# STAT347: Generalized Linear Models
# Lecture 7

Today's topics: Chapter 6.1

- Nominal response: baseline-category logit model
  - Model setup
  - Multivariate GLM
  - Model fitting

Multinomial response variables:

- Nominal response: $c$ categories without orders. For instance the response can be the answer to: which major does an undergraduate student choose?
- Ordinal response: categories with orders: not satisfied, satisfied, very satisfied

How to model their relationship with the covariates?

## Nominal responses: Baseline-Category logit model

For the nominal response variable, a natural choice of the distribution is the multinomial distribution. Specifically, we assume that for each sample, the multinomial response variable is

$$y_i = (y_{i1}, y_{i2}, \cdots, y_{ic}) \sim \text{Multinomial}\left(n_i, p = (p_{i1}, p_{i2}, \cdots, p_{ic})\right)$$

where $c$ is the total number of choices. $y_{ij} = 1$ for sample $i$ choose level $j$ and $y_{ij'} = 0$ for all $j' \neq j$.

Treat the multinomial response variable as multiple responses and build a model for each of these responses.

## 1 Why using the logit link?

We can build a Binary GLM model for each pair of categories.

Select a baseline category (say category $c$), then we can build a binary GLM for each of $1, 2, \cdots, c-1$ categories compared with category $c$. Basically, we assume

$$\frac{p_{ik}}{p_{ik} + p_{ic}} = F(X_i^T \beta_k)$$

However, not every $F$ is good to use. When we think that these categories are "exchangeable", since the choice of baseline category $c$ is arbitrary, a desired property is that the model does not depend on which category you choose as the baseline. Specifically, it means that if we switch to a category $c'$, for any $k' \neq c'$, from (1) we can find some $\tilde{\beta}_{k'}$

$$\frac{p_{ik'}}{p_{ik'} + p_{ic'}} = F(X_i^T \tilde{\beta}_{k'})$$

- If $F$ corresponds to the logit link, then we have

$$\frac{p_{ik}}{p_{ic}} = e^{X_i^T \beta_k}$$

  This is called the baseline-category logit model.

  - for $k \neq c$, $\tilde{\beta}_k = \beta_k - \beta_{c'}$.

$$\frac{p_{ik}}{p_{ic'}} = e^{X_i^T (\beta_k - \beta_{c'})}$$

  - for $k = c$, $\tilde{\beta}_c = -\beta_{c'}$ ($\beta_c = 0$)

- If there is a natural baseline category in some applications (categories not "exchangeable"), other links can still be used.

Under the baseline-category logit model, we have

$$p_{ik} = \frac{e^{X_i^T \beta_k}}{1 + \sum_{h=1}^{c-1} e^{X_i^T \beta_h}}$$

# 2 Multivariate GLM

Treating each pair is a seperate logistic regression, we can get the asymptotic distribution of each $\hat{\beta}_k$.

- The $\hat{\beta}_k$ for $k = 1, 2, \cdots, c$ categories are not independent (as $y_{ik}$ are not)
- The $\hat{\beta}_k$ may not be efficient ignoring other categories
- How to calculate the distribution of some function $h(\hat{\beta}_1, \cdots, \hat{\beta}_k)$ if needed? (For example, we may want to know the distribution of $\hat{p}_{i1} - \hat{p}_{i2}$)

We can generalize the univariate GLM to a multivariate GLM where $y_i = (y_{i1}, y_{i2}, \cdots, y_{i,c-1})$ follows a multivariate exponential dispersion family distribution

$$f(y_i; \theta_i) = e^{\frac{y_i^T \theta_i - b(\theta_i)}{a(\phi)}} f_0(y_i; \phi)$$

where $\theta_i = (\theta_{i1}, \cdots, \theta_{ic})$.

- We drop $y_{ic}$ as $y_{ic} = 1 - \sum_{k \neq c} y_{ik}$
- The mean vector is $\mu_i = (\mu_{i1}, \cdots, \mu_{i,c-1}) = (p_{i1}, \cdots, p_{i,c-1})$
- The link function is $g(\mu_i) = \boldsymbol{X}_i \beta$ where

$$\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_c \end{pmatrix}, \boldsymbol{X}_i = \begin{pmatrix} X_i^T & 0 & \cdots & 0 \\ 0 & X_i^T & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & X_i^T \end{pmatrix}$$

- The form of the link function is $g_k(\mu_i) = \log\left[\mu_{ik} / (1 - \sum_{k'} \mu_{ik'})\right]$

# 3 Fitting baseline-category logit model

Consider the ungrouped data format and let $N = \sum_{i'} n_{i'}$.
The joint log-likelihood for the multivariate GLM is

$$
\begin{aligned}
L(\beta; y) &= \log\left[\prod_{i=1}^{N}\left(\prod_{k=1}^{c} p_{ik}^{y_{ik}}\right)\right] \\
&= \sum_{i=1}^{N}\left\{\sum_{k=1}^{c-1} y_{ik}\log\frac{p_{ik}}{p_{ic}} + \log p_{ic}\right\} \\
&= \sum_{i=1}^{N}\left\{\sum_{k=1}^{c-1} y_{ik}X_i^T\beta_k - \log\left(1+\sum_{h=1}^{c-1} e^{X_i^T\beta_h}\right)\right\} \\
&= \sum_{k=1}^{c-1}\left\{\sum_{j=1}^{p}\beta_{kj}\left(\sum_{i=1}^{N} y_{ik}x_{ij}\right)\right\} - \sum_{i=1}^{N}\left\{\log\left(1+\sum_{h=1}^{c-1} e^{X_i^T\beta_h}\right)\right\}
\end{aligned}
$$

The score equations are

$$
\frac{\partial L}{\partial \beta_{kj}} = \sum_{i=1}^{N} y_{ik}x_{ij} - \sum_{i=1}^{N}\frac{e^{X_i^T\beta_k}x_{ij}}{1+\sum_{h=1}^{c-1} e^{X_i^T\beta_h}} = \sum_{i=1}^{N}(y_{ik}-p_{ik})x_{ij} = 0
$$

which have the same forms as we saw before for canonical link.

For computation, we can find that Fisher-scoring is the same as Newton's method (details omitted, see Chapter 6.1.3).