CSCI 5304 HW 6

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Problem 1

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ s & 0 & 2 \end{bmatrix}$, with $s = -10^{-6}$

Problem a

Compute all the eigenvalues λ_i and corresponding right eigenvectors x_i and left eigenvectors y_i.

Solution

$$\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3) = \begin{bmatrix} 2.0000 & 0 & 0 \\ 0 & 1.0010 & 0 \\ 0 & 0 & 0.9990 \end{bmatrix}$$

$$X = (x_1, x_2, x_3) = \begin{bmatrix} -0.8165 & 1.0000 & -1.0000 \\ -0.4082 & 0.0010 & 0.0010 \\ -0.4082 & 0.0000 & -0.0000 \end{bmatrix}$$

$$Y = (y_1, y_2, y_3) = \begin{bmatrix} 0.0000 & 0.0007 & -0.0007 \\ 0.0000 & 0.7064 & 0.7078 \\ -1.0000 & -0.7078 & -0.7064 \end{bmatrix}$$

Problem b

Compute the "overall condition number" for this eigenproblem based on the perturbation formula

$$|\lambda_{A+E} - \lambda_A| \leq ||X|| \cdot ||X^{-1}|| \cdot ||E||$$

where X is the matrix of right eigenvectors and E is some generic perturbation matrix.

Solution

$\text{cond}_0 = ||X|| \cdot ||X^{-1}|| = 1.6608 \times 10^3$

Problem c

Compute the condition numbers for every individual eigenvalue based on the perturbation approximation (ignoring higher order terms)

$$|\lambda_{A+E} - \lambda_A| \approx \frac{O(||E||)}{\cos\theta} = \frac{||y_i||_2 ||x_i||_2}{y_i^T x_i} O(||E||)$$

where y_i, x_i are the left and right eigenvectors corresponding to the eigenvalue λ_A for the matrix A. Do this for each eigenvalue of A..

Solution

$\text{cond}_1 = (2.4495, 707.8165, 706.4023)^T$

Problem d

Compute the eigenvalues of the matrix $\tilde{A}$ defined as the same as the matrix A above but with $s = 0$. How do the eigenvalues of $\tilde{A}$ compare with what would be expected based on the condition number bounds?

Solution

Let's define $E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -s & 0 & 0 \end{bmatrix}$, then we can apply the above result to guess the eigenvalues of $\tilde{A}$. Note that $||E||_2 = -s$.

First consider the "overall condition number", it suggests that for each eigenvalue λ_{A+E} of $A + E$ there exists an eigenvalue λ_A of A such that

$$|\lambda_{A+E} - \lambda_A| \leq ||X|| \cdot ||X^{-1}|| \cdot ||E|| = -s * 1.6608 \times 10^3 = 1.6608 \times 10^{-3}$$

Then consider the "condition numbers for every individual eigenvalue", we know that

$$\begin{aligned} |\lambda^{(1)}_{A+E} - \lambda^{(1)}_A| &\approx -s * 2.4495 = 2.4495 * 10^{-6} \\ |\lambda^{(2)}_{A+E} - \lambda^{(2)}_A| &\approx -s * 707.8165 = 7.0782 * 10^{-4} \\ |\lambda^{(3)}_{A+E} - \lambda^{(3)}_A| &\approx -s * 706.4023 = 7.0640 * 10^{-4} \end{aligned}$$

Note that the above two "estimations" are consistent with each other.

Problem 2

Compute all the eigenvalues and eigenvectors of the complex symmetric matrix

$$A = \begin{bmatrix} 2i & 1 \\ 1 & 0 \end{bmatrix}$$

Solution

$\lambda_1 + \lambda_2 = 2i$, $\lambda_1 \cdot \lambda_2 = -1$, thus $\lambda_1 = \lambda_2 = i$. Then we consider the null space of $A - i \cdot I$, thus $u_1 = (i, 1)^T$, $u_2 = (1, -i)^T$

Problem 3

Consider the block upper triangular matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

Problem a

Suppose $A_{11}u = \lambda u$, but λ is not an eigenvalue of A_{22}. Find a vector v (in terms of $A_{i,j}$, u) such that the vector $\begin{bmatrix} u \\ v \end{bmatrix}$ is an eigenvector of A. What is the corresponding eigenvalue?

Solution

$$A \cdot \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} A_{11}u + A_{12}v \\ A_{22}v \end{bmatrix} = \begin{bmatrix} \lambda u + A_{12}v \\ A_{22}v \end{bmatrix}$$

Let $v = 0$, then the above equation holds, which indicates that $\begin{bmatrix} u \\ 0 \end{bmatrix}$ is an eigenvector of A, with corresponding eigenvalue λ which is given by $A_{11}u = \lambda u$.

Problem b

Suppose $A_{22}v = \lambda v$, but λ is not an eigenvalue of A_{22}. Find a vector u (in terms of $A_{i,j}$, v) such that the vector $\begin{bmatrix} u \\ v \end{bmatrix}$ is an eigenvector of A. What is the corresponding eigenvalue?

Solution

$$A \cdot \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} A_{11}u + A_{12}v \\ A_{22}v \end{bmatrix} = \begin{bmatrix} A_{11}u + A_{12}v \\ \lambda v \end{bmatrix}$$

Then we know that

$$\begin{aligned} & A_{11}u + A_{12}v = \lambda u \\ \Rightarrow & (A_{11} - \lambda I)u = -A_{12}v \\ \Rightarrow & u = -(A_{11} - \lambda I)^{-1} A_{12}v \end{aligned}$$

Thus if $A_{11} - \lambda I$ is invertible, $\begin{bmatrix} -(A_{11} - \lambda I)^{-1} A_{12}v \\ v \end{bmatrix}$ is an eigenvector of A with corresponding eigenvalue λ.

Problem c

Repeat the above assuming λ is an eigenvalue for both A_{11} and A_{22}. Do any of the above cases fail?

Solution

$$A \cdot \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} A_{11}u + A_{12}v \\ A_{22}v \end{bmatrix} = \begin{bmatrix} \lambda u + A_{12}v \\ \lambda v \end{bmatrix}$$

For case (a), if u is given, then set v as 0, $\begin{bmatrix} u \\ 0 \end{bmatrix}$ is still an eigenvector of A with corresponding eigenvalue λ. Hence case (a) does not fail.

For case (b), the problem leads to the following equation

$$\lambda u = \lambda u + A_{12}v \Rightarrow A_{12}v = 0$$

Thus if v is happened to be in the null space of A_{12}, case (b) still holds and $\begin{bmatrix} 0 \\ v \end{bmatrix}$ is the eigenvector with corrfesponding eigenvalue λ, otherwise case (b) fails.