

Lecture 5

Neyman's repeated sampling approach for completely randomized experiments

Outline

- Neyman's repeated sampling approach
 - Motivation
 - Variance calculation
 - CI and hypothesis testing
- Fisher VS Neyman
- Suggested reading: Imbens and Rubin Chapter 6, Peng's book Chapter 4

Motivation

- Limitations of the Fisher's randomization inference
 - Do not allow heterogeneity of causal effects across individuals
 - Do not have inference for the population
- Completely randomized experiments: can we use two sample test?
- Neyman's approach
 - Allow heterogeneity of causal effects across individuals
 - Focus on estimation and inference for the **average treatment effect**: either just for the N samples or for the whole population (PATE)
 - Repeated sampling: randomization distribution of assignment vector \mathbf{W} , and sampling generated by drawing from the population units if inferring PATE

Example: Duflo-Hanna-Ryan teacher-incentive experiment

- Conducted in rural India, designed to study the effect of financial incentives on teacher performance
- In total $N = 107$ single-teacher schools, 53 schools are randomly chosen and are given a salary that's tied to their attendance
- One outcome: open (proportion of times the school is open during a random visit)

Table 6.1. *Summary Statistics for Duflo-Hanna-Ryan Teacher-Incentive Observed Data*

Variable		Control ($N_c = 54$)		Treated ($N_t = 53$)		Min	Max
		Average	(S.D.)	Average	(S.D.)		
Pre-treatment	pctprewritten	0.19	(0.19)	0.16	(0.17)	0.00	0.67
Post-treatment	open	0.58	(0.19)	0.80	(0.13)	0.00	1.00
	pctpostwritten	0.47	(0.19)	0.52	(0.23)	0.05	0.92
	written	0.92	(0.45)	1.09	(0.42)	0.07	2.22
	written_all	0.46	(0.32)	0.60	(0.39)	0.04	1.43

Example: Duflo-Hanna-Ryan teacher-incentive experiment

Standard two-sample test:

$$\hat{t}^{\text{dif}} = 0.80 - 0.58 = 0.22$$

$$s.e. = \sqrt{\frac{0.19^2}{54} + \frac{0.13^2}{53}} \approx 0.032$$

$$95\% \text{ CI: } [0.22 - 1.96 * 0.032, 0.22 + 1.96 * 0.032]$$

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- This calculation ignores the randomization procedure of the treatment assignment
- Can we justify this standard difference-in-means analysis from the randomization perspective?

Estimation of the sample average treatment effect

- Causal estimand: **SATE** = $\tau_{fs} = \frac{1}{N} \sum_{i=1}^N \{Y_i(1) - Y_i(0)\}$ for the sampled N units
- Difference-in-means estimator:

$$\hat{\tau}^{\text{dif}} = \bar{Y}_t^{\text{obs}} - \bar{Y}_c^{\text{obs}}$$

where $\bar{Y}_c^{\text{obs}} = \frac{1}{N_c} \sum_{i:W_i=0} Y_i^{\text{obs}}$ and $\bar{Y}_t^{\text{obs}} = \frac{1}{N_t} \sum_{i:W_i=1} Y_i^{\text{obs}}$.

- Under complete randomization (random W) and treat the potential outcomes as fixed (fixed $\mathbf{Y}(0) = \{Y_i(0), i = 1, \dots, N\}$ and $\mathbf{Y}(1) = \{Y_i(1), i = 1, \dots, N\}$), this estimator is unbiased

$$\begin{aligned} \mathbb{E}_W \left[\hat{\tau}^{\text{dif}} \mid \mathbf{Y}(0), \mathbf{Y}(1) \right] &= \frac{1}{N} \sum_{i=1}^N \left(\frac{\mathbb{E}_W[W_i] \cdot Y_i(1)}{N_t/N} - \frac{\mathbb{E}_W[1 - W_i] \cdot Y_i(0)}{N_c/N} \right) \\ &= \frac{1}{N} \sum_{i=1}^N (Y_i(1) - Y_i(0)) = \tau_{fs}. \end{aligned}$$

First, we can re-write $\hat{\tau}^{dif}$:

$$\begin{aligned}
\hat{\tau}^{dif} &= \overline{Y}_t^{obs} - \overline{Y}_c^{obs} \\
&= \frac{1}{N_t} \sum_{i:W_i=1} Y_i^{obs} - \frac{1}{N_c} \sum_{i:W_i=0} Y_i^{obs} \\
&= \frac{1}{N_t} \sum_{i:W_i=1} W_i Y_i(1) - \frac{1}{N_c} \sum_{i:W_i=0} (1 - W_i) Y_i(0) \\
&= \frac{1}{N} \sum_{i=1}^N \left(\frac{N}{N_t} W_i Y_i(1) - \frac{N}{N_c} (1 - W_i) Y_i(0) \right)
\end{aligned}$$

And now we can take an expectation:

$$\begin{aligned}
E(\hat{\tau}^{dif}) &= \frac{1}{N} \sum_{i=1}^N \left(\frac{N}{N_t} E(W_i) Y_i(1) - \frac{N}{N_c} E(1 - W_i) Y_i(0) \right) \\
&= \frac{1}{N} \sum_{i=1}^N \left(\frac{N}{N_t} \frac{N_t}{N} Y_i(1) - \frac{N}{N_c} \frac{N_c}{N} Y_i(0) \right) \\
&= \frac{1}{N} \sum_{i=1}^N (Y_i(1) - Y_i(0)) \\
&= \tau
\end{aligned}$$

Calculate the variance of the estimator

- Causal estimand: **SATE** = $\tau_{fs} = \frac{1}{N} \sum_{i=1}^N \{Y_i(1) - Y_i(0)\}$ for the sampled N units
- Difference-in-means estimator: $\hat{\tau}^{\text{dif}} = \bar{Y}_t^{\text{obs}} - \bar{Y}_c^{\text{obs}}$
- Under complete randomization and fixed potential outcomes, we can also calculate the variance of $\hat{\tau}^{\text{dif}}$ (if you are interested in the proof, see Appendix A of Chapter 6 in Rubin's book or Section 4.3 in Peng's book)

$$V_W[\hat{\tau}^{\text{dif}} | \mathbf{Y}(0), \mathbf{Y}(1)] = \frac{S_c^2}{N_c} + \frac{S_t^2}{N_t} - \frac{S_{ct}^2}{N}$$

where

$$S_c^2 = \frac{1}{N-1} \sum_{i=1}^N (Y_i(0) - \bar{Y}(0))^2, \quad \text{and} \quad S_t^2 = \frac{1}{N-1} \sum_{i=1}^N (Y_i(1) - \bar{Y}(1))^2$$

Sample variance
of $Y_i(0)$ and $Y_i(1)$

$$\begin{aligned} S_{ct}^2 &= \frac{1}{N-1} \sum_{i=1}^N (Y_i(1) - Y_i(0) - (\bar{Y}(1) - \bar{Y}(0)))^2 \\ &= \frac{1}{N-1} \sum_{i=1}^N (Y_i(1) - Y_i(0) - \tau_{fs})^2. \end{aligned}$$

Sample variance
of the unit-level
treatment effects

Some explanation of the variance

$$V_W[\hat{\tau}^{\text{dif}} | \mathbf{Y}(0), \mathbf{Y}(1)] = \frac{S_c^2}{N_c} + \frac{S_t^2}{N_t} - \frac{S_{ct}^2}{N}$$

- Where does the randomness come from? Treatment assignment
 - Potential outcomes are fixed (conditioned on)
- How is it different from the classical variance formula of $\hat{\tau}^{\text{dif}}$?
 - Classical formula treats Y_i i.i.d. within group and the group indicators W_i fixed
- How is it different from the setting in Fisher's randomization test?
 - The formula allows for arbitrary treatment effect sizes and heterogeneity
 - This formula only works for completely randomized experiment
- How to estimate these quantities with observed variables?

Conservative approximation of the variance of the estimator

- $\mathbb{V}_W[\hat{\tau}^{\text{dif}} | \mathbf{Y}(0), \mathbf{Y}(1)] = \frac{S_c^2}{N_c} + \frac{S_t^2}{N_t} - \frac{S_{ct}^2}{N}$

where

$$S_c^2 = \frac{1}{N-1} \sum_{i=1}^N (Y_i(0) - \bar{Y}(0))^2, \quad \text{and} \quad S_t^2 = \frac{1}{N-1} \sum_{i=1}^N (Y_i(1) - \bar{Y}(1))^2$$

Sample variance of $Y_i(0)$ and $Y_i(1)$

$$S_{ct}^2 = \frac{1}{N-1} \sum_{i=1}^N (Y_i(1) - Y_i(0) - (\bar{Y}(1) - \bar{Y}(0)))^2$$

Sample variance of the unit-level treatment effects

$$= \frac{1}{N-1} \sum_{i=1}^N (Y_i(1) - Y_i(0) - \tau_{fs})^2.$$

- Estimate S_c^2 and S_t^2 by sample variance of observed outcomes

$$S_c^2 = \frac{\sum_{i:W_i=0} (Y_i^{\text{obs}} - \bar{Y}_c^{\text{obs}})^2}{N_c - 1}, \quad S_t^2 = \frac{\sum_{i:W_i=1} (Y_i^{\text{obs}} - \bar{Y}_t^{\text{obs}})^2}{N_t - 1}$$

- S_{ct}^2 is not **identifiable**

- No heterogeneity of treatment effects across individuals $S_{ct}^2 = 0$
- In general, $S_{ct}^2 \geq 0$ though the exact value is unknown

Conservative approximation of the variance of the estimator

- $\mathbb{V}_W[\hat{\tau}^{\text{dif}} | \mathbf{Y}(0), \mathbf{Y}(1)] = \frac{S_c^2}{N_c} + \frac{S_t^2}{N_t} - \frac{S_{ct}^2}{N}$

where

$$S_c^2 = \frac{1}{N-1} \sum_{i=1}^N (Y_i(0) - \bar{Y}(0))^2, \quad \text{and} \quad S_t^2 = \frac{1}{N-1} \sum_{i=1}^N (Y_i(1) - \bar{Y}(1))^2$$

Sample variance
of $Y_i(0)$ and $Y_i(1)$

$$\begin{aligned} S_{ct}^2 &= \frac{1}{N-1} \sum_{i=1}^N (Y_i(1) - Y_i(0) - (\bar{Y}(1) - \bar{Y}(0)))^2 \\ &= \frac{1}{N-1} \sum_{i=1}^N (Y_i(1) - Y_i(0) - \tau_{fs})^2. \end{aligned}$$

Sample variance
of the unit-level
treatment effects

- A conservative estimator of $\text{Var}_W[\hat{\tau}^{\text{dif}} | \mathbf{Y}(0), \mathbf{Y}(1)]$

$$\mathbb{V}_W[\hat{\tau}^{\text{dif}} | \mathbf{Y}(0), \mathbf{Y}(1)] \leq \frac{S_c^2}{N_c} + \frac{S_t^2}{N_t} = \mathbb{E}_W \left[\frac{S_c^2}{N_c} + \frac{S_t^2}{N_t} \mid \mathbf{Y}(0), \mathbf{Y}(1) \right]$$

Neyman's estimator of the variance, same as
s.e. on slide 5

Estimation of the population average treatment effect

- Causal estimand: **PATE** = $\tau_{sp} = \mathbb{E}(Y_i(1) - Y_i(0)) = \mathbb{E}(\text{SATE}) = \mathbb{E}(\tau_{fs})$
- We assume that $(Y_i(0), Y_i(1))$ are jointly i.i.d samples from a super population with variance σ_c^2 and σ_s^2
- We still use difference-in-means estimator:

$$\hat{\tau}^{\text{dif}} = \bar{Y}_t^{\text{obs}} - \bar{Y}_c^{\text{obs}}$$

- $\hat{\tau}^{\text{dif}}$ is still unbiased for τ_{sp} : $\mathbb{E}(\hat{\tau}^{\text{dif}}) = \mathbb{E}(\mathbb{E}_W[\hat{\tau}^{\text{dif}} | \mathbf{Y}(0), \mathbf{Y}(1)]) = \mathbb{E}(\tau_{fs}) = \tau_{sp}$
- The variance of $\hat{\tau}^{\text{dif}}$ (variance decomposition formula):
 - Check Wikipedia if you do not know the variance decomposition formula
https://en.wikipedia.org/wiki/Law_of_total_variance

$$\mathbb{V}(\hat{\tau}^{\text{dif}}) = \mathbb{E}(\mathbb{V}_W[\hat{\tau}^{\text{dif}} | \mathbf{Y}(0), \mathbf{Y}(1)]) + \mathbb{V}(\mathbb{E}_W[\hat{\tau}^{\text{dif}} | \mathbf{Y}(0), \mathbf{Y}(1)])$$

Variance calculation for the population

$$\mathbb{V}(\hat{\tau}^{\text{dif}}) = \mathbb{E}(\mathbb{V}_W[\hat{\tau}^{\text{dif}} | \mathbf{Y}(0), \mathbf{Y}(1)]) + \mathbb{V}(\mathbb{E}_W[\hat{\tau}^{\text{dif}} | \mathbf{Y}(0), \mathbf{Y}(1)])$$

- $\mathbb{V}_W[\hat{\tau}^{\text{dif}} | \mathbf{Y}(0), \mathbf{Y}(1)] = \frac{S_c^2}{N_c} + \frac{S_t^2}{N_t} - \frac{S_{ct}^2}{N}$, with
 $\mathbb{E}(S_c^2) = \sigma_c^2, \mathbb{E}(S_t^2) = \sigma_t^2, \mathbb{E}(S_{ct}^2) = \mathbb{V}(Y_i(1) - Y_i(0))$
- $\mathbb{V}(\mathbb{E}_W[\hat{\tau}^{\text{dif}} | \mathbf{Y}(0), \mathbf{Y}(1)]) = \mathbb{V}(\tau_{\text{fs}}) = \mathbb{V}\left(\frac{1}{N} \sum_{i=1}^N \{Y_i(1) - Y_i(0)\}\right) = \frac{1}{N} \mathbb{V}(Y_i(1) - Y_i(0))$
- So $\mathbb{V}_W(\hat{\tau}^{\text{dif}}) = \frac{\sigma_c^2}{N_c} + \frac{\sigma_t^2}{N_t}$ **exactly the same as in two-sample testing**
 - In two-sample testing, we assume that observed outcome Y_i are i.i.d. in the treatment group and Y_i are i.i.d. in the control group
 - Under complete randomization, $Y_i = Y_i(W_i)$ are not i.i.d. even with the treatment/control group because W_i are negatively correlated across i

Construct confidence intervals for τ_{fs} or τ_{sp}

- We have the same estimator $\hat{\tau}^{\text{dif}}$ and the same variance approximation of $\hat{\tau}^{\text{dif}}$

$$\widehat{V}(\hat{\tau}^{\text{dif}}) = \frac{s_c^2}{N_c} + \frac{s_t^2}{N_t}$$

no matter we are interested about SATE τ_{fs} or PATE τ_{sp}

- When N is large enough, we can approximate the distribution of $\hat{\tau}^{\text{dif}}$ by a normal distribution
- Then the 95% CI for either τ_{fs} or τ_{sp} is
 $[\hat{\tau}^{\text{dif}} - 1.96 \times \sqrt{\widehat{V}(\hat{\tau}^{\text{dif}})}, \hat{\tau}^{\text{dif}} + 1.96 \times \sqrt{\widehat{V}(\hat{\tau}^{\text{dif}})}]$
same as what we had earlier

Hypothesis testing for τ_{fs} or τ_{sp}

- We have the same estimator $\hat{\tau}^{\text{dif}}$ and the same variance approximation of $\hat{\tau}^{\text{dif}}$

$$\widehat{V}(\hat{\tau}^{\text{dif}}) = \frac{s_c^2}{N_c} + \frac{s_t^2}{N_t}$$

no matter we are interested about SATE τ_{fs} or PATE τ_{sp}

- When N is large enough, we can approximate the distribution of $\hat{\tau}^{\text{dif}}$ by a normal distribution
- When can test for the null hypothesis $H_0: \tau_{fs} = 0$ or $H_0: \tau_{sp} = 0$
- The t-statistics: $t = \frac{\hat{\tau}^{\text{dif}}}{\sqrt{\widehat{V}(\hat{\tau}^{\text{dif}})}}$
- Under H_0 and when N is large, we have t approximately follows a $N(0, 1)$ distribution
- Two-sided p-value: $2(1 - \phi(|t|))$

Application to the Duflo-Hanna-Ryan data

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Confidence interval for each of the four outcomes:

$\hat{\tau}$	$\widehat{(s. e.)}$	95% C.I.
0.22	(0.03)	(0.15,0.28)
0.05	(0.04)	(−0.03,0.13)
0.17	(0.08)	(0.00,0.34)
0.14	(0.07)	(0.00,0.28)

Application to the Duflo-Hanna-Ryan data

Analysis on two different subgroups:

- Check if the treatment effect is the same for the subset of schools with 0 proportion of students attending the exam before treatment, and for the subset of other schools
- Conditional on the assignment results of other groups, within each subgroup we still have complete randomization of assignments
 - For more explanations, wait until the later lecture on post-stratification

Variable	<code>pctpre = 0</code>			<code>pctprewritten > 0</code>			Difference		
	$(N = 40)$			$(N = 67)$					
	$\hat{\tau}$	$\widehat{(s.e.)}$	95% C.I.	$\hat{\tau}$	$\widehat{(s.e.)}$	95% C.I.	EST	$\widehat{(s.e.)}$	95% C.I.
<code>open</code>	0.23	(0.05)	(0.14,0.32)	0.21	(0.04)	(0.13,0.29)	0.02	(0.06)	(−0.10,0.14)
<code>pctpost written</code>	−0.004	(0.06)	(−0.16,0.07)	0.11	(0.05)	(0.01,0.21)	−0.15	(0.08)	(−0.31,0.00)
<code>written</code>	0.20	(0.10)	(0.00,0.40)	0.18	(0.10)	(−0.03,0.38)	0.03	(0.15)	(−0.26,0.31)
<code>written _all</code>	0.04	(0.07)	(−0.10,0.19)	0.22	(0.09)	(0.04,0.40)	−0.18	(0.12)	(−0.41,0.05)

Fisher v.s. Neyman

- Like Fisher, Neyman proposed randomization-based inference
- Unlike Fisher,
 - estimands are average treatment effects
 - heterogeneous treatment effects are allowed
 - population as well as sample inference is possible
 - asymptotic approximation is required for inference
- Fisher's approach can easily be applied to deal with any randomization mechanism in an experiment, but it can be much harder for Neyman's approach