

# Lecture 8

## Pairwise randomized experiments

---

# Outline

- pairwise randomized experiment
  - Fisher's exact p-value
  - Neyman's repeated sampling approach
  - Regression analysis
  - How to find strata / pairs?
  - R example
- Suggested reading: Imbens and Rubin Section 10.1 -10.6; Peng's book Section 7.1-7.6

# Pairwise randomized experiment

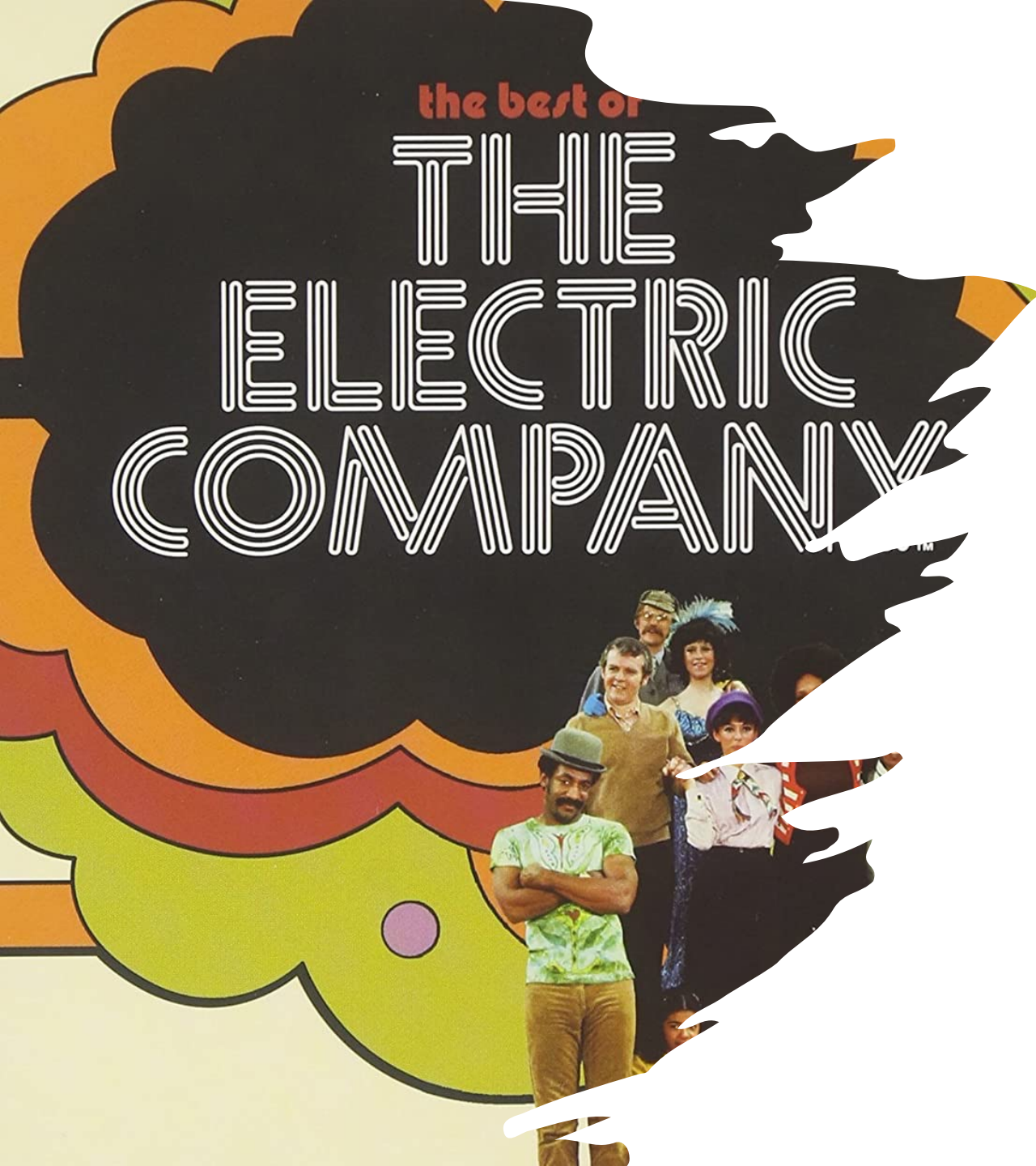
- Procedure:

1. Create  $J = N/2$  pairs of similar units
2. Randomize treatment assignment within each pair

- Assignment probability

A special case of stratified randomized experiment where  $N(j) = 2$  and  $N_t(j) = 1$

$$P(\mathbf{W} = \mathbf{w} | \mathbf{X}) = \begin{cases} \prod_{j=1}^J \binom{N(j)}{N_t(j)}^{-1} = 2^{-N/2} & \text{if } \sum_{i: B_i=j}^N w_i = 1 \text{ for } j = 1, \dots, J \\ 0 & \text{otherwise} \end{cases}$$



## The Children's television workshop experiment

[Ball, Bogatz, Rubin and Beaton, 1973.]

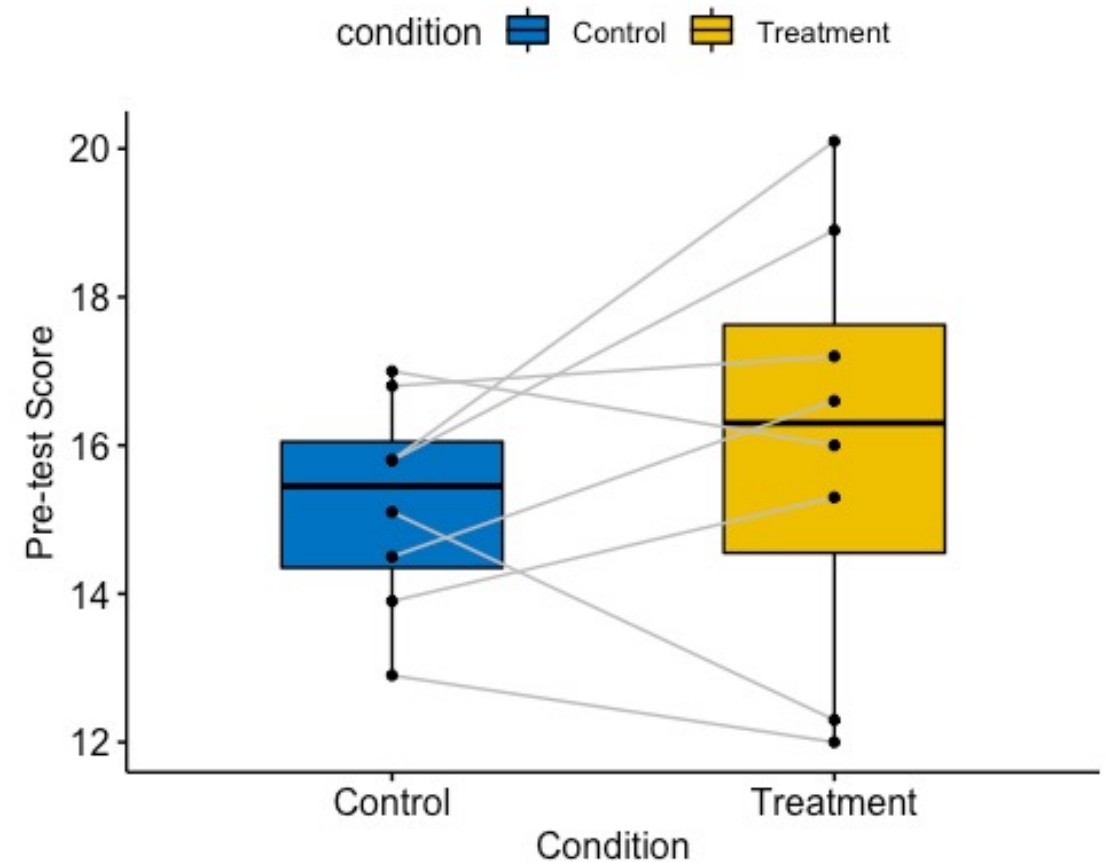
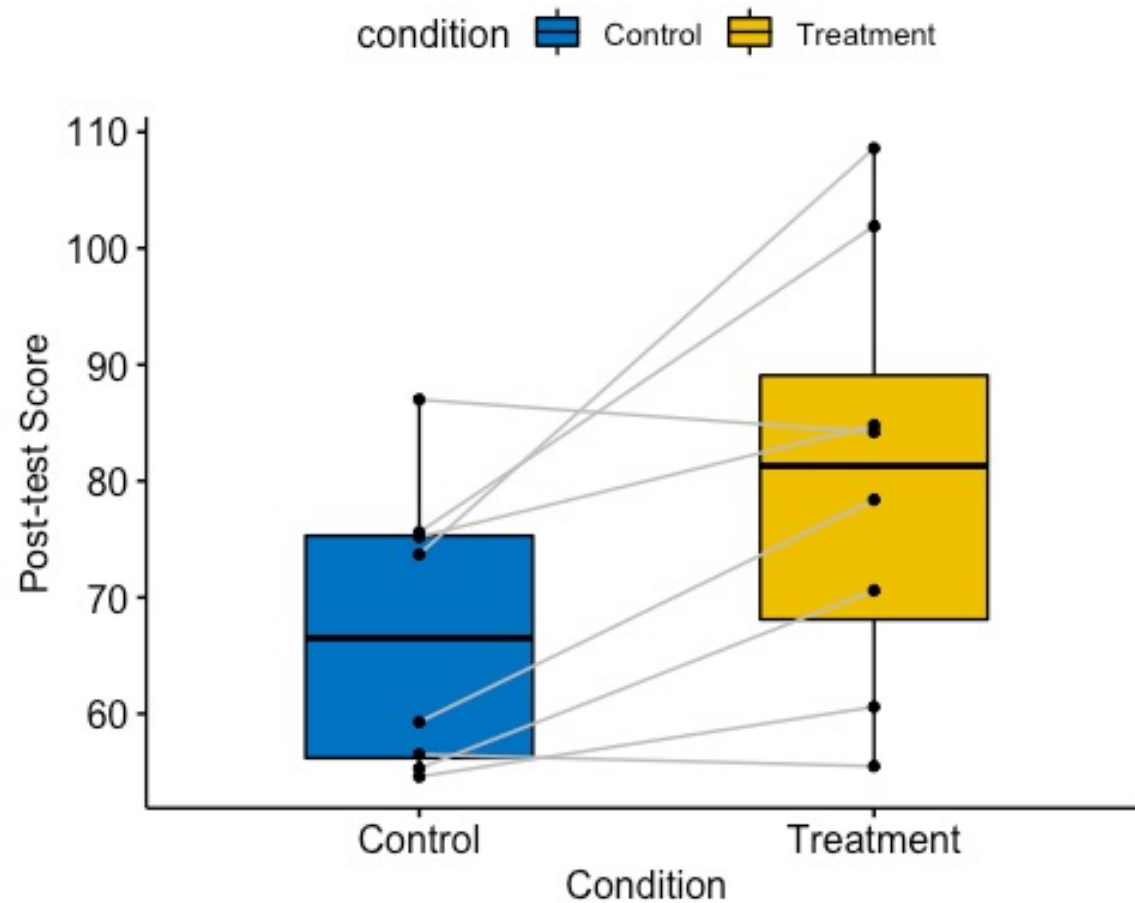
- The Educational Testing Service (ETS) wanted to evaluate *The Electric Company*, an American educational children's television series aimed at improving reading skills for young children
- Two sites, Yongstown, Ohio and Fresno, California where the show was not broadcast on local television, were selected to evaluate the effect of watching the show at school
- Within each school, a pair of two classes are selected
  - One class randomly assigned to watch the show
  - Another class continue with regular reading curriculum

# Data from Youngstown

Pair $G_i$	Treatment $W_i$	Pre-Test Score $X_i$	Post-Test Score $Y_i^{\text{obs}}$
1	0	12.9	54.6
1	1	12.0	60.6
2	0	15.1	56.5
2	1	12.3	55.5
3	0	16.8	75.2
3	1	17.2	84.8
4	0	15.8	75.6
4	1	18.9	101.9
5	0	13.9	55.3
5	1	15.3	70.6
6	0	14.5	59.3
6	1	16.6	78.4
7	0	17.0	87.0
7	1	16.0	84.2
8	0	15.8	73.7
8	1	20.1	108.6

- Two first-grade classes from each of eight schools participate in the experiment
- ETS performed reading ability tests to the kids both before the program started and after it finished.

# Data from Youngstown



# Some notations

Pair	Unit A					Unit B				
	$Y_{i,A}(0)$	$Y_{i,A}(1)$	$W_{i,A}$	$Y_{i,A}^{\text{obs}}$	$X_{i,A}$	$Y_{i,B}(0)$	$Y_{i,B}(1)$	$W_{i,B}$	$Y_{i,B}^{\text{obs}}$	$X_{i,B}$
1	54.6	?	0	54.6	12.9	?	60.6	1	60.6	12.0
2	56.5	?	0	56.5	15.1	?	55.5	1	55.5	13.9
3	75.2	?	0	75.2	16.8	?	84.8	1	84.8	17.2
4	76.6	?	0	75.6	15.8	?	101.9	1	101.9	18.9
5	55.3	?	0	55.3	13.9	?	70.6	1	70.6	15.3
6	59.3	?	0	59.3	14.5	?	78.4	1	78.4	16.6
7	87.0	?	0	87.0	17.0	?	84.2	1	84.2	16.0
8	73.7	?	0	73.7	15.8	?	108.6	1	108.6	20.1

- Average treatment effect within pair  $j$

$$\tau^{\text{pair}}(j) = \frac{1}{2} \sum_{i:G_i=j} (Y_i(1) - Y_i(0)) = \frac{1}{2} ((Y_{j,A}(1) - Y_{j,A}(0)) + (Y_{j,B}(1) - Y_{j,B}(0))).$$

- Observed outcomes for both treatment and control groups

$$Y_{j,c}^{\text{obs}} = \begin{cases} Y_{j,A}^{\text{obs}} & \text{if } W_{i,A} = 0, \\ Y_{j,B}^{\text{obs}} & \text{if } W_{i,A} = 1, \end{cases} \quad \text{and} \quad Y_{j,t}^{\text{obs}} = \begin{cases} Y_{j,B}^{\text{obs}} & \text{if } W_{i,A} = 0, \\ Y_{j,A}^{\text{obs}} & \text{if } W_{i,A} = 1. \end{cases}$$

# Fisher's exact p-value

- We still focus on the **Sharp null:  $H_0: Y_i(0) \equiv Y_i(1)$  for all  $i = 1, \dots, N$**
- Choice of test statistics:
  - Average group mean differences across pairs

$$T^{\text{dif}} = \left| \frac{1}{J} \sum_{j=1}^J (Y_{j,t}^{\text{obs}} - Y_{j,c}^{\text{obs}}) \right| = |\bar{Y}_t^{\text{obs}} - \bar{Y}_c^{\text{obs}}|$$

As each pair has exactly one treatment and one control

- We don't need to consider different weights
  - No worry of Simpson's paradox
- Rank statistics
    - Use population ranks:  $T = |\overline{\text{rank}}(Y_t^{\text{obs}}) - \overline{\text{rank}}(Y_c^{\text{obs}})|$
    - Use within-pair ranks

$$T^{\text{rank,pair}} = \left| \frac{2}{N} \sum_{j=1}^{N/2} \left( \mathbf{1}_{Y_{j,1}^{\text{obs}} > Y_{j,0}^{\text{obs}}} - \mathbf{1}_{Y_{j,1}^{\text{obs}} < Y_{j,0}^{\text{obs}}} \right) \right|$$



# Application to the television workshop data

- Fisher's exact p-values
  - Mean differences:  $T = 13.4$ , pvalue = 0.031
  - Rank mean differences:  $T = 3.75$ , pvalue = 0.031
  - Within-pair rank differences:  $T = 0.5$ , pvalue = 0.29
- Rank v.s. within-pair rank
  - Both can reduce the sensitivity to outliers
  - Using within-pair ranks can have more power when there is substantial variation in the level of the outcomes between pairs
  - Otherwise, using within-pair ranks loses power as it treats small within-pair differences (which may be due to random noises) equally with large within-pair differences
  - Using within-pair ranks is more appropriate for large, heterogenous population

# Neyman's repeated sampling approach

- **Target:** PATE or SATE  $\tau = \sum_j \frac{N(j)}{N} \tau(j)$  where  $\tau(j)$  is the PATE or SATE for strata  $j$
- **Point estimate:**

$$\hat{\tau}^{\text{pair}}(j) = Y_{j,t}^{\text{obs}} - Y_{j,c}^{\text{obs}} \quad \hat{\tau}^{\text{dif}} = \frac{1}{N/2} \sum_{j=1}^{N/2} \hat{\tau}^{\text{pair}}(j) = \bar{Y}_t^{\text{obs}} - \bar{Y}_c^{\text{obs}}$$

- We can not estimate the within-pairs variances as there are only two units per pair
- Use the following empirical estimate of the uncertainty (paired t-test)

$$\hat{\mathbb{V}}^{\text{pair}}(\hat{\tau}^{\text{dif}}) = \frac{4}{N \cdot (N - 2)} \cdot \sum_{j=1}^{N/2} \left( \hat{\tau}^{\text{pair}}(j) - \hat{\tau}^{\text{dif}} \right)^2$$

- Above estimate is conservative
  - $\hat{\tau}^{\text{pair}}(j)$  has mean  $\tau^{\text{pair}}(j)$  instead of  $\tau$

$$\mathbb{E} \left[ \hat{\mathbb{V}}^{\text{pair}}(\hat{\tau}^{\text{dif}}) \right] = \mathbb{V}_W(\hat{\tau}^{\text{dif}}) + \frac{4}{N \cdot (N - 2)} \cdot \sum_{j=1}^{N/2} \left( \tau^{\text{pair}}(j) - \tau \right)^2$$

# Application to the television workshop data

- Est. = 13.4, sd. = 4.6, 95% CI: [4.3, 22.5]
- As we have 8 pairs, Gaussian approximation is inaccurate and it's better to compare with a t-distribution with  $df = 7$
- 95% CI comparing with t-distribution: [2.5, 24.3]
- If we treat the data as from completely randomized experiment, then sd. = 7.8

Pair	Outcome for Control Unit	Outcome for Treated Unit	Difference
1	54.6	60.6	6.0
2	56.5	55.5	-1.0
3	75.2	84.8	9.6
4	75.6	101.9	26.3
5	55.3	70.6	15.3
6	59.3	78.4	19.1
7	87.0	84.2	-2.8
8	73.7	108.6	34.9
Mean	67.2	80.6	13.4
(S.D.)	(12.2)	(18.6)	(13.1)

# Linear regression

- We can not run separate linear regressions within each pair, as there are only 2 units per pair

## How to build a reasonable regression framework?

- For each pair  $j$ ,  $Y_{j,k}(w) = Y_{j,k}(0) + \tau_{j,k}$  for  $k = A$  or  $B$
- We assume that

$$\mathbb{E}(Y_{j,k}(0)|\mathbf{X}) = \alpha_j + \boldsymbol{\beta}^T \mathbf{X}_{j,k}, \quad \mathbb{E}(\tau_{j,k}|\mathbf{X}) = \tau + \boldsymbol{\gamma}^T (\mathbf{X}_{j,k} - \bar{\mathbf{X}})$$

- Then

$$\mathbb{E}(Y_{j,k}(w)|\mathbf{X}_{j,k}) = \alpha_j + \tau_j w + \boldsymbol{\beta}^T \mathbf{X}_{j,k}$$

- Unconfoundedness property:

$$(Y(0), Y(1)) \perp \mathbf{W} | \mathbf{X}$$

- Then we have

$$\begin{aligned} \mathbb{E}(Y_{j,t}^{\text{obs}} - Y_{j,c}^{\text{obs}} | \mathbf{W} = \mathbf{w}, \mathbf{X} = \mathbf{x}) &= \mathbb{E}(Y_{j,t}(1) - Y_{j,c}(0) | \mathbf{W} = \mathbf{w}, \mathbf{X} = \mathbf{x}) \\ &= \mathbb{E}(Y_{j,t}(1) - Y_{j,c}(0) | \mathbf{X} = \mathbf{x}) \end{aligned}$$

where  $Y_{j,t}^{\text{obs}}$  and  $Y_{j,c}^{\text{obs}}$  are observed responses for the treated and control unit in the  $j$ th pair

# Linear regression

- We finally have the regression model:

$$\begin{aligned}\mathbb{E}(Y_{j,t}^{\text{obs}} - Y_{j,c}^{\text{obs}} | \mathbf{W} = \mathbf{w}, \mathbf{X} = \mathbf{x}) &= \mathbb{E}(Y_{j,t}(1) - Y_{j,c}(0) | \mathbf{X} = \mathbf{x}) \\ &= \tau + \boldsymbol{\gamma}^T (\mathbf{X}_{j,t} - \bar{\mathbf{X}}) + \boldsymbol{\beta}^T (\mathbf{X}_{j,t} - \mathbf{X}_{j,c}) \\ &= \tau + \boldsymbol{\gamma}^T (\bar{\mathbf{X}}_j - \bar{\mathbf{X}}) + \left( \boldsymbol{\beta} + \frac{\boldsymbol{\gamma}}{2} \right)^T (\mathbf{X}_{j,t} - \mathbf{X}_{j,c})\end{aligned}$$

- $\tau$  is still the PATE
- We still implicitly condition on the pair indicators variables
- If  $\boldsymbol{\gamma} = \mathbf{0}$ , then  $\mathbb{E}(Y_{j,t}^{\text{obs}} - Y_{j,c}^{\text{obs}} | \mathbf{W} = \mathbf{w}, \mathbf{X} = \mathbf{x}) = \tau + \boldsymbol{\beta}^T (\mathbf{X}_{j,t} - \mathbf{X}_{j,c})$  we only need to include the covariates differences in the linear regression model
- We can assume homoscedastic errors in the linear regression even if  $\mathbb{V}(Y_i(0)) \neq \mathbb{V}(Y_i(1))$ 
  - We assume the pairs are i.i.d.

# How to perform stratification / pairing

- Implementation based on convenience
- Univariate blocking: discrete or discretized variable
- Multivariate blocking: Mahalanobis distance

$$D(\mathbf{X}_i, \mathbf{X}_j) = \sqrt{(\mathbf{X}_i - \mathbf{X}_j)^\top \widehat{\mathbb{V}}(\mathbf{X})^{-1} (\mathbf{X}_i - \mathbf{X}_j)}$$

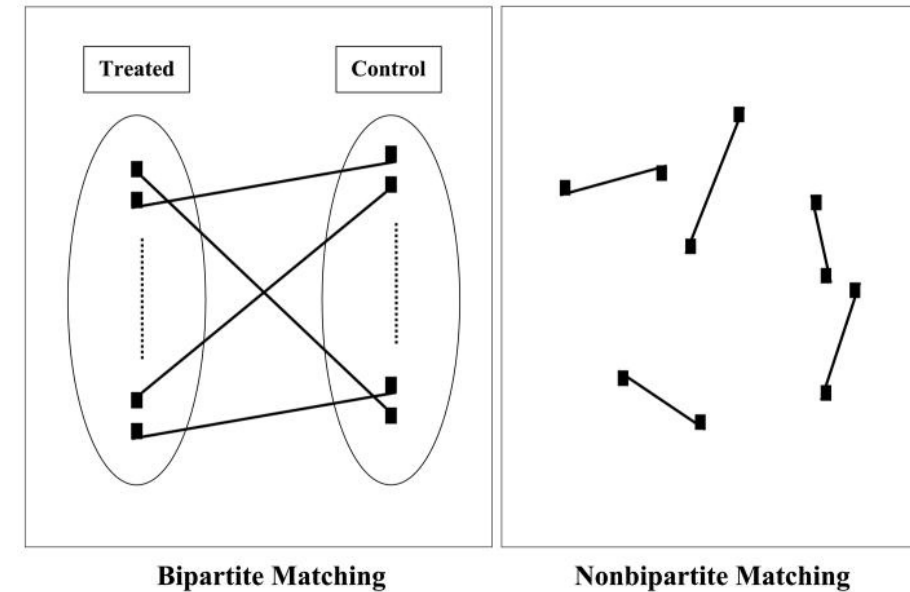
## Greedy algorithms

- Matching: pair two units with the shortest distance, set them aside, and repeat
- Blocking: randomly choose one unit and choose  $N_j$  units with the shortest distances, set them aside, and repeat

But the resulting matches may not be optimal

# Optimal matching

- $D$ :  $N \times N$  matrix of pairwise distance or a cost matrix
- Optimal matching
  - Binary  $N \times N$  matching matrix:  $M$  with  $M_{ij} \in \{0,1\}$
  - Optimization problem:  
$$\min_M \sum_{i=1}^N \sum_{j=1}^N M_{ij} D_{ij} \quad \text{subject to } \sum_{i=1}^N M_{ij} = 1 \text{ for all } j$$
where we set  $D_{ii} = \infty$  for all  $i$
  - $M$  also need to be symmetric
- Nonbipartite matching
- Computational cost  $O(n^3)$
- Derigs' algorithm: implemented in the R package nbpMatching  
<https://cran.r-project.org/web/packages/nbpMatching/>



# Example: evaluation of health insurance policy

[Public policy for the poor? A randomised assessment of the Mexican universal health insurance programme. *The lancet*, 2009.]

- Seguro Popular, a programme aimed to deliver health insurance, regular and preventive medical care, medicines, and health facilities to 50 million uninsured Mexicans
- Units: health clusters = predefined health facility catchment areas
- 4 pre-treatment cluster-average covariates: age, education, household size, household assets
- 100 clusters, 50 pairs

