VALIDATION OF THE KNOCK-IN PROBABILITY FORMULA VIA MONTE CARLO SIMULATION

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Abstract

This note reports the validation results of the knock-in probability formula. Section one looks into the knock-in probability values and simulation results with distinct levels of volatility, number of instants, and S_T value. Section two introduces the Monte Carlo simulation scheme applied in the validation.

1 Main findings

In another paper, we proposed an analytical solution to the discrete monitoring knock-in probability problem. The formula itself is a corollary from Ross's conclusion[Ros14]. The setting of the problem is as follows: suppose we observe the price dynamics of a stock over a month, that is, from time t_0 to time T. The price of the stock constitutes a 'knock-in' when it is observed to breach a lower bound with the value KI. We denote the price of the stock at time t to be S_t. The term structure is flat and the continuous risk-free interest rate stays at r. We also assume constant volatility as well as a constant dividend payout rate q. We denote the stock return volatility to be σ. We also use β to represent a constant s.t.

$$\beta = \frac{-\zeta(\frac{1}{2})}{\sqrt{2\pi}}$$

where ζ is the Riemann Zeta function. The the knock-in probability should be

$$exp(-\frac{2}{T\sigma^2}log(\frac{KIexp(-\beta\sigma\sqrt{\Delta t})}{S_0})log(\frac{KIexp(-\beta\sigma\sqrt{\Delta t})}{S_1})) \tag{1}$$

To justify the validity of formula (1), one proper measure is to compare the results produced by the formula to those produced by simulation. The reasons are as follows. If our assumptions for the dynamics of the simulation were correct, then the simulation results should not be far from the true ones, with the former converging to the latter as more paths are considered. And if it so happens that our approximation formula were correct, then the simulated results and the ones by the formula should not be significantly far from each other. It is worth noting though, that since the formula itself yields an *approximation* of the knock-in probability instead of an exact result, we rightfully expect a certain level of discrepancy between the two types of results for any case where the number of instants $m < \infty$.

2 Do simulation results agree with the formula?

They do, at least in most scenarios. A more detailed discussion is as follows. For code of simulation please refer to `https://github.com/jingxiangzou/knock_in-probability`

To see whether the two types of results coincide, we look at 1. The approximation error is their absolute discrepancy. The error ratio which is the approximation error over the analytical probability, serves as another meaningful criterion. For simulation, we set $1.S_0 = 100$ $2.KI = 90$. In the meantime, we take an interest in its sensitivity towards 1. Volatility 2.The number of instants per month 3. The end-of-month stock price S_T. Eventually, 5 million simulated paths are leveraged for the calculation of each of the 3200 probabilities.

In another paper, we have provided theoretical justification for the approximation formula. The more frequent our monitoring is, the more our discrete model approaches the continuous case. This is observed in our validation results(see Figure 1 and Figure 2)

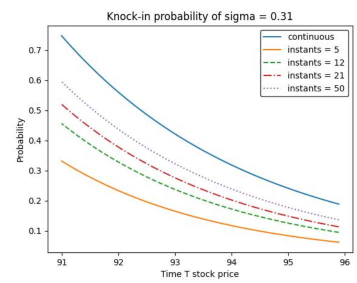

Figure 1: Knock-in probability of $\sigma = 0.31$

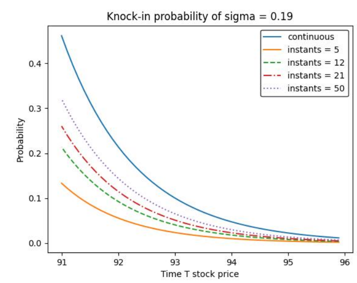

Figure 2: Knock-in probability of $\sigma = 0.19$

Table 1: Average discrepancies of different instants m (per month) per volatility $\sigma = 0.31$

Instants	Approximation Error	Error Ratio
5	0.01031	0.06209
12	0.00322	0.01251
21	0.00128	0.00467
50	0.00030	0.00149

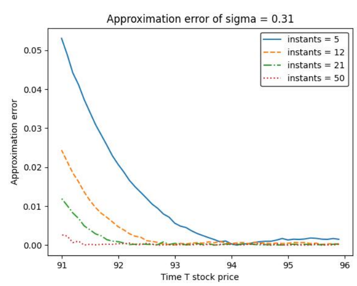

Figure 3: Approximation error

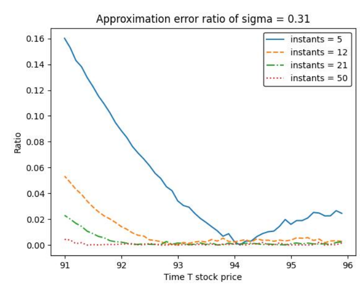

Figure 4: Approximation error ratio

How does approximation performance change as m changes? Well, the relations between discrepancies and S_T, m came out just as what we would expect them to be. The approximation performance of the analytical expression is very poor when m is low. We can see from 2 that as the time interval between two consecutive monitoring instants declines, so does the approximation error and the error ratio. This stems directly from the fact that the expression itself is a first-order Taylor approximation [MBK97]. Approximation errors surge as S_T approaches KI, but it shrinks drastically as the number of instants escalates, as we can observe from Figure 3 and Figure 4.

Table 2: Average discrepancies of different instants m (per month) per instants $m = 21$

Volatility	Approximation Error	Error Ratio
0.15	0.00045	0.04411
0.23	0.00202	0.01902
0.31	0.00400	0.01554
0.39	0.00593	0.01622

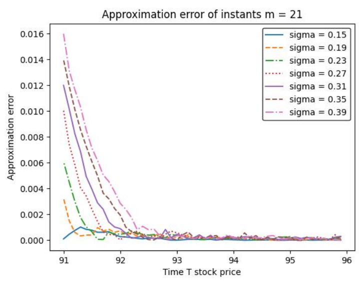

Figure 5: Approximation error

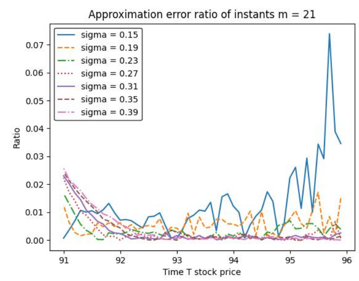

Figure 6: Approximation error ratio

We proceed to look into how approximation results may be affected by volatility. Our readings of Table 2 tell us that higher volatility corresponds to higher discrepancies. This should not be a major concern because the absolute value of the probability is also larger with high volatility (see Figure 7). If we look at the error ratio of Table 2 we can immediately come to the conclusion that our formula is quite robust w.r.t. volatility change: when volatility σ surges from 0.23 to 0.39, the error ratio stays below 2 percent, which means the analytical approximation stays close to the true value regardless of volatility change.

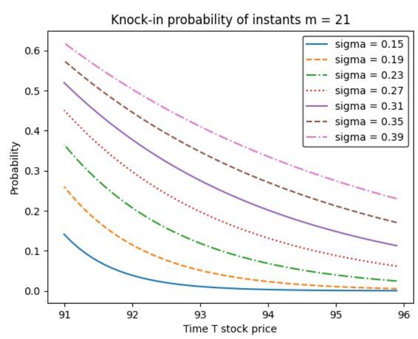

Figure 7: Knock-in probability with different σ

3 How Monte Carlo simulation is conducted

For simulation, one has to make assumptions about the dynamics of the stock. The classic Geometric Brownian Motion suggests the following dynamics for the time t stock price S_t given constant risk-free rate r and dividend rate q:

$$\frac{dS_t}{S_t} = (r - q)dt + \sigma dW_t \tag{2}$$

where Brownian Motion W_t is defined on the filtration $\mathcal{F}_t$. With Ito's Lemma, 2 is equivalent to

$$S_t = S_0 exp\{(r - q - \frac{1}{2}\sigma^2)t + \sigma W_t\} \tag{3}$$

Our assumptions fail to comply with such dynamics: A stock price process that follows a Geometric Brownian Motion would have countless possible values at time T. In our setting, however, time T

price S_T is known at time 0. Knowing S_T at time 0 would violate the definition of W_t, which asserts that W_T is not F_t measurable for $t < T$.

We propose a proper dynamics for S_t that incorporates our knowledge of S_T at the very beginning. Define X_t to be the time t value of a $0 \to b$ Brownian Bridge [Shr04]

$$X_t = \frac{bt}{T} + W_t - \frac{t}{T} W_T \tag{4}$$

This Brownian Bridge would be 0 at time 0 and becomes b at time T. Moreover, we assume that

$$\frac{dS_t}{S_t} = (r-q)dt + \sigma dX_t \tag{5}$$

Apparently

$$\begin{aligned} \frac{dS_t}{S_t} &= (r-q)dt + \sigma dX_t \\ &= (r - q + \frac{\sigma(b - W_T)}{T})dt + \sigma dW_t \end{aligned} \tag{6}$$

This is where things become tricky: W_T is by definition the time T price of Brownian Motion W_t, a random variable not a deterministic function of t. However, as per our assumptions, $S_T = S_1$ is $\mathcal{F}_0$ measurable, so should W_T. With Ito's lemma, 6 is equivalent to

$$S_t = S_0 exp\{(r - q + \frac{\sigma(b - W_T)}{T} - \frac{1}{2}\sigma^2)t + \sigma W_t\} \tag{7}$$

What do we know about b? Well, we know nothing about it, but what we do know is S_T. Nevertheless, it suffice to obtain a unique value of b if we plug in $S_T = S_1$ into 7

$$b = \frac{log(\frac{S_1}{S_0}) - (r - q - \frac{1}{2}\sigma^2)T}{\sigma} \tag{8}$$

We can use 8 to simplify 7:

$$S_t = S_0 exp\{\frac{t}{T} log(\frac{S_1}{S_0}) + W_t - \frac{t}{T} W_T\} \tag{9}$$

9 is a feasible dynamics per which we develop our simulation scheme. Specifically, for the simulated path of discretely monitored price process of m instants evenly distributed within a month (1/12 years) with volatility σ, we would first generate the values of the underlying Brownian Motion W_t at the m instants as a summation of $\{0, \sigma\sqrt{\frac{1}{12m}}\}$ Gaussian random variables before obtaining X_t as a linear combination of W_t and W_T.

References

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[Ros14] Sheldon M. Ross. *Introduction to Probability Models.* Academic Press, Los Angeles, California, 11 edition, 2014.

[Shr04] Steven E Shreve. *Stochastic calculus for finance 2, Continuous-time models.* Springer, New York, NY; Heidelberg, 2004.