to nearly satisfy $f''(x_0) = f''(x_n) = 0$. An alternative to the natural boundary condition that does not require knowledge of the derivative of $f$ is the *not-a-knot* condition, (see [Deb2], pp. 55–56). This condition requires that $S'''(x)$ be continuous at $x_1$ and at $x_{n-1}$.

## EXERCISE SET 3.5

**1.** Determine the natural cubic spline $S$ that interpolates the data $f(0) = 0$, $f(1) = 1$, and $f(2) = 2$.

**2.** Determine the clamped cubic spline $s$ that interpolates the data $f(0) = 0$, $f(1) = 1$, $f(2) = 2$ and satisfies $s'(0) = s'(2) = 1$.

**3.** Construct the natural cubic spline for the following data.

**a.**

| $x$ | $f(x)$ |
|---|---|
| 8.3 | 17.56492 |
| 8.6 | 18.50515 |

**b.**

| $x$ | $f(x)$ |
|---|---|
| 0.8 | 0.22363362 |
| 1.0 | 0.65809197 |

**c.**

| $x$ | $f(x)$ |
|---|---|
| −0.5 | −0.0247500 |
| −0.25 | 0.3349375 |
| 0 | 1.1010000 |

**d.**

| $x$ | $f(x)$ |
|---|---|
| 0.1 | −0.62049958 |
| 0.2 | −0.28398668 |
| 0.3 | 0.00660095 |
| 0.4 | 0.24842440 |

**4.** Construct the natural cubic spline for the following data.

**a.**

| $x$ | $f(x)$ |
|---|---|
| 0 | 1.00000 |
| 0.5 | 2.71828 |

**b.**

| $x$ | $f(x)$ |
|---|---|
| −0.25 | 1.33203 |
| 0.25 | 0.800781 |

**c.**

| $x$ | $f(x)$ |
|---|---|
| 0.1 | −0.29004996 |
| 0.2 | −0.56079734 |
| 0.3 | −0.81401972 |

**d.**

| $x$ | $f(x)$ |
|---|---|
| −1 | 0.86199480 |
| −0.5 | 0.95802009 |
| 0 | 1.0986123 |
| 0.5 | 1.2943767 |

**5.** The data in Exercise 3 were generated using the following functions. Use the cubic splines constructed in Exercise 3 for the given value of $x$ to approximate $f(x)$ and $f'(x)$, and calculate the actual error.

**a.** $f(x) = x \ln x$; approximate $f(8.4)$ and $f'(8.4)$.

**b.** $f(x) = \sin(e^x - 2)$; approximate $f(0.9)$ and $f'(0.9)$.

**c.** $f(x) = x^3 + 4.001x^2 + 4.002x + 1.101$; approximate $f(-\frac{1}{3})$ and $f'(-\frac{1}{3})$.

**d.** $f(x) = x \cos x - 2x^2 + 3x - 1$; approximate $f(0.25)$ and $f'(0.25)$.

**6.** The data in Exercise 4 were generated using the following functions. Use the cubic splines constructed in Exercise 4 for the given value of $x$ to approximate $f(x)$ and $f'(x)$, and calculate the actual error.

**a.** $f(x) = e^{2x}$; approximate $f(0.43)$ and $f'(0.43)$.

**b.** $f(x) = x^4 - x^3 + x^2 - x + 1$; approximate $f(0)$ and $f'(0)$.

**c.** $f(x) = x^2 \cos x - 3x$; approximate $f(0.18)$ and $f'(0.18)$.

**d.** $f(x) = \ln(e^x + 2)$; approximate $f(0.25)$ and $f'(0.25)$.

**7.** Construct the clamped cubic spline using the data of Exercise 3 and the fact that

**a.** $f'(8.3) = 3.116256$ and $f'(8.6) = 3.151762$

**b.** $f'(0.8) = 2.1691753$ and $f'(1.0) = 2.0466965$

**c.** $f'(-0.5) = 0.7510000$ and $f'(0) = 4.0020000$

**d.** $f'(0.1) = 3.58502082$ and $f'(0.4) = 2.16529366$

**8.** Construct the clamped cubic spline using the data of Exercise 4 and the fact that

**a.** $f'(0) = 2$ and $f'(0.5) = 5.43656$

**b.** $f'(-0.25) = 0.437500$ and $f'(0.25) = -0.625000$

**c.** $f'(0.1) = -2.8004996$ and $f'(0) = -2.9734038$

**d.** $f'(-1) = 0.15536240$ and $f'(0.5) = 0.45186276$

**9.** Repeat Exercise 5 using the clamped cubic splines constructed in Exercise 7.

**10.** Repeat Exercise 6 using the clamped cubic splines constructed in Exercise 8.

**11.** A natural cubic spline $S$ on $[0, 2]$ is defined by

$$S(x) = \begin{cases} S_0(x) = 1 + 2x - x^3, & \text{if } 0 \le x < 1, \\ S_1(x) = 2 + b(x-1) + c(x-1)^2 + d(x-1)^3, & \text{if } 1 \le x \le 2. \end{cases}$$

Find $b$, $c$, and $d$.

**12.** A clamped cubic spline $s$ for a function $f$ is defined on $[1, 3]$ by

$$s(x) = \begin{cases} s_0(x) = 3(x-1) + 2(x-1)^2 - (x-1)^3, & \text{if } 1 \le x < 2, \\ s_1(x) = a + b(x-2) + c(x-2)^2 + d(x-2)^3, & \text{if } 2 \le x \le 3. \end{cases}$$

Given $f'(1) = f'(3)$, find $a$, $b$, $c$, and $d$.

**13.** A natural cubic spline $S$ is defined by

$$S(x) = \begin{cases} S_0(x) = 1 + B(x-1) - D(x-1)^3, & \text{if } 1 \le x < 2, \\ S_1(x) = 1 + b(x-2) - \frac{3}{4}(x-2)^2 + d(x-2)^3, & \text{if } 2 \le x \le 3. \end{cases}$$

If $S$ interpolates the data $(1, 1)$, $(2, 1)$, and $(3, 0)$, find $B$, $D$, $b$, and $d$.

**14.** A clamped cubic spline $s$ for a function $f$ is defined by

$$s(x) = \begin{cases} s_0(x) = 1 + Bx + 2x^2 - 2x^3, & \text{if } 0 \le x < 1, \\ s_1(x) = 1 + b(x-1) - 4(x-1)^2 + 7(x-1)^3, & \text{if } 1 \le x \le 2. \end{cases}$$

Find $f'(0)$ and $f'(2)$.

**15.** Construct a natural cubic spline to approximate $f(x) = \cos \pi x$ by using the values given by $f(x)$ at $x = 0, 0.25, 0.5, 0.75,$ and $1.0$. Integrate the spline over $[0, 1]$, and compare the result to $\int_0^1 \cos \pi x \, dx = 0$. Use the derivatives of the spline to approximate $f'(0.5)$ and $f''(0.5)$. Compare these approximations to the actual values.

**16.** Construct a natural cubic spline to approximate $f(x) = e^{-x}$ by using the values given by $f(x)$ at $x = 0$, 0.25, 0.75, and 1.0. Integrate the spline over $[0, 1]$, and compare the result to $\int_0^1 e^{-x} \, dx = 1 - 1/e$. Use the derivatives of the spline to approximate $f'(0.5)$ and $f''(0.5)$. Compare the approximations to the actual values.

**17.** Repeat Exercise 15, constructing instead the clamped cubic spline with $f'(0) = f'(1) = 0$.

**18.** Repeat Exercise 16, constructing instead the clamped cubic spline with $f'(0) = -1$, $f'(1) = -e^{-1}$.

**19.** Suppose that $f(x)$ is a polynomial of degree 3. Show that $f(x)$ is its own clamped cubic spline, but that it cannot be its own natural cubic spline.

**20.** Suppose the data $\{x_i, f(x_i)\}_{i=1}^n$ lie on a straight line. What can be said about the natural and clamped cubic splines for the function $f$? [*Hint:* Take a cue from the results of Exercises 1 and 2.]

**21.** Given the partition $x_0 = 0$, $x_1 = 0.05$, and $x_2 = 0.1$ of $[0, 0.1]$, find the piecewise linear interpolating function $F$ for $f(x) = e^{2x}$. Approximate $\int_0^{0.1} e^{2x} \, dx$ with $\int_0^{0.1} F(x) \, dx$, and compare the results to the actual value.

**22.** Let $f \in C^2[a, b]$, and let the nodes $a = x_0 < x_1 < \cdots < x_n = b$ be given. Derive an error estimate similar to that in Theorem 3.13 for the piecewise linear interpolating function $F$. Use this estimate to derive error bounds for Exercise 21.

**23.** Extend Algorithms 3.4 and 3.5 to include as output the first and second derivatives of the spline at the nodes.

**24.** Extend Algorithms 3.4 and 3.5 to include as output the integral of the spline over the interval $[x_0, x_n]$.

**25.** Given the partition $x_0 = 0$, $x_1 = 0.05$, $x_2 = 0.1$ of $[0, 0.1]$ and $f(x) = e^{2x}$:

**a.** Find the cubic spline $s$ with clamped boundary conditions that interpolates $f$.

**b.** Find an approximation for $\int_0^{0.1} e^{2x} \, dx$ by evaluating $\int_0^{0.1} s(x) \, dx$.

**c.** Use Theorem 3.13 to estimate $\max_{0\le x\le 0.1} |f(x) - s(x)|$ and

$$\left|\int_0^{0.1} f(x)\,dx - \int_0^{0.1} s(x)\,dx\right|.$$

**d.** Determine the cubic spline $S$ with natural boundary conditions, and compare $S(0.02)$, $s(0.02)$, and $e^{0.04} = 1.04081077$.

**26.** Let $f$ be defined on $[a, b]$, and let the nodes $a = x_0 < x_1 < x_2 = b$ be given. A quadratic spline interpolating function $S$ consists of the quadratic polynomial

$$S_0(x) = a_0 + b_0(x - x_0) + c_0(x - x_0)^2 \quad \text{on } [x_0, x_1]$$

and the quadratic polynomial

$$S_1(x) = a_1 + b_1(x - x_1) + c_1(x - x_1)^2 \quad \text{on } [x_1, x_2],$$

such that

**i.** $S(x_0) = f(x_0), S(x_1) = f(x_1)$, and $S(x_2) = f(x_2)$,

**ii.** $S \in C^1[x_0, x_2]$.

Show that conditions (i) and (ii) lead to five equations in the six unknowns $a_0$, $b_0$, $c_0$, $a_1$, $b_1$, and $c_1$. The problem is to decide what additional condition to impose to make the solution unique. Does the condition $S \in C^2[x_0, x_2]$ lead to a meaningful solution?

**27.** Determine a quadratic spline $s$ that interpolates the data $f(0) = 0$, $f(1) = 1$, $f(2) = 2$ and satisfies $s'(0) = 2$.

**28.** **a.** The introduction to this chapter included a table listing the population of the United States from 1950 to 2000. Use natural cubic spline interpolation to approximate the population in the years 1940, 1975, and 2020.

**b.** The population in 1940 was approximately 132,165,000. How accurate do you think your 1975 and 2020 figures are?

**29.** A car traveling along a straight road is clocked at a number of points. The data from the observations are given in the following table, where the time is in seconds, the distance is in feet, and the speed is in feet per second.

| Time | 0 | 3 | 5 | 8 | 13 |
|---|---|---|---|---|---|
| Distance | 0 | 225 | 383 | 623 | 993 |
| Speed | 75 | 77 | 80 | 74 | 72 |

**a.** Use a clamped cubic spline to predict the position of the car and its speed when $t = 10$ s.

**b.** Use the derivative of the spline to determine whether the car ever exceeds a 55-mi/h speed limit on the road; if so, what is the first time the car exceeds this speed?

**c.** What is the predicted maximum speed for the car?

**30.** The 2009 Kentucky Derby was won by a horse named Mine That Bird (at more than 50:1 odds) in a time of 2:02.66 (2 minutes and 2.66 seconds) for the $1\frac{1}{4}$-mile race. Times at the quarter-mile, half-mile, and mile poles were 0:22.98, 0:47.23, and 1:37.49.

**a.** Use these values together with the starting time to construct a natural cubic spline for Mine That Bird's race.

**b.** Use the spline to predict the time at the three-quarter-mile pole, and compare this to the actual time of 1:12.09.

**c.** Use the spline to approximate Mine That Bird's starting speed and speed at the finish line.

**31.** It is suspected that the high amounts of tannin in mature oak leaves inhibit the growth of the winter moth (*Operophtera bromata L., Geometridae*) larvae that extensively damage these trees in certain years. The following table lists the average weight of two samples of larvae at times in the first 28 days after birth. The first sample was reared on young oak leaves, whereas the second sample was reared on mature leaves from the same tree.

**a.** Use a natural cubic spline to approximate the average weight curve for each sample.