Section 3.5

1\. (b)

$$(-1)^{2/3} = \exp(\tfrac{2}{3}\log 1)\cdot\exp(i\tfrac{2}{3}\cdot \mathrm{Arg}(-1)+2k\pi) \qquad k=0,1,2$$

$$= e^{\frac{2}{3}i(\pi+2k\pi)}$$

$$= e^{\frac{2}{3}i\pi},\ e^{2i\pi},\ e^{\frac{10}{3}i\pi}.$$

(c) $$2^{\pi i} = e^{\pi i \log 2}$$

$$= e^{\pi i(\mathrm{Log}2+2k\pi i)} \qquad k=0,\pm1,\pm2\cdots$$

$$= e^{\pi i \log 2}e^{-2\pi^2 k}$$

(e) $$(1+i)^3 = e^{3(\log\sqrt{2}+\frac{\pi}{4}i)} = e^{3\log\sqrt{2}}\,e^{3\cdot\frac{\pi}{4}i}$$

$$= (\sqrt{2})^3 e^{\frac{3}{4}\pi i} = -2+2i$$

4\. No.

$$1^{\alpha} = e^{\alpha\log 1} = e^{\alpha(\mathrm{Log}1+i2\pi k)}$$

$$= e^{\alpha i\cdot 2\pi k}$$

Thus 1 is not the only value for 1^{α} if α is not an integer.

For example $1^{\frac{1}{2}}=\pm1$, -1 is also one of the values of $1^{\frac{1}{2}}$

5\. let $z_1=i$ $z_2=i-1$, then $z_1z_2=-1-i$

$$(z_1z_2)^{\frac{1}{2}} = (-1-i)^{\frac{1}{2}} = e^{\frac{1}{2}\log(-1-i)} = e^{\frac{1}{2}(\log\sqrt{2}-\frac{3}{4}\pi i)}$$

$$z_1^{\frac{1}{2}}\cdot z_2^{\frac{1}{2}} = e^{\frac{1}{2}(\frac{\pi}{2}i)}\,e^{\frac{1}{2}(\log\sqrt{2}+\frac{3}{4}\pi i)} = e^{\frac{1}{2}(\log\sqrt{2}+\frac{5}{4}\pi i)}$$

Thus $(z_1z_2)^{\frac{1}{2}} \neq z_1^{\frac{1}{2}}z_2^{\frac{1}{2}}$.

6. (a) $z^{-\alpha} = e^{-\alpha \log z} = \dfrac{1}{e^{\alpha \log z}} = \dfrac{1}{z^{\alpha}}$

(b) $z^{\alpha} z^{\beta} = e^{\alpha \log z} \cdot e^{\beta \log z} = e^{\alpha \log z + \beta \log z}$

$$= e^{(\alpha+\beta)\log z} = z^{\alpha+\beta}.$$

(c) $\dfrac{z^{\alpha}}{z^{\beta}} = \dfrac{e^{\alpha \log z}}{e^{\beta \log z}} = e^{\alpha \log z - \beta \log z} = e^{(\alpha-\beta)\log z} = z^{\alpha-\beta}.$

10. $\cos z = 2i$

Then $z = \cos^{-1}(2i) = -i \log((2 \pm \sqrt{5})i)$

$$= -i\left[\mathrm{Log}(2\pm\sqrt{5}) + \arg(2\pm\sqrt{5}) + 2k\pi i\right)$$

$$= -i\left[\mathrm{Log}(\sqrt{5}\pm 2) + \arg(\sqrt{5}\pm 2)(\pm i) + 2k\pi i\right]$$

$$= -i\left[\mathrm{Log}(\sqrt{5}\pm 2) + \frac{\pi}{2} + 2k\pi\right] \text{ or } -i\left[\mathrm{Log}(\sqrt{5}\pm 2) - \frac{\pi}{2} + 2k\pi\right)$$

Then $z = -i\mathrm{Log}(\sqrt{5}+2) + \frac{\pi}{2} + 2k\pi$ ~~it's equivalent to~~ $\cancel{z = i\mathrm{Log}(\sqrt{5}+2) - \frac{\pi}{2} + 2k\pi}$

or $z = -i\mathrm{Log}(\sqrt{5}-2) - \frac{\pi}{2} + 2k\pi = i\mathrm{Log}(\sqrt{5}+2) - \frac{\pi}{2} + 2k\pi$

$k = 0, \pm 1, \pm 2 \cdots$

11. $\sin z = \cos z \Rightarrow \frac{1}{2}(e^{iz} + e^{-iz}) = \frac{1}{2i}(e^{iz} - e^{-iz})$

Then $(1-i)e^{iz} = (1+i)e^{-iz}$

$$e^{2iz} = \frac{1+i}{1-i} = i$$

$$2iz = \log(i) = i\left(\frac{\pi}{2} + 2k\pi\right) \quad k = 0, \pm 1, \pm 2 \cdots$$

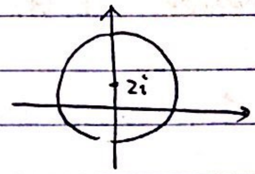
$$z = \frac{\pi}{4} + k\pi \quad k = 0, \pm 1, \pm 2 \cdots$$

Section 4.1.

1. (a) $z(t) = (1+i)\ t(-2-3i-(1+i)) \qquad 0 \le t \le 1$

$= (1-3t) + (1-4t)i$

(b) $|z-2i| = 4$



$z(t) = 2i + 4e^{-it} \qquad 0 \le t \le 2\pi$

(c) $|z| = R$

$z(t) = Re^{it} \qquad \frac{\pi}{2} \le t \le \pi$

(d) $y = x^2 \qquad z(t) = t + it^2 \qquad 1 \le t \le 3.$

3. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

parameterize this function $z(t) = a\cos t + ib\sin t$ and $0 \le t \le 2\pi$.

Then $z' = -a\sin t + ib\cos t$

① Since $\sin t$ and $\cos t$ are continuous. $z(t)$ is continuous. and

② $z' \neq 0$ on $0 \le t \le 2\pi$.

③ If $z(t_1) = z(t_2)$ then $\begin{cases} a\cos t_1 = a\cos t_2 \\ b\sin t_1 = b\sin t_2 \end{cases}$

since $0 \le t_1, t_2 \le 2\pi$, $t_1 = t_2$ which implies $z(t)$ is one-to-one.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is a smooth curve.

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4. $z(t) = t^3 + it^6 \quad -1 \le t \le 1$

It is the same curve with $y = x^2$, which is a smooth curve.

However, $z'(t) = 3t^2 + 6it^5$, when $t=0$ $z'(0) = 0$

Thus $z(t)$ is not admissible.

6. let $z_1(t) = z(g(t)) \quad g(t) = \frac{b-a}{d-c} t + \frac{ad-bc}{d-c}$

~~since $a \le t \le b \Rightarrow c \le g(t)$~~

since $c \le t \le d \Rightarrow$ then $a \le g(t) \le b$.

①. since $z(t)$, $g(t)$ are continuous on $c \le t \le d$.
we know that $z_1(t) = z(g(t))$ is also continuous.

② $z_1'(t) = z'(g(t)) \cdot g'(t) \Rightarrow z_1'(t)$ will not vanish.

since $g'(t) = \frac{b-a}{d-c}$ and $z'(t)$ is not vanish.

③ assume $z_1(t_1) = z_2(t_2)$
then we have $z(g(t_1)) = z(g(t_2))$

since $z(t)$ is one to one, we have $g(t_1) = g(t_2)$

$g(t)$ as a linear function is also one to one.

Thus $t_1 = t_2$ and $z_1(t)$ is one to one.

Therefore. $z_1(t)$ is admissible.

8.

$$\Gamma = z(t) = \begin{cases} -2+2i + t(1-2i) & 0 \le t \le 1 \\ e^{-i\pi t} & 1 \le t \le 2 \end{cases}$$

$$-\Gamma = z_1(t) = \begin{cases} -2+2i + t(1-2i) & -1 \le t \le 0 \\ e^{i\pi t} & -2 \le t \le -1. \end{cases}$$

11. $z = 5e^{3it}$ $\quad 0 \le t \le \pi.$

$$\frac{dz}{dt} = 15e^{3it}$$

$$\int_0^{\pi} \left|\frac{dz}{dt}\right| dt = 15\Big|_0^{\pi} = 15\pi.$$

Section 4.2.

3. (a) $$\int_0^1 (2t + it^2)\,dt = t^2 + \frac{1}{3}it^3 \Big|_0^1 = 1 + \frac{i}{3}.$$

(b) $$\int_{-2}^{0} (1+i)\cos(it)\,dt = (1+i)\int_{-2}^{0} \cos(it)\,dt$$

$$= (1+i)\,\frac{1}{i}\sin(it)\Big|_{-2}^{0}$$

$$= \frac{1+i}{i}\sin(2i)$$

6. (a) $z(t) = 2e^{it}$ $\quad 0 \le t \le 2\pi.$

$$\frac{dz}{dt} = 2ie^{it}$$

$$\int_0^{2\pi} \overline{2e^{it}}\, 2ie^{it}\,dt = 4i\Big|_0^{2\pi} = 8i\pi$$

(b). $z = 2e^{-it} \qquad 0 \le t \le 2\pi.$

$z' = -2e^{-it}$

$$\int_0^{2\pi} \overline{2e^{-it}} \cdot (-2e^{-it})\,dt = -4i\Big|_0^{2\pi} = -8i\pi.$$

(c) $$\int_0^{6\pi} \overline{2e^{-it}} \,(-2e^{-it})\,dt = -4i\Big|_0^{6\pi} = -24i\pi.$$

7. $\int_\Gamma \operatorname{Re} z\, dz = \int_\Gamma \operatorname{Re} z \cdot z'\, dt$

$\Gamma = z(t) = t(1+2i) \qquad 0 \le t \le 1.$

$\operatorname{Re} z = t \qquad z' = 1+2i$

$$\int_\Gamma \operatorname{Re} z\, dz = \int_0^1 t(1+2i)\,dt = (1+2i)\,\frac{1}{2}t^2\Big|_0^1$$

$$= \frac{1}{2} + i.$$

9. $\int_\Gamma (x - 2xyi)\, dz \qquad \Gamma: \; z = t + it^2 \quad 0 \le t \le 1$

$x = t \quad y = t^2$

$$\int_0^1 (t - 2t^3 i)(1 + 2it)\,dt$$

$$= \int_0^1 (t + 2it^2 - 2t^3 i + 4t^4)\,dt$$

$$= \frac{1}{2}t^2 + \frac{2}{3}it^3 - \frac{2}{4}t^4 i + \frac{4}{5}t^5\Big|_0^1$$

$$= \frac{1}{2} + \frac{2}{3}i - \frac{2}{4}i + \frac{4}{5} = \frac{13}{10} + \frac{1}{6}i$$

11.

(a) $z(t) = -i + t(1+i) \quad 0 \le t \le 1$

$$\int_\Gamma f(z)\,dz = \int_0^1 \left(2(-i + t(1+i)) + 1\right)(1+i)\,dt$$

$$= (1+i)\int_0^1 (1 - 2i + 2(1+i)t)\,dt$$

$$= (1+i)(1 - 2i + 1 + i)$$

$$= 3 + i$$

(b) ~~$z_1(t) = i + i dt$~~

$z_1(t) = -i + it \quad 0 \le t \le 1$

$z_2(t) = t \quad 0 \le t \le 1$

$$\int_{\Gamma_1} (2z_1 + 1)\,dz = \int_0^1 (-2i + 2it + 1)\,i\,dt$$

$$= 2t - t^2 + i \Big|_0^1 = 1 + i$$

$$\int_{\Gamma_2} (2z_2 + 1)\,dz = \int_0^1 (2t+1)\,dt = 2$$

$$\int_\Gamma f(z)\,dz = 3 + i + 2 = 3 + i$$

(c) $z(t) = e^{it} \quad -\frac{\pi}{2} \le t \le 0$

$$\int_\Gamma f(z)\,dz = \int_{-\frac{\pi}{2}}^0 (2e^{it} + 1)\,ie^{it}\,dt$$

$$= i\int_{-\frac{\pi}{2}}^0 (2e^{2it} + e^{it})\,dt$$

$$= e^{2it} + e^{it} \Big|_{-\frac{\pi}{2}}^0 = 2 - e^{i\pi} - e^{-i\frac{\pi}{2}} = 3 + i$$

14. (c)

$\text{Log}\, z = \log|z| + i\arg(z) \Rightarrow |\text{Log}\, z| \leq \frac{\pi}{2}$ since $0 \leq \arg z \leq \frac{\pi}{2}$

$= i\arg(z)$

Then $\left|\int_{\Gamma} \text{Log}\, z\, dz\right| \leq \max|\text{Log}\, z| \cdot \ell(\Gamma)$

$= \frac{\pi}{2} \cdot \ell(\Gamma) = \frac{\pi}{2} \cdot \frac{1}{4} \cdot 2\pi = \frac{\pi^2}{4}$

(d) $z=0$ $z=i$ $z(t) = it$ $0 \leq t \leq 1$

$\left|\int_{\gamma} e^{\sin z} dz\right| \leq \max|e^{\sin z}| \cdot \ell(\gamma) = \max|e^{\sin z}|$

$\ell(\gamma) = |i - 0| = 1$

$\max|e^{\sin z}| = \max|e^{\sin it}|$