CSCI 5304 HW 1

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Problem 1

Given two vectors $u, v \in \mathbb{R}^n$, and real scalars α, β, let $A = I + \alpha uv^T$, $B = I + \beta uv^T$.

Problem a

If u, v, α are given, find β such that $B = A^{-1}$

Solution

$\because B = A^{-1}$
$\therefore A \cdot B = I$
$\Rightarrow (I + \alpha uv^T)(I + \beta uv^T) = I$
$\Rightarrow \alpha uv^T + \beta uv^T + \alpha\beta uv^T uv^T = 0$
$\Rightarrow \text{trace}(\alpha uv^T + \beta uv^T + \alpha\beta uv^T uv^T) = 0$
$\Rightarrow \alpha\text{trace}(uv^T) + \beta\text{trace}(uv^T) + \alpha\beta\text{trace}(uv^T uv^T)$
$\Rightarrow \alpha v^T u + \beta v^T u + \alpha\beta v^T uv^T u$
if $v^T u \neq 0$
then $\alpha + \beta + \alpha\beta v^T u = 0$
$\therefore \beta = -\frac{\alpha}{1+\alpha v^T u}$

Problem b

For which values of α is A singular, if any? For that particular value of α, give a non-zero vector x in the right nullspace of A. Write x in terms of u, v, α.

Solution

$\because A$ is singular
$\therefore |I + \alpha uv^T| = 0$
Following the Sylvester's determinant theorem, we have
$|I + \alpha uv^T| = 1 + \alpha v^T u = 0$
$\Rightarrow \alpha = -\frac{1}{v^T u}$

Set $x = u$ We have $Ax = u - \frac{uv^T u}{v^T u} = u - u = 0$

Problem c

Prove of disprove: for any given pair of vector u, v, there always exists a value α such that A singular. To prove, show such an α always exists, giving a formula in terms of u, v. To disprove, give an example of a pair of non-zero vectors u, v for which no such α exists. In the latter case, what general property do u, v satisfy to prevent the existence of α? You can illustrate your answer with a 2×2 example.

Solution

Disprove: When $v^T u = 0$, no such α exists.
Because when $v^T u = 0$, $|I + \alpha uv^T| = 1 + \alpha v^T u = 1 > 0$;
For example, let $u = (1,1)^T$, $v = (1,-1)^T$, Then A is always non-singular.

Problem d

Give a value of α (in terms of u, v) such that $A^2 = A$ (i.e., A is a projector).

Solution

$\because A^2 = A$
$\therefore (I + \alpha uv^T)^2 = I + \alpha uv^T$
$\Rightarrow I + \alpha uv^T + \alpha uv^T + \alpha^2 uv^T uv^T = I + \alpha uv^T$
$\Rightarrow \alpha uv^T + \alpha^2 uv^T uv^T = 0$
Thus $\alpha = 0$ is one solution.
Otherwise, $\mathrm{trace}(uv^T) + \alpha \mathrm{trace}(uv^T uv^T) = 0$
$\Rightarrow v^T u + \alpha v^T uv^T u = 0$
if $v^T u \neq 0$
$\alpha = -\frac{1}{v^T u}$
So that we have $\alpha = 0$ or $\alpha = -\frac{1}{v^T u}$.

Problem 2

Let $f_p(v) = \max_{||u||_p=1} |u^T v|$, where $||v||_p$ denotes the p-norm.

Problem a

Prove or disprove: f_p is a vector norm. (check each property, or show one is violated).

Solution

Proof.

1. $f_p(v) = \max_{||u||_p=1} |u^T v| > 0$
2. $\forall \in \mathbb{R},\ f_p(av) = \max_{||u||_p=1} |u^T av| = \max_{||u||_p=1} a|u^T v| = af_p(v)$
3. $f_p(x+y) = \max_{||u_1||_p=1} |u^T(x+y)| \le |u_1^T x| + |u_1^T y| \le \max_{||u_x||_p=1} |u_x^T x| + \max_{||u_y||_p=1} |u_y^T y| = f_p(x) + f_p(y)$

Therefore f_p is a vector norm.

Problem b

Give a formula for f_p for $p = 1, 2$. Hint, the answers can be written in terms of $||\cdot||_2$, $||\cdot||_\infty$. For $p = 2$, use the Cauchy-Schwartz inequality.

Solution

$f_1(v) = \max_{||u||_1=1} |u^T v| = ||v||_\infty$
$f_2(v) = \max_{||u||_2=1} |u^T v| \le ||u||_2 \cdot ||v||_2 = ||v||_2$

Problem 3

Define the inner product among square matrices by $< A, B >= \text{trace}(A^T B)$, where A, B are $n \times n$ matrices.

Problem a

What is the norm induced by this inner product: $||A|^2 =< A, A >$? Answer this question for the general case for any A.

Solution

$||A|| = \sqrt{\text{trace}(A^T A)}$

Problem b

Now answer the remaining questions below using this specific matrix:

$$A = \begin{pmatrix} 6 & -2 & 1 \\ 7 & -7 & 3 \\ -4 & 5 & -2 \end{pmatrix}$$

For this specific matrix A, what is the value of $< A, A >$ and the corresponding induced norm $||A||^2 =< A, A >$ from part (a)?

Solution

$< A, A >= \text{trace}(A^T A) = 193$
$||A|| = \sqrt{< A, A >} = \sqrt{193}$

Problem c

What is the p norm of A, for $p = 1$? Find a vector x s.t. $||A||_p = ||Ax||_p$.

Solution

Assume the dimension of A is $n \times m$
$||A||_1 = \max_{||x||_1=1} ||Ax||_1 = 1_{1\times n} \cdot A \cdot x = \max_{||x||_1=1}(s_1, \ldots, s_m) \times x$
Where s_i is the ith column sum of absolute value of matrix A. Since $||x||_1 = 1$, we have
$||A||_1 = \max_i s_i = 17$
$x = (1, 0, 0)^T$

Problem d

Repeat the above for $p = 2$. Use Matlab and write the result to 4 decimal places. Show your Matlab commands.

Solution

```
format short;
A = [6, -2, 1; 7, -7, 3;-4, 5, -2];
[EVECT,EVAL]=eig(A' * A);
sqrt(max(abs(diag(EVAL))))
>
13.595
x = EVECT(:,3)
>
0.72496
-0.63330
0.27086
```

$||A||_2 = 13.595$
$x = (0.7250, -0.6333, 0.2709)^T$

Problem e

Use Matlab to help solve this problem: Find a vector x achieving the minimum in $\min_{||x||_p=1} ||Ax||_p$. Do this for $p = 1, 2$.

Solution

For $p = 1$, similar to the maximum problem, the minimum result should be the smallest value of sum of absolute column values.

```
min(sum(abs(A),1))
> 6
```

$\min_{||x||_1=1} ||Ax||_1 = 6$
$x = (0, 0, 1)^T$

For $p = 2$, similar to the maximum problem, the minimum result should be the smallest square root of absolute eigenvalue.

```
[EVECT,EVAL]=eig(A' * A);
sqrt(min(abs(diag(EVAL))))
>
0.077258

x = EVECT(:,1)
>
-0.040332
0.353531
0.934553
```

$\min_{||x||_p=2} ||Ax||_2 = 0.0773$
$x = (-0.0403, 0.3535, 0.9346)^T$