Math 132

Section 2.1

1. (c) $h(z) = \frac{z+i}{z^2+1} = \frac{z+i}{(z+i)(z-i)} = \frac{1}{z-i} = \frac{1}{x+yi-i} = \frac{1}{x+(y-1)i}$

$$= \frac{x-(y-1)i}{x^2-(y-1)^2i^2} = \frac{x}{x^2+(y-1)^2} - \frac{y-1}{x^2+(y-1)^2}i$$

(f) $G(z) = e^z + e^{-z}$

$$= e^{x+yi} + e^{-x-yi}$$

$$= e^x(\cos y + i\sin y) + e^{-x}(\cos y - i\sin y)$$

$$= (e^x + e^{-x})\cos y + i(e^x - e^{-x})\sin y.$$

6.(b) plugging $z = e^{i\theta}$ into $u = J(z) = \frac{1}{2}(z + \frac{1}{z})$

Then $u = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \cos\theta$

Since $\theta \in [0, 2\pi)$, we have $u \in [-1, 1]$

Section 2.2.

3. "$\Rightarrow$" if $z_n \to z_0$, we have $\forall \varepsilon > 0$, $\exists N$ s.t. $\forall n > N$ $|z_n - z_0| < \varepsilon$

Then since $|x_n - x_0| < |z_n - z_0| < \varepsilon$, $x_n \to x_0$ as $n \to \infty$
and $|y_n - y_0| < |z_n - z_0| < \varepsilon$, $y_n \to y_0$ as $n \to \infty$

"$\Leftarrow$" since $x_n \to x_0$ and $y_n \to y_0$, $\forall \varepsilon > 0$ $\exists N$ s.t. $\forall n \geq N$ $|x_n - x_0| < \frac{\varepsilon}{2}$
and $|y_n - y_0| < \frac{\varepsilon}{2}$

Then $|z_n - z_0| < |x_n - x_0| + |y_n - y_0| < \varepsilon$, which implies that $z_n \to z_0$ as $n \to \infty$

7. (a)

$||z_n| - 0| = |\frac{i}{n}| = \frac{1}{n} \to 0$ as $n \to \infty$

Thus $z_n = \frac{i}{n} \to 0$ as $n \to \infty$

(c) $z_n = \arg(-1 + \frac{i}{n})$

$-1 + \frac{i}{n} \to -1$ as $n \to \infty$

Then $z_n \to \arg(-1) = \pi$ as $n \to \infty$

(e) $z_n = (\frac{1-i}{4})^n \to 0$ as $n \to \infty$

$||z_n| - 0| = |\frac{1-i}{4}|^n = |\frac{1}{4} - \frac{1}{4}i|^n = (\frac{\sqrt{2}}{4})^n \to 0$ as $n \to \infty$

(f) $z_n = \exp(\frac{2n\pi i}{5}) = \cos(\frac{2\pi n}{5}) + i \sin(\frac{2\pi n}{5})$

z_n does not converge.

11. (c) $\lim\limits_{z \to 3i} \frac{z^2+9}{z-3i} = \lim\limits_{z \to 3i} \frac{(z+3i)(z-3i)}{z-3i} = \lim\limits_{z \to 3i} z+3i = 6i$

(d) $\lim\limits_{z \to i} \frac{z^2+i}{z^4-1} = \frac{i-1}{0}$ this limit does not exist.

15. Since $f(z)$ is continuous at z_0.

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $0 < |z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$

(i) Then $|\overline{f(z)} - \overline{f(z_0)}| = |\overline{f(z) - f(z_0)}| = |f(z) - f(z_0)| < \varepsilon$

and we know $\overline{f(z)}$ is continuous at z_0.

(ii) $|\operatorname{Re} f(z) - \operatorname{Re} f(z_0)| = |\operatorname{Re}(f(z) - f(z_0))| \le |f(z) - f(z_0)| < \varepsilon$

Then $\operatorname{Re} f(z)$ is continuous at z_0.

(iii) $|\operatorname{Im} f(z) - \operatorname{Im} f(z_0)| = |\operatorname{Im}(f(z) - f(z_0))| \leq |f(z) - f(z_0)| < \varepsilon$

Then $\operatorname{Im} f(z)$ is continuous at z_0.

(iv).

$||f(z)| - |f(z_0)|| \leq |f(z) - f(z_0)|$ by reverse triangle inequality $< \varepsilon$

Then $|f(z)|$ is continuous at z_0.

Section 2.3.

7. (c)

$$f'(z) = \frac{(iz^3+2z+\pi)(z^2-9)' - (z^2-9)(iz^3+2z+\pi)'}{(iz^3+2z+\pi)^2}$$

$$= \frac{(iz^3+2z+\pi)\cdot 2z - (z^2-9)(3iz^2+2)}{(iz^3+2z+\pi)^2}$$

$$= \frac{-iz^4+2z^2+27iz^2+2\pi z+18}{(iz^3+2z+\pi)^2}$$

(e)

$$f'(z) = 6i\left(\left((z^3-1)^4\right)'(z^2+iz)^{100} + (z^3-1)^4\left((z^2+iz)^{100}\right)'\right)$$

$$= 6i\left(4(z^3-1)^3 \cdot 3z^2 (z^2+iz)^{100} + (z^3-1)^4 \cdot 100 \cdot (z^2+iz)^{99} \cdot (2z+i)\right)$$

$$= 6i\left(12z^2(z^3-1)^3(z^2+iz)^{100} + 100(2z+i)(z^3-1)^4(z^2+iz)^{99}\right)$$

9. (b) $\dfrac{iz^3+2z}{z^2+1}$ is not analytic at $z = \pm i$

$z^2+1=0 \Rightarrow z = \pm i$

(c) $\dfrac{3z-1}{z^2+z+4}$ is not analytic at $z = \dfrac{-1 \pm \sqrt{15}\, i}{2}$

$z^2+z+4=0 \Rightarrow z = \dfrac{-1 \pm \sqrt{15}\, i}{2}$

10. $\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \frac{|z_0+\Delta z|^2 - |z_0|^2}{\Delta z}$

$$= \frac{(z_0+\Delta z)(\overline{z_0}+\overline{\Delta z}) - z_0\overline{z_0}}{\Delta z} = \overline{z_0} + \overline{\Delta z} + z_0\frac{\overline{\Delta z}}{\Delta z}.$$

if $z_0 = 0$ then $\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \overline{\Delta z} = 0$

if $z_0 \neq 0$ $\frac{\overline{\Delta z}}{\Delta z} = 1$ if approach from real side

$\frac{\overline{\Delta z}}{\Delta z} = -1$ if approach from imaginary side.

thus $\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \begin{cases} 2\overline{z_0} & \text{if from imaginary side} \\ 0 & \text{if from real side.} \end{cases}$

$f(z) = |z|^2$ is not differentiable at any point other than $z=0$.

13. (a) T

$(f(z)^3)' = 3f(z)^2 \cdot f'(z)$ is analytic then $(f(z)^3)'$ is differentiable on complex numbers.

Then $f(z)^3$ is entire.

(b) T.

$(f(z)g(z))' = f'(z)g(z) + f(z)g(z)$ is analytic, then

$(f(z)g(z))'$ is differentiable on complex ~~numble~~ numbers.

and $f(z)g(z)$ is entire.

(c) F. $\left(\frac{f(z)}{g(z)}\right)' = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$ is not differentiable

when $g(z)=0$.

thus $\frac{f(z)}{g(z)}$ is not entire.

T

(d) $(5f(z)+ig(z))' = 5f'(z)+ig'(z)$ is differentiable on complex numbers.

Then $5f(z)+ig(z)$ is entire

(e) F

$(f(\frac{1}{z}))' = \frac{1}{z^2} f'(\frac{1}{z})$ is not differentiable when $z=0$

Thus $f(\frac{1}{z})$ is not entire.

(f) T $(g(z^2+2))' = 2z\, g'(z^2+2)$ is differentiable on complex numbers.

Thus $g(z^2+2)$ is entire.

(g) T $(f(g(z)))' = f'(g(z)) \cdot g'(z)$ is differentiable on complex numbers.

Then $f(g(z))$ is entire.

Section 2.4. let $z = x+iy$.

1. (b) $w = \operatorname{Re} z = u(x,y) + iv(x,y)$

$u(x,y) = x \qquad v(x,y) = 0$

$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$

Cauchy-Riemann equations does not satisfy at any point. Then $w = \operatorname{Re} z$ is no where differentiable.

(c) $w = 2y - ix$

$u(x,y) = 2y \qquad v(x,y) = -x$

$\frac{\partial u}{\partial y} = 2 \qquad \frac{\partial v}{\partial x} = -1$

$\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x} \Rightarrow$ Cauchy-Riemann equations does not satisfy and $w = 2y - ix$ is no where differentiable.

2. $u(x,y) = x^3 + 3xy^2 - 3x$

$v(x,y) = y^3 + 3x^2y - 3y.$

$\frac{\partial u}{\partial x} = 3x^2 + 3y^2 - 3 \qquad \frac{\partial u}{\partial y} = 6xy.$

$\frac{\partial v}{\partial x} = 6xy \qquad \frac{\partial v}{\partial y} = 3y^2 + 3x^2 - 3.$

Then $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ if and only if $6xy = 0.$

Then $h(z)$ is differentiable if $x=0$ or $y=0$ $\Leftrightarrow$ on coordinate axes.

Thus $h(z)$ is not analytic any where.

4. $\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(\Delta x, 0) - u(0,0)}{\Delta x} = 0$

$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(0,\Delta y) - u(0,0)}{\Delta y} = 0.$

$\frac{\partial v}{\partial x} = \lim_{\Delta x \to 0} \frac{v(\Delta x, 0) - v(0,0)}{\Delta x} = 0$

$\frac{\partial v}{\partial y} = \lim_{\Delta y \to 0} \frac{v(0,\Delta y) - v(0,0)}{\Delta y} = 0$

Thus $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, satisfies Cauchy-Riemann equation. at $z=0$

Then $\lim_{\Delta z \to 0} \frac{f(\Delta z)}{\Delta z} = 0$ if we approach from real axis.

$\lim_{\Delta z \to 0} \frac{f(\Delta z)}{\Delta z} = \frac{(\Delta x^{4/3} \cdot \Delta x^{5/3} + i \Delta x^{5/3} \Delta x^{4/3})}{2\Delta x^2}$ if we approach from line $y=x$

Thus f is not differentiable at $z=0$.

5. $f(z) = e^{x^2-y^2}(\cos(2xy) + i\sin(2xy))$

Then $u(x,y) = e^{x^2-y^2} \cdot \cos(2xy)$

$v(x,y) = e^{x^2-y^2} \sin(2xy)$

Then $\frac{\partial u}{\partial x} = e^{x^2-y^2} \cdot 2x\cos(2xy) - e^{x^2-y^2} \cdot 2y\sin(2xy)$

$\frac{\partial u}{\partial y} = e^{x^2-y^2} \cdot (-2x)\sin(2xy) - e^{x^2-y^2} \cdot 2y\cos(2xy)$

$\frac{\partial v}{\partial x} = e^{x^2-y^2} \cdot 2y\cos(2xy) + e^{x^2-y^2} 2x\sin(2xy)$

$\frac{\partial v}{\partial y} = e^{x^2-y^2} 2x \cdot \cos(2xy) - e^{x^2-y^2} \cdot 2y\sin(2xy)$

Therefore. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$f(z)$ satisfies Cauchy-Riemann equations and $f(z)$ is entire.

11. $f(z)$ and $\overline{f(z)}$ are analytic $\Rightarrow$ $f(z)$ is constant.

let $h(z) = \frac{1}{2}(f(z) + \overline{f(z)}) = \operatorname{Re} f(z)$

$\cancel{\operatorname{Im} f(z)} =$

Then $\operatorname{Im} h(z) = 0 \cancel{= v(x,y)}$

Thus $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$

Also $h'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y} = 0 \Rightarrow h(z)$ is constant

Then $f(z) = c_1 + i\,v(x,y)$

since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 0$

we know $v(x,y)$ is also a constant

Thus $f(z) = C_1 + iC_2$, C_1, C_2 are constant.

Section 2.5

3. (b) $u = e^x \sin y$.

$$\frac{\partial^2 u}{\partial x^2} = e^x \sin y \qquad \frac{\partial^2 u}{\partial y^2} = -e^x \sin y.$$

Then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow u = e^x \sin y$ is harmonic.

$$\frac{\partial v}{\partial y} = e^x \sin y. \text{ and } \frac{\partial v}{\partial x} = -e^x \sin y.$$

$$v(x,y) = -e^x \cos y + g(x)$$

$$\frac{\partial v}{\partial x} = -e^x \cos y + g'(x) \Rightarrow g'(x) = 0 \Rightarrow g(x) = c$$

$$v(x,y) = -e^x \cos y + c$$

(d) $u(x,y) = \sin x \cosh y$.

$$\frac{\partial^2 u}{\partial x^2} = -\sin x \cosh y \qquad \frac{\partial^2 u}{\partial y^2} = \sin x \cosh y$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow u(x,y) = \sin x \cosh y \text{ is harmonic.}$$

$$\frac{\partial v}{\partial y} = \cos x \cosh y. \text{ and } \frac{\partial v}{\partial x} = -\sin x \sinh y.$$

Then $v = \cos x \sinh y + g(x)$

$$\frac{\partial v}{\partial x} = -\sin x \sinh y + g'(x)$$

Then $g'(x) = 0 \Rightarrow g(x) = c$

$v(x,y) = \cancel{\sin x \sinh y + c}$
$\cos x \sinh y + c.$

(f) $u = \operatorname{Im} e^{z^2} = \operatorname{Im} e^{x^2-y^2}(\cos 2xy + i \sin 2xy)$

$= e^{x^2-y^2}\sin(2xy)$

$\frac{\partial^2 u}{\partial x^2} = \sin(2xy)e^{x^2-y^2}(4x^2-4y^2+2) + e^{x^2-y^2} \cdot 8xy\cos(2xy)$

$\frac{\partial^2 u}{\partial y^2} = \sin(2xy)e^{x^2-y^2}(4y^2-4x^2-2) - e^{x^2-y^2} \cdot 8xy\cos(2xy)$

Thus $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \Rightarrow$ u is harmonic.

Then $\frac{\partial v}{\partial x} = e^{x^2-y^2} \cdot 2y \cdot \sin(2xy) - e^{x^2-y^2} \cdot 2x \cdot \cos(2xy)$

$\frac{\partial v}{\partial y} = e^{x^2-y^2} \cdot 2y\cos(2xy) + e^{x^2-y^2} \cdot 2x \cdot \sin(2xy)$

Thus. $v(x,y) = -\cos(2xy)e^{x^2-y^2} + c.$

5. $f(z) = u(x,y) + iv(x,y)$ is analytic

Then $g(z) = -if(z) = v(x,y) + i(-u(x,y))$ is also ~~ang~~ analytic.

Thus $-u$ is a harmonic conjugate of v.

18. Assume $\phi(x,y)$ is harmonic

Then $\frac{\partial \phi_x}{\partial x} = \frac{\partial \phi_v}{\partial y}$ and $\frac{\partial \phi_x}{\partial y} = -\frac{\partial \phi_y}{\partial x}$

Thus ϕ_x and ϕ_y satisfies Cauchy-Riemann equation.
and $\phi_x - i\phi_y$ is analytic.

Section 3.2.

9. (b) $w = \cos(2z) + i\sin(\frac{1}{z})$

$$\frac{dw}{dz} = -\sin(2z)\cdot 2 + i\cos(\tfrac{1}{z})\,(\tfrac{1}{z})'$$

$$= -2\sin(2z) - \frac{i}{z^2}\cos(\tfrac{1}{z})$$

(c) $w = \exp(\sin 2z)$

$$\frac{dw}{dz} = \exp(\sin 2z)\cdot\cos(2z)\cdot 2$$

$$= 2e^{\sin 2z}\cos 2z$$

(e) $w = (\sinh z + 1)^2$

$$\frac{dw}{dz} = 2(\sinh z + 1)\cdot\cosh z$$

13. (a) $\sin(x+iy)$

$$= \sin x\cos iy + \cos x\sin iy$$
$$= \sin x\cosh y + i\cos x\sinh y.$$

$\cos iy = \cosh y$
$\sin iy = \cancel{i\sinh y}\ . i\sin y.$

14. (a) $\cancel{e^{iz}}$

$$e^{i(z+2\pi)} = e^{iz}\cdot e^{i\cdot 2\pi}$$

$$= e^{iz}\cdot(\cos 2\pi + i\sin 2\pi)$$

$$= e^{iz}$$

e^{iz} is periodic with period 2π.

~~(b)~~ ~~$\tan z =$~~

~~$\tan(z+2\pi) =$~~

(b) $\tan(z+\pi) = \dfrac{\sin(z+\pi)}{\cos(z+\pi)} = \dfrac{-\sin z}{-\cos z} = \dfrac{\sin z}{\cos z} = \tan z$

Thus $\tan z$ is periodic with period π.

Section 3.3.

1. (b) ~~$\log(1-i) = \log\left(\sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)\right)$~~

$1-i = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

$= \sqrt{2}\, e^{-\frac{\pi}{4}i}$

Then $\log(1-i) = \log\sqrt{2} + \left(-\frac{\pi}{4}i\right)$

(c) $-i = \cos\frac{\pi}{2} + i\sin\left(\frac{\pi}{2}\right)$

$= e^{-\frac{\pi}{2}i}$

Thus $\log(-i) = -\frac{\pi}{2}i$

3. $z_1 = i \quad z_2 = i-1$

$\log z_1 z_2 \neq \log z_1 + \log z_2.$

$\log z_1 z_2 = \log(-1-i) = \log\left(\sqrt{2}\left(\cos\left(-\frac{3}{4}\pi\right) + i\sin\left(-\frac{3}{4}\pi\right)\right)\right)$

$= \log\left(\sqrt{2} \cdot e^{-\frac{3}{4}\pi i}\right)$

$= \log\sqrt{2} + \left(-\frac{4}{3}\pi i\right)$

$\log z_1 + \log z_2 = \frac{\pi}{2}i + \log\left(\sqrt{2}\, e^{\frac{3}{4}\pi i}\right)$

$= \frac{\pi}{2}i + \frac{3}{4}\pi i + \log\sqrt{2} = \frac{5}{4}\pi i + \log\sqrt{2}$

Thus $\log z_1 + \log z_2 \neq \log z_1 z_2.$

Scanned with CamScanner

5. (a) $e^z = 2i$

$\log e^z = \log 2i$

$z = \log 2 + \log i$

$= \log 2 + (\frac{\pi}{2} + 2k\pi) i \quad k = 0, \pm 1, \pm 2 \cdots$

(b) $\log(z^2 - 1) = \frac{i\pi}{2}$

$z^2 - 1 = e^{i\frac{\pi}{2}} = i$

$z^2 = 1 + i \Rightarrow z = \pm\sqrt{1+i}$

9. $f(z) = \log(4 + i - z)$

$4 + i - z = \cancel{4+i+x+yi = 4+x}$

$4 + i - x - yi = (4-x) + (1-y)i$

$\begin{cases} 4 - x \le 0 \\ 1 - y > 0 \end{cases} \Rightarrow \begin{cases} x \ge 4 \\ y < 1 \end{cases}$

The domain of analyticity is all complex number except for $\{x \ge 4, y < 1\}$

$f'(z) = -\frac{1}{4 + i - z}$

12. $f(z) = \log(z^2 + 1) \quad z = 0.$ ~~[illegible]~~.

~~choose branch $Ln(g(z))$~~

Scanned with CamScanner

Section 3.4.

the center of circle is $1+i$

consider the function of circle.

For inner circle. $A\log|z-(1+i)|+B=0.$

$\left.\begin{array}{l}\\ \text{plug in } z=1 \text{ and } z=i\end{array}\right\} \Rightarrow B=0.$

plug in $z=-1+i \Rightarrow A=\frac{10}{\log 2}.$

Thus. $\phi(x,y)=\frac{10}{\log 2}\log|z-(1+i)|.$

Than $\phi(0,0)=\frac{10}{\log 2}\log|-1-i| = \frac{10}{\log 2}\cdot\log\sqrt{2} = 5$