Academic integrity pledge:

Upon my honor, I affirm that I did not solicit nor did I receive the help of any individual in writing my answers to this exam.

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1. $\forall z, w \in \mathbb{C}$, we have $z = a+bi$ $w = c+di$ $a,b,c,d \in \mathbb{R}$

Then $LHS = |z+w|^2 = |a+bi+c+di|^2 = |(a+c)+i(b+d)|^2$

$= (a+c)^2 + (b+d)^2 = a^2+b^2+c^2+d^2+2ac+2bd$

$RHS = |z|^2 + |w|^2 + 2Re(z\bar{w})$

$= |a+bi|^2 + |c+di|^2 + 2Re((a+bi)(c-di))$

$= a^2+b^2+c^2+d^2 + 2Re(ac + ibc - iad + bd)$

$= a^2+b^2+c^2+d^2 + 2Re((ac+bd)+i(bc-ad))$

$= a^2+b^2+c^2+d^2 + 2(ac+bd)$

Thus $LHS = RHS$ and $|z+w|^2 = |z|^2+|w|^2+2Re(z\bar{w})$ for any $z,w \in \mathbb{C}$

2.

(a)

$f(z)$ is analytic at z_0 means that $f(z)$ is analytic in some neighborhood of z_0, ~~Implie~~ It implies that $f(z)$ has a derivative at every point of such neighborhood.

(b) let $f(z) = u(x,y) + v(x,y) i$

Thus $u(x,y) = x^3+3xy^2-6y^2+x$ $\quad v(x,y) = y^3+3x^2y+y$

Then $\frac{\partial u}{\partial x} = 3x^2+3y^2+1$ $\quad \frac{\partial v}{\partial x} = 6xy$

$\frac{\partial u}{\partial y} = 6xy-12y$ $\quad \frac{\partial v}{\partial y} = 3y^2+3x^2+1$

Since the first partial derivatives of u, v exists and are continuoust on $\mathbb{C}$, Cauchy-Riemann equations need to be satisfied to make $f(z)$ differentiable.

Then $\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \Rightarrow \begin{cases} 3x^2+3y^2+1 = 3y^2+3x^2+1 \\ 6xy-12y = -6xy \end{cases} \Rightarrow xy = y \Rightarrow x = 1 \text{ or } y = 0$

By Theorem 3.5, $f(z)$ is differentiable on $\{z = x+iy \mid x=1 \text{ or } y=0\}$

Since $f(z)$ is only differentiable on lines $x=1$ or $y=0$,

It is not analytic any where on $\mathbb{C}$

3.

(a) $u(x,y) = xy^2 - x^2y$

$$\frac{\partial u}{\partial x} = y^2 - 2xy \qquad \frac{\partial^2 u}{\partial x^2} = -2y$$

$$\frac{\partial u}{\partial y} = 2xy - x^2 \qquad \frac{\partial^2 u}{\partial y^2} = 2x$$

Then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2x - 2y \neq 0$, u does not satisfy Laplace's equation.

Thus $u(x,y)$ is not harmonic and by Theorem 39 there is no such a ~~funct~~ function $v(x,y)$ to make $f(z) = u(x,y) + iv(x,y)$ analytic.

(b) $u(x,y) = y^3 - 3x^2y$

$$\frac{\partial u}{\partial x} = -6xy \qquad \frac{\partial^2 u}{\partial x^2} = -6y$$

$$\frac{\partial u}{\partial y} = 3y^2 - 3x^2 \qquad \frac{\partial^2 u}{\partial y^2} = 6y$$

Thus $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6y + 6y = 0$, u satisfies Laplace's equation.

Then $u(x,y)$ is harmonic on $\mathbb{C}$ and $v(x,y)$ could be its harmonic conjugate.

u, v should satisfy Cauchy-Riemann equations, then

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = -6xy \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 3x^2 - 3y^2$$

antiderivative $\frac{\partial v}{\partial y}$ w.r.t y, we have $V(x,y) = -3xy^2 + \phi(x)$

$$\frac{\partial v}{\partial x} = -3y^2 + \phi'(x) \Rightarrow \phi'(x) = 3x^2 \Rightarrow \phi(x) = x^3 + C \qquad C \in \mathbb{R}$$

Thus $v(x,y) = -3xy^2 + x^3 + C$ and $f(z) = u(x,y) + iv(x,y)$ is analytic on $\mathbb{C}$

(c) $u(x,y) = \text{Arg}(x+iy) = \arctan\left(\frac{y}{x}\right)$ for $x>0$

Then $\frac{\partial u}{\partial x} = -\frac{y}{x^2+y^2}$ $\quad \frac{\partial^2 u}{\partial x^2} = \frac{2xy}{(x^2+y^2)^2}$

$\frac{\partial u}{\partial y} = \frac{x}{x^2+y^2}$ $\quad \frac{\partial^2 u}{\partial y^2} = \frac{-2xy}{(x^2+y^2)^2}$

Thus $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ satisfies Laplace's equation.

Then $u(x,y)$ is harmonic and $v(x,y)$ is its harmonic conjugate.

Since $\text{Log}(z)$ is analytic for $x>0$

Then let $f(z) = -i\,\text{Log}(z) = -i(\log|z| + i\text{Arg}\,z)$

$= \text{Arg}(x+iy) - i\log|x+iy|$

Since $f(z) = -i\,\text{Log}(z)$ is analytic on ~~x>0~~ $\{z = x+iy \mid x>0\}$

we have $v(x,y) = -\log|x+iy| = -\frac{1}{2}\log(x^2+y^2)$

4.

$$\int_\Gamma \frac{1}{z^2-2z}\,dz$$

$$= \int_\Gamma \frac{1}{(z-1)^2-1}\,dz \qquad \text{let } u = z-1$$

$$= \int_\Gamma \frac{1}{u^2-1}\,du \qquad \text{since } \int \frac{1}{1-x^2}\,dx = \tanh^{-1}(x)$$

$$= -\tanh^{-1}(u)\Big|_{u_i}^{u_t}$$

$$= -\tanh^{-1}(z-1)\Big|_{z_i}^{z_t} = \tanh^{-1}(1-z)\Big|_{z_i}^{z_t} = \tanh^{-1}(1-z)\Big|_{1-i}^{1+i}$$

$$= \tanh^{-1}(1-i) - \tanh^{-1}(1+i)$$

$$= \frac{1}{2}\log(2-i) - \frac{i\pi}{4} - \left(\frac{1}{2}\log(2+i) + \frac{i\pi}{4}\right)$$

$$= -\frac{i\pi}{2}$$

5.

(a) $0<|z-i|<2$

$$f(z)=\frac{2z}{z^2+1}=\frac{2z}{z-i}\cdot\frac{1}{z+i}=\frac{2z}{z-i}\cdot\frac{1}{z+2i-i}$$

$$=\frac{2z}{z-i}\cdot\frac{1}{2i}\cdot\frac{1}{1+\frac{z-i}{2i}}$$

$$=\frac{2z}{z-i}\cdot\frac{1}{2i}\cdot\frac{1}{1-\frac{i}{2}(z-i)}$$

Since $\left|\frac{i}{2}(z-i)\right|=\left|\frac{1}{2}(z-i)\right|=\frac{1}{2}|z-i|<1$

Then we have $f(z)=\frac{2z}{z-i}\cdot\frac{1}{2i}\sum_{n=0}^{\infty}\left(\frac{i}{2}(z-i)\right)^n$

$$=\sum_{n=0}^{\infty}z\cdot i^{n-1}\cdot\left(\frac{1}{2}\right)^n(z-i)^{n-1}$$

(b) $1<|z|$

$$f(z)=\frac{2z}{z^2+1}=\frac{1}{z+i}+\frac{1}{z-i}=\frac{1}{z}\cdot\frac{1}{1-(-\frac{i}{z})}+\frac{1}{z}\cdot\frac{1}{1-\frac{i}{z}}$$

Since $\left|-\frac{i}{z}\right|=\left|\frac{1}{z}\right|=\frac{1}{|z|}<1$ and $\left|\frac{i}{z}\right|=\left|\frac{1}{z}\right|=\frac{1}{|z|}<1$

Then $f(z)=\frac{1}{z}\cdot\sum_{n=0}^{\infty}\left(-\frac{i}{z}\right)^n+\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{i}{z}\right)^n$

$$=\sum_{n=0}^{\infty}(-i)^n z^{-n-1}+\sum_{n=0}^{\infty}i^n z^{-n-1}$$

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6. Fix $z \in \mathbb{C}$ with $|z| \geq R_0$, there exists a large enough r such that $r > |z|$

Consider a circle C_r of radius r centered at z

Then for any w on C_r, we have $|f(w)| \leq |w| \leq |z| + r < 2r$

Since f is bounded by $2r$ on C_r,

$$|f^{(2)}(z)| \leq \frac{2! \cdot 2r}{r^2} \quad \text{by Theorem 50}$$

Thus $|f^{(2)}(z)| \leq \frac{4}{r}$

Since r can be infinitely large, we have $|f^{(2)}(z)| \leq 0$

i.e. $f^{(2)}(z) = 0$. Then $f(z) = f(z_0) + f'(z_0)z + \frac{f''(z_0)}{2!}z^2 + \cdots$

$$= f(z_0) + f'(z_0)z$$

Hence $f(z)$ is a first degree polynomial $f(z) = a + bz$

7.

(a) $f(z) = \dfrac{z-i}{z(z-\pi)^3(\operatorname{Arg} z - \frac{\pi}{4})}$

For $z=0$, the denominator of $f(z)$ will be zero, and by corollary 26 $f(z)$ is not continuous at $z=0$

For $z=\pi$, the denominator of $f(z)$ will be zero and $f(z)$ is not continuous at $z=\pi$.

let $g(z) = \dfrac{z-i}{z(\operatorname{Arg} z - \frac{\pi}{4})}$, $f(z) = \dfrac{g(z)}{(z-\pi)^3}$

Since $g(z)$ is analytic at π and $g(\pi) \neq 0$

$z=\pi$ is a pole of order 3 of $f(z)$ by Lemma 32

For $z=i$,

let $h(z) = \dfrac{1}{z(z-\pi)^3(\operatorname{Arg} z - \frac{\pi}{4})}$, $f(z) = g(z)\cdot(z-i)$

since $h(z)$ is analytic at i and $h(i) \neq 0$,

$z=i$ is a simple zero of $f(z)$ by proposition 27.

For all z with $\operatorname{Arg} z = \frac{\pi}{4}$, the denominator of $f(z)$ will be zero, and by corollary 26, $f(z)$ is discontinuous at these z.

(b) True.

If $h(z)$ has an essential singularity at z_0

Then by definition 29. $h(z) = \sum_{-\infty}^{\infty} a_j (z-z_0)^j$ and $a_j \neq 0$ for infinite number of negative j.

$h(z) = \sum_{-\infty}^{-1} a_j (z-z_0)^j + a_0 + \sum_{1}^{\infty} a_j (z-z_0)^j$

Then let $g(z) = \sum_{-\infty}^{-1} a_j (z-z_0)^j$ $a_j \neq 0$ for infinite number of j ~~of negative~~

Since $h'(z) = \sum_{-\infty}^{\infty} j a_j (z-z_0)^{j-1} = \sum_{-\infty}^{-1} j a_j (z-z_0)^{j-1} + \sum_{1}^{\infty} j a_j (z-z_0)^{j-1}$

In $\sum_{-\infty}^{-1} j a_j (z-z_0)^{j-1}$, if $a_j \neq 0$ $ja_j \neq 0$. there are infinite number of j such that $ja_j \neq 0$

Thus in $h'(z) = \sum_{-\infty}^{\infty} b_j (z-z_0)^j$, $b_j \neq 0$ for infinitely number of negative integer j.

Hence, proved.

8. $f(z) = \frac{1}{(z+1)^2(z^3+1)} + \frac{1}{\cos z}$

(a)

$(z+1)^2 = 0 \Rightarrow z = -1$

$z^3+1 = 0 \Rightarrow z = -1, \frac{1}{2}+\frac{\sqrt{3}}{2}i, \frac{1}{2}-\frac{\sqrt{3}}{2}i$

$\cos z = 0 \Rightarrow z = \frac{\pi}{2} + n\pi \quad n \in \mathbb{Z}$

Then $f(z) = \frac{1}{(z+1)^3(z-\frac{1}{2}-\frac{\sqrt{3}}{2}i)(z-\frac{1}{2}+\frac{\sqrt{3}}{2}i)} + \frac{1}{\cos z}$

For $z=-1$, ~~leg~~ let $g(z) = \frac{1}{(z-\frac{1}{2}-\frac{\sqrt{3}}{2}i)(z-\frac{1}{2}+\frac{\sqrt{3}}{2}i)} + \frac{(z+1)^3}{\cos z}$

Then $f(z) = \frac{g(z)}{(z+1)^3}$

since $g(-1) \neq 0$ and $g(z)$ is analytic at -1, $z=-1$ is a pole of order 3 for $f(z)$

For $z = \frac{1}{2}+\frac{\sqrt{3}}{2}i$, let $g(z) = \frac{1}{(z+1)^3(z-\frac{1}{2}+\frac{\sqrt{3}}{2}i)} + \frac{(z-\frac{1}{2}-\frac{\sqrt{3}}{2}i)}{\cos z}$

Then $f(z) = \frac{g(z)}{(z-\frac{1}{2}-\frac{\sqrt{3}}{2}i)}$

since $g(\frac{1}{2}+\frac{\sqrt{3}}{2}i) \neq 0$ and $g(z)$ is analytic at $\frac{1}{2}+\frac{\sqrt{3}}{2}i$,

$z = \frac{1}{2}+\frac{\sqrt{3}}{2}i$ is a pole of order 1 for $f(z)$

For $z = \frac{1}{2}-\frac{\sqrt{3}}{2}i$, let $g(z) = \frac{1}{(z+1)^3(z-\frac{1}{2}-\frac{\sqrt{3}}{2}i)} + \frac{z-\frac{1}{2}+\frac{\sqrt{3}}{2}i}{\cos z}$

and $f(z) = \frac{g(z)}{(z-\frac{1}{2}+\frac{\sqrt{3}}{2}i)}$

Since $g(z)$ is analytic on $\frac{1}{2}-\frac{\sqrt{3}}{2}i$ and $g(\frac{1}{2}-\frac{\sqrt{3}}{2}i) \neq 0$

$z = \frac{1}{2}-\frac{\sqrt{3}}{2}i$ is a pole of order 1 for $f(z)$

For $z=\frac{\pi}{2}+n\pi \quad n\in\mathbb{Z}$

since $((\cos z))' = -\sin z \neq 0$, we know $\cos z$ has simple zeros at $z=\frac{\pi}{2}+n\pi, n\in\mathbb{Z}$

Then by Lemma 34, $\frac{1}{\cos z}$ has simple poles at $z=\frac{\pi}{2}+n\pi, n\in\mathbb{Z}$

Thus $f(z)$ has simple poles at $z=\frac{\pi}{2}+n\pi$, $n\in\mathbb{Z}$

In summary:

$z=-1$ pole of order 3

$z=\frac{1}{2}+\frac{\sqrt{3}}{2}i, \frac{1}{2}-\frac{\sqrt{3}}{2}i, \frac{\pi}{2}+n\pi$ simple poles.

(b) since Γ is simply closed contour and oriented positively,

we have $\int_\Gamma f(z)\,dz = 2\pi i\left(\text{Res}\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) + \text{Res}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) + \text{Res}\left(\frac{\pi}{2}\right)\right)$

$$\text{Res}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = \lim_{z\to\frac{1}{2}+\frac{\sqrt{3}}{2}i} \frac{1}{(z+1)^3\left(z-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)} + \frac{z-\frac{1}{2}-\frac{\sqrt{3}}{2}i}{\cos z}$$

$$= \frac{1}{\left(\frac{\sqrt{3}}{2}i+\frac{3}{2}\right)^3(\sqrt{3}i)} = -\frac{1}{9}$$

$$\text{Res}\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) = \lim_{z\to\frac{1}{2}-\frac{\sqrt{3}}{2}i} \frac{1}{(z+1)^3\left(z-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)} + \frac{z-\frac{1}{2}+\frac{\sqrt{3}}{2}i}{\cos z}$$

$$= \frac{1}{\left(\frac{3}{2}-\frac{\sqrt{3}}{2}i\right)^3(-\sqrt{3}i)} = -\frac{1}{9}$$

$$\text{Res}\left(\frac{\pi}{2}\right) = \lim_{z\to\frac{\pi}{2}} \frac{z-\frac{\pi}{2}}{(z+1)^3\left(z-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\left(z-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)} + \frac{z-\frac{\pi}{2}}{\cos z} = -1$$

Thus $\int_\Gamma f(z)\,dz = 2\pi i\left(-\frac{1}{9}-\frac{1}{9}-1\right) = -\frac{22}{9}\pi i$