Math 381 HW6: Buffon's Needle Problem and its Variation

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Due 5/22/2020

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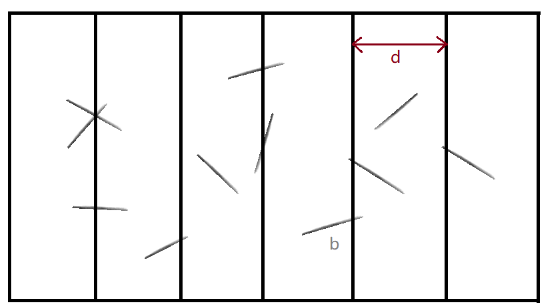
1 Introduction

Buffon's Needle problem was first posted by an French mathematician *Georges-Louis Leclerc, Comte de Buffon* in 18th century. Buffon's Needle problem discusses the problem where we draw parallel lines on a plan, and the distance between any two adjacent lines is the same. Then, we drop a needle onto the plan, and want to know the probability that the needle will cross a line.

In this week's writing, we will first examine the Buffon's Needle problem described above. Then, we will apply the result to a new situation where we drop triangles instead of needles.

2 Drop Needles Onto Parallel Lines Separated Plan

Suppose we have a surface lined with equally-spaced parallel lines d units apart. We throw the needle of length b onto the surface. We want to know what is the probability that it will cross a line upon landing. Throwing the needle many times may result in them landing like this:



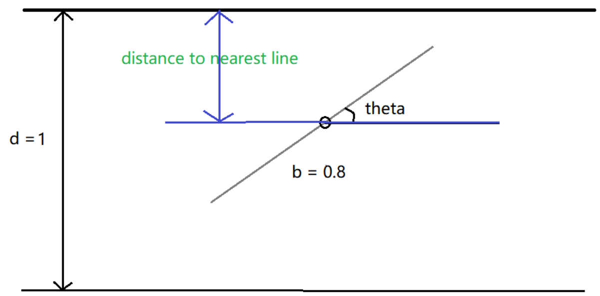
Below is a diagram after it is dropped. We set the distance between each two parallel lines is 1 and the length of the needle is 0.8. dis is the distance from the midpoint of the needle to the nearest line. The range of dis is from 0 to $\frac{1}{2}$, since the distance cannot be more than $\frac{1}{2}$ to the nearest parallel line. Another variable is theta, describing the angle between the needle and a line which is parallel to the lines. The range of theta is from 0 to π (If the angle is greater than $\frac{\pi}{2}$, the diagram will be symmetric to the below one). The needle will cross the line if the distance dis is less than or equal to $\frac{0.8 \cdot 2}{sin(\theta)}$.

What we need to do in simulation is to generate two ends of the needle: x and y. Next, we get the angle θ by calculating the quotient of $\frac{y}{x}$. Next, we generate the distance from the midpoint to the nearest line. Then, we can use the condition: $dis \leq \frac{0.8 \cdot 2}{sin(\theta)}$ to count for the needles crossing the lines.

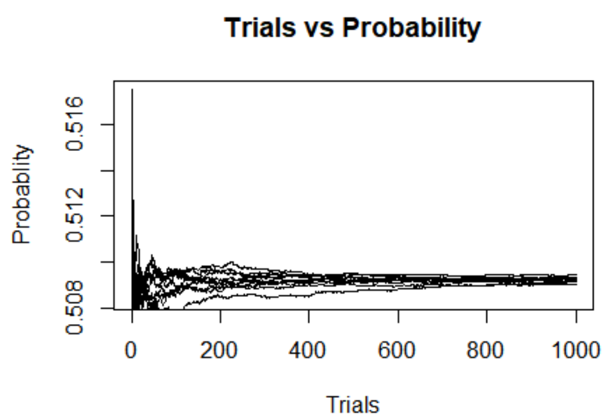
By Buffon's Needle theorem, we actually know that the probability of a needle crossing the line is

$$P = \frac{2 \cdot b}{\pi \cdot d} = \frac{2 \cdot 0.8}{\pi \cdot 1} \approx 0.5092.$$

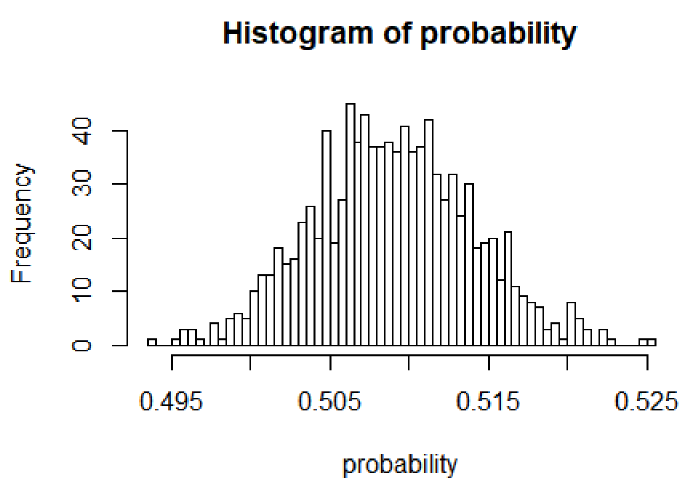
We will simulate this problem and check if the probability is around 0.509.



We simulate this problem in R and we run 1000 runs of 10000 needle throws. Here is what happens with ten runs of the simulation:



We notice that the probability converges to around 0.51 after one thousand runs. We see that it matches the answer get from the theorem; the error is really small. We also record the estimated probability from each run (i.e., the average of those runs), yielded the following histograms:

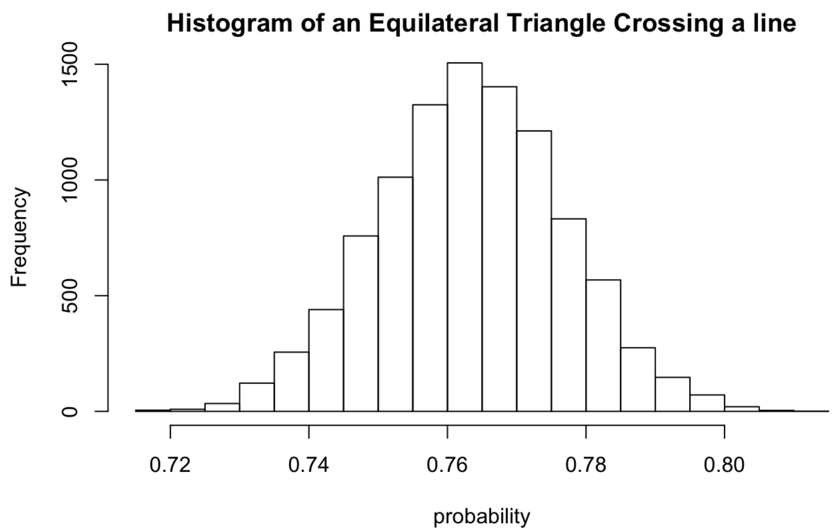


3 Drop Triangles Onto Parallel Lines Separated Plan

Now we switch to the new situation. Assume the distance between any two adjacent lines is d. At each step, we drop a triangle with sides A, B, C each has a length smaller than d onto the plan, and we want to know what is the probability that a randomly dropped triangle will cross a line. Figure ** shows a possible outcome when we dropped 3 triangles onto the plan.

3.1 Dropping Equilateral Triangles

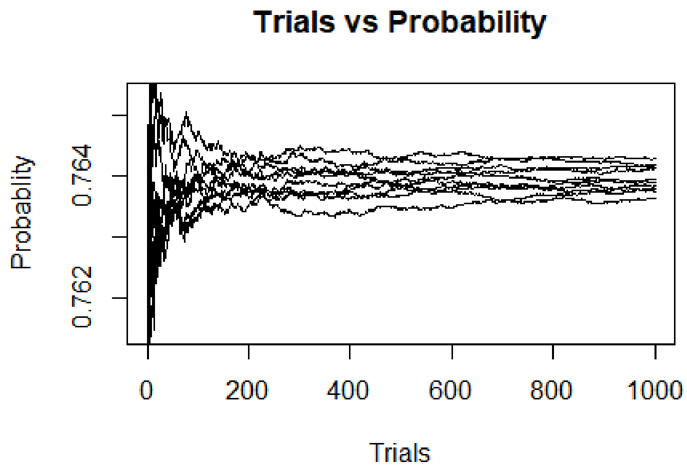
Before we look for the mathematical expression of the solution, we start with a simulation of the equilateral triangles. We simulated 10000 runs of 10000 triangle throws. We calculate the proportion of triangles crossing a line in each run, and yield the following histograms:



We generate the simulation using the following procedure: On a plan equally divided into 6 vertical stripes, we fix each side of the equilateral triangle to be 0.8. First, uniformly pick a random point on the plan as the centroid of the equilateral triangle; second, pick a random degree as the degree from one of the vertices to the horizontal line. Now, with the centroid, the randomly picked degree, and the length of the sides, we can determine the position of the equilateral triangle.

To determine if the triangle crosses a line, we take the floor function of each x-coordinate of the vertices. If the 3 x-coordinates yield same results, then the triangle doesn't cross a line; otherwise, it crosses.

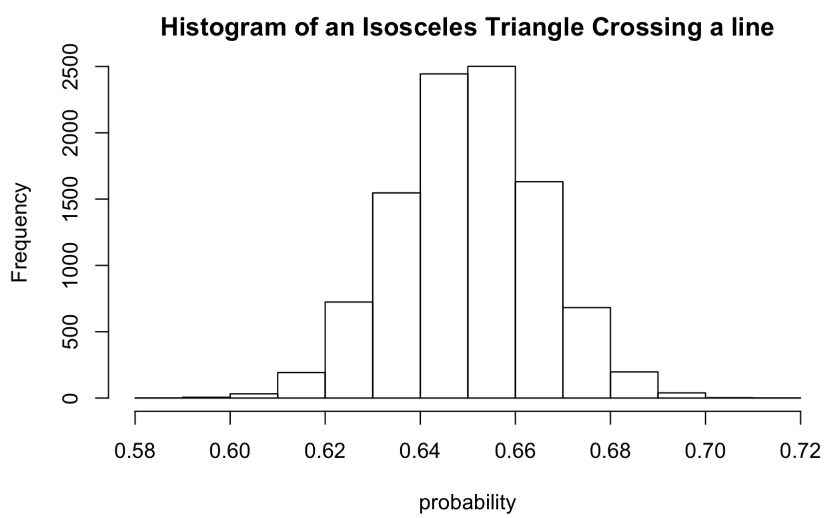
We simulate this problem in R and we run 10000 runs of 10000 equilateral triangles. Here is what happens with ten runs of the simulation:



We can observe that the probability of the equilateral triangle crossing the line converges to around 0.764 after 1000 runs of the simulation.

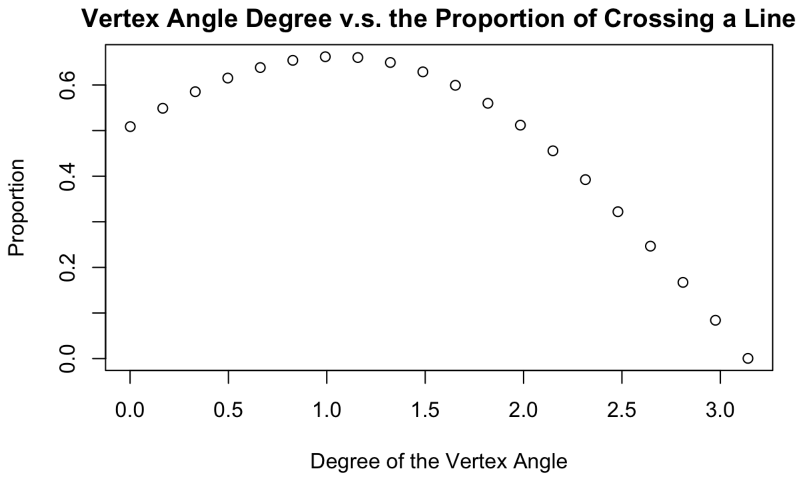
3.2 Dropping Isosceles Triangles

Following the same procedure above, we set the vertex angle to be $\frac{\pi}{4}$, and set the distance from centroid to each vertex to be 0.4. Then we produce the following histogram for isosceles triangles.



We also simulate isosceles triangles with other vertex angle. The following graph plot different vertex angle degree with the sample mean of that trial. Notice that when the vertex angle of an isosceles triangle approaches zero, the triangle approaches a needle with a length of 0.8. Record from the previous section, we calculate the proportion of a needle with length 0.8 crossing a line, which is around 0.51. From the graph above, we see that when the vertex angle is close to 0 degree, the proportion of crossing a line is slightly above 0.5, which matches our result for the needle simulation.

Note, when the vertex angle goes to π, since we fixed the distance from centroid to each vertices, the triangle approached to a point, and thus has zero possibility to cross a line.



3.3 Extend to General cases

Examining the situation, we have the following observations:

1. : A triangle crosses a line if and only if two of its sides cross the line
2. : The events side A crossing a line, side B crossing a line, and side C crossing a line are not independent.

Now, define the following events. Let

P(A) be the event where side A crossing a line;
P(B) be the event where side B crossing a line;
P(C) be the event where side C crossing a line.

To find out the probability of a triangle crossing a line, we want to calculate $P(A \cup B \cup C)$. We derive the following equations:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \tag{1}$$

$$P(A \cap B \cap C) = 0 \tag{2}$$

$$P(A) = P(A \cap B) + P(A \cap C) \tag{3}$$

$$P(B) = P(B \cap A) + P(B \cap C) \tag{4}$$

$$P(C) = P(C \cap A) + P(C \cap B) \tag{5}$$

$$P(A \cap B) + P(A \cap C) + P(B \cap C) = \frac{1}{2}[P(A) + P(B) + P(C)] \tag{6}$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - [P(A \cap B) + P(A \cap C) + P(B \cap C)] + 0 \tag{7}$$

Substitute equation(6) to equation(7),

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - \frac{1}{2}[P(A) + P(B) + P(C)] \tag{8}$$

Thus, we have

$$P(A \cup B \cup C) = \frac{1}{2}[P(A) + P(B) + P(C)] \tag{9}$$

Now, we check if our proportions from simulations match with theoretical probability we derived.

Consider the case in section 3.1, we write the probability for an equilateral triangle to cross a line as $P(A) + P(B) + P(C)$. Since the sides of the triangles are 0.8, the distance between each pair of lines is 1, then, according to the probability for a needle to cross a line, we know:

$$P(A) = P(B) = P(C) = \frac{2}{\pi}\frac{l}{t} = \frac{2(0.8)}{\pi(1)} = \frac{1.6}{\pi} \tag{10}$$

$$P(A \cup B \cup C) = \frac{1}{2}[P(A) + P(B) + P(C)] = \frac{3}{2}\frac{1.6}{\pi} \approx 0.764 \tag{11}$$

That is, the probability for a randomly dropped equilateral triangle with side length 0.8 to cross a line is approximately 0.764, which matches our observation from the convergence test in section 3.1.

For the isosceles triangle case we present in section 3.2, we first calculate the sides lengths from its distance between centroid to vertices and its vertex angle. The distance is 0.4, the vertex angle is $\frac{\pi}{4}$. Let A, B denote the two legs of the isosceles triangle, and C denote the base.

$$A = B = 2 \cdot 0.4 \cos\left(\left(\frac{1}{2}\right)\left(\frac{\pi}{4}\right)\right) = 0.8 \cos\left(\frac{\pi}{8}\right) \tag{12}$$

$$C = 2 \cdot 0.4 \sin\left(\left(\frac{\pi}{4}\right)\right) = 0.8 \sin\left(\frac{\pi}{4}\right) \tag{13}$$

$$P(A) = P(B) = \frac{2}{\pi}\frac{l}{t} = \frac{2}{\pi}0.8 \cos\left(\frac{\pi}{8}\right) \tag{14}$$

$$P(C) = \frac{2}{\pi}\frac{l}{t} = \frac{2}{\pi}0.8 \sin\left(\frac{\pi}{4}\right) \tag{15}$$

$$P(A \cup B \cup C) = \frac{1}{2}[P(A) + P(B) + P(C)] \approx 0.651 \tag{16}$$

That is, the probability for a randomly dropped isosceles triangle with vertex angle $\frac{\pi}{4}$ and two legs $0.8 \cos\left(\frac{\pi}{8}\right)$ to cross a line is approximately 0.651, which matches our observation from the histogram in section 3.2.

4 Conclusion

Central Limit Theorem states that if we sample many values from an unknown distribution and average them, and do this multiple times, the averages will be approximately normally distributed around the mean of distribution. From the histograms of each cases, after we run multiple times, we notice a normal shape. What these histograms suggest corresponds to the central limit theorem: the distribution of these values is very nearly normal. We can utilize this information to estimate the true probability.

4.1 Dropping Needles

For the needle case, based on 1000 runs of 10000 simulated throwing, we get the mean of the 1000 values is

$$\bar{x} = \frac{1}{1000}\sum_{i=1}^{1000} x_i = 0.5091378$$

and the standard deviation is

$$s = \sqrt{\frac{1}{1000-1}\sum_{i=1}^{1000}(x_i - \bar{x})^2} = 0.005173205$$

For a true normal distribution, 99.73% of the values of the population will be within 3 standard deviations of the mean. Therefore, we are 99.73% confident that true probability is in this range: $[0.4936182, 0.5246574]$.

4.2 Dropping Equilateral Triangles

Based on the 1000 runs of 10000 simulated throwing, we get the mean is

$$\bar{x} = \frac{1}{1000} \sum_{i=1}^{1000} x_i = 0.763793$$

and the standard deviation is

$$s = \sqrt{\frac{1}{1000-1} \sum_{i=1}^{1000} (x_i - \bar{x})^2} = 0.004066242$$

We can feel 95% confident that the true probability of a equilateral triangle crossing the line is in this range: [0.7556605, 0.7719255].

5 R code for Needles and Triangles

5.1 Dropping Needles

```
buffoneedle <- function(n, L, a){
  hit = 0
  for(i in 1:n) {
    x <- runif(1, 0, 1)
    y <- runif(1, 0, 1)
    if(x^2 + y^2 > 1) {
      x <- runif(1, 0, 1)
      y <- runif(1, 0, 1)
    }
    theta <- atan(y/x) # the random angle
    d <- runif(1, 0, (L/2)) # distance of needle midpoint to nearest line
    if(d <= (a/2)*sin(theta)) {
      hit <- hit + 1
    }
  }
  prob = hit/n
}
numGame = 1000
ans <- rep(0, numGame)
for(i in 1:numGame){
  ans[i] <- buffoneedle(10000, 1, 0.8)
}
hist(ans, breaks = 50, xlab = "probability", main = "Histogram of probability")
mean <- rep(0, numGame)
for(i in 1:numGame){
  mean[i] = mean(ans[0:i])
}
plot(c(1:1000), mean, xlab = "Trials", ylab = "Probablity",
     main = "Trials vs Probability", type = "l")
lines(c(1:1000), mean)
```

5.2 Dropping Equilateral Triangle

```
```{r}
N = 10000;
```
```

````
prop <- rep(0, N);
for(j in 1:N){
  n = 1000
  cross = 0;
  for(k in 1:n){
    center.x = runif(1, 0, 6);
    center.y = runif(1, 0, 6);

    pt.x = c(0, 0, 0);
    alpha = runif(1, 0, 2*pi/3);
    l = 0.8*sqrt(3)/3;
    pt.x[1] = center.x + l*cos(alpha);
    pt.x[2] = center.x + l*cos(2*pi/3+alpha);
    pt.x[3] = center.x + l*cos(4*pi/3+alpha);
    if(floor(pt.x[1]) != floor(pt.x[2]) | floor(pt.x[1]) != floor(pt.x[3]) | floor(pt.x[3]
      cross = cross + 1;
    }
  }
  prop[j] = cross/n;
}
```
````
````