Exploration 1: Finding the pathway and speed of Red, when air resistance is negligible

Assuming that air resistance is negligible and the only force acting on the object is gravity, the object travels through the air with projectile motion.

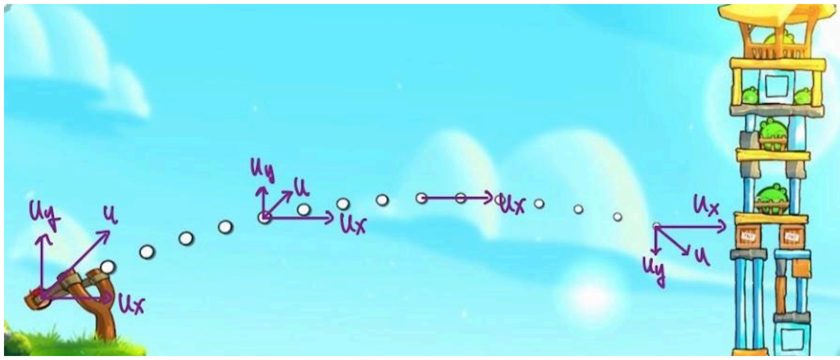


Figure 7. Projectile motion of the Red

In a projectile motion, the motion of the Red can be analyzed separately by looking at the horizontal and vertical components. Figure 7 shows the variations of velocity of the Red as he is projected. Since there is no acceleration upon the horizontal component of the velocity, both direction and magnitude of the vector u_x does not change. In the vertical direction, the magnitude of the velocity vector u_y decreases as Red reaches the vertex of the parabola and the acceleration of the gravity repels his motion. The vertical velocity vector also changes direction, initially directing upwards whereas directing downwards after Red passes his maximum height of the flight.

Since Kinematics is the study of motion, there are four already-existing formulas that we can use to describe motions of an object in two dimensions. These four equations are called "SUVAT" equations.

Table 1. Four SUVAT Equations

<table>
<tr><td>$v = u + at$</td><td>$s = ut + \frac{1}{2}at^2$</td><td>$s = \left(\frac{u+v}{2}\right)t$</td><td>$v^2 = u^2 + 2as$</td></tr>
<tr><td colspan="4">v : final velocity $(\frac{\mathrm{m}}{\mathrm{s}})$
u : intial velocity $(\frac{\mathrm{m}}{\mathrm{s}})$
a : acceleration $\left(\frac{\mathrm{m}}{\mathrm{s}^2}\right)$
t : time (s)
s : displacement (m)</td></tr>
</table>

From physics class, I learned that the four SUVAT equations, shown in table 1, comprise only the displacement, velocity, and acceleration. Learning in a higher-level mathematics course allowed me to understand that displacement, velocity, and acceleration are all related by derivatives and antiderivatives. Therefore, while I didn't get a chance to prove these four equations by hand, learning calculus allows me to prove one of the equations in table 1, which I will constantly reinforce throughout my exploration.

First, the acceleration of the Red can be also expressed as the rate of change of his velocity:

$$a = \frac{\mathrm{d}v}{\mathrm{d}t}$$

Integrating both sides with respect to time t, we get:

$$\int a\,dt = \int \frac{\mathrm{d}v}{\mathrm{d}t}\,dt$$

$$at + c_1 = v \text{......(1.1)}$$

When an object is at rest, at $t = 0$, the initial velocity is equal to the final velocity, $v = u$. Hence, by substituting the values, we can solve for constant c_1:

$$0 + c_1 = u$$

$$c_1 = u$$

Substituting the value of C_1 to equation (1.1), we get

$$v = u + at \text{......(1.2)}$$

As seen in table 1, the equation (1.2) is one of the SUVAT equations. By using the relationship of derivatives and antiderivatives to express velocity, displacement, and acceleration, I was able to prove mathematically, by hand, the equations that I have been using in my physics class. Additionally, using equation (1.2), I can also solve for other SUVAT equations, which I will be constantly reinforcing throughout my exploration.

Since velocity is the rate of change of displacement, we get:

$$v = \frac{\mathrm{d}s}{\mathrm{d}t}$$

Hence by substituting the rate of change of displacement to equation (1.2), we get:

$$u + at = \frac{\mathrm{d}s}{\mathrm{d}t}$$

Integrating both sides in respect to time, we get:

$$\int (u + at)dt = \int \frac{\mathrm{d}s}{\mathrm{d}t} dt$$

$$ut + \frac{1}{2}at^2 + c_2 = s \quad ......(1.3)$$

When an object is at rest, at $t = 0$, the object hasn't been displaced, and therefore $s = 0$. Hence, by substituting 0 for variables t and s to equation (1.3), we can solve for constant c_2:

$$0 + 0 + c_2 = 0$$
$$c_2 = 0$$

This gives another SUVAT equation on table 1, which is a useful equation to solve for the displacement of an object, and which I will be constantly referring t throughout my exploration:

$$s = ut + \frac{1}{2}at^2 \quad ......(1.4)$$

In our physics class, students simply learned the application of SUVAT equations, where we focused more on how to apply each of four SUVAT equations based on different physical scenarios given in the question. However, through my math IA, I got the chance to derive the SUVAT equations using fundamental concepts of calculus. In doing so, I was not only able to understand the SUVAT equations better, but also gained graphical interpretation skills of displacement, velocity, and acceleration —Since velocity is the rate of change of displacement, in a displacement versus time graph, the slope is ultimately the velocity; Since acceleration is the rate of change of displacement, the area under acceleration versus time graph is the velocity.

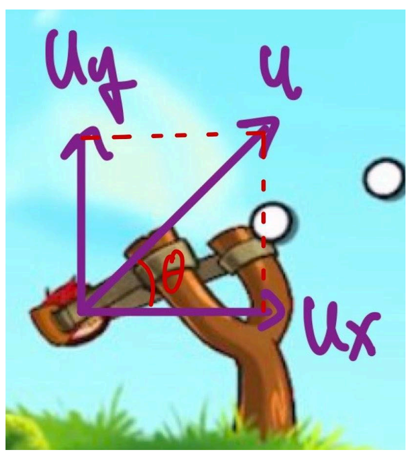


Figure 8. The Red is launched at an angle θ to the horizontal speed u

As shown in figure 8, the initial velocity u of the Red can be separated into vertical and horizontal components. This can be written as:

$$u_x = u\cos(\theta)$$
$$u_y = u\sin(\theta)$$

By separating the initial velocity into horizontal and vertical components, I can also deduce equation (1.4) into both horizontal and vertical components. For the horizontal displacement of the Red, we get:

$$s_x = u_x t + \frac{1}{2} a_x t^2$$

However, since no force is acted on the Red in the horizontal direction due to the negligible air resistance, the horizontal acceleration would be zero. Hence, the horizontal displacement of the Red could be written as:

$$s_x = u_x t$$

And since $u_x = u \cos(\theta)$, we can re-express the equation as:

$$s_x = (u \cos(\theta))t \text{......(2.1)}$$

For the vertical displacement of the Red, we can again deduce equation (1.4), which can be expressed as:

$$s_y = u_y t + \frac{1}{2} a_y t^2$$

Since $u_y = u \sin(\theta)$, we can re-express this equation as:

$$s_y = (u \sin(\theta))t - \frac{1}{2} g t^2 \text{......(2.2)}$$

The value g is also known as the acceleration of free fall. When the air resistance is negligible, an object thrown into the air will experience the acceleration of free fall, and the magnitude of this acceleration is approximately 9.8 ms^{-2} on earth. (cambridge) Since Angry Birds are fantastical characters, the planet that they live on does not exist in real life. However, to continue on my exploration, I will be treating the value of g as a constant with a magnitude of 10 ms^{-2}.

The equation (2.1) and (2.2) shows the horizontal and vertical component of Red's displacement using the SUVAT equation (1.4) that I deduced. However, by manipulating the equation (2.1) and (2.2), I can further express the vertical component of Red's displacement in terms of his horizontal displacement.

The data collected through primary data collection gave me the horizontal and vertical displacement of the Red, as seen in Appendix B. Therefore, by deducing an equation of Red's displacement that composes both horizontal and vertical components, I would be then able to deduce the initial displacement of Red as well as his initial velocity, with no components for consideration. In doing so, I would also be able to calculate the projection angle of the Red and compare it to the angle calculated through GeoGebra as shown in figure 6.

By manipulating the equation (2.1), we get:

$$t = \frac{s_x}{u \cos(\theta)}$$

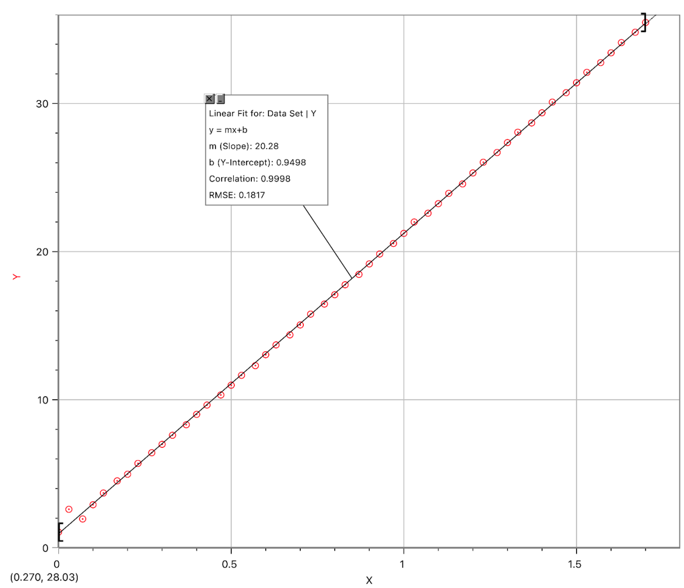
By substituting equation (2.1) to the variable t for equation (2.2), we get:

$$s_y = (u\sin(\theta))\left(\frac{s_x}{u\cos(\theta)}\right) - \frac{1}{2}g\left(\frac{s_x}{u\cos(\theta)}\right)^2$$
$$= \left(\frac{(u\sin(\theta))}{(u\cos(\theta))}\right)s_x - \left(\frac{g}{2u^2\cos^2(\theta)}\right)s_x^2$$
$$= -\left(\frac{g}{2u^2\cos^2(\theta)}\right)s_x^2 + (\tan(\theta))s_x \quad \text{......(2.3)}$$

By separating the initial velocity of the Red into horizontal and vertical components, I was also able to derive an equation of vertical displacement of the Red composing of his horizontal displacement. This exercise would be beneficial for mathematically verifying that the vertical pathway projected by Red follows a perfect parabola, as assumed for an object traveling with negligible air resistance. Observing equation (2.3), we can deduce that this equation follows the quadratic formula:

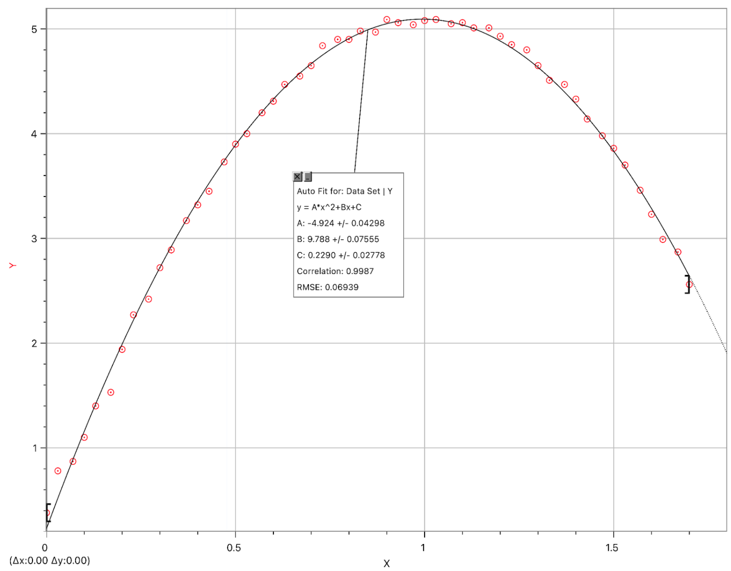
$$s_y = -\left(\frac{g}{2u^2\cos^2(\theta)}\right)s_x^2 + (\tan(\theta))s_x$$
$$y = ax^2 + bx + c \quad \text{......(2.4)}$$

Appendix C shows how the equation (2.3) follows a quadratic equation, and the relationship between their coefficients.



Graph 1. Horizontal displacement of the Red against time

Type of Graph	Slope (ms^{-1})	Y-intercept (m)
Linear	20.28	0.9498



Graph 2. Vertical displacement of the Red against time

Type of Graph	Coefficient (a)	Coefficient (b)	Coefficient (c)
Quadratic	-4.924	9.788	0.2290

Graph 1 shows the horizontal displacement of the Red against time. Graph 2 shows vertical displacement of the Red against time. Both graphs were calculated by using the data values from the processed data table in Appendix B. As expected, the horizontal displacement of the Red increases linearly as no force is present against the motion of the Red in a horizontal direction. For vertical displacement, since gravity acts on Red during his flight, and it's the only force present assuming that air resistance is negligible, the vertical pathway is parabolic.

I can further deduce horizontal velocity of the Red by using the equation of graph 1, which is:

$$y = 20.28x + 0.95 \quad \text{......(3.1)}$$

$$y : \text{horizontal displacement of the Red (m)}$$

$$x : \text{time (s)}$$

Since velocity is the rate of change of displacement, differentiating equation (3.1) gives us the value of horizontal velocity of the Red:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 20.28 \quad \text{......(3.2)}$$

Hence, the horizontal velocity of the Red is constant throughout his flight, which is 20.28 ms^{-1}. Similarly, the equation of graph 2 can be expressed as:

$$y = -4.90x^2 + 9.8x + 0.23 \quad \text{......(3.3)}$$

$$y : \text{vertical displacement of the Red (m)}$$

$$x : \text{time (s)}$$

Therefore, since velocity is the rate of change of displacement, and using equation (3.3), we get:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -4.90(2)x + 9.8 \\ &= -9.8x + 9.8 \qquad \text{......(3.4)}\end{aligned}$$

Hence, when the Red is initially at rest, $t = 0$, which is equivalent to $x = 0$ In terms of equation (3.4), the vertical velocity of the Red is 9.8 ms^{-1}, which is also the y-intercept.

By using both equation (3.2) and (3.4), manipulation of vertical and horizontal components of both displacement and velocity of the Red allows me to solve for the initial velocity u of the Red. Since the initial velocity u of the Red is divided into horizontal and vertical components as seen in figure 8, using pythagoras theorem and inverse tangent law, I can both solve for the initial velocity of the Red as well as his projected angle at $t = 0$.

Using pythagoras theorem, we get:

$$\begin{aligned} u_x^2 + u_y^2 &= u^2 \\ u &= \sqrt[2]{u_x^2 + u_y^2} \\ &= \sqrt[2]{(2.28)^2 + (9.8)^2} \\ &= 10.0617 \\ &\approx 10.06 \frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

This shows that the initial velocity of the Red at $t = 0$ is 10.06 ms^{-1}. Using the inverse tangent law, we get:

$$\begin{aligned} \tan(\theta) &= \frac{u_y}{u_x} \\ \theta &= \arctan\left(\frac{u_y}{u_x}\right) \\ &= \arctan\left(\frac{9.8}{20.28}\right) \\ &= 25.7914 \\ &\approx 25.8^{\circ} \end{aligned}$$

Hence, by using the inverse tangent law, the angle at which the Red is projected is 25.8 degrees. However, in figure 6, the GeoGebra initially calculated the angle at which the Red was projected to be 33.9 degrees. The deviation between the two values shows that my methods to initially calculate the angle of the Red were inaccurate compared to separating the components of the Red's initial velocity and applying the inverse tangent law. This adds on to the significance of my exploration to separate the components of the initial velocity, so as to derive to more valid results in comparison with primary data collection. In order to continue on with my exploration, I will be using the value of initial velocity of the Red as 10.06 ms^{-1} and the angle of projection as 25.8 degrees. While both value may contain an error, it is possible to at least reduce systematic error by using a set of values that were derived from common variables such as u_x and u_y.

I will now recall the equation (2.3), which is:

$$s_y = -\left(\frac{g}{2u^2\cos^2(\theta)}\right)s_x^2 + (\tan(\theta))s_x$$

By substituting the initial velocity u and the projection angle θ for 10.06 ms^{-1} followed by 25.8 degrees as calculated. Then I graph the equation (2.3) through Logger Pro and compare it with the graph plotted by the Logger Pro Video Analysis as shown in graph 2. In doing so, I will be able to comprehend the accuracy of my mathematical equation (2.3) to model the actual pathway and speed of the Red, when air resistance is negligible.

By substituting the values for initial velocity u and the projection angle θ into equation (2.3), we get:

$$\begin{aligned} s_y &= -\left(\frac{9.8}{2(10.06)^2\cos^2(25.8)}\right)s_x^2 + (\tan(25.8))s_x \\ &= -(0.059732)s_x^2 + 0.483419s_x \text{......(3.5)} \end{aligned}$$

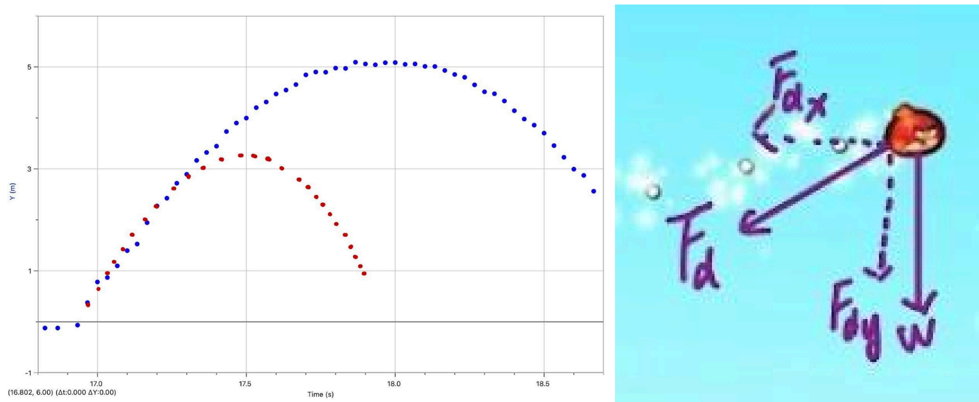
The equation (3.5) will not be rounded to nearest second decimal places. This is because the more decimal places I leave, the more accurate the equation will become, allowing me to interpret with high degrees of validity and accuracy.

To conclude, the final equation (3.5) represents the vertical displacement of the Red as a function of his horizontal displacement. Derving to this equation involved various complicated processes. For one, separating the displacement and velocity of the Red into horizontal and vertical components allowed me to gain a deeper understanding in magnitude and direction vectors, when describing a motion of a moving object. The understanding of vector was further enhanced through the reversible process of using pythagoras theorem to combine the two separated components into one initial velocity as well as calculating for the projected angle of the Red and comparing it with the value that I initially obtained using GeoGebra. Using the Logger Pro to plot the graph with the collected primary data allowed me to gain analytical skills involving various mathematical tools other than my own calculator.

Exploration 2: What if air resistance is not negligible?

Real-life application in the Angry Birds requires the consideration of the air resistance force acting on the Red. The air resistance force is also called a drag force (fluid resistance force). The equation of the drag force is:

$$F_{drag} = kv$$
$$k : \text{constant}$$
$$v : \text{velocity} \left(\frac{\text{m}}{\text{s}}\right)$$



(Figure 9. The effect of air resistance on the Red's projectile motion)

The effect of drag force on a projectile motion is pronounced, as shown in figure 9. I only tried to renounce the effect through comparing the shape of the Red's pathway. The drag force will both decrease the range and height of Red's vertical displacement, removing the parabolic characteristics of the pathway when the air resistance was negligible. The free-body diagram shown in figure 9 further illustrates the effect of drag force on Red's flight. By separating the drag force into horizontal and vertical components, the only force acting on the Red in horizontal direction is the drag force, while it is the sum of drag force and Red's weight that is affecting the flight of Red in vertical direction.

Exploration 2.1: Horizontal net force of the Red

For the horizontal net force acting on the Red, we can describe it as:

$$F_{drag} = F_{net} = -kv$$

$$ma = -kv$$

where m, the mass of the Red, and k are both treated as a constant value. In order to derive the equation for horizontal displacement of the Red when air resistance is present, we can define acceleration as the rate of change of velocity:

$$a = \frac{\mathrm{d}v}{\mathrm{d}t}$$

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -kv \quad ......(4.1)$$

Noticing the separable variable for when differential equation, we can manipulate the equation (4.1) to:

since $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = g(y)$ are special cases of separable differentiation equations

$$\frac{1}{v}dv = -\frac{k}{m}dt$$

Integrating both sides in respect to time, we get:

$$\int \frac{1}{v}dv = \int -\frac{k}{m}dt$$

$$\ln(|v|) = -\frac{k}{v}t + c_1$$

In the horizontal direction, the velocity of the Red is always greater than 0 as the Red is always moving forward. Therefore, the horizontal velocity of the Red can be expressed as:

$$v = e^{-\frac{k}{m}t + c_1}$$

Using the law of natural logarithm, we can express the constant c_1 as:

$$c_1 = \ln(u\cos(\theta))$$

Therefore, the final equation of the horizontal velocity of the Red becomes:

$$v = e^{-\frac{k}{m}t + \ln(u\cos(\theta))} \text{......(4.2)}$$

Using equation (4.2), since the velocity is the rate of change of displacement, we get:

$$v = \frac{\mathrm{d}x}{\mathrm{d}t}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = e^{-\frac{k}{m}t + \ln(u\cos(\theta))}$$

Noticing that this differential equation can be manipulated into separable variable form, we get:

$$dx = e^{-\frac{k}{m}t + \ln(u\cos(\theta))}dt$$

Integrating both sides in respect to time, we get:

$$\int 1dx = \int e^{-\frac{k}{m}t + \ln(u\cos(\theta))}dt$$

$$x = -\frac{m}{k}e^{-\frac{k}{m}t}(u\cos(\theta)) + c_2 \text{......(4.3)}$$

When $t = 0$, the Red's displacement is also equal to zero. Therefore substituting $t = 0$ and $x = 0$ for equation (4.3), we get:

$$c_2 = \frac{m}{k}(u\cos(\theta))$$

By substituting the value of constant c_2 to equation (4.3), we can derive to the final equation for the horizontal displacement of the Red:

$$x=-\frac{m}{k}e^{-\frac{k}{m}t}(u\cos(\theta))+\frac{m}{k}(u\cos(\theta))$$ ......(4.4)

Exploration 2.2: Vertical net force of the Red

As seen in figure 9, the vertical net force acting on the Red is the sum of drag force and his weight, downwards. In physics, the signs in front of a number are treated as a direction. Using this direction concept, the vertical net force can be described as:

$$F_{net}=-mg-kv$$

$$ma=-mg-kv$$ ......(5.1)

Since acceleration is the rate of change of velocity, we get:

$$a=\frac{\mathrm{d}v}{\mathrm{d}t}$$

By substituting the rate of change of velocity to equation (5.1), we get:

$$m\frac{\mathrm{d}v}{\mathrm{d}t}=-mg-kv$$

Noticing that this differential equation can be manipulated to find the integrating factor:

$$\frac{\mathrm{d}y}{\mathrm{d}x}+P(x)y=Q(x)$$

$$I.F=e^{\int P(x)dx}$$

Similarly, we get:

$$\frac{\mathrm{d}v}{\mathrm{d}t}=-g-\frac{kv}{m}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t}+\left(\frac{k}{m}\right)v=-g$$

If: $e^{\int\frac{k}{m}dt}=e^{\frac{k}{m}t}$, we can multiply the integrating factor to both sides:

$$e^{\frac{k}{m}t}\left(\frac{\mathrm{d}v}{\mathrm{d}t}+\frac{k}{m}v\right)=(-g)e^{\frac{k}{m}t}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t}e^{\frac{k}{m}t}+e^{\frac{k}{m}t}\left(\frac{k}{m}v\right)=(-g)e^{\frac{k}{m}t}$$ ......(5.2)

By reversely using the product rule of differentiation, we can express the equation (5.2) as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(v \times e^{\frac{k}{m}t}\right) = (-g)e^{\frac{k}{m}t}$$

By integrating both sides in respect to time, we get:

$$\int \frac{\mathrm{d}}{\mathrm{d}t}\left(v \times e^{\frac{k}{m}t}\right)dt = \int -g \times e^{\frac{k}{m}t}$$

$$ve^{\frac{k}{m}t} = -\frac{m}{k}ge^{\frac{k}{m}t} + c_1 \text{......(5.3)}$$

Before continuing on to solve for the constant c_1, it is necessary to first define terminal velocity. Recalling the equation of drag force, $F_{drag} = kv$ and figure 9, the drag force will become equal to the weight of the Red. Eventually, the Red is assumed to move at a constant speed, which is also called terminal velocity, v_T. Hence, the equation of drag force can be re-written as $F_{drag} = kv_T$.

When the weight of the Red becomes equal to the drag force, we get:

$$mg = kv_T$$

$$v_T = \frac{mg}{k} \text{......(5.4)}$$

To simplify the equation (5.3) by substituting the terminal velocity (5.4), we get:

$$ve^{\frac{k}{m}t} = -v_Te^{\frac{k}{m}t} + c_1$$

By dividing both sides by $e^{\frac{k}{m}t}$, we get:

$$v = -v_T + c_1e^{-\frac{k}{m}t} \text{......(5.5)}$$

We know that when $t = 0$, the final velocity of the Red is the same as his initial velocity as his motion does not change. Therefore, we get:

$$v = u\sin(\theta) \text{......(5.6)}$$

Therefore by substituting the equation (5.6) to equation (5.5), we get:

$$u\sin(\theta) = -v_T + c_1$$

$$c_1 = u\sin(\theta) + v_T$$

By substituting the constant c_1 to equation (5.5) we get:

$$v = (u\sin(\theta) + v_T)e^{-\frac{k}{m}t} - v_T$$

Since vertical component of the velocity can also be expressed as the rate of change of vertical displacement, we get:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = (u\sin(\theta) + v_T)e^{-\frac{k}{m}t} - v_T$$

Noticing that this differential equation is in separable form, we can multiply dt on both sides to get:

$$dy = \left((u\sin(\theta) + v_T)e^{-\frac{k}{m}t} - v_T\right)dt$$

By integrating both sides in respect to time, we get:

$$\int 1dy = \int \left((u\sin(\theta) + v_T)e^{-\frac{k}{m}t} - v_T\right)dt$$

$$y = -\frac{m}{k}(u\sin(\theta) + v_T)e^{-\frac{k}{m}t} - v_T t + c_2 \quad \text{......(5.7)}$$

At the initial condition of the Red when $t = 0$, the vertical displacement does not change for the Red. Therefore, we can express this as:

$$0 = -\frac{m}{k}(u\sin(\theta) + v_T) + c_2$$

$$c_2 = \frac{m}{k}(u\sin(\theta) + v_T)$$

By substituting the value of constant c_2 to equation (5.7), we get:

$$y = -\frac{m}{k}(u\sin(\theta) + v_T)e^{-\frac{k}{m}t} - v_T t + \frac{m}{k}(u\sin(\theta) + v_T)$$

$$y = -\frac{m}{k}(u\sin(\theta) + v_T)\left(e^{-\frac{k}{m}t} - 1\right) - v_T t$$

And by substituting the equation for terminal velocity (5.4), we get;

$$y = -\frac{m}{k}\left(u\sin(\theta) + \left(\frac{mg}{k}\right)\right)\left(e^{-\frac{k}{m}t} - 1\right) - \left(\frac{mg}{k}\right)t \quad \text{......(5.8)}$$

To conclude, the equation (4.4) and (5.8) shows the horizontal and vertical component of the displacement of the Red, when air resistance is not negligible. To derive both equations, several difficult procedures were involved. For one, I had to consider the presence of drag force which affected the motion of flight of Red after released from his initial conditions. Two more variables of mass m and constant k, which are both treated as a constant, were included in both of the equations due to the presence of drag force. Difficult process of finding a form of integrating factor and the form of separable variables, and constantly removing the constant c_1 and c_2, I had a chance through my mathematics IA to gain a deeper analytical skills of calculus.

Findings:

$$s_y = -\left(\frac{g}{2u^2\cos^2(\theta)}\right)s_x^2 + (\tan(\theta))s_x$$

$$s_y = -\left(\frac{9.8}{2(10.06)^2\cos^2(25.8)}\right)s_x^2 + (\tan(25.8))s_x$$

$$= -(0.059732)s_x^2 + 0.483419s_x$$

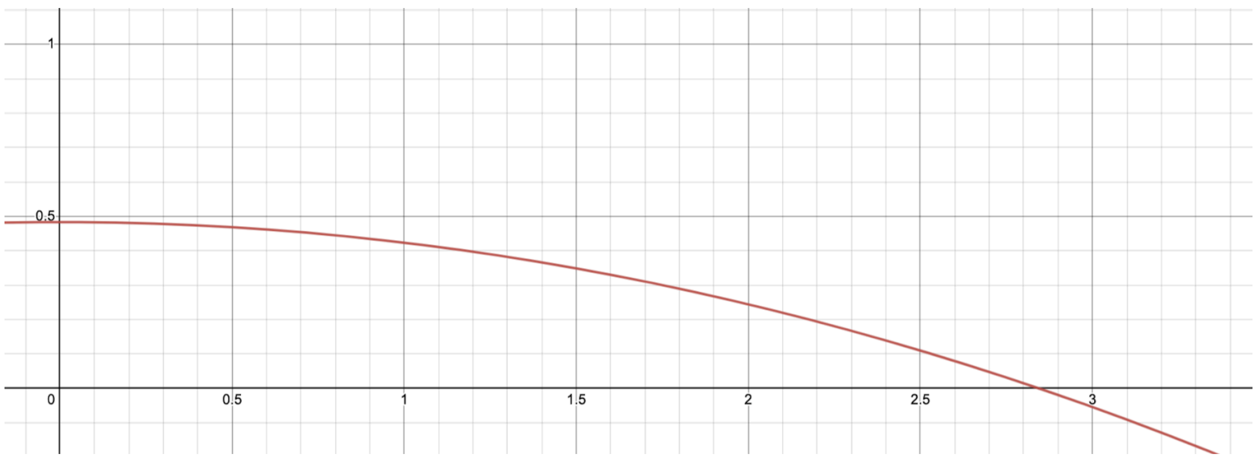


Figure. Graph of equation of vertical displacement of the Red as a function of his horizontal displacement, when air resistance is negligible

$$x = -\frac{m}{k}e^{-\frac{k}{m}t}(u\cos(\theta)) + \frac{m}{k}(u\cos(\theta))$$

Equation of horizontal displacement of the Red, when air resistance is not negligible

$$y = -\frac{m}{k}\left(u\sin(\theta) + \left(\frac{mg}{k}\right)\right)\left(e^{-\frac{k}{m}t} - 1\right) - \left(\frac{mg}{k}\right)t$$

Equation of vertical displacement of the Red, when air resistance is not negligible