MATH 327 HW3

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Exercise 1 $\lim_{n\to\infty} \frac{1+n}{1-2n} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{\frac{1}{n}-2}$ and $\lim_{n\to\infty} 1+\frac{1}{n} = \lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n} = 1+0 = 1$

$\lim_{n\to\infty} \frac{1}{n} - 2 = \lim_{n\to\infty} \frac{1}{n} - \lim_{n\to\infty} 2 = 0 - 2 = -2.$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} a_n \cdot \frac{1}{b_n} = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} \frac{1}{b_n}$ and we want to prove that $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}$.

$|b| = |b + b_n - b_n| \leq |b_n - b| + |b_n| < \frac{|b|}{2} + |b_n|$ and thus, $|b_n| > \frac{|b|}{2}$.

Since $\lim_{n\to\infty} b_n = b$, $\forall \epsilon > 0$, there exists N if $n > N$, $|b_n - b| < \frac{\epsilon |b|^2}{2}$.

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b-b_n}{b_n b}\right| = \frac{|b_n - b|}{|b_n|\,|b|} = |b_n - b| \cdot \frac{1}{|b_n|\,|b|} < \frac{\epsilon\,|b|^2}{2} \cdot \frac{1}{\frac{|b|}{2}\,|b|} = \epsilon.$$

Thus, $\lim_{n\to\infty} \frac{1+n}{1-2n} = \lim_{n\to\infty} (1+\frac{1}{n}) \cdot \lim_{n\to\infty} (\frac{1}{\frac{1}{n}-2}) = 1 \cdot \frac{1}{-2} = \frac{-1}{2}.$

Exercise 2 Goal: $\forall A$, there exists a N, if $n > N$, $\frac{n^2+1}{2n-1} > A$.

$$\frac{n^2+1}{2n-1} > \frac{n^2}{2n} = \frac{n}{2} > A$$

$\forall A$, there exists $N = 2A$, if $n > N$, $\frac{n^2+1}{2n-1} > \frac{n}{2} > A$ and thus, $\lim_{n\to\infty} \frac{n^2+1}{2n-1} = \infty$.

Exercise 3 Since a_n converges to L, $\forall \epsilon > 0$, there exists a $n \in \mathbb{N}$, $|a_n - L| < \epsilon$, thus $L - \epsilon < a_n < L + \epsilon$.

Take $\epsilon = \frac{L}{2} > 0$, there exists a $n \in \mathbb{N}$, $a_n > L - \epsilon = L - \frac{L}{2} = \frac{L}{2} > 0$.

Exercise 4 Since u_n is a convergent sequence, $\lim_{n\to\infty} u_n = u$ and $\forall \epsilon > 0$, there exists a N, if $n > N$, $|u_n - u| < \epsilon$.

Thus, $-\epsilon < u_n - u < \epsilon$ and $u - \epsilon < u_n < u + \epsilon$.

If $n < N$, then define $S = \{u_1, u_2, \dots, u_{N-1}\}$ is a finite set and a maximum value u_m exists such that $u_m > u_i$ $(i = 1, \dots, N-1)$.

Define $M = max\{u + \epsilon, u_m\}$; thus, $\forall n$, $u_n < M$ that u_n are bounded.

$|u_n| = |u_n - u + u| < |u_n - u| + |u|$

$|u_n| - |u| < |u_n - u| < \epsilon$

$|u_n| < |u| + \epsilon$

If $n < N$, then define $S = \{|u_1|, |u_2|, \dots, |u_{N-1}|\}$ is a finite set and a maximum value $|u_m|$ exists such that $|u_m| > |u_i|$ $(i = 1, \dots, N-1)$.

Define $M = max\{|u| + \epsilon, |u_m|\}$; thus, $\forall n$, $|u_n| < M$ that $|u_n|$ are bounded.

Exercise 5 Goal: $\lim\limits_{n\to\infty} |a_n| = |a|$ and we want to prove $\forall\epsilon > 0$, there exists a N, if $n > N$, $||a_n| - |a|| < \epsilon$.

Since a_n is a convergent sequence, $\lim_{n\to\infty} a_n = a$ and $\forall\epsilon > 0$, there exists a N, if $n > N$, $|a_n - a| < \epsilon$.

By the proposition, $||a_n| - |a|| < |a_n - a|$.

Thus, $||a_n| - |a|| < |a_n - a| < \epsilon$ and $\forall\epsilon > 0$, there exists a N, if $n > N$, $||a_n| - |a|| < \epsilon$, $\lim\limits_{n\to\infty} |a_n| = |a|$.

Exercise 6 1. Since $\lim\limits_{n\to\infty} b_n = B$, $\forall\epsilon > 0$, there exists a N, if $n > N$, $|b_n - B| < \epsilon$.

$\forall\epsilon < \dfrac{B}{2}$, there exists a N_1, if $n > N_1$, $|b_n - B| < \epsilon < \dfrac{B}{2}$.

$|b_n - B| < \dfrac{B}{2}$, then $B - \dfrac{B}{2} < b_n < B + \dfrac{B}{2}$ and $b_n > \dfrac{B}{2}$.

2. $|B| = |B + b_n - b_n| \le |b_n - B| + |b_n| < \frac{|B|}{2} + |b_n|$ and thus, $|b_n| > \frac{|B|}{2}$.

Since $\lim\limits_{n\to\infty} b_n = B$, $\forall\epsilon > 0$, there exists N if $n > N$, $|b_n - B| < \frac{\epsilon|B|^2}{2}$.

$\left|\dfrac{1}{b_n} - \dfrac{1}{B}\right| = \left|\dfrac{B - b_n}{b_n B}\right| = \dfrac{|b_n - B|}{|b_n|\,|B|} = |b_n - B| \cdot \dfrac{1}{|b_n|\,|B|} < \dfrac{\epsilon\,|B|^2}{2} \cdot \dfrac{1}{\frac{|B|}{2}\,|B|} = \epsilon$.

Exercise 7 goal: $\forall A$, there exists $n \in \mathbb{N}$, if $n > N$, $a_n + b_n > A$.

Since $\lim\limits_{n\to\infty} a_n = L$, $\forall\epsilon > 0$, there exists a $n \in \mathbb{N}$, if $n > N_2$, $|a_n - L| < \epsilon$, thus, $L - \epsilon < a_n < L + \epsilon$.

Since $\lim\limits_{n\to\infty} b_n = \infty$, $\forall A_1$, there exists a $n \in \mathbb{N}$, if $n > N_1$, $b_n > A_1$.

Since $a_n > L - \epsilon$, for any b_n, $a_n + b_n > L - \epsilon + b_n$, since $b_n > A_1$, $a_n + b_n > L - \epsilon + A_1$.

$\forall A < L - \epsilon + A_1$, there exists $N = max\{N_1, N_2\}$, if $n > N$, $a_n + b_n > L - \epsilon + A_1 > A$.