

# STAT347: Generalized Linear Models

## Lecture 2

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# Today's topics: Agresti Chapters 4.1-4.2

- The exponential dispersion family
- Exponential family distribution for GLM
- Likelihood score equations for parameter estimation

# The exponential dispersion family

- A random variable  $Y$  follows an exponential dispersion family distribution and has the density  $f(y; \theta, \phi)$  of the form

$$f(y; \theta, \phi) = e^{\frac{y\theta - b(\theta)}{a(\phi)}} f_0(y; \phi)$$

Terminologies:

- $\theta$ : natural or canonical parameters
- $b(\theta)$ : normalizing or cumulant function
- $\phi$ : dispersion parameter with  $a(\phi) > 0$
- Typically  $a(\phi) \equiv 1$  and  $f_0(y; \phi) = f_0(y)$ . An exception is the Gaussian distribution where  $a(\phi) = \sigma^2$
- “density” here includes the possibility of discrete atoms.
- Above definition is not the most general form of the exponential family distribution

# Some well-known examples

- Normal distribution for continuous data

$$f(y; \mu, \sigma) = e^{\frac{y\mu - \mu^2/2}{\sigma^2}} \left[ \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}} \right]$$

- $\theta = \mu, b(\theta) = \theta^2/2, a(\phi) = \sigma^2$
- Bernoulli distribution for binary data

$$\begin{aligned} f(y; p) &= p^y (1-p)^{1-y} = e^{y \log \frac{p}{1-p} + \log(1-p)} \\ &= e^{y\theta - \log[1+e^\theta]} \end{aligned}$$

- $\theta = \log(\frac{p}{1-p}), b(\theta) = \log[1 + e^\theta], a(\phi) = 1$

# Some well-known examples

- Binomial distribution for counts data

$$\begin{aligned} f(y; p, n) &= \binom{n}{y} p^y (1-p)^{n-y} = e^{y \log \frac{p}{1-p} + n \log(1-p)} \binom{n}{y} \\ &= e^{y\theta - n \log[1+e^\theta]} \binom{n}{y} \end{aligned}$$

- $\theta = \log(\frac{p}{1-p})$ ,  $b(\theta) = n \log[1 + e^\theta]$ ,  $a(\phi) = 1$

- Poisson distribution for counts data

$$f(y; \lambda) = \frac{e^{-\lambda} \lambda^y}{y!} = e^{y \log \lambda - \lambda} \frac{1}{y!} = e^{y\theta - e^\theta} \frac{1}{y!}$$

- $\theta = \log(\lambda)$ ,  $b(\theta) = e^\theta$ ,  $a(\phi) = 1$

# Moment relationships

- The exponential family has some special properties that can make our calculation easier
  - Calculate mean and variance of  $Y$

$$\mu = \mathbb{E}(y) = b'(\theta)$$

$$V_\theta = \text{Var}(y) = b''(\theta)a(\phi)$$

- Why? As  $\int f(y; \theta, \phi) = 1$ , we have

$$e^{b(\theta)/a(\phi)} = \int e^{y\theta/a(\phi)} f_0(y; \phi) dy$$

- Take derivatives with respect to  $\theta$

# Moment relationships

- The exponential family has some special properties that can make our calculation easier
  - Calculate mean and variance of  $Y$

$$\mu = \mathbb{E}(Y) = b'(\theta)$$

$$V_\theta = \text{Var}(Y) = b''(\theta)a(\phi)$$

- The above relationship also indicates that

$$\frac{\partial \mu}{\partial \theta} = \frac{\text{Var}(Y)}{a(\phi)} > 0$$

- Mapping from  $\theta$  to  $\mu$  is one to one increasing

# Exponential family distribution for GLM

- Assume that each observation  $y_i$  follows an exponential family with the canonical parameter  $\theta_i$  and a shared dispersion parameter  $\phi$
- $\mu_i = \mathbb{E}(y_i)$  is a function of  $X_i$ , so  $\theta_i$  is also a function of  $X_i$
- Canonical link function

$$g(\mu_i) = \theta_i = X_i^T \beta$$

As  $\mu_i = b'(\theta_i)$ , the link function has the form

$$g(\cdot) = (b')^{-1}(\cdot)$$

# Canonical link function examples

- Gaussian:  $\theta_i = \mu_i = X_i^T \beta$
- Binomial and Bernoulli distribution:  $\theta_i = \log\left(\frac{p_i}{1-p_i}\right) = X_i^T \beta$ 
  - Called the logit function
- Poisson distribution:  $\theta_i = \log(\mu_i) = X_i^T \beta$

# Likelihood score equations

- We now use the maximum likelihood method to solve for the GLM and estimate  $\beta$

Assume each observation  $y_i$  follows an exponential dispersion distribution

$$f(y_i; \theta_i, \phi) = e^{\frac{y_i \theta_i - b(\theta_i)}{a(\phi)}} f_0(y_i; \phi)$$

and the link function  $g(\mu_i) = X_i^T \beta$ . Then for  $n$  independent observations, the log likelihood is

$$L = \sum_i L_i = \sum_i \frac{y_i \theta_i - b(\theta_i)}{a(\phi)} + \sum_i \log f_0(y_i; \phi)$$

# Likelihood score equation for the canonical link

If  $g(\mu_i) = \theta_i = X_i^T \beta$ , then

$$L = \frac{1}{a(\phi)} \left[ \sum_j \left( \sum_i y_i x_{ij} \right) \beta_j - \sum_i b(X_i^T \beta) \right] + \sum_i \log f_0(y_i; \phi)$$

- Score equation for  $\beta_j$

$$\frac{\partial L}{\partial \beta_j} = \frac{1}{a(\phi)} \left[ \sum_i y_i x_{ij} - \sum_i b'(X_i^T \beta) x_{ij} \right] = \frac{1}{a(\phi)} \left[ \sum_i (y_i - \mu_i) x_{ij} \right] = 0$$

which is equivalent to

$$\sum_i (y_i - \mu_i) x_{ij} = 0$$

# Likelihood score equation for the canonical link

- Examples

Gaussian model:

$$\sum_i (y_i - X_i^T \beta) x_{ij} = 0$$

Poisson model:

$$\sum_i (y_i - e^{X_i^T \beta}) x_{ij} = 0$$

- $L$  is a concave function of  $\beta = (\beta_1, \dots, \beta_p)$

$$\frac{\partial}{\partial \beta} \left[ \sum_i (y_i - \mu_i) X_i \right] = - \sum_i \frac{\partial \mu_i}{\partial \theta_i} \frac{\partial \theta_i}{\partial \beta} X_i^T = - \sum_i \frac{\text{Var}(y_i)}{a(\phi)} X_i X_i^T \prec 0$$

- Easy optimization to find the solution (will discuss computation later)

# Likelihood score equation for a general link

Let  $\eta_i = g(\mu_i) = X_i^T \beta$  Then

$$\frac{\partial L_i}{\partial \beta_j} = \frac{\partial L_i}{\partial \theta_i} \frac{\partial \theta_i}{\partial \mu_i} \frac{\partial \mu_i}{\partial \eta_i} \frac{\partial \eta_i}{\partial \beta_j}$$

We have

- $\frac{\partial L_i}{\partial \theta_i} = \frac{y_i - b'(\theta_i)}{a(\phi)} = \frac{y_i - \mu_i}{a(\phi)}$
- $\frac{\partial \theta_i}{\partial \mu_i} = \frac{1}{b''(\theta_i)} = \frac{a(\phi)}{\text{Var}(y_i)}$
- $\frac{\partial \mu_i}{\partial \eta_i} = \frac{\partial \mu_i}{\partial g(\mu_i)} = \frac{1}{g'(\mu_i)}$
- $\frac{\partial \eta_i}{\partial \beta_j} = x_{ij}$

# Likelihood score equation for a general link

- The score equations can be written as

$$\frac{\partial L}{\partial \beta_j} = \sum_i \frac{(y_i - \mu_i)x_{ij}}{\text{Var}(y_i)} \frac{1}{g'(\mu_i)} = 0$$

- $\mu_i$  and  $\text{Var}(y_i)$  are both functions of  $\beta = (\beta_1, \dots, \beta_p)$
- The score equations only depend on the mean and variance of  $y_i$
- Matrix form of the score equation:

$$\dot{L}(\beta) = X^T D V^{-1} (y - \mu) = 0$$

where  $V = \text{diag}(\text{Var}(y_1), \dots, \text{Var}(y_n))$  and  $D = \text{diag}(g'(\mu_1), \dots, g'(\mu_n))^{-1}$ ,  $y = (y_1, \dots, y_n)$  and  $\mu = (\mu_1, \dots, \mu_n)$ .

- $L$  is not necessarily a concave function of  $\beta$

# Likelihood score equation for a general link

## Special cases

- If the link function is the canonical link, then  $D = \frac{1}{a(\phi)}V$ , thus the score equation becomes

$$\frac{1}{a(\phi)} X^T (y - \mu) = 0$$

the same as we derived earlier

- If we assume that  $g(\mu_i) = \mu_i = X_i^T \beta$ , then the estimating (score) equation becomes

$$\sum_i \frac{(y_i - X_i^T \beta) X_i}{\text{Var}(y_i)} = 0$$

which looks like weighted least square (difference: weights can depend on  $\beta$ )