

# STAT347: Generalized Linear Models

## Lecture 7

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# Today's topics:

- Nominal response: baseline-category logit model
  - Model setup
  - Multivariate GLM
  - Model fitting
  - Data example

# Nominal response: Alligator Food Choice

```
Alligators <- read.table("Alligators.dat", header = T)  
Alligators
```

lake <int>	size <int>	y1 <int>	y2 <int>	y3 <int>	y4 <int>	y5 <int>
1	1	23	4	2	2	8
1	0	7	0	1	3	5
2	1	5	11	1	0	3
2	0	13	8	6	1	0
3	1	5	11	2	1	5
3	0	8	7	6	3	5
4	1	16	19	1	2	3
4	0	17	1	0	1	3

8 rows

- y1: Fish, y2: Invertebrate, y3: Reptile, y4: Bird, y5: Other
- lake: 1 = Hancock, 2 = Ocklawaha, 3 = Trafford, 4 = George
- size: 1 = Alligator size > 2.3m, 0 = Alligator size <= 2.3m

This is a dataset with grouped multinomial responses

# Ordinal response: Mental impairment

Mental impairment is ordinal: 1 = well, 2 = mild symptom formation, 3 = moderate symptom formation, 4 = impaired

```
Mental <- read.table("Mental.dat", header = T)
Mental
```

impair <int>	ses <int>	life <int>
1	1	1
1	1	9
1	1	0
1	1	4
1	1	3
1	0	2
1	0	1
1	1	3
1	1	3
1	1	7

1-10 of 40 rows

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```
#table(Mental$impair)
```

- life: life events index, measuring the number of severity of important life events happen in the past three years
- ses: socioeconomic status 1 = high 0 = low

# Multinomial response variables

- Nominal response:  
 $c$  categories without orders. For instance, the response can be the answer to: which major does an undergraduate student choose?
- Ordinal response:  
categories with orders: not satisfied, satisfied, very satisfied

How to model their relationship with the covariates?

A natural choice is the multinomial distribution for the response:

$$y_i = (y_{i1}, y_{i2}, \dots, y_{ic}) \sim \text{Multinomial}(n_i, p_i = (p_{i1}, p_{i2}, \dots, p_{ic}))$$

where  $c$  is the total number of choices.  $y_{ij} = 1$  for sample  $i$  choose level  $j$  and  $y_{ij'} = 0$  for all  $j' \neq j$ .

# Nominal response: model setup

- We can build a Binary GLM model for each pair of categories.
- Select a base category (say category  $c$ )
- We can build a binary GLM for each of  $1, 2, \dots, c - 1$  categories to compare with category  $c$ 
  - Basically, we assume

$$\frac{p_{ik}}{p_{ik} + p_{ic}} = F(X_i^T \beta_k)$$

- Not every  $F$  is good, we need to make our modeling assumption irrelevant to the choice of base category

# Nominal response: choose a good $F$

$$\frac{p_{ik}}{p_{ik} + p_{ic}} = F(X_i^T \beta_k)$$

Not every  $F$  is good, we need to make our modeling assumption irrelevant to the choice of base category

- If we switch to another base category  $c'$ , for any  $k' \neq c'$ , we can find some  $\tilde{\beta}_{k'}$

$$\frac{p_{ik'}}{p_{ik'} + p_{ic'}} = F(X_i^T \tilde{\beta}_{k'})$$

# Nominal response: Baseline-Category logit model

- If  $F$  corresponds to the logit link, then we have

$$\frac{p_{ik}}{p_{ic}} = e^{X_i^T \beta_k}$$

This is called the baseline-category logit model.

- for  $k \neq c$ ,  $\tilde{\beta}_k = \beta_k - \beta_c$ .

$$\frac{p_{ik}}{p_{ic'}} = e^{X_i^T (\beta_k - \beta_{c'})}$$

- for  $k = c$ ,  $\tilde{\beta}_c = -\beta_{c'}$  ( $\beta_c = 0$ )

Under the baseline-category logit model, we have

$$p_{ik} = \frac{e^{X_i^T \beta_k}}{1 + \sum_{h=1}^{c-1} e^{X_i^T \beta_h}}$$

# Multivariate GLM

Treating each pair as a separate logistic regression, we can get the asymptotic distribution of each  $\hat{\beta}_k$ .

- The  $\hat{\beta}_k$  for  $k = 1, 2, \dots, c - 1$  categories are not independent (as  $y_{ik}$  are not across  $k$  and all models share the same base category data)
- The estimation of  $\hat{\beta}_k$  may not be efficient ignoring other categories
- How to calculate the distribution of some function  $h(\hat{\beta}_1, \dots, \hat{\beta}_{c-1})$  if needed? (For example, we may want to know the distribution of  $\hat{p}_{i1} - \hat{p}_{i2}$ )
- We can write down the joint likelihood of across all categories

# Multivariate GLM

Generalize the univariate GLM to a multivariate GLM where

$$y_i = (y_{i1}, \dots, y_{i,c-1})$$

- Assume that  $y_i$  follows a multivariate exponential dispersion family distribution

$$f(y_i; \theta_i) = e^{\frac{y_i^T \theta_i - b(\theta_i)}{a(\phi)}} f_0(y_i; \phi)$$

where  $\theta_i = (\theta_{i1}, \dots, \theta_{i,c-1})$ .

- For the multinomial response:

- We drop  $y_{ic}$  as  $y_{ic} = n_i - \sum_{k \neq c} y_{ik}$
- The mean vector is  $\mu_i = (\mu_{i1}, \dots, \mu_{i,c-1}) = (n_i p_{i1}, \dots, n_i p_{i,c-1})$

# Multivariate GLM

- We drop  $y_{ic}$  as  $y_{ic} = n_i - \sum_{k \neq c} y_{ik}$
- The mean vector is  $\mu_i = (\mu_{i1}, \dots, \mu_{i,c-1}) = (n_i p_{i1}, \dots, n_i p_{i,c-1})$
- The link function is  $g(\mu_i) = \mathbf{X}_i \beta$  where

$$\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{c-1} \end{pmatrix}, \mathbf{X}_i = \begin{pmatrix} \mathbf{X}_i^T & 0 & \cdots & 0 \\ 0 & \mathbf{X}_i^T & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & \mathbf{X}_i^T \end{pmatrix}$$

- The form of the link function is  $g_k(\mu_i) = \log [\mu_{ik}/(n_i - \sum_{k'} \mu_{ik'})]$

# Fitting baseline-category logit model

Consider the ungrouped data format and let  $N$  be the total sample size

- The joint log-likelihood for the multivariate GLM is

$$\begin{aligned} L(\beta; y) &= \log \left[ \prod_{i=1}^N \left( \prod_{k=1}^c p_{ik}^{y_{ik}} \right) \right] \\ &= \sum_{i=1}^N \left\{ \sum_{k=1}^{c-1} y_{ik} \log \frac{p_{ik}}{p_{ic}} + \log p_{ic} \right\} \\ &= \sum_{i=1}^N \left\{ \sum_{k=1}^{c-1} y_{ik} X_i^T \beta_k - \log \left( 1 + \sum_{h=1}^{c-1} e^{X_i^T \beta_h} \right) \right\} \\ &= \sum_{k=1}^{c-1} \left\{ \sum_{j=1}^p \beta_{kj} \left( \sum_{i=1}^N y_{ik} x_{ij} \right) \right\} - \sum_{i=1}^N \left\{ \log \left( 1 + \sum_{h=1}^{c-1} e^{X_i^T \beta_h} \right) \right\} \end{aligned}$$

# Fitting baseline-category logit model

The score equations are

$$\frac{\partial L}{\partial \beta_{kj}} = \sum_{i=1}^N y_{ik} x_{ij} - \sum_{i=1}^N \frac{e^{X_i^T \beta_k} x_{ij}}{1 + \sum_{h=1}^{c-1} e^{X_i^T \beta_h}} = \sum_{i=1}^N (y_{ik} - p_{ik}) x_{ij} = 0$$

which have the same forms as we saw before for canonical link.

# R data example for nominal response

- Check Example 4\_1 R notebook