Theorem (First Fundamental Theorem of Asset Pricing):
A market is free of arbitrage if and only if there exists a strictly positive state-price density (or risk-neutral probability measure).

Proof:

We'll approach this in two parts:

1. If there exists a strictly positive state-price density, then the market is free of arbitrage.
2. If the market is free of arbitrage, then there exists a strictly positive state-price density.

Part 1: If there exists a strictly positive state-price density, then the market is free of arbitrage.

Assume there exists a strictly positive state-price density, denoted by $\lambda(s)$, for each state s.

Consider a portfolio with a current price of zero (meaning it doesn't require any initial investment). Let the payoff of this portfolio in state s be $X(s)$. If there were an arbitrage, this would mean that $X(s) \geq 0$ for all s, and $X(s) > 0$ for at least one state s.

However, the expected value of this portfolio under the risk-neutral measure defined by $\lambda(s)$ would be:

$$E_\lambda[X] = \sum_s \lambda(s) X(s)$$

Given our assumption of strictly positive state-price densities and the fact that $X(s) \geq 0$ for all states, this expected value is strictly positive. But this contradicts the fact that the portfolio has a current price of zero. Hence, an arbitrage opportunity cannot exist.

Part 2: If the market is free of arbitrage, then there exists a strictly positive state-price density.

This part of the proof involves more sophisticated mathematics. The idea is to construct a probability measure under which the discounted asset prices are martingales. This is done using the separating hyperplane theorem from convex analysis.

Intuitively:

- Consider the set of all possible payoffs that can be achieved with non-negative investments (this set is convex).
- If there is no arbitrage, the origin (representing a zero-cost, zero-payoff investment) cannot be an interior point of this set.
- By the separating hyperplane theorem, there exists a hyperplane that separates the origin from this set. The normal vector to this hyperplane gives us the state-price density.