In the context of stochastic processes, martingales, local martingales, and semi-martingales are concepts that help to formalize different types of random behavior, particularly with respect to the concept of "fair game" in financial mathematics and time series analysis.

Martingale

A **martingale** is a stochastic process that represents a fair game, where the expected future value, conditional on the present and the past, is equal to the current value. Mathematically, a process $(X_t)_{t\geq 0}$ is a martingale with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ and a probability measure P if:

1. X_t is $\mathcal{F}_t$-measurable for each t.
2. $E[|X_t|] < \infty$ for each t.
3. $E[X_{t+s}|\mathcal{F}_t] = X_t$ for all $s \geq 0$, and for all t.

This definition implies that, given the current information, the expected change in the process's value is zero, making it a "fair game."

Local Martingale

A **local martingale** relaxes the conditions of a martingale. A process $(X_t)_{t\geq 0}$ is a local martingale if there exists a sequence of stopping times $(T_n)_{n\in\mathbb{N}}$ with $T_n \uparrow \infty$ as $n \to \infty$ such that the stopped process $(X_{t\wedge T_n})_{t\geq 0}$ is a martingale for each n. In other words, a local martingale behaves like a martingale up to a certain time and may not be a martingale thereafter.

Semi-Martingale

A **semi-martingale** is a generalization of a martingale and includes many processes that can be used to model asset prices. A process $(X_t)_{t\geq 0}$ is a semi-martingale if it can be decomposed into the sum of a local martingale $(M_t)_{t\geq 0}$ and an adapted process of finite variation $(A_t)_{t\geq 0}$:

$$X_t = M_t + A_t$$

Every martingale is a semi-martingale, and every semi-martingale is a local martingale, but the converse statements are not true. Semi-martingales are particularly important in the theory of stochastic integration and are the most general class of processes for which the Itô integral is defined.

In finance, these concepts are integral to the modeling of price processes. Martingales represent idealized models where prices reflect all available information, local martingales allow for some temporary deviations from this idealized behavior, and semi-martingales are capable of capturing a wider range of possible price dynamics, including trends and other systematic changes over time.