4.3.

1.(b) $\int_\Gamma e^z dz = \int_1^{-1} e^z dz = e^z \Big|_1^{-1} = e^{-1} - e$

(c) $\int_\Gamma \frac{1}{z} dz = \int_{-3i}^{3i} \frac{1}{z} dz = \log z \Big|_{-3i}^{3i} = \log(3i) - \log(-3i)$

$= \log\left(\frac{3i}{-3i}\right) = \log(-1) = i\pi$

(e) $\int_\Gamma \sin^2 z \cos z \, dz = \int_\pi^i \sin^2 z \, d\sin z = \frac{1}{3} \sin^3 z \Big|_\pi^i$

$= \frac{1}{3} \sin^3(i)$

(g) $\int_\Gamma z^{\frac{1}{2}} dz = \int_\pi^i z^{\frac{1}{2}} dz = \frac{2}{3} z^{\frac{3}{2}} \Big|_\pi^i = \frac{2}{3} \left(i^{\frac{3}{2}} - \pi^{\frac{3}{2}} \right)$

(h) $\int_\Gamma (\mathrm{Log} z)^2 dz = \int_1^i (\mathrm{Log}\, z)^2 dz = (z\log^2 z - 2 \cdot z \log z + 2z) \Big|_1^i$

$= i\log^2 i - 2i\log i + 2i - 2$

(i) $\int_\Gamma \frac{1}{1+z^2} dz = \int_1^{1+i} \frac{1}{1+z^2} dz = \arctan(z) \Big|_1^{1+i}$

$= \arctan(1+i) - \arctan(1)$

$= \arctan(1+i) - \frac{\pi}{4}$

2. Suppose $Q(z)' = P(z)$

Since Γ is closed contour, for any corresponding initial and terminal points $z_1\ z_2$, we have $Q(z_1) = Q(z_2)$

Thus, $\int_\Gamma P(z) dz = Q(z) \Big|_{z_1}^{z_2} = Q(z_2) - Q(z_1) = 0.$

Hence $\int_\Gamma P(z) dz = 0.$

4. False

for $f(z)=\frac{1}{z}$

$\int_{|z|=1} \frac{1}{z} dz = 2\pi i \neq 0.$

5. suppose $f(z)=\frac{1}{z}$ has a anti-derivative $F(z)$

then $F'=f(z)$

Then for a closed contour $\Gamma : |z|=1.$, $\oint$

$\int_\Gamma \frac{1}{z} dz = 0$. However, by example 2 $\int_\Gamma \frac{1}{z} dz = 2\pi i \neq 0.$

Contradiction.

6.

$$\int_C \frac{1}{z-z_0} dz = \lim_{\alpha\beta \to \tau} \int_\alpha^\beta \frac{1}{z-z_0} dz$$

$$= \lim_{\alpha,\beta\to\tau} [\text{Log}(z-z_0)]\Big|_\alpha^\beta$$

$$= \text{Log}(\tau - z_0) + i\pi - (\text{Log}(\tau - z_0) - i\pi)$$

$$= 2\pi i$$

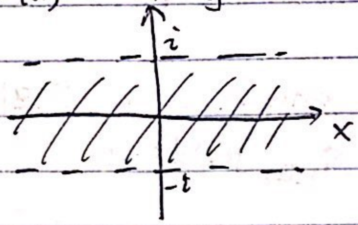
12.

$$|f(z_2)-f(z_1)| = \left|\int_{z_1}^{z_2} f'(z) dz\right| \leq \left|\int_{z_1}^{z_2} M\, dz\right|$$

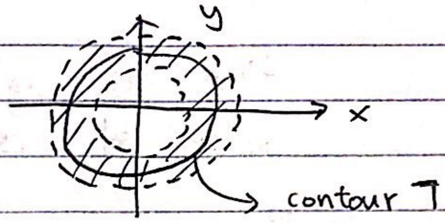
$$= \left| M z\Big|_{z_1}^{z_2}\right| = |M(z_2 - z_1)|$$

$$= M|z_2 - z_1|$$

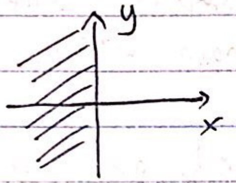
4.4 9. (a) simply connected.



(b) not simply connected.

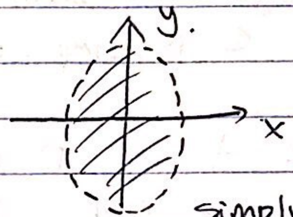


(c)



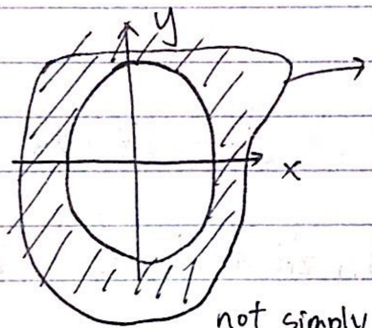
Simply connected.

(d)



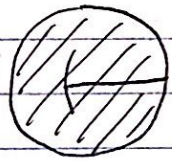
simply connected.

(e)



not simply connected

(f)



Simply connected.

10.(b) since e^{-z} and $2z+1$ is analytic on $\mathbb{C}$

$f(z) = e^{-z}(2z+1)$ is analytic on $|z| \le 2$, which is a simply connected domain.

Since $|z|=2$ is in D, we have $\oint_{|z|=2} f(z)\,dz = 0$.

(c) $\cos z$ is entire

$z^2 - 6z + 10$ is zero at $z = 3 \pm i$

Thus $f(z)$ is analytic on everywhere except for $z = 3 \pm i$

Then $f(z)$ is analytic on $D = |z| \le 2$, D is simply connected.
Since $|z|=2$ is ~~off~~ in D, we have $\oint_{|z|=2} f(z)\,dz = 0$.

(d) $\text{Log}(z+3)$ is analytic on $\mathbb{C}$ except for $x<-3$, $y=0$

Thus $f(z)=\text{Log}(z+3)$ is analytic on $D: |z|\le 2$. D is simply connected.
Since $|z|=2$ is in D, $\oint_{|z|=2} f(z)\,dz=0$

11. e^{z^2} is analytic on entire complex plane $\mathbb{C}$ and since $\mathbb{C}$ is simply connected, we have e^{z^2} has a anti-derivative.

12. Assume D is simply connected, by Cauchy's integral theorem, we have $\int_\Gamma \frac{1}{z-z_0}\,dz=0$. contradiction.
Thus D is not simply connected.

13. (a)

$$\int_{\Gamma_1} \frac{1}{(z^2+1)}\,dz = \frac{i}{2}\int_{\Gamma_1}\left(\frac{1}{z+i}-\frac{1}{z-i}\right)dz$$

$$= \frac{i}{2}\int_{\Gamma_1}\frac{1}{z+i}\,dz - \frac{i}{2}\int_{\Gamma_1}\frac{1}{z-i}\,dz$$

$$= \frac{i}{2}\cdot 0 - \frac{i}{2}\cdot(2\pi i) = \pi.$$

(b) $$\int_{\Gamma_2}\frac{1}{z^2+1}\,dz = \frac{i}{2}\int_{\Gamma_2}\frac{1}{z+i}\,dz - \frac{i}{2}\int_{\Gamma_2}\frac{1}{z-i}\,dz$$

$$= \frac{i}{2}\cdot 2\pi i - \frac{i}{2}2\pi i = 0.$$

(c) $$\int_{\Gamma_3}\frac{1}{z^2+1}\,dz = \frac{i}{2}\int_{\Gamma_3}\frac{1}{z+i}\,dz - \frac{i}{2}\int_{\Gamma_3}\frac{1}{z-i}\,dz$$

$$= \frac{i}{2}\cdot 2\pi i - \frac{i}{2}\cdot 0$$

$$= -\pi.$$

15. $\frac{z}{(z+2)(z-1)} = \frac{a}{z+2} + \frac{b}{z-1} = \frac{a(z-1)+b(z+2)}{(z+2)(z-1)}$

$$\begin{cases} a = \frac{2}{3} \\ b = \frac{1}{3} \end{cases}$$

$$\int_\Gamma \frac{z}{(z+2)(z-1)}\, dz = \frac{2}{3} \int_\Gamma \frac{1}{z+2}\, dz + \frac{1}{3} \int_\Gamma \frac{1}{z-1}\, dz$$

$$= \frac{2}{3} \cdot (-4\pi i) + \frac{1}{3}(-4\pi i) = -4\pi i$$

4.5.

1. since z_0 is not on contour Γ, $z - z_0$ is nonzero

$\frac{f(z)}{z-z_0}$ is analytic in and on Γ.

Thus $\frac{1}{2\pi i} \int \frac{f(z)}{z-z_0}\, dz = 0$

2. ~~$\forall z_0$ in Γ~~ $\forall z_0$ inside Γ, we have

$$f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-z_0}\, dz = \frac{1}{2\pi i} \int_\Gamma \frac{g(z)}{z-z_0} = g(z_0)$$

Thus $f(z_0) = g(z_0)$ for all z_0 inside Γ

3. (b) $\int_C \frac{ze^z}{2z-3}\, dz = \int_C \frac{\frac{1}{2} z e^z}{z - \frac{3}{2}}\, dz = 2\pi i \, \frac{1}{2}\left(\frac{3}{2}\right) e^{\frac{3}{2}}$

$$= \frac{3}{2} \pi i \, e^{\frac{3}{2}}$$

(d) $\int_C \frac{5z^2+2z+1}{(z-i)^3}\, dz = \frac{2\pi i}{2!} (5z^2+2z+1)'' \Big|_{z=i}$ by Cauchy's integral formula.

$$= \frac{2\pi i}{2} \cdot 10 = 10\pi i$$

(e) $f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz.$

Thus $\int_C \frac{e^{-z}}{(z+1)^2}\,dz = \frac{2\pi i}{1!}\left(e^{-z}\right)'\Big|_{z=-1}$

$= -2\pi i e$

4. (a) $\int_C \frac{z+i}{z^3+2z^2}\,dz = \int_C \frac{(z+i)/(z+2)}{z^2}\,dz$

$= \frac{2\pi i}{1!}\left(\frac{z+i}{z+2}\right)'\Big|_{z=0}$

$= 2\pi i\left(\frac{2-i}{4}\right) = \frac{(2-i)\pi i}{2}$

(b) $\int_C \frac{(z+i)/z^2}{z+2}\,dz = 2\pi i\left(\frac{z+i}{z^2}\right)\Big|_{z=-2} = \frac{(i-2)\pi i}{2}$

(c) $\int_C \frac{z+i}{z^2(z+2)}\,dz$

since $z=0$ and $z=-2$ is not in C,

$\int_C \frac{z+i}{z^3+2z^2}\,dz = 0.$

6. $\int_\Gamma \frac{e^{iz}}{(z^2+1)^2}\,dz = \int_\Gamma \frac{e^{iz}/(z+i)^2}{(z-i)^2}\,dz + \int_\Gamma \frac{e^{iz}/(z-i)^2}{(z+i)^2}\,dz$

$= \frac{2\pi i}{1!}\left(\frac{e^{iz}}{(z+i)^2}\right)'\Big|_{z=i} + \frac{2\pi i}{1!}\left(\frac{e^{iz}}{(z-i)^2}\right)\Big|_{z=-i}$

$= 2\pi i\left(-\frac{i}{2e}\right) + 2\pi i\cdot 0$

$= \frac{\pi}{e}$

7. $\int_\Gamma \frac{\cos z}{z^2(z-3)}\,dz = \int_\Gamma \frac{\cos z/(z-3)}{z^2}\,dz$

$= \frac{2\pi i}{1!}\left(\frac{\cos z}{z-3}\right)'\Big|_{z=0}$

$= 2\pi i\left(\frac{-(z-3)\sin z - \cos z}{(z-3)^2}\right)\Big|_{z=0}$

$= 2\pi i\cdot\left(-\frac{1}{9}\right) = -\frac{2}{9}\pi i$

9. $|f(0)| = \left|\frac{1}{2\pi i}\int_T \frac{f(z)}{z}\,dz\right| \le \left|\frac{1}{2\pi i}\int_T \frac{M}{z}\,dz\right|$

$= \left|\frac{M}{2\pi i}\int_T \frac{1}{z}\,dz\right| = \left|\frac{M}{2\pi i}\cdot 2\pi i\right|$

$= M$

$|f'(0)| = \left|\frac{1!}{2\pi i}\int_T \frac{f(z)}{z^2}\,dz\right| \le \left|\frac{1}{2\pi i}\int_T \frac{M}{z^2}\,dz\right|$

$= \left|\frac{M}{2\pi i}\,2\pi i\right| = M$

$|f^{(n)}(0)| = \left|\frac{n!}{2\pi i}\int_T \frac{f(z)}{z^n}\,dz\right| \le \left|\frac{n!}{2\pi i}\cdot M\cdot 2\pi i\right|$

$= n!M$

$|f^{(n)}(0)| \le n!\,M.$

16. (a) Since f, f' are analytic on D and f is nonzero,

$\frac{f'}{f}$ is also analytic on D.

(b) $\frac{f'(z)}{f(z)}$ is analytic on a simply connected domain D

Thus, $\frac{f'(z)}{f(z)}$ has an anti-derivative $H(z)$

such that $H'(z) = \frac{f'(z)}{f(z)}$

(c) $\frac{d}{dz}\left(f(z) e^{-H(z)} \right)$

$= f'(z) \cdot e^{-H(z)} + (-f(z) e^{-H(z)} H'(z))$

$= e^{-H(z)} (f'(z) - f(z) H'(z))$

$= e^{-H(z)} (f'(z) - f'(z)) = 0$

Thus $f(z) e^{-H(z)}$ is constant. and $f(z) = c\, e^{H(z)}$

(d) let $\alpha = \log c$

$\log f(z) = \log (c\, e^{H(z)})$

$= \log c + \log e^{H(z)}$

$= \alpha + H(z) + 2\pi i \cdot n$

Thus for $n=0$, $\log f(z) = \alpha + H(z)$ is single-valued.

Hence, $\alpha + H(z)$ is a branch of $\log f(z)$.

4.6

1. $f(z) = \frac{1}{(1-z)^2}$

on $|z| = R$, we have $|f(z)| \le \left|\frac{1}{(1-z)^2}\right| \le \frac{1}{(1-|z|)^2} \le \frac{1}{(1-R)^2}$

Thus $\max_{|z|=R} |f(z)| = \frac{1}{(1-R)^2}$

$f'(z) = \frac{2}{(1-z)^3}$ $\quad f''(z) = \frac{2\cdot 3}{(1-z)^4}$ $\quad \cdots$

$f^{(n)}(z) = \frac{2\cdot 3\cdot 4 \cdots (n+1)}{(1-z)^{n+2}} = \frac{(n+1)!}{(1-z)^{n+2}}$

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Then $f^{(n)}(0) = (n+1)!$

and by Cauchy estimates $(n+1)! = f^{(n)}(0) \le \dfrac{n! \frac{1}{(1-R)^2}}{R^n}$

$$= \frac{n!}{R^n (1-R)^2}$$

4. $P(z) = a_0 + a_1 z + \cdots + a_n z^n$.

let $T = |z| = 1$.

Then $a_n = \dfrac{1}{2\pi i} \int_T \dfrac{P(z)}{z^{n+1}} dz$

$$|a_k| = \left| \frac{1}{2\pi i} \int_T \frac{P(z)}{z^{k+1}} dz \right| \le \left| \frac{1}{2\pi i} \int_T \frac{M}{z^{k+1}} dz \right|$$

$$= \left| \frac{M}{2\pi i} \cdot 2\pi i \right| = M$$

Thus $|a_k| \le M$.

5. let $f(z) = u(x,y) + v(x,y)i$

Then $|e^{f(z)}| = |e^{u+iv}| = |e^u| |\cos v + i \sin v|$

$$\le |e^M| = e^M$$

Thus $e^{f(z)}$ is bounded by e^M and by Liouville's theorem $e^{f(z)}$ is constant

Hence $f(z)$ is a constant.

6. since $f^{(5)}(z)$ is bounded. by Liouvilles theorem,

$f^{(5)}(z)$ is constat, say $f^{(5)}(z)=c$

Thus $f^{(6)}(z)=0 \Rightarrow f(z)$ cannot be degree 6 or higher.

7. let $R > |z_0| + r_0$

Thus $|f^{(n)}(z_0)| \leq \dfrac{n! \max_{|z_0|=R} |z|^2}{R^n} = \dfrac{n! (|z_0|+R)^2}{R^n}.$

Thus $\forall n > 2$ if $R \to \infty$ we have

$$\lim_{R\to\infty} \frac{n!(|z_0|+R)^2}{R^n} = 0$$

Then $|f^{(n)}(z_0)| \leq 0 \Rightarrow f^{(n)}(z_0) = 0.$

$f(z)$ is a polynomial of degree at most 2.

14. since $f(z)$ is nonzero ~~and~~, $\frac{1}{f(z)}$ is analytic on D.

Thus $\left|\frac{1}{f(z)}\right|$ attain its maximum on the boundary of D by maxinal value principal.

Then $|f(z)|$ attain its maximum on the boundary of D.

Example: for $f(z)=z$ and $D: |z|\leq 1$

The minimum of $|f(z)|$ is 0 at $z=0$, which is not on the boundary of $|z|=1$

15. suppose $f(z)$ is nonzero on D.
it attains maximum and minimum on the boundary.
Thus $f(z)$ is a constant on the boundary B.