In a similar manner, we now obtain a three-digit rounding answer of -14.3. The new relative errors are

$$\text{Three-digit (chopping):} \quad \left|\frac{-14.263899 + 14.2}{-14.263899}\right| \approx 0.0045;$$

$$\text{Three-digit (rounding):} \quad \left|\frac{-14.263899 + 14.3}{-14.263899}\right| \approx 0.0025.$$

Nesting has reduced the relative error for the chopping approximation to less than 10% of that obtained initially. For the rounding approximation the improvement has been even more dramatic; the error in this case has been reduced by more than 95%. □

Polynomials should *always* be expressed in nested form before performing an evaluation, because this form minimizes the number of arithmetic calculations. The decreased error in the Illustration is due to the reduction in computations from four multiplications and three additions to two multiplications and three additions. One way to reduce round-off error is to reduce the number of computations.

EXERCISE SET 1.2

1. Compute the absolute error and relative error in approximations of p by p^*.

a. $p = \pi$, $p^* = 22/7$
b. $p = \pi$, $p^* = 3.1416$
c. $p = e$, $p^* = 2.718$
d. $p = \sqrt{2}$, $p^* = 1.414$
e. $p = e^{10}$, $p^* = 22000$
f. $p = 10^{\pi}$, $p^* = 1400$
g. $p = 8!$, $p^* = 39900$
h. $p = 9!$, $p^* = \sqrt{18\pi}(9/e)^9$

2. Find the largest interval in which p^* must lie to approximate p with relative error at most 10^{-4} for each value of p.

a. π
b. e
c. $\sqrt{2}$
d. $\sqrt[3]{7}$

3. Suppose p^* must approximate p with relative error at most 10^{-3}. Find the largest interval in which p^* must lie for each value of p.

a. 150
b. 900
c. 1500
d. 90

4. Perform the following computations (i) exactly, (ii) using three-digit chopping arithmetic, and (iii) using three-digit rounding arithmetic. (iv) Compute the relative errors in parts (ii) and (iii).

a. $\dfrac{4}{5} + \dfrac{1}{3}$
b. $\dfrac{4}{5} \cdot \dfrac{1}{3}$
c. $\left(\dfrac{1}{3} - \dfrac{3}{11}\right) + \dfrac{3}{20}$
d. $\left(\dfrac{1}{3} + \dfrac{3}{11}\right) - \dfrac{3}{20}$

5. Use three-digit rounding arithmetic to perform the following calculations. Compute the absolute error and relative error with the exact value determined to at least five digits.

a. $133 + 0.921$
b. $133 - 0.499$
c. $(121 - 0.327) - 119$
d. $(121 - 119) - 0.327$
e. $\dfrac{\frac{13}{14} - \frac{6}{7}}{2e - 5.4}$
f. $-10\pi + 6e - \dfrac{3}{62}$
g. $\left(\dfrac{2}{9}\right) \cdot \left(\dfrac{9}{7}\right)$
h. $\dfrac{\pi - \frac{22}{7}}{\frac{1}{17}}$

6. Repeat Exercise 5 using four-digit rounding arithmetic.

7. Repeat Exercise 5 using three-digit chopping arithmetic.

8. Repeat Exercise 5 using four-digit chopping arithmetic.

9. The first three nonzero terms of the Maclaurin series for the arctangent function are $x - (1/3)x^3 + (1/5)x^5$. Compute the absolute error and relative error in the following approximations of π using the polynomial in place of the arctangent:

a. $4\left[\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right)\right]$

b. $16\arctan\left(\frac{1}{5}\right) - 4\arctan\left(\frac{1}{239}\right)$

10. The number e can be defined by $e = \sum_{n=0}^{\infty}(1/n!)$, where $n! = n(n-1)\cdots 2\cdot 1$ for $n \neq 0$ and $0! = 1$. Compute the absolute error and relative error in the following approximations of e:

a. $\displaystyle\sum_{n=0}^{5}\frac{1}{n!}$ **b.** $\displaystyle\sum_{n=0}^{10}\frac{1}{n!}$

11. Let

$$f(x) = \frac{x\cos x - \sin x}{x - \sin x}.$$

a. Find $\lim_{x\to 0} f(x)$.

b. Use four-digit rounding arithmetic to evaluate $f(0.1)$.

c. Replace each trigonometric function with its third Maclaurin polynomial, and repeat part (b).

d. The actual value is $f(0.1) = -1.99899998$. Find the relative error for the values obtained in parts (b) and (c).

12. Let

$$f(x) = \frac{e^x - e^{-x}}{x}.$$

a. Find $\lim_{x\to 0}(e^x - e^{-x})/x$.

b. Use three-digit rounding arithmetic to evaluate $f(0.1)$.

c. Replace each exponential function with its third Maclaurin polynomial, and repeat part (b).

d. The actual value is $f(0.1) = 2.003335000$. Find the relative error for the values obtained in parts (b) and (c).

13. Use four-digit rounding arithmetic and the formulas (1.1), (1.2), and (1.3) to find the most accurate approximations to the roots of the following quadratic equations. Compute the absolute errors and relative errors.

a. $\frac{1}{3}x^2 - \frac{123}{4}x + \frac{1}{6} = 0$

b. $\frac{1}{3}x^2 + \frac{123}{4}x - \frac{1}{6} = 0$

c. $1.002x^2 - 11.01x + 0.01265 = 0$

d. $1.002x^2 + 11.01x + 0.01265 = 0$

14. Repeat Exercise 13 using four-digit chopping arithmetic.

15. Use the 64-bit long real format to find the decimal equivalent of the following floating-point machine numbers.

a. 0 10000001010 1001001100000000000000000000000000000000000000000000

b. 1 10000001010 1001001100000000000000000000000000000000000000000000

c. 0 01111111111 0101001100000000000000000000000000000000000000000000

d. 0 01111111111 0101001100000000000000000000000000000000000000000001

16. Find the next largest and smallest machine numbers in decimal form for the numbers given in Exercise 15.

17. Suppose two points (x_0, y_0) and (x_1, y_1) are on a straight line with $y_1 \neq y_0$. Two formulas are available to find the x-intercept of the line:

$$x = \frac{x_0 y_1 - x_1 y_0}{y_1 - y_0} \quad \text{and} \quad x = x_0 - \frac{(x_1 - x_0)y_0}{y_1 - y_0}.$$

a. Show that both formulas are algebraically correct.

b. Use the data $(x_0, y_0) = (1.31, 3.24)$ and $(x_1, y_1) = (1.93, 4.76)$ and three-digit rounding arithmetic to compute the x-intercept both ways. Which method is better and why?

18. The Taylor polynomial of degree n for $f(x) = e^x$ is $\sum_{i=0}^{n}(x^i/i!)$. Use the Taylor polynomial of degree nine and three-digit chopping arithmetic to find an approximation to e^{-5} by each of the following methods.

a. $e^{-5} \approx \sum_{i=0}^{9} \frac{(-5)^i}{i!} = \sum_{i=0}^{9} \frac{(-1)^i 5^i}{i!}$

b. $e^{-5} = \frac{1}{e^5} \approx \frac{1}{\sum_{i=0}^{9} \frac{5^i}{i!}}$.

c. An approximate value of e^{-5} correct to three digits is 6.74×10^{-3}. Which formula, (a) or (b), gives the most accuracy, and why?

19. The two-by-two linear system

$$ax + by = e,$$
$$cx + dy = f,$$

where a, b, c, d, e, f are given, can be solved for x and y as follows:

$$\text{set } m = \frac{c}{a}, \quad \text{provided } a \neq 0;$$
$$d_1 = d - mb;$$
$$f_1 = f - me;$$
$$y = \frac{f_1}{d_1};$$
$$x = \frac{(e - by)}{a}.$$

Solve the following linear systems using four-digit rounding arithmetic.

a. $1.130x - 6.990y = 14.20$
$1.013x - 6.099y = 14.22$

b. $8.110x + 12.20y = -0.1370$
$-18.11x + 112.2y = -0.1376$

20. Repeat Exercise 19 using four-digit chopping arithmetic.

21. a. Show that the polynomial nesting technique described in Example 6 can also be applied to the evaluation of

$$f(x) = 1.01e^{4x} - 4.62e^{3x} - 3.11e^{2x} + 12.2e^x - 1.99.$$

b. Use three-digit rounding arithmetic, the assumption that $e^{1.53} = 4.62$, and the fact that $e^{nx} = (e^x)^n$ to evaluate $f(1.53)$ as given in part (a).

c. Redo the calculation in part (b) by first nesting the calculations.

d. Compare the approximations in parts (b) and (c) to the true three-digit result $f(1.53) = -7.61$.

22. A rectangular parallelepiped has sides of length 3 cm, 4 cm, and 5 cm, measured to the nearest centimeter. What are the best upper and lower bounds for the volume of this parallelepiped? What are the best upper and lower bounds for the surface area?

23. Let $P_n(x)$ be the Maclaurin polynomial of degree n for the arctangent function. Use Maple carrying 75 decimal digits to find the value of n required to approximate π to within 10^{-25} using the following formulas.

a. $4\left[P_n\left(\frac{1}{2}\right) + P_n\left(\frac{1}{3}\right)\right]$

b. $16P_n\left(\frac{1}{5}\right) - 4P_n\left(\frac{1}{239}\right)$