C161 Project Report: Gravitational Lens Simulation

Jingxu Xie, Yihao Zhou

We code an application that can calculate the trajectory of light around a Schwarzschild black hole or other massive object. User input an original image of an object, the scale of the image, the mass M of the black hole, the distance L_1 between the object and the black hole, and distance L_2 between the black hole and the observer. Then the application will calculate the effect of the black hole and output a simulation image observed when there is a black hole between the object and the observer. This report illustrates the formalism of calculating the trajectory, the method of very precise numerical integration and the usage of this application.

1 Formalism

We code the program based on the formalism developed in Ref [1], here we just states some important formulas we use to determine the trajectory of light around a Schwarzschild black hole.

The original equations of motion in Schwarzschild metric for a particle of mass $m > 0$ around a mass M [1] are:

$$\frac{dr}{d\tau} = \left(\frac{E}{m}\right)^2 - \left(1 - \frac{2M}{r}\right)\left[1 + \frac{(L/m)^2}{r^2}\right] \tag{1}$$

$$\frac{d\phi}{d\tau} = \frac{L/m}{r^2} \tag{2}$$

$$\frac{d\tau}{dt} = \frac{(1 - 2M/r)}{(E/m)} \tag{3}$$

where E is the total energy of the particle and L is the angular momentum.

Using equation (3), we can eliminate the proper time $d\tau$ in equations (1) and (2) and yield two equations:

$$\left(\frac{dr}{dt}\right)^2 = \left(1 - \frac{2M}{r}\right)^2 - \left(1 - \frac{2M}{r}\right)^3 \left[\frac{m^2}{E^2} + \frac{1}{r^2}\left(\frac{L}{E}\right)^2\right]$$

$$\frac{d\phi}{dt} = \left(\frac{L}{E}\right)\frac{1 - 2M/r}{r^2}$$

take the limit $m \to 0$, we obtain

$$\left(\frac{dr}{dt}\right)^2 = \left(1 - \frac{2M}{r}\right)^2 - \left(1 - \frac{2M}{r}\right)^3 \frac{b^2}{r^2}$$

$$\left(\frac{d\phi}{dt}\right)^2 = \frac{b^2}{r^4}\left(1 - \frac{2M}{r}\right)^2$$

where $b = L/p = L/E$ is the impact parameter for light. By eliminating dt, we have the relationship between $d\phi$ and dr:

$$|d\phi| = \frac{|dr|}{r^2\left[\frac{1}{b^2} - \frac{1}{r^2} + \frac{2M}{r^3}\right]^{1/2}} \tag{4}$$

To obtain the total deflection ϕ, in principle we could simply integrate equation (4) from $r = (L_1^2 + d^2)^{1/2}$ to $r = R$, the distance of closest approach, add the integral from $r = L_2$ to $r = R$. To determine R, substitute $r = R$ into equation (4) and set the denominator equal to zero, we then obtain:

$$R^3 - b^2 R + 2Mb^2 = 0 \tag{5}$$

then we can solve $R = R(b)$ for given b and thus integrate equation (4) to find the total deflection ϕ

However, L_1 and L_2 in practice are very large, which are around 1 ly $\sim 10^{16}$ meters, causing some difficulties in integration. To take the integral, we first make the substitution $u = R/r$, then $dr = -r^2 du/R$ and the equation (4) becomes

$$|d\phi| = \frac{|du|}{\left[\frac{R^2}{b^2} - u^2 + 2\frac{M}{R}u^3\right]^{1/2}} \tag{6}$$

Then we can integrate from $u = R/L_i \simeq 0$, $i = 1, 2$ to $u = R/R = 1$,

$$\phi = \phi(b) = \left(\int_{R(b)/(L_1^2+d^2)^{1/2}}^{1} + \int_{R(b)/L_2}^{1}\right)|d\phi|$$

which is easier to handle the integration. However, the denominator in equation (6) goes to zero as $u \to 1$, taking equation (5) into account, i.e. the integrand $|d\phi|$ will go to infinity as $u \to 1$. To deal with this, we use the trapezoidal rule to approximate the integral and use smaller and smaller intervals as u goes closer and closer to 1.

[1] Here we use $M = M_0 G/c^2$, where M_0 is the actual mass of the black hole. By using this convention, M has the unit of length, i.e., meter.

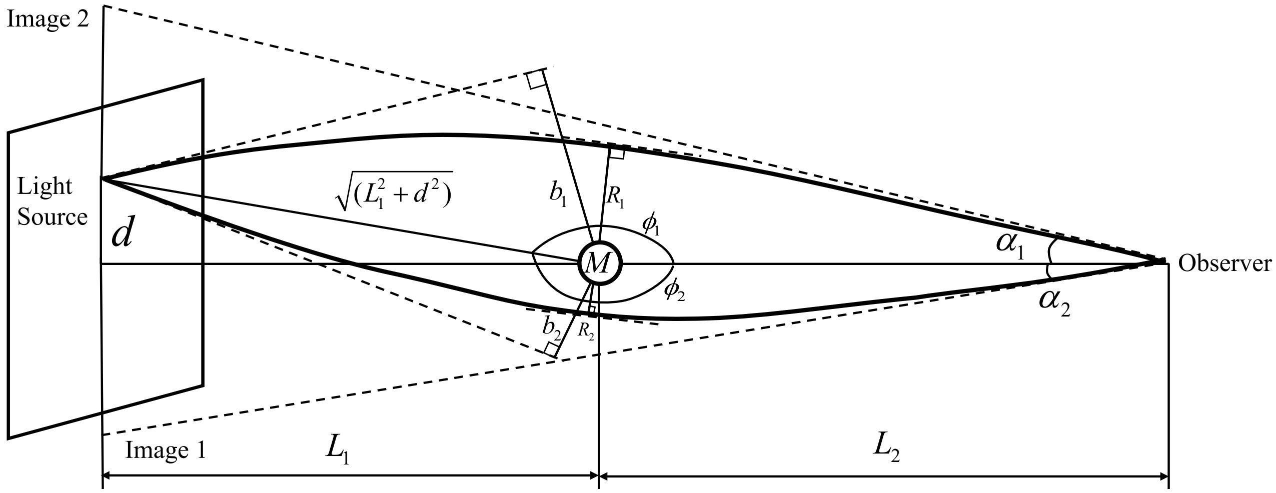


Figure 1: Light trajectory around a Schwarzschild black hole. Light source, the black hole with mass M, and the observer, are separated by distance L_1 and L_2. $\phi_{1,2} = \pi \mp \arctan(d/L_1)$ are two deflection angles required for light to reach observer from the source. $b_{1,2}$ are the impact parameters for two trajectories and $R_{1,2}$ are the shortest distance between black hole and points on the trajectories. $\alpha_{1,2}$ are two observation angles result in two images for one source point. The system has rotational symmetry around the axis connects observer, black hole, and the center of light source.

For every point on the original image, we first need to determine the impact parameter b, see Fig 1. To do this, we first calculate the two target deflection angles $\phi_{1,2} = \pi \mp \arctan(d/L_1)$ for light to go from the origin point to the observer, where d is the distance between the point and the center of the image. Then we choose an initial value $b = \sqrt{27}M$, to guarantee equation (5) to have positive solution, and integrate equation (6) to check if the integration $\phi(b)$ equals to the deflection angles $\phi_{1,2}$ we want. Since the integral $\phi(b)$ is monotonous depends on the impact parameter b, i.e., $\phi(b)$ decreases as b increases, we use the binary search algorithm to search the correct impact parameters $b_{1,2}$ which satisfy $\phi(b_{1,2}) = \phi_{1,2}$. This method is very effective and can reach a precision of $|\phi(b_{1,2}) - \phi_{1,2}| < 10^{-8}$ within 100 iterations.

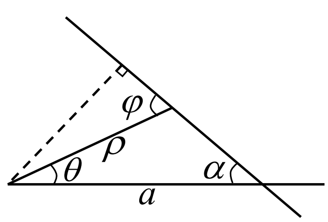


Figure 2: Schematic of line equation in polar coordinate.

After calculating the impact parameter $b_{1,2}$ for the point, we then calculate the observation angle $\alpha_{1,2}$. For a line passing point $(a, 0)$ and has angle α with the horizontal line, see Fig. 2, we know the equation in polar coordinate is

$$a \sin\alpha = \rho \sin\varphi = \rho \sin(\alpha + \theta) \tag{7}$$

differentiate it respect to ρ, we have

$$\tan\alpha = a \left|\frac{d\theta}{d\rho}\right|_{\rho=a,\theta=0}$$

therefore we get two observation angles

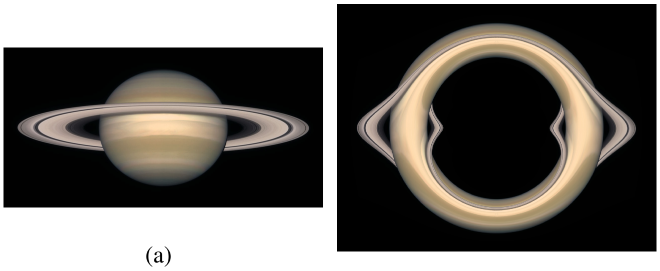
$$\begin{aligned} \alpha_{1,2} &= \arctan\left(L_2 \left|\frac{d\phi}{dr}(b_{1,2})\right|_{r=L_2}\right) \\ &= \frac{L_2}{\sqrt{1/b_{1,2}^2 - 1/L_2^2 + 2M/L_2^3}} \end{aligned} \tag{8}$$

then there are two images for one point on the light source.

2 Simulation Results

Here we give two examples of our simulation program. Fig. 3 shows the original[2] (a) and simulation (b) images of Saturn.

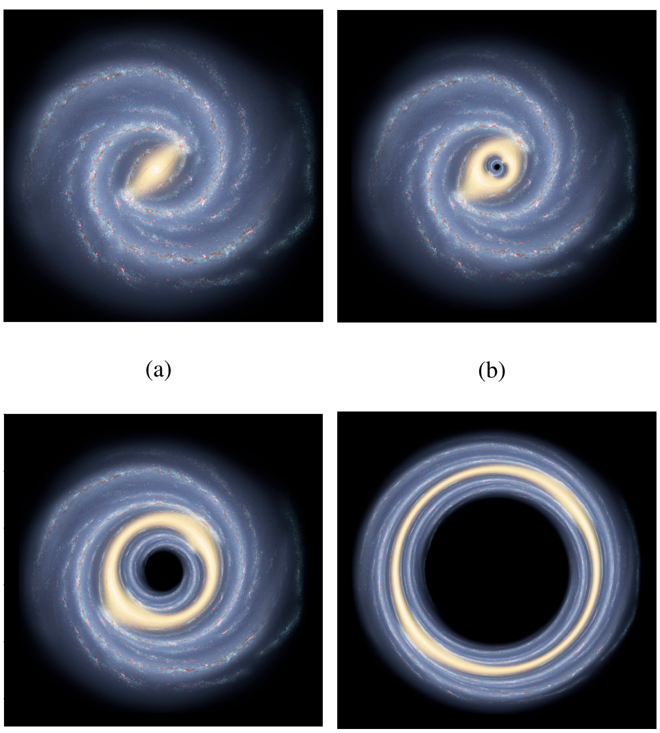
[2]Taken by Hubble Space Telescope on October, 1997. https://www.spacetelescope.org/images/opo0115d/



(a)

(b)

Figure 3: Image of Saturn (a), taken by Hubble Space Telescope on October, 1997. and the simulation image (b) by using parameter: the distance $L_1 = L_2 = 7\times10^{10}$ meters $\simeq 7\times10^{-6}ly$, the black hole $M = 1.5\times10^5$ meters $\simeq 10M_\odot$, and the diameter of the Saturn $D = 10^8$ meters $\simeq 10^{-8}ly$. In this simulation, Saturn is distorted into a donut (torus), and there are two distorted images, one lies along the top of the outer surface of the donut, while the other reversed right-for-left and top-for-bottom [1].



(a)

(b)

(c)

(d)

Figure 4: Image of the Milky Way Galaxy (a), taken by NASA/JPL-Caltech/R. Hurt. And simulation of gravitational lens effect with parameter: the distance $L_1 = L_2 = 5\times10^5 ly$, the scale of the original image $D = 10^5 ly$, and the mass of black hole $M = 10^{14}M_\odot$ (b), $M = 10^{15}M_\odot$ (c), $M = 10^{16}M_\odot$ (d).

In the simulation we use the parameter as follows: the distance between the Saturn and the black hole $L_1 = 7\times10^{10}$ meters $\simeq 7\times10^{-6}ly$, and the distance between the black hole and the observer $L_2 = 7\times10^{10}$ meters $\simeq 7\times10^{-6}ly$, too. The mass of the black hole $M = 1.5\times10^5$ meters $\simeq 10M_\odot$[3]. And the diameter of the Saturn $D = 10^8$ meters $\simeq 10^{-8}ly$. In the simulation image, Saturn is distorted into a donut (torus), and there are two distorted images, one lies along the top the outer surface of the donut, while the other reversed right-for-left and top-for-bottom [1].

Fig. 4 shows the original[4] (a) and simulation (b-d) images of the Milky Way Galaxy. In this simulation, we use the parameter $L_1 = L_2 = 5\times10^{21}$ meters $\simeq 5\times10^5$ light years. The diameter of the galaxy $D = 10^{21}$ meters $\simeq 10^5$ light years. And the mass of black hole $M = 10^{14}M_\odot$ (b), $M = 10^{15}M_\odot$ (c), $M = 10^{16}M_\odot$ (d). From simulation images, we see that with the increase of the mass of the black hole, the effect of gravitational lens becomes stronger and stronger, and finally the whole galaxy become a ring around the black hole.

3 Software Usage

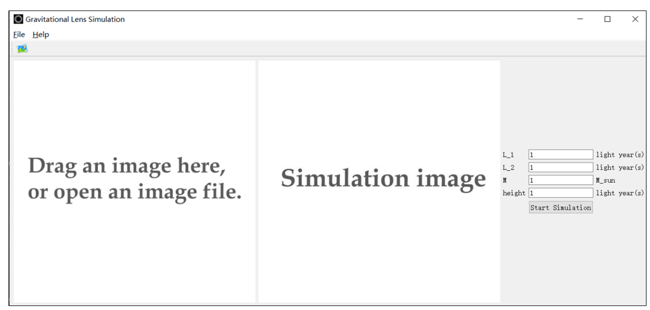


Figure 5: User interface of gravitational lens simulation program.

The user interface of our software is shown in Fig. 5. To use it, first open drag an image file (only *.jpg, *.png, *.bmp formats are supported) into the left the region. Then set the parameters in the right panel. L_1 is the distance between the object and the black hole, L_2 is the distance between the black hole and the observer. They both use light year as unit (1 ly $\sim 10^{16}$ meters). Then set the mass of the black hole, which is in $M_\odot$ unit. Finally set the height of the image, which is also in unit of light year. After setting up the parameters,

[3] $M_{\odot 0} = 1.989\times10^{30}$kg, $M_\odot = M_{\odot 0}G/c^2 = 1477$ meters

[4] Downloaded from https://solarsystem.nasa.gov/resources/285/the-milky-way-galaxy/

click start simulation and then wait for about 3 minutes. The simulation image will appear in the middle.

Fig. 6 is a sample usage of the program. Here we use an image of the sun, with the height is the diameter of the sun, i.e., $1.392e-7p$ly, and the distance $L_1 = L_2 = 7.5e-6$ ly, and the black hole has the mass of 1000 $M_\odot$. The upper limit of the distance can go up to at least 10^{30} light years ($L_1 = L_2 = 5 \times 10^{30} ly, M = 10^{40} M_\odot$, height $= 10^{30} ly$ is tested to be valid), which is much larger than the known universe. When setting the parameters, be cautious that they should be of similar order, otherwise the program may crash for some unknown reason. All the codes are at https://github.com/jingxuxie/Gravitational-Lens-Simulation

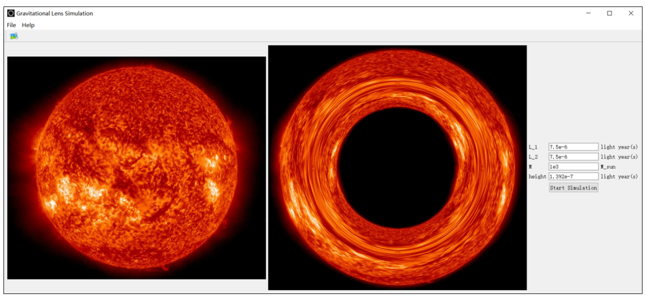


Figure 6: User interface of gravitational lens simulation program.

References

[1] Taylor and Wheeler, Exploring Black Holes (2000).