MATH 327 HW 1

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1 Sheet 1

1. Let A and Λ be two sets. Using Λ as an indexing set, for each $\lambda \in \Lambda$ let E_λ be as subset of A. Let $E \subset A$ also. Show that:

   (i) $E \cap \left(\bigcup_{\lambda \in \Lambda} E_\lambda \right) = \bigcup_{\lambda \in \Lambda} (E \cap E_\lambda)$

   Let $x \in E \cap \left(\bigcup_{\lambda \in \Lambda} E_\lambda \right)$. Then $x \in E$ and $x \in \bigcup_{\lambda \in \Lambda} E_\lambda$. Thus, $x \in E_\lambda$, for some $\lambda \in \Lambda$. Thus, $x \in (E \cap E_\lambda)$ for some $\lambda \in \Lambda$ and $x \in \bigcup_{\lambda \in \Lambda} (E \cap E_\lambda)$. Thus, $E \cap \left(\bigcup_{\lambda \in \Lambda} E_\lambda \right) = \bigcup_{\lambda \in \Lambda} (E \cap E_\lambda)$.

   (iii) $\left(\bigcup_{\lambda \in \Lambda} E_\lambda \right)^c = \bigcap_{\lambda \in \Lambda} {E_\lambda}^c$

   Let $x \in \left(\bigcup_{\lambda \in \Lambda} E_\lambda \right)^c$, so $x \notin \bigcup_{\lambda \in \Lambda} E_\lambda$. Thus for each $\lambda \in \Lambda$, $x \notin E_\lambda$. Thus, for each $\lambda \in \Lambda$, $x \in E_\lambda^c$. Thus, $x \in \bigcap_{\lambda \in \Lambda} {E_\lambda}^c$ and $\left(\bigcup_{\lambda \in \Lambda} E_\lambda \right)^c = \bigcap_{\lambda \in \Lambda} {E_\lambda}^c$.

3. Let $f : A \to B$ and $g : B \to C$ be functions.

   (a) Prove that if f and g are injective then so is $g \circ f$.

   Assume $a, a' \in A$, $c \in C$ and $b, b' \in B$. If f is injective, then if $f(a) = f(a') = b$, so $a = a'$. If g is injective, then if $g(b) = g(b') = c$, so $b = b'$. Then since $f(a) = f(a') = b$ and $g(b) = c$, $g(f(a) = g(f(a') = g \circ f(a) = g \circ f(a') = c$. Since $a = a'$, $g \circ f$ is injective.

   (b) Suppose $g \circ f$ is injective. Does it follow that f is injective?

   Let $x, y \in A$ such that $f(x) = f(y)$. Since $g \circ f$ is injective, then if $g(f(x)) = g(f(y))$ and $(g \circ f)(x) = (g \circ f)(y)$, then $x = y$. Thus, f is injective.

   (c) Suppose $g \circ f$ is injective. Does it follow that g is injective?

   g is not necessarily injective. The counterexample map could be $f : \mathbb{N} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{N}$ given by $f(n) = 2n$ and $g(m) = |m|$. Then since $n \in \mathbb{N}$, $g \circ f(n) = |2n| = 2n$. This map is injective, but g is not injective since $g(-1) = 1 = g(1)$.

4. Let $f : A \to B$ and $g : B \to C$ be functions.

(a) Prove that if f and g are surjective then so is $g \circ f$.

Let $c \in C$. Since g is surjective, there exists $b \in B$ such that $g(b) = c$. Since f is surjective, there exists $a \in A$ such that $f(a) = b$. Then $g \circ f(a) = g(f(a)) = g(b) = c$. Thus for every $c \in C$, there exists $a \in A$ such that $(g \circ f)(a) = c$ and $g \circ f$ is surjective.

(b) Suppose $g \circ f$ is surjective. Does it follow that f is surjective?

f is not necessarily surjective. The counterexample map could be $f : \mathbb{N} \to \mathbb{N}$ given that $f(x) = 2x$ and $g : \mathbb{N} \to \{0\}$ given that $g(y) = 0$. $g \circ f$ is surjective, but f is not surjective since there is no number in $\mathbb{N}$ can be mapped to 3 by $f(x) = 2x$.

(c) Suppose $g \circ f$ is surjective. Does it follow that g is surjective?

Let $c \in C$. Since $g \circ f$ is surjective, there exists $a \in A$ such that $(g \circ f)(a) = g(f(a)) = c$. Assume $f(a) = b$ and $b \in B$. Then $g(b) = c$. Thus g is surjective.

5. Let $f : A \to B$. Recall that, for a set $F \subset B$, the set $f^{-1}(F) \subset A$ is defined by $f^{-1}(F) = \{a \in A : f(a) \in F\}$ and called the inverse image of F.

(i) Let $F \subset B$. Show that $f^{-1}(F^c) = f^{-1}(F)^c$.

$f^{-1}(F^c) = \{x \in A | f(x) \in F^c\} = \{x \in A | f(x) \in B \setminus F\}$

$f^{-1}(F)^c = A \setminus \{x \in A | f(x) \in F\} = \{x \in A | f(x) \in B \setminus F\}$

(ii) Let Λ be a set, and for each $\lambda \in \Lambda$ let F_λ be a subset of B. Check that

$$f^{-1}\Big(\bigcup_{\lambda \in \Lambda} F_\lambda\Big) = \bigcup_{\lambda \in \Lambda} f^{-1}(F_\lambda) \text{ and}$$
$$f^{-1}\Big(\bigcap_{\lambda \in \Lambda} F_\lambda\Big) = \bigcap_{\lambda \in \Lambda} f^{-1}(F_\lambda).$$

a. Let $x \in f^{-1}\Big(\bigcup_{\lambda \in \Lambda} F_\lambda\Big)$ so there exist a $y \in \bigcup_{\lambda \in \Lambda} F_\lambda$ such that $f(x) = y$. Thus, $x \in f^{-1}F_\lambda$ and $y \in F_\lambda$ for some λ. So $x \in \bigcup_{\lambda \in \Lambda} f^{-1}F_\lambda$ and $f^{-1}\Big(\bigcup_{\lambda \in \Lambda} F_\lambda\Big) \subset \bigcup_{\lambda \in \Lambda} f^{-1}F_\lambda$.

Similarly, let $x \in \bigcup_{\lambda \in \Lambda} f^{-1}(F_\lambda)$ and $x \in f^{-1}(F_\lambda)$ for some λ. So $f(x) \in F_\lambda$ and $f(x) \in \bigcup_{\lambda \in \Lambda} \Big(F_\lambda\Big)$. Thus, $x \in f^{-1}\Big(\bigcup_{\lambda \in \Lambda} F_\lambda\Big)$ and $\bigcup_{\lambda \in \Lambda} f^{-1}(F_\lambda) \subset f^{-1}\Big(\bigcup_{\lambda \in \Lambda} F_\lambda\Big)$. Thus, $f^{-1}\Big(\bigcup_{\lambda \in \Lambda} F_\lambda\Big) = \bigcup_{\lambda \in \Lambda} f^{-1}(F_\lambda)$.

b. Let $x \in f^{-1}\Big(\bigcap_{\lambda \in \Lambda} F_\lambda\Big)$ so there exists a $y \in \bigcap_{\lambda \in \Lambda} F_\lambda$ such that $f(x) = y$. Thus, $x \in f^{-1}F_\lambda$ and $y \in F_\lambda$ for every $\lambda \in \Lambda$. So $x \in \bigcap_{\lambda \in \Lambda} f^{-1}(F_\lambda)$ and $f^{-1}\Big(\bigcap_{\lambda \in \Lambda} F_\lambda\Big) \subset \bigcap_{\lambda \in \Lambda} f^{-1}(F_\lambda)$.

Similarly, let $x \in \bigcap_{\lambda \in \Lambda} f^{-1}(F_\lambda)$ and $x \in f^{-1}F_\lambda$ for every $\lambda \in \Lambda$. Thus $f(x) \in F_\lambda$ and $f(x) \in \bigcap_{\lambda \in \Lambda} F_\lambda$. Thus, $x \in f^{-1} \bigcap_{\lambda \in \Lambda} \Big(F_\lambda\Big)$. So, $\bigcap_{\lambda \in \Lambda} f^{-1}(F_\lambda) \subset f^{-1} \bigcap_{\lambda \in \Lambda} \Big(F_\lambda\Big)$.

Thus, $f^{-1}\Big(\bigcap_{\lambda\in\Lambda} F_\lambda\Big) = \bigcap_{\lambda\in\Lambda} f^{-1}(F_\lambda)$.

6. Let $f : A \to B$. Recall that, for a set $E \subset A$, the set $f(E) \subset B$ is defined by $f(E) = \{f(a) : a \in E\}$ and called the image of E.

   (i) Let $E \subset A$. Check that $E \subset f^{-1}(f(E))$. Give an example to show that this inclusion may be strict. What happens when f is injective?
   Let $a \in E$ and then by definition, $f(a) \in f(A)$ $a \in f^{-1}(f(E))$. Thus, $E \subset f^{-1}(f(E))$. The example could be $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$; specifically, $f(1) = f(-1) = 1$ and $-1 \neq 1$, so $\{1, -1\} \subset f^{-1}(f(\{1\}))$. Thus, $f^{-1}(f(\{1\})) \not\subset \{1\}$ and $E \subset f^{-1}(f(E))$. Assume f is injective and let $x \in f^{-1}(f(E))$. Thus, by definition, $f(x) \in f(E)$. Thus, there exists a $y \in E$ such that $f(x) = f(y)$. Since f is injective, $x = y$. Thus, $E = f^{-1}(f(E))$.

   (ii) Let Λ be a set, and for each $\lambda \in \Lambda$ let E_λ be a subset of A. Check that

$$f\Big(\bigcup_{\lambda\in\Lambda} E_\lambda\Big) = \bigcup_{\lambda\in\Lambda} f(E_\lambda) \text{ and}$$
$$f\Big(\bigcap_{\lambda\in\Lambda} E_\lambda\Big) \subset \bigcap_{\lambda\in\Lambda} f(E_\lambda)$$

   Give an example in which the last inclusion is proper. What happens when f is injective?

   a. Let $y \in f\Big(\bigcup_{\lambda\in\Lambda} E_\lambda\Big)$ so there exists $x \in \bigcup_{\lambda\in\Lambda} E_\lambda$ such that $f(x) = y$. Thus, $y \in f(E_\lambda)$ and $x \in E_\lambda$ for some $\lambda \in \Lambda$. Thus, $y \in \bigcup_{\lambda\in\Lambda} f(E_\lambda)$ and $f\Big(\bigcup_{\lambda\in\Lambda} E_\lambda\Big) \subset \bigcup_{\lambda\in\Lambda} f(E_\lambda)$.
   Similarly, let $y \in \bigcup_{\lambda\in\Lambda} f(E_\lambda)$ and so $y \in f(E_\lambda)$ for some $\lambda \in \Lambda$. Thus, there exists $x \in E_\lambda$ and then $x \in \bigcup_{\lambda\in\Lambda} E_\lambda$. Thus, $y = f(x) \in \bigcup_{\lambda\in\Lambda} f(E_\lambda)$ and $\bigcup_{\lambda\in\Lambda} f(E_\lambda) \subset f\Big(\bigcup_{\lambda\in\Lambda} E_\lambda\Big)$.
   Thus, $f\Big(\bigcup_{\lambda\in\Lambda} E_\lambda\Big) = \bigcup_{\lambda\in\Lambda} f(E_\lambda)$.
   b. Let $y \in f\Big(\bigcap_{\lambda\in\Lambda} E_\lambda\Big)$ so there exists a $x \in \bigcap_{\lambda\in\Lambda} E_\lambda$ such that $f(x) = y$. Thus $y \in f(E_\lambda)$ and $x \in E_\lambda$ for every λ. Thus, $y \in \bigcap_{\lambda\in\Lambda} f(E_\lambda)$ and $f\Big(\bigcap_{\lambda\in\Lambda} E_\lambda\Big) \subset \bigcap_{\lambda\in\Lambda} f(E_\lambda)$. The example could be $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. If f is injective, $f\Big(\bigcap_{\lambda\in\Lambda} E_\lambda\Big) = \bigcap_{\lambda\in\Lambda} f(E_\lambda)$.

2 Sheet 2

1. Let F be an ordered field, and let x and y be positive elements of F.

   (a) Use induction to prove that if $x < y$ then $x^n < y^n$ for all $n \in \mathbb{N}$.
   Base case: n = 1. So $x^1 = x$ and $y^1 = y$ and since $x < y$, $x^n < y^n$.

Assume true for $k > 1$, i.e. $x^k < y^k$. We want to show that $x^{k+1} < y^{k+1}$. Since $x \in \mathbb{N}$, $x \times x^k < x \times y^k$ and $x^{k+1} < xy^k$. Since $x < y$ and $y^k \in \mathbb{N}$, $x \times y^k < y \times y^k = y^{k+1}$. Then, since $x^{k+1} < xy^k$ and $xy^k < y^{k+1}$, $x^{k+1} < y^{k+1}$. Thus, if $x < y$ then $x^n < y^n$ for all $x \in \mathbb{N}$.

(b) Deduce that if $x^n < y^n$ for some $n \in \mathbb{N}$ then $x < y$.

Using contrapositive: Assume $x \geq y$. Based on (a), if $x > y$, we know $x^n > y^n$. If $x = y, n \in \mathbb{N}$, $x^n = y^n$. Thus, $x^n \geq y^n$. Thus, if $x^n < y^n$ for some $n \in \mathbb{N}$, then $x < y$.