Problem 00 First of all, carefully reread Chapter 4 and read Chapter 5 in Eldén.

Problem 01 Let $A \in \mathbb{R}^{m\times m}$ be a *symmetric* matrix. As you have already learned in MAT 22A or MAT 67, an *eigenvector* of A is a nonzero vector $\mathbf{x} \in \mathbb{C}^m$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some $\lambda \in \mathbb{C}$, the corresponding *eigenvalue*. Here $\mathbb{C}$ denotes the complex numbers; i.e. all numbers of the form $z = a + b\,i$ where a and b are real numbers and $i = \sqrt{-1}$ is the square root of -1. Also, you will also need to know that $\overline{z} = a - b\,i$ is the *complex conjugate* of z and if z_1 and z_2 are any two complex numbers we have $\overline{z_1\,z_2} = \overline{z_1}\;\overline{z_2}$.

In the following problem, you may assume that all of the eigenvalues of A are *distinct*.

(a) Prove that all of the eigenvalues of A are real.

[Hint: If λ is a (complex) eigenvalue of A with eigenvector $\mathbf{x}$, then it's complex conjugate $\overline{\lambda}$ is also an eigenvalue of A with eigenvector $\overline{\mathbf{x}}$.]

(b) Prove that if $\mathbf{x}_1$ and $\mathbf{x}_2$ are eigenvectors corresponding to distinct eigenvalues, then $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal.

Problem 02 Let

$$A = \begin{bmatrix} 1 & 1 \\ \epsilon & 0 \\ 0 & \epsilon \end{bmatrix},$$

where ϵ is a small positive number (e.g., 10^{-8}) so that ϵ^2 can be ignored numerically.

(a) Compute the reduced QR factorization $A = \widehat{Q}\widehat{R}$ using the classical Gram-Schmidt algorithm by hand.

(b) Compute the reduced QR factorization $A = \widehat{Q}\widehat{R}$ using the modified Gram-Schmidt algorithm by hand.

(c) Compute the full QR factorization $A = QR$ using the Householder triangularization by hand.

(d) Check the quality of these results by computing the Frobenius norm of $\|\widehat{Q}^T\widehat{Q} - I\|_F$ for the results obtained by the CGS and MGS algorithms and $\|Q^TQ - I\|_F$ for the result obtained by the Householder triangularization.

Problem 03 Let $E \in \mathbb{R}^{m\times m}$ that extracts the "even part" of an m-vector: $E\mathbf{x} = (\mathbf{x} + F\mathbf{x})/2$, where $F \in \mathbb{R}^{m\times m}$ flips $\mathbf{x} = [x_1, \ldots, x_m]^T$ to $\mathbf{x} = [x_m, \ldots, x_1]^T$.

(a) Is E an orthogonal projector, an oblique projector, or not a projector at all?

(b) What are its entries?

Problem 04 Take $m = 50$, $n = 12$. Using MATLAB's `linspace`, define t to be the m-vector corresponding to linearly spaced grid points from 0 to 1. Using MATLAB's `vander` and `fliplr`, define A to be the $m \times n$ matrix associated with least squares fitting on this grid by a polynomial of order $n-1$. Take $\mathbf{b}$ to be the function $\cos(4t)$ evaluated on the grid. Now, calculate and print (to 16 digit precision) the least squares coefficient vector $\mathbf{x}$ by the following three methods.

(a) Solving the normal equation explicitly computing $(A^TA)^{-1}$.

(b) Using the MATLAB implementation CGS.m of the classical Gram-Schmidt algorithm `CGS`, which can be downloaded from CANVAS.

(c) Using the MATLAB implementation MGS.m of the modified Gram-Schmidt algorithm `MGS`, which can be downloaded from CANVAS.

(d) QR factorization using MATLAB's `qr`, which is based on the Householder triangularization.

(e) `x = A \ b` in MATLAB, which is also based on QR factorization.

(f) The calculations above will produce five lists of twelve coefficients. In each list, use the "\textcolor{color}{words}" function in LaTeX

\textcolor{red}{This sentence will be in red.} → This sentence will be in red.

to highlight the digits that appear to be incorrect; i.e., affected by rounding error.

- Comment on the differences you observe.
- Do the normal equations exhibit instability?

Although, explanations for what you observe are welcome, you are not *required* to explain your observations.