**Problem 01** (100 points) This is a MATLAB exercise.

**(a)** Download the data file: HW_01.mat from Canvas to your working directory, and load it into your MATLAB session by:

```
>> load HW_01;
```

Then, draw the signal $\mathbf{x}$ in the data file using the following commands:

```
>> figure(1);
>> stem(x); hold on; plot(x); grid;
```

Note that this signal $\mathbf{x}$ consists of only 8 points, i.e., a very short signal (vector).

**(b)** In a different figure window, draw the 8 basis vectors stored as column vectors of the matrix $U$ as follows:

```
>> figure(2);
>> for k=1:8
     subplot(8,1,k);
     stem(U(:,k)); axis([0 9 -0.5 0.5]); axis off; hold on;
   end
>> for k=1:8
     subplot(8,1,k);
     plot(U(:,k));
   end
```

You may need to see the details of these 8 plots by enlarging the window to a full screen. Print this figure and attach it to your HW submission.

**(c)** Compute the expansion coefficients (i.e., the weights of the linear combination) of $\mathbf{x}$ with respect to the basis vectors $U(:,1),\ldots,U(:,8)$ via

```
>> a=U'*x;
```

**(d)** Check the values of the entries of the coefficient vector $\mathbf{a}$ and create a new vector $\mathbf{a}2$ of length 8 whose only nonzero entries are the two largest entries of $\mathbf{a}$ in terms of their absolute values.

**(e)** Construct an approximation $\mathbf{x}2$ of $\mathbf{x}$ using $\mathbf{a}2$. Then, plot $\mathbf{x}2$ over Figure 1 as follows:

```
>> figure(1); stem(x2,'r*'); plot(x2,'r');
```

**(f)** Now, instead of $\mathbf{a}2$, let's construct $\mathbf{a}4$ of length 8 whose only nonzero entries are the four largest entries of $\mathbf{a}$ in terms of their absolute values. Then,

**(g)** Construct an approximation $\mathbf{x}4$ of $\mathbf{x}$ using $\mathbf{a}4$. Then, plot $\mathbf{x}4$ over Figure 1 as follows (note using the different color from $\mathbf{x}2$):

```
>> figure(1); stem(x4,'gx'); plot(x4,'g');
```

Then, print out Figure 1, and attach it to your HW submission.

**(h)** Consider now $\mathbf{x}8$, which is just a full reconstruction without throwing out any coefficients, i.e.,

```
>> x8=U*a;
```

Finally, compute the relative error of $\mathbf{x}8$ by

```
>> sqrt(sum((x-x8).^2)/sum(x.^2))
```

and report the result. Similarly compute the relative error of $\mathbf{x}4$ and $\mathbf{x}2$, and report the results.

**(i)** Write a detailed explanation of what this MATLAB program does.

**Solution to Problem 01:** Here is one possible MATLAB script used to solve this problem. The data on the following page comes from running this MATLAB script.

Code Included: 20 points
-5 points If your code runs with an error

```
clear, clc
% Homework 1: Math 167
% Problem 1: Change of Basis to the Discrete Cosine Basis
% Related to Music Sampling

%P1-Part (a)
load hw01.mat                   %Download Data
figure(1)                       %Initialize figure 1
stem(x);                        %plot stems
hold on;                        %allow for multiple graphs on same figure
plot(x);                        %plot piecewise linear funciton
grid;                           %print grid on figure 1

%P1-Part(b): View Basis vectors
figure(2);                      %Initialize figure 2
for k = 1:8
    subplot(8,1,k);
    stem(U(:,k));
    axis([0 9 -.5 .5]);
    axis off;
    hold on;
end %for(k)

for k = 1:8
    subplot(8,1,k);
    plot(U(:,k));
end %for(k)

%P1-Part(c): Compute Expansion coefficients
a = U'*x;

%Part(d-e): Create a2: isolate 2 largest magnitude entries (thresholding)
[tmp, I] = sort(abs(a),'descend');  %find indices: largest to smallest mag.
csa = size(a,1);                    %column size of a
a2 = a;                             %initialize a2
NLM_d = 2;                          %Num of Largest Mag. of coeff part d
for k = NLM_d+1:csa,
    a2(I(k,1)) = 0;                 %zero out 6 smallest magnitude coeff.
end %for(k)
x2 = U*a2;
figure(1); stem(x2,'r*'); plot(x2,'r');
```

```
%P1-Part(f-g): Create a4: isolate 4 largest magnitude entries
a4 = a;                                 %initialize a4
NLM_e = 4;                              %Num of Largest Mag. of coeff part e
for k = NLM_e+1:csa
     a4(I(k)) = 0;                      %zero out 4 smallest magnitude coeff.
end %for(k)
x4 = U*a4;
figure(1); stem(x4,'gx'); plot(x4,'g');

%Part(h)
x8 = U*a;
rel_err_x8 = sqrt(sum((x-x8).^2)/sum(x.^2));
rel_err_x4 = sqrt(sum((x-x4).^2)/sum(x.^2));
rel_err_x2 = sqrt(sum((x-x2).^2)/sum(x.^2));
```

## REPORT

Explaining Output 20 points -5 points If your explanation makes no sense or does not demonstrate that you tried to explain the results in your own words.

From the data given in the file HW_01.mat MATLAB generates the following graphs. Figure 1 is a graphical representation of the data represented in the vectors $\mathbf{x}$ (blue), $\mathbf{x}_2$ (red), and $xvec_4$ (green). This figure illustrates how well the *thresholding* method works for approximating the original signal.

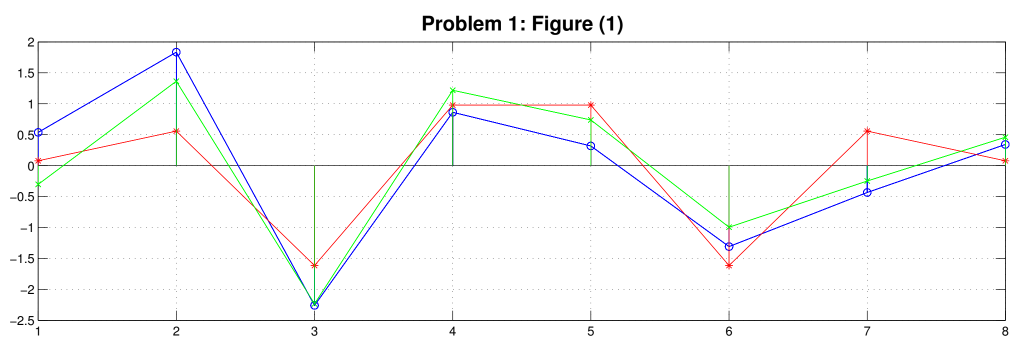



Figure 1: Problem 01

Output Data 20 points -10 points If your graph didn't match the solution. In other words, incorrect relative error.

Properly generated output figures:

Figure 2 two is a graphical representation of the Discrete Cosine Transform (DCT) basis vectors stored in the $8 \times 8$ matrix $U$. Notice that each of these looks like a cosine function with increasing frequency $cos(kx)$, for $k = 0, 2, \ldots, 7$ The Discrete Cosine Transform is a beautiful concept in applied mathematics and, as is partially apparent in this problem, it is intimately connected to a wide variety of linear algebra algorithms.

```
Relative error of x2 = 0.5757
Relative error of x4 = 0.34403
Relative error of x8 = 1.5193e-16
```

Correct figures 40 points -20 points for each missing figure

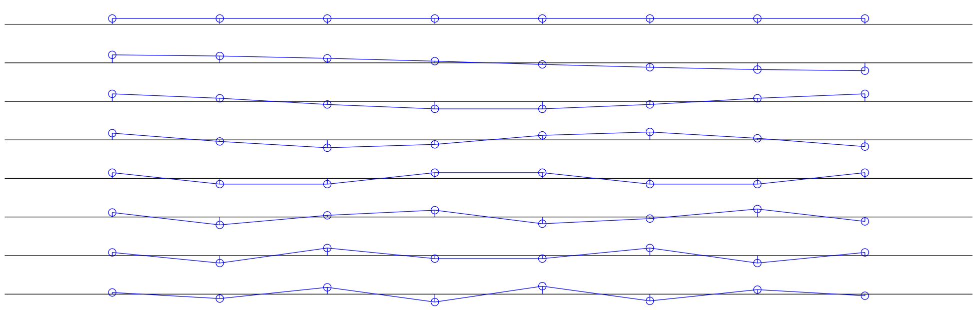
It is apparent that,

(a) To machine zero, the vector of coefficients $\mathbf{x}_8$ is identical to the original coefficient vector $\mathbf{x}$.

(b) The vector $\mathbf{x}_4$ has a relative error $0.34403$ or about 34% to the original signal.

(c) The approximate coefficient vector $\mathbf{x}_2$ has a relative error $0.5757$, or about 57% to the original signal.

Figure 2: Problem 01



---

**COMMENTS TO THE STUDENTS:**

A) Some of you did not use the naming convention for the files This made grading your work more time consuming on my end. For future homework, and other files such as the Programming Project, I will take off 10 points for failure to properly name your file.

B) A large number of you generated line plots where two of the lines were light green, so it is difficult to see them. I did not take off points for this, but please use a proper color palate.

C) A lot of you did not explain their program in MATLAB and just restated almost verbatim the homework prompt. That will also lose points in the future. Please look at the Professor's sample **REPORT** above for an example of how to properly write a report.

**POINTS FOR EACH OF THE FOLLOWING QUESTIONS**

**Problem 02:** 10 points each for (a)–(c)

**Problem 03:** 10 points

**Problem 04:** 20 points each for (a)–(b); 5 points per correct $\mathbf{q}_1$, $\mathbf{q}_1, \ldots, \mathbf{q}_4$

**Problem 05:** 05 points each for (a)–(h)

**Problem 06:** 02 points for (a) (i)–(v) and the same for (b)

**Problem 02** Consider the following set of terms (words) and documents (or rather book titles):

| | Terms | | Documents |
|---|---|---|---|
| T1: | Book (Handbook, BOOK) | D1: | The Princeton Companion to Mathematics |
| T2: | Equation (Equations) | D2: | NIST Handbook of Mathematical Functions |
| T3: | Function (Functions) | D3: | Table of Integrals, Series, and Products |
| T4: | Integral (Integrals) | D4: | Linear Integral Equations |
| T5: | Linear | D5: | Proofs from THE BOOK |
| T6: | Mathematics (Mathematical) | D6: | The Book of Numbers |
| T7: | Number (Numbers) | D7: | Number Theory in Science and Communication |
| T8: | Series | D8: | Green's Functions and Boundary Value Problems |
| | | D9: | Discourse on Fourier Series |
| | | D10: | Basic Linear Partial Differential Equations |
| | | D11: | Mathematical Physics, An Advanced Course |

**(a)** Construct $8 \times 11$ term-document matrix.

**(b)** Suppose we want to query "Integral Equation." Construct the query vector $\mathbf{a}$.

**(c)** Find three closest documents for the query in (b).

**Solution to Problem 02:** Recall from Lecture that to set up the term-document matrix the row indices represent the terms while the column entries represent the documents. Looking down a column, we consider the document title that the column represents. Then, focusing on a single document (a single column), we mark a 1 in the row corresponding to terms that arise in the document title. All other rows are filled in with zero.

Term-Document Matrix

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ | $D_6$ | $D_7$ | $D_8$ | $D_9$ | $D_{10}$ | $D_{11}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_1$ | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| $T_2$ | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| $T_3$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| $T_4$ | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $T_5$ | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| $T_6$ | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| $T_7$ | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| $T_8$ | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |

To query the term “integral equation,” note that $T_2$ is equations and $T_4$ is integral. Then, the query vector is given by

$$\mathbf{q}^T = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The three closest documents for this query are given by $D_4$, $D_3$ and $D_{10}$.

**Problem 03** (This is a review problem.) At the beginning of 2009, the population of California was 36,453,973. The population living in the United States but outside of California was 271,491,582. During that year, 458,682 people moved to California from another state. Similarly, 545,921 people moved from California to elsewhere in the United States. Set up a matrix vector multiplication problem whose solution shows the population changes in California and in the rest of the United States for 2009.

**First Possible Solution to Problem 03:** First, translate the word problem into a formal problem statement:

**(a)** Given the population of California at the beginning of 2009 and changes in the population throughout the year, find the population of California at the end of 2009.

**(b)** Given the population of the USA outside of California at the beginning of 2009 and changes in the population throughout the year, find the population of the USA outside of California at the end of 2009.

From these problem statements, formulate how to calculate the population of California at the end of 2009 given our data by starting with problem (a):

$$\begin{aligned} \text{Population of California at the end of 2009} = {} & \text{Population of California at the start of 2009} \\ & + \text{Total-In Migration} - \text{Total Out-Migration} \end{aligned} \quad (1)$$

Let $C_1$ be the population of California at the end of 2009. Then

$$C_1 = 36453973 + 458682 + (-1) \cdot 545921.$$

Now, by analogy, let $U_1$ be the population of US outside of California at the end of 2009. Then,

$$U_1 = 271491582 + 545921 + (-1) \cdot 458682.$$

Then, recognizing this as a linear system, the following equation holds:

$$\begin{bmatrix} C_1 \\ U_1 \end{bmatrix} = \begin{bmatrix} 36453973 & 458682 & 545921 \\ 271491582 & 545921 & 458682 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

**Second Possible Solution to Problem 03:** First, translate the word problem into a formal problem statement:

**(i)** Given the population of California at the beginning of 2009 and changes in the population throughout the year, find the population change in California at the end of 2009.

**(ii)** Given the population of the USA outside of California at the beginning of 2009 and changes in the population throughout the year, find the population change in the USA at the end of 2009.

From these problem statements, formulate an intuitive guess to how to calculate the population of California at the end of 2009 given our data by starting with problem **(i)**

$$\text{Population change in CA at the end of 2009} = \text{Total-In Migration } - \text{Total Out-Migration }.$$

Let $\delta C$ be the population of California at the end of 2009. Then

$$\delta C = 458682 + (-1) \cdot 545921.$$

Now, by analogy, let $\delta U$ be the population of the US outside of CA at the end of 2009. Then,

$$\delta U = 545921 + (-1) \cdot 458682.$$

Recognizing this as a linear system, the following equation holds,

$$\begin{bmatrix} \delta C \\ \delta U \end{bmatrix} = \begin{bmatrix} 458682 & 545921 \\ 545921 & 458682 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

**Problem 04** (This is another review problem.) Use the Gram-Schmidt Process to construct an orthonormal basis for $\boldsymbol{R}^4$ starting with the following set of vectors.

**Solution to Problem 04:** Given a set of linearly independent vectors

$$\mathbf{x}_1\,, \qquad \mathbf{x}_2\,, \qquad \mathbf{x}_3\,, \qquad \mathbf{x}_4\,,$$

We first determine a set of mutually orthogonal vectors with the following procedure. $\tilde{\mathbf{v}}_1$ , $\tilde{\mathbf{v}}_2$ , $\tilde{\mathbf{v}}_3$ , and $\tilde{\mathbf{v}}_4$:

$$\begin{aligned}
\tilde{\mathbf{v}}_1 &= \mathbf{x}_1 \\
\tilde{\mathbf{v}}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2^T \tilde{\mathbf{v}}_1}{\tilde{\mathbf{v}}_1^T \tilde{\mathbf{v}}_1} \tilde{\mathbf{v}}_1 \\
\tilde{\mathbf{v}}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3^T \tilde{\mathbf{v}}_1}{\tilde{\mathbf{v}}_1^T \tilde{\mathbf{v}}_1} \tilde{\mathbf{v}}_1 - \frac{\mathbf{v}_3^T \tilde{\mathbf{v}}_2}{\tilde{\mathbf{v}}_2^T \tilde{\mathbf{v}}_2} \tilde{\mathbf{v}}_2 \\
\tilde{\mathbf{v}}_4 &= \mathbf{x}_4 - \frac{\mathbf{x}_4^T \tilde{\mathbf{v}}_1}{\tilde{\mathbf{v}}_1^T \tilde{\mathbf{v}}_1} \tilde{\mathbf{v}}_1 - \frac{\mathbf{x}_4^T \tilde{\mathbf{v}}_2}{\tilde{\mathbf{v}}_2^T \tilde{\mathbf{v}}_2} \tilde{\mathbf{v}}_2 - \frac{\mathbf{x}_4^T \tilde{\mathbf{v}}_3}{\tilde{\mathbf{v}}_3^T \tilde{\mathbf{v}}_3} \tilde{\mathbf{v}}_3
\end{aligned}$$

The vectors $\tilde{\mathbf{v}}_1$, $\tilde{\mathbf{v}}_2$, $\tilde{\mathbf{v}}_3$, and $\tilde{\mathbf{v}}_4$ are orthogonal, but they don't have unit length, so we *normalize* them to obtain

$$\mathbf{q}_1 = \frac{\tilde{\mathbf{v}}_1}{\|\tilde{\mathbf{v}}_1\|}, \qquad \mathbf{q}_2 = \frac{\tilde{\mathbf{v}}_2}{\|\tilde{\mathbf{v}}_2\|}, \qquad \mathbf{q}_3 = \frac{\tilde{\mathbf{v}}_3}{\|\tilde{\mathbf{v}}_3\|}, \qquad \mathbf{q}_4 = \frac{\tilde{\mathbf{v}}_4}{\|\tilde{\mathbf{v}}_4\|}.$$

Thus, $\mathscr{B} = \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4\}$ is an orthonormal basis for $\boldsymbol{R}^4$.

**(a)** Use the Gram-Schmidt procedure to construct an orthonormal basis for $\boldsymbol{R}^4$ starting with the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, and $\mathbf{x}_4$.

$$\mathbf{x}_1 = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix}0\\1\\0\\-1\end{bmatrix}, \qquad \mathbf{x}_3 = \begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \qquad \mathbf{x}_4 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

In this case the orthonormal basis that results from the Gram-Schmidt procedure is

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \qquad \mathbf{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\-1\end{bmatrix}, \qquad \mathbf{u}_3 = \frac{1}{2}\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}, \qquad \mathbf{u}_4 = \frac{1}{2}\begin{bmatrix}-1\\1\\1\\1\end{bmatrix}.$$

**(b)** Use the Gram-Schmidt procedure to construct an orthonormal basis for $\boldsymbol{R}^4$ starting with the vectors $\mathbf{y}_1$, $\mathbf{y}_2$, $\mathbf{y}_3$, and $\mathbf{y}_4$.

$$\mathbf{y}_1 = \begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \qquad \mathbf{y}_2 = \begin{bmatrix}4\\1\\0\\0\end{bmatrix}, \qquad \mathbf{y}_3 = \begin{bmatrix}1\\0\\2\\1\end{bmatrix}, \qquad \mathbf{y}_4 = \begin{bmatrix}0\\2\\0\\1\end{bmatrix}$$

In this case the orthonormal basis that results from Gram-Schmidt process shown above yields

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \qquad \mathbf{v}_2 = \frac{1}{3}\begin{bmatrix}2\\1\\0\\-2\end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \qquad \mathbf{v}_4 = \frac{1}{3\cdot\sqrt{2}}\begin{bmatrix}-1\\4\\0\\1\end{bmatrix}$$

**NOTE:** If you are looking at these solutions and do not understand how these vectors were calculated, please get in contact with the instructor or TA ASAP and speak to them about these ideas. The following MATLAB function runs the Modified Gram-Schmidt algorithm. It assumes that the input matrix $A$ has full column rank.

```
function [q,r] = mgs(A)

m = size(A,1);
n = size(A,2);
q = zeros(m, n);
r = zeros(n, n);

for i = 1:n
    v(:,i)=A(:,i);
end %for (i)

for i = 1:n
```

```
    r(i,i) = norm(v(:,i),2);
    q(:,i) = v(:,i)/r(i,i);
    for k = i+1:n
        r(i,k) = q(:,i)'*v(:,k);
        v(:,k) = v(:,k) - r(i,k)*q(:,i);
    end %for(k)
end %for(i)
```

**(c)** Check your work using MATLAB by writing your own Gram-Schmidt algorithm. **NOTE:** This was NOT a part of HW 01.

**Problem 05** (This is also a review problem.) Consider the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

**(a)** View this matrix as a linear transformation, $T$, between two vector spaces. What are the *domain* and the *codomain* of $T$?

**Solution to Problem 05 (a):** The matrix $A \in \boldsymbol{R}^{7\times 4}$ has 7 rows and 4 columns. Thus, $A$ represents a linear transformation $T$, with *domain* $\boldsymbol{R}^4$ and *codomain* $\boldsymbol{R}^7$; i.e., $T : \boldsymbol{R}^4 \to \boldsymbol{R}^7$.

**(b)** Identify the column space, $C(A)$, or *image*, of the linear transformation $T$.

**Solution to Problem 05 (b):** While $A$ maps vectors from $\boldsymbol{R}^4$ to $\boldsymbol{R}^7$, this linear transformation is not *onto*. In fact, the column space, $C(A)$, or *image* of the matrix $A$ under vector multiplication is the space

$$C(A) = \left\{ \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & 0 & 0 & 0 \end{bmatrix}^T \mid x_i \in \boldsymbol{R} \text{ for each } i = 1, 2, 3, 4 \right\}$$

**(c)** Find a basis for the column space, $C(A)$, of $T$.

**Solution to Problem 05 (c):** One basis for $C(A)$ is

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

In other words, $C(A) = \text{ span } \{\mathbf{e}_1, \mathbf{e}_1, \mathbf{e}_1, \mathbf{e}_1\}$ and, since the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ are linearly independent,

$$\mathscr{B}_{C(A)} = \{ \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4 \} \tag{2}$$

is a basis for $C(A)$.

**(d)** Identify the nullspace, $N(A)$, which is also known as the kernel of $T$.

**First solution to Problem 05 (d):** This uses the rank $r$ of the matrix $A$, which is the solution to **Problem 05 (h)**, to obtain the solution to this problem immediately from our understanding of the structure of the four fundamental subspaces of $A$. (See §3.6 of Strang or the handout entitled 'The Four Fundamental Subspaces' for a review.)

Since $A$ is an $m \times n = 7 \times 4$ matrix we know that $A : \boldsymbol{R}^4 \to \boldsymbol{R}^7$ Furthermore, we know that $N(A)$ is a subspace of the domain of $A$, which, From **Problem 05 (a)**, is $R^m = R^4$. Thus, $dim\,(N(A)) = m - r$, where $r$ is the rank of $A$ and hence, $dim\,(N(A)) = 4 - 4 = 0$. The only vector space $V$ that is a subspace of $\boldsymbol{R}^4$ with $dim\,(V)$ is the set consisting of just the zero vector. Hence,

$$N(A) \;=\; \{\, \mathbf{0} \,\}\,. \tag{3}$$

**(e)** Write a basis for the nullspace, $N(A)$, of $T$.[1]

**First solution to Problem 05 (e):** From **Problem 05 (d)** and, in particular, (3), we know that $dim\,(N(A)) = 0$ and hence, has no basis.

**A more general solution to Problem 05 (e):** By definition, the nullspace of $A$ is

$$N(A) \;=\; \{\, \mathbf{0} \,\}\,.$$

To determine which vectors $\mathbf{x}$ satisfy this relation, consider the matrix vector multiplication of $A$ with an arbitrary vector $\mathbf{x}$

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \;=\; x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ 7 \\ 6 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\,.$$

This corresponds to the linear system with four equations given by:

$$\begin{aligned} x_1 + 2 \cdot x_2 + 3 \cdot x_3 + 4 \cdot x_4 &= 0 \\ x_2 + 5 \cdot x_3 + 7 \cdot x_4 &= 0 \\ x_3 + 6 \cdot x_4 &= 0 \\ x_4 &= 0 \end{aligned}$$

Backsolving this system of equations yields $x_4 = x_3 = x_2 = x_1 = 0$. Thus, the nullspace of $A$ consists only of the zero vector $\mathbf{0} \in \boldsymbol{R}^4$,

$$N(A) \;=\; \{\, \mathbf{0} \,\}$$

and hence, it has no basis

[1] At the top of page 173 in the Fifth Edition of Strang's *Introduction to Linear Algebra* the author adopts the convention that the basis for the subspace $\mathbf{Z}\}$ is the empty set $\{\;\;\}$. This is what I call an 'edge case', I don't entirely like this solution to how define a basis for $\mathbf{Z}$, when there is no good definition that satisfies all of the appropriate definitions required of a basis. I am removing this problem from the point total. Please make sure your points add up 237 points without problem 5(e).

**(f)** What is the row space of this linear transformation?

**First solution to Problem 05 (f):** The row space, $C(A^T)$, of $A$ is a subspace of the domain $\boldsymbol{R}^m = \boldsymbol{R}^4$. The Fundamental Theorem of Linear Algebra states that the dimension of the row space $C(A^T)$ of $A$ is equal to the rank of $A$, so $dim\left(C(A^T)\right) = r = 4$. But the only subspace of $\boldsymbol{R}^4$ with dimension $4$ is $\boldsymbol{R}^4$ itself, and hence $C(A^T) = \boldsymbol{R}^4$.

**Second solution to Problem 05 (f):** The row space, $C(A^T)$, of $A$ is the subspace of the domain $\boldsymbol{R}^4$ of $A$ that is spanned by the rows of $A$. Notice that there are four linearly independent rows of $A$ and thus, the dimension of the row space, $C(A^T)$, is $4$. Thus, as we reasoned at the end of the first solution to Problem 05 (f) above, we must have $C(A^T) = \boldsymbol{R}^4$.

**(g)** What is the left null space, $N(A^T)$, of $T$.

**Solution to Problem 05 (g):** From our understanding of the four fundamental subspaces of $A$ we know $N(A^T)$ is a subspace of the codomain $\boldsymbol{R}^7$ of $A$ and is the orthogonal complement of the column space $C(A)$ in $R^7$. (See §4.1 of Strang for a review of orthogonal complements.) Since by Problem 05 (c)

$$\mathscr{B}_{C(A)} \;=\; \{\, \mathbf{e}_1 \,, \mathbf{e}_2 \,, \mathbf{e}_3 \,, \mathbf{e}_4 \,\}$$

is a basis for $C(A)$, if follows that one basis for the left nullspace $N(A^T)$ of $A$ consists of the remaining standard, or *canonical*, basis vectors in $R^7$,

$$\mathscr{B}_{N(A^T)} \;=\; \{\, \mathbf{e}_5 \,, \mathbf{e}_6 \,, \mathbf{e}_7 \,\}$$

**(h)** What is the rank of the matrix A?

**Solution to Problem 05 (h):** By definition, the rank $r$ of $A$ is the number of linear independent columns in $A$ or, equivalently, the dimension of the column space of $A$, $r = dim\,(C(A))$. Since, by problem Problem 05 (c), the basis $\mathscr{B}_{C(A)}$ in equation (2) $r = dim\,(C(A))$. In this case the rank of $A$ is $r = 4$. The Fundamental Theorem of Linear Algebra states the rank of $A$ is also equal to the dimension of the column space which is also equal to the dimension of the row space $C(A^T)$ of $A$; i.e., $dim\,(C(A)) = dim\left(C(A^T)\right) = r$.

There is much more to be said about these relations. Please check out the *Fredholm alternative* in Linear Algebra which concerns the fourth fundamental subspace of $A$; namely, the left nullspace of $A$, $N((A^T)$, for more on this subject or ask the professor or TA about this in office hours.

**Problem 06** Convert the following numbers to from base X to base Y as specified below.

**(a)** Convert the following binary numbers to decimal:

**Solution to Problem 06 (a):** Start with the first five numbers represented in binary form. Using the format discussed in lecture, notice that an $n+1$ digit binary number can be expanded as

$$\underbrace{b_n b_{n-1} ... b_2 b_1 b_0}_{\text{base 2 representation}} \;=\; \underbrace{b_n \cdot 2^n + b_{n-1} \cdot 2^{n-1} + \cdots + b_2 \cdot 2^2 + b_1 \cdot 2^1 + b_0 \cdot 2^0}_{\text{base 10 representation}}$$

where the $i$th binary digit $b_i$ is either $0$ or $1$ for each $i \in \{0, 1, ...n\}$. Then, with this is mind, we have

(i) $11111 = 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = 16 + 8 + 4 + 2 + 1 = 31$

(ii) $1000000 = 1 \cdot 2^6 + 0 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0 = 64$

(iii) $1001101101 = 1 \cdot 2^9 + 0 \cdot 2^8 + 0 \cdot 2^7 + 1 \cdot 2^6 + 1 \cdot 2^5 + 0 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$

To finish this exercise, use the concept of a dot product from linear algebra to concisely illustrate the binary to decimal conversion process.

(iv)
$$10101010 = \begin{bmatrix} 2^7 & 2^6 & 2^5 & 2^4 & 2^3 & 2^2 & 2^1 & 2^0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 2^7 + 2^5 + 2^3 + 2^1$$

(v)
$$000011110000 = \begin{bmatrix} 2^{11} & 2^{10} & 2^9 & 2^8 & 2^7 & 2^6 & 2^5 & 2^4 & 2^3 & 2^2 & 2^1 & 2^0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 2^7 + 2^6 + 2^5 + 2^4$$

**(b)** Convert the following binary numbers to decimal.

**Solution to Problem 06 (b):** Write out a table of the powers of two from $2^0$ through $2^{14}$:

Powers of Two

| $n$ | $2^n$ |
|---|---|
| 0 | 1 |
| 1 | 2 |
| 2 | 4 |
| 3 | 8 |
| 4 | 16 |
| 5 | 32 |
| 6 | 64 |
| 7 | 128 |
| 8 | 256 |
| 9 | 512 |
| 10 | 1024 |
| 11 | 2048 |
| 12 | 4096 |
| 13 | 8192 |
| 14 | 16384 |

Then, we see that

(i) $73 = 64 + 8 + 1 = 2^6 + 2^3 + 2^0 = 1001001$

(ii) $127 = 64 + 32 + 16 + 8 + 4 + 2 + 1 = 2^6 + 2^5 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0 = 1111111$

(iii) $402 = 256 + 128 + 16 + 2 = 2^8 + 2^7 + 2^4 + 2^1 = 110010010$

(iv) $512 = 2^9 = 1000000000$

(v) $1000 = 512 + 256 + 128 + 64 + 32 + 8$

(vi) 
$$\begin{aligned} 32767 &= 16384+8192+4096+2048+1024+512+256+128+64+32+16+8+4+2+1 \\ &= 2^{15}+2^{14}+2^{13}+2^{12}+2^{11}+2^{10}+2^9+2^8+2^7+2^6+2^5+2^4+2^3+2^2+2^1+2^0 \\ &= 111111111111111 \end{aligned}$$