Systematic review of SysID - Subspace identification

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% Matlab live script for introducing subspace identification
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Part 4. Subspace State Space System IDentification (4SID)

For stochastic system!

4.1 The state space model:

$x_{k+1} = A\,x_k + B\,u_k + w_k$

$y_k = C\,x_k + D\,u_k + v_k$

$E[w_k w_k] = Q$

$E[v_k v_k] = R$

$E[w_k v_k] = S$

It can be split up into a deterministic and a stochastic subsystem. Deterministic subsystem:

$x^d_{k+1} = A\,x^d_k + B\,u_k$

$y^d_k = C\,x^d_k + D\,u_k$

Stochastic subsystem:

$x^s_{k+1} = A\,x^s_k + w_k$

$y^s_k = C\,x^s_k + v_k$

This process can be also represent by Kalman filter:

$$\begin{aligned} \hat{x}_{k+1} &= A\,\hat{x}_k + B\,u_k + K\left(y_k - C\hat{x}_k + D\,u_k\right) \\ y_k &= C\,\hat{x}_k + D\,u_k + \left(y_k - C\hat{x}_k + D\,u_k\right) \end{aligned},$$

where K is the Kalman gain.

Define $e_k = \left(y_k - C\hat{x}_k + D\,u_k\right)$

Then, we also have the following form:

$x_{k+1} = A_K x_k + B_k\,z_k$

$y_k = C\,x_k + D\,u_k + e_k$

where,

$z_k = \left[u_k^T, y_k^T\right],\ A_k = A - K\,C, B_k = \left[B - K\,D, K\right].$

4.2 Define matrices

The extended observability matrix Γ:

$$\Gamma_i = \begin{bmatrix} C \\ C\ A \\ C\ A^2 \\ \cdots \\ C\ A^{i-1} \end{bmatrix}$$

The reversed extended controllability matrix Δ_i^d:

$$\Delta_i^d = \begin{bmatrix} A^{i-1}B & A^{i-2}B & \cdots & A\,B & B \end{bmatrix}$$

The lower block triangular Toeplitz matrix H_i^d:

$$H_i^d = \begin{bmatrix} D & 0 & 0 & \cdots & 0 \\ C\ B & D & 0 & \cdots & 0 \\ C\ A\ B & C\ B & D & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ C\ A^{i-2}B & C\ A^{i-3}B & C\ A^{i-4}B & \cdots & D \end{bmatrix}$$

The past input and output block Hankel matrices are:

$$U_{0|i-1} = \begin{bmatrix} u_0 & u_1 & u_2 & \cdots & u_{j-1} \\ u_1 & u_2 & u_3 & \cdots & u_j \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ u_{i-1} & u_i & u_{i+1} & \cdots & u_{i+j-2} \end{bmatrix}$$

$$Y_{0|i-1} = \begin{bmatrix} y_0 & y_1 & y_2 & \cdots & y_{j-1} \\ y_1 & y_2 & y_3 & \cdots & y_j \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_{i-1} & y_i & y_{i+1} & \cdots & y_{i+j-2} \end{bmatrix}$$

Similarly, the future input and output block Hankel matrices are:

$U_{i|2i-1},\ Y_{i|2i-1}.$

The past and future deterministic state matrices:

$$X_0^d = \begin{bmatrix} x_0^d & x_1^d & x_2^d & \cdots & x_{j-1}^d \end{bmatrix},$$
$$X_i^d = \begin{bmatrix} x_i^d & x_{i+1}^d & x_{i+2}^d & \cdots & x_{i+j-1}^d \end{bmatrix}.$$

Then, we have the matrix input-output equations:

$$Y_{0|i-1} = \Gamma_i X_0^d + H_i^d U_{0|i-1} + Y_{0|i-1}^s,$$
$$Y_{i|2i-1} = \Gamma_i X_i^d + H_i^d U_{i|2i-1} + Y_{i|2i-1}^s,$$
$$X_i^d = A^i X_0^d + \Delta_i^d U_{0|i-1}.$$

In Kalman form, the first equation above can be written as:

$$Y_{i|2i-1} = \Gamma_i X_i^d + H_i^d U_{i|2i-1} + G_{i|2i-1} E_{i|2i-1}$$

where,

$$G_{i|2i-1} = \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ C\ K & I & 0 & \cdots & 0 \\ C\ A\ K & C\ K & I & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ C\ A^{i-2}K & C\ A^{i-3}K & C\ A^{i-4}K & \cdots & I \end{bmatrix}$$

Also, for a large *i*, $A^i X_0^d \approx 0$, thus,

$$Y_{i|2i-1} = \Gamma_i \Delta_i^d U_{0|i-1} + H_i^d U_{i|2i-1} + G_{i|2i-1} E_{i|2i-1}$$

4.3 The main projection

We all know the least squares solution for $Y = \beta X + w$ is:

$$\hat{\beta} = Y X^T (X X^T)^{-1}.$$

The model prediction is:

$$\hat{Y} = \hat{\beta} X = Y X^T (X X^T)^{-1} X.$$

Then, the main projection is defined as

$$Y / X = Y X^T (X X^T)^{-1} X$$

It corresponds to the optimal prediction of Y given X in a sense that

$$\|Y - Y/X\|^2.$$

4.4 The N4SID algorithm

Estimate the observability matrix and the state by projecting future observations onto past observations:

$$Y_{i|2i-1}/U_{i|2i-1}^{\perp} = \Gamma_i \Delta_i^d U_{0|i-1}/U_{i|2i-1}^{\perp} + H_i^d U_{i|2i-1}/U_{i|2i-1}^{\perp} + G_{i|2i-1} E_{i|2i-1}/U_{i|2i-1}^{\perp}.$$

Since the past input are uncorrelated with future input and future observation noise, thus,

$$U_{i|2i-1}/U_{i|2i-1}^{\perp} = 0$$
$$E_{i|2i-1}/U_{i|2i-1}^{\perp} = E_{i|2i-1}\left(I - U_{i|2i-1}^T \left(U_{i|2i-1} U_{i|2i-1}^T\right)^{-1} U_{i|2i-1}\right) = E_{i|2i-1}$$

We can eliminate the above equation as:

$$Y_{i|2i-1}/U^{\perp}_{i|2i-1} = \Gamma_i \Delta_i^d U_{0|i-1}/U^{\perp}_{i|2i-1} + G_{i|2i-1} E_{i|2i-1}.$$

Then, we can remove the noise term by multiplying $U_{0|i-1}$:

$$Y_{i|2i-1}/U^T_{i|2i-1} U_{0|i-1} = \Gamma_i \Delta_i^d U_{0|i-1}/U^T_{i|2i-1} U_{0|i-1} + G_{i|2i-1} E_{i|2i-1} U_{0|i-1} = \Gamma_i \Delta_i^d U_{0|i-1}/U^T_{i|2i-1} U_{0|i-1}.$$

We can calcualte LHS using data, then perform SVD:

$$Y_{i|2i-1}/U^T_{i|2i-1} U_{0|i-1} = U\ S\ V^T$$

And choose $\Gamma_i = U\ S^{1/2}$.

Then we can calculate the transition matrix $\hat{A}$ from Γ_i.

4.5 Modal analysis

After obtaining A, we can perform an eigenvalue decomposition of it:

$$A = \Phi[\lambda_i]\Phi^{-1}$$

Then, find the discrete-time values, which are also modal frequencies:

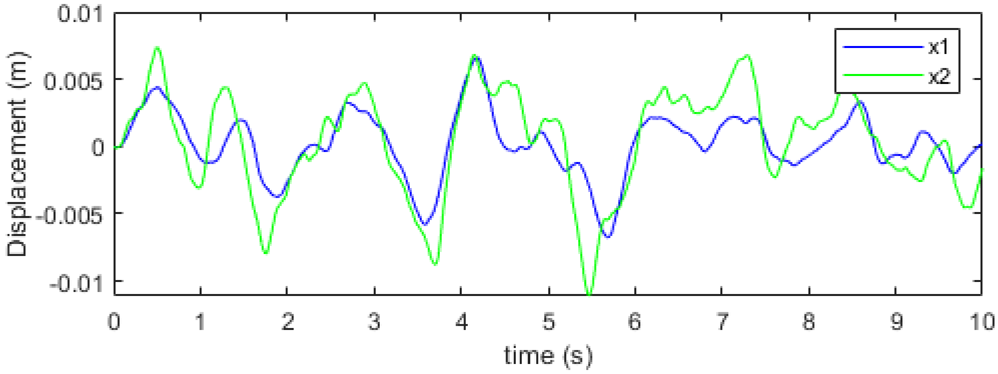
$$\omega_i = \frac{\ln(\lambda_i)}{\Delta t}$$

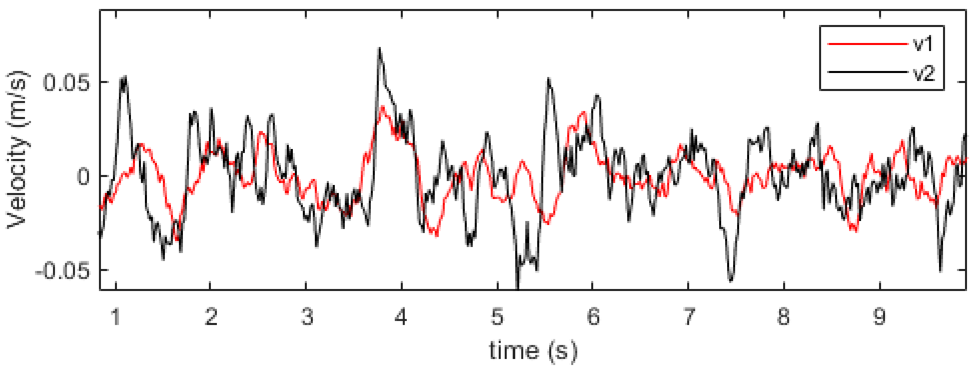
Also, mode shape matrix:

$$\psi = C\Phi.$$

4.6 Implementation

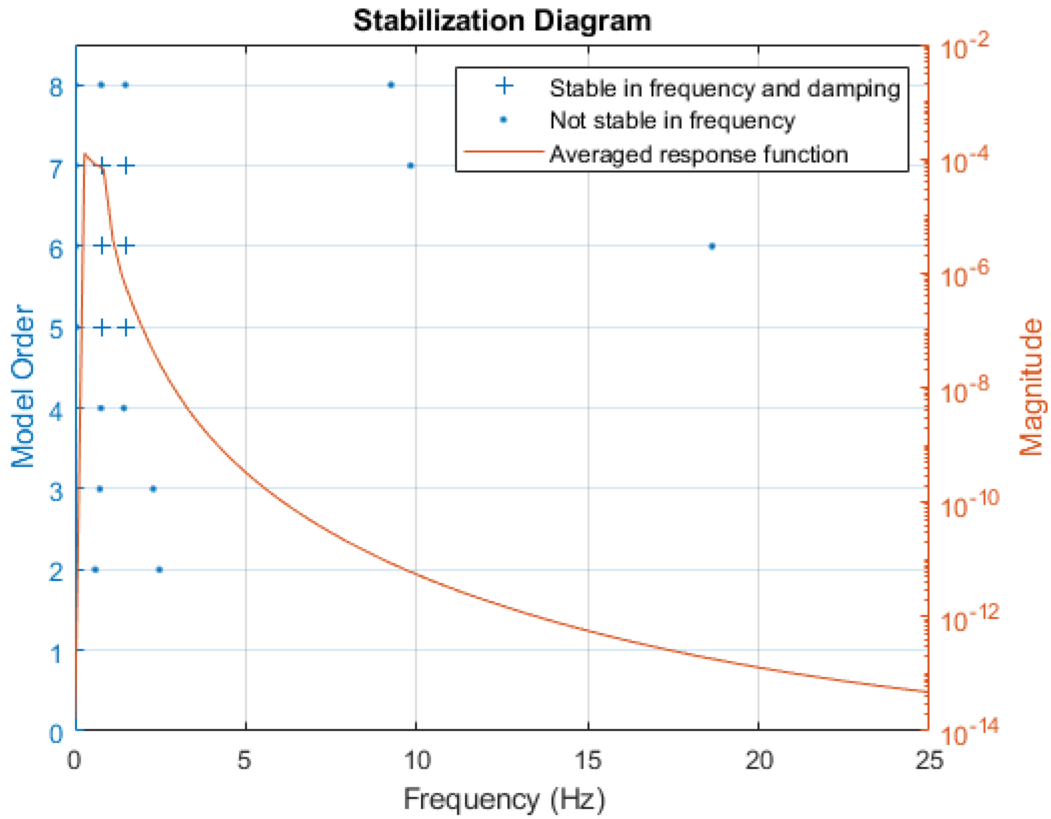
```
clc;
clear;
close all;
addpath ../functions/
T = 10; %length of time duration
nt = 500; %number of time stamps
Fs = nt/T;
tspan=linspace(0,T,nt);
F = csvread(['../data/ambient.csv']); %load data
ysol = csvread(['../data/2dof.csv']); %load data
tspan=linspace(0,T,nt);
figure(1)
plot_dv(tspan,ysol')
```





Matlab has a function to help us calculate the modal parameters:

```
[frf,f] = modalfrf(F',ysol(3:4,:)',Fs,180,'Estimator','subspace','Order',30);
figure(2)
modalsd(frf,f,Fs,'MaxModes',8)
```



Stabilization Diagram
Stable in frequency and damping
Not stable in frequency
Averaged response function
Model Order
Magnitude
Frequency (Hz)
0
1
2
3
4
5
6
7
8
0
5
10
15
20
25
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}