The parameter 0.000001 tells Maple to truncate the series when the remaining coefficients divided by the largest coefficient is smaller that 0.000001. Maple returns

$$1.266065878T(0,x) - 1.130318208T(1,x) + .2714953396T(2,x) - 0.044336849 85T(3,x)$$
$$+ 0.005474240442T(4,x) - 0.0005429263119T(5,x) + 0.00004497732296T(6,x)$$
$$- 0.000003198436462T(7,x)$$

The approximation to $e^{-0.8} = 0.449328964$ is found with

$evalf(subs(x = .8, g))$

$$0.4493288893$$

To obtain the Chebyshev rational approximation enter

$gg := convert(chebyshev(e^{-x}, x, 0.00001), ratpoly, 3, 2)$

resulting in

$$gg := \frac{0.9763521942 - 0.5893075371x + 0.1483579430x^2 - 0.01643823341x^3}{0.9763483269 + 0.3870509565x + 0.04730334625x^2}$$

We can evaluate $g(0.8)$ by

$evalf(subs(x = 0.8, g))$

which gives 0.4493317577 as an approximation to $e^{-0.8} = 0.449328964$.

The Chebyshev method does not produce the best rational function approximation in the sense of the approximation whose maximum approximation error is minimal. The method can, however, be used as a starting point for an iterative method known as the second Remez' algorithm that converges to the best approximation. A discussion of the techniques involved with this procedure and an improvement on this algorithm can be found in [RR], pp. 292–305, or in [Pow], pp. 90–92.

In 1930, Evgeny Remez (1896–1975) developed general computational methods of Chebyshev approximation for polynomials. He later developed a similar algorithm for the rational approximation of continuous functions defined on an interval with a prescribed degree of accuracy. His work encompassed various areas of approximation theory as well as the methods for approximating the solutions of differential equations.

EXERCISE SET 8.4

1. Determine all degree 2 Padé approximations for $f(x) = e^{2x}$. Compare the results at $x_i = 0.2i$, for $i = 1, 2, 3, 4, 5$, with the actual values $f(x_i)$.
2. Determine all degree 3 Padé approximations for $f(x) = x\ln(x+1)$. Compare the results at $x_i = 0.2i$, for $i = 1, 2, 3, 4, 5$, with the actual values $f(x_i)$.
3. Determine the Padé approximation of degree 5 with $n = 2$ and $m = 3$ for $f(x) = e^x$. Compare the results at $x_i = 0.2i$, for $i = 1, 2, 3, 4, 5$, with those from the fifth Maclaurin polynomial.
4. Repeat Exercise 3 using instead the Padé approximation of degree 5 with $n = 3$ and $m = 2$. Compare the results at each x_i with those computed in Exercise 3.
5. Determine the Padé approximation of degree 6 with $n = m = 3$ for $f(x) = \sin x$. Compare the results at $x_i = 0.1i$, for $i = 0, 1, \dots, 5$, with the exact results and with the results of the sixth Maclaurin polynomial.
6. Determine the Padé approximations of degree 6 with (a) $n = 2, m = 4$ and (b) $n = 4$, $m = 2$ for $f(x) = \sin x$. Compare the results at each x_i to those obtained in Exercise 5.
7. Table 8.10 lists results of the Padé approximation of degree 5 with $n = 3$ and $m = 2$, the fifth Maclaurin polynomial, and the exact values of $f(x) = e^{-x}$ when $x_i = 0.2i$, for $i = 1, 2, 3, 4,$

and 5. Compare these results with those produced from the other Padé approximations of degree five.

a. $n = 0, m = 5$ **b.** $n = 1, m = 4$ **c.** $n = 3, m = 2$ **d.** $n = 4, m = 1$

8. Express the following rational functions in continued-fraction form:

a. $\dfrac{x^2 + 3x + 2}{x^2 - x + 1}$ **b.** $\dfrac{4x^2 + 3x - 7}{2x^3 + x^2 - x + 5}$

c. $\dfrac{2x^3 - 3x^2 + 4x - 5}{x^2 + 2x + 4}$ **d.** $\dfrac{2x^3 + x^2 - x + 3}{3x^3 + 2x^2 - x + 1}$

9. Find all the Chebyshev rational approximations of degree 2 for $f(x) = e^{-x}$. Which give the best approximations to $f(x) = e^{-x}$ at $x = 0.25$, 0.5, and 1?

10. Find all the Chebyshev rational approximations of degree 3 for $f(x) = \cos x$. Which give the best approximations to $f(x) = \cos x$ at $x = \pi/4$ and $\pi/3$?

11. Find the Chebyshev rational approximation of degree 4 with $n = m = 2$ for $f(x) = \sin x$. Compare the results at $x_i = 0.1i$, for $i = 0, 1, 2, 3, 4, 5$, from this approximation with those obtained in Exercise 5 using a sixth-degree Padé approximation.

12. Find all Chebyshev rational approximations of degree 5 for $f(x) = e^x$. Compare the results at $x_i = 0.2i$, for $i = 1, 2, 3, 4, 5$, with those obtained in Exercises 3 and 4.

13. To accurately approximate $f(x) = e^x$ for inclusion in a mathematical library, we first restrict the domain of f. Given a real number x, divide by $\ln\sqrt{10}$ to obtain the relation

$$x = M \cdot \ln\sqrt{10} + s,$$

where M is an integer and s is a real number satisfying $|s| \le \frac{1}{2}\ln\sqrt{10}$.

a. Show that $e^x = e^s \cdot 10^{M/2}$.

b. Construct a rational function approximation for e^s using $n = m = 3$. Estimate the error when $0 \le |s| \le \frac{1}{2}\ln\sqrt{10}$.

c. Design an implementation of e^x using the results of part (a) and (b) and the approximations

$$\frac{1}{\ln\sqrt{10}} = 0.8685889638 \quad \text{and} \quad \sqrt{10} = 3.162277660.$$

14. To accurately approximate $\sin x$ and $\cos x$ for inclusion in a mathematical library, we first restrict their domains. Given a real number x, divide by π to obtain the relation

$$|x| = M\pi + s, \quad \text{where } M \text{ is an integer and } |s| \le \frac{\pi}{2}.$$

a. Show that $\sin x = \operatorname{sgn}(x) \cdot (-1)^M \cdot \sin s$.

b. Construct a rational approximation to $\sin s$ using $n = m = 4$. Estimate the error when $0 \le |s| \le \pi/2$.

c. Design an implementation of $\sin x$ using parts (a) and (b).

d. Repeat part (c) for $\cos x$ using the fact that $\cos x = \sin(x + \pi/2)$.

8.5 Trigonometric Polynomial Approximation

The use of series of sine and cosine functions to represent arbitrary functions had its beginnings in the 1750s with the study of the motion of a vibrating string. This problem was considered by Jean d'Alembert and then taken up by the foremost mathematician of the time, Leonhard Euler. But it was Daniel Bernoulli who first advocated the use of the infinite sums of sine and cosines as a solution to the problem, sums that we now know as Fourier series. In the early part of the 19th century, Jean Baptiste Joseph Fourier used these series to study the flow of heat and developed quite a complete theory of the subject.