# Linear least square (final)

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- Static Estimation:

$$\begin{cases} y = h(x) + v, & v \text{ is the noise} \\ \dot{x} = 0 \end{cases}$$

- Goal: Estimate $x$ given one or more $y$.

$$\hat{x} = g(y) \quad \text{or} \quad \hat{x} = g(\tilde{y}), \quad \tilde{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

- Least square estimation:

$$J = \|y - h(x)\|^2 \qquad \hat{x} = \underset{x}{\mathrm{Argmin}}\, J = \underset{x}{\mathrm{Argmin}} \|y - h(x)\|^2$$

- Weighted least square estimation:

$$J = \|y - h(x)\|^2_{A^{-1}} = \underset{1\times m}{(y - h(x))^T} \underset{m\times m}{A^{-1}} \underset{m\times 1}{(y - h(x))} \longmapsto \text{scalar} \quad y \in \mathbb{R}^m$$

- Incorporate the a priori state (previous information or belief on $x$) $x \sim ?(\bar{x}_0, \cdots)$

$$J = (y - h(x))^T A^{-1} (y - h(x)) + (x - \bar{x}_0)^T B^{-1} (x - \bar{x}_0)$$ Complete W.L.S.E.

- Linear WLSE.

System $\begin{cases} y = Hx + v, & v \sim N(0, R) \\ \dot{x} = 0, & x \sim N(\bar{x}_0, P_0) \end{cases}$

The best choise for $J$:

$$J = (y - Hx)^T R^{-1} (y - Hx) + (x - \bar{x}_0)^T P_0^{-1} (x - \bar{x}_0)$$

$$\hat{x}? \quad \hat{x} \Leftarrow \frac{\partial J}{\partial x} = 0$$

$$\frac{\partial J}{\partial x} = (y - Hx)^T R^{-1} H + (x - \bar{x}_0)^T P_0^{-1} = 0$$

$$\hat{x} = (H^T R^{-1} H + P_0^{-1})^{-1} (H^T R^{-1} y + P_0^{-1} \bar{x}_0)$$

$$\overset{\text{Matrix Inversion}}{=} \bar{x}_0 + \underbrace{P_0 H^T (H P_0 H^T + R)^{-1}}_{K} (y - H\bar{x}_0) \longmapsto \text{Update fcn for Linear K.F.}$$

$$= \bar{x}_0 + K(y - H\bar{x}_0)$$

- What is the variance of the estimation error?

Recall: Estimation Error $\tilde{x} = x - \hat{x}$

Error Covariance $P = E[\tilde{x}\tilde{x}^T]$

Derivation:

$$x - \bar{x}_0 = \tilde{x}_0$$

$$\begin{aligned} \tilde{x} = x - \hat{x} &= x - \bar{x}_0 - K(y - H\bar{x}_0) \\ &= \tilde{x}_0 - K(Hx + v - H\bar{x}_0) \\ &= (I - KH)\tilde{x}_0 - Kv \end{aligned}$$

$$\begin{aligned} \tilde{x}\tilde{x}^T = &(I - KH)\tilde{x}_0\tilde{x}_0^T(I - KH) + Kvv^TK^T \\ &- (I - KH)\tilde{x}_0 v^T K - Kv\tilde{x}_0^T(I - KH)^T \end{aligned}$$

$$\begin{aligned} P = E[\tilde{x}\tilde{x}^T] &= (I - KH)P_0(I - KH)^T + KRK^T \quad \text{Joseph form} \\ &= P_0 - P_0H^T[HP_0H^T + R^{-1}]^{-1}HP_0 \\ &= [H^TR^{-1}H + P_0^{-1}]^{-1} \end{aligned}$$

$E[\tilde{x}v^T] = 0$

$E[\tilde{x}_0 v^T] = 0$

$E[\tilde{x}_0\tilde{x}_0^T] = P_0$

$E[vv^T] = R$

- Multiple measurements:

$$y_i = H_i x + v_i, \quad v_i \sim N(0, R_i), \quad i = 1, \cdots, n$$

- Solution after construction:

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \quad H = \begin{bmatrix} H_1 \\ \vdots \\ H_n \end{bmatrix} \quad v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \Rightarrow y = Hx + v, \quad v \sim N(0, R)$$

$$R = \begin{bmatrix} R_1 & & & \\ & R_2 & & \\ & & \ddots & \\ & & & R_n \end{bmatrix}$$

$$\hat{x} = \bar{x}_0 + P_0H^T(HP_0H^T + R)^{-1}(y - H\bar{x}_0)$$

$$P = [H^TR^{-1}H + P_0^{-1}]^{-1}$$ drawback: high dimension.

- Alternate Solution:

$$\hat{x}_i = [H_i^T R_i^{-1} H_i + P_0^{-1}]^{-1}[H_i^T R_i^{-1} y_i + P_0^{-1}\bar{x}_0]$$

$$P_i = [H_i^T R_i^{-1} H_i + P_0^{-1}]^{-1}$$

$$P = [\sum_{i=1}^{n} P_i^{-1}]^{-1}$$

$$\hat{x} = P \sum_{i=1}^{n} P_i^{-1}\hat{x}_i$$ low dimensional, flexible in adding new measure.

- Matlab program ☺