Causal Inference Methods and Case Studies

STAT24630

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Lecture 8

Topic: pairwise randomized experiment

- pairwise randomized experiment
  - Fisher’s exact p-value
  - Neyman’s repeated sampling approach
  - Regression analysis
  - How to find strata / pairs?

Pairwise randomized experiment

- Basic procedure:
  1. Blocking (Stratification): create groups of similar units based on pre-treatment covariates, let $B_i \in \{1, \cdots, J\}$ be the block indicator
  2. Block (Stratified) randomization: completely randomize treatment assignment within each group
- Blocking can improve the efficiency by minimizing the variance of the potential outcomes within each strata

"Block what you can and randomize what you cannot"

Box, et al. (2005). Statistics for Experimenters. 2nd eds. Wiley

- Assignment probability

$$P(\boldsymbol{W} = \boldsymbol{w}|\boldsymbol{X}) = \begin{cases} \prod_{j=1}^{J} \binom{N(j)}{N_t(j)}^{-1} & \text{if } \sum_{i:B_i=j}^{N} w_i = N_t(j) \text{ for } j = 1, \cdots, J \\ 0 & \text{otherwise} \end{cases}$$

The Children's television workshop experiment

[Ball, Bogatz, Rubin and Beaton, 1973.]

- The Educational Testing Service (ETS) wanted to evaluate *The Electric Company,* an American educational children's television series aimed at improving reading skills for young children

- Two sites, Yongstown, Ohio and Fresno, California where the show was not broadcast on local television, were selected to evaluate the effect of watching the show at school

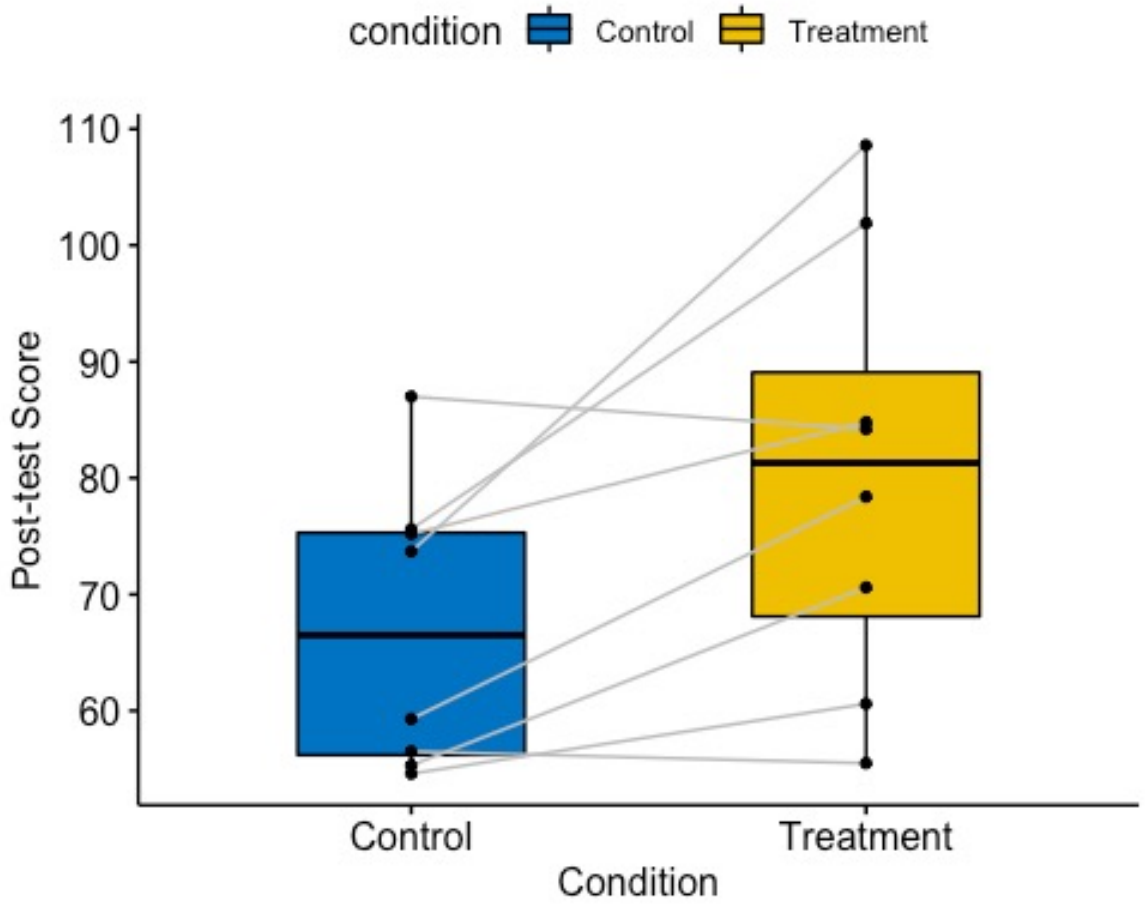
- Within each school, a pair of two classes are selected
  - One class randomly assigned to watch the show
  - Another class continue with regular reading curriculum

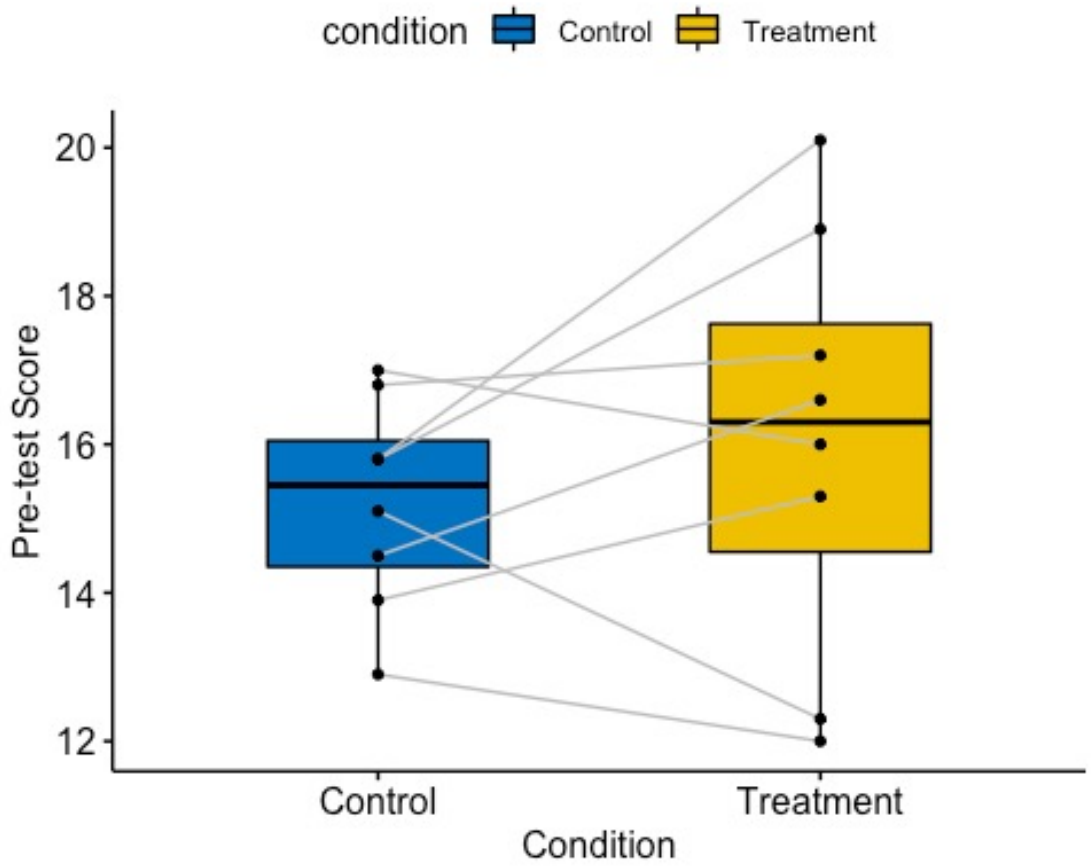
Data from Youngstown

Pair G_i	Treatment W_i	Pre-Test Score X_i	Post-Test Score Y_i^{obs}
1	0	12.9	54.6
1	1	12.0	60.6
2	0	15.1	56.5
2	1	12.3	55.5
3	0	16.8	75.2
3	1	17.2	84.8
4	0	15.8	75.6
4	1	18.9	101.9
5	0	13.9	55.3
5	1	15.3	70.6
6	0	14.5	59.3
6	1	16.6	78.4
7	0	17.0	87.0
7	1	16.0	84.2
8	0	15.8	73.7
8	1	20.1	108.6

- Two first-grade classes from each of eight schools participate in the experiment
- ETS performed reading ability tests to the kids both before the program started and after it finished.

Data from Youngstown





Some notations

Pair	Unit A					Unit B				
	$Y_{i,A}(0)$	$Y_{i,A}(1)$	$W_{i,A}$	$Y_{i,A}^{\text{obs}}$	$X_{i,A}$	$Y_{i,B}(0)$	$Y_{i,B}(1)$	$W_{i,B}$	$Y_{i,B}^{\text{obs}}$	$X_{i,B}$
1	54.6	?	0	54.6	12.9	?	60.6	1	60.6	12.0
2	56.5	?	0	56.5	15.1	?	55.5	1	55.5	13.9
3	75.2	?	0	75.2	16.8	?	84.8	1	84.8	17.2
4	76.6	?	0	75.6	15.8	?	101.9	1	101.9	18.9
5	55.3	?	0	55.3	13.9	?	70.6	1	70.6	15.3
6	59.3	?	0	59.3	14.5	?	78.4	1	78.4	16.6
7	87.0	?	0	87.0	17.0	?	84.2	1	84.2	16.0
8	73.7	?	0	73.7	15.8	?	108.6	1	108.6	20.1

- Average treatment effect within pair j

$$\tau^{\text{pair}}(j) = \frac{1}{2}\sum_{i:G_i=j}\big(Y_i(1) - Y_i(0)\big) = \frac{1}{2}\big(\big(Y_{j,A}(1) - Y_{j,A}(0)\big) + \big(Y_{j,B}(1) - Y_{j,B}(0)\big)\big).$$

- Observed outcomes for both treatment and control groups

$$Y_{j,c}^{\text{obs}} = \begin{cases} Y_{j,A}^{\text{obs}} & \text{if } W_{i,A} = 0, \\ Y_{j,B}^{\text{obs}} & \text{if } W_{i,A} = 1, \end{cases} \quad \text{and} \quad Y_{j,t}^{\text{obs}} = \begin{cases} Y_{j,B}^{\text{obs}} & \text{if } W_{i,A} = 0, \\ Y_{j,A}^{\text{obs}} & \text{if } W_{i,A} = 1. \end{cases}$$

Fisher's exact p-value

- We still focus on the **Sharp null:** $H_0: Y_i(0) \equiv Y_i(1)$ for all $i = 1, \cdots, N$

- Choice of test statistics:
  - Average group mean differences across pairs

$$T^{\text{dif}} = \left| \frac{1}{J} \sum_{j=1}^{J} \left(Y_{j,t}^{\text{obs}} - Y_{j,c}^{\text{obs}} \right) \right| = \left| \bar{Y}_t^{\text{obs}} - \bar{Y}_c^{\text{obs}} \right|$$

    As each pair has exactly one treatment and one control
    - We don't need to consider different weights
    - No worry of Simpson's paradox

  - Rank statistics
    - Use population ranks: $T = |\overline{\text{rank}}(Y_t^{\text{obs}}) - \overline{\text{rank}}(Y_c^{\text{obs}})|$
    - Use within-pair ranks

$$T^{\text{rank,pair}} = \left| \frac{2}{N} \sum_{j=1}^{N/2} \left(\mathbf{1}_{Y_{j,1}^{\text{obs}} > Y_{j,0}^{\text{obs}}} - \mathbf{1}_{Y_{j,1}^{\text{obs}} < Y_{j,0}^{\text{obs}}} \right) \right|$$

Application to the television workshop data

- Fisher's exact p-values
  - Mean differences: $T = 13.4$, $\text{pvalue} = 0.031$
  - Rank mean differences: $T = 3.75$, $\text{pvalue} = 0.031$
  - Within-pair rank differences: $T = 0.5$, $\text{pvalue} = 0.29$

- Rank v.s. within-pair rank
  - Both can reduce the sensitivity to outliers
  - Using within-pair ranks can have more power when there is substantial variation in the level of the outcomes between pairs
  - Otherwise, using within-pair ranks loses power as it treats small within-pair differences (which may be due to random noises) equally with large within-pair differences

  - Using within-pair ranks is more appropriate for large, heterogenous population

Neyman's repeated sampling approach

- Target: PATE or SATE $\tau = \sum_j \frac{N(j)}{N} \tau(j)$ where $\tau(j)$ is the PATE or SATE for strata j

- Point estimate:

$$\hat{\tau}^{\text{pair}}(j) = Y_{j,t}^{\text{obs}} - Y_{j,c}^{\text{obs}} \qquad \hat{\tau}^{\text{dif}} = \frac{1}{N/2} \sum_{j=1}^{N/2} \hat{\tau}^{\text{pair}}(j) = \overline{Y}_{\text{t}}^{\text{obs}} - \overline{Y}_{\text{c}}^{\text{obs}}$$

- We can not estimate the within-pairs variances as there are only two units per pair
- Use the following empirical estimate of the uncertainty (paired t-test)

$$\hat{\mathbb{V}}^{\text{pair}}\left(\hat{\tau}^{\text{dif}}\right) = \frac{4}{N \cdot (N-2)} \cdot \sum_{j=1}^{N/2} \left(\hat{\tau}^{\text{pair}}(j) - \hat{\tau}^{\text{dif}}\right)^2$$

- Above estimate is conservative

$$\mathbb{E}\left[\hat{\mathbb{V}}^{\text{pair}}\left(\hat{\tau}^{\text{dif}}\right)\right] = \mathbb{V}_W(\hat{\tau}^{\text{dif}}) + \frac{4}{N \cdot (N-2)} \cdot \sum_{j=1}^{N/2} \left(\tau^{\text{pair}}(j) - \tau\right)^2$$

Application to the television workshop data

- Est. = 13.4, sd. = 4.6, 95% CI: $[4.3, 22.5]$
- As we have 8 pairs, Gaussian approximation is inaccurate and it's better to compare with a t-distribution with df = 7
- 95% CI comparing with t-distribution: $[2.5, 24.3]$
- If we treat the data as from completely randomized experiment, then sd. = 7.8

Pair	Outcome for Control Unit	Outcome for Treated Unit	Difference
1	54.6	60.6	6.0
2	56.5	55.5	−1.0
3	75.2	84.8	9.6
4	75.6	101.9	26.3
5	55.3	70.6	15.3
6	59.3	78.4	19.1
7	87.0	84.2	−2.8
8	73.7	108.6	34.9
Mean	67.2	80.6	13.4
(S.D.)	(12.2)	(18.6)	(13.1)

Linear regression

- We can not run separate linear regressions within each pair, as there are only 2 units per pair

- We assume that $Y_i(w) = \alpha_j + \tau_i w + \boldsymbol{\beta}^T \boldsymbol{X}_i + \varepsilon_i^*$ where $\mathbb{E}(\tau_i - \tau \mid \boldsymbol{X}_i) = \boldsymbol{\gamma}^T (\boldsymbol{X}_i - \overline{\boldsymbol{X}})$
- Then we have

$$\mathbb{E}\left(Y_{j,t}^{\text{obs}} - Y_{j,c}^{\text{obs}} | \boldsymbol{W} = \boldsymbol{w}, \boldsymbol{X} = \boldsymbol{x}\right) = \tau + \boldsymbol{\gamma}^T \left(\overline{\boldsymbol{X}}_j - \overline{\boldsymbol{X}}\right) + \left(\boldsymbol{\beta} + \frac{\boldsymbol{\gamma}}{2}\right)^T \left(\boldsymbol{X}_{j,t} - \boldsymbol{X}_{j,c}\right)$$

  where $\boldsymbol{X}_{j,t}$ and $\boldsymbol{X}_{j,c}$ are the covariates for the treated and control unit of the jth pair, and $\overline{\boldsymbol{X}}_j$ is the average between the two
- τ is still the PATE
- We still implicitly condition on the pair indicators variables
- If $\boldsymbol{\gamma} = \boldsymbol{0}$, then $\mathbb{E}\left(Y_{j,t}^{\text{obs}} - Y_{j,c}^{\text{obs}} | \boldsymbol{W} = \boldsymbol{w}, \boldsymbol{X} = \boldsymbol{x}\right) = \tau + \boldsymbol{\beta}^T \left(\boldsymbol{X}_{j,t} - \boldsymbol{X}_{j,c}\right)$ we only need to include the covariates differences in the linear regression model

- We can assume homoscedastic errors in the linear regression even if $\mathbb{V}(Y_i(0)) \neq \mathbb{V}(Y_i(1))$

How to perform stratification / pairing

- Univariate blocking: discrete or discretized variable
- Multivariate blocking: Mahalanobis distance

$$D(\mathbf{X}_i, \mathbf{X}_j) = \sqrt{(\mathbf{X}_i - \mathbf{X}_j)^\top \widehat{\mathbb{V}(\mathbf{X})}^{-1} (\mathbf{X}_i - \mathbf{X}_j)}$$

Greedy algorithms

- Matching: pair two units with the shortest distance, set them aside, and repeat
- Blocking: randomly choose one unit and choose N_j units with the shortest distances, set them aside, and repeat

But the resulting matches may not be optimal

Optimal matching

- D: $N \times N$ matrix of pairwise distance or a cost matrix
- Select N elements of D such that there is only one element in each row and one element in each column and the sum of pairwise distances is minimized

- Linear Sum Assignment Problem (LSAP)
  - Binary $N \times N$ matching matrix: M with $M_{ij} \in \{0,1\}$
  - Optimization problem

$$\min_{M} \sum_{i=1}^{N} \sum_{j=1}^{N} M_{ij} D_{ij} \quad \text{subject to} \sum_{i=1}^{N} M_{ij} = N, \sum_{j=1}^{N} M_{ij} = N$$

    where we set $D_{ii} = \infty$ for all i
  - can apply the Hungarian algorithm

Example: evaluation of health insurance policy

[Public policy for the poor? A randomised assessment of the Mexican universal health insurance programme. *The lancet*, *2009*.]

- Seguro Popular, a programme aimed to deliver health insurance, regular and preventive medical care, medicines, and health facilities to 50 million uninsured Mexicans
- Units: health clusters = predefined health facility catchment areas
- 4 pre-treatment cluster-average covariates: age, education, household size, household assets
- 100 clusters, 50 pairs