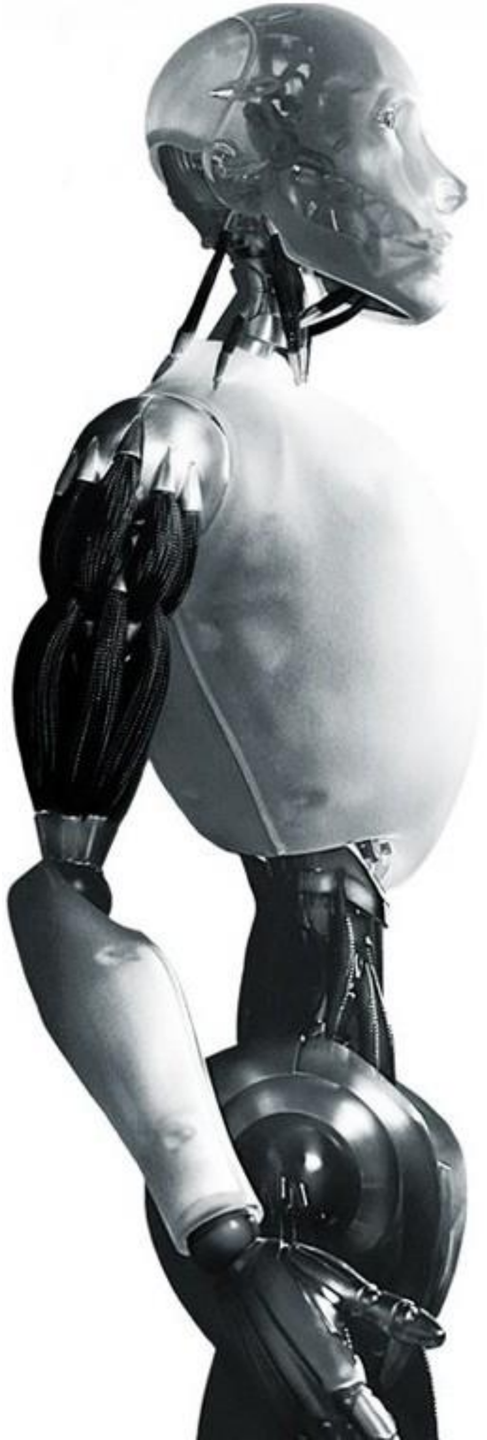
Disclaimer

These slides are intended as presentation aids for the lecture. They contain information that would otherwise be to difficult or time-consuming to reproduce on the board. But they are incomplete, not self-explanatory, and are not always used in the order they appear in this presentation. As a result, these slides should not be used as a script for this course. I recommend you take notes during class, maybe on the slides themselves. It has been shown that taking notes improves learning success.

Reading for this set of slides

- Craig – Intro to Robotics (3rd Edition)
  - 1 Introduction
  - 2 Spatial descriptions and transformations (2.1 – 2.9)
  - 3 Manipulator kinematics (3.1 – 3.6)
  - 7 Trajectory generation

Please note that this set of slides is intended as support for the lecture, not as a stand-alone script. If you want to study for this course, please use these slides in conjunction with the indicated chapters in the text books. The textbooks are available online or in the TUB library (many copies that can be checked out for the entire semester. There are also some aspects of the lectures that will not be covered in the text books but can still be part of the homework or exam. For those It is important that you attend class or ask somebody about what was covered in class.



Robotics

Achieving Critical Dampening

TU Berlin

Oliver Brock

Solving the Equation of Motion (EOM) of a Second Order Linear System

Find trajectory $x(t)$ depending on parameters m, b, k

such that the following holds at all times: $m\ddot{x} + b\dot{x} + kx = 0$

We assume

$$x = e^{st}$$

Therefore

$$\dot{x} = se^{st}$$

$$\ddot{x} = s^2 e^{st}$$

$$ms^2 e^{st} + bse^{st} + ke^{st} = 0$$

$$e^{st}(ms^2 + bs + k) = 0$$

$$ms^2 + bs + k = 0$$

Solve for s (easy), then $x = e^{st}$

Solving the Equation of Motion (EOM) of a Second Order Linear System

$$m\ddot{x} + b\dot{x} + kx = 0$$

Characteristic equation:

$$ms^2 + bs + k = 0$$

Roots (poles):

$$s_1 = -\frac{b}{2m} + \frac{\sqrt{b^2 - 4mk}}{2m}$$

$$s_2 = -\frac{b}{2m} - \frac{\sqrt{b^2 - 4mk}}{2m}$$

Solutions:

1. $b^2 > 4mk$
   real and unequal roots
   overdamped
2. $b^2 < 4mk$
   complex roots
   underdamped
3. $b^2 = 4mk$
   real and equal roots
   critically damped

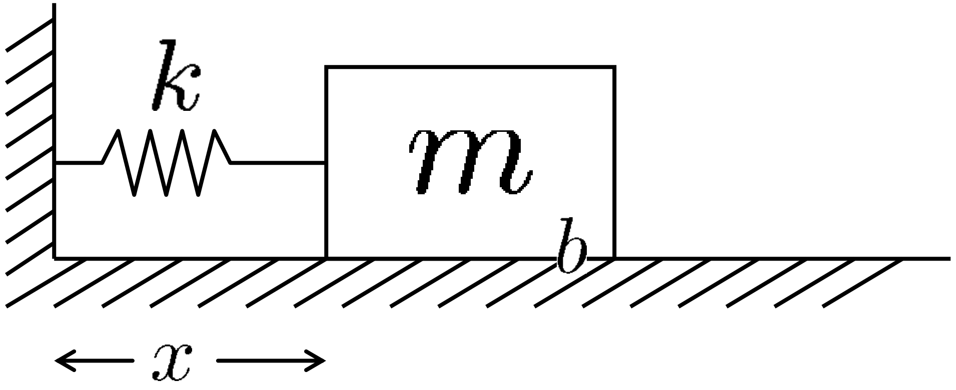
Solution: $x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$

Solution with real, unequal roots (case 1)

(overdamped)

$$x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$$

Example (real, unequal roots, case 1)



$m = 1,\ b = 5,\ k = 6,\ x(0) = -1,\ \dot{x}(0) = 0$

$$\ddot{x} + 5\dot{x} + 6x = 0$$

$$s^2 + 5s + 6 = 0$$

$$s_1 = -2, s_2 = -3$$

$$x(t) = c_1 e^{-2t} + c_2 e^{-3t}$$

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x(0) = -1$$

$$c_1 + c_2 = -1$$

$$\dot{x}(0) = 0$$

$$-2c_1 - 3c_2 = 0$$

$$c_1 = -3 \quad c_2 = 2$$

$$x(t) = -3e^{-2t} + 2e^{-3t}$$

Solution with complex roots (case 2)

(underdamped)

$$s_1 = \lambda + \mu i$$
$$s_2 = \lambda - \mu i$$

$$x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$$

Euler's formula: $e^{ix} = \cos x + i \sin x$

$$x(t) = c_1 e^{\lambda t} \cos(\mu t) + c_2 e^{\lambda t} \sin(\mu t)$$

with

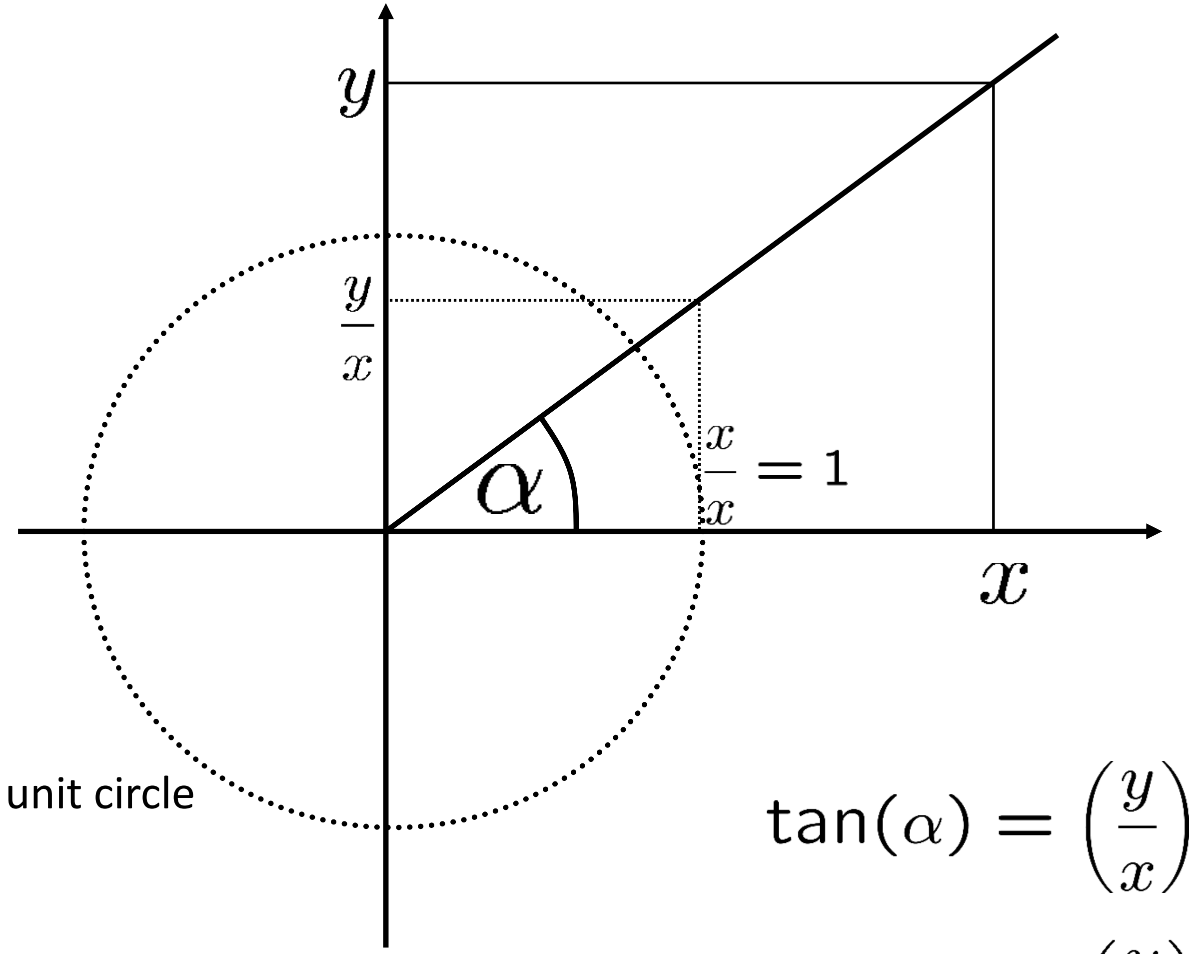
$$c_1 = r \cos \delta$$
$$c_2 = r \sin \delta$$

becomes $x(t) = r e^{\lambda t} \cos(\mu t - \delta)$

where

$$r = \sqrt{c_1^2 + c_2^2}$$
$$\delta = \operatorname{atan2}(c_2, c_1)$$

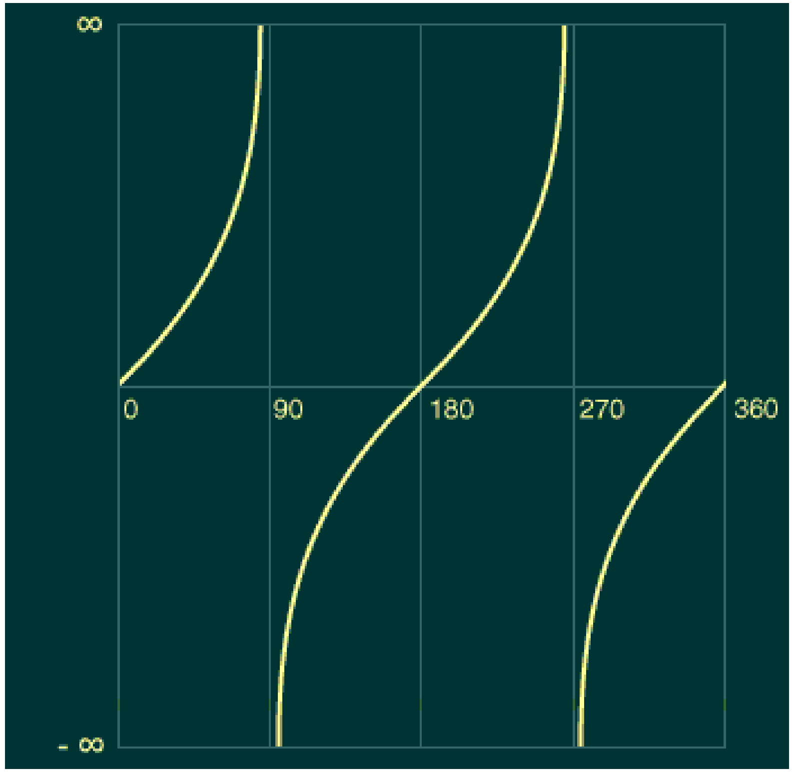
tangent and arctangent

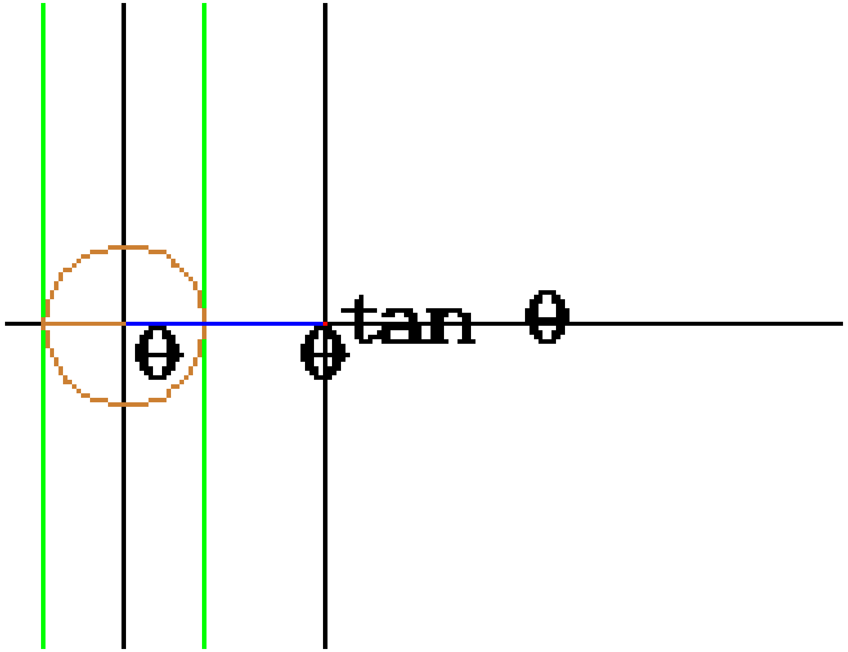


$$\tan(\alpha) = \left(\frac{y}{x}\right)$$

$$\alpha = \arctan\left(\frac{y}{x}\right)$$

tangent





atan2(y,x)

$$\text{atan2}(y,x) = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0 \\ \text{sign}(y)\left(\pi - \arctan(|\frac{y}{x}|)\right) & \text{if } x < 0 \\ \text{sign}(y)\frac{\pi}{2} & \text{if } x = 0 \text{ and } y \neq 0 \\ \text{undefined} & \text{otherwise} \end{cases}$$

Solution with real, repeated roots (case 3)

(critically damped)

Solution: $x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$

with: $s_1 = s_2 = -\frac{b}{2m} t$

$$x(t) = (c_1 + c_2 t)\ e^{-\frac{b}{2m} t}$$

L'Hôpital's rule: if $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$ or $\pm\infty$ and $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists

then $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$

$$\lim_{t \to \infty} (c_1 + c_2 t) e^{-at} = 0 \quad \text{for any } c_1, c_2, a$$

Solving the Equation of Motion (EOM) of a Second Order Linear System

$$m\ddot{x} + b\dot{x} + kx = 0$$

Characteristic equation:

$$ms^2 + bs + k = 0$$

Roots (poles):

$$s_1 = -\frac{b}{2m} + \frac{\sqrt{b^2 - 4mk}}{2m}$$

$$s_2 = -\frac{b}{2m} - \frac{\sqrt{b^2 - 4mk}}{2m}$$

Solution for cases 1 & 2:

$$x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$$

Solutions:

1. $b^2 > 4mk$
   real and unequal roots
   overdamped
2. $b^2 < 4mk$
   complex roots
   underdamped
3. $b^2 = 4mk$
   real and equal roots
   critically damped

$$b^2 = 4mk$$

$$\frac{b^2}{m^2} = 4\frac{k}{m}$$

$$\frac{b}{m} = 2\sqrt{\frac{k}{m}} = 2\omega_n$$

Damping Ratio & Natural Frequency

Original characteristic equation: $$ms^2 + bs + k = 0$$

Alternative characteristic equation: $$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

with:

$$\zeta = \frac{b}{2\sqrt{km}}$$ damping ratio

$$\omega_n = \sqrt{\frac{k}{m}}$$ natural frequency

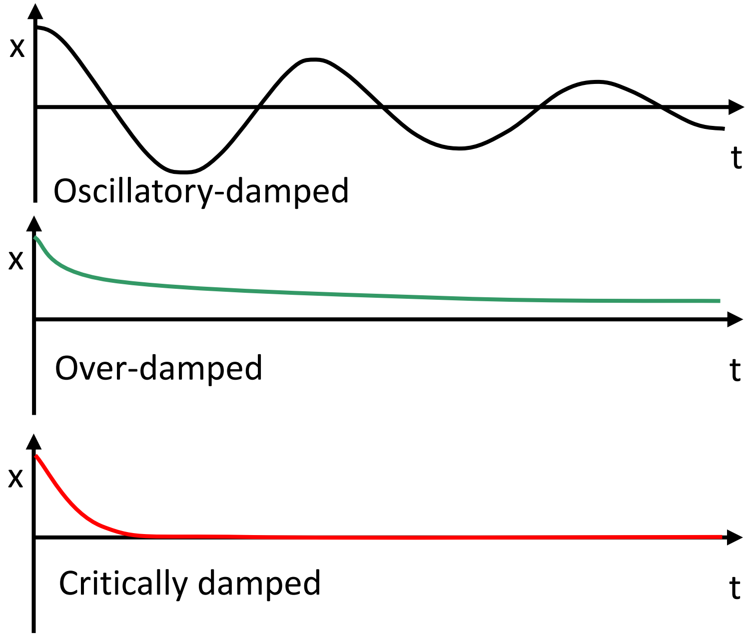
Relationship to λ and μ from first characteristic eqn.:

$$\lambda = -\zeta\omega_n$$

$$\mu = \omega_n\sqrt{1-\zeta^2}$$ damped natural frequency

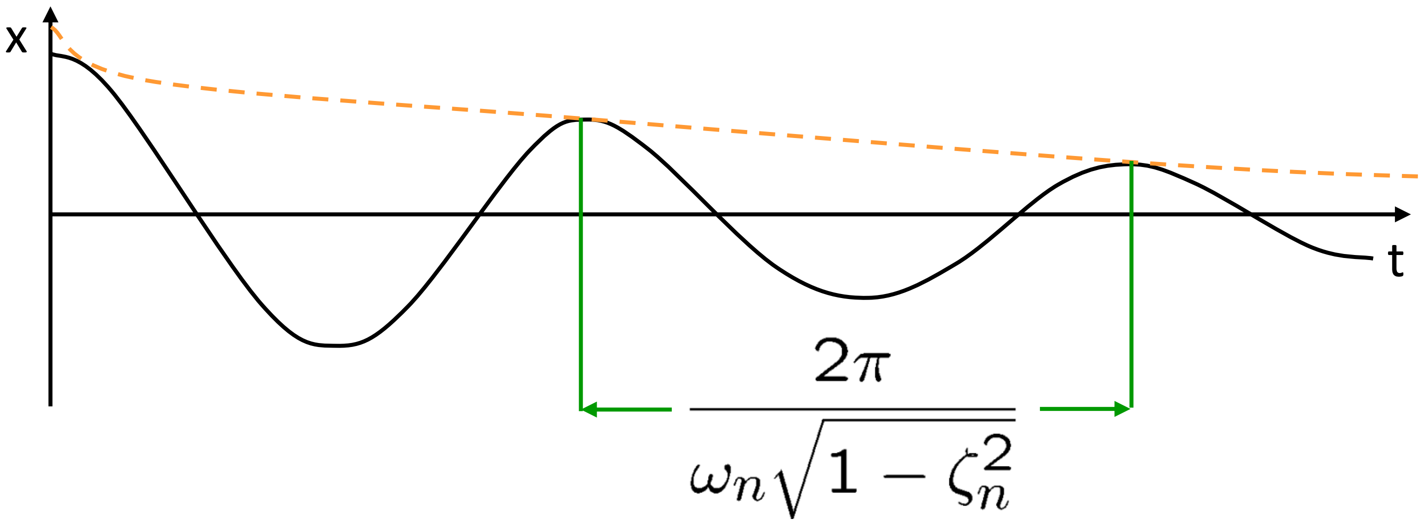
Values of ζ

- $\zeta > 1$: over-damped
- $\zeta = 1$: critically damped
- $\zeta < 1$: under-damped



Damped Natural Frequency ω

$$x(t) = ce^{-\zeta_n t} \cos\left(t\,\omega_n \sqrt{1-\zeta_n^2} + \phi\right)$$



Damped Natural Frequency: $\omega = \omega_n\sqrt{1-\zeta_n^2}$

Natural Dampening Ratio

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0 \qquad \omega_n = \sqrt{\frac{k}{m}} \qquad \frac{b}{m} = 2\,\omega_n$$

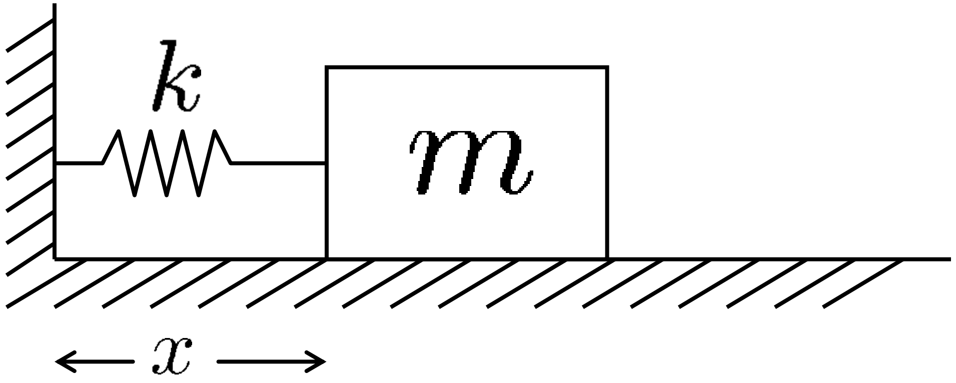
Natural dampening ratio

$$\zeta_n \triangleq \frac{b}{2\,\omega_n m} = \frac{b}{2\,\sqrt{\frac{k}{m}}\,m} = \frac{b}{2\sqrt{km}}$$

$$\ddot{x} + 2\,\zeta_n\,\omega_n\,\dot{x} + \omega_n^2 x = 0$$

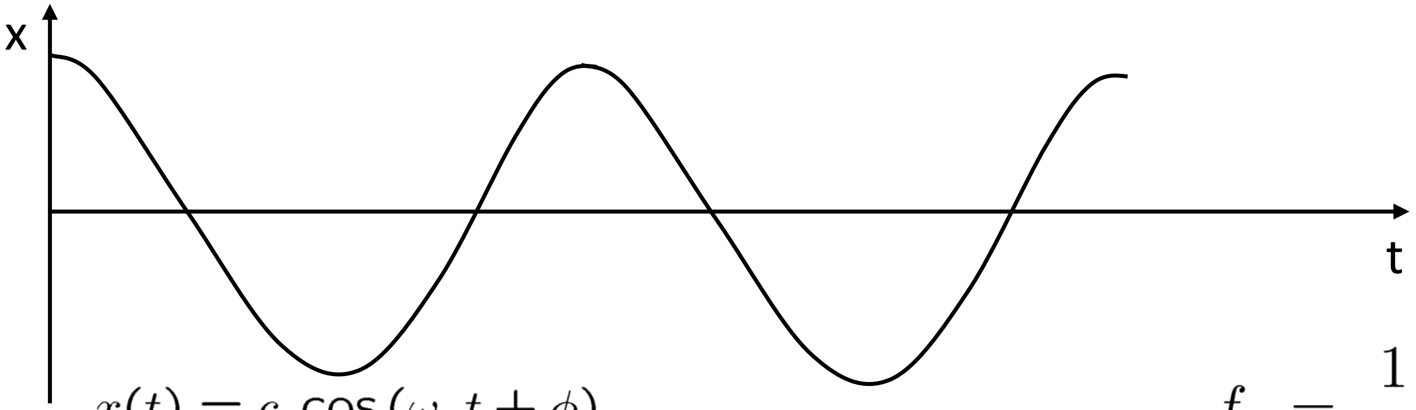
Natural Frequency

$$\omega_n = \sqrt{\frac{k}{m}}$$



$$m\ddot{x} + kx = 0$$

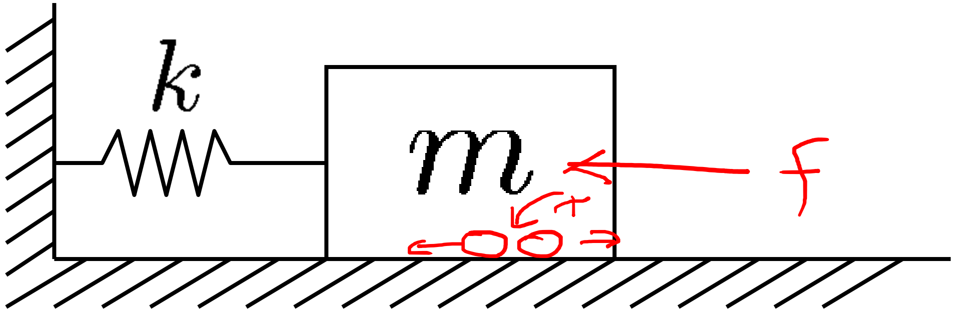
$$\ddot{x} + \omega_n^2 x = 0$$



$$x(t) = c \cos(\omega_n t + \phi)$$

$$f_n = \frac{1}{2\pi}\omega_n$$

Designing a Linear Controller



$$\boxed{f} = \boxed{m\ddot{x} + b\dot{x} + kx}$$

$$f = -k_p x - k_v \dot{x}$$

$$m\ddot{x} + b\dot{x} + kx = -k_p x - k_v \dot{x}$$

$$m\ddot{x} + (b + k_v)\dot{x} + (k + k_p)x = 0$$

physical system

control parameters

Linear Controller cont.

$$m\ddot{x} + \overbrace{(b + k_v)}^{b'}\dot{x} + \overbrace{(k + k_p)}^{k'}x = 0$$

$$m\ddot{x} + b'\dot{x} + k'x = 0$$

determines damping

closed-loop stiffness

for critical damping: $b' = 2\sqrt{mk'}$

Example: $m = b = k = 1 \qquad k' = 16$

$b' = 2\sqrt{mk'} = 2\sqrt{1 \cdot 16} = 8$ for critical damping

$$\Rightarrow \qquad k_p = 15 \qquad k_v = 7$$

Proportional Derivative (PD) Control

$$f = -k_p(x - x_d) - k_v\,\dot{x}$$

Propotional to reduce error
Derivative (velocity) to introduce dissipation

$$m\ddot{x} + k_v\dot{x} + k_p x = k_p x_d$$

$$\ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = \omega^2 x_d$$

closed-loop frequency

$$\omega^2 = \frac{k_p}{m}$$

closed-loop damping ratio

$$\zeta = \frac{k_v}{2\sqrt{k_p m}}$$

$$k_p = m\,\omega^2$$

$$k_v = m\,(2\zeta\omega)$$

1) DIFF EQN

↓

3 TYPES OF SOLN.

↓

2 EQUAL, REAL ROOT

≡ CRITICAL DAMP.

↓

PHYSICAL PROPs K_V, K_P, ω, ω_n ...