University of California, Los Angeles
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Instructor: T. Arant
Date: July 8, 2020

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MATH 132: Complex Analysis
Midterm 1

This exam contains 7 pages (including this cover page) and 5 problems. Your solutions to the problems must be uploaded to Gradescope before 8am PST on July 9, 2020.

This is a take-home exam. The following rules regarding the take-home format apply:

- The exam is open-book/open-notes/open-internet exam.

  You **cannot** collaborate in any way with any individual on the exam. Any form of communication/consultation/collaboration with another person about the exam is expressly prohibited—this includes, but is not limited to, Zoom meetings, email, telephone calls, texting, making posts on stack exchanges, etc. Violation of the no-collaboration policy is a violation of the UCLA code of student conduct and will come with serious consquences.

- The instructor reserves the right to ask any student for clarification regarding any of the student's exam answers at any time during a two week period after the day of the exam. This may require a Zoom meeting with the instructor.

- Please sign the pledge on the next page and upload an image of the signed pledge onto Gradescope when uploading your exam.

You are required to show your work on each problem of this exam. The following rules apply:

- **All answers must be justified. Mysterious or unsupported answers will not receive credit.** A correct answer, unsupported by calculations and/or explanation will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

- Although you may use a calculator, it is not necessary. If you do use one, you still need to show your work and justify the computation, as you would on a test without calculators. Be aware that calculators often produce rational approximations to numerical answers rather than the precisely correct answer, e.g., $\pi \neq 3.14$.

- **If you use a theorem or proposition from class or the notes or the textbook or a result established in the homework, you must indicate this** and explain why the theorem may be applied.

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.

Good luck!

Academic integrity pledge:

Upon my honor, I affirm that I did not solicit nor did I receive the help of any individual in writing my answers to this exam.

Signature: ____________

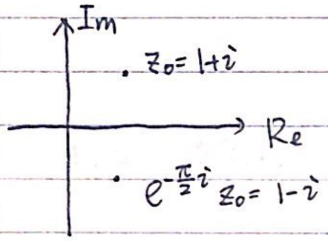
Print name: JINGYI ZUO

1. (a)

$$e^{-\frac{\pi}{2}i} z_0$$

$$= e^{-\frac{\pi i}{2}} (1+i) = \left(\cos(-\tfrac{\pi}{2}) + i\sin(-\tfrac{\pi}{2})\right)(1+i)$$

$$= -i\,(1+i) = 1-i$$



(b) $f(z) = e^{i\theta_0} z$

since $z = |z| \cancel{\cos\theta} (\cos\phi + i\sin\phi)$ $\quad \phi = \arg z$

Then $f(z) = e^{i\theta_0} |z| (\cos\phi + i\sin\phi)$

$$= |z|\, e^{i\theta_0} e^{i\phi}$$

$$= |z|\, e^{i(\theta_0+\phi)}$$

$$= |z| \left(\cos(\theta_0+\phi) + i\sin(\theta_0+\phi)\right)$$

Thus $f(z)$ is obtained by rotating z by angle θ_0.

2. (a) $-1+i = \sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = \sqrt{2}\,(\cos\theta + i\sin\theta) \Rightarrow \theta = \frac{3}{4}\pi$

$= \sqrt{2}\,\left(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi\right)$

$= \sqrt{2}\,\left(\cos\left(\frac{3}{4}\pi + 2k\pi\right) + i\sin\left(\frac{3}{4}\pi + 2k\pi\right)\right) \quad k \in \mathbb{Z}$

$= \sqrt{2}\, e^{\left(\frac{3}{4}\pi + 2k\pi\right)i}$

Then $\log(-1+i) = \log\sqrt{2} + \left(\frac{3}{4}\pi + 2k\pi\right)i \quad k \in \mathbb{Z}$

(b) $F(z) = \mathcal{L}_{2\pi}(z+i) = \mathrm{Log}\,|z+i| + i\arg_{2\pi}(z+i)$

- By theorem 4 in textbook section 3.3, $\mathrm{Log}\,(z+i)$ is differentiable at z for which $z+i$ lies in Domain D^*. This implies that $\mathrm{Log}\,(z+i)$ is not differentiable on $\{z_0 \in \mathbb{C} \mid z_0 + i = x \geq 0\} = \{(x, 1) \mid x \geq 0\}$

  Then $F(z)$ is not differentiable on $\{z_0 \in \mathbb{C} \mid z_0 = x - i,\ x \geq 0\}$
  Thus, $F(z)$ is not differentiable at all complex numbers except for $z_0 = x - i$, $x \geq 0$.

- $F(-1-i) = \mathrm{Log}\,|-1| + i\arg_{2\pi}(-1)$

  $= 0 + i\,(\pi + 2\pi)$

  $= 3\pi i$

  $\arg(-1) = \theta$

  $-1 = \cos\theta + i\sin\theta$

  $\begin{cases} \cos\theta = -1 \\ \sin\theta = 0 \end{cases} \Rightarrow \theta = \pi$

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3.

(a) $u(x,y) = e^{2y} \sin(ax)$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(a e^{2y} \cos(ax)\right) = -a^2 e^{2y} \sin(ax)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(2e^{2y} \sin(ax)\right) = 4 e^{2y} \sin(ax)$$

If $u(x,y)$ is harmonic $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Then $(4-a)^2 (e^{2y} \sin(ax)) = 0 \Rightarrow a = \pm 2$

(b) $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = a e^{2y} \cos(ax)$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -2e^{2y} \sin(ax)$$

Then $v(x,y) = \frac{a}{2} e^{2y} \cos(ax) + g(x)$

$$\frac{\partial v}{\partial x} = -\frac{a^2}{2} e^{2y} \sin(ax) + g'(x)$$

$$= -2e^{2y} \sin(ax) + g'(x) \Rightarrow g'(x) = 0 \Rightarrow g(x) = c$$

$$v(x,y) = \frac{a}{2} e^{2y} \cos(ax) + c.$$

Thus for $a = 2$ $v(x,y) = e^{2y} \cos(2x) + c_1$

for $a = -2$ $v(x,y) = -e^{2y} \cos(-2x) + c_2$

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4. (a)

By theorem 35, $f(z) = u(x,y) + iv(x,y)$ is differentiable at z_0 when the first partial derivatives of u and v exist and are continuous. and it should satisfy Cauchy-Riemann equations.

$$f(z) = \overline{z^2} = \overline{(x+iy)(x+iy)} = x^2 - y^2 - 2xyi$$

Then $u(x,y) = x^2 - y^2 \quad v(x,y) = -2xy$.

$$\frac{\partial u}{\partial x} = 2x \quad \frac{\partial u}{\partial y} = -2y \quad \frac{\partial v}{\partial x} = -2y \quad \frac{\partial v}{\partial y} = -2x$$

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ must be satisfied, we have $x = y = 0$

Thus $f(z)$ is only differentiable at $z = 0$.

(b) $g(z) = xy + (x-2xy)i \Rightarrow u(x,y) = xy \quad v(x,y) = x - 2xy$

Then $\frac{\partial u}{\partial x} = y \quad \frac{\partial u}{\partial y} = x \quad \frac{\partial v}{\partial x} = 1-2y \quad \frac{\partial v}{\partial y} = -2x$

All first partial derivatives exist and are continuous.

To satisfy Cauchy-Riemann equations, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow \begin{cases} y = -2x \\ x = 2y - 1 \end{cases} \Rightarrow \begin{cases} y = \frac{2}{5} \\ x = -\frac{1}{5} \end{cases}$$

Then $g(z) = xy + (x-2xy)i$ is only differentiable

at $z = -\frac{1}{5} + \frac{2}{5}i$

5. (a) e^z is continuous on $\mathbb{C}$

$\sin z$ and (z^3-1) are continuous on $\mathbb{C}$

Then $f(z)$ is continuous on $\mathbb{C}$ if $\sin z\,(z^3-1) \neq 0$.

$\sin z = 0 \Rightarrow z = n\pi \quad n \in \mathbb{Z}$

$z^3 - 1 = 0 \Rightarrow z^3 = 1 = (x+iy)^3 \Rightarrow \begin{cases} x^3 - 3xy^2 = 1 \\ 3x^2y - y^3 = 0 \end{cases}$

Then we solved for $\begin{cases} x = 1 & y = 0 \\ x = -\frac{1}{2} & y = \frac{\sqrt{3}}{2}i \\ x = -\frac{1}{2} & y = -\frac{\sqrt{3}}{2}i \end{cases} \Rightarrow \begin{cases} z = 1 \\ z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \\ z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i \end{cases}$

Thus $f(z)$ is continuous on $\mathbb{C} \setminus S$, where $S =$
$\{n\pi \mid n \in \mathbb{Z}\} \cup \{1, -\frac{1}{2}+\frac{\sqrt{3}}{2}i, -\frac{1}{2}-\frac{\sqrt{3}}{2}i\}$

(b) Since $\operatorname{Re} g(z)$ and $\operatorname{Im} g(z)$ are continuous on all of complex plane,
$\forall z_0 \in \mathbb{C}$, $\lim_{z \to z_0} \operatorname{Re} g(z) = \operatorname{Re} g(z_0)$ and $\lim_{z \to z_0} \operatorname{Im} g(z) = \operatorname{Im} g(z_0)$

Then by definition, $\forall \varepsilon > 0$, $\exists \delta_1, \delta_2 > 0$ such that

if $|z - z_0| < \delta_1$, then $|\operatorname{Re} g(z) - \operatorname{Re} g(z_0)| < \frac{\varepsilon}{2}$
if $|z - z_0| < \delta_2$, then $|\operatorname{Im} g(z) - \operatorname{Im} g(z_0)| < \frac{\varepsilon}{2}$

Thus $\forall \varepsilon > 0$ $\exists \delta = \min(\delta_1, \delta_2)$ and $|z - z_0| < \delta$
$|g(z) - g(z_0)| \leq |\operatorname{Re}(g(z) - g(z_0))| + |\operatorname{Im}(g(z) - g(z_0))|$

$= |\operatorname{Re} g(z) - \operatorname{Re} g(z_0)| + |\operatorname{Im} g(z) - \operatorname{Im} g(z_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

This implies $\lim_{z \to z_0} g(z) = g(z_0)$

Then $\forall z_0 \in \mathbb{C}$ $\lim_{z \to z_0} g(z) = g(z_0)$ and by the definition of continuity, $g(z)$ is continuous on all of the complex plane.