AN ANALYTICAL EXPRESSION OF KNOCK-IN PROBABILITY WITH DISCRETE MONITORING INSTANTS

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Abstract

This paper provides an analytical solution to the conditional knock-probability problem under discrete monitoring instants which might prove useful in the practical pricing of discrete barrier options or other derivatives.

1 Basic Setting And Assumptions

Suppose we observe the price dynamics of a stock over the period of a month, that is, from time t_0 to time T. The price of the stock constitutes a 'knock-in' when it is observed to breach a lower bound with the value KI. We denote the price of the stock at time t to be S_t. Term structure is flat and continuous risk-free interest rate stays at r. We also assume constant volatility as well as constant dividend payout rate q . The stock starts from time 0 price $S_0 > KI$. It follows Geometric Brownian within the interval $0 \leq t < T$ and have value $S_1 > KI$ at time T.

We also assume that the stock price is merely monitored at discrete instants within the period, i.e., once a day, at times $i\Delta t, i = 0, 1, 2, ..., m$, where $\Delta t = T/m$. Let us write S_i for $S_{i\Delta t}$ and thus $S_i, i = 0, 1, 2, ...$ is the stock price at the monitoring instants

2 Main Result

2.1 Irrelevance of the drift term

As per the assumptions, our analysis is based on the modified version of the classic Black-Scholes framework. Under the risk-neutral measure, for $0 \leq t < T$, the stock price follows the stochastic differential equation

$$\frac{dS_t}{S_t} = (r - q)dt + \sigma dW_t \tag{1}$$

where W_t is a standard Brownian Motion, r and q are defined as in Section 1, and $\sigma > 0$ is the constant volatility. Both S_0 and $S_T = S_1$ are fixed.

From Ito's Lemma, we have

$$S_t = S_0 e^{(r-q-\frac{1}{2}\sigma^2)t+\sigma W_t} \tag{2}$$

Each stock price in the price process corresponds to a value of the Brownian Motion. At each of the monitoring instants, each $\bar{S}_i$ corresponds to a value of W_t which we denote as W_i. Define

$$X_t = \frac{1}{\sigma}(r - q - \frac{1}{2}\sigma^2)t + W_t \tag{3}$$

Analogously we can define the value of X_t at these instants as X_i.

We also define that $Y_i = X_i - X_{i-1}$, thus we have

$$Y_i = \frac{1}{\sigma}(r - q - \frac{1}{2}\sigma^2)\Delta t + (W_i - W_{i-1}) \tag{4}$$

Per the definition of the Brownian motion, $W_0 = 0$ and each $X_i - X_{i-1}$ are identically and independently distributed following a Gaussian Distribution with mean $\frac{1}{\sigma}(r - q - \frac{1}{2}\sigma^2)\Delta t$and volatility $\sqrt{\Delta t}$.

Ross proved that if $Y_1, ..., Y_n$ are independent and identically distributed normal random variables with mean θ and variance ν^2, then the conditional density of $Y_1, ..., Y_{n-1}$ given that $\Sigma_{i=1}^n Y_i = z$ is not dependent on θ.

In our settings, this proposition suggests that $f_{Y_1,...,Y_{n-1}|z}(y_1, ..., y_{n-1}|Z)$ does not depend on the what $\frac{1}{\sigma}(r - q - \frac{1}{2}\sigma^2)\Delta t$ is.

Since

$$Y_1 = y_1, ..., Y_{m-1} = y_{m-1}, z = Z \iff Y_1 = y_1, ..., Y_{m-1} = y_{m-1}, Y_m = z - \Sigma_{i=1}^{m-1} \tag{5}$$

It follows that

$$\begin{aligned} f_{Y_1,...,Y_{m-1}|z}(y_1, ..., y_{m-1}|Z) &= \frac{f_{Y_1,...,Y_{m-1},z}(y_1, ..., y_{m-1}, z)}{f_Z(z)} \\ &= \frac{f_{Y_1,...,Y_{m-1},Y_m}(y_1, ..., y_{m-1}, z - \Sigma_{i=1}^{m-1} y_i)}{f_{Y_m}(z - \Sigma_{i=1}^{m-1} y_i)} \\ &= f_{Y_1,...,Y_{m-1}|Y_m}(y_1, ..., y_{m-1}|z - \Sigma_{i=1}^{m-1}) \end{aligned} \tag{6}$$

We are given S_T, which means Y_m should be known to us via 2. Since the knock-in probability is in essence a multi-dimensional integral of the conditional density, for the purpose of obtaining knock-in probability, we can safely ignore the drift term in the stock dynamics.

2.2 Continuity Correction

Broadie, Glasserman, and Kou [MBK97]showed that by applying a simple continuity correction to the barrier, the discrete barrier options can be priced with remarkable accuracy. For simplicity, we define the 'hitting time' using the same notation as in their work.

Under continuous monitoring, the moment of the stock's fist knock-in is denoted as τ and is defined below

$$\tau_H = inf\{t > 0 : S_t = H\}$$

while under discrete monitoring the first knock-in time of the stock is

$$\tilde{\tau_H} = inf\{n > 0 : S_n < H\}$$

Their paper [MBK97]has proved that for a given execution price K,

$$Prob(S_m < K, \max_{0 \le n \le m} S_n \ge H) = Prob(S_T < K, \tau_{Hexp(\beta\sigma\sqrt{\Delta t})} \le T) \tag{7}$$

It is worth noting that when deriving the continuity correction, the authors did not fix the price at time T. To incorporate fixed $S_T = S_m$, we think of conditional probability. Apparently,

$$Prob(S_m < K, \max_{0 \le n \le m} S_n \ge H|S_T) = Prob(S_m < K, \max_{0 \le n \le m} S_n \ge H|S_m) \tag{8}$$

Given the knowledge of S_m, $Prob(S_m < K$ shall be immediately known. In Kou's paper, K is a given value, yet for the purpose of getting knock-in probability, K is not necessarily given. When S_m is given and K is not, $Prob(S_m < K)$ becomes a binary function of K whose value is solely dependent on K. On the contrary, $Prob(\max_{0 \le n \le m} S_n \ge H|S_T)$ depends on the path of S_t from 0 till T and has nothing to do with K whatsoever. The two conditional probabilities are independent. Thus

$$\begin{aligned} Prob(S_m < K, \max_{0 \le n \le m} S_n \ge H|S_m) &= Prob(S_m < K|S_m) \cdot Prob(\max_{0 \le n \le m} S_n \ge H|S_m) \\ &= I_{S_m < K} \cdot Prob(\max_{0 \le n \le m} S_n \ge H|S_m) \\ &= I_{S_T < K} \cdot Prob(\max_{0 \le n \le m} S_n \ge H|S_m) \end{aligned} \tag{9}$$

From 7 a similar argument can be made:

$$\begin{aligned} Prob(S_m < K, \max_{0\le n\le m} S_n \ge H|S_m) &= Prob(S_T < K, \tau_{Hexp(\beta\sigma\sqrt{\Delta t})} \le T|S_T) \\ &= I_{S_T<K} \cdot Prob(\tau_{Hexp(\beta\sigma\sqrt{\Delta t})} \le T|S_T) \end{aligned} \tag{10}$$

Compare 9 and 13, we have

$$Prob(\max_{0\le n\le m} S_n \ge H|S_m) = Prob(\tau_{Hexp(\beta\sigma\sqrt{\Delta t})} \le T|S_T) \tag{11}$$

where β is a constant which is approximately 0.5826. Define

$$M(t) = \min_{0\le y\le t} W_t$$

Then

$$Prob(M(t) \ge y|X(t) = x) = e^{-2y(y-x)/t\sigma^2}, y \ge 0 \tag{12}$$

Apply the continuity correction to 13, we will obtain an analytical representation of knock-in probability, which is

$$\begin{aligned} Prob(\max_{0\le n\le m} S_n \ge H|S_m) &= Prob(\tau_{Hexp(-\beta\sigma\sqrt{\Delta t})} \le T|S_T) \\ &= Prob(M(t) \ge He^{-\beta\sigma\sqrt{\Delta t}}|X(T) = \frac{log(\frac{S_1}{S_0})}{\sigma}) \\ &= exp(-\frac{2}{T\sigma^2}log(\frac{KIexp(-\beta\sigma\sqrt{\Delta t})}{S_0})log(\frac{KIexp(-\beta\sigma\sqrt{\Delta t})}{S_1})) \end{aligned} \tag{13}$$

3 Performance Of Approximation

We evaluate how well the formula in 2.2approximates the knock-in probability. To that end, for different levels of m and different levels of (S_0, KI) we use Monte Carlo simulation to provide a numerical approximation to the knock-in probability and see whether they coincide with those given by 2.2.

References

[MBK97] Paul Glasserman Mark Broadie and Steven Kou. A continuity correction for discrete barrier options. *Mathematical Finance*, 7(4):325–348, 1997.