Disclaimer

These slides are intended as presentation aids for the lecture. They contain information that would otherwise be to difficult or time-consuming to reproduce on the board. But they are incomplete, not self-explanatory, and are not always used in the order they appear in this presentation. As a result, these slides should not be used as a script for this course. I recommend you take notes during class, maybe on the slides themselves. It has been shown that taking notes improves learning success.

Reading for this set of slides

- None

Please note that this set of slides is intended as support for the lecture, not as a stand-alone script. If you want to study for this course, please use these slides in conjunction with the indicated chapters in the text books. The textbooks are available online or in the TUB library (many copies that can be checked out for the entire semester. There are also some aspects of the lectures that will not be covered in the text books but can still be part of the homework or exam. For those It is important that you attend class or ask somebody about what was covered in class.

Robotics

Quaternions

TU Berlin

Oliver Brock

Problems with Rotation

- Rotation matrices are used most often, but
  - numerical error buildup
  - interpolation
  - complexity of multiplication
- Angle representations address
  - numerical error buildup
- Arbitrary axis representation introduces
  - zero rotation problem

Quaternions

- Address:
  - numerical error buildup
  - interpolation
  - complexity of multiplication (among themselves)
- But:
  - eliminate homogeneity
  - are inefficient when many points are rotated
    (then we convert them into a matrix before performing the rotation)

Problems with Rotation

- Rotation matrices are used most often, but
  – numerical error buildup
  – interpolation
  – complexity of multiplication
- Angle representations
  – address numerical error buildup
  – but have degeneracies (Gimbal problem)
- Arbitrary axis representation introduces
  – zero rotation problem

Quaternions

- Address:
  - numerical error buildup
  - interpolation
  - complexity of multiplication (among themselves)
- But:
  - eliminate homogeneity
  - are inefficient when many points are rotated
    (then we convert them into a matrix before performing the rotation)

But then what are Quaternions?

- 1844, Sir William Rowan Hamilton
- Scratched into stone of Brougham bridge
- Can be used as rotation representations
- “Four-dimensional complex numbers”
- Sometimes called Euler parameters

Parameterizing Rotations

$$(w, x, y)$$

$$w^2 + x^2 + y^2 = 1$$

$$\alpha = 2\cos^{-1} w = 2\sin^{-1} \sqrt{x^s + y^2}$$

$$(w, x, y, z)$$

$$w^2 + x^2 + y^2 + z^2 = 1$$

$$\alpha = 2\cos^{-1} w = 2\sin^{-1} \sqrt{x^s + y^2 + z^2}$$

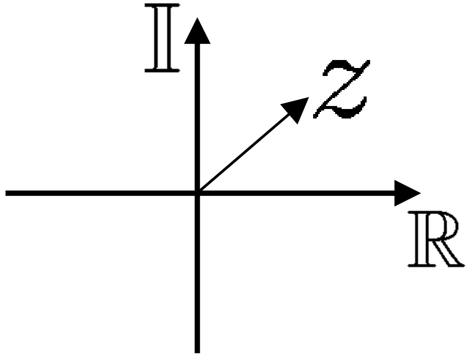
Unit Quaternion to Rotation Matrix

$$q = (a, b, c, d) = (a, \begin{pmatrix} b \\ c \\ d \end{pmatrix})$$

$$R = \begin{bmatrix} 1 - 2c^2 - 2d^2 & 2bc - 2ad & 2db + 2ac \\ 2bc + 2ad & 1 - 2b^2 - 2d^2 & 2cd - 2ab \\ 2bd - 2ac & 2cd + 2ab & 1 - 2b^2 - 2c^2 \end{bmatrix}$$

Complex Numbers

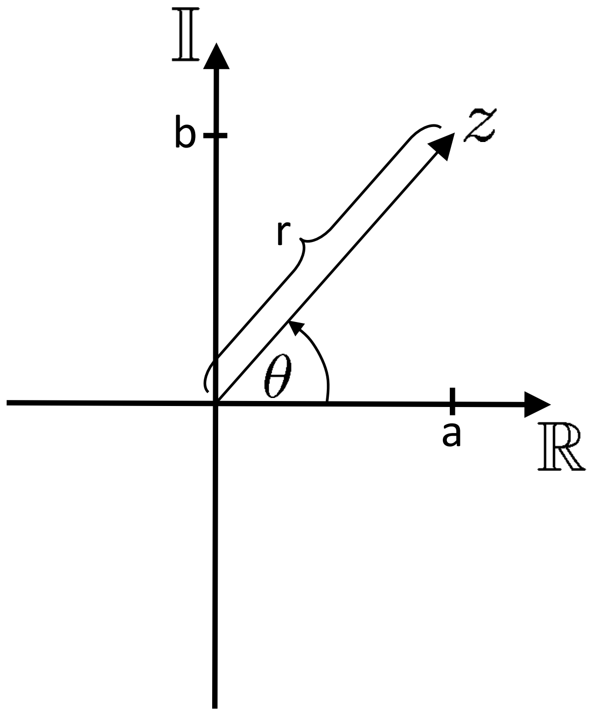
$$\sqrt{-1} = i$$



$$\begin{aligned} z &= a + bi \\ \bar{z} &= a - bi \\ \|z\| &= \sqrt{z \cdot \bar{z}} = \sqrt{a^2 + b^2} \end{aligned}$$

$$\begin{aligned} z_1 &= a_1 + b_1 i \\ z_2 &= a_2 + b_2 i \\ z_1 \cdot z_2 &= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2) i \end{aligned}$$

Sidebar I on Complex Numbers

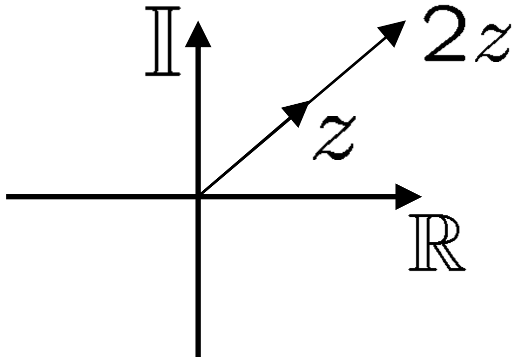


$$
\begin{aligned}
z &= a + i\, b \quad \|z\| = r \\
&= r\cos\theta + r\, i\ \sin\theta \\
&= r(cos\theta + i\ \sin\theta) \\
&= re^{i\theta}
\end{aligned}
$$

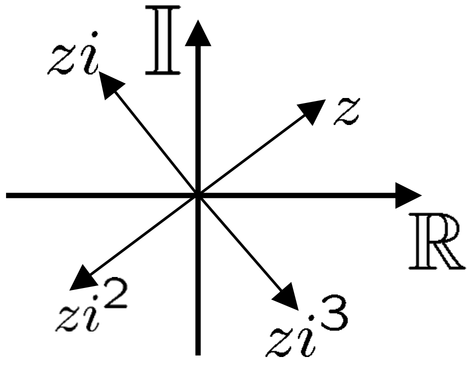
Euler's Identity

Sidebar II on Complex Numbers

$r \cdot z \quad r \in \mathbb{R}, z \in \mathbb{C}$

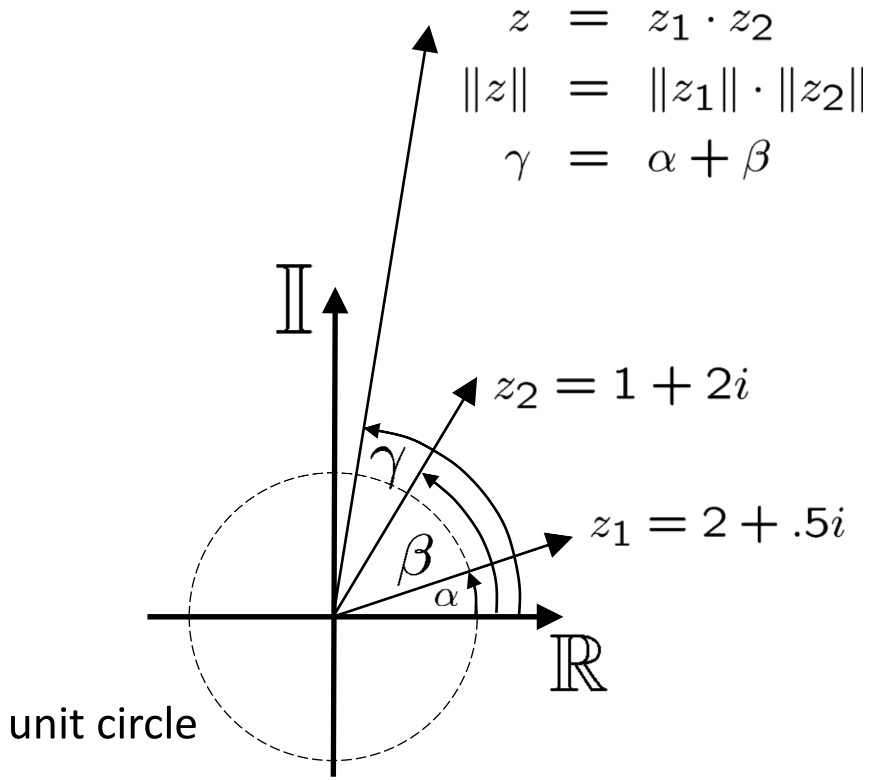


$z \cdot i \quad z \in \mathbb{C}$



Sidebar III on Complex Numbers

$$z_1 \cdot z_2 \quad z_1, z_2 \in \mathbb{C}$$

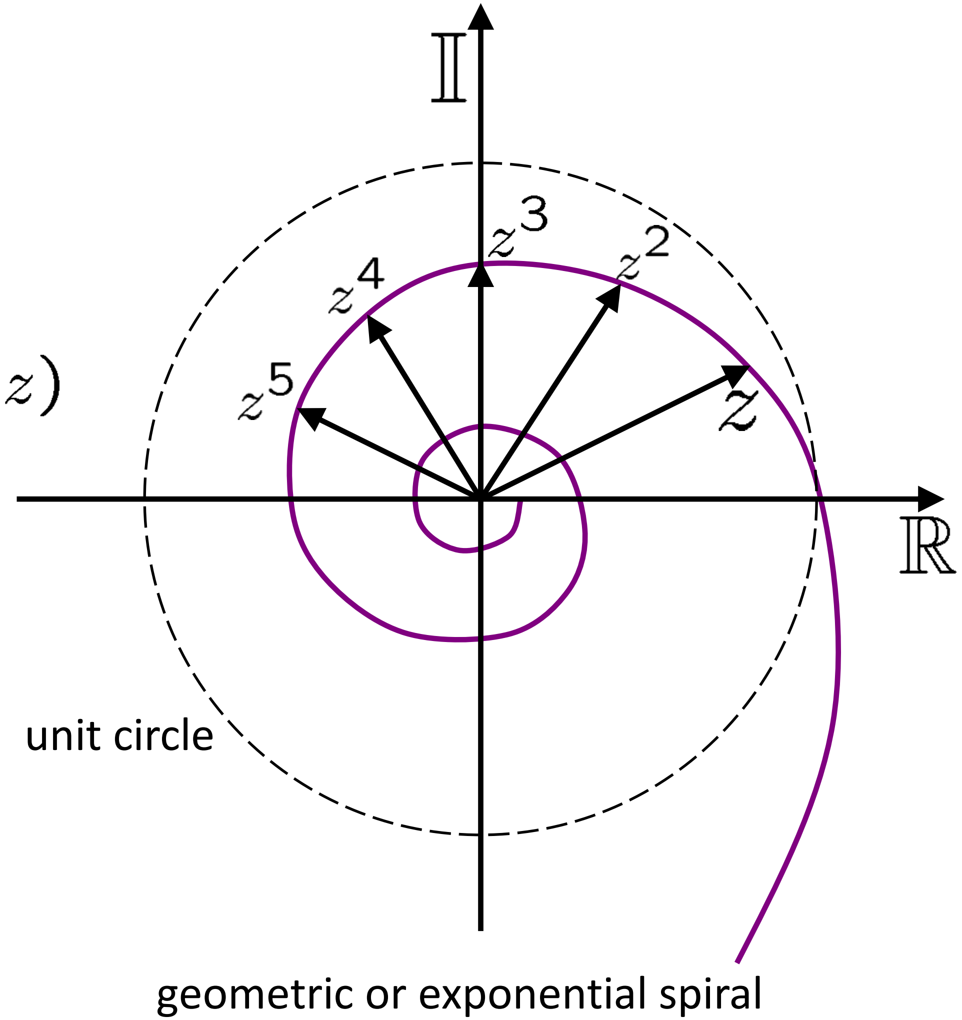


α, β, γ are called the argument of z_1, z_2, and z, respectively
$\alpha = \arg(z_1)$, $\beta = \arg(z_2)$, $\gamma = \arg(z)$

Sidebar IV on Complex Numbers

$$z^n \quad z \in \mathbb{C}, n \in \mathbb{N}$$

$$\begin{aligned} \|z^n\| &= \|z\|^n \\ \arg(z^n) &= n \cdot \arg(z) \end{aligned}$$



Quaternions

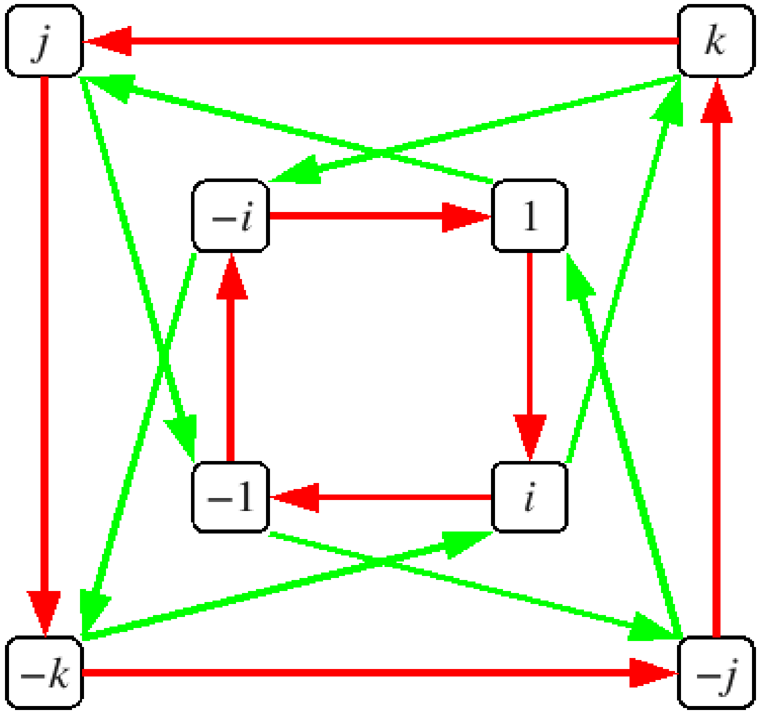
$$\mathbb{C}: \quad a + b\,\mathbf{i} \quad \rightsquigarrow \quad \mathbb{H}: \quad a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$$

$$\begin{aligned} \mathbf{i}\cdot\mathbf{i} &= -1 \\ \mathbf{j}\cdot\mathbf{j} &= -1 \\ \mathbf{k}\cdot\mathbf{k} &= -1 \end{aligned} \qquad \begin{aligned} \mathbf{i}\cdot\mathbf{j} &= -\mathbf{j}\cdot\mathbf{i} &= k \\ \mathbf{j}\cdot\mathbf{k} &= -\mathbf{k}\cdot\mathbf{j} &= i \\ \mathbf{k}\cdot\mathbf{i} &= -\mathbf{i}\cdot\mathbf{k} &= j \end{aligned} \qquad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

$$\begin{aligned} q &= a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k} \\ \bar{q} &= a - b\,\mathbf{i} - c\,\mathbf{j} - d\,\mathbf{k} \\ \|q\| &= \sqrt{q\cdot\bar{q}} = \sqrt{a^2 + b^2 + c^2 + d^2} \end{aligned}$$

Quaternions are associative but **not** commutative!

Cayley Graph



multiplication on the right by **i**

multiplication on the right by **j**

$$\begin{aligned} \mathbf{i}\cdot\mathbf{j} &= -\mathbf{j}\cdot\mathbf{i} &= k \\ \mathbf{j}\cdot\mathbf{k} &= -\mathbf{k}\cdot\mathbf{j} &= i \\ \mathbf{k}\cdot\mathbf{i} &= -\mathbf{i}\cdot\mathbf{k} &= j \end{aligned}$$

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

Representation of Quaternions

$$q = a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$$

$$q = \begin{pmatrix} b \\ c \\ d \\ a \end{pmatrix}$$

$$q = (s, \mathbf{v}) = (a, \begin{pmatrix} b \\ c \\ d \end{pmatrix})$$

$$q = (s + \mathbf{v}) = a + \begin{pmatrix} b \\ c \\ d \end{pmatrix}$$

Quaternion to Rotation Matrix

$$q = a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$$

$$\begin{pmatrix} a^2+b^2-c^2-d^2 & 2bc-2ad & 2bd+2ac \\ 2bc+2ad & a^2-b^2+c^2-d^2 & 2cd-2ab \\ 2bd-2ac & 2cd+2ab & a^2-b^2-c^2+d^2 \end{pmatrix}.$$

The opposite direction is slightly more complicated, see
[Wikipedia page on quaternions as spatial rotations]

Operations on Quaternions

Inversion

$$q^{-1} = \frac{\bar{q}}{||q||}$$

Multiplication

$$\begin{aligned} q_1 &= (s_1, \mathbf{v}_1) \quad q_2 = (s_2, \mathbf{v}_2) \\ q_1 \cdot q_2 &= (s_1 s_2 - \mathbf{v}_1 \cdot \mathbf{v}_2,\ \ s_1 \mathbf{v}_2 + s_2 \mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2) \end{aligned}$$

Operations on **Unit** Quaternions

Inversion $q^{-1} = \frac{\bar{q}}{\|q\|} \quad \rightsquigarrow \quad \text{if } \|q\| = 1 \Rightarrow \hat{q}^{-1} = \bar{\hat{q}}$

Exponential

$$
\begin{aligned}
\hat{q} &= s + \mathbf{v} = \cos\theta + \hat{u}\sin\theta \\
e^{i\theta} &= \cos\theta + i\sin\theta \\
e^{\hat{u}\theta} &= \cos\theta + \hat{u}\sin\theta
\end{aligned}
$$

Euler's Identity

Exponentiation

$$
\begin{aligned}
q^t &= (cos\theta + \hat{u}\sin\theta)^t \\
&= e^{\hat{u}t\theta} = \cos(t\theta) + \hat{u}\sin(t\theta)
\end{aligned}
$$

Logarithm $\log q = \log(\cos\theta + \hat{u}\sin\theta) = \log(e^{\hat{u}\theta}) = \hat{u}\theta$

Finally! **Unit** Quaternions as Rotations

q representing a rotation by angle θ about axis û

$$\begin{aligned} q &= (s, \mathbf{v}) \\ s &= \cos\frac{\theta}{2} \\ \mathbf{v} &= \hat{u}\sin\frac{\theta}{2} \end{aligned}$$

rotating *p* using quaternion *q*

$$\begin{aligned} P &= (0, \mathbf{p}) \\ p' &= q\;P\;q^{-1} \\ &= q\;P\;\bar{q} \end{aligned}$$

performing multiple rotations

$$\begin{aligned} & q_2\,(q_1\;P\;q_1^{-1})\;q_2^{-1} \\ = \;& (q_2\,q_1)\;P\;(q_1^{-1}\;q_2^{-1}) \\ = \;& (q_2\,q_1)\;P\;(q_2\,q_1)^{-1} \end{aligned}$$

Why are Quaternions Rotations?

Remember?
This is almost the same as it was with the complex numbers!

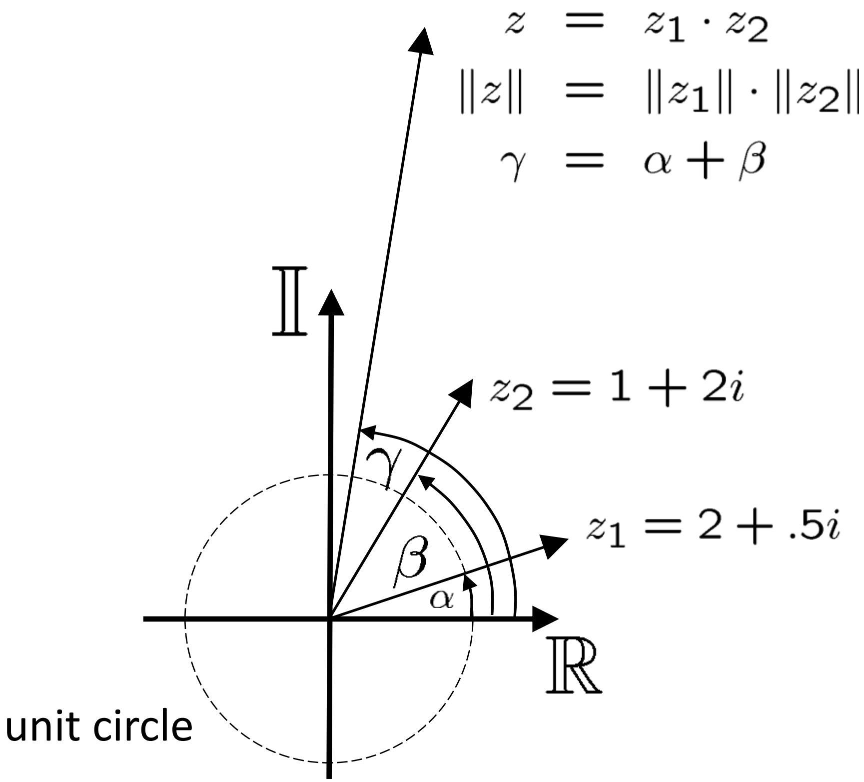
Unit complex number means no length change! This is slightly different with quaternions:

$$p' = q\, p\, \bar{q}$$

$$q = (s, \mathbf{v})$$

$$s = \cos\frac{\theta}{2}$$

$$\mathbf{v} = \hat{u}\sin\frac{\theta}{2}$$



Quaternions as Rotations

- Efficient inversion
- Efficient concatenation
- Efficient consistency check (normalize q)
- What about interpolation?

Interpolation of **Unit** Quaternions

Interpolate between rotations represented by *a* and *b*:

$$q = (b\, a^{-1})^t\, a \quad \text{with} \quad t : 0 \rightarrow 1$$

Exponentiation is easy and efficient

$$\begin{aligned} q^t &= (cos\theta + \widehat{u} \sin\theta)^t \\ &= e^{\widehat{u} t \theta} = \cos(t\theta) + \widehat{u} \sin(t\theta) \end{aligned}$$

Why Quaternions?

- Often ignored
- Powerful tool for some types of problems
  - Interpolation
  - Numerical stability
  - Efficiency
- We will stick to homogeneous transforms

Summary

- Frames
- Homogeneous Transforms
- Rotations
  - Matrix
  - Fixed angles
  - Euler angles
  - Arbitrary axis
  - Quaternions