University of California, Los Angeles
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Date: July 22, 2020

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MATH 132: Complex Analysis
Midterm 2

This exam contains 7 pages (including this cover page) and 5 problems. Your solutions to the problems must be uploaded to Gradescope before 8am PST on July 23, 2020.

This is a take-home exam. The following rules regarding the take-home format apply:

- The exam is open-book/open-notes/open-internet exam.

  You **cannot** collaborate in any way with any individual on the exam. Any form of communication/consultation/collaboration with another person about the exam is expressly prohibited—this includes, but is not limited to, Zoom meetings, email, telephone calls, texting, making posts on stack exchanges, etc. Violation of the no-collaboration policy is a violation of the UCLA code of student conduct and will come with serious consquences.

- The instructor reserves the right to ask any student for clarification regarding any of the student's exam answers at any time during a two week period after the day of the exam. This may require a Zoom meeting with the instructor.

- Please sign the pledge on the next page and upload an image of the signed pledge onto Gradescope when uploading your exam.

You are required to show your work on each problem of this exam. The following rules apply:

- **All answers must be justified. Mysterious or unsupported answers will not receive credit.** A correct answer, unsupported by calculations and/or explanation will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

- Although you may use a calculator, it is not necessary. If you do use one, you still need to show your work and justify the computation, as you would on a test without calculators. Be aware that calculators often produce rational approximations to numerical answers rather than the precisely correct answer, e.g., $\pi \neq 3.14$.

- **If you use a theorem or proposition from class or the notes or the textbook or a result established in the homework, you must indicate this** and explain why the theorem may be applied.

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.

Good luck!

Academic integrity pledge:

Upon my honor, I affirm that I did not solicit nor did I receive the help of any individual in writing my answers to this exam.

Signature: [signature]

Print name: JINGYI ZUO

1. Let $p(z)$ be the principal branch of z^{1-i}. Let $D^* = \mathbb{C} \setminus (-\infty, 0]$ be all the complex numbers except for the non-positive real numbers.

   (a) (4 points) Find a function which is an antiderivative of $p(z)$ on D^*.

   (b) (6 points) Let Γ be a contour such that (i) Γ is contained in D^* and (ii) the initial point of Γ is 1 and the terminal point of Γ is i. Compute

$$\int_\Gamma p(z)dz.$$

Justify your answers.

(a) let $q(z) = \dfrac{z^{2-i}}{2-i}$

The $q(z)' = \dfrac{z^{2-i}}{2-i}' = (2-i)(z^{2-i})' = (2-i)\left(e^{(2-i)\log z}\right)'$

$= (2-i)\, e^{(2-i)\log z} \cdot \dfrac{2-i}{z}$

$= (2-i) \cdot z^{2-i} \cdot \dfrac{2-i}{z} = z^{1-i}$

Thus $q(z)' = p(z)$ on D^*

(b) By theorem 30. in leture, since $p(z)$ is continuous on D^* and $q(z)$ is its antiderivative

$\int_\Gamma p(z)dz = q(i) - q(1)$

$= \dfrac{i^{2-i}}{2-i} - \dfrac{1^{2-i}}{2-i}$

$= \dfrac{1}{2-i}\left(i^{2-i} - 1^{2-i}\right)$

$= \dfrac{1}{2-i}\left(e^{(2-i)\log i} - e^{(2-i)\log 1}\right)$ since we are taking the principal branch

$= \dfrac{1}{2-i}\left(e^{(2-i)\frac{\pi i}{2}} - 1\right)$

2. Let $f(z)$ be the function
$$f(z) = \frac{2}{z+i} + \frac{3}{z-i}.$$

(a) (4 points) Find a directed smooth curve γ_1 such that $\int_{\gamma_1} f(z)dz = 4\pi i$.

(b) (4 points) Find a directed contour Γ_2 such that $\int_{\Gamma_2} f(z)dz = -12\pi i$.

(c) (4 points) Find a directed smooth curve γ_3 such that $\int_{\gamma_3} f(z)dz = 10\pi i$.

Justify your answers (and don't forget to specify the direction of each of your smooth curves/contour).

(a) $\gamma_1 : |z+i| = 1$ in counterclockwise direction

$$\int_{\gamma_1} f(z)dz = 2\int_{\gamma_1} \frac{1}{z+i}dz + 3\int_{\gamma_1} \frac{1}{z-i}dz$$

since $-i$ is in $|z+i|=1$ and i is outside of $|z+i|=1$

$$\int_{\gamma} f(z)dz = 2\cdot 2\pi i + 3\cdot 0 = 4\pi i$$

(b) Γ_2 : traverse $|z-i|=1$ twice in clockwise direction

$$\int_{\Gamma_2} f(z)dz = 2\int_{\Gamma_2} \frac{1}{z+i}dz + 3\int_{\Gamma_2} \frac{1}{z-i}dz$$

since i is in $|z-i|=1$ and $-i$ is not inside of $|z-i|=1$

$$\int_{\Gamma_2} f(z)dz = 2\cdot 0 + 3\cdot 2(-2\pi i) = -12\pi i$$

(c) $\gamma_3 : |z| = 2$ in counterclockwise direction

$$\int_{\gamma_3} f(z)dz = 2\int_{\gamma_3} \frac{1}{z+i}dz + 3\int_{\gamma_3} \frac{1}{z-i}dz$$

since both i and $-i$ are in $|z|=2$

$$\int_{\gamma_3} f(z)dz = 2\cdot 2\pi i + 3\cdot 2\pi i = 10\pi i$$

3. Let γ_1, γ_2 and γ_3 be the smooth curves with admissible parameterizations

$$z_1(t) = e^{it}, \quad 0 \le t \le \pi,$$
$$z_2(t) = -1 + (1-i)t, \quad 0 \le t \le 1$$
$$z_3(t) = 1 - (1+i)t, \quad 0 \le t \le 1,$$

respectively. Assume that γ_1, γ_2, and γ_3 are directed in a way consistent with the parameterizations $z_1(t), z_2(t)$, and $z_3(t)$, respectively.

(a) (3 points) Sketch the directed curves $\gamma_1, \gamma_2, \gamma_3$ in the complex plane and draw arrows on each curve to indicate the direction of the curve.

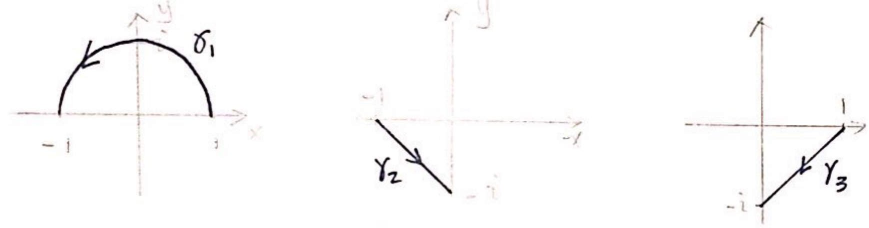
(b) (4 points) Compute

$$\int_{\gamma_2} \frac{1}{z} dz.$$

(c) (5 points) Let $\Gamma = \gamma_1 + \gamma_2 - \gamma_3$. Compute

$$\int_\Gamma \frac{e^{2z}}{(z-2i)(2z-1)^2}.$$

(a)



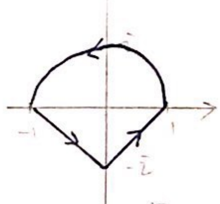
(b)

$$\int_{\gamma_2} \frac{1}{z} dz = \int_0^1 \frac{1}{-1+(1-i)t} \cdot z_2'(t) dt$$

$$= \int_0^1 \frac{1}{-1+(1-i)t} \cdot (1-i) dt$$

$$= \log(-i + (1+i)t)\Big|_0^1 = \log(1) - \log(-i) = \frac{\pi i}{2}$$

(c)



$\Gamma = \gamma_1 + \gamma_2 - \gamma_3$ is a closed curve on $\mathbb{C}$ and it's simply closed

$$\int_\Gamma \frac{e^{2z}}{(z-2i)(2z-1)^2} = \int_\Gamma \frac{e^{2z}/(z-2i)}{(2z-1)^2} dz = \int_\Gamma \frac{\frac{1}{4}e^{2z}/(z-2i)}{(z-\frac{1}{2})^2} dz.$$

By Generalized Cauchy's formula, since $z=\frac{1}{2}$ is in the interior of Γ

$$\int_\Gamma \frac{\frac{1}{4}e^{2z}/(z-2i)}{(z-\frac{1}{2})^2} dz = \frac{2\pi i}{1!} \left(\tfrac{1}{4}e^{2z}/(z-2i)\right)'\Big|_{z=\frac{1}{2}}$$

$$= \frac{\pi i}{2}\left(\frac{e^{2z}(2z-1-4i)}{(z-2i)^2}\right)\Big|_{z=\frac{1}{2}} = \frac{\pi i}{2} \cdot \frac{e(-4i)}{(\frac{1}{2}-2i)^2} = \frac{2\pi e}{(\frac{1}{2}-2i)^2}$$

4. (6 points) Let C be the circle of radius 2 centered at the origin, oriented positively. Suppose f is continuous on C and that $|f(z)| \le 2$ for all z on C. Prove the upper bound

$$\left| \int_C \frac{e^{f(z)}}{z^2+2} dz \right| \le 2\pi e^2.$$

Since $|f(z)| \le 2$, we have $|e^{f(z)}| \le e^2$

by triangle inequality, $|z^2+2-2| \le |z^2+2| + |2|$

Thus $|z^2| \le |z^2+2| + 2$

$|z^2+2| \ge |z^2| - 2 = |z|^2 - 2 = 2^2 - 2 = 2$ since $C: |z| = 2$

Then we have $\frac{1}{|z^2+2|} \le \frac{1}{2}$

$$\left| \frac{e^{f(z)}}{z^2+2} \right| = \frac{|e^{f(z)}|}{|z^2+2|} \le \frac{e^2}{2}$$

Then by the property of contour integral

$$\left| \int_C \frac{e^{f(z)}}{z^2+2} dz \right| \le \frac{e^2}{2} \cdot \ell(C)$$

since $C: |z| = 2$ $\ell(C) = 2\pi \cdot 2 = 4\pi$

$$\left| \int_C \frac{e^{f(z)}}{z^2+2} dz \right| \le \frac{e^2}{2} \cdot 4\pi = 2\pi e^2$$

5. (a) (4 points) Let $h(z)$ be an entire function. Suppose there exists a real $M > 0$ such that $|\mathrm{Im}\, h(z)| \leq M$ for all $z \in \mathbb{C}$. Prove that $h(z)$ is a constant function.

(b) (6 points) Let $f(z)$ and $g(z)$ be entire functions such that $|f(z)| < |g(z)|$ for all $z \in \mathbb{C}$. Prove that there is a constant $c \in \mathbb{C}$ with $|c| < 1$ such that $f(z) = cg(z)$ for all $z \in \mathbb{C}$.

(a) let $h(z) = u(x,y) + iv(x,y)$ $\quad |\mathrm{Im}\, h(z)| = |v(x,y)| \leq M$

Then $\left|e^{-ih(z)}\right| = \left|e^{v(x,y) - iu(x,y)}\right| = \left|e^{v(x,y)}\right| \cdot \left|e^{-iu(x,y)}\right|$

$\leq |e^M| = e^M$

Since $\left|e^{-ih(z)}\right|$ is bounded by e^M, then by Liouville's Theorem $e^{-ih(z)}$ is a constant

Hence $h(z)$ is also a constant.

(b)

Since $|f(z)| < |g(z)|$, we have $\left|\frac{f(z)}{g(z)}\right| = \frac{|f(z)|}{|g(z)|} < 1$

let $p(z) = \frac{f(z)}{g(z)}$, since $f(z), g(z)$ are entire functions on $\mathbb{C}$

$p(z)$ is also an entire fuction. and $p(z)$ is bounded by 1

Thus by Liouville's theorem $p(z)$ is a constant on $\mathbb{C}$

$p(z) = c = \frac{f(z)}{g(z)}$ and $|c| = \left|\frac{f(z)}{g(z)}\right| < 1$ on $\mathbb{C}$

Hence $\exists\, c \in \mathbb{C}$ with $|c| < 1$ such that $f(z) = cg(z)$ for all $z \in \mathbb{C}$