# Homework 1

Jingwei Zhang 201528013229095

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# 1 Problem 1

## 1.1 a

Since:

$$P(error|x) = \begin{cases} P(\omega_2|x) & \text{if } x > \theta \\ P(\omega_1|x) & \text{if } x \leq \theta \end{cases}$$

Then:

$$\begin{aligned} P(error) &= \int_{-\infty}^{+\infty} p(error, x)\ \mathrm{d}x \\ &= \int_{-\infty}^{\theta} p(\omega_1, x)\ \mathrm{d}x + \int_{\theta}^{+\infty} p(\omega_2, x)\ \mathrm{d}x \\ &= \int_{-\infty}^{\theta} p(x|\omega_1)P(\omega_1)\ \mathrm{d}x + \int_{\theta}^{+\infty} p(x|\omega_2)P(\omega_2)\ \mathrm{d}x \\ &= P(\omega_1)\int_{-\infty}^{\theta} p(x|\omega_1)\ \mathrm{d}x + P(\omega_2)\int_{\theta}^{+\infty} p(x|\omega_2)\ \mathrm{d}x \end{aligned}$$

## 1.2 b

Since $x$ is defined on $(-\infty, +\infty)$, if $P(error)$ has minimum, $P'(error)$ must be 0.

$$\begin{aligned} P'(error) &= [P(\omega_1)\int_{-\infty}^{\theta} p(x|\omega_1)\ \mathrm{d}x + P(\omega_2)\int_{\theta}^{+\infty} p(x|\omega_2)\ \mathrm{d}x]' \\ &= P(\omega_1)[\int_{-\infty}^{\theta} p(x|\omega_1)\ \mathrm{d}x]' - P(\omega_2)[\int_{+\infty}^{\theta} p(x|\omega_2)\ \mathrm{d}x]' \\ &= P(\omega_1)p(\theta|\omega_1) - P(\omega_2)p(\theta|\omega_2) \\ &= 0 \end{aligned}$$

We can get:

$$P(\omega_1)p(\theta|\omega_1) = P(\omega_2)p(\theta|\omega_2)$$

## 1.3 c

No.$P'(error) = 0$ for specific $x$ only guarantees that $x$ is an stagnation point. It can be a local maximum, a local minimum or just nothing. And there may be multiple $\theta$ that satisfy the equation.

### 1.4 d

Suppose:

$$\begin{aligned}
P(\omega_1) &= P(\omega_2) = \frac{1}{2} \\
P(x|\omega_1) &= \frac{1}{\sqrt{2\pi}} e^{-(x+1)^2/2} \\
P(x|\omega_2) &= \frac{1}{\sqrt{2\pi}} e^{-(x-1)^2/2}
\end{aligned}$$

Then:

$$\begin{aligned}
P(error) &= \frac{1}{2}\int_{-\infty}^{\theta} \frac{1}{\sqrt{2\pi}} e^{-(x+1)^2/2}\,\mathrm{d}x + \frac{1}{2}\int_{-\infty}^{\theta} \frac{1}{\sqrt{2\pi}} e^{-(x-1)^2/2}\,\mathrm{d}x \\
P'(error) &= \frac{1}{2\sqrt{2\pi}}(e^{-(\theta+1)^2/2} + e^{-(\theta-1)^2/2}) \\
P''(error) &= \frac{-\theta}{\sqrt{2\pi}}(e^{-(\theta+1)^2/2} + e^{-(\theta-1)^2/2})
\end{aligned}$$

So, $P(\omega_1)p(\theta|\omega_1) = P(\omega_2)p(\theta|\omega_2)(P'(error) = 0)$ iff $\theta = 0$. And $P''(error) < 0, \forall\, \theta \in R$, which means that $P'(error) < 0, \forall\, \theta \in (-\infty, 0)$ and $P'(error) > 0, \forall\, \theta \in (0, +\infty)$. So that when $\theta = 0, P(error)$gets its global maximum.

## 2 Problem 3

### 2.1 a

Suppose $\mu_1 \le \mu_2$:

From the probability function we can get that:

$$\begin{cases} p(x|\omega_1) \ge p(x|\omega_1), \text{if } x \le \frac{\mu_2+\mu_1}{2} \\ p(x|\omega_1) < p(x|\omega_1), \text{if } x > \frac{\mu_2+\mu_1}{2} \end{cases}$$

So, we decide $\omega_1$ if $x \le (\mu_2 + \mu_1)/2$, otherwise decide $\omega_2$, which minimize $P_e$. Then the probability of error would be:

$$\begin{aligned}
P_e &= \int_{-\infty}^{t} p(\omega_2, x)\,\mathrm{d}x + \int_{t}^{+\infty} p(\omega_1, x)\,\mathrm{d}x \quad (t = \frac{\mu_2 + \mu_1}{2}) \\
&= P(\omega_2)\int_{-\infty}^{t} p(x|\omega_2)\,\mathrm{d}x + P(\omega_1)\int_{t}^{+\infty} p(x|\omega_1)\,\mathrm{d}x \\
&= \frac{1}{2\sqrt{2\pi}\sigma}(\int_{-\infty}^{t} e^{-\frac{(x-\mu_2)^2}{2\sigma^2}}\,\mathrm{d}x + \int_{t}^{+\infty} e^{-\frac{(x-\mu_1)^2}{2\sigma^2}}\,\mathrm{d}x) \\
&\text{Let } u_2 = \frac{x-\mu_2}{\sigma}, u_1 = \frac{x-\mu_1}{\sigma}: \\
&= \frac{1}{2\sqrt{2\pi}}(\int_{-\infty}^{\frac{\mu_1-\mu_2}{2}} e^{-\frac{u_2^2}{2}}\,\mathrm{d}u_2 + \int_{\frac{\mu_2-\mu_1}{2}}^{+\infty} e^{-\frac{u_1^2}{2}}\,\mathrm{d}u_1) \\
&\text{Let } u = u_1 = -u_2 \\
&= \frac{1}{2\sqrt{2\pi}}(\int_{\frac{\mu_2-\mu_1}{2}}^{+\infty} e^{-\frac{u^2}{2}}\,\mathrm{d}u + \int_{\frac{\mu_2-\mu_1}{2}}^{+\infty} e^{-\frac{u^2}{2}}\,\mathrm{d}u) \\
&= \frac{1}{\sqrt{2\pi}}\int_{a}^{+\infty} e^{-\frac{u^2}{2}}\,\mathrm{d}u \quad (a = \frac{\mu_2 - \mu_1}{\sigma})
\end{aligned}$$

When $\mu_1 > \mu_2$, the process is similar, we can get:

$$P_e = \frac{1}{\sqrt{2\pi}} \int_a^{+\infty} e^{-\frac{u^2}{2}} \mathrm{d}u \quad (a = \frac{\mu_1 - \mu_2}{\sigma})$$

Above all, the equation$P_e = \frac{1}{\sqrt{2\pi}} \int_a^{+\infty} e^{-\frac{u^2}{2}} \mathrm{d}u \quad (a = \frac{|\mu_1 - \mu_2|}{\sigma})$ can be proven.

## 2.2 b

Since:

$$\lim_{a\to\infty} e^{-a^2/2} = 0$$
$$\lim_{a\to\infty} \frac{1}{\sqrt{2\pi}a} = 0$$

We can get:

$$\begin{aligned}
\lim_{a\to\infty} P_e &= \lim_{a\to\infty} \frac{1}{\sqrt{2\pi}} \int_a^{+\infty} e^{-\frac{u^2}{2}} \mathrm{d}t \\
&\leq \lim_{a\to\infty} \frac{1}{\sqrt{2\pi}a} e^{-a^2/2} \\
&= \lim_{a\to\infty} e^{-a^2/2} \lim_{a\to\infty} \frac{1}{\sqrt{2\pi}a} \\
&= 0
\end{aligned}$$

# 3 Problem 4

## 3.1 a

Since these samples are independent, the joint probability density function is:

$$\begin{aligned}
P(x_1\omega_1, x_2\omega_3, x3\omega_3, x4\omega_2) &= P(x_1\omega_1)P(x_2\omega_3)P(x3\omega_3)P(x4\omega_2) \\
&= P(x_1|\omega_1)P(x_2|\omega_3)P(x3|\omega_3)P(x4|\omega_2)P(\omega_1)P(\omega_2)P(\omega_3)^2 \\
&= z(\frac{0.6}{1})z(\frac{0.1-1}{1})z(\frac{0.9-1}{1})z(\frac{1.1-1}{1})\frac{1}{2}\frac{1}{4}(\frac{1}{4})^2
\end{aligned}$$

## 3.2 b

## 3.3 c

# 4 Problem 5