Rocket Attitude

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1 Introduction

In this document, I will discuss the use of the Kalman Filter for estimating the attitude of a rocket during launch. The Kalman Filter is a powerful algorithm that combines predictions from a system model with measurements from sensors to provide an optimal estimate of the state variables, which in this case include the rocket's attitude and angular velocity.

2 Given Data

The following data is given for the rocket launch: Ignition (t = 0):

- Acceleration: $(0, 0, 0)$ m/s^2
- Rolling rate: $(0, 0, 0)$ deg/s
- Pitch angle: 0 rad
- Roll angle: 0 rad

Time step 1 (t = 1):

- Acceleration: $(9, 1, 1)$ m/s^2
- Rolling rate: $(10, 0, 10)$ deg/s
- Pitch angle: -0.1 rad
- Roll angle: 0.05 rad

Time step 2 (t = 2):

- Acceleration: $(8, 1, 1)$ m/s^2
- Rolling rate: $(20, 1, 20)$ deg/s
- Pitch angle: -0.2 rad
- Roll angle: 0.1 rad

3 State Vector and Covariance Matrix Initialization

The state vector $\boldsymbol{x}$ represents the current estimate of the rocket's attitude and angular velocity. It consists of the quaternion components (q_0, q_1, q_2, q_3) and the angular velocity components $(\omega_x, \omega_y, \omega_z)$. Initialize the state vector $\boldsymbol{x}$ and covariance matrix $\boldsymbol{P}$:

$$
\begin{aligned}
\boldsymbol{x} &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^\top \\
\boldsymbol{P} &= \boldsymbol{I}^{7\times 7}
\end{aligned}
$$

where $\boldsymbol{I}^{7\times 7}$ is the 7×7 identity matrix.

4 Prediction Step

The prediction step uses the system model to predict the state estimate and covariance matrix at the current time step based on the previous state estimate. Define the state transition matrix $\boldsymbol{A}$ and process noise covariance matrix $\boldsymbol{Q}$:

$$
\boldsymbol{A} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0.5\Delta t & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0.5\Delta t & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0.5\Delta t \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\quad \boldsymbol{Q} \quad = \mathrm{diag}([0.001, 0.001, 0.001, 0.001, 0.01, 0.01, 0.01])
$$

where Δt is the time step between measurements. Predict the state estimate $\hat{\boldsymbol{x}}^-$ and covariance matrix $\boldsymbol{P}^-$:

$$
\begin{aligned}
\hat{\boldsymbol{x}}^- &= \boldsymbol{A}\hat{\boldsymbol{x}} \\
\boldsymbol{P}^- &= \boldsymbol{A}\boldsymbol{P}\boldsymbol{A}^\top + \boldsymbol{Q}
\end{aligned}
$$

The state transition matrix $\boldsymbol{A}$ and the process noise covariance matrix $\boldsymbol{Q}$ are important components of the Kalman Filter prediction step. Let's dive into the details of each matrix and explain their elements and meanings. **State Transition Matrix $\boldsymbol{A}$:** The state transition matrix $\boldsymbol{A}$ describes how the state variables evolve over time based on the system model. It relates the state at the previous time step to the state at the current time step, assuming no noise or external influences. In this case, the state vector consists of the quaternion components (q_0, q_1, q_2, q_3) and the angular velocity components $(\omega_x, \omega_y, \omega_z)$. The matrix $\boldsymbol{A}$

is a 7×7 matrix:

$$
\boldsymbol{A} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0.5\Delta t & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0.5\Delta t & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0.5\Delta t \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

The elements of matrix $\boldsymbol{A}$ have the following meanings:

- The diagonal elements $(1, 1, 1, 1)$ in the first four rows and columns represent the quaternion components (q_0, q_1, q_2, q_3). These elements indicate that the quaternion remains unchanged if there is no rotation.
- The off-diagonal elements $(0.5\Delta t)$ in the first four rows and the last three columns represent the coupling between the quaternion and the angular velocity. These elements describe how the quaternion components change based on the angular velocity over the time step Δt. The factor of 0.5 comes from the quaternion kinematics equation.
- The diagonal elements $(1, 1, 1)$ in the last three rows and columns represent the angular velocity components $(\omega_x, \omega_y, \omega_z)$. These elements assume a constant angular velocity model, meaning that the angular velocity remains unchanged over the time step.

Process Noise Covariance Matrix $\boldsymbol{Q}$: The process noise covariance matrix $\boldsymbol{Q}$ represents the uncertainty or noise associated with the system model. It accounts for the inherent uncertainties and inaccuracies in the mathematical model used to describe the system dynamics. In this example, $\boldsymbol{Q}$ is defined as a diagonal matrix using the `diag` function:

$$
\boldsymbol{Q} = \mathrm{diag}([0.001, 0.001, 0.001, 0.001, 0.01, 0.01, 0.01])
$$

The diagonal elements of $\boldsymbol{Q}$ represent the variances (squared uncertainties) of the corresponding state variables:

- The first four diagonal elements $(0.001, 0.001, 0.001, 0.001)$ represent the variances of the quaternion components (q_0, q_1, q_2, q_3). These values indicate a relatively small uncertainty in the quaternion estimates.
- The last three diagonal elements $(0.01, 0.01, 0.01)$ represent the variances of the angular velocity components $(\omega_x, \omega_y, \omega_z)$. These values indicate a higher uncertainty in the angular velocity estimates compared to the quaternion estimates.

The off-diagonal elements of $\boldsymbol{Q}$ are zero, assuming that the process noise for each state variable is independent of the others. The specific values chosen for the elements of $\boldsymbol{Q}$ depend on the characteristics of the system being modeled and are typically determined based on prior knowledge, experimentation, or tuning to achieve the desired performance of the Kalman Filter. In summary, the state transition matrix $\boldsymbol{A}$ describes how the state variables evolve over time based on the system model, while the process noise covariance matrix $\boldsymbol{Q}$ represents the uncertainties associated with the system model. These matrices are crucial for propagating the state estimate and its uncertainty in the prediction step of the Kalman Filter.

5 Update Step

The update step combines the predicted state estimate with the measurements to obtain an improved state estimate. Convert the rolling rates from deg/s to rad/s:

- Rolling rates at t=1: $(10, 0, 10)$ deg/s = $(0.1745, 0, 0.1745)$ rad/s
- Rolling rates at t=2: $(20, 1, 20)$ deg/s = $(0.3491, 0.0175, 0.3491)$ rad/s

Construct the measurement vector $\boldsymbol{z}$:

- Measurement vector at t=1: $\boldsymbol{z} = \begin{bmatrix}0.1745 & 0 & 0.1745 & -0.1 & 0.05\end{bmatrix}^\top$
- Measurement vector at t=2: $\boldsymbol{z} = \begin{bmatrix}0.3491 & 0.0175 & 0.3491 & -0.2 & 0.1\end{bmatrix}^\top$

The measurement vector $\boldsymbol{z}$ consists of the angular velocity measurements $(\omega_x, \omega_y, \omega_z)$ obtained from the gyroscope and the pitch and roll angles (θ, ϕ) obtained from the accelerometer. The measurement vector $\boldsymbol{z}$ at time step $t = 1$ is given by:

$$\boldsymbol{z} = \begin{bmatrix}0.1745 & 0 & 0.1745 & -0.1 & 0.05\end{bmatrix}^\top$$

The elements of the measurement vector $\boldsymbol{z}$ have the following meanings:

- $\omega_x = 0.1745$ rad/s: The measured angular velocity along the x-axis.
- $\omega_y = 0$ rad/s: The measured angular velocity along the y-axis.
- $\omega_z = 0.1745$ rad/s: The measured angular velocity along the z-axis.
- $\theta = -0.1$ rad: The measured pitch angle.
- $\phi = 0.05$ rad: The measured roll angle.

Define the measurement matrix $\boldsymbol{H}$ and measurement noise covariance matrix $\boldsymbol{R}$:

$$\boldsymbol{H} = \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0\end{bmatrix}$$

$$\boldsymbol{R} = \text{diag}([0.01, 0.01, 0.01, 0.01, 0.01])$$

Calculate the Kalman gain $\boldsymbol{K}$:

$$\boldsymbol{S} = \boldsymbol{H}\boldsymbol{P}^{-}\boldsymbol{H}^{\top} + \boldsymbol{R}$$
$$\boldsymbol{K} = \boldsymbol{P}^{-}\boldsymbol{H}^{\top}\boldsymbol{S}^{-1}$$

Update the state estimate $\hat{\boldsymbol{x}}$ and covariance matrix $\boldsymbol{P}$:

$$\hat{\boldsymbol{x}} = \hat{\boldsymbol{x}}^{-} + \boldsymbol{K}(\boldsymbol{z} - \boldsymbol{H}\hat{\boldsymbol{x}}^{-})$$
$$\boldsymbol{P} = (\boldsymbol{I} - \boldsymbol{K}\boldsymbol{H})\boldsymbol{P}^{-}$$

6 Numerical Example

Here's a numerical example of the Kalman Filter calculations for the given data:

6.1 Time Step 1

Predicted state estimate:

$$\hat{\boldsymbol{x}}^{-} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{\top}$$

Predicted covariance matrix:

$$\boldsymbol{P}^{-} = \begin{bmatrix} 1.001 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 1.001 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 1.001 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 1.001 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 1.01 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 1.01 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 1.01 \end{bmatrix}$$

Innovation covariance:

$$\boldsymbol{S} = \begin{bmatrix} 1.011 & 0 & 0 & 0 & 0.5 \\ 0 & 1.011 & 0 & 0 & 0 \\ 0 & 0 & 1.011 & 0 & 0 \\ 0 & 0 & 0 & 1.011 & 0 \\ 0.5 & 0 & 0 & 0 & 1.02 \end{bmatrix}$$

Kalman gain:

$$\boldsymbol{K} = \begin{bmatrix} 0.9901 & 0 & 0 & 0 & 0.4902 \\ 0 & 0.9901 & 0 & 0 & 0 \\ 0 & 0 & 0.9901 & 0 & 0 \\ 0 & 0 & 0 & 0.9901 & 0 \\ 0.4902 & 0 & 0 & 0 & 0.9804 \\ 0 & 0.4902 & 0 & 0 & 0 \\ 0 & 0 & 0.4902 & 0 & 0 \end{bmatrix}$$

Updated state estimate:

$$\hat{\boldsymbol{x}} = \begin{bmatrix}0.9973 & 0 & 0.0027 & -0.0991 & 0.1745 & 0 & 0.1745\end{bmatrix}^\top$$

Updated covariance matrix:

$$\boldsymbol{P} = \begin{bmatrix} 0.0108 & 0 & 0 & 0 & 0.0054 & 0 & 0 \\ 0 & 0.0108 & 0 & 0 & 0 & 0.0054 & 0 \\ 0 & 0 & 0.0108 & 0 & 0 & 0 & 0.0054 \\ 0 & 0 & 0 & 0.0108 & 0 & 0 & 0 \\ 0.0054 & 0 & 0 & 0 & 0.0196 & 0 & 0 \\ 0 & 0.0054 & 0 & 0 & 0 & 0.0196 & 0 \\ 0 & 0 & 0.0054 & 0 & 0 & 0 & 0.0196 \end{bmatrix}$$

6.2 Time Step 2

Predicted state estimate:

$$\hat{\boldsymbol{x}}^- = \begin{bmatrix}0.9973 & 0 & 0.0027 & -0.0991 & 0.2618 & 0 & 0.2618\end{bmatrix}^\top$$

Predicted covariance matrix:

$$\boldsymbol{P}^- = \begin{bmatrix} 0.0118 & 0 & 0 & 0 & 0.0081 & 0 & 0 \\ 0 & 0.0118 & 0 & 0 & 0 & 0.0081 & 0 \\ 0 & 0 & 0.0118 & 0 & 0 & 0 & 0.0081 \\ 0 & 0 & 0 & 0.0118 & 0 & 0 & 0 \\ 0.0081 & 0 & 0 & 0 & 0.0296 & 0 & 0 \\ 0 & 0.0081 & 0 & 0 & 0 & 0.0296 & 0 \\ 0 & 0 & 0.0081 & 0 & 0 & 0 & 0.0296 \end{bmatrix}$$

Innovation covariance:

$$\boldsymbol{S} = \begin{bmatrix} 0.0228 & 0 & 0 & 0 & 0.0081 \\ 0 & 0.0228 & 0 & 0 & 0 \\ 0 & 0 & 0.0228 & 0 & 0 \\ 0 & 0 & 0 & 0.0228 & 0 \\ 0.0081 & 0 & 0 & 0 & 0.0396 \end{bmatrix}$$

Kalman gain:

$$\boldsymbol{K} = \begin{bmatrix} 0.5180 & 0 & 0 & 0 & 0.2046 \\ 0 & 0.5180 & 0 & 0 & 0 \\ 0 & 0 & 0.5180 & 0 & 0 \\ 0 & 0 & 0 & 0.5180 & 0 \\ 0.2046 & 0 & 0 & 0 & 0.7431 \\ 0 & 0.2046 & 0 & 0 & 0 \\ 0 & 0 & 0.2046 & 0 & 0 \end{bmatrix}$$

Updated state estimate:

$$\hat{\boldsymbol{x}} = \begin{bmatrix} 0.9885 & 0.0091 & 0.0206 & -0.1512 & 0.3490 & 0.0175 & 0.3490 \end{bmatrix}^\top$$

Updated covariance matrix:

$$\boldsymbol{P} = \begin{bmatrix} 0.0057 & 0 & 0 & 0 & 0.0023 & 0 & 0 \\ 0 & 0.0057 & 0 & 0 & 0 & 0.0023 & 0 \\ 0 & 0 & 0.0057 & 0 & 0 & 0 & 0.0023 \\ 0 & 0 & 0 & 0.0057 & 0 & 0 & 0 \\ 0.0023 & 0 & 0 & 0 & 0.0219 & 0 & 0 \\ 0 & 0.0023 & 0 & 0 & 0 & 0.0219 & 0 \\ 0 & 0 & 0.0023 & 0 & 0 & 0 & 0.0219 \end{bmatrix}$$

7 Attitude Representation

The quaternion components (q_0, q_1, q_2, q_3) in the state vector represent the attitude of the rocket. Quaternions provide a compact and efficient way to describe the orientation of an object in three-dimensional space. The quaternion components have the following interpretation:

- q_0 represents the scalar part of the quaternion, also known as the real part.
- q_1, q_2, and q_3 represent the vector part of the quaternion, also known as the imaginary parts.

The quaternion satisfies the unit norm constraint:

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$$

The attitude represented by the quaternion can be visualized as a rotation of angle θ about an axis defined by the unit vector (q_1, q_2, q_3). The relationship between the quaternion components and the rotation angle and axis is given by:

$$\begin{aligned} q_0 &= \cos(\theta/2) \\ q_1 &= a_x \sin(\theta/2) \\ q_2 &= a_y \sin(\theta/2) \\ q_3 &= a_z \sin(\theta/2) \end{aligned}$$

where (a_x, a_y, a_z) is the unit vector representing the axis of rotation. In the example, the estimated quaternion components at the end of Time step 2 are:

$$[q_0, q_1, q_2, q_3] = [0.9885, 0.0091, 0.0206, -0.1512]$$

This quaternion represents the estimated attitude of the rocket at that specific time step.

8 Angular Velocity and Linear Velocity

Angular velocity ($\boldsymbol{\omega}$) and linear velocity ($\boldsymbol{v}$) are two different quantities that describe different aspects of motion. Angular velocity represents the rate of change of the rocket's orientation. It is a vector quantity that describes the rotational speed and direction of the rocket's body frame. The angular velocity is typically expressed in radians per second (rad/s) or degrees per second (deg/s) and is represented by the components $(\omega_x, \omega_y, \omega_z)$ in the body frame. Linear velocity represents the rate of change of the rocket's position. It is a vector quantity that describes the speed and direction of the rocket's motion in space. The linear velocity is typically expressed in meters per second (m/s) or any other appropriate unit of speed and is represented by the components (v_x, v_y, v_z) in a chosen reference frame. To determine the linear velocity of the rocket, additional information beyond just the angular velocity is needed. The relationship between angular velocity and linear velocity depends on the specific point of interest on the rocket and its distance from the center of rotation. The linear velocity ($\boldsymbol{v}$) of a point on the rocket can be calculated using the following equation:

$$\boldsymbol{v} = \boldsymbol{v}_c + \boldsymbol{\omega} \times \boldsymbol{r}$$

where:

- $\boldsymbol{v}_c$ is the linear velocity of the rocket's center of mass.
- $\boldsymbol{\omega}$ is the angular velocity of the rocket.
- $\boldsymbol{r}$ is the position vector from the center of mass to the point of interest.
- $\times$ denotes the cross product operation.

To determine the linear velocity of the rocket, you would need to know:

1. The linear velocity of the rocket's center of mass ($\boldsymbol{v}_c$).
2. The position vector ($\boldsymbol{r}$) from the center of mass to the point of interest.

These quantities are not provided by the angular velocity alone.

9 Conclusion

The Kalman Filter combines predictions from a system model with measurements from sensors to provide an optimal estimate of the state variables, which include the rocket's attitude represented by quaternion components and angular velocity.

The numerical example demonstrated the calculation steps involved in the prediction and update stages of the Kalman Filter. The updated state estimate at each time step represents the best estimate of the rocket's attitude based on the available information.

The quaternion components in the state vector represent the attitude of the rocket, providing a compact and efficient way to describe its orientation in three-dimensional space. Angular velocity and linear velocity are two different quantities, and additional information beyond angular velocity is needed to determine the linear velocity of the rocket.

The Kalman Filter is a powerful tool for attitude estimation, as it recursively updates the state estimate and covariance matrix by combining the predictions from the system model with the measurements at each time step, taking into account the uncertainties associated with both the model and the measurements.