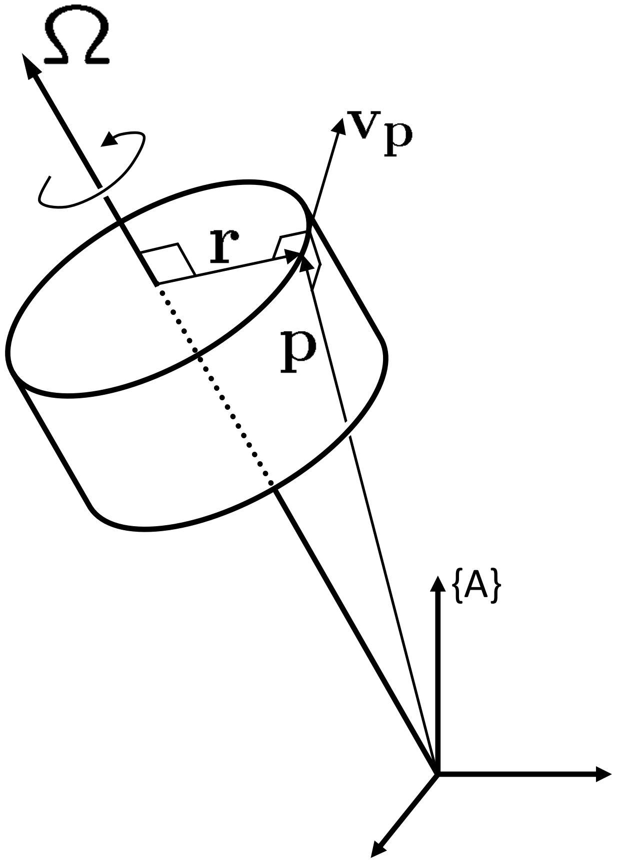
$$\|\mathbf{v_p}\| = \|\Omega\|\ \|\mathbf{p}\| \sin\phi = \|\Omega\|\ \|\mathbf{r}\|$$

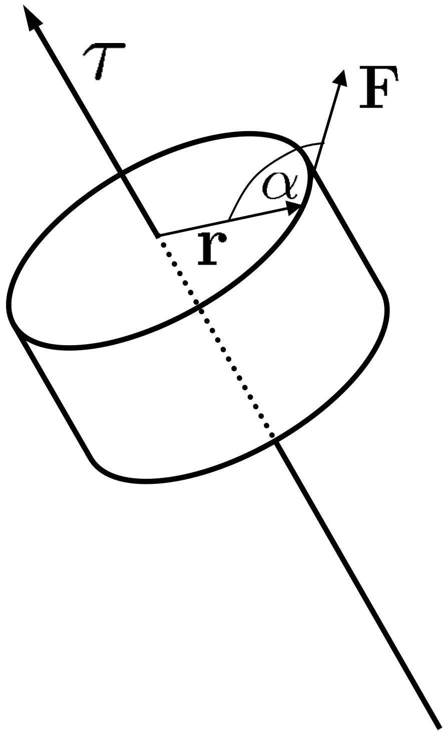
$$\|\Omega\| = \frac{\|\mathbf{v_p}\|}{\|\mathbf{r}\|}$$

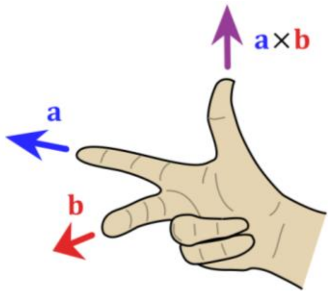
$$\|\mathbf{v_p}\| = \|\Omega\|\ \|\mathbf{r}\|$$

$$\Omega = \frac{\mathbf{r} \times \mathbf{v_p}}{\|\mathbf{r}\|^2}$$

Side by Side







$$\mathbf{v_p} = \Omega \times \mathbf{p}$$

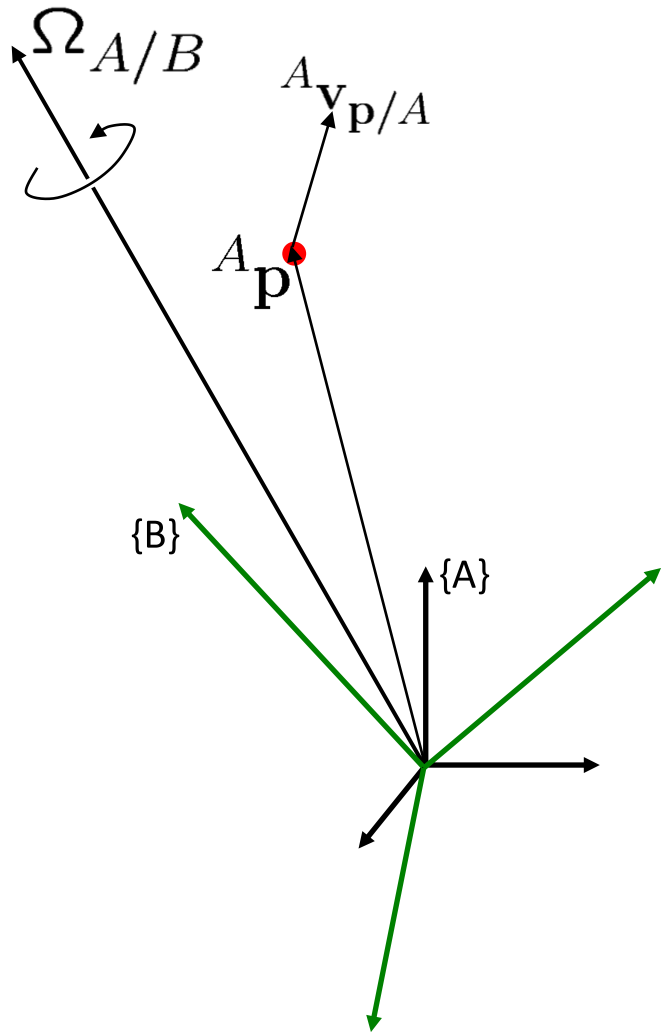
$$\tau = \mathbf{r} \times \mathbf{F}$$

Velocity Between Rotating Frames

Considering rotation between {A} and {B}:

$$ {}^{A}\mathbf{v}_{\mathbf{p}} = {}^{A}\Omega_{A/B} \times {}^{A}\mathbf{p} $$

$$ {}^{B}\mathbf{v}_{\mathbf{p}} = {}^{B}_{A}R\, {}^{A}\Omega_{A/B} \times {}^{B}\mathbf{p} $$



Linear and Angular Motion

From before: $$ {}^{B}\mathbf{v}_{\mathbf{p}} = \underbrace{{}^{B}_{A}R\, {}^{A}\Omega_{A/B}}_{{}^{B}\Omega_{A/B}} \times {}^{B}\mathbf{p} $$

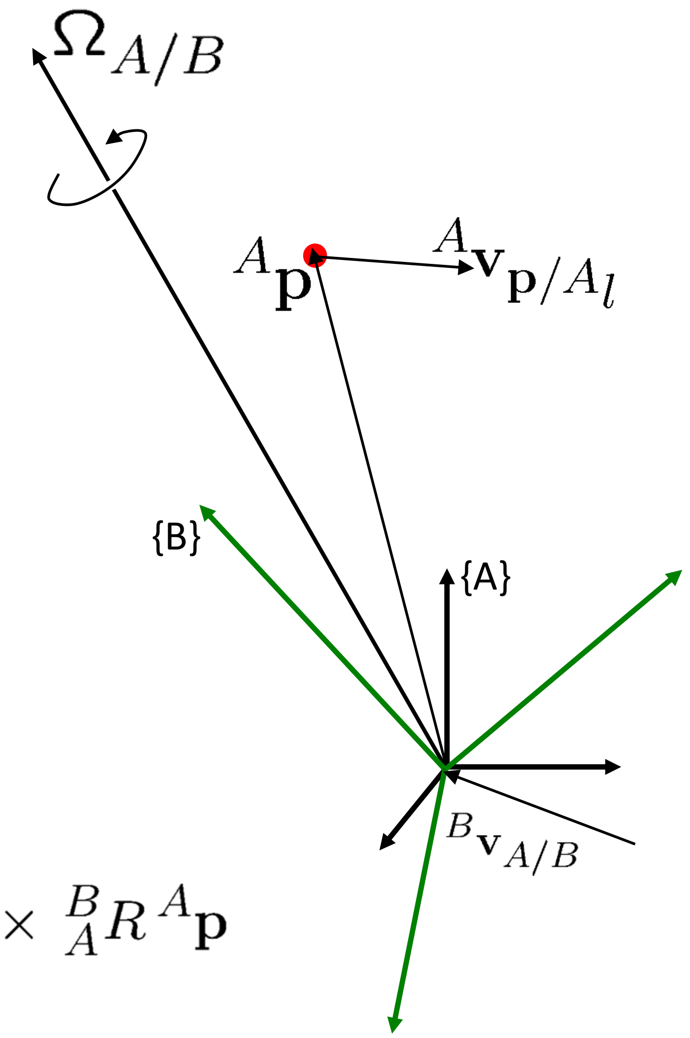
Adding linear motion of **p** (possibly the result of linear and angular motion in {A}):

$$ {}^{B}\mathbf{v}_{\mathbf{p}} = {}^{B}({}^{A}\mathbf{v}_{\mathbf{p}/A_l}) + {}^{B}\Omega_{A/B} \times {}^{B}\mathbf{p} $$

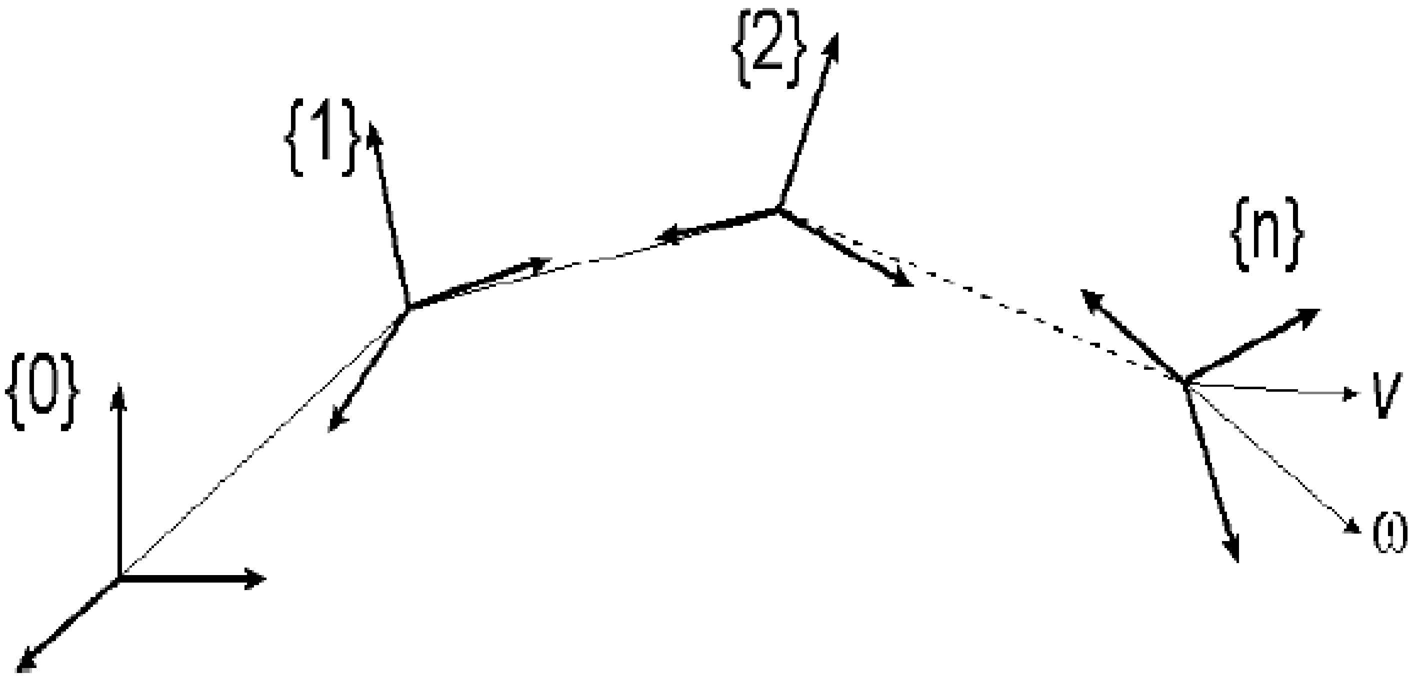
$$ {}^{B}\mathbf{v}_{\mathbf{p}} = {}^{B}_{A}R\, {}^{A}\mathbf{v}_{\mathbf{p}/A_l} + {}^{B}\Omega_{A/B} \times {}^{B}_{A}R\, {}^{A}\mathbf{p} $$

Adding linear motion between frames A and B:

$$ {}^{B}\mathbf{v}_{\mathbf{p}} = {}^{B}\mathbf{v}_{A/B} + {}^{B}_{A}R\, {}^{A}\mathbf{v}_{\mathbf{p}/A_l} + {}^{B}\Omega_{A/B} \times {}^{B}_{A}R\, {}^{A}\mathbf{p} $$



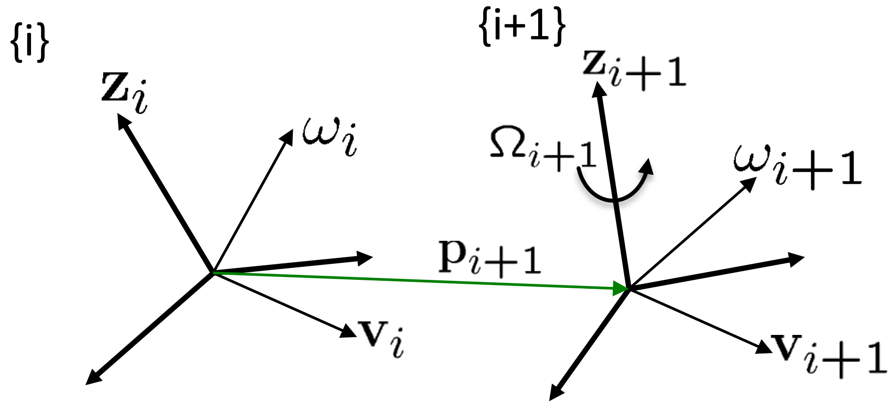
Velocity Propagation



Velocity Propagation

$$\omega_{i+1} = \omega_i + \Omega_{i+1}$$

$$\Omega_{i+1} = \dot{\theta}_{i+1} \, \mathbf{z}_{i+1}$$



$$\mathbf{v}_{i+1} = \mathbf{v}_i + \omega_i \times \mathbf{p}_{i+1} + \dot{d}_{i+1}\mathbf{z}_{i+1}$$

$$\begin{pmatrix} {}^0\mathbf{v} \\ {}^0\omega \end{pmatrix} = \begin{bmatrix} {}^0_nR & 0 \\ 0 & {}^0_nR \end{bmatrix} \begin{pmatrix} {}^n\mathbf{v} \\ {}^n\omega \end{pmatrix}$$

Huh?

- Velocity propagation gives us end-effector velocities based on joint velocities
- That is exactly what the Jacobian does!
- Velocity propagation only involved simple operations – no differentiation of the forward kinematics!
- Can we use velocity propagation to easily compute the Jacobian?

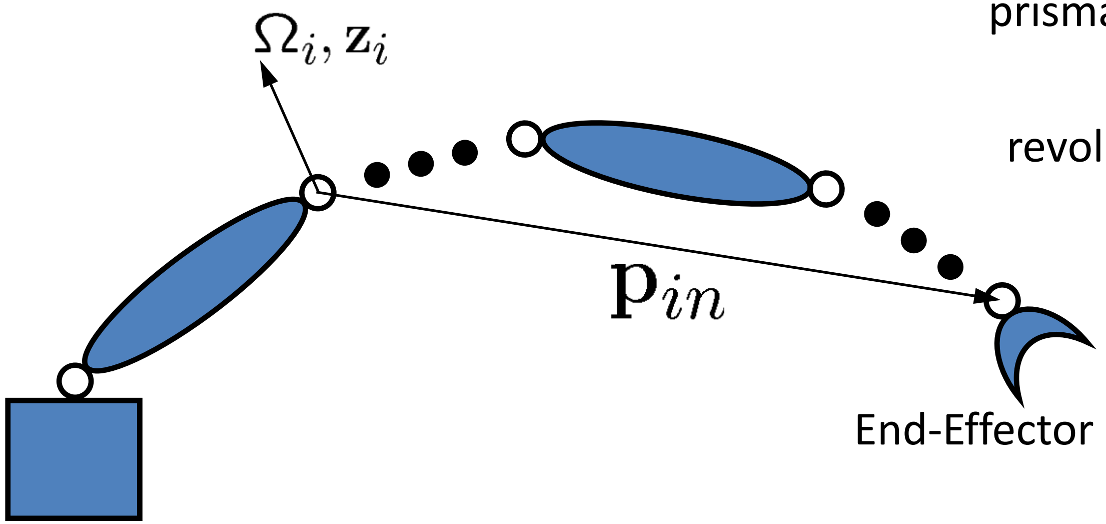
q revisited: Joint Coordinates

$$q_i = \bar{\epsilon}\theta_i + \epsilon d_i$$

$$\epsilon = \begin{cases} 0 & \text{for revolute joints} \\ 1 & \text{for prismatic joints} \end{cases}$$

$$\bar{\epsilon} = 1 - \epsilon$$

End-Effector Velocities



prismatic: $\mathbf{v}_{i/i-1} = \mathbf{z}_i \dot{q}_i$

revolute: $\Omega_{i/i-1} = \mathbf{z}_i \dot{q}_i$

we'll drop these

$$\mathbf{v} = \sum_{i=1}^{n} \left[\epsilon_i \mathbf{v}_i + \bar{\epsilon}_i (\Omega_i \times \mathbf{p}_{in})\right]$$

$$\omega = \sum_{i=1}^{n} \bar{\epsilon}_i \mathbf{z}_i \dot{q}_i$$

Linear End-Effector Velocities

$$\begin{aligned}
\dot{\mathbf{x}}_p = \mathbf{v} &= \sum_{i=1}^{n} \left[\epsilon_i \mathbf{v}_i + \bar{\epsilon}_i (\Omega_i \times \mathbf{p}_{in})\right] \\
&= (\epsilon_1 \mathbf{z}_1 + \bar{\epsilon}_1 (\mathbf{z}_1 \times \mathbf{p}_{1n}))\, \dot{q}_1 + \\
&\quad (\epsilon_2 \mathbf{z}_2 + \bar{\epsilon}_2 (\mathbf{z}_2 \times \mathbf{p}_{2n}))\, \dot{q}_2 + \\
&\quad \vdots \\
&\quad (\epsilon_n \mathbf{z}_n + \bar{\epsilon}_n (\mathbf{z}_n \times \mathbf{p}_{nn}))\, \dot{q}_n \\
&= J_v\, \dot{\mathbf{q}}
\end{aligned}$$

i-th column of J_v

$$\begin{aligned}
\mathbf{v}_{i/i-1} &= \mathbf{z}_i \dot{q}_i \\
\Omega_{i/i-1} &= \mathbf{z}_i \dot{q}_i
\end{aligned}$$

Angular End-Effector Velocities

$$\dot{\mathbf{x}}_r = \omega = \sum_{i=1}^{n} \bar{\epsilon}_i \mathbf{z}_i \dot{q}_i$$

$$= [\bar{\epsilon}_1 \mathbf{z}_1]\ \dot{q}_1 +$$

$$[\bar{\epsilon}_2 \mathbf{z}_2]\ \dot{q}_2 +$$

$$\vdots$$

$$[\bar{\epsilon}_n \mathbf{z}_n]\ \dot{q}_n$$

$$= J_\omega\ \dot{\mathbf{q}}$$

i-th column of J_ω

Jacobian Computation

$$J = \begin{bmatrix} \frac{\partial \mathbf{x_p}}{\partial q_1} & \frac{\partial \mathbf{x_p}}{\partial q_2} & \cdots & \frac{\partial \mathbf{x_p}}{\partial q_n} \\ \bar{\epsilon}_1 \, \mathbf{z}_1 & \bar{\epsilon}_2 \, \mathbf{z}_2 & \cdots & \bar{\epsilon}_n \, \mathbf{z}_n \end{bmatrix}$$

$${}^0J = \begin{bmatrix} \frac{\partial \, {}^0\mathbf{x_p}}{\partial q_1} & \frac{\partial \, {}^0\mathbf{x_p}}{\partial q_2} & \cdots & \frac{\partial \, {}^0\mathbf{x_p}}{\partial q_n} \\ \bar{\epsilon}_1 \, {}^0\mathbf{z}_1 & \bar{\epsilon}_2 \, {}^0\mathbf{z}_2 & \cdots & \bar{\epsilon}_n \, {}^0\mathbf{z}_n \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial \, {}^0\mathbf{x_p}}{\partial q_1} & \frac{\partial \, {}^0\mathbf{x_p}}{\partial q_2} & \cdots & \frac{\partial \, {}^0\mathbf{x_p}}{\partial q_n} \\ \bar{\epsilon}_1 \, ({}^0_1R\mathbf{z}) & \bar{\epsilon}_2 \, ({}^0_2R\mathbf{z}) & \cdots & \bar{\epsilon}_n \, ({}^0_nR\mathbf{z}) \end{bmatrix}$$

Jacobian Computation

$$J = \begin{bmatrix} (\epsilon_1 \mathbf{z}_1 + \bar{\epsilon}_1 \mathbf{z}_1 \times \mathbf{p}_{1n}) & \cdots & \left(\epsilon_{n-1} \mathbf{z}_{n-1} + \bar{\epsilon}_{n-1} \mathbf{z}_{n-1} \times \mathbf{p}_{(n-1)n}\right) & \epsilon_n \mathbf{z}_n \\ \bar{\epsilon}_1 \mathbf{z}_1 & \cdots & \bar{\epsilon}_{n-1} \mathbf{z}_{n-1} & \bar{\epsilon}_n \mathbf{z}_n \end{bmatrix}$$

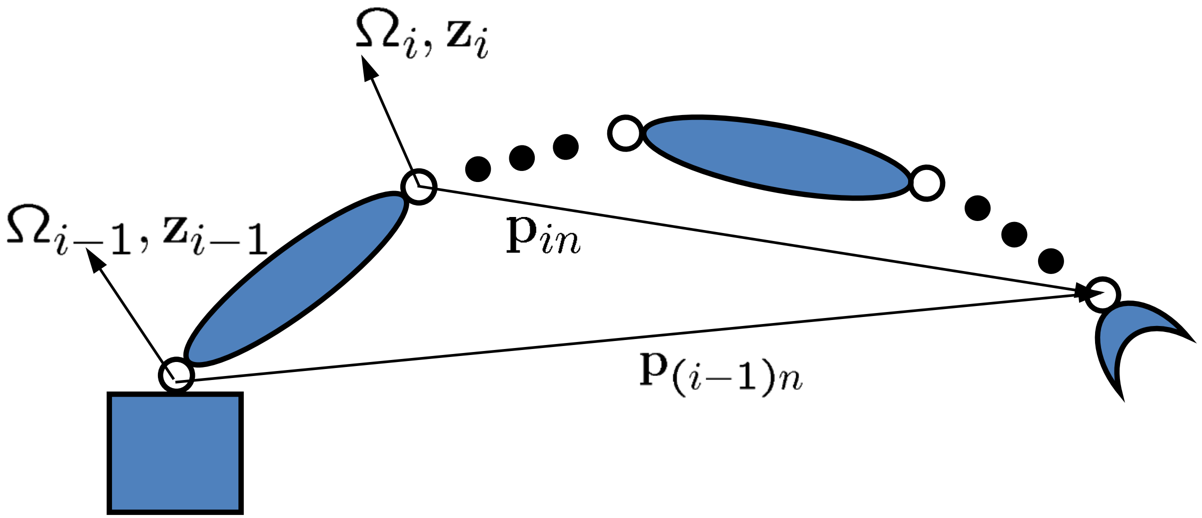
$${}^0J = \begin{bmatrix} {}^0_1R\,(\epsilon_1 \mathbf{z}_1 + \bar{\epsilon}_1 \mathbf{z}_1 \times \mathbf{p}_{1n}) & \cdots & {}^0_{n-1}R\left(\epsilon_{n-1} \mathbf{z}_{n-1} + \bar{\epsilon}_{n-1} \mathbf{z}_{n-1} \times \mathbf{p}_{(n-1)n}\right) & {}^0_nR\epsilon_n \mathbf{z}_n \\ {}^0_1R\bar{\epsilon}_1 \mathbf{z}_1 & \cdots & {}^0_{n-1}R\bar{\epsilon}_{n-1} \mathbf{z}_{n-1} & {}^0_nR\bar{\epsilon}_n \mathbf{z}_n \end{bmatrix}$$

Example: Jacobian for *n* revolute joint robot

$${}^0J_{allrev} = \begin{bmatrix} \left({}^0\mathbf{z}_1 \times {}^0\mathbf{p}_{1n}\right) & \left({}^0\mathbf{z}_2 \times {}^0\mathbf{p}_{2n}\right) & \cdots & {}^0\mathbf{z}_n \\ {}^0\mathbf{z}_1 & {}^0\mathbf{z}_2 & \cdots & {}^0\mathbf{z}_n \end{bmatrix}$$

All quantities are known from forward kinematics!!!

A Word about Reference Frames



Jacobian in Different Frames

$$ {}^{n}J = \begin{bmatrix} \frac{\partial \mathbf{x_p}}{\partial q_1} & \frac{\partial \mathbf{x_p}}{\partial q_2} & \cdots & \frac{\partial \mathbf{x_p}}{\partial q_n} \\ \bar{\epsilon}_1 \, \mathbf{z}_1 & \bar{\epsilon}_2 \, \mathbf{z}_2 & \cdots & \bar{\epsilon}_n \, \mathbf{z}_n \end{bmatrix} $$

$$ {}^{0}J = \begin{bmatrix} \frac{\partial \, {}^{0}\mathbf{x_p}}{\partial q_1} & \frac{\partial \, {}^{0}\mathbf{x_p}}{\partial q_2} & \cdots & \frac{\partial \, {}^{0}\mathbf{x_p}}{\partial q_n} \\ \bar{\epsilon}_1 \, {}^{0}\mathbf{z}_1 & \bar{\epsilon}_2 \, {}^{0}\mathbf{z}_2 & \cdots & \bar{\epsilon}_n \, {}^{0}\mathbf{z}_n \end{bmatrix} $$

So far…

- Mathematically motivated we saw the Jacobian for
  - infinitesimal displacements
  - velocities
  - forces
- Some more Jacobian definitions
- Physical understanding

Moving *J* Between Frames

$$ {}^{B}J = \begin{bmatrix} {}^{B}_{A}R & 0 \\ 0 & {}^{B}_{A}R \end{bmatrix} {}^{A}J $$