EXERCISE SET 8.2

1. Find the linear least squares polynomial approximation to $f(x)$ on the indicated interval if
   - **a.** $f(x) = x^2 + 3x + 2, \quad [0, 1];$
   - **b.** $f(x) = x^3, \quad [0, 2];$
   - **c.** $f(x) = \frac{1}{x}, \quad [1, 3];$
   - **d.** $f(x) = e^x, \quad [0, 2];$
   - **e.** $f(x) = \frac{1}{2}\cos x + \frac{1}{3}\sin 2x, \quad [0, 1];$
   - **f.** $f(x) = x \ln x, \quad [1, 3].$
2. Find the linear least squares polynomial approximation on the interval $[-1, 1]$ for the following functions.
   - **a.** $f(x) = x^2 - 2x + 3$
   - **b.** $f(x) = x^3$
   - **c.** $f(x) = \dfrac{1}{x+2}$
   - **d.** $f(x) = e^x$
   - **e.** $f(x) = \frac{1}{2}\cos x + \frac{1}{3}\sin 2x$
   - **f.** $f(x) = \ln(x + 2)$
3. Find the least squares polynomial approximation of degree two to the functions and intervals in Exercise 1.
4. Find the least squares polynomial approximation of degree 2 on the interval $[-1, 1]$ for the functions in Exercise 3.
5. Compute the error E for the approximations in Exercise 3.
6. Compute the error E for the approximations in Exercise 4.
7. Use the Gram-Schmidt process to construct $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, and $\phi_3(x)$ for the following intervals.
   - **a.** $[0, 1]$
   - **b.** $[0, 2]$
   - **c.** $[1, 3]$
8. Repeat Exercise 1 using the results of Exercise 7.
9. Obtain the least squares approximation polynomial of degree 3 for the functions in Exercise 1 using the results of Exercise 7.
10. Repeat Exercise 3 using the results of Exercise 7.
11. Use the Gram-Schmidt procedure to calculate L_1, L_2, and L_3, where $\{L_0(x), L_1(x), L_2(x), L_3(x)\}$ is an orthogonal set of polynomials on $(0, \infty)$ with respect to the weight functions $w(x) = e^{-x}$ and $L_0(x) \equiv 1$. The polynomials obtained from this procedure are called the **Laguerre polynomials**.
12. Use the Laguerre polynomials calculated in Exercise 11 to compute the least squares polynomials of degree one, two, and three on the interval $(0, \infty)$ with respect to the weight function $w(x) = e^{-x}$ for the following functions:
    - **a.** $f(x) = x^2$
    - **b.** $f(x) = e^{-x}$
    - **c.** $f(x) = x^3$
    - **d.** $f(x) = e^{-2x}$
13. Suppose $\{\phi_0, \phi_1, \ldots, \phi_n\}$ is any linearly independent set in $\prod_n$. Show that for any element $Q \in \prod_n$, there exist unique constants $c_0, c_1, \ldots, c_n$, such that
    $$Q(x) = \sum_{k=0}^{n} c_k \phi_k(x).$$
14. Show that if $\{\phi_0, \phi_1, \ldots, \phi_n\}$ is an orthogonal set of functions on $[a, b]$ with respect to the weight function w, then $\{\phi_0, \phi_1, \ldots, \phi_n\}$ is a linearly independent set.
15. Show that the normal equations (8.6) have a unique solution. [*Hint:* Show that the only solution for the function $f(x) \equiv 0$ is $a_j = 0, j = 0, 1, \ldots, n$. Multiply Eq. (8.6) by a_j, and sum over all j. Interchange the integral sign and the summation sign to obtain $\int_a^b [P(x)]^2 dx = 0$. Thus, $P(x) \equiv 0$, so $a_j = 0$, for $j = 0, \ldots, n$. Hence, the coefficient matrix is nonsingular, and there is a unique solution to Eq. (8.6).]

8.3 Chebyshev Polynomials and Economization of Power Series

The Chebyshev polynomials $\{T_n(x)\}$ are orthogonal on $(-1, 1)$ with respect to the weight function $w(x) = (1 - x^2)^{-1/2}$. Although they can be derived by the method in the previous