STAT 5701: Statistical Computing

Homework 2

Jingxiang Li

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Note that if $u \sim \mathrm{Unif}(0, 1)$, $\mathrm{P}(u < \alpha) = \alpha, \ \forall \alpha \in [0, 1]$.
Let p be the p-value, under H_0 we have

$$\begin{aligned} & \mathrm{P}(p < \alpha) \\ = \ & \mathrm{P}(t > T_{1-\alpha/2} \text{ or } t < T_{\alpha/2}) \\ = \ & \mathrm{P}(t > T_{1-\alpha/2}) + \mathrm{P}(t < T_{\alpha/2}) \\ = \ & \alpha/2 + \alpha/2 \\ = \ & \alpha \end{aligned}$$

i.e., $\forall \alpha \in [0, 1]$ $\mathrm{P}(p < \alpha) = \alpha$, which implies that p-value has a uniform distribution under H_0.
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$$l(x_i|\mu) = \log L(x_i|\mu) = -\log \mu - \frac{x}{\mu}$$

$$\Rightarrow l(x|\mu) = \sum_{i=1}^{n} l(x_i|\mu) = -n \log \mu - n\frac{\bar{x}}{\mu}$$

Then set

$$\begin{aligned} &\frac{\partial}{\partial \mu} l(x|\mu) = 0 \\ &\Rightarrow -\frac{1}{\mu} + \frac{\bar{x}}{\mu^2} = 0 \\ &\Rightarrow \hat{\mu} = \bar{x} \end{aligned}$$

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$$\begin{aligned} & L(a) = \mathrm{E}\{(a\bar{x} - \mu)^2\} \\ \Rightarrow \quad & L(a) = \mathrm{E}\{(a(\bar{x} - E(\bar{x})) + a\mathrm{E}(\bar{x}) - \mu)^2\} \\ \Rightarrow \quad & L(a) = \mathrm{E}\{a^2(\bar{x} - E(\bar{x}))^2\} + \mathrm{E}\{(a\mathrm{E}(\bar{x}) - mu)^2\} \\ \Rightarrow \quad & L(a) = a^2\mathrm{Var}(\bar{x}) + a^2\mu^2 - 2a\mu^2 + \mu^2 \\ \Rightarrow \quad & L(a) = a^2\frac{\mu^2}{n} + a^2\mu^2 - 2a\mu^2 + \mu^2 \end{aligned}$$

Then set

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\begin{aligned}
&\frac{\partial}{\partial a} L(a) = 0 \\
\Rightarrow \quad & 2a\frac{\mu^2}{n} + 2a\mu^2 - 2\mu^2 = 0 \\
\Rightarrow \quad & \hat{a} = \frac{\mu^2}{\frac{\mu^2}{n} + \mu^2} = \frac{n}{n+1}
\end{aligned}
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\begin{aligned}
\mathrm{E}\{(\bar{x} - \mu)^2\} &= \mathrm{Var}(\bar{x}) = \frac{\mu^2}{n} \\
\mathrm{E}\{(a\bar{x} - \mu)^2\} &= a^2\mathrm{Var}(\bar{x}) + (a\mu - \mu)^2 = \frac{1}{n+1}\mu^2
\end{aligned}
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