MATH 327 HW #4

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Exercise 1

(1)

I use partial fraction decomposition of $\frac{1}{k(k+1)}$.

$$\begin{aligned}
S_n &= \sum_{k=1}^{n} \frac{1}{k(k+1)} \\
&= \sum_{k=1}^{n} \frac{k+1-k}{k(k+1)} \\
&= \sum_{k=1}^{n} (\frac{1}{k} - \frac{1}{k+1}) \\
&= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} \cdots - \frac{1}{n-1+1} + \frac{1}{n} - \frac{1}{n+1} \\
&= 1 - \frac{1}{n+1}
\end{aligned}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{n+1} = 1 - 0 = 1.$$

Thus, S_n is convergent and S_n is a Cauchy sequence.

(2)

Take the partial sum $\sum_{k=1}^{n} \frac{1}{k^2}$ of the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

$$\sum_{k=1}^{n} \frac{1}{k^2} = \frac{1}{1} + \frac{1}{2 \cdot 2} + \frac{1}{3 \cdot 3} + \cdots + \frac{1}{n^2} < 1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n-1)}$$

$$1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n-1)} = 1 + 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n-1} - \frac{1}{n} = 2 - \frac{1}{n}$$

Thus, $\sum_{k=1}^{n} \frac{1}{k^2} < 2 - \frac{1}{n} < 2$ and $\sum_{k=1}^{n} \frac{1}{k^2}$ is bounded by 0 and 2. Since $\forall k$, $\frac{1}{k^2} > 0$ and $\sum_{k=1}^{n} \frac{1}{k^2}$ is bounded, $\sum_{k=1}^{n} \frac{1}{k^2}$ is convergent. Thus, $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent.

(3)

Goal: We want to prove $\sum_{k=1}^{n} \frac{1}{k^p}$ is bounded, and then it is convergent. Take partial sum $S_{2n+1} = \sum_{k=1}^{2n+1} \frac{1}{k^p}$

$$\begin{aligned}
S_{2n+1} &= 1 + \sum_{k=1}^{2n+1} \frac{1}{k^p} = 1 + (\frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{(2n)^p} + \frac{1}{(2n+1)^p}) \\
&= 1 + \sum_{i=1}^{n} (\frac{1}{(2i)^p} + \frac{1}{(2i+1)^p}) \\
&< 1 + \sum_{i=1}^{n} (\frac{1}{(2i)^p} + \frac{1}{(2i)^p}) = 1 + \sum_{i=1}^{n} \frac{2}{(2i)^p} \\
&= 1 + 2^{1-p} \sum_{i=1}^{n} \frac{1}{i^p} = 1 + 2^{1-p} S_n \\
&< 1 + 2^{1-p} S_{2n+1}
\end{aligned}$$

Thus, $S_{2n+1} < \frac{1}{1 - 2^{1-p}}$. Since $\frac{1}{k^p} > 0$ and $\sum_{k=1}^{2n+1} \frac{1}{k^p}$ is bounded above, $\sum_{k=1}^{2n+1} \frac{1}{k^p}$ is convergent. Thus, $\sum_{k=1}^{\infty} \frac{1}{k^p}$ is convergent.

Exercise 2

(1)

$\sum_1^\infty a^n$ can be $\frac{(-1)^{n+1}}{\sqrt{n}}$ and b_n can be $(-1)^n$. b_n is bounded by -1 and 1.

$\sum_1^\infty b_n a_n = \sum_1^\infty (-1)^n \frac{(-1)^{n+1}}{n} = (-1)^{2n+1} \sum_1^\infty \frac{1}{n}$. $(-1)^{2n+1} = (-1)^1 \cdot (-1)^{2n} = -1$, $\forall n$.

Since $\sum_1^\infty \frac{(-1)^{n+1}}{\sqrt{n}}$ is an alternating series, let $a_n = \frac{1}{\sqrt{n}}$ and $a_{n+1} = \frac{1}{\sqrt{n+1}}$.

Since $\sqrt{n+1} > \sqrt{n}$, $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$ and thus, $a_{n+1} < a_n$.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, thus $\sum_1^\infty a_n$ is convergent.

Thus, $(-1)^{2n+1} \sum_1^\infty \frac{1}{n} = \sum_1^\infty \frac{-1}{n}$ is divergent.

(2)

Take the partial sum $\sum_{n=1}^k |a_n b_n|$ and we want to show it is bounded above. Assume B be the upper bound of $|b_n|$, thus $\forall n, |b_n| \leq B$ and $\sum_1^k |a_n b_n| \leq B \sum_1^k |a_n|$. Assume $\sum a_n$ is convergent to A. Since $a_n > 0$ for any n and $\sum a_n$ is convergent to A, $\sum_1^k |a_n b_n| \leq AB$. Since $|a_n b_n| \leq 0$ for every n and $\sum_1^k |a_n b_n|$ is bounded by AB, $\sum_1^k |a_n b_n|$ is convergent. Thus, $\sum_1^n a_n b_n$ is absolutely convergent, and thus, convergent. Therefore, $\sum_1^\infty a_n b_n$ is convergent.

Exercise 3

(1)

Let $b_n = n$ and $a_n = \frac{n^3 + 2n - 7}{5n^2 - 2}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{n^3 + 2n - 7}{5n^2 - 2} \cdot \frac{1}{n} = \frac{n^3 + 2n - 7}{5n^3 - 2n} = \frac{1 + \frac{2}{n^2} - \frac{7}{n^3}}{5 - \frac{2}{n^2}} = \frac{1}{5}$$

Since $\sum b_n$ is divergent and $\sum a_n$ and $\sum b_n$ are both convergent or divergent, $\sum a_n$ is divergent.

(2)

Since $n \geq 1$, $n^4 \geq n^3 \geq n^2 \geq n \geq 1$ and $2n^4 > 2n^2 > 2n$.

$$\frac{7n^2 - 2n + 4}{n^4 + 2n + 1} \leq \frac{7n^2 + 4}{n^4} = \frac{7}{n^2} + \frac{4}{n^4}$$

By p-test, $\sum_{n=1}^{\infty} \frac{7}{n^2}$ is convergent and $\sum_{n=1}^{\infty} \frac{4}{n^4}$ is convergent. Thus, $\sum(\frac{7}{n^2} + \frac{4}{n^4})$ is convergent and then, $\sum \frac{7n^2 - 2n + 4}{n^4 + 2n + 1}$ is convergent.

Since $n \geq 1$, $\frac{7}{n^2} > 0$ and $|\frac{7}{n^2}| = \frac{7}{n^2}, |\frac{4}{n^4}| = \frac{4}{n^4}$.
Thus, $|\frac{7n^2 - 2n + 4}{n^4 + 2n + 1}| \leq |\frac{5}{4n^2}| + |\frac{4}{n^4}| = \frac{7}{n^2} + \frac{4}{n^4} > 0$. Thus, $|\frac{7n^2 - 2n + 4}{n^4 + 2n + 1}|$ is convergent.

Thus, the series is convergent and absolutely convergent.

(3)

The range of $cos(3n)$ is $[-1, 1]$ and since $n \geq 1$, $n \geq > cos(3n)$.

$$\frac{1}{n} \leq \frac{4n}{4n^2} = \frac{5n - n}{4n^2} \leq \frac{5n - cos(3n)}{4n^2} \leq \frac{5n - cos(3n)}{4n^2 - 6n}$$

Since $\sum \frac{1}{n}$ is divergent, $\sum_{1}^{\infty} \frac{5n - cos(3n)}{4n^2 - 6n}$ is divergent.

(4)

Let $a_n = \frac{(-1)^n(n+3)}{n+1}$ and thus, $|a_n| = |\frac{n+3}{n+1}|$.

$|\frac{n+3}{n+1}| \leq |\frac{n+3n}{n+n}| = |\frac{4n}{2n}| = |2|$. Let $\sum_{1}^{\infty} b_n = \sum_{1}^{\infty} |2|$ which is convergent. Since $0 \leq |a_n| \leq |b_n|$, $\sum_{1}^{\infty} |a_n|$ is convergent, thus $\sum_{1}^{\infty} a_n$ is absolutely convergent and convergent.

(5)

Let $a_n = \frac{(-1)^n n}{n^2+1}$ and thus, $|a_n| = |\frac{n}{n^2+1}|$. $|\frac{n}{n^2+1}| \geq |\frac{n}{n^2+n^2}| = |\frac{n}{2n^2}| = \frac{1}{2n}$, $\forall n \geq 1$. Since $|\frac{n}{2n^2}| = \frac{1}{2n}$ is divergent, $\sum_{1}^{\infty} a_n$ is divergent.