

# STAT347: Generalized Linear Models

## Lecture 5

Winter, 2024  
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# Today's topics:

- GLM computation
- Binary / Binomial data model
  - Data input
  - Link functions
  - R example

# GLM computation

- Only discuss the case of  $a(\phi) = 1$  to simplify notation
- If  $a(\phi)$  is not a constant, one can get  $\hat{\beta}$  from the score equations first, and then estimate  $\phi$  from MLE with  $\hat{\beta}$  plugged in

Score equation:

$$\dot{L}(\beta) = X^T D V^{-1} (y - \mu) = 0$$

where

$$L(\beta) = \sum [y_i \theta_i - b(\theta_i)]$$

- Newton's method
- Fisher scoring method
- Iteratively reweighted least squares (IRLS): equivalent to Fisher scoring

# Newton's method

Second-order approximation of  $L(\beta)$

$$L(\beta) \approx L(\beta^{(t)}) + \dot{L}(\beta^{(t)})^T (\beta - \beta^{(t)}) + \frac{1}{2} (\beta - \beta^{(t)})^T \ddot{L}(\beta^{(t)}) (\beta - \beta^{(t)})$$

at  $t$ th iteration. If  $\ddot{L}(\beta^{(t)}) \preceq 0$ , then maximizing the second-order approximation is equivalent to solving

$$\dot{L}(\beta) \approx \dot{L}(\beta^{(t)}) + \ddot{L}(\beta^{(t)}) (\beta - \beta^{(t)}) = 0$$

We have

$$\beta^{(t+1)} = \beta^{(t)} - \ddot{L}(\beta^{(t)})^{-1} \dot{L}(\beta^{(t)})$$

# Newton's method

- Newton's method is a general algorithm for optimizing twice-differentiable functions.
- Generally, it converges to the global maximum if  $L(\beta)$  is strongly concave
  - If  $g(\cdot)$  is the canonical link, then  $L(\beta)$  is concave in  $\beta$

$$-\ddot{L}(\beta^{(t)}) = X^T W^{(t)} X = \frac{1}{a(\phi)} X^T V^{(t)} X = -\mathbb{E} \left( \ddot{L}(\beta^{(t)}) \right) \succeq 0$$

- If  $g(\cdot)$  is a general link, then  $L(\beta)$  is NOT guaranteed to be concave in  $\beta$
- If  $-\ddot{L}(\beta^{(t)})$  is not non-negative, then step  $t$  does not maximize the quadratic approximation and Newton's method may not converge.

# Fisher scoring method

- In lecture 2, we showed that  $-\mathbb{E}(\ddot{L}(\beta)) \succcurlyeq 0$  for any  $\beta$ .
- Instead of using the Hessian  $\ddot{L}(\beta^{(t)})$  itself in the second order approximation, we use its expectation

$$J^{(t)} = \mathbb{E} \left( \ddot{L}(\beta^{(t)}) \right) = -X^T W^{(t)} X$$

Each iteration becomes:

$$\beta^{(t+1)} = \beta^{(t)} - \left( J^{(t)} \right)^{-1} \dot{L}(\beta^{(t)})$$

- For the canonical link, Fisher scoring = Newton's method
- For a general link, Fisher scoring works better in practice

# Iteratively reweighted least squares

- We can make a connection between the optimization for GLM and weighted least squares estimation.

Recall the score equation:

$$\dot{L}(\beta) = X^T D V^{-1} (y - \mu) = 0$$

where  $V = \text{diag}(\text{Var}(y_1), \dots, \text{Var}(y_n))$  and  $D = \text{diag}(g'(\mu_1), \dots, g'(\mu_n))^{-1}$ ,  $y = (y_1, \dots, y_n)$  and  $\mu = (\mu_1, \dots, \mu_n)$ .

Also in lecture 2, we used the notation  $\eta_i = X_i^T \beta = g(\mu_i)$ . Thus,  $D = \text{diag}\left(\frac{\partial \mu_1}{\partial \eta_1}, \dots, \frac{\partial \mu_n}{\partial \eta_n}\right)$ . We also defined the diagonal matrix  $W = D^2 V^{-1}$ . Thus,

$$\dot{L}(\beta) = X^T D V^{-1} (y - \mu) = X^T W D^{-1} (y - \mu)$$

We can make a first order approximation of  $\mu$

$$\mu = \mu^{(t)} + D^{(t)} (\eta - \eta^{(t)})$$

then

$$\dot{L}(\beta) \approx X^T W^{(t)} (z^{(t)} - X\beta)$$

where

$$z^{(t)} = X\beta^{(t)} + \left(D^{(t)}\right)^{-1} (y - \mu^{(t)})$$

is a linear approximation of  $\eta$  at the  $t$ th iteration.

# Iteratively reweighted least squares (IRLS)

- At the  $t+1$  th iteration, we solve the “approximated score equation”:

$$X^T W^{(t)} (z^{(t)} - X\beta) = 0$$

which can be considered as a weighted linear regression with observations  $z_i^{(t)}$  and weight  $w_i$  for each sample  $i$ .

- IRLS is equivalent to Fisher scoring. The  $t$ th step of Fisher scoring satisfy

$$\begin{aligned}(X^T W^{(t)} X) \beta^{(t+1)} &= X^T W^{(t)} X \beta^{(t)} + X^T D^{(t)} (V^{(t)})^{-1} (y - \mu^{(t)}) \\ &= X^T W^{(t)} \left[ X \beta^{(t)} + (D^{(t)})^{-1} (y - \mu^{(t)}) \right] \\ &= X^T W^{(t)} z^{(t)}\end{aligned}$$

- Weight matrix  $W^{(t)} \approx \text{Var} (z^{(t)})^{-1}$



# Binary / binomial data model

If the observation  $y_i$  is binomial

$$y_i \sim \text{Binomial}(n_i, p_i)$$

and probability function:

$$f(y_i) = \binom{n_i}{y_i} p_i^{y_i} (1 - p_i)^{n_i - y_i} = \binom{n_i}{y_i} \left( \frac{p_i}{1 - p_i} \right)^{y_i} (1 - p_i)^{n_i}$$

If  $n_i = 1$ , then  $y_i$  is a 0/1 binary data point (follows a Bernoulli distribution).

- Link function:  $g(p_i) = X_i^T \beta$
- logistic regression:  $\log\left(\frac{p_i}{1-p_i}\right) = X_i^T \beta$

# Data input for binary model

If  $X_i$  are categorical variables, then we may have samples with the same  $X_i$  and we can group them together

- ungrouped data: each  $n_i = 1$  and some samples have the same  $X_i$ , thus they share the same  $p_i$
- a grouped sample  $\tilde{y}_k$  for group  $k$  where all observations in the group share the same  $X_i$ 
  - Define  $n_k$  as the number of binary observations
  - The grouped response for group  $k$  is

$$\tilde{y}_k = \sum_{i \in I_k} y_i \sim \text{Binomial}(n_k, p_k)$$

- The grouped data follows the Binomial distribution because we assume that the samples are independent within each group

# Likelihood for grouped and ungrouped data

- Let  $N = \sum_k n_k$  The likelihood for the ungrouped data is:

$$\begin{aligned} f(y_1, y_2, \dots, y_N) &= \prod_i p_i^{y_i} (1 - p_i)^{1-y_i} \\ &= \prod_k \prod_{i \in I_k} p_k^{\tilde{y}_k} (1 - p_k)^{n_k - \tilde{y}_k} \end{aligned}$$

The likelihood for the corresponding grouped data is:

$$f(\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_K) = \prod_k \binom{n_k}{y_k} \prod_{i \in I_k} p_k^{\tilde{y}_k} (1 - p_k)^{n_k - \tilde{y}_k}$$

- The likelihood is not the same between the grouped data and ungrouped data. However, the log-likelihood function only differs by a constant, thus the GLM solution does not change.

# Link function for binary / binomial GLM

The expectation of each sample is  $\mathbb{E}(y_i) = n_i p_i$  where  $n_i$  is a known constant. Thus we define the link function as a function of  $p_i$

$$g(p_i) = X_i^T \beta$$

Equivalently,

$$p_i = g^{-1}(X_i^T \beta) \in [0, 1]$$

- If  $g$  is a one-to-one mapping and continuous function, then  $g^{-1}$  should be monotone.
- one natural choice of  $g^{-1}$  is to make it as a cdf of some distribution.
- Denote  $F(z) = g^{-1}(z)$  as some cdf function
  - Let  $\epsilon_i \stackrel{i.i.d.}{\sim} F(\cdot)$

# Latent variable threshold models

- Denote  $F(z) = g^{-1}(z)$  as some cdf function

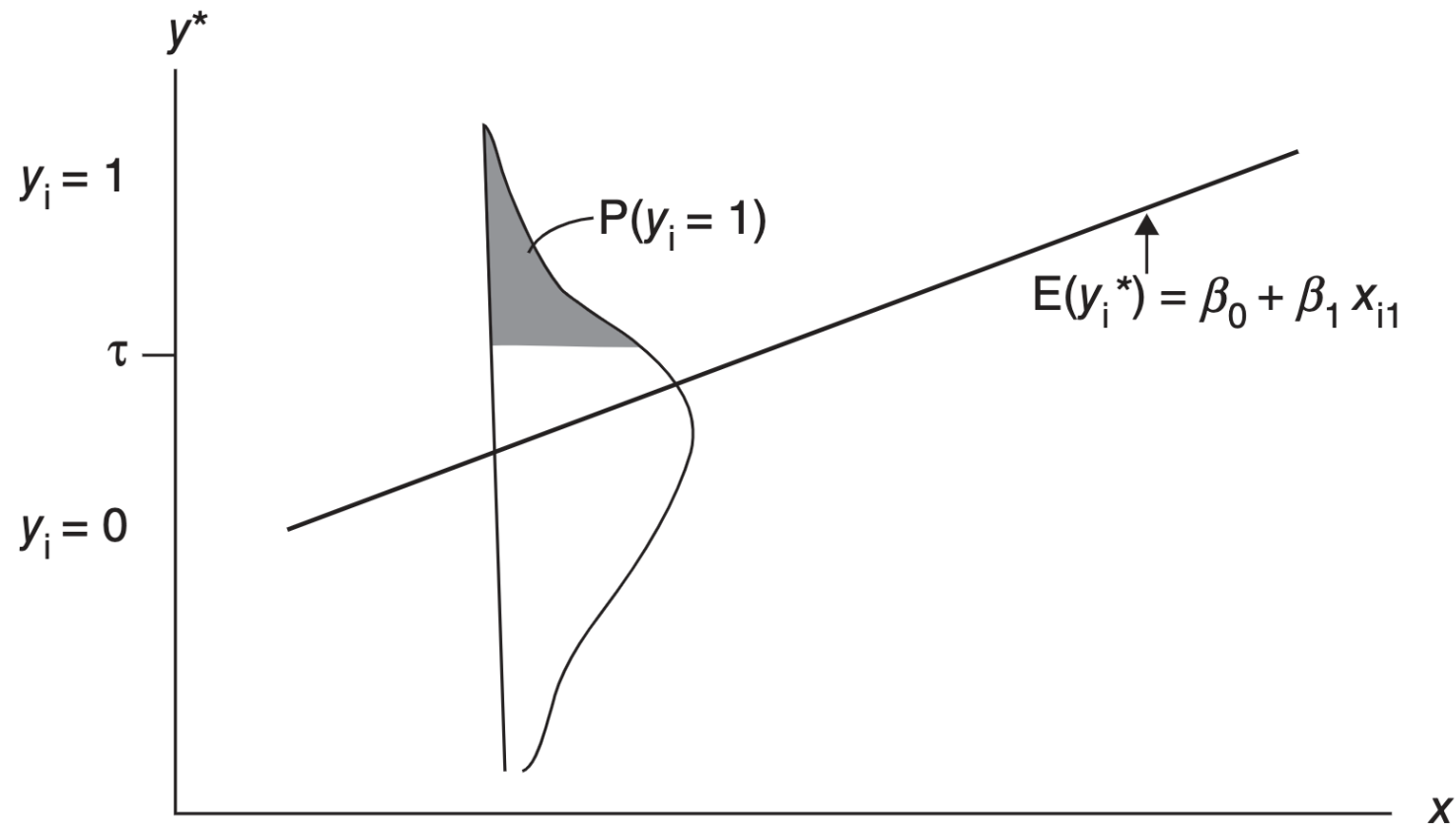
- Let  $\epsilon_i \stackrel{i.i.d.}{\sim} F(\cdot)$

- Then

$$p_i = F(X_i^T \beta) = \mathbb{P}(\epsilon_i \leq X_i^T \beta) = \mathbb{P}(X_i^T \beta - \epsilon_i \geq 0)$$

- This is called a latent variable threshold models and  $X_i^T \beta - \epsilon_i$  are the “latent variables”
  - It does not make any essential modeling difference choosing the cutoff to be 0 or any other value  $\tau$

# Latent variable threshold models



**Figure 5.1** Threshold latent variable model, for which we observe  $y_i = 1$  when underlying latent variable  $y_i^* > \tau$ .

# The probit link

- The probit link:  $F(z)$  is the cdf of a standard Gaussian distribution

$$p_i = \mathbb{P}(X_i^T \beta - \epsilon_i \geq 0) = \mathbb{P}(X_i^T \beta + \epsilon_i \geq 0)$$

where  $\epsilon_i \sim N(0, 1)$ . Let the hidden variable be  $y_i^* = X_i^T \beta + \epsilon_i$ , then it goes to the definition of the probit link that some of you may be more familiar with:

$$Y_i = \begin{cases} 1 & \text{if } y_i^* \geq 0 \\ 0 & \text{else} \end{cases}$$

# The logit link

- The logit link:  $F(z)$  is the cdf of a standard logistic distribution

$$F(z) = \frac{e^z}{1 + e^z}$$

- The link function is called the logit link:  $g(p_i) = \text{logit}(p_i) = \log\left(\frac{p_i}{1-p_i}\right)$
- The logit link is the canonical link of the Binomial distribution



# The identity link

- The identity link:  $F(z)$  is the cdf of a uniform  $[0, 1]$  distribution and  $p_i = X_i^T \beta$ 
  - The identity link corresponds to a uniform cdf only when  $X_i^T \beta \in [0, 1]$  for all samples.
  - Because of the range issue, when using R to solve a binomial GLM with identity link, there can often be numerical problems (such as the error we saw in the earlier data example in Section 1.4, Data Example 1).

# The log-log link

- All previous links assume a symmetric  $\epsilon_i$  around 0
  - A corresponding restriction is that the response curve is symmetric at 0.5
  - We should use some other link functions if this assumption is severely violated
- The log-log link:  $F(z)$  is the cdf of a standard double-exponential distribution (Gumbel distribution)

$$F(z) = e^{-e^{-z}}$$

- The link function is called the log-log link:

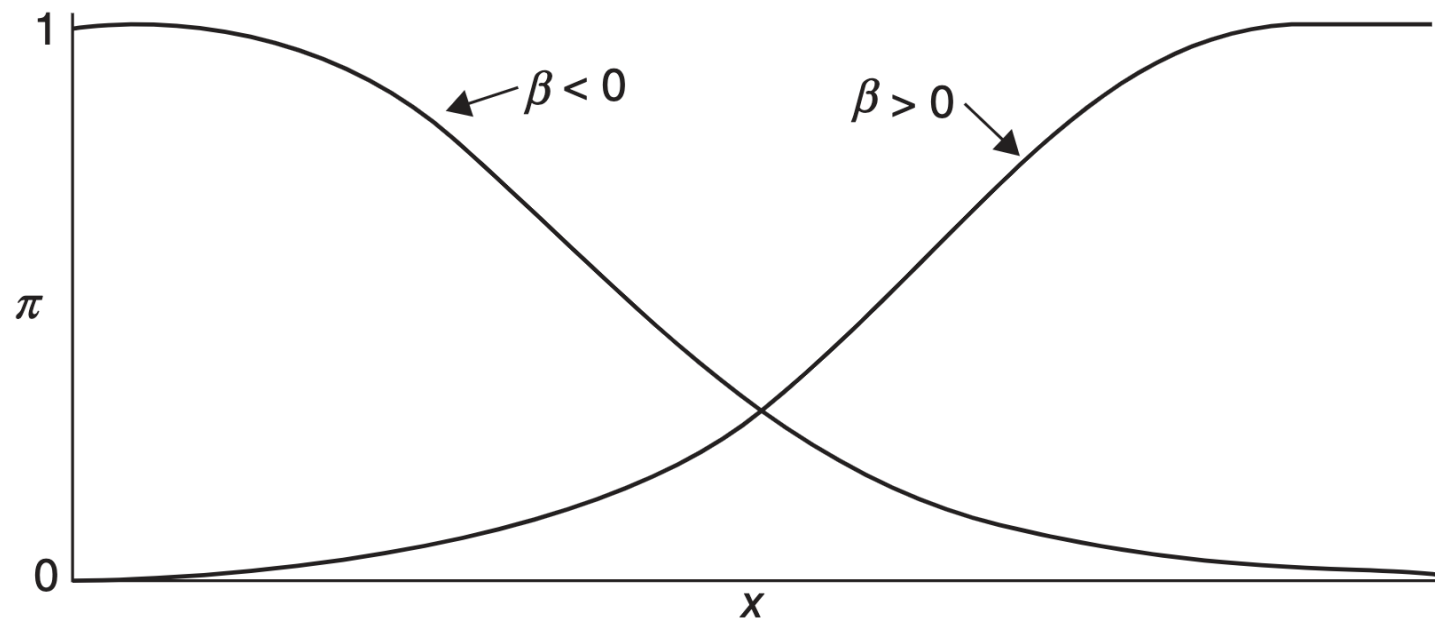
$$g(p_i) = -\log[-\log(p_i)] = X_i^T \beta$$

- With the log-log link,  $p_i$  approaches 0 sharply but approaches 1 slowly

# The complementary log-log link

- With a complementary log-log link,  $p_i$  approaches 1 sharply but approaches 0 slowly

$$g(p_i) = \log[-\log(p_i)] = X_i^T \beta$$



**Figure 5.4** GLM for binary data using complementary log-log link function.

# R data example for binary / binomial GLM (part I)

- Check Example3\_1 R notebook