Topics in Causal Inference

STAT41530

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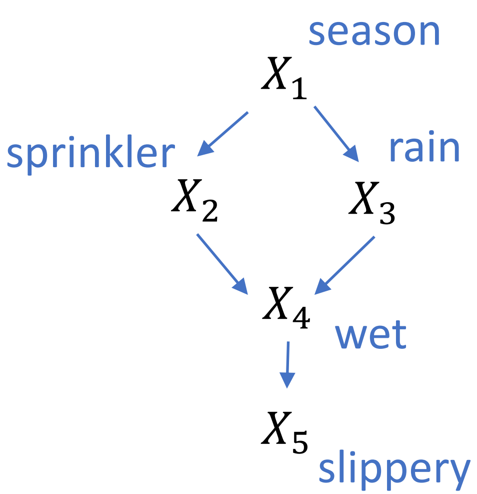
Lecture 4

Topic: causal directed acyclic graph (DAG)

- Do-operator
- Confounding
    - Backdoor criterion
    - Frontdoor criterion

Do-operator



- Assume that we have a causal DAG with structural equations

$$X_j = f_j\left(PA_j, E_{X_j}\right), \qquad j = 1, \cdots n$$

- Do-operator: $P(Y \mid do(X))$ to describe a causal effect

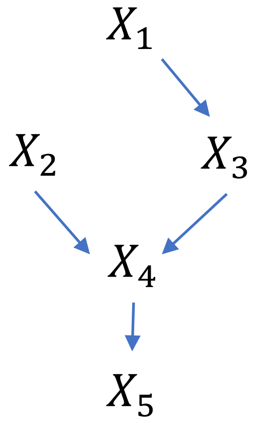
Definition (pearl 1995)

Given two disjoint sets of variables, X and Y, the causal effect of X on Y, denoted as $P(Y \mid do(X))$, is a function from X to the probability distribution on Y. For each realization x of X, $P(Y \mid do(X))$ gives the probability of $Y = y$ induced on deleting from the above set of structural equations all equations corresponding to variable X and substituting x for X in the reminder

$$P(X_i \mid do\left(X_j\right) = x) = \sum_{V/X_i} \frac{P(V)}{P(X_j|PA_j)} 1_{X_j=x}$$

$P(X_5 \mid do(X_2))$?

$= P(X_5 \mid X_2)$ in the DAG after intervention



An illustration of the do-operator

$$\begin{tikzcd} & A_2 \arrow[dr] & \\ A_1 \arrow[ur] \arrow[rr] & & Y \end{tikzcd}$$

$$P(A_1, A_2, Y) = P(A_1)P(A_2|A_1)P(Y|A_1, A_2)$$

In general,

$$P(X_i = x_i|\, do(X_j) = x_j) = P(\underline{X_i(x_j)} = x_i)$$

Potential outcome

- $P(Y|do(A_1, A_2)) = P(Y|A_1, A_2)$
- $P(Y|do(A_1)) = \sum_{a_2} P(A_2 = a_2|A_1)P(Y|A_1, A_2 = a_2) = P(Y|A_1)$
- $P(Y|do(A_2)) = \sum_{a_1} P(A_1 = a_1)P(Y|A_1 = a_1, A_2) \neq P(Y|A_2)$

Different from conditional probability

A fundamental theorem

(Adjustment of direct causes)
Let Y be any set of variables disjoint of $\{X_i \cup PA_i\}$. The causal effect of X_i on Y is given by

$$P(Y|do(X_i)) = \sum_{pa_i} P(PA_i = pa_i)P(Y|X_i, PA_i = pa_i)$$

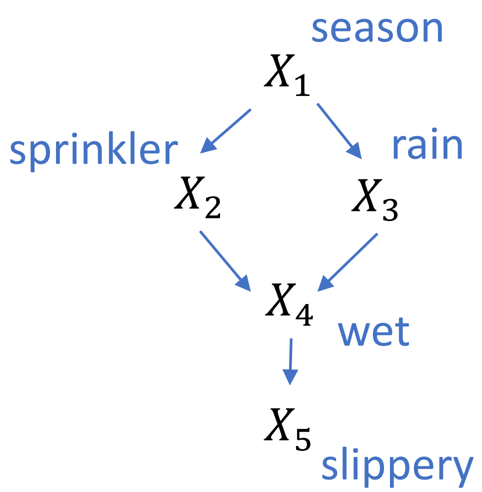
Where $P(PA_i = pa_i)$ and $P(Y|X_i, PA_i = pa_i)$ represent pre-interventional probabilities

(Before we change the DAG)

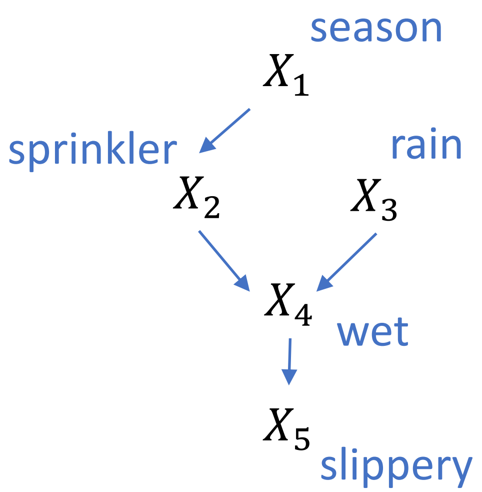
Define $W = V/\{X_i, Y, PA_i\}$

$$\begin{aligned}
\mathbb{P}\left[Y \mid do(X_i) = x\right] &= \sum_{W}\sum_{PA_i}\sum_{X_i} \mathbb{P}\left[V \mid do(X_i) = x\right] \\
&= \sum_{W}\sum_{PA_i}\sum_{X_i} \frac{\mathbb{P}\left[V\right]}{\mathbb{P}\left[X_i \mid PA_i\right]} 1_{X_i = x} \\
&= \sum_{PA_i}\sum_{X_i} \frac{\mathbb{P}\left[Y, PA_i, X_i\right]}{\mathbb{P}\left[X_i \mid PA_i\right]} 1_{X_i = x} \\
&= \sum_{PA_i}\sum_{X_i} \frac{\mathbb{P}\left[Y|PA_i, X_i\right]\mathbb{P}\left[X_i \mid PA_i\right]\mathbb{P}\left[PA_i\right]}{\mathbb{P}\left[X_i \mid PA_i\right]} 1_{X_i = x} \\
&= \sum_{pa_i} \mathbb{P}\left[Y|PA_i = pa_i, X_i = x\right]\mathbb{P}\left[PA_i = pa_i\right]
\end{aligned}$$

Example



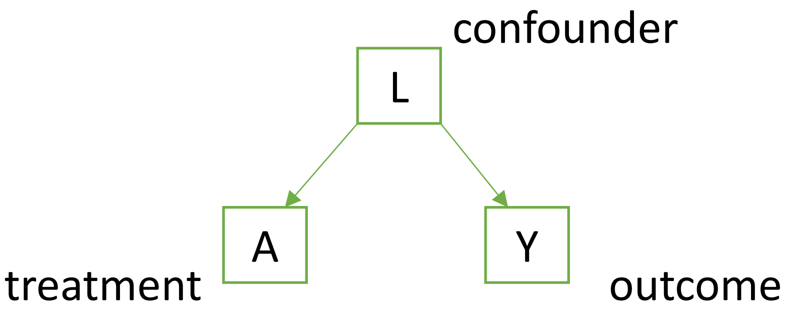
DAG before intervention



DAG after intervention

$$P(X_5|do(X_3)) = \sum_{x_1} P(X_1 = x_1)P(X_5|X_3, X_1 = x_1)$$

Do-operator to explain Simpson's paradox



$$P(Y|do(A)) = \sum_{l} P(Y|L = l, A)P(L = l)$$

$$P(Y|A) = \sum_{l} P(Y|L = l, A)P(L = l|A)$$

$$P(Y|do(X_i)) = \sum_{pa_i} P(PA_i = pa_i)P(Y|X_i, PA_i = pa_i)$$

$$P(Y|X_i) = \sum_{pa_i} P(PA_i = pa_i|X_i)P(Y|X_i, PA_i = pa_i)$$

Structural classification of bias

Bias:

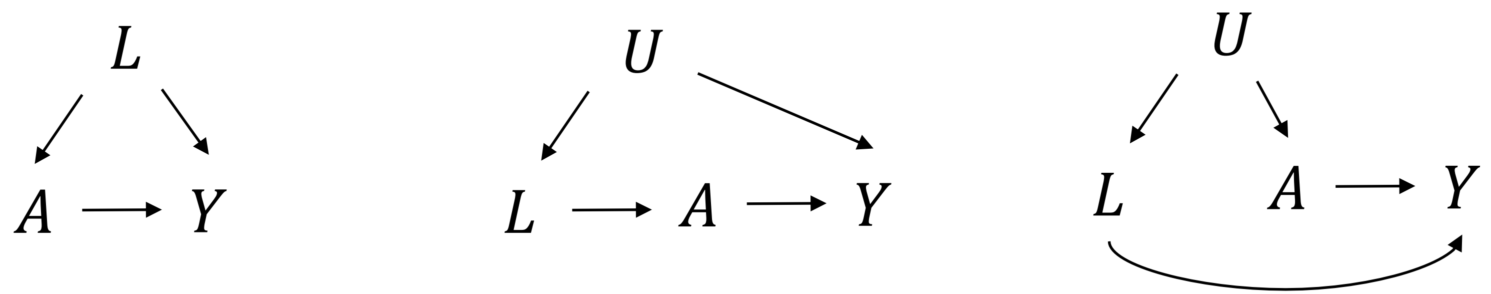
- Systematic bias: bias that exists even when the true distribution of observed data is known
- Estimation bias: bias in statistical estimation

$A \perp Y(a) \mid L$ for all a not true: introduce systematic bias

Two types of systematic bias:

- Confounding: treatment and outcome share a common cause
- Selection bias: conditioning on common effects

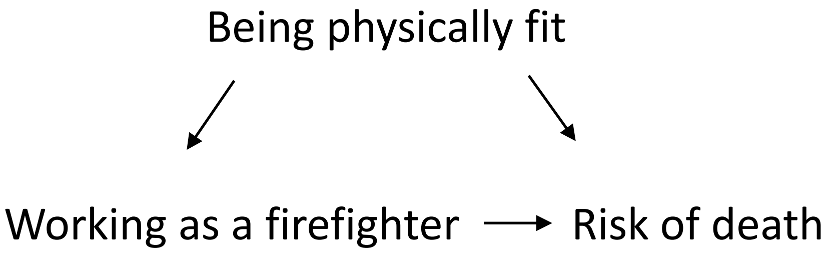
Confounding



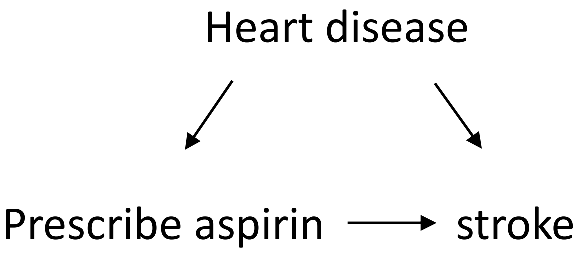
If we believe that confounding is likely, how to determine L for which $A \perp Y(a) \mid L$ for all a holds?

Backdoor path: non-causal path between A and Y which has an arrow pointing to the assignment A

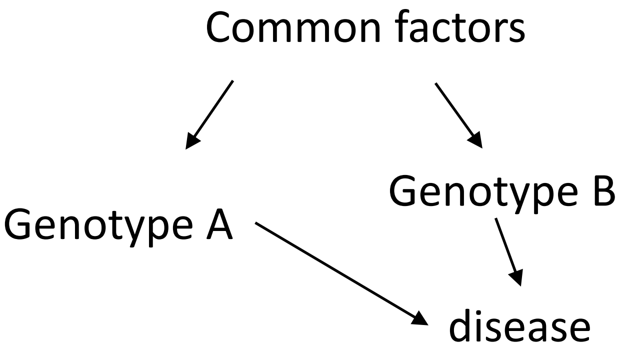
Examples of confounding:



Healthy working bias



Confounding by indication



Linkage disequilibrium / population stratification

The backdoor criterion

A set of covariates L satisfies the backdoor criterion if all backdoor paths between A and Y are blocked by conditioning on L and L contains no variables that are descendant of treatment A

Backdoor adjustment

If L satisfies the backdoor criterion, then

$$P(Y|do(A)) = \sum_l P(Y|A, L = l)P(L = l)$$

Or in other words, $A \perp Y(a) \mid L$ for all a

Proof idea of the backdoor adjustment

$$\mathbb{P}\left[Y \mid do(A)\right] = \sum_{pa} \mathbb{P}\left[Y \mid A, PA = pa\right] \mathbb{P}\left[PA = pa\right] = \sum_{pa} \left(\sum_{l} \mathbb{P}\left[Y, L = l \mid A, PA = pa\right] \right) \mathbb{P}\left[PA = pa\right]$$

$$= \sum_{pa} \left(\sum_{l} \mathbb{P}\left[Y \mid L = l, A, PA = pa\right] \mathbb{P}\left[L = l \mid A, PA = pa\right] \right) \mathbb{P}\left[PA = pa\right]$$

As L satisfies the backdoor criterion:

1. L are no descents of A, PA are all direct parents of A

$$(L \perp\!\!\!\perp A \mid PA)_G \implies L \perp A \mid PA$$

2. $(L \perp\!\!\!\perp PA \mid A, L)_G \implies L \perp PA \mid A, L$

If this is not true, then there is an open path not through A and L. This introduces a backdoor path between A and Y that is not blocked by L

$$\mathbb{P}\left[Y \mid do(A)\right]$$

$$= \sum_{pa} \left(\sum_{l} \mathbb{P}\left[Y \mid L = l, A\right] \mathbb{P}\left[L = l \mid PA = pa\right] \right) \mathbb{P}\left[PA = pa\right]$$

$$= \sum_{l} \mathbb{P}\left[Y \mid L = l, A\right] \sum_{pa} \left(\mathbb{P}\left[L = l \mid PA = pa\right] \mathbb{P}\left[PA = pa\right] \right)$$

$$= \sum_{l} \mathbb{P}\left[Y \mid L = l, A\right] \mathbb{P}\left[L = l\right]$$

L satisfies the backdoor criterion $\underset{\not\Longleftarrow}{\Longrightarrow}$ $A \perp Y(a) \mid L$ for all a

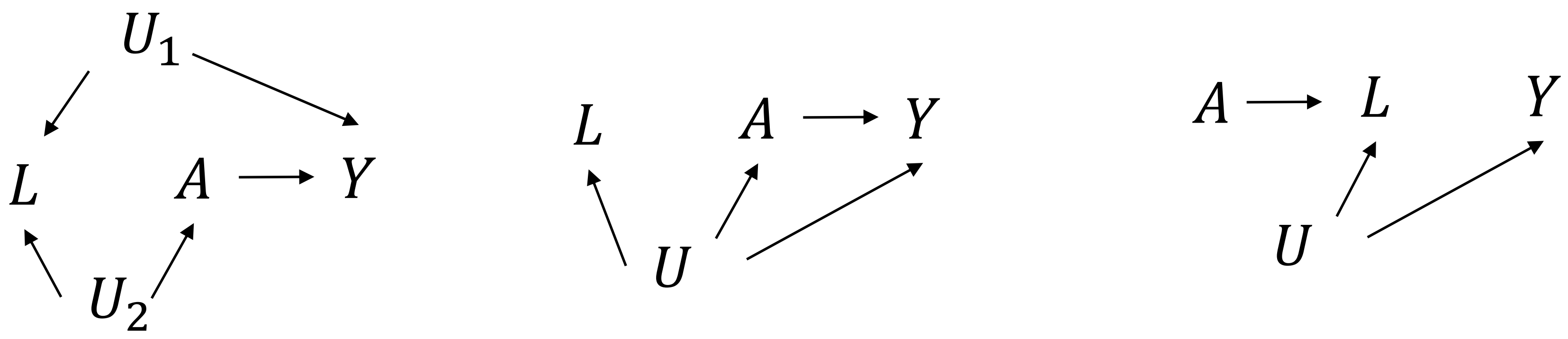
(need faithfulness assumption)

Traditional way to find confounders may introduce bias

The backdoor criterion states that there is no unmeasured confounding after adjusting for L

A traditional way to find confounding covariates L

1. L is associated with A; and
2. A is associated with Y conditional on L



In all three cases, L is not a confounder!

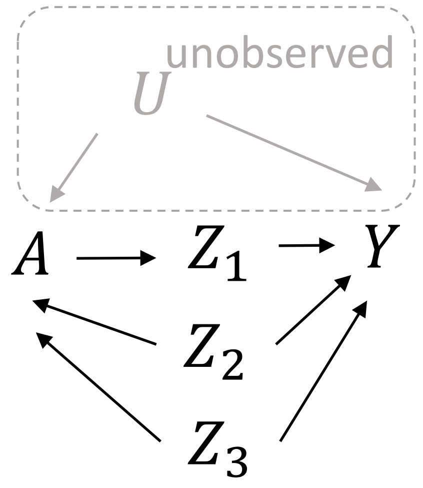
The frontdoor criterion

Can we identify causal effects when conditional exchangeability fails?

Definition (Pearl 1995)

A set of variables Z satisfies the front-door criterion if:

1. Z intercepts all directed paths from A to Y; and
2. There is no unblocked backdoor path from A to Z; and
3. All backdoor paths from Z to Y are blocked by A



$Z = (Z_1, Z_2, Z_3)$

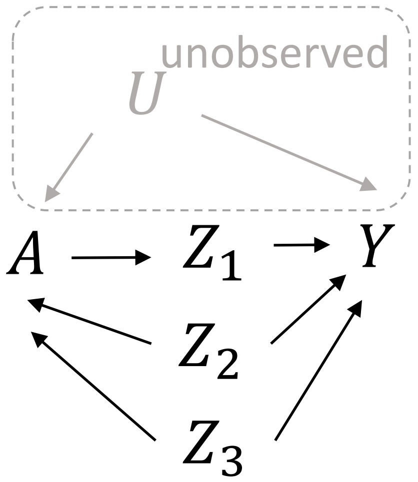
Frontdoor adjustment

If Z satisfies the frontdoor criterion and if P(A, Z) > 0, then the causal effect of A on Y is identifiable:

$$P(Y|do(A)) = \sum_{z} P(Z=z|A) \sum_{a'} P(Y|A=a', Z=z) P(A=a')$$

Frontdoor adjustment

$$P(Y|do(A)) = \sum_z P(Z = z|A) \sum_{a'} P(Y|A = a', Z = z) P(A = a')$$

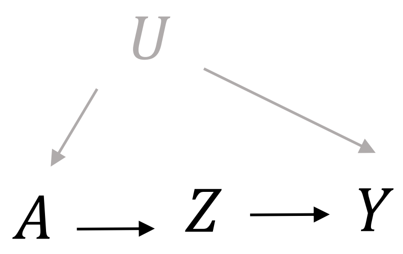


$Z = (Z_1, Z_2, Z_3)$

Intuitive idea of the frontdoor adjustment:

1. $P(Z|do(A))$ is identifiable
2. $P(Y|do(Z))$ is identifiable
3. $P(Y|do(A)) = \sum_z P(Y|do(Z) = z) P(Z = z|do(A))$

Proof idea of the frontdoor adjustment



$$\mathbb{P}\left[Y \mid do(A)\right] = \sum_u \mathbb{P}\left[Y \mid A, U = u\right] \mathbb{P}\left[U = u\right] = \sum_u \left(\sum_z \mathbb{P}\left[Y, Z = z \mid A, U = u\right] \right) \mathbb{P}\left[U = u\right]$$

$$= \sum_u \left(\sum_z \mathbb{P}\left[Y \mid Z = z, A, U = u\right] \mathbb{P}\left[Z = z \mid A, U = u\right] \right) \mathbb{P}\left[U = u\right]$$

As Z satisfies the frontdoor criterion:

1. $(Z \perp\!\!\!\perp U \mid A)_G \implies Z \perp U \mid A$

   Otherwise, there is an unblocked backdoor path from A to Z
2. $(A \perp\!\!\!\perp Y \mid Z, U)_G \implies A \perp Y \mid Z, U$

   Z blocks paths from A and U blocks paths to A
3. U blocks all backdoor paths from Z to Y

$$\begin{aligned}
&\mathbb{P}\left[Y \mid do(A)\right] \\
&= \sum_u \left(\sum_z \mathbb{P}\left[Y \mid Z = z, U = u\right] \mathbb{P}\left[Z = z \mid A\right] \right) \mathbb{P}\left[U = u\right] \\
&= \sum_z \mathbb{P}\left[Z = z \mid A\right] \left(\sum_u \mathbb{P}\left[Y \mid Z = z, U = u\right] \mathbb{P}\left[U = u\right] \right) \\
&= \sum_z \mathbb{P}\left[Z = z \mid A\right] \mathbb{P}\left[Y \mid do(Z)\right] \\
&= \sum_z \mathbb{P}\left[Z = z \mid A\right] \left(\sum_{a'} \mathbb{P}\left[Y \mid Z = z, A = a'\right] \mathbb{P}\left[A = a'\right] \right)
\end{aligned}$$