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2. Section 9.6 - 1, 3. Question 1 should be solved by hand. For constrained optimization problems, you can use Newton's method on R to find the critical points if the gradient of the Lagrangian is not solvable by hand. Use l12 and l13 as reference to see how to write your solutions.

9.6 1. Find the max & min of the function

$$f(x_1, x_2, x_3, x_4) = x_1 - 2x_2 + 3x_3 - 4x_4$$

subject to the constraints $x_1^2 + x_2^2 = x_4^2 + 1$

and $x_1^2 + 2x_2^2 + 3x_3^2 = 6$

$g_1(x) = x_1^2 + x_2^2 - x_4^2 - 1$

$g_2(x) = x_1^2 + 2x_2^2 + 3x_3^2 - 6$

$$Df(x) = \begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \end{bmatrix}$$

$$Dg(x) = \begin{bmatrix} \partial g_1/x_1 & \partial g_1/x_2 & \partial g_1/x_3 & \partial g_1/x_4 \\ \partial g_2/x_1 & \partial g_2/x_2 & \partial g_2/x_3 & \partial g_2/x_4 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 & 2x_2 & 0 & -2x_4 \\ 2x_1 & 4x_2 & 6x_3 & 0 \end{bmatrix}$$

$$F(X) = \begin{bmatrix} 2x_1\lambda_1 + 2x_1\lambda_2 + 1 \\ 2x_2\lambda_1 + 4x_2\lambda_2 - 2 \\ 6x_3\lambda_2 + 3 \\ -2x_4\lambda_1 - 4 \\ x_1^2 + x_2^2 - x_4^2 - 1 \\ x_1^2 + 2x_2^2 + 3x_3^2 - 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$DF = \begin{bmatrix} 2\lambda_1 + 2\lambda_2 & 0 & 0 & 0 & 2x_1 & 2x_1 \\ 0 & 2\lambda_1 + 4\lambda_2 & 0 & 0 & 2x_2 & 4x_2 \\ 0 & 0 & 6\lambda_2 & 0 & 0 & 6x_3 \\ 0 & 0 & 0 & -2\lambda_1 & -2x_4 & 0 \\ 2x_1 & 2x_2 & 0 & -2x_4 & 0 & 0 \\ 2x_1 & 4x_2 & 6x_3 & 0 & 0 & 0 \end{bmatrix}$$

pls see R code.

3. Consider five tradable assets with the following expected values and standard deviations of the rates of return of the assets:

$$\mu_1 = 0.08;\ \mu_2 = 0.10;\ \mu_3 = 0.13;\ \mu_4 = 0.15;\ \mu_5 = 0.20;$$
$$\sigma_1 = 0.14;\ \sigma_2 = 0.18;\ \sigma_3 = 0.23;\ \sigma_4 = 0.25;\ \sigma_5 = 0.35.$$

The correlation matrix of the rates of return is

$$\Omega = \begin{pmatrix} 1 & -0.3 & 0.4 & 0.25 & -0.2 \\ -0.3 & 1 & -0.1 & -0.2 & 0.15 \\ 0.4 & -0.1 & 1 & 0.35 & 0.25 \\ 0.25 & -0.2 & 0.35 & 1 & -0.15 \\ -0.2 & 0.15 & 0.25 & -0.15 & 1 \end{pmatrix}$$

Assume that it is possible to take both long and short positions of arbitrary size in these assets.

(i) Find the asset allocation for a minimal variance portfolio with 15% expected rate of return and the corresponding minimal standard deviation of the rate of return of the portfolio;

(ii) Find the asset allocation for a maximum expected return portfolio with 25% standard deviation of the rate of return and the corresponding maximal expected rate of return of the portfolio.

(i) min : $w^T \Sigma w$ $\qquad \frac{\partial(w^T\Sigma w)}{\partial w} = 2(\Sigma w)^T$

subject to: $e^T w = 1$ $\quad g_1$ $\qquad \frac{\partial(w^T\Sigma w)\partial}{\partial\lambda_1} = \overline{\partial\lambda_2} = 0$

$u^T w = \mu_p$ $\quad g_2$

$$F(w,\lambda) = w^T\Sigma w + \lambda_1(e^T w - 1) + \lambda_2(u^T w - \mu_p)$$

$$Dg(x) = \begin{bmatrix} \partial g_1 / w & \partial g_1 / \lambda_1 & \partial g_1 / \lambda_2 \\ \partial g_2 / w & \partial g_2 / \lambda_1 & \partial g_2 / \lambda_2 \end{bmatrix}$$

$$= \begin{bmatrix} e^T & 0 & 0 \\ u^T & 0 & 0 \end{bmatrix}$$

$$G(x) = DF(x) = \begin{bmatrix} 2w^T\Sigma + e^T\lambda_1 + u^T\lambda_2 \\ \\ e^T w - 1 \\ u^T w - \mu_p \end{bmatrix}$$

$$
DG(w,\lambda) = \begin{array}{c} \\ \\ \\ \\ \end{array}
\begin{array}{ccc} w & \lambda_1 & \lambda_2 \end{array}
$$

$$
DG(w,\lambda)=\begin{bmatrix} 2\Sigma & e & u \\ e^T & 0 & 0 \\ u^T & 0 & 0 \end{bmatrix}
$$

$r_{xy} = \dfrac{Cov(x,y)}{S_x S_y}$ we need to transform from correlation matrix to covariance.

$Cov(1,2) = -0.3 \times 0.14 \times 0.18$

$Cov(1,3) = 0.4 \times 0.14 \times 0.23$

$Cov(1,4) = 0.25 \times 0.14 \times 0.25$

$Cov(1,5) = -0.2 \times 0.14 \times 0.35$

$Cov(2,3) = -0.1 \times 0.18 \times 0.23$

$Cov(2,4) = -0.2 \times 0.18 \times 0.25$

$Cov(2,5) = 0.15 \times 0.18 \times 0.35$

$Cov(3,4) = 0.35 \times 0.23 \times 0.25$

$Cov(3,5) = 0.25 \times 0.23 \times 0.35$

$Cov(4,5) = -0.15 \times 0.25 \times 0.35$

$Cov(1,1) = Var(1) = 0.14^2$ $\quad$ $Cov(4,4) = 0.25^2$

$Cov(2,2) = 0.18^2$ $\quad$ $Cov(5,5) = 0.35^2$

$Cov(3,3) = 0.23^2$

iii) : $F(w,\lambda) = u^T w + \lambda_1 (e^T w - 1) + \lambda_2 (w^T \Sigma w - \sigma_p^2)$

$$\max: \ u^T w$$
$$\text{s.t.:} \quad e^T w = 1$$
$$w^T \Sigma w = \sigma_p^2$$

$$\nabla F(w, \lambda) = \begin{bmatrix} u + \lambda_1 e + 2\lambda_2 \Sigma w \\ e^T w - 1 \\ w^T \Sigma w - \sigma_p^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$= G(w, \lambda)$$

$$DG = \begin{bmatrix} 2\lambda_2 \Sigma & e & 2\Sigma w \\ e^T & 0 & 0 \\ 2(\Sigma w)^T & 0 & 0 \end{bmatrix}$$

3. Recall finding implied volatility from given price of a call option is equivalent coupon bond w/ face value 100 & coupon rate 8%. If the zero rate curve is given by $r(0,t) = 0.05 + 0.01\ln(1+\frac{t}{2})$ where

$$f(x) = Se^{-qT}N(d_1(x)) - ke^{-rT}N(d_2(x)) - c$$

$$d_1(x) = \frac{\ln(\frac{S}{K}) + (r-q+\frac{x^2}{2})T}{x\sqrt{T}},$$

$$d_2(x) = \frac{\ln(\frac{S}{K}) + (r-q-\frac{x^2}{2})T}{x\sqrt{T}}$$

(i) show that $\lim_{x\to\infty} d_1(x) = \infty$, $\lim_{x\to\infty} d_2(x) = -\infty$

and concludes $\lim_{x\to\infty} f(x) = Se^{-qT} - c$

$\lim_{x\to\infty} x\sqrt{T} = \infty$ $\quad \lim_{x\to\infty} (\ln(\frac{S}{K}) + (r-q+\frac{x^2}{2})T)$

$= \infty$

$\frac{d(x\sqrt{T})}{x} = \sqrt{T} \neq 0$ Thus, with using L.hospital rule,

$\frac{d(\ln(\frac{S}{K}) + (r-q+\frac{x^2}{2})T)}{dx}$

$= TX$

$\lim_{x\to\infty} = \lim_{x\to\infty} \frac{TX}{\sqrt{T}} = \infty$

$$\lim_{x\to\infty}\left(\ln\left(\frac{S}{K}\right)+\left(r-q-\frac{x^2}{2}\right)T\right) \qquad \frac{d\left(\ln\left(\frac{S}{K}\right)+\left(r-q-\frac{x^2}{2}\right)T\right)}{dx}$$

$$= -\infty \qquad\qquad = -Tx$$

with using l hospital rule

$$\lim_{x\to\infty} = \lim_{x\to\infty}\frac{-Tx}{\sqrt{T}} = -\infty \qquad N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{z^2}{2}}dx$$

$$\Rightarrow N(d_2(x)) = 0$$

$$\Rightarrow \lim_{x\to\infty} f(x) = Se^{-qT} - 0 - C \qquad N(d_1(x)) = 1$$

$$= Se^{-qT} - C$$

(ii) show that

$$\lim_{x\searrow 0} d_1(x) = \lim_{x\searrow 0} d_2(x) = \begin{cases} -\infty & Se^{(r-q)T} < K \\ 0 & = \\ \infty & > \end{cases}$$

let $a = \ln\left(\frac{S}{K}\right)+(r-q)T$

$$d_1(x) = \frac{a + \left(\frac{x^2}{2}\right)T}{x\sqrt{T}} \qquad d_2(x) = \frac{a - \left(\frac{x^2}{2}\right)T}{x\sqrt{T}}$$

① $a > 0$

$$\lim_{x\searrow 0} d_1(x) = \lim \frac{a}{x\sqrt{T}} + \lim_{x\searrow 0}\frac{\left(\frac{x^2}{2}\right)T}{x\sqrt{T}}$$

$$= \infty$$

$$\lim_{x\searrow 0} d_2(x) = \infty - 0 = \infty$$

$$\ln\left(\frac{S}{K}\right) + (r-q)T > 0$$

$$\ln\left(\frac{S}{K}\right) > -(r-q)T$$

$$\frac{S}{K} > e^{-(r-q)T}$$

$$\frac{S}{e^{-(\cancel{r}-q)T}} > K$$

$$Se^{(r-q)T} > K$$

② $a<0$ which means $Se^{(r-q)T} < K$

$$\lim_{x\searrow 0} d_1(x) = -\infty \qquad \lim_{x\searrow 0} d_2(x) = -\infty$$

③ $a=0$ $Se^{(r-q)T} = K$

$$\lim_{x\searrow 0} d_1(x) = \lim_{x\searrow 0} \frac{\frac{x^2}{2}T}{x\sqrt{T}} = \frac{x}{2}\sqrt{T} = 0$$

$$\lim_{x\searrow 0} d_2(x) = \lim_{x\searrow 0} \frac{-\frac{x^2}{2}T}{x\sqrt{T}} = \lim_{x\searrow 0} -\frac{x}{2}\sqrt{T} = 0$$

If $a\leq 0$ $Se^{(r-q)T} \leq K$ $\lim_{x\searrow 0} d_1(x) = -\infty$

$$\lim_{x\searrow 0} N(d_1(x)) = \lim_{x\searrow 0} \int_{-\infty}^{d_1(x)} e^{-\frac{z^2}{2}}\, d\,(d_1(x))$$
$$= 0$$

$$\lim_{x\searrow 0} Se^{-qT}N(d_1(x)) - Ke^{-rt}N(d_2(x)) - c$$

$$= 0 - 0 - c = -c$$

If $a>0$ $\lim_{x\searrow 0} d_1(x) = \infty = \lim_{x\searrow 0} d_2(x) = \infty$

$$\lim_{x\searrow 0} N(d_1(x)) = \lim_{x\searrow 0} N(d_2(x)) = 1$$

$$\lim_{x \to 0} f(x) = Se^{-qT} \cdot 1 - Ke^{-rT} \cdot 1 - C$$

$$= Se^{-qT} - Ke^{-rT} - C$$

4. A three months at-the-money call on an underlying asset with spot price 30 paying dividends continuously at a 2% rate is worth \$2.5. Assume that the risk free interest rate is constant at 6%.

   (i) Compute the implied volatility with six decimal digits accuracy, using the bisection method on the interval [0.0001, 1], ~~the secant method with initial guess 0.5~~, and Newton's method with initial guess 0.5.

please see R code

6.11 Exercises

1. Show that the cubic Taylor approximation of $\sqrt{1+x}$ $= (1+x)^{\frac{1}{2}}$ around 0 is

$$\sqrt{1+x} \approx 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}.$$

$$T(x) = \sum_{k=0}^{\infty} \frac{(x-0)^k}{k!} f^{(k)}(0)$$

$$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$$

$$f''(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}}$$

$$f'''(x) = \frac{3}{8}(1+x)^{-\frac{5}{2}}$$

$\vdots$

$$\frac{d}{dx}(1+x)^{-\frac{n}{2}} = \frac{-n}{2}(1+x)^{\frac{-n-2}{2}}$$

nth derivative of $f(x) = \sqrt{x+1}$

$$\frac{1}{2} \cdot \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) \cdots \left(-\frac{2n-3}{2}\right)(1+x)^{-\frac{1-2n}{2}}$$

$$f^{(n)}(0) = (-1)^{n-1} \frac{1 \cdot 3 \cdots (2n-3)}{2^n}$$

$$(\ ) = \sum_{k=0}^{\infty} \frac{(x-0)^k}{k!} f^{(k)}(0)$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!} f^{(k)}(0)$$

$$\approx 1 + x f^{(1)}(0) + \frac{x^2}{2} f^{(2)}(0) + \frac{x^3}{6} f^{(3)}(0)$$

$$\approx 1 + x \cdot \frac{1}{2} - \frac{x^2}{2} \cdot \frac{1}{2^2} + \frac{x^3}{6} \cdot \frac{3}{8}$$

$$\approx 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$$

2. Use the Taylor series expansion of the function e^x to find the value of $e^{0.25}$ with six decimal digits accuracy.

$f'(x) = e^x$

$\vdots$

$f^{(n)}(x) = e^x$

$f^{(k)}(a) = e^a$

We know e^x centered at 0 is

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3}$$

$$e^{0.25} = 1 + 0.25 + \frac{0.25^2}{2} + \frac{0.25^3}{3}$$

$$= 1.25 + 0.03125 + \frac{0.25^3}{3}$$

$$\approx 1.286458$$

3. Show that

$$e^{-x} - \frac{1}{1+x} = O(x^2), \quad \text{as } x \to 0.$$

$\frac{1}{1+x}$ has the following Maclaurin series

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

e^{-x} has the following Maclaurin series

$$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \cdots$$

$$e^{-x} - \frac{1}{1+x} = \left(1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \cdots\right) - \left(1 - x + x^2 - x^3 + \cdots\right)$$

$$= -\frac{x^2}{2} + \frac{2x^3}{3} - \cdots$$

$$\left|e^{-x} - \frac{1}{1+x}\right| = \left|-\frac{x^2}{2} + \frac{2x^3}{3} - \cdots\right|$$

when $|x-a| \leq \delta$

$0 < x \leq 1$

$\cdot\ x^2 \geq x^3 \geq x^4 \cdots$

$$\left|e^{-x} - \frac{1}{1+x}\right| \leq \left|-\frac{x^2}{2}\right| + \left|\frac{2x^3}{3}\right| + \cdots$$

$$\leq x^2 + x^2 + \cdots$$

Thus, $g(x) = x^2$ $\qquad e^{-x} - \frac{1}{1+x}$ is $O(x^2)$

4 (i) Let $g(x)$ be an infinitely differentiable function. Find the linear and quadratic Taylor approximations of $e^{g(x)}$ around the point 0.

(ii) Use the result above to compute the quadratic Taylor approximation around 0 of $e^{(x+1)^2}$.

(iii) Compute the quadratic Taylor approximation around 0 of $e^{(x+1)^2}$ by using Taylor approximations of e^x and e^{x^2}.

$f(x) = e^{g(x)}$ $\qquad e^{2x^2}$

$f'(x) = g'(x)\, e^{g(x)}$

$f''(x) = g'(x)\, g'(x)\, e^{g(x)}$

$\vdots$

$f^{(n)}(x) = [g'(x)]^n\, e^{g(x)}$ $\qquad f^{(n)}(0) = [g'(0)]^n\, e^{g(0)}$

linear: $f(x) = f(0) + (x-0) f'(0) + O((x-0)^2)$ as $x \to 0$

$= e^{g(0)} + x \cdot g'(0)\, e^{g(0)} + O(x^2)$

$= e^{g(0)} (1 + x \cdot g'(0)) + O(x^2)$

quardratic: $f(x) = e^{g(0)} + x \cdot g'(0)\, e^{g(0)} + \frac{x^2}{2} f''(0) + O(x^3)$

$= e^{g(0)} + x \cdot g'(0)\, e^{g(0)} + \frac{[g'(0)]^2 e^{g(0)}\, x^2}{2} + O(x^3)$

$g(x) = (x+1)^2$ $\qquad g(0) = 1$ $\qquad e^{g(0)} = e$ $\quad g'(0) = 2$

$g'(x) = 2x+2$ $\quad f'(0) = g'(0)\, e = 2e$

$f''(0) = g'(0) \cdot g'(0) \cdot e = 4e$

$f(x) = e + x \cdot 2e + \frac{x^2}{2} \cdot 4e + O(x^3)$

$= e + 2ex + 2ex^2 + O(x^3)$

$$e^x = 1 + x + \frac{x^2}{2} + O(x^3)$$

$$e^{x^2} = 1 + x^2 + O(x^3)$$

$$e^{(x+1)^2} = e^{x^2+2x+1} = e^{x^2} e^{2x} e$$

$$= (1+x^2+\cdots)(1+x+\frac{x^2}{2}+\cdots)^2 e$$

$$= (1+x+\frac{x^2}{2} + x^2 + x^3 + \frac{x^4}{2} + \cdots)(1+x+\frac{x^2}{2}+\cdots)\, e$$

$$= e(1+x+\frac{x^2}{2}+x^2+x^3+\frac{x^4}{2}+x+x^2+\frac{x^3}{2}+x^3+x^4+\frac{x^5}{2}$$
$$+\frac{x^2}{2}+\frac{x^3}{2}+\frac{x^4}{4}+\frac{x^4}{2}+\frac{x^5}{2}+\frac{x^6}{4}+\cdots)$$

$$= e(3x^2+2x+O(x^3))$$

5. Find the Taylor series expansion of the functions

$$\ln(1-x^2) \quad \text{and} \quad \frac{1}{1-x^2}$$

around the point 0, using the Taylor series expansions (6.48) and (6.49) of $\ln(1-x)$ and $\frac{1}{1-x}$ around 0.

$$f(x) = \ln(1-x^2)$$

$$\ln(1-x) = x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots$$

$$\ln(1-x^2) = x^2 - \frac{x^4}{2} - \frac{x^6}{3} - \frac{x^8}{4} - \cdots$$

$$\frac{1}{1-x} = 1+x+x^2+x^3+x^4+\cdots$$

$$\frac{1}{1-x^2} = 1+x^2+x^4+x^6+x^8\cdots$$

9. Let

$$T(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\, x^k}{k}$$

be the Taylor series expansion of $f(x) = \ln(1+x)$. In section 6.3.1, we showed that $T(x) = f(x)$ if $|x| \le \frac{1}{2}$. In this exercise, we show that $T(x) = f(x)$ for all x such that $|x| < 1$.

Let $P_n(x)$ be the Taylor polynomial of degree n corresponding to $f(x)$. Since $T(x) = \lim_{n\to\infty} P_n(x)$, it follows that $f(x) = T(x)$ for all $|x| < 1$ if and only if

$$\lim_{n\to\infty} |f(x) - P_n(x)| = 0, \quad \forall\, |x| < 1. \tag{6.128}$$

(i) Use (6.51) and the integral formula (6.4) for the Taylor approximation error to show that, for any x,

$$f(x) - P_n(x) = \int_0^x \frac{(-1)^{n+2}\,(x-t)^n}{(1+t)^{n+1}}\, dt.$$

(ii) Show that, for any $0 \le x < 1$,

$$|f(x) - P_n(x)| \le \int_0^x \left(\frac{x-t}{1+t}\right)^n \frac{1}{1+t}\, dt \le x^n \ln(1+x). \tag{6.129}$$

Use (6.129) to prove that (6.128) holds for all x such that $0 \le x < 1$.

(iii) Assume that $-1 < x \le 0$. Let $s = -x$. Show that

$$|f(x) - P_n(x)| = \int_0^s \frac{(s-z)^n}{(1-z)^{n+1}}\, dz.$$

Note that $\frac{s-z}{1-z} \le s$, for all $0 \le z \le s < 1$, and obtain that

$$|f(x) - P_n(x)| \le s^n |\ln(1-s)| = (-x)^n |\ln(1+x)|.$$

Conclude that (6.128) holds true for all x such that $-1 < x \le 0$.

(i) $f(x) - P_n(x)$ is the remainder term

$$f(x) - P_n(x) = \frac{1}{n!}\int_0^x f^{(n+1)}(t)\,(x-t)^n\, dt$$

$$f(x) = \ln(1+x)$$

$$f'(x) = (1+x)^{-1}$$

$$f''(x) = -1\,(1+x)^{-2}$$

$$\vdots$$

$$f^{(n+1)}(x) = (-1)^{n+2}\,(1+x)^{-(n+1)}$$

$$f(x) - P_n(x) = \frac{1}{n!} \int_0^x \frac{(-1)^{n+2}(x-t)^n}{(1+t)^{n+1}} dt$$

(ii) range of $(-1)^{n+2}$ is -1 or 1

$$|f(x) - P_n(x)| \le \int_0^x \frac{1 \cdot (x-t)^n}{(1+t)^{n+1}} dt$$

$$= \int_0^x \left(\frac{x-t}{1+t}\right)^n \cdot \frac{1}{1+t} dt$$

$$\le \int_0^x (x-t)^n \cdot \frac{1}{1+t} dt$$

$$\frac{d(x^n \ln(1+x))}{dx} = n x^{n-1} \ln(1+x) + x^n \frac{1}{1+x}$$

$$= n t^{n-1} \ln(1+t) + t^n \frac{1}{1+t}$$

$$t^n \ge (x-t)^n$$

$$\Rightarrow n t^{n-1} \ln(1+t) + t^n \frac{1}{1+t} \ge (x-t)^n \cdot \frac{1}{1+t}$$

$$\ge \left(\frac{x-t}{1+t}\right)^n \frac{1}{1+t}$$

Thus, $|f(x) - P_n(x)| \le \int_0^x \left(\frac{x-t}{1+t}\right)^n \frac{1}{1+t}$

$$\le x^n \ln(1+x)$$

$\lim_{n\to\infty} x^n \ln(1+x) = 0$

$x^n \ln(1+x)$ converges

by comparison theorem.

$|f(x) - P_n(x)|$ converges

$\lim_{n\to\infty} |f(x) - P_n(x)| = 0$

(iii) Assume that $-1 < x \le 0$. Let $s = -x$. Show that

$$|f(x) - P_n(x)| = \int_0^s \frac{(s-z)^n}{(1-z)^{n+1}}\, dz.$$

Note that $\frac{s-z}{1-z} \le s$, for all $0 \le z \le s < 1$, and obtain that

$$|f(x) - P_n(x)| \le s^n |\ln(1-s)| = (-x)^n |\ln(1+x)|.$$

Conclude that (6.128) holds true for all x such that $-1 < x \le 0$.

$$|f(x) - P_n(x)| = \left| \int_0^x \frac{(-1)^{n+2}(x-t)^n}{(1+t)^{n+1}}\, dt \right|$$

$$= \left| \int_0^x \frac{(-1)^{n+2}(x+z)^n}{(1-z)^{n+1}}\, dz \right|$$

$0.5^2 \quad 0.5^4$

$0.25 \quad 0.625$

$$= \left| \int_0^x \frac{(x+z)^n}{(1-z)^{n+1}}\, dz \right|$$

$$= \left| \int_0^s \frac{(-s+z)^n}{(1-z)^{n+1}}\, dz \right|$$

$-1 < x \le 0 \Rightarrow 0 < s \le 1 \quad 0 < 1-s \le 1$

$\Rightarrow s > 0 \Rightarrow s > z \quad \ln(0) < \ln(1-s) \le 0$

$$= \int_0^s \frac{(s-z)^n}{(1-z)^{n+1}}\, dz$$

$$\frac{d(s^n |\ln(1-s)|)}{ds} = \frac{d(-s^n \ln(1-s))}{ds}$$

$$= -ns^{n-1}\ln(1-s) + \frac{-s^n}{1-s}$$

$$= -nz^{n-1}\ln(1-z) + \frac{-z^n}{1-z}$$

$$\int_0^s \frac{(s-z)^n}{(1-z)^{n+1}} \le -nz^{n-1}\ln(1-z) + \frac{-z^n}{1-z}$$

$$|f(x) - P_n(x)| \le s^n \ln(1-s)$$

$$= (-x)^n \ln(1-s)$$

$$\lim_{n\to\infty} (-x)^n \ln(1-s) = 0$$

by comparison theorem.

$|f(x) - P_n(x)|$ converges when $-1 < x \le 0$

$$\Rightarrow \lim_{n\to\infty} |f(x) - P_n(x)| = 0$$