KNOCK-IN PROBABILITY UNDER DISCRETE MONITORING

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1 Background

The problem goes like this: suppose a stock follows Geometric Brownian Motion, with a dividend rate of q, the risk-free interest rate being r and the stock's volatility being σ. Define 'knock-in price' KI as the price the stock reaches when it constitutes a 'knock-in'. Suppose the time zero price of the stock is S_0 and the price in a month is $S_1; S_1 > KI$; The goal is to calculate the knock-in price under discrete monitoring.

We are first enlightened by the Theorem 10.3 from [Ros14] which states that for

$$M(t) = \max_{0 \leq y \leq t} X(y) \tag{1}$$

where $X(y)$ follows Brownian Motion. For $y > x$ the conditional distribution of $M(t)$ should satisfy

$$P(M(t) \geq y | X(t) = x) = e^{-2y(y-x)/t\sigma^2} \tag{2}$$

Ross proved that the conditional distribution of $M(t)$ given the value of $X(t)$ does not depend on value of the drift term.

We also learn from the symmetry of Brownian Motion that for the scenario where the 'knock-in price' being $-KI$, the time zero price being $-S_0$ and the price in one month being $-S_1$, the knock-in probability would be the same. Now that the drift term is irrelevant, we can obtain y and x in (1) from:

$$-KI = -S_0 e^y$$

$$-S_1 = -S_0 e^x$$

thus we have the knock-in probability under continuous monitoring:

$$Prob(knock\, in) = e^{-\frac{2}{t\sigma^2} log(\frac{KI}{S_0}) log(\frac{KI}{S_1})} \tag{3}$$

However, the question features monitoring in a discrete manner. It came to our attention that the 1997 paper by Glasserman, Broadie and Kou ([MBK97])addressed this issue by proposing a continuity correction for discrete barrier option pricing.

We show, however, that discrete barrier options can be priced with remarkable accuracy using continuous barrier formulas by applying a simple continuity correction to the barrier. The correction shifts the barrier away from the underlying by a factor of $exp(\beta\sigma\sqrt{\Delta t})$, where $\beta = 0.5826$, σ is the underlying volatility, and Δt is the time between monitoring instants.

We thus apply the continuity correction to 3and we arrive at the following expression:

$$Prob(knock\, in) = e^{-\frac{2}{t\sigma^2} log(\frac{KI exp(-\beta\sigma\sqrt{\frac{T}{m}})}{S_0}) log(\frac{KI exp(-\beta\sigma\sqrt{\frac{T}{m}})}{S_1})} \tag{4}$$

It is fair to hypothesize that applying the continuity correction to the conditional probability under the continuous case would arrive at a solution to the original problem, but a convincing justification awaits rigorous theoretical validation yet to be completed.

To justify the validity of 4, one proper measure is to compare the results produced by the formula to those produced by simulation. The reasons are as follows. If our assumptions for the dynamics of the simulation were correct, then the simulation results should not be far from the true ones, with the former converging to the latter as more paths are considered. And if it so happens that our approximation formula were correct, then the simulated results and the ones by the formula should not be significantly far from each other. It is worth noting though, that since the formula itself yields an *approximation* of the knock-in probability instead of an exact result, we rightfully expect a certain level of discrepancy between the two types of results for any case where the number of instants $m < \infty$.

2 Do simulation results agree with the formula?

It turns out that they do. A detailed discussion is as follows.

To see whether the two types of results coincide, we look at

1. The approximation error, which is the absolute discrepancy between the probability generated by simulation and that suggested by the hypothesis.

2. The error ratio, which is the approximation error over the analytical probability, serves as another meaningful criterion.

For simulation, we set $S_0 = 100$ as well as $KI = 90$. In the meantime, we take an interest in its sensitivity towards

1. volatility σ

2. The number of instants per month m

3. The end-of-month stock price S_T

Eventually, 5 million simulated paths are leveraged for the calculation of each of the 3200 knock in probabilities.

The more frequent our monitoring is, the more our discrete model approaches the continuous case. This is observed in our validation results(see Figure 1 and Figure 2)

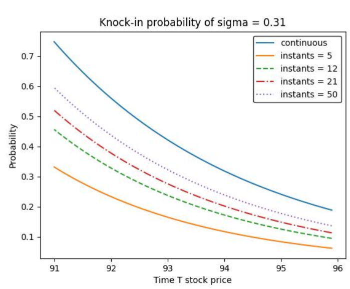

Figure 1: Knock-in probabilities of $\sigma = 0.31$

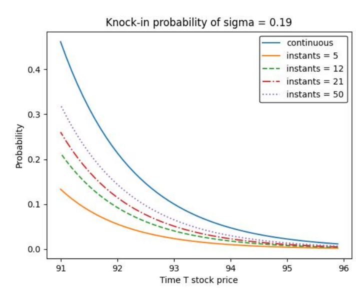

Figure 2: Knock-in probabilities of $\sigma = 0.19$

Table 1: Average discrepancies of different m per month when $\sigma = 0.31$

Instants	Approximation Error	Error Ratio
5	0.01031	0.06209
12	0.00322	0.01251
21	0.00128	0.00467
50	0.00030	0.00149

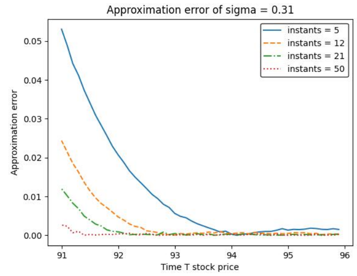

Figure 3: Approximation errors when $\sigma = 0.31$

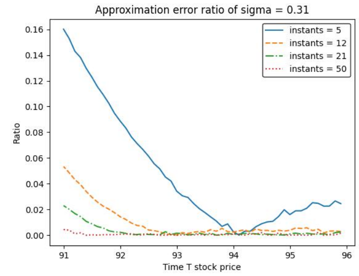

Figure 4: Error ratios when $\sigma = 0.31$

How does approximation performance change as m changes? Well, the relations between discrepancies and S_T, m came out just as what we would expect them to be. The approximation performance of the analytical expression is very poor when m is low. We can see from 2 that as the time interval between two consecutive monitoring instants declines, so does the approximation error and the error ratio. This stems directly from the fact that the expression itself is a first-order Taylor approximation [MBK97]. Approximation errors surge as S_T approaches KI, but it shrinks drastically as the number of instants escalates, as we can observe from Figure 3 and Figure 4.

Table 2: Average discrepancies per different σ when $m = 21$

Volatility	Approximation Error	Error Ratio
0.15	0.00045	0.04411
0.23	0.00202	0.01902
0.31	0.00400	0.01554
0.39	0.00593	0.01622

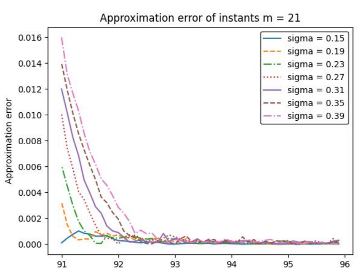

Figure 5: Approximation errors when $m = 21$

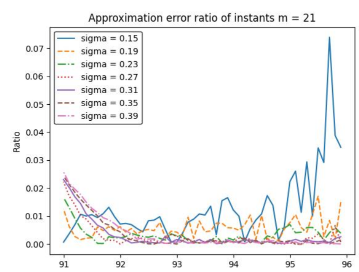

Figure 6: Error ratios when $m = 21$

We proceed to look into how approximation results may be affected by volatility. Our readings of Table 2 tell us that higher volatility corresponds to higher discrepancies. This should not be a major concern because the absolute value of the probability is also larger with high volatility (see Figure 7). If we look at the error ratio of Table 2 we can immediately come to the conclusion that our formula is quite robust w.r.t. volatility change: when volatility σ surges from 0.23 to 0.39, the error ratio stays below 2 percent, which means the analytical approximation stays close to the true value regardless of volatility change.

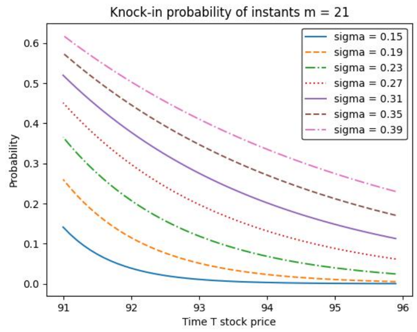

Figure 7: Knock-in probability per different σ when $m = 21$

3 How Monte Carlo simulation is conducted

For simulation, one has to make assumptions about the dynamics of the stock. The classic Geometric Brownian Motion suggests the following dynamics for the time t stock price S_t given constant risk-free rate r and dividend rate q:

$$\frac{dS_t}{S_t} = (r-q)dt + \sigma dW_t \tag{5}$$

where Brownian Motion W_t is defined on the filtration $\mathcal{F}_t$. With Ito's Lemma, 5 is equivalent to

$$S_t = S_0 exp\{(r-q-\frac{1}{2}\sigma^2)t + \sigma W_t\} \tag{6}$$

Our assumptions fail to comply with such dynamics: A stock price process that follows a Geometric Brownian Motion would have countless possible values at time T. In our setting, however, time T price S_T is known at time 0. Knowing S_T at time 0 would violate the definition of W_t, which asserts that W_T is not F_t measurable for $t < T$.

We propose a proper dynamics for S_t that incorporates our knowledge of S_T at the very beginning. Define X_t to be the time t value of a $0 \to b$ Brownian Bridge [Shr04]

$$X_t = \frac{bt}{T} + W_t - \frac{t}{T}W_T \tag{7}$$

This Brownian Bridge would be 0 at time 0 and becomes b at time T. Moreover, we assume that

$$\frac{dS_t}{S_t} = (r-q)dt + \sigma dX_t \tag{8}$$

Apparently

$$\begin{aligned} \frac{dS_t}{S_t} &= (r-q)dt + \sigma dX_t \\ &= (r-q+\frac{\sigma(b-W_T)}{T})dt + \sigma dW_t \end{aligned} \tag{9}$$

This is where things become tricky: W_T is by definition the time T price of Brownian Motion W_t, a random variable not a deterministic function of t. However, as per our assumptions, $S_T = S_1$ is $\mathcal{F}_0$ measurable, so should W_T. With Ito's lemma, 9 is equivalent to

$$S_t = S_0 exp\{(r-q+\frac{\sigma(b-W_T)}{T} - \frac{1}{2}\sigma^2)t + \sigma W_t\} \tag{10}$$

What do we know about b? Well, we know nothing about it, but what we do know is S_T. Nevertheless, it suffice to obtain a unique value of b if we plug in $S_T = S_1$ into 10

$$b = \frac{log(\frac{S_1}{S_0}) - (r-q-\frac{1}{2}\sigma^2)T}{\sigma} \tag{11}$$

We can use 11 to simplify 10:

$$S_t = S_0 exp\{\frac{t}{T} log(\frac{S_1}{S_0}) + W_t - \frac{t}{T} W_T\} \tag{12}$$

12 is a feasible dynamics per which we develop our simulation scheme. Specifically, for the simulated path of discretely monitored price process of m instants evenly distributed within a month (1/12 years) with volatility σ, we would first generate the values of the underlying Brownian Motion W_t at the m instants as a summation of $\{0, \sigma\sqrt{\frac{1}{12m}}\}$ Gaussian random variables before obtaining X_t as a linear combination of W_t and W_T.

References

[MBK97] Paul Glasserman Mark Broadie and Steven Kou. A continuity correction for discrete barrier options. *Mathematical Finance*, 7(4):325–348, 1997.

[Ros14] Sheldon M. Ross. *Introduction to Probability Models*. Academic Press, Los Angeles, California, 11 edition, 2014.

[Shr04] Steven E Shreve. *Stochastic calculus for finance 2, Continuous-time models*. Springer, New York, NY; Heidelberg, 2004.