5.1.

(a) $\sum_{j=0}^{\infty} \left(\frac{i}{3}\right)^j$

since $\left|\frac{i}{3}\right| = \frac{1}{3} < 1$, $\sum_{j=0}^{\infty}\left(\frac{i}{3}\right)^j = \frac{1}{1-\frac{i}{3}} = \frac{3}{3-i}$

(b) $\sum_{k=0}^{\infty} \frac{3}{(1+i)^k} = 3\sum_{k=0}^{\infty}\left(\frac{1}{1+i}\right)^k = 3\sum_{k=0}^{\infty}\left(\frac{1}{2}-\frac{i}{2}\right)^k$

since $\left|\frac{1}{2}-\frac{i}{2}\right| = \frac{\sqrt{2}}{2} < 1$, $\sum_{k=0}^{\infty}\frac{3}{(1+i)^k} = 3\cdot\frac{1}{1-\frac{1}{2}+\frac{i}{2}} = 3\cdot\frac{1}{\frac{1}{2}+\frac{i}{2}} = 3-3i$

(d) since $\left|\frac{1}{2i}\right| = \frac{1}{2} < 1$, we have $\sum_{k=0}^{13}\left(\frac{1}{2i}\right)^k = \frac{1-\left(\frac{1}{2i}\right)^{14}}{1-\frac{1}{2i}}$

Then $\sum_{k=14}^{\infty}\left(\frac{1}{2i}\right)^k = \sum_{k=0}^{\infty}\left(\frac{1}{2i}\right)^k - \sum_{k=0}^{13}\left(\frac{1}{2i}\right)^k$

$$= \frac{1}{1-\frac{1}{2i}} - \frac{1-\left(\frac{1}{2i}\right)^{14}}{1-\frac{1}{2i}} = \frac{\left(\frac{1}{2i}\right)^{14}}{1-\frac{1}{2i}}$$

$$= \left(\frac{i}{2}\right)^{14}\cdot\left(1+\frac{i}{2}\right)$$

$$= \left(\frac{i}{2}\right)^{14} + \left(\frac{i}{2}\right)^{15}$$

8. (a) since $\sum_{j=0}^{\infty} c_j = S$, we have

$\forall \varepsilon > 0\ \exists N$ s.t. $\forall n > N$ $\left|\sum_{j=0}^{n} c_j - S\right| < \varepsilon$

Then $\left|\sum_{j=0}^{n}\overline{c_j} - \overline{S}\right| = \left|\overline{\sum_{j=0}^{n} c_j} - \overline{S}\right| = \left|\overline{\sum_{j=0}^{n} c_j - S}\right| = \left|\sum_{j=0}^{n} c_j - S\right| < \varepsilon$

Thus $\sum_{j=0}^{\infty}\overline{c_j} = \overline{S}$

"$\Rightarrow$"

9. Since $\sum_{j=0}^{\infty} z_j = S$ and $\sum_{j=0}^{\infty}\overline{z_j} = \overline{S}$, then we have.

$\sum_{j=0}^{\infty}\mathrm{Re}(z_j) = \sum_{j=0}^{\infty}\frac{1}{2}(z_j+\overline{z_j}) = \frac{1}{2}(S+\overline{S})$

$\sum_{j=0}^{\infty}\mathrm{Im}(z_j) = \sum_{j=0}^{\infty}\frac{1}{2i}(z_j-\overline{z_j}) = \frac{1}{2i}(S-\overline{S})$

Thus $\sum_{j=0}^{\infty}\mathrm{Re}(z_j)$ and $\sum_{j=0}^{\infty}\mathrm{Im}(z_j)$ are convergent.

"$\Leftarrow$" since $\sum_{j=0}^{\infty} Re(z_j)$ and $\sum_{j=0}^{\infty} Im(z_j)$ are convergent.

$\sum_{j=0}^{\infty} Re(z_j) = A$ and $\sum_{j=0}^{\infty} Im(z_j) = B$

Thus $\sum_{j=0}^{\infty} z_j = \sum_{j=0}^{\infty} \cancel{A+Bi}(Re(z_j) + i\, Im(z_j)) = A + Bi$, implies

$\sum_{j=0}^{\infty} z_j$ is also convergent.

11.(a) $\lim_{j\to\infty} \left| \frac{(j+1)z^{j+1}}{j z^j} \right| = \lim_{j\to\infty} \left| \frac{j+1}{j} z \right| = |z|$

Then $\sum_{j=0}^{\infty} j z^j$ is convergent when $|z|<1$.

(b) $\lim_{k\to\infty} \left| \frac{(z-i)^{k+1}/z^{k+1}}{(z-i)^k/z^k} \right| = \left| \frac{\cancel{z+i}\ z-i}{z} \right|$

Then $\sum_{k=0}^{\infty} \frac{(z-i)^k}{z^k}$ is convergent if $\left|\frac{z-i}{z}\right| < 1$, which is $|z-i|<2$

(c) $\lim_{j\to\infty} \left| \frac{z^{j+1}/(j+1)!}{z^j/j!} \right| = \left| \frac{z}{j+1} \right| = 0 < 1$

Thus $\sum_{j=0}^{\infty} \frac{z^j}{j!}$ is convergent for all $z \in \mathbb{C}$

(d) $\lim_{k\to\infty} \left| \frac{(z+5i)^{\cancel{2k}2k+2}\ (k+2)^2}{(z+5i)^{2k}\ (k+1)^2} \right| = |(z+5i)^2| = |z+5i|^2$

Then $\sum_{k=0}^{\infty} (z+5i)^{2k}(k+1)^2$ is convergent if $|z+5i|^2<1$, which is $|z+5i|<1$

18. $\forall z \in T \quad |F(z)| \geq \rho > 0$

Then by definition 2, let $\varepsilon = \frac{\rho}{2}$ $\exists N$ s.t. $\forall n > N$

$|F_n(z) - F(z)| < \varepsilon = \frac{\rho}{2}$

Then $|F_n(z)| = |F_n(z) - F(z) + F(z)| \geq |F(z)| - |F_n(z) - F(z)|$

$\geq \rho - \frac{\rho}{2} = \frac{\rho}{2}$

Hence $|F_n(z)| \geq \frac{\rho}{2}$

5.2.

1. (a) $f^{(j)}(0) = (-1)^j e^0 = (-1)^j$

Then $f(z) = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} f''(0) + \cdots$

$= 1 + \frac{z}{1!}(-1) + \frac{z^2}{2!}(-1)^2 + \frac{z^3}{3!}(-1)^3 + \cdots$

$= \sum_{j=0}^{\infty} \frac{(-z)^j}{j!}$

(b) $f(z) = \cosh(z) \Rightarrow f^{(j)}(z) = \begin{cases} \sinh(z) & j \text{ is odd} \\ \cosh(z) & j \text{ is even.} \end{cases}$

Then $f^{(j)}(0) = \begin{cases} 0 & j \text{ is odd} \\ 1 & j \text{ is even.} \end{cases}$

$f(z) = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} f''(0) + \frac{z^3}{3!} f'''(0) \cdots$

$= 1 + 0 + \frac{z^2}{2!} + 0 + \frac{z^4}{4!} \cdots$

$= \sum_{j=0}^{\infty} \frac{z^{2j}}{(2j)!}$

(e) $f(z) = \text{Log}(1-z) \Rightarrow f^{(j)}(0) = -\frac{(j-1)!}{(1-0)^j} = -(j-1)!$

$f(z) = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} f''(0) + \frac{z^3}{3!} f'''(0) + \cdots$

$= 0 + \frac{z}{1!}(-1) + \frac{z^2}{2!}(-1) \cdot 1! + \frac{z^3}{3!}(-1) \cdot 2! + \cdots = \sum_{j=1}^{\infty} -\frac{z^j}{j}$

4. $(1+z)^{\alpha} = e^{\alpha \operatorname{Log}(1+z)}$

since $e^{z} = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ we have.

$$(1+z)^{\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha \operatorname{Log}(1+z))^n}{n!} \quad ①$$

Since $\operatorname{Log}(1+z) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n}$, plug it into ①

$$(1+z)^{\alpha} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!}\left(\sum_{k=1}^{\infty}(-1)^{k+1}\frac{z^k}{k}\right)^n$$

$$= 1+\alpha\left(z-\frac{z^2}{2}+\frac{z^3}{3}+\cdots\right)+\frac{\left(\alpha\left(z-\frac{z^2}{2}+\frac{z^3}{3}-\cdots\right)\right)^2}{2!}+\cdots$$

$$= 1+\alpha z+\frac{z^2}{2}(\alpha^2-\alpha)+\frac{z^3}{2\cdot 3}\alpha(\alpha-1)(\alpha-2)+\cdots$$

$$= 1+\alpha z+\frac{z^2}{2}(\alpha-1)\alpha+\frac{z^3}{2\cdot 3}\alpha(\alpha-1)(\alpha-2)+\cdots$$

Hence proved.

7. $\operatorname{Log}\left(\frac{1+z}{1-z}\right) = \log\left(\frac{1+z}{1-z}\right) + i \operatorname{Arg}\left(\frac{1+z}{1-z}\right)$

$$= \log(1+z) - \log(1-z) + i \operatorname{Arg}\left(\frac{1+z}{1-z}\right)$$

Since $\operatorname{Arg}\left(\frac{1+z}{1-z}\right) = 0$ when $|z|<1$

we have $\operatorname{Log}\left(\frac{1+z}{1-z}\right) = \operatorname{Log}(1+z) - \operatorname{Log}(1-z)$

$$\operatorname{Log}(1+z) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1} z^j}{j} \qquad \operatorname{Log}(1-z) = \sum_{j=1}^{\infty} -\frac{z^j}{j}$$

Then $\operatorname{Log}\left(\frac{1+z}{1-z}\right) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1} z^j}{j} - \frac{z^j}{j} = 2\sum_{j=1}^{\infty} \frac{z^{2j-1}}{2j-1}$

13. $(1-z)^{-1} = \sum_{k=0}^{\infty} z^k$

$(1-z)^{-2} = \sum_{k=0}^{\infty} k z^{k-1}$ differentiate both sides

$z(1-z)^{-2} = \sum_{k=0}^{\infty} k z^k$

differentiate both sides

$(1-z)^{-2} + z(-2)(1-z)^{-3}(-1) = \sum_{k=0}^{\infty} k^2 z^{k-1}$

$z(1-z)^{-2} + 2z^2(1-z)^{-3} = z(1+z)(1-z)^{-3} = \sum_{k=0}^{\infty} k^2 z^k$

Thus $\sum_{k=0}^{\infty} k^2 z^k = z(1+z)(1-z)^{-3}$

14. $f^{(k)}(z_0) = 0$, $k = 0, 1, 2 \ldots$ in D

Taylor expand $f(z)$, we have $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$

plug in $f^{(k)}(z_0) = 0$

$f(z) = \sum_{k=0}^{\infty} \frac{0}{k!} \cdot (z-z_0)^k = 0$ for all z in D

15. $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k = f(0) + \frac{f'(0)}{1!} z + \frac{f''(0)}{2!} z^2 + \cdots$

Since $f'(0) = f'''(0) = f^{(5)}(0) = \cdots = 0$

$= f(0) + \frac{f''(0)}{2!} z^2 + \frac{f^{(4)}(0)}{4!} z^4 + \cdots$

$= \sum_{j=0}^{\infty} \frac{f^{(2j)}(0)}{(2j)!} z^{2j}$

Thus $f(-z) = \sum_{j=0}^{\infty} \frac{f^{(2j)}(0)}{(2j)!} (-z)^{2j} = \sum_{j=0}^{\infty} \frac{f^{(2j)}(0)}{(2j)!} z^{2j} = f(z)$

Thus $f(z) = f(-z)$

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5.3.

2. $\lim_{j\to\infty}\left|\frac{a_{j+1}}{a_j}\right| = L$

Then $\lim_{j\to\infty}\left|\frac{a_{j+1}(z-z_0)^{j+1}}{a_j(z-z_0)^j}\right| = \lim_{j\to\infty}\left|\frac{a_{j+1}}{a_j}(z-z_0)\right| = L|z-z_0|$

Then $\sum_{j=0}^{\infty} a_j(z-z_0)^j$ is convergent if $L|z-z_0|<1 \Rightarrow |z-z_0|<\frac{1}{L}$

Hence radius of convergence is $R=\frac{1}{L}$

3.(a) $a_j = j^3$ $z_0=0$

Then $L=\lim_{j\to\infty}\left|\frac{(j+1)^3}{j^3}\right| = 1$, by problem 2, we have $R=\frac{1}{L}=1$
~~and~~
Hence $\sum_{j=0}^{\infty} j^3 z^j$ is convergent when $|z|<1$

(c) ~~$a_j = (3-i)^j/j^2$~~
$a_j = j!$ $z_0=0$
Then $\lim_{j\to\infty}\left|\frac{(j+1)!}{j!}\right| = \lim_{j\to\infty}|j+1| = \infty \Rightarrow R=\frac{1}{L}=0$

Thus $\sum_{j=0}^{\infty} j!z^j$ will never converge.

(e) $a_j = (3-i)^j/j^2$ $z_0=-2$

$L=\lim_{j\to\infty}\left|\frac{(3-i)^{j+1}/(j+1)^2}{(3-i)^j/j^2}\right| = |3-i| = \sqrt{10}$

Then $R=\frac{1}{L}=\frac{1}{\sqrt{10}}$
Thus $\sum_{k=0}^{\infty}\frac{(3-i)^k}{k^2}(z+2)^k$ is convergent when $|z+2|<\frac{1}{\sqrt{10}}$

5. (a) $\frac{j^3}{3^j} = \frac{f^{(j)}(0)}{j!}$

Then $f^{(j)}(0) = \frac{j^3}{3^j} j!$

$$f^{(6)}(0) = \frac{6^3}{3^6} 6! = \frac{640}{3}$$

(b) by Cauchy integral formula

$$\oint_{|z|=1} \frac{f(z)}{z^4} dz = \frac{2\pi i}{3!} f^{(3)}(0)$$

$$= \frac{2\pi i}{3!} \frac{3^3}{3^3} 3! = 2\pi i$$

(c) $\oint_{|z|=1} e^z f(z) dz = 0$ since e^z and $f(z)$ are analytic in domain $|z| \le 1$

(d) $\oint_{|z|=1} \frac{f(z)\sin z}{z^2} dz = \frac{2\pi i}{1!} (f(z)\sin(z))'\big|_{z=0}$ by Cauchy integral formula.

$$= \frac{2\pi i}{1!} (f(z)\cos z + f'(z) \sin z)\big|_{z=0}$$

$\left.\begin{array}{l} f(0)=0 \\ \sin 0 = 0 \end{array}\right\} \Rightarrow$

$$= 0$$

6. (a) Since $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$

Then when $z \neq 0$

$f(z) = \frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$

when $z = 0$

$f(z) = 1 = 1 - \frac{0^2}{3!} + \frac{0^4}{5!} - \frac{0^6}{7!} + \cdots$

Hence, $f(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$ proved.

(b) $f(0)=1$

$f(z)$ is differentiable when $z\neq 0$ since $\frac{\sin z}{z}$ is differentiable when $z\neq 0$

$$f'(0) = \lim_{\Delta z\to 0} \frac{f(\Delta z)-f(0)}{\Delta z} = \lim_{\Delta z\to 0}\left(\frac{1-\frac{\Delta z^2}{3!}+\frac{\Delta z^4}{5!}-\frac{\Delta z^6}{7!}\cdots -1}{\Delta z}\right)$$

$$= \lim_{\Delta z\to 0}\left(-\frac{\Delta z}{3!}+\frac{\Delta z^3}{5!}-\frac{\Delta z^5}{7!}+\cdots\right)$$

$$=0$$

Since $f'(0)$ is defined. $f(z)$ is analytic at origin.

(c) Taylor expand $f(z)$ at $z=0$

$$f(z) = 1+\frac{f'(0)}{1!}z+\frac{f^{(2)}(0)}{2!}z^2+\frac{f^{(3)}(0)}{3!}z^3+\frac{f^{(4)}(0)}{4!}z^4+\cdots$$

Compare it with part (a)'s formula

we have $f^{(3)}(0)=0$ $\quad f^{(4)}(0)=\frac{1}{5!}4! = \frac{1}{5}$

9. Let $\{f_n(z)\}$ be the sequence converge uniformly to g on C

Then $\int_C f_n(z)\,dz = 0$ for all n

Thus $\oint g(z)\,dz = \lim_{n\to\infty}\int f_n(z)\,dz = 0$ since $\{f_n(z)\}$ converge uniformly to g.

10. Let $R = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|$ be the radius of convergence of $\sum_{k=0}^{\infty} a_k z^k$

Then $\lim_{k\to\infty}\left|\frac{(k+1)a_{k+1}z^k}{k\,a_k z^{k-1}}\right| = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| \lim_{k\to\infty}\left|\frac{k+1}{k}\right| = R.$

Thus $\sum_{k=0}^{\infty} a_k z^k$ and $\sum_{k=0}^{\infty} k a_k z^{k-1}$ have same radius of convergence.

5.5.

1. (a) $\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n$ if $|w|<1$

Then $f(z) = \frac{1}{z+z^2} = \frac{1}{z} \cdot \frac{1}{1+z} = \frac{1}{z} \cdot \frac{1}{1-(-z)}$

since $|-z| = |z| < 1$, we have $f(z) = \frac{1}{z} \cdot \sum_{n=0}^{\infty} (-z)^n$

$$= \sum_{n=0}^{\infty} (-1)^n z^{n-1}$$

(b)

$$f(z) = \frac{1}{z^2} \frac{1}{\frac{1}{z}+1} = \frac{1}{z^2}\left(\frac{1}{1-(-\frac{1}{z})}\right)$$

Since $|-\frac{1}{z}| = |\frac{1}{z}| = \frac{1}{|z|} < 1$, we have

$$f(z) = \frac{1}{z^2} \sum_{n=0}^{\infty} (-\tfrac{1}{z})^n = \sum_{n=0}^{\infty} (-1)^n z^{-n-2}$$

(c) $f(z) = -\frac{1}{1+z} \cdot \frac{1}{1-(z+1)}$

since $|z+1|<1$, we have $f(z) = -\frac{1}{1+z} \sum_{n=0}^{\infty} (z+1)^n$

$$= \sum_{n=0}^{\infty} -(z+1)^{n-1}$$

(d) $f(z) = \frac{1}{(z+1-1)(1+z)} = \frac{1}{(z+1)^2} \cdot \frac{1}{1-\frac{1}{z+1}}$

since $|\frac{1}{z+1}| = \frac{1}{|z+1|} < 1$

$$f(z) = \frac{1}{(z+1)^2} \cdot \sum_{n=0}^{\infty} \left(\frac{1}{z+1}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{z+1}\right)^{n+2}$$

3. (a) $|z|<1 \Rightarrow |-z| = |z| < 1$

$$f(z) = \frac{z}{z-2} \cdot \frac{1}{1-(-z)} = \frac{z}{z-2} \sum_{n=0}^{\infty} (-z)^n = \sum_{n=0}^{\infty} \frac{(-1)^n z^{n+1}}{z-2}$$

(b) since $|z|<2 \Rightarrow |\frac{z}{2}|<1$

$$f(z) = -\frac{z}{z+1} \cdot \frac{1}{2} \cdot \frac{1}{1-\frac{z}{2}} = -\frac{z}{z+1} \cdot \frac{1}{2} \sum_{n=0}^{\infty} (\tfrac{z}{2})^n = \sum_{n=0}^{\infty} -(\tfrac{1}{2})^{n+1} z^{n+1} \frac{1}{z+1}$$

(c) $2 < |z| \Rightarrow \left|\frac{2}{z}\right| = \frac{2}{|z|} < 1$

Then $f(z) = \frac{z}{z+1} \cdot \frac{1}{z-2} = \frac{1}{z+1} \cdot \frac{1}{1-\frac{2}{z}}$

Then $f(z) = \frac{1}{z+1} \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n$

$= \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n \cdot \frac{1}{z+1}$

4. when $|z| > 0$, $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} \cdots\cdots$

Then $\sin 2z = 2z - \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} \cdots\cdots$

Then $\frac{\sin 2z}{z^3} = \frac{2}{z^2} - \frac{2^3}{3!} + \frac{2^5 z^2}{5!} - \frac{2^7 z^4}{7!} \cdots\cdots$

5. $|z-4| < 4 \Rightarrow \left|\frac{z-4}{4}\right| < 1$

Then $f(z) = -\frac{z+1}{(z-4)^3} \cdot \frac{1}{4-(z-4)}$

$= -\frac{z+1}{(z-4)^3} \cdot \frac{\frac{1}{4}}{1-\frac{z-4}{4}}$

$= -\frac{z+1}{4(z-4)^3} \sum_{n=0}^{\infty} \left(\frac{z-4}{4}\right)^n$

$= \sum_{n=0}^{\infty} -(z+1)(z-4)^{n-3}\left(\frac{1}{4}\right)^{n+1}$

9. $\sum_{j=-\infty}^{\infty} \frac{z^j}{2^{|j|}} = \sum_{j=1}^{\infty} \frac{1}{2^j z^j} + \sum_{j=0}^{\infty} \frac{z^j}{2^j}$

Then $\sum_{j=-\infty}^{\infty} \frac{z^j}{2^{|j|}}$ converges if $\left|\frac{1}{2z}\right| < 1$ and $\left|\frac{z}{2}\right| < 1$

Thus $\frac{1}{2} < |z| < 2$

5.6.

1. (a) $z^2(z+1)=0 \Rightarrow z=0, -1$

for $z=0$. $f(z)=g(z)\frac{1}{z^2}$ with $g(z)=\frac{z^3+1}{z+1}$ is analytic and $g(0)\neq 0$.

Thus $z=0$ is a pole of order 2 of $f(z)$

for $z=-1$ $f(z)=\frac{z^3+1}{z^2}\left(\frac{1}{1+z}\right) = \frac{z^3+1}{z^2}\sum_{n=0}^{\infty}(-z)^n$

Thus $f(z)\to 0$ as $z\to -1$, $\therefore z=-1$ is removable singularity of $f(z)$

~~(b)~~(c) $\frac{\cos z}{z^2+1}+4z$.

$z^2+1=0 \Rightarrow z=\pm i$.

for $z=i$ $f(z)=\frac{\cos z}{z+i}\cdot\frac{1}{z-i}+4z$ since $g(z)=\frac{\cos z}{z+i}$ is analytic and $g(i)\neq 0$

~~we have~~

we have $z=i$ is pole of order 1 of $f(z)$

Similarly, $z=-i$ is also a pole of order 1 of $f(z)$

(d) $\frac{1}{e^z-1}$

$e^z-1=0 \Rightarrow z=2n\pi i \quad n\in\mathbb{Z}$

since $f'(2n\pi i)=e^{2n\pi i}=1\neq 0$

$z=2n\pi i$, $n\in\mathbb{Z}$ are all pole of order 1 of $f(z)$

(f) $\cos(1-\frac{1}{z}) = 1-\frac{1}{2!}(1-\frac{1}{z})^2+\frac{1}{4!}(1-\frac{1}{z})^4+\cdots$

$z=0$

There are infinitly many negative power of z in $f(z)$

Thus $z=0$ is a essential singularity of $f(z)$

2. $\frac{1}{f(z)} = (z\cos z - z + z^2)^2$

$= \left[z\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!}\right) - z + z^2\right]^2 = 0$

$(z^4)^2 \left(\frac{z}{4!} - \frac{z z^2}{6!} + \cdots \right)^2 = 0$

Then $z=0$ is pole of order 8 of $f(z)$

3. (a) $f(z) = \frac{1}{(z-i)^2 (z-2+3i)^5}$

(b) $f(z) = z\cos\left(1 - \frac{1}{1-z}\right)$

(c) $f(z) = \frac{\sin\left(1 - \frac{1}{z-1}\right)\sin z}{z(z-1)^6}$

(d) $f(z) = \frac{\sin\left(1 - \frac{1}{z-1}\right)\sin\left(1 - \frac{1}{z}\right)}{(z-1-i)^2}$

5. (a) F

(b) T

(c) T

(d) F

(e) T

6. If $f(z)$ has a pole of order m

Then $f(z) = (z-z_0)^{-m} g(z)$ for $g(z_0) \neq 0$

Then $f'(z) = -m(z-z_0)^{-m-1} g(z) + g'(z)(z-z_0)^{-m}$

$= (z-z_0)^{-m-1}\left(-g(z)m + (z-z_0)g'(z)\right)$

let $= (z-z_0)^{-m-1} h(z)$

since $h(z_0) \neq 0$, $f'(z)$ has a pole of order $m+1$