第 2 周阅读资料

The Proof of Theory 2.4.

We will show that a subset P of $\mathbb{R}^n$ is a polytope if and only if it is a bounded polyhedron. This might be intuitively clear, but a strictly mathematical proof requires some work.

We first give a definition. Let P be a convex set. A point $z \in P$ is called a *vertex* of P if z is *not* a convex combination of two other points in P. That is, there do not exist points x, y in P and a λ with $0 < \lambda < 1$ such that $x \neq z, y \neq z$ and $z = \lambda x + (1-\lambda)y$.

To characterize vertices we introduce the following notation. Let $P = \{x \mid Ax \leq b\}$ be a polyhedron and let $z \in P$. Then A_z is the submatrix of A consisting of those rows a_i of A for which $a_i z = b_i$.

Then we can show:

Theorem 2.2. *Let $P = \{x \mid Ax \leq b\}$ be a polyhedron in $\mathbb{R}^n$ and let $z \in P$. Then z is a vertex of P if and only if* $\text{rank}(A_z) = n$.

Proof. *Necessity.* Let z be a vertex of P and suppose $\text{rank}(A_z) < n$. Then there exists a vector $c \neq 0$ such that $A_z c = 0$. Since $a_i z < b_i$ for every a_i that does not occur in A_z, there exists a $\delta > 0$ such that:

$$(12) \qquad a_i(z + \delta c) \leq b_i \text{ and } a_i(z - \delta c) \leq b_i$$

for every row a_i of A not occurring in A_z. Since $A_z c = 0$ and $Az \leq b$ it follows that

$$(13) \qquad A(z + \delta c) \leq b \text{ and } A(z - \delta c) \leq b.$$

So $z + \delta c$ and $z - \delta c$ belong to P. Since z is a convex combination of these two vectors, this contradicts the fact that z is a vertex of P.

Sufficiency. Suppose $\text{rank}(A_z) = n$ while z is not a vertex of P. Then there exist points x and y in P such that $x \neq z \neq y$ and $z = \frac{1}{2}(x + y)$. Then for every row a_i of A_z:

$$(14) \qquad \begin{aligned} a_i x \leq b_i = a_i z &\implies a_i(x - z) \leq 0, \text{ and} \\ a_i y \leq b_i = a_i z &\implies a_i(y - z) \leq 0. \end{aligned}$$

Since $y - z = -(x - z)$, this implies that $a_i(x - z) = 0$. Hence $A_z(x - z) = 0$. Since $x - z \neq 0$, this contradicts the fact that $\text{rank}(A_z) = n$. ∎

Theorem 2.2 implies that a polyhedron has only a finite number of vertices: For each two different vertices z and z' one has $A_z \neq A_{z'}$, since $A_z x = b_z$ has only one solution, namely $x = z$ (where b_z denotes the part of b corresponding to A_z). Since there exist at most 2^m collections of subrows of A, P has at most 2^m vertices.

From Theorem 2.2 we derive:

Theorem 2.3. *Let P be a bounded polyhedron, with vertices $x_1, \ldots, x_t$. Then* $P = \text{conv.hull}\{x_1, \ldots, x_t\}$.

Proof. Clearly

(15) $\quad \text{conv.hull}\{x_1, \ldots, x_t\} \subseteq P,$

since $x_1, \ldots, x_t$ belong to P and since P is convex.

The reverse inclusion amounts to:

(16) $\quad$ if $z \in P$ then $z \in \text{conv.hull}\{x_1, \ldots, x_t\}$.

We show (16) by induction on $n - \text{rank}(A_z)$.

If $n - \text{rank}(A_z) = 0$, then $\text{rank}(A_z) = n$, and hence, by Theorem 2.2, z itself is a vertex of P. So $z \in \text{conv.hull}\{x_1, \ldots, x_t\}$.

If $n - \text{rank}(A_z) > 0$, then there exists a vector $c \neq 0$ such that $A_z c = 0$. Define

(17)
$$\mu_0 := \max\{\mu \mid z + \mu c \in P\},$$
$$\nu_0 := \max\{\nu \mid z - \nu c \in P\}.$$

These numbers exist since P is compact. Let $x := z + \mu_0 c$ and $y := z - \nu_0 c$.

Now

(18)
$$\mu_0 = \min\{\frac{b_i - a_i z}{a_i c} \mid a_i \text{ is a row of } A;\ a_i c > 0\}.$$

This follows from the fact that μ_0 is the largest μ such that $a_i(z + \mu c) \leq b_i$ for each $i = 1, \ldots, m$. That is, it is the largest μ such that

(19)
$$\mu \leq \frac{b_i - a_i z}{a_i c}$$

for every i with $a_i c > 0$.

Let the minimum (18) be attained by i_0. So for i_0 we have equality in (18). Therefore

(20)
$$\text{(i) } A_z x = A_z z + \mu_0 A_z c = A_z z,$$
$$\text{(ii) } a_{i_0} x = a_{i_0}(z + \mu_0 c) = b_{i_0}.$$

So A_x contains all rows in A_z, and moreover it contains row a_{i_0}. Now $A_z c = 0$ while $a_{i_0} c \neq 0$. This implies $\text{rank}(A_x) > \text{rank}(A_z)$. So by our induction hypothesis, x belongs to $\text{conv.hull}\{x_1, \ldots, x_t\}$. Similarly, y belongs to $\text{conv.hull}\{x_1, \ldots, x_t\}$. Therefore, as z is a convex combination of x and y, z belongs to $\text{conv.hull}\{x_1, \ldots, x_t\}$. ∎

As a direct consequence we have:

Corollary 2.3a. *Each bounded polyhedron is a polytope.*

Proof. Directly from Theorem 2.3. ∎

Conversely:

Theorem 2.4. *Each polytope is a bounded polyhedron.*

Proof. Let P be a polytope in $\mathbb{R}^n$, say

$$(21) \qquad P = \text{conv.hull}\{x_1, \ldots, x_t\}.$$

We may assume that $t \geq 1$. We prove the theorem by induction on n. Clearly, P is bounded.

If P is contained in some affine hyperplane, the theorem follows from the induction hypothesis.

So we may assume that P is not contained in any affine hyperplane. It implies that the vectors $x_2 - x_1, \ldots, x_t - x_1$ span $\mathbb{R}^n$. It follows that there exist a vector x_0 in P and a real $r > 0$ such that the ball $B(x_0, r)$ is contained in P.

Without loss of generality, $x_0 = 0$. Define P^* by

$$(22) \qquad P^* := \{y \in \mathbb{R}^n \mid x^T y \leq 1 \text{ for each } x \in P\}.$$

Then P^* is a polyhedron, as

$$(23) \qquad P^* = \{y \in \mathbb{R}^n \mid x_j^T y \leq 1 \text{ for } j = 1, \ldots, t\}.$$

This follows from the fact that if y belongs to the right hand set in (23) and $x \in P$ then $x = \lambda_1 x_1 + \cdots + \lambda_t x_t$ for certain $\lambda_1, \ldots, \lambda_t \geq 0$ with $\lambda_1 + \cdots + \lambda_t = 1$, implying

$$(24) \qquad x^T y = \sum_{j=1}^{t} \lambda_j x_j^T y \leq \sum_{j=1}^{t} \lambda_j = 1.$$

So y belongs to P^*.

Moreover, P^* is bounded, since for each $y \neq 0$ in P^* one has that $x := r \cdot \|y\|^{-1} \cdot y$ belongs to $B(0, r)$ and hence to P. Therefore, $x^T y \leq 1$, and hence

$$(25) \qquad \|y\| = (x^T y)/r \leq 1/r.$$

So $P^* \subseteq B(0, 1/r)$.

This proves that P^* is a bounded polyhedron. By Corollary 2.3a, P^* is a polytope. So there exist vectors $y_1, \ldots, y_s$ in $\mathbb{R}^n$ such that

$$(26) \qquad P^* = \text{conv.hull}\{y_1, \ldots, y_s\}.$$

We show:

(27) $$P = \{x \in \mathbb{R}^n \mid y_j^T x \leq 1 \text{ for all } j = 1, \ldots, s\}.$$

This implies that P is a polyhedron.

To see the inclusion $\subseteq$ in (27), it suffices to show that each of the vectors x_i belongs to the right hand side in (27). This follows directly from the fact that for each $j = 1, \ldots, s$, $y_j^T x_i = x_i^T y_j \leq 1$, since y_j belongs to P^*.

To see the inclusion $\supseteq$ in (25), let $x \in \mathbb{R}^n$ be such that $y_j^T x \leq 1$ for all $j = 1, \ldots, s$. Suppose $x \notin P$. Then there exists a hyperplane separating x and P. That is, there exist a vector $c \neq 0$ in $\mathbb{R}^n$ and a $\delta \in \mathbb{R}$ such that $c^T x' < \delta$ for each $x' \in P$, while $c^T x > \delta$. As $0 \in P$, $\delta > 0$. So we may assume $\delta = 1$. Hence $c \in P^*$. So there exist $\mu_1, \ldots, \mu_s \geq 0$ such that $c = \mu_1 y_1 + \cdots \mu_s y_s$ and $\mu_1 + \cdots + \mu_s = 1$. This gives the contradiction:

(28) $$1 < c^T x = \sum_{j=1}^{s} \mu_j y_j^T x \leq \sum_{j=1}^{s} \mu_j = 1. \blacksquare$$