MATH 327 HW1

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Exercise 1

1. Assume that $x \geq z$, then $x - z \geq 0$. Since z is a positive number, $z > 0$. Thus, $z(x-z) \geq 0$ and $zx - z^2 \geq 0$. Thus, $zx \geq z^2$. Similarly, since x is a positive number, $x > 0$. Thus, $x(x-z) \geq 0$ and $x^2 - zx \geq 0$. Thus, $x^2 \geq zx$ and $x^2 \geq zx \geq z^2$.

2. Assume $m > 1$ and so $1 \in L$, since $1^2 = 1 < m$. Thus, L is not empty. Since $m > 1$ and $m > 0$, $m \cdot m > 1 \cdot m$ and so $m^2 > m$. If $x \in L$, $x^2 < m < m^2$. From part 1, since $x^2 < m^2$, then $x < m$. Thus, L is bounded above by m.

3. According to Axiom 3, since L is bounded above, then L has least upper bound, c. Thus, c exists.
   Taking $b = \frac{1}{z} > 0$ and $a = 1$. By Archimedean Law, there exists $n \in \mathbb{N}$ such that $n > \frac{1}{z}$. Hence, $nz > 1$ and $z > \frac{1}{n} > 0$. Thus, if $z > 0$, there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < z$.
   Initially, assume $m > 1$. Thus, $1 \in L$ since $1 > 0$ and $1^2 = 1 < m$. Thus, $c > 1$ and $c^2 > c$.
   Suppose $c^2 > m$. For $k \in \mathbb{N}$, $(c - \frac{1}{k})^2 = c^2 - \frac{2c}{k} + \frac{1}{k^2} > c^2 - \frac{2c}{k}$. Since $c^2 > m$, $c^2 - m > 0$. Since $c > 0$, $\frac{c^2 - m}{2c} > 0$. Thus, by the proof above, there exists a $k \in \mathbb{N}$ such that $\frac{c^2-m}{2c} > \frac{1}{k}$. Thus, $(c - \frac{1}{k})^2 > c^2 - \frac{2c}{k} > c^2 - 2c \cdot \frac{c^2-m}{2c} = c^2 - c^2 + m = m$. Suppose $y \in L$ and then $y^2 < m < (c - \frac{1}{k})^2 < c^2$ which implies that $y < c - \frac{1}{k}$ contradicting that c is the elast upper bound of L.
   Suppose $c^2 < m$. For $k \in \mathbb{N}$, $(c + \frac{1}{k})^2 = c^2 + \frac{2c}{k} + \frac{1}{k^2} < c^2 + \frac{2c}{k} + \frac{1}{k} = c^2 + \frac{2c+1}{k}$. Since $c^2 < m$, $m - c^2 > 0$. Since $c > 0$, $2c + 1 > 0$ and $\frac{m-c^2}{2c+1} > 0$. Thus, by the proof above, there exists a $k \in \mathbb{N}$ such that $0 < \frac{1}{k} < \frac{m-c^2}{2c+1}$. Thus, $(c + \frac{1}{k})^2 < c^2 + \frac{2c+1}{k} < c^2 + (2c+1) \cdot \frac{m-c^2}{2c+1} = c^2 + m - c^2 = m$. Thus, we get $c + \frac{1}{k} \in L$ contradicting that c is the least upper bound since $c + \frac{1}{k} > c$.
   Thus, $c^2 = m$. If $m = 1$, then $x = 1$ works and if $0 < m < 1$, then $\frac{1}{m} > 1$, thus there exists $c > 0$ such that $c^2 = \frac{1}{m}$ and $m = \frac{1}{c^2} = (\frac{1}{c})^2$.

Exercise 2

1 Suppose F satisfies the Archimedean Law, we want to prove another two propositions.

Since $1 > 0$, $z > 0$, there exists $n \in \mathbb{N}$ such that $n \cdot 1 > z$. Thus, there exists n in $\mathbb{N}$ such that $n > z$.

Since $z > 0$, $z^{-1} > 0$. Since $1 > 0$ and $z^{-1} > 0$, there exists $n \in \mathbb{N}$ such that $n \cdot 1 > z^{-1}$. Since $n > z^{-1}$ and $z > 0$, $n \cdot z > z^{-1} \cdot z = 1$. Since $n \in \mathbb{N}$, $n^{-1} > 0$. Thus, $n \cdot n^{-1} \cdot z > 1 \cdot n^{-1}$. Since $n \cdot n^{-1} = 1$ and $\frac{1}{n} = n^{-1}$, $1 \cdot z = z > 1 \cdot n^{-1} = \frac{1}{n}$ and so $\frac{1}{n} < z$.

2 Suppose 2 is true, we want to prove that 1 and 3 are true.

Assume $z = \frac{b}{a} > 0$. Thus, there exists a $n \in \mathbb{N}$ such that $n > \frac{b}{a}$ which implies $na > b$. 1 has been proven true and we can use Archimedean Law for proving $2 \to 3$.

Since $z > 0$, $z^{-1} > 0$. Since $1 > 0$ and $z^{-1} > 0$, there exists $n \in \mathbb{N}$ such that $n \cdot 1 > z^{-1}$. Since $n > z^{-1}$ and $z > 0$, $n \cdot z > z^{-1} \cdot z = 1$. Since $n \in \mathbb{N}$, $n^{-1} > 0$. Thus, $n \cdot n^{-1} \cdot z > 1 \cdot n^{-1}$. Since $n \cdot n^{-1} = 1$ and $\frac{1}{n} = n^{-1}$, $1 \cdot z = z > 1 \cdot n^{-1} = \frac{1}{n}$ and so $\frac{1}{n} < z$.

3 Suppose 3 is true, and we want to prove 1 and 2 are true.

Assue $z = \frac{a}{b} > 0$. Thus, there exists a rational number in the form $\frac{1}{n}$ such that $o < \frac{1}{n} < \frac{a}{b}$. Since $\frac{1}{n} < \frac{a}{b}$, then $(\frac{1}{n})^{-1} > (\frac{a}{b})^{-1}$ and $n > \frac{b}{a}$. Thus, $na > b$ which implies that 1 is true. We can use Archimedean Law for proving that 2 is right.

Since $z > 0$ and $1 > 0$, then use the Archimedean Law, there exists $n \in \mathbb{N}$ such that $n \cdot 1 > z$ and thus, $n > z$.

Exercise 3

1. $P(m)$: $\forall n, m$, $a^m a^n = a^{m+n}$.
   Base case: $m = 1$. $a^1 a^n = aa^n = a^{n+1}$.
   Induction process: Assume for a given $m \geq 1$, $P(m)$ is true.
   Goal prove $P(m+1)$ is true.
   Assume $a^n a^m = a^{m+n}$ is true. $a^n a^{m+1} = a^n a^m a = a^{m+n} a$. Since $n, m \in \mathbb{N}$, $m + n \in \mathbb{N}$ and so $a^{m+n} a = a^{m+n+1}$. Thus, $a^n a^{m+1} = a^{m+n+1}$. $P(m+1)$ is true.
   Conclusion: $\forall m, n$, $a^m a^n = a^{m+n}$.

2. $P(n)$: $\forall n$, $a^n b^n = (ab)^n$.
   Base case: $n = 1$. $a^1 b^1 = ab = (ab)^1 = ab$.
   Induction process: Assume for a given $n \geq 1$, $P(n)$ is true.
   Goal prove $P(n+1)$ is true.
   Assume $a^n b^n = (ab)^n$ is true. $a^{n+1} b^{n+1} = a^n a b^n b = (ab)^n ab = (ab)^{n+1}$. $P(n+1)$ is true.
   Conclusion: $\forall n$, $a^n b^n = (ab)^n$.