

STAT347: Generalized Linear Models

Lecture 4

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Jingshu Wang

Today's topics:

- Deviance analysis
- Model checking with the residuals
- Example: Building a GLM
- Reading: Agresti Chapters 4.4, 4.7, Faraway Chapters 8.3-8.4

Deviance analysis in GLM

```
## Call:  
## glm(formula = y ~ weight + factor(color), family = poisson(),  
##       data = Crabs)  
##  
## Deviance Residuals:  
##      Min        1Q    Median        3Q       Max  
## -2.9833  -1.9272  -0.5553   0.8646   4.8270  
##  
## Coefficients:  
##              Estimate Std. Error z value Pr(>|z|)  
## (Intercept) -0.04978  0.23315 -0.214   0.8309  
## weight       0.54618  0.06811  8.019 1.07e-15 ***  
## factor(color)2 -0.20511  0.15371 -1.334   0.1821  
## factor(color)3 -0.44980  0.17574 -2.560   0.0105 *  
## factor(color)4 -0.45205  0.20844 -2.169   0.0301 *  
## ---  
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1  
##  
## (Dispersion parameter for poisson family taken to be 1)  
##  
## Null deviance: 632.79 on 172 degrees of freedom  
## Residual deviance: 551.80 on 168 degrees of freedom  
## AIC: 917.1  
##  
## Number of Fisher Scoring iterations: 6
```

- In linear regression, we use

$$R^2 = 1 - \frac{\sum_i (y_i - \hat{y}_i)^2}{\sum_i (y_i - \bar{y})^2} = \frac{\sum_i (\hat{y}_i - \bar{y})^2}{\sum_i (y_i - \bar{y})^2}$$

To evaluate how well the model fits the data. We have an analogy in GLM, which is the deviance analysis.

$$\sum_i (y_i - \bar{y})^2$$

$$\sum_i (y_i - \hat{y}_i)^2$$

Definition of deviance

Consider density function $f(y; \theta) = e^{\frac{y\theta - b(\theta)}{a(\phi)}} f_0(y; \phi)$ at two values θ_1 and θ_2 . Measure the “distance” between two distributions:

$$\begin{aligned} D(\theta_1, \theta_2) &= 2\mathbb{E}_{\theta_1} \left\{ \log \frac{f(y; \theta_1)}{f(y; \theta_2)} \right\} = 2\mathbb{E}_{\theta_1} \{y(\theta_1 - \theta_2) - b(\theta_1) + b(\theta_2)\} / a(\phi) \\ &= 2 [\mu_1(\theta_1 - \theta_2) - b(\theta_1) + b(\theta_2)] / a(\phi) \end{aligned}$$

Remember the 1-to-1 mapping between μ and θ , we also write $D(\mu_1, \mu_2) = D(\theta_{\mu_1}, \theta_{\mu_2})$

- $D(\mu_1, \mu_2) \geq 0$ and the equality holds only when $\mu_1 = \mu_2$
- Generally, $D(\mu_1, \mu_2) \neq D(\mu_2, \mu_1)$
- KL divergence: $D(\mu_1, \mu_2)/2$
- If f is the normal density, then $D(\mu_1, \mu_2) = (\mu_1 - \mu_2)^2/\sigma^2$

Residual deviance

- Saturated model: imagine the case that we collect an infinite number of covariates, then we can perfectly fit the data and obtain $\hat{\mu}_i = y_i$ for all samples.
- For a particular sample i , Deviance between the saturated model $\hat{\mu}_i = y_i$ and another model with μ_i (corresponding canonical parameter θ_i)

$$\begin{aligned} D(\theta_1, \theta_2) &= 2\mathbb{E}_{\theta_1} \left\{ \log \frac{f(y; \theta_1)}{f(y; \theta_2)} \right\} = 2\mathbb{E}_{\theta_1} \{y(\theta_1 - \theta_2) - b(\theta_1) + b(\theta_2)\} / a(\phi) \\ &= 2 [\mu_1(\theta_1 - \theta_2) - b(\theta_1) + b(\theta_2)] / a(\phi) \end{aligned}$$

$$\begin{aligned} D(y_i, \mu_i) &= \frac{2[y_i(\theta_{y_i} - \theta_i) - b(\theta_{y_i}) + b(\theta_i)]}{a(\phi)} \\ &= -2\log[f(y_i, \theta_i)/f(y_i, \theta_{y_i})] \end{aligned}$$

- $\theta_{y_i} = (b')^{-1}(y_i)$ [As $\mu_i = b'(\theta_i)$]

Residual deviance

- Residual deviance (total deviance):
deviance between the fitted saturated model and the proposed model

$$\begin{aligned} D_+(y, \hat{\mu}) &= \sum_i D(y_i, \hat{\mu}_i) \\ &= -2 \sum_i \log \left[f(y_i, \hat{\theta}_i) / f(y_i, \theta_{y_i}) \right] \end{aligned}$$

- $\theta_{y_i} = (b')^{-1}(y_i)$
- Example: for Gaussian linear model $D_+(y, \hat{\mu}) = \sum_i (y_i - \hat{\mu}_i)^2 / \sigma^2$

Null deviance

- Null model: the linear model that only includes intercept. Thus,

$$\mu_i \equiv \mu$$

- MLE estimate of μ from the null model will be $\hat{\mu} = \bar{y} = \sum_i y_i / n$
- Null deviance: deviance between the fitted saturated model and the null model

$$\sum_i D(y_i, \bar{y})$$

- “ R^2 ” in GLM:

$$1 - \frac{D_+(y, \hat{\mu})}{\sum_i D(y_i, \bar{y})}$$

Deviance analysis for nested models

Let $\beta = \begin{pmatrix} \beta^{(1)} \\ \beta^{(2)} \end{pmatrix}$ where $\beta^{(1)} \in \mathbb{R}^{p_1}$ and $X = (X^{(1)} \quad X^{(2)})$.

We call $\mathcal{M}^{(1)}$ with

$$g(\mu_i) = X^{(1)}\beta^{(1)}$$

a nested model of the full model \mathcal{M} where

$$g(\mu_i) = X\beta.$$

- Test for whether the nested model is already enough:

$$H_0: \beta^{(2)} = 0$$

Deviance analysis for nested models

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Let $\hat{\beta}^{(1)}$ be the MLE solution of the model $\mathcal{M}^{(1)}$ and $\hat{\mu}^{(1)}$ be the corresponding estimated expectations of y in the fitted model.

Then,

$$D_+(\hat{\mu}, \hat{\mu}^{(1)}) = D_+(y, \hat{\mu}^{(1)}) - D_+(y, \hat{\mu}) = -2 \left[L(\hat{\beta}^{(1)}) - L(\hat{\beta}) \right]$$

Deviance analysis for nested models

Let $\hat{\beta}^{(1)}$ be the MLE solution of the model $\mathcal{M}^{(1)}$ and $\hat{\mu}^{(1)}$ be the corresponding estimated expectations of y in the fitted model.

$$D_+(\hat{\mu}, \hat{\mu}^{(1)}) = D_+(y, \hat{\mu}^{(1)}) - D_+(y, \hat{\mu}) = -2 \left[L(\hat{\beta}^{(1)}) - L(\hat{\beta}) \right]$$

- Deviance additivity theorem (Efron, Annals of Statistics 1978)
- This is the likelihood ratio between the full and nested models
- Likelihood ratio test:
If both p and p_1 are fixed, then asymptotically under $H_0: \beta^{(2)} = 0$

$$D_+(y, \hat{\mu}^{(1)}) - D_+(y, \hat{\mu}) \rightarrow \chi^2_{p-p_1}$$

Deviance analysis table for model comparisons

Say we partition our covariates as

$$X = (1, X_{(1)}, X_{(2)}, \dots, X_{(J)})$$

and $X_{(j)} \in \mathbb{R}^{d_j}$. We can sequentially add each partition of covariates into the model (in some pre-determined order) and understand each partition's “relative contribution” with a deviance analysis table.

- $\hat{\beta}^{(j)}$ is the MLE solution of the GLM model with covariates $X^{(j)} = (1, X_{(1)}, X_{(2)}, \dots, X_{(j)})$
- $\hat{\mu}^{(j)}$ is the corresponding vector of expectations of $y = (y_1, \dots, y_n)$ in the fitted model.

Deviance analysis table in R

Model	twice log-likelihood	residual deviance	difference	df	Compare with
$\hat{\beta}^{(0)}$ (null)	$2L(\hat{\beta}^{(0)})$	$D_+(y, \hat{\mu}^{(0)}) = \sum_i D(y_i, \bar{y})$			
$\hat{\beta}^{(1)}$	$2L(\hat{\beta}^{(1)})$	$D_+(y, \hat{\mu}^{(1)})$	$D_+(y, \hat{\mu}^{(0)}) - D_+(y, \hat{\mu}^{(1)})$	d_1	$\chi^2_{d_1}$
$\hat{\beta}^{(2)}$	$2L(\hat{\beta}^{(2)})$	$D_+(y, \hat{\mu}^{(2)})$	$D_+(y, \hat{\mu}^{(1)}) - D_+(y, \hat{\mu}^{(2)})$	d_2	$\chi^2_{d_2}$
\vdots					
$\hat{\beta}^{(J)}$	$2L(\hat{\beta}^{(J)})$	$D_+(y, \hat{\mu}^{(J)})$	$D_+(y, \hat{\mu}^{(J-1)}) - D_+(y, \hat{\mu}^{(J)})$	d_J	$\chi^2_{d_J}$

- Add variables sequentially to check if larger models are necessary
- Similar to the analysis of variable table in linear regression
- Typically the full model can not be the saturated model as df in a saturated model is too large

Deviance analysis table

- R output for the election counts example in Lecture 1

```
> result.glm <- glm(cbind(undercountNumber, votes) ~ pergore + factor(rural) + factor(econ) +
  factor(atlanta) + factor(equip), data = gavote, family = "binomial")
> anova(result.glm, test = "LRT")
```

Analysis of Deviance Table

Model: binomial, link: logit

Response: cbind(undercountNumber, votes)

Terms added sequentially (first to last)

	Df	Deviance	Resid. Df	Resid. Dev	Pr(>Chi)
NULL			158	36829	
pergore	1	5031.0	157	31798	< 2.2e-16 ***
factor(rural)	1	4197.2	156	27601	< 2.2e-16 ***
factor(econ)	2	7248.1	154	20353	< 2.2e-16 ***
factor(atlanta)	1	534.6	153	19818	< 2.2e-16 ***
factor(equip)	4	4150.5	149	15668	< 2.2e-16 ***

Signif. codes:	0	***	0.001	**	0.01 *
					0.05 .
					0.1 ' '
					1

equip: the [young metro](#), takes five values LEVER , US-CU (optical scan, central county), US-PC (optical scan, precinct county), "Paper", "PUNCH" (punch card)
econ: the economic level of the county, takes three values "middle", "poor" and "rich"
perAA: the percentage of African Americans
rural: whether the county is rural or urban
atlanta: whether the county is part of the Atlanta metropolitan area
gore: number of votes for Al Gore
bush: number of votes for George Bush
other: number of votes for other candidates
votes: total vote counts
ballots: number of ballots issued

This analysis is reliable only when model assumptions for each corresponding null hold

Model checking with the residuals

- As in the linear models, we can examine the residuals to help us check whether a model fits poor or not, and whether there are any outliers in the observations.
- Three types of residuals
 - Pearson residual

$$e_i = \frac{y_i - \hat{\mu}_i}{\sqrt{v(\hat{\mu}_i)}} \quad v(\hat{\mu}_i) = \widehat{\text{Var}}(y_i)$$

- Standardized residual (similar as in linear regression)

$$r_i = \frac{e_i}{\sqrt{1 - \hat{h}_{ii}}}$$

where h_{ii} is the i th diagonal element of the H_W defined equation (4.19) of the Agresti chapter 4.4.5.

Model checking with the residuals

- Three types of residuals

- Pearson residual

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- Deviance residual

$$d_i = \sqrt{D(y_i, \hat{\mu}_i)} \times \text{sign}(y_i - \hat{\mu}_i)$$

Residuals examples

- For Gaussian linear model
 - Pearson residual

$$e_i = \frac{y_i - \hat{\mu}_i}{\hat{\sigma}}$$

- Deviance residual

$$d_i = \frac{y_i - \hat{\mu}_i}{\hat{\sigma}} = e_i$$

Some intuition related to deviance residuals

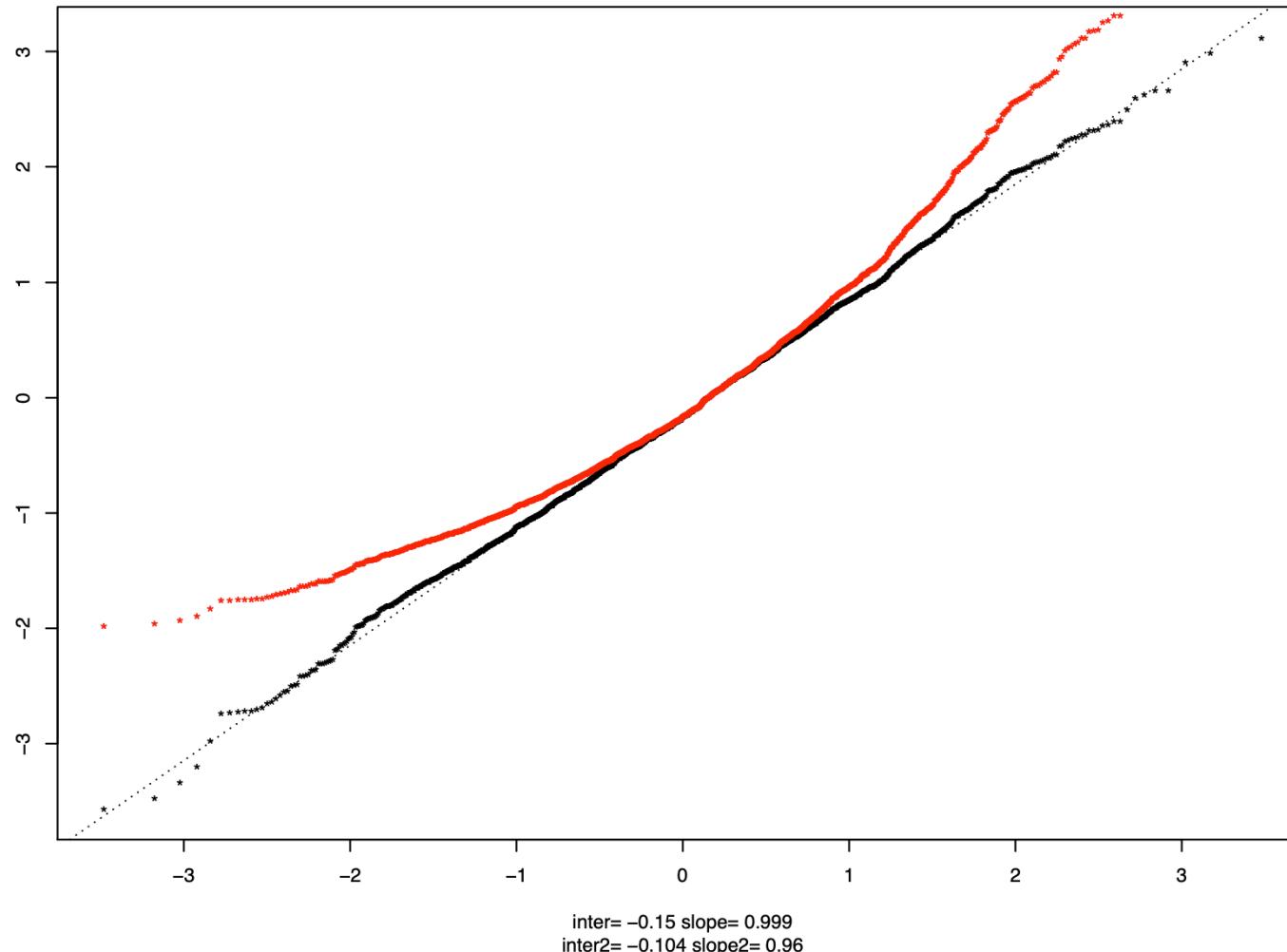
- Deviance residuals are considered more “normal” than Pearson residuals

- Consider deviance residual of i.i.d samples

$$R = \text{sign}(\bar{y} - \mu) \sqrt{D(\bar{y}, \mu)}.$$

- It has been shown that R converges to $N(0,1)$ when sample size $n \rightarrow \infty$, and has better third order accuracy than corresponding Pearson residuals
- You can check Appendix C of McCullagh and Nelder, *Generalized Linear Models* for more math details

Some intuition related to deviance residuals



$$y \sim \text{Gamma}(k = 5, \mu = 5)$$

Deviance residual

$$\text{sign}(y - \mu) \sqrt{D(y, \mu)}$$

Pearson residual

$$\frac{y - \mu}{\sqrt{V(y)}}$$

qq comparison of deviance residuals (black) with Pearson residuals (red);
Gamma distribution $k = 1, \theta = 1, n = 5$; B = 2000 simulations.

Example: Building a GLM

- Check Example2 R notebook