MATH 327 HW1

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Exercise 1

1. $a + a = a$, $b + a = b$, $c + a = c$, $d + a = d$. Thus, a is the additive identity of F.
   $A + A = A$, $B + A = B$, $C + A = C$. Thus, A is the additive identity of G.
2. $a \cdot b = a$, $b \cdot b = b$, $c \cdot b = c$, $d \cdot b = d$. Thus, b is the multiplicative identity of F.
   $A \cdot B = A$, $B \cdot B = B$, $C \cdot B = C$. Thus, B is the multiplicative identity of G.
3. additive inverse

a	a	$a+? = a$
b	d	$b+? = a$
c	c	$c+? = a$
d	b	$d+? = a$

additive inverse

A	A	$A+? = A$
B	C	$B+? = A$
C	B	$C+? = A$

4. multiplicative inverse

b	b	$b \cdot ? = b$
c		$c \cdot ? = b$
d	d	$d \cdot ? = b$

There is no multiplicative inverse for c.

multiplicative inverse

B	B	$B \cdot ? = B$
C	C	$C \cdot ? = B$

5. G is the commutative field, since it satisfies seven properties of commutative field: associative, distributive, commutative and it has additive identities, multiplicative identities, additive inverse and multiplicative inverse of every element.
   F is not the commutative fields, since even though it is associative, distributive, commutative and it has additive identities, multiplicative identities and additive inverse of elements, it does not satisfy that every element has multiplicative element.

Exercise 2

1. Assume 1_1 and 1_2 are 2 different multiplicative identities. According to the commutative property, $1_1 \cdot 1_2 = 1_2 \cdot 1_1$. Since 1_1 is the multiplicative identity, $1_2 \cdot 1_1 = 1_2$ and since 1_2 is the multiplicative identity, $1_1 \cdot 1_2 = 1_1$. Thus, $1_1 = 1_2$ and the multiplicative identity is unique.

2. According to the multiplicative inverse property, $ab \cdot (ab)^{-1} = 1_F$. Thus, $a \cdot b \cdot (ab)^{-1} = 1$ and $a^{-1} \cdot b^{-1} \cdot a \cdot b \cdot (ab)^{-1} = 1_F \cdot a^{-1} \cdot b^{-1} = a^{-1} \cdot b^{-1}$. According to the commutative property, $a^{-1} \cdot b^{-1} \cdot a \cdot b \cdot (ab)^{-1} = a \cdot a^{-1} \cdot b \cdot b^{-1} \cdot (ab)^{-1}$. According to the multiplicative inverse property, $a \cdot a^{-1} \cdot b \cdot b^{-1} \cdot (ab)^{-1} = 1_F \cdot 1_F \cdot (ab)^{-1} = (ab)^{-1}$.
Thus, $(ab)^{-1} = a^{-1} \cdot b^{-1}$.

3. According to the multiplicative inverse property, $((b)^{-1}) \cdot ((b)^{-1})^{-1} = 1_F$. Thus, $b \cdot (b)^{-1} \cdot ((b)^{-1})^{-1} = b \cdot 1_F$. According to the multiplicative property, $b \cdot b^{-1} = 1_F$. Thus, $b \cdot (b)^{-1} \cdot ((b)^{-1})^{-1} = 1_F \cdot ((b)^{-1})^{-1} = ((b)^{-1})^{-1}$. Thus, $b \cdot 1_F = ((b)^{-1})^{-1} = b$.

Exercise 3

1. If $a < b$, $b - a > 0$. Thus, $b - a + (c - c) > 0$ and according to the commutative property, $b - a + (c - c) = b + c - a - c = b + c + (-1) \cdot a + (-1) \cdot c$. According to the distributive property, $b + c + (-1) \cdot a + (-1) \cdot c = (b + c) + (-1) \cdot (a + c) > 0$. Thus, $b + c > a + c$.

2. Since $a > 0$, $a^{-1} > 0$ and since $b > 0$, $b^{-1} > 0$. Thus, $a^{-1} \cdot b^{-1} > 0$. Since $a < b$ and $a^{-1} \cdot b^{-1} > 0$, $a \cdot a^{-1} \cdot b^{-1} < b \cdot a^{-1} \cdot b^{-1} = b \cdot b^{-1} \cdot a^{-1}$. According to the multiplicative inverse property, $a \cdot a^{-1} \cdot b^{-1} = 1_F \cdot b^{-1} = b^{-1}$ and $b \cdot b^{-1} \cdot a^{-1} = 1_F \cdot a^{-1} = a^{-1}$. Therefore, $b^{-1} < a^{-1}$.

3. Since $a < b$, $b - a > 0$. Since $c < 0$, $-c > 0$. Since $b - a > 0$ and $-c > 0$, $(-c) \cdot (b - a) > 0$. According to the distributive property, $-c \cdot (b - a) = -b \cdot c + (-c) \cdot (-a) = -b \cdot c + (-1) \cdot a \cdot (-1) \cdot c - b \cdot c + (ca) > 0$. Thus, $ac > b \cdot c$.