CDS pricing formula: A probabilistic understanding

Jingxiang Zou*

1. A probabilistic view in to the pricing formula of Credit Default Swap. .

We assume that credit event can happen at any arbitrary time within (t_0, t_N).

Define t^* to be the time of the CDS default.

Define $f(t)$ to be the probability density function that credit event happens at time t.

Define $P(t_n)$ to be the probability that the credit event does not happen from inception till time t_n, then naturally $f(t)$ is the derivative of $-P(t)$ because

$$Prob[t^* \in (t_{n-1}, t_n)] = \int_{t_{n-1}}^{t_n} f(t)\,\mathrm{d}t = P(t_{n-1}) - P(t_n)$$

We first consider the protection leg of the CDS formula. The t_0 - expectation of the 'protection' the seller of CDS is providing is essentially the discounted expected value of loss in a credit event.

Define the rate of recovered value to be R.

Define $DF(t_n)$ to be the discount factor corresponding to time t_n. As we assumed, that credit event can happen at any arbitrary time within (t_0, t_N), we can write the expected loss (as per $ 1 nominal) as the following:

$$\int_0^{t_N} (1-R)\cdot DF(t)\cdot f(t)\,\mathrm{d}t$$

If we overlook the difference among discount factors for all $t \in (t_{n-1}, t_n)$ and set their discount factors to be $DF(t_n)$, then the previous formula is equivalent to:

$$\sum_{n=1}^{N}(1-R)\cdot DF(t_n)\cdot\int_{t_{n-1}}^{t_n} f(t)\,\mathrm{d}t$$

which is equivalent to

$$\sum_{n=1}^{N}(1-R)\cdot DF(t_n)\cdot[P(t_{n-1}) - P(t_n)]$$

which is exactly the protection leg of the CDS pricing formula.

The premium leg is a little more complicated. It has two parts: the 'realized premium payments' and the 'unrealized premium'. In mathematical terms:

Define $t_n, n = 1, 2, 3, 4...N$ to be the dates when premiums are supposed to be payed.

Define S to be the annualized premium, also known as the contractual spread.

Define $\delta(t_{n-1}, t_n)$ to be the fraction of a year according to some day count convention.

The expected discounted value of the $t_n$realized premium payments equals the discounted value of the realized premium payments given $t^* > t_n$ multiplied by the probability of $t^* > t_n$, which is:

$$\sum_{n=1}^{N} S\cdot\delta(t_{n-1}, t_n)\cdot DF(t_n)\cdot Prob(t^* > t_n)$$

* The author is a BU msmft 2024 candidate. This short article is an updated version of March 28th 2023.

which is equivalent to

$$\sum_{n=1}^{N} S \cdot \delta(t_{n-1}, t_n) \cdot DF(t_n) \cdot P(t_n)$$

We now discuss the 'unrealized premium payments', which refers to a fraction of the premium originally due at the nearest payment date which is never collected because of the default. It is analogous to part of the accumulated interest as a fraction of the next coupon payment, which is a part of the bond's dirty price. The unrealized premium is not trivial because the seller of the CDS is obliged to provide protection though out the time from inception till default (or maturity) instead of just till premium payment dates.

The existence of the 'unrealized premium payments' is almost inevitable because it is not very likely that CDS defaults at exactly each premium payment dates.

The expected discounted value of the unrealized premium payment that corresponds to the time interval (equals the discounted value of the realized premium payments given $t^* > t_n$ multiplied by the probability of $t^* > t_n$, which is: (given that $t \in (t_{n-1}, t_n)$)

$$\sum_{n=1}^{N} S \cdot \delta(t_{n-1}, t^*) \cdot DF(t_n) \cdot \int_{t_{n-1}}^{t_n} f(t^*) \, \mathrm{d}t^*$$

if we assume that $\delta(t_{n-1}, t^*)$ is approximately half of the time interval (t_{n-1}, t_n), then this formula is approximate to

$$1/2 \cdot \sum_{n=1}^{N} S \cdot \delta(t_{n-1}, t) \cdot DF(t_n) \cdot (P(t_{n-1}) - P(t_n))$$

The 'realized' and 'unrealized' part of the premium leg thus add up to

$$1/2 \cdot \sum_{n=1}^{N} S^{(t_{n-1},t)} \cdot \delta(t_{n-1}, t) \cdot DF(t_n) \cdot (P(t_{n-1}) + P(t_n))$$

which is the premium leg of a CDS.

2. A little note on the pricing of insurance and credit derivatives. .

Credit swap is like insurance, and the pricing of insurance products follows a common intuition. From the perspective of the insurer(protection seller, protection provider, etc.), it receives premium payments and it pays for the credit event happening from time to time. The 'fair' situation, the situation where it approximately neither gains nor lose money on aggregate, is that the insurance premiums it receives equals the payments it pays due to protection obligation. A 'fair' or 'par price' for the insurance premium it charges would make the expected sum of these two kinds of cash flows equal. This is exactly the logic behind the pricing of any insurance product.

In the case of CDS pricing, since we do not know when the default happens, we equate (on discounted terms) the sum of insurance premiums for each time (simplified as mid point of all intervals from inception till expiry) to the sum of expected loss at each time (which amusingly, is a mathematically exact representation).