A generic explanation of filtration and sigma field

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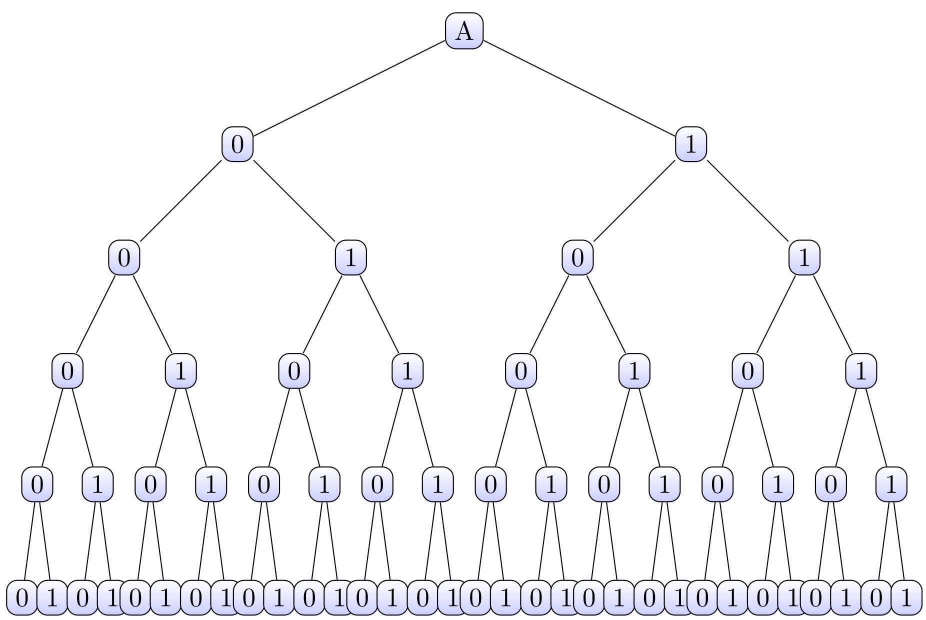
This short text is inspired by [DB06].

1 The model

Consider a simple binary code generating process which goes like this.

The code has a constant prefix 'A', followed by an 5 digit binary numbers which shall be given in a sequential generating process. Each step the process spits out a number, which is either 0 or 1, and this shall be filled in the corresponding digit of the code. Here are some sample codes:
A01011
A11101
A01000
For purpose of illustration let us denote the code as $Aa_1a_2a_3a_4a_5$
The whole process can be illustrated in by the following binary tree model:



We specified the corresponding time steps to be $t_i, i \in \{0, 1, 2, 3, 4, 5\}$

The sigma algebra at each time step is:

1. $F_{t_0} = \{A(\_)(\_)(\_)(\_)(\_)\}$, which is also the set of all possible codes that can be finally generated. This sigma field has 1 element.
2. $F_{t_1} = \{A(\_)(\_)(\_)(\_)(\_), A(a_1 = 0)(\_)(\_)(\_)(\_), A(a_1 = 1)(\_)(\_)(\_)(\_)\}$ with $1 + 2 = 3$ elements

3. $F_{t_2} = \{A(\_)(\_)(\_)(\_)(\_), A(a_1 = 0)(\_)(\_)(\_)(\_), A(a_1 = 1)(\_)(\_)(\_)(\_), A(a_1 = 0)(a_2 = 0)(\_)(\_)(\_), A(a_1 = 0)(a_2 = 1)(\_)(\_)(\_)\}, A(a_1 = 1)(a_2 = 0)(\_)(\_)(\_), A(a_1 = 1)(a_2 = 1)(\_)(\_)(\_)\}$ with $1 + 2 + 2^2 = 7$ elements.

4. $F_{t_4} = ...$

5. $F_{t_5} = ...$, which is also the sample space Ω, with $2^5 - 1 = 31$ elements.

Consider event $E_1 \coloneqq \{A(a_1 = 0)(a_2 = 0)(a_3 = 0)(\_)(\_)\}$, this event is not F_{t_1} measurable (or mbl. in short) because F_{t_1} tells us nothing about information pertaining to a_2 and a_3, which is depicted mathematically by the fact that $E_1 \notin F_{t_1}$.

The sigma field contains events of which you can verdict that whether it happens or it does not happen by the corresponding time step. For example, we know that

$$F_{t_2} = \{A(\_)(\_)(\_)(\_)(\_), ..., A(a_1 = 1)(a_2 = 1)(\_)(\_)(\_)\}$$

thus for any random time we go through the process, we can verdict at time step 2 that whether $E_2 \coloneqq A(a_1 = 0)(\_)(\_)(\_)(\_)$ happens or not. The very fact that we can make such verdict implies that $A(a_1 = 0)(\_)(\_)(\_)(\_)$ is F_{t_2} mbl, as depicted by the fact that $E_2 \in F_{t_2}$.

Similarly, we can not verdict at time step 1 whether $E_1 \coloneqq \{A(a_1 = 0)(a_2 = 0)(a_3 = 0)(\_)(\_)\}$ happens or not, the fact that we may not make such verdict implies that E_1 is not F_{t_1} mbl.

We have $F \supseteqq F_{t_j} \supseteqq F_{t_i}$ for all $i \leq j$, meaning that "the information increases in time", never exceeding the whole set of events F. The family of sigma fields $(F_{t_i})\ i \in \{0, 1, 2, 3, 4, 5\}$ is called filtration.

References

[DB06] Fabio Mercurio Damiano Brigo. *Interest Rate Models – Theory and Practice With Smile, Inflation and Credit.* Springer, New York, NY, 2006.