Chapter4 Numerical methods for nonlinear equation

Zero point of Legendre polynomial

Problem：

$$P_{N+1}(x) = 0,$$

$$P_N\,'(x) = 0,$$

find x=?

$$f(x) = 0,$$ X=?

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- §4.2 Fixed-point iteration
- §4.3 Newton's Method
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- §4 Introduction
- §4.1 Bisection method

§ 4 Introduction

- 1. Definition of the nonlinear function

The equation $f(x)=0$ is a linear equation when f(x) is a polynomial of first degree; otherwise it is called a nonlinear equation.

§ 4 Introduction

- Example:

Algebraic equation

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0, \quad n > 1$$

Transcendental equation

$$f(x) = e^x + \sin x = 0$$

§ 4 Introduction

- Finding the root of the nonlinear equation
  - 1. existence
  - 2. isolation
  - 3. refinement

$P_N^{(M)}(x) = 0$ has N-M roots on the interval (-1, 1)

If f (x) is continuous on the interval [a, b], and strictly monotonous with

f (a) * f (b)<0,

thus f (x)=0 has a root in [a, b]

§ 4 Introduction

- 3. refinement

If f (x) is continuous on the interval [a, b], and strictly monotonous with

f (a) * f (b)<0,

thus f (x)=0 has a root in [a, b]

If f (x) is continuous on the interval [a, b], and has only one root x^* : $f(x^*)$=0, with f (a) * f (b)<0, x^* =?

§ 4.1 Bisection method

- Bisection method is a direct method, which is intuitive and simple.

If f (x) is continuous on the interval [a, b], and has only one root x^* : $f(x^*)$=0, with f (a) * f (b)<0, x^* =?

§ 4.1 Bisection method

$$\text{while}(|b-a|>\varepsilon)$$

$$\quad \text{if } f(a)f(\frac{a+b}{2})<0$$

$$\qquad b=\frac{a+b}{2};$$

$$\quad \text{else}$$

$$\qquad a=\frac{a+b}{2};$$

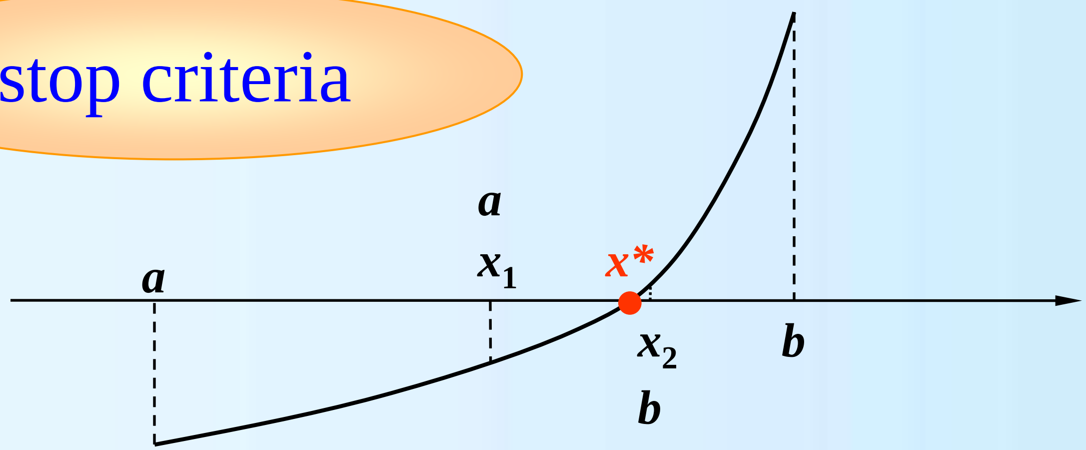
$$\quad \text{end}$$

$$\text{end}$$

$$x^*=\frac{a+b}{2};$$

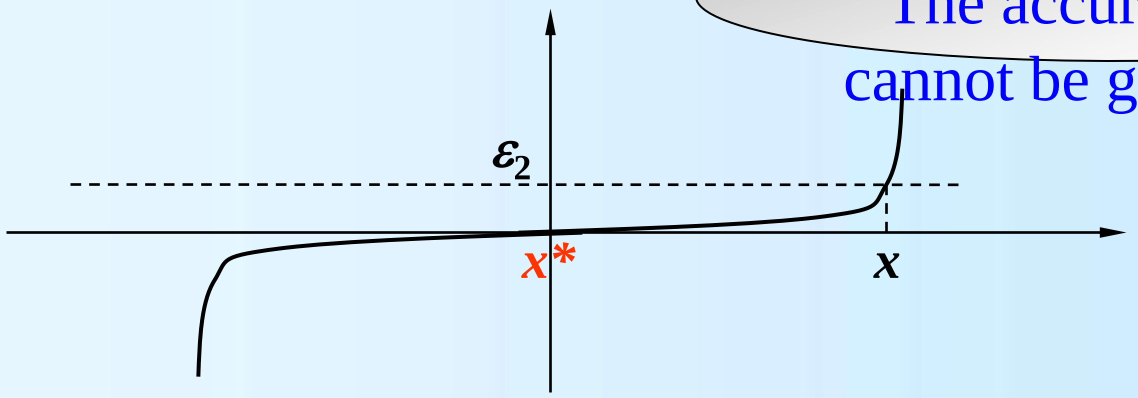
If f (x) is continuous on the interval [a, b], and has only one root $x^*: f(x^*)=0$, with f (a) * f (b)<0， $x^* =?$

stop criteria



$$\left|x_{k+1}-x_k\right|<\varepsilon_1 \ \text{ or } \ \left|f(x)\right|<\varepsilon_2$$

The accuracy of x cannot be guaranteed



§ 4.1 Bisection method

- Convergence of Bisection Method

In a interval where a root lies

$[a_1, b_1] \supset [a_2, b_2] \supset \cdots \supset [a_k, b_k]$, the length of $[a_k, b_k]$ is

$$b_k - a_k = \frac{1}{2}(b_{k-1} - a_{k-1}) = \cdots = \frac{1}{2^{k-1}}(b_1 - a_1)$$

when k is big enough, thus we have $x_k = \frac{a_k + b_k}{2}$

and the error $\left| x^* - x_k \right| \le \frac{b_k - a_k}{2} = \frac{b-a}{2^k}$

simple calculation, easy error estimation but slow convergence

§ 4.1 Bisection method

- Convergence of Bisection Method

Given the stop criteria $\epsilon > 0$, finding the iteration times k that

$$\left|x^* - x_k\right| \le \frac{b-a}{2^k} < \varepsilon$$

and as $2^{-k} < \frac{\epsilon}{b-a}$, we have

$$k > \frac{\ln(b-a) - \ln \varepsilon}{\ln 2}$$

```
>> -log2(1e-14)

ans =

   46.5070
```

§ 4.1 Bisection method

Function : $x = \text{legendregauss\_bisection}(N, M)$

Input : N, M

Calculation : $\left.P_N^{(M)}(\xi) = 0, \xi \in (-1,1)\right]$

Output : $x = (x_1, \ldots, x_{N-M})$

§ 4.2 fixed-point iteration

- 1. Iteration formula
- 2. Geometric representation of iteration
- 3. Convergence of the iteration
- 4. Global convergence
- 5. Local convergence

fixed-point iteration to solve $x=g(x)$

§ 4.1 Fixed-point iteration

- /* Fixed-Point Iteration */

Equivalent transformation

$$f(x) = 0 \quad \Longleftrightarrow \quad x = g(x)$$

root of $f(x)$ $\quad \Longleftrightarrow \quad$ **Fixed point of g (x)**

- §4.2 Fixed-point iteration

§ 4.2 fixed-point iteration

■ 1. Iteration formula

change: f(x)=0 ⟺ x=g(x), g(x) is continuous, then construct the iteration formula: $x_{k+1} = g(x_k)$, where $\{x_k\}$ is the iteration sequence.

$$x_{k+1} = \sqrt{2x_k + 3}$$

$$x_0 = 4 \to x_1 = 3.316 \to x_2 = 3.104 \to$$
$$x_3 = 3.034 \to x_4 = 3.011 \to x_5 = 3.004 \ \ldots$$

x_k diverges or converges

§ 4.2 fixed-point iteration

1. Iteration formula

change: f(x)=0 ⟺ x=g(x), g(x) is continuous, then construct the iteration formula: $x_{k+1} = g(x_k)$, where $\{x_k\}$ is the iteration sequence.

if $\{x_k\}$ converges to x^*, then it converges to the root of f(x) :

$$\lim_{k\to\infty} x_{k+1} = \lim_{k\to\infty} g(x_k) = g(\lim_{k\to\infty} x_k)$$

$$x^* = g(x^*) \Rightarrow f(x^*) = 0$$

§ 4.2 fixed-point iteration

- Example 1:

Finding the root of equation

$$f(x) = x^2 - 2x - 3 = 0 \; (x_1 = 3, x_2 = -1)$$

by fixed-point iteration

§ 4.2 fixed-point iteration

- Solution：

  (1)change f(x) into $x=(2x+3)^{1/2}$

  then construct the iteration formula

  $x_{k+1}=(2x_k+3)^{1/2} \quad (k=0,1,2\ldots)$,

  take $x_0=4$， $x_1=3.316$， $x_2= 3.104$， $x_3=3.034$，

  $x_4= 3.011$， $x_5= 3.004$

  when $k\to\infty$， $x_k \to 3$， convergence；

§ 4.2 fixed-point iteration

- Solution：

(2) change f(x) into $x=1/2*(x^2-3)$

then construct the iteration formula

$$x_k= 1/2 *(x_k{}^2-3) \quad (k=0,1,2\ldots),$$

take $x_0=4$, $x_1=6.5$, $x_2=19.625$, $x_3= 191.0$

when $k\to\infty$, $x_k \to \infty$, divergence .

§ 4.2 fixed-point iteration

- Example 2：

Find the root x^* of $f(x)=x^3-x-1=0$ near the point $x_0=1.5$.

Solution：

(1)change f(x) into $x=(x+1)^{1/3}$, convergence;

while

(2)change it into $x=(x^3-1)$, divergence.

§ 4.2 fixed-point iteration

Question :

$$\text{solve } f(x) = 0$$

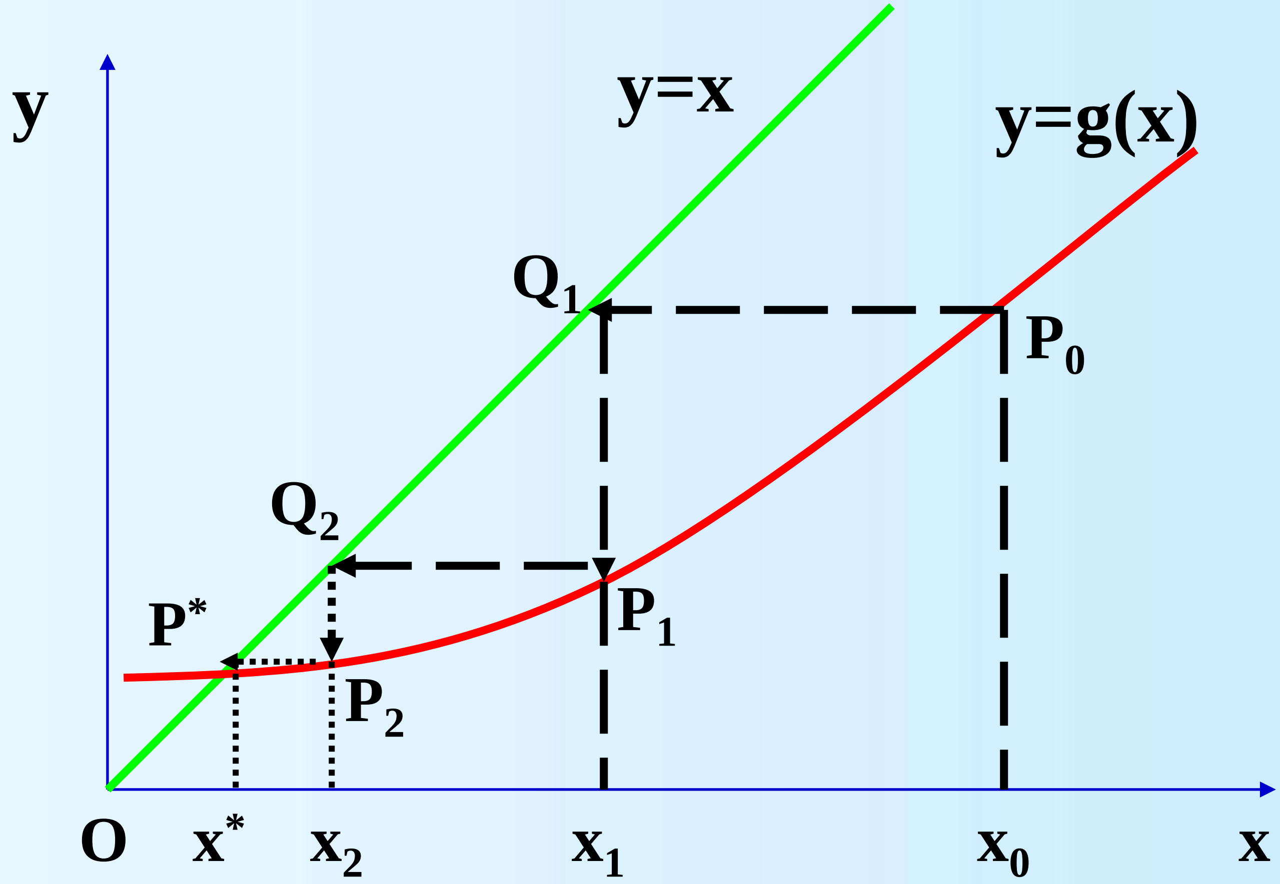
Formula(algorithm):

$$\begin{cases} 1、x = g(x) \\ 2、x_{k+1} = g(x_k) \end{cases}$$

- 2. Geometric representation of iterative process

$$x = g(x) \Rightarrow \begin{cases} y = g(x) \\ x = y \end{cases}$$

Intersection means the root



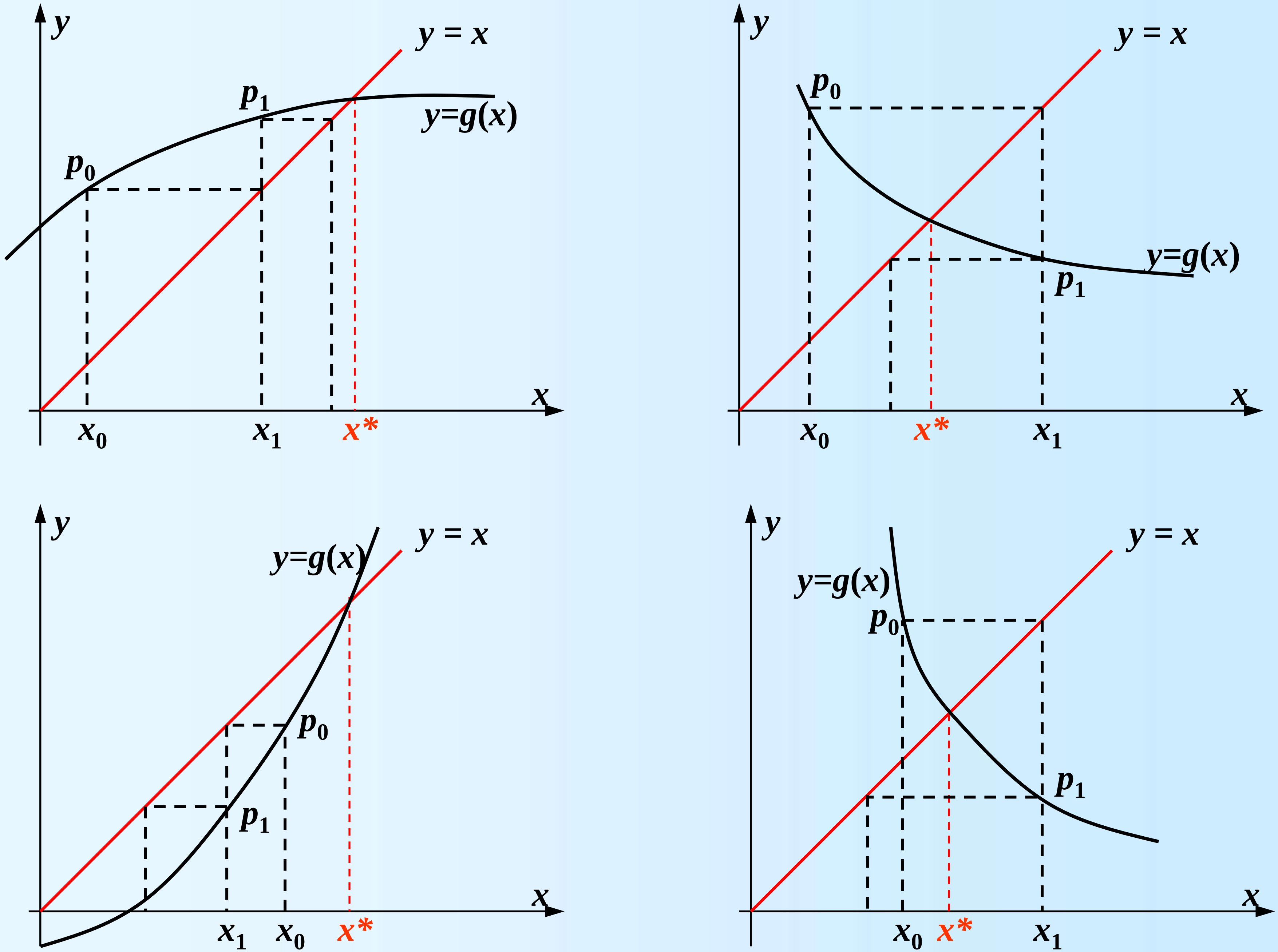
§ 4.2 fixed-point iteration

- 3. Convergence of the iteration：

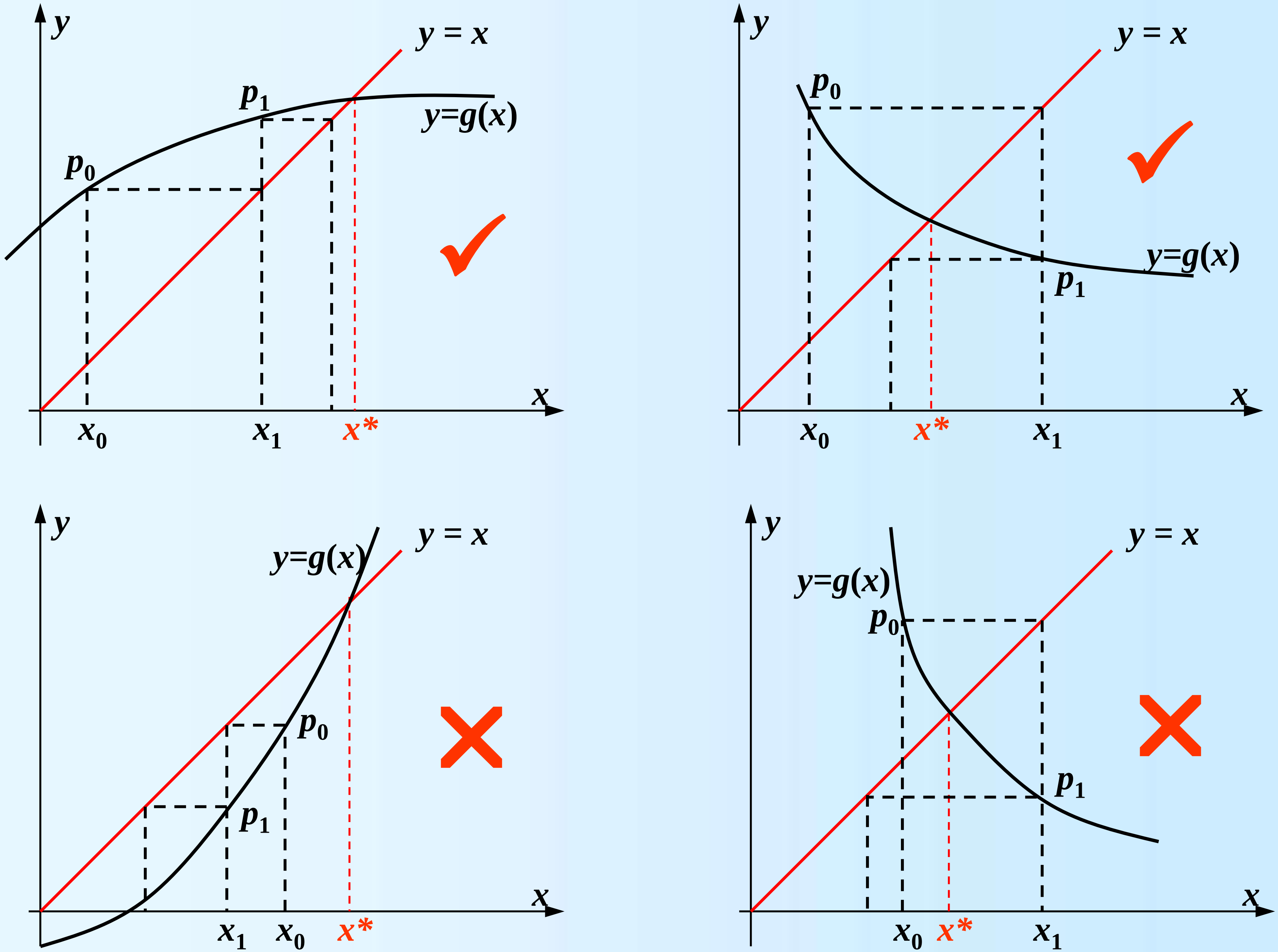
Basic question :

$$\begin{cases} 1、\ how\ to\ construct\ x = g(x) \\ \quad 2、\ convergence\ of\ \{x_{k+1}\} \end{cases}$$

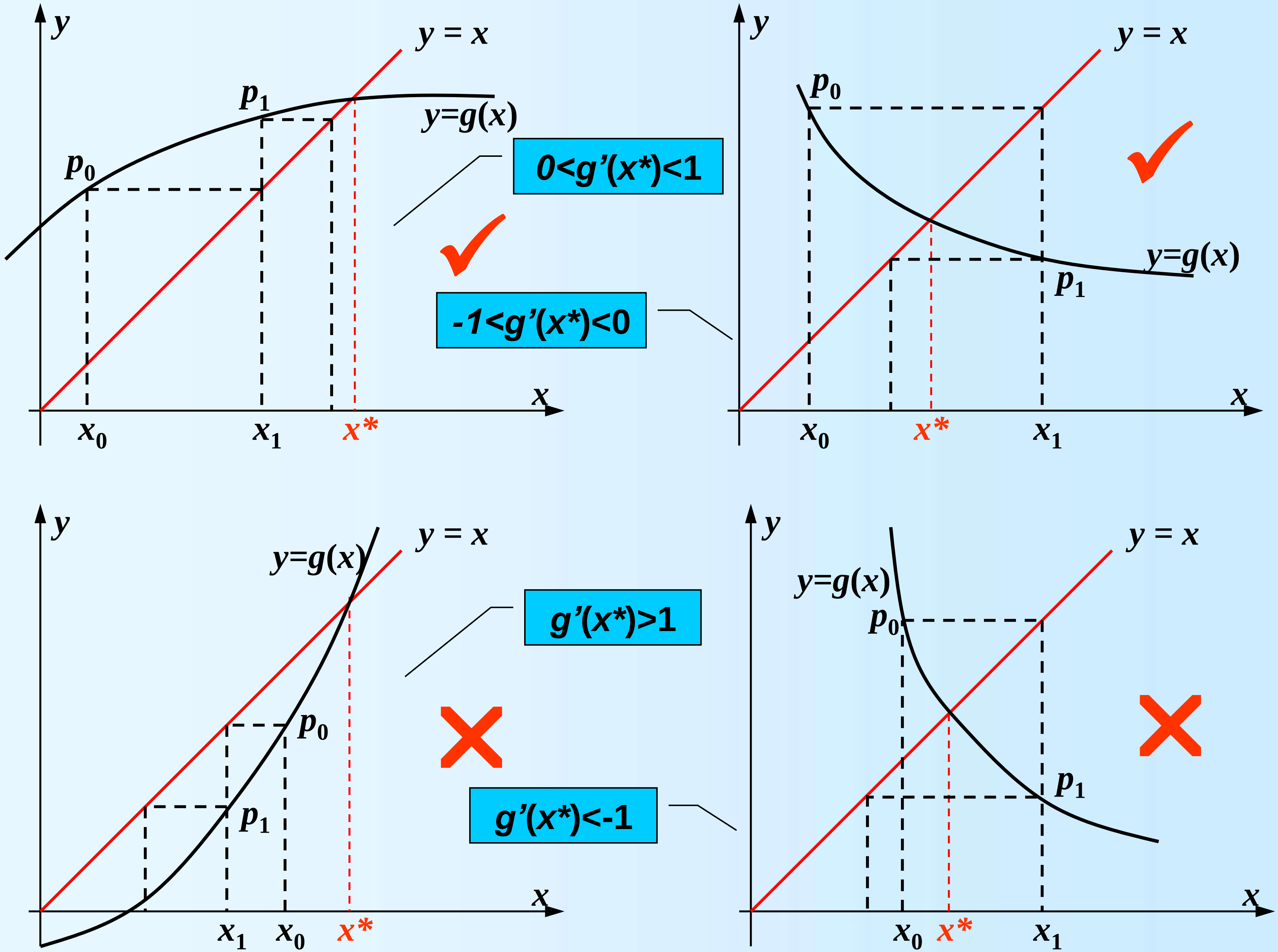
Check the convergence of the fixed-point iteration geometrically, see the following



y
y = x
y=g(x)
p0
p1
x
x0
x1
x*
y
y = x
p0
y=g(x)
p1
x
x0
x*
x1
y
y = x
y=g(x)
p0
p1
x
x1
x0
x*
y
y = x
y=g(x)
p0
p1
x
x0
x*
x1



y
y = x
p1
y=g(x)
p0
x
x0
x1
x*
y
y = x
p0
y=g(x)
p1
x
x0
x*
x1
y
y = x
y=g(x)
p0
p1
x
x1
x0
x*
y
y = x
y=g(x)
p0
p1
x
x0
x*
x1



y
y = x
p0
p1
y=g(x)
0<g'(x*)<1
x
x0
x1
x*
y
y = x
p0
p1
y=g(x)
-1<g'(x*)<0
x
x0
x*
x1
y
y = x
y=g(x)
p0
p1
g'(x*)>1
x
x1
x0
x*
y
y = x
y=g(x)
p0
p1
g'(x*)<-1
x
x0
x*
x1

§ 4.2 fixed-point iteration

- Example 2：

find the root x^* of $f(x)=x^2-2x-3=0$.

Solution：

(1) change f(x) into $x=(2x+3)^{1/2}$, convergence;

(2) change it into $x=1/2(x^2-3)$, divergence.

(1) $g'(x=3)=0.3333...$; (2) $g'(x=3)=3>1$

the indicator $|g'(x)|<1$?

§ 4.2 fixed-point iteration

- from the geometrical representation we know that under the condition (1), (2) $\{x_k\}$ converges to x^*, and (3), (4) does not converge.

- Necessary condition for the convergence is that $|g'(x)|<1$, or otherwise there are several roots in the interval [a, b] which leads to divergence.

§ 4.2 fixed-point iteration

- 4. Global convergence：

considering the equation $x=g(x)$, $g(x) \in C[a,b]$, and if

(1) when $x \in [a,b]$, $g(x) \in [a,b]$;

(2) $\exists 0 \leq L \leq 1$ thus $|g^*(x)| \leq L<1$ for any $x \in C[a,b]$.

thus for any $x_0 \in C[a,b]$, iteration $x_{k+1} = g(x_k)$ converges to the only fixed point of g(x) in [a, b], and

$$\text{(a)} |x^* - x_k| \leq 1/(1-L)|x_{k+1} - x_k|$$

$$\text{(b)} |x^* - x_k| \leq L^k/(1-L)|x_1 - x_0|$$

§ 4.2 fixed-point iteration

For the equation $x=g(x)$, $g(x)\in C[a,b]$, and if

(1) when $x\in [a,b]$, $g(x)\in [a,b]$;

(2) $\exists 0 \le L \le 1$ thus $|g^*(x)| \le L<1$ for any $x\in C[a,b]$.

thus for any $x_0 \in C[a,b]$, iteration $x_{k+1} = g(x_k)$ converges to the only fixed point of g(x) in [a, b].

Question : 1. x=g(x) has root in [a, b];

2. x=g(x) has only one root in [a, b];

3. $|x^* - x_k| \to \infty (k \to \infty)$

prove:

① $g(x)$ has fixed point in [a, b]?

set $h(x) = g(x) - x \quad \because a \le g(x) \le b$

$$\therefore h(a) = g(a) - a \ge 0, \quad h(b) = g(b) - b \le 0$$

$\Rightarrow h(x)$ has root in [a, b]

② Fixed point is unique?

If not, suppose there also exist $\tilde{x} = g(\tilde{x})$, then

$$x^* - \tilde{x} = g(x^*) - g(\tilde{x}) = g'(\xi)(x^* - \tilde{x}), \ \xi \text{ between } \tilde{x} \text{ and } x^*,$$

$$\Rightarrow \quad (x^* - \tilde{x})(1 - g'(\xi)) = 0 \ \text{而} \ |g'(\xi)| < 1 \quad \therefore \quad x^* = \tilde{x}$$

③ when $k \to \infty$, x_k converges x^*?

$$|x^* - x_k| = |g(x^*) - g(x_{k-1})| = |g'(\xi_{k-1})| \cdot |x^* - x_{k-1}|$$

$$\le L|x^* - x_{k-1}| \le \underbrace{\ldots\ldots}_{\text{and so forth}} \le L^k |x^* - x_0| \to 0$$

④ $|x^*-x_k| \le \dfrac{1}{1-L}|x_{k+1}-x_k|$?

Control the convergence by $|x_{k+1}-x_k|$

$$|x_{k+1}-x_k| \ge |x^*-x_k| - |x^*-x_{k+1}| \ge |x^*-x_k| - L|x^*-x_k|$$

⑤ $|x^*-x_k| \le \dfrac{L^k}{1-L}|x_1-x_0|$?

The smaller L, the faster the convergence

$$|x_{k+1}-x_k| = |g(x_k)-g(x_{k-1})| = |g'(\xi_k)(x_k-x_{k-1})|$$
$$\le L|x_k-x_{k-1}| \le \ldots\ldots \le L^k|x_1-x_0|$$

⑥ $\lim\limits_{k\to\infty}\dfrac{x^*-x_{k+1}}{x^*-x_k} = g'(x^*)$?

$$\lim_{k\to\infty}\frac{x^*-x_{k+1}}{x^*-x_k} = \lim_{k\to\infty}\frac{g'(\xi_k)(x^*-x_k)}{x^*-x_k} = g'(x^*)$$

§ 4.2 fixed-point iteration

- 5. Local convergence：

If $g(x)$ is continuous and derivative in the neighborhood $O(x^*, \delta^*)$ of x^* , and x^* is the root of $x=g(x)$, $|g'(x)|<1$.

Thus, there exists $0<\delta \leq \delta^*$, that for any $x_0 \in[x^*- \delta, x^*+ \delta]$, sequence $x_{k+1}= g(x_k)$ converges to x^*

Idea of proof： make use of the conditions in the global convergence, and set $[x^*- \delta, x^*+ \delta]=[a, b]$, check the condition $x=g(x)$, $g(x)\in [a,b]$ holds or not

Prove:

As $g'(x)$ is continuous in $O(x^*, \delta^*)$ and $|g'(x)|$<1. There exist L<1and δ< δ^* that for any

$$x \in[x^*- \delta, x^*+ \delta], |g'(x)| \leq L<1.$$

On the other hand, as $g(x^*) = x^*$, then

$$|g(x)- x^*|= |g(x)- g(x^*)| \leq L| x - x^*|< \delta,$$

thus we have $g(x) \in [x^*- \delta, x^*+ \delta]$.

According to the global convergence, the iteration sequence $x_{k+1}= g(x_k)$ converges to x^*

In practice, we could use the bisection method to find a good initial point x^0

§ 4.2 fixed-point iteration

Finding the root of $f(x) = x^2 - x - 1 = 0$

solve: $f(1.5) = -0.25 < 0, \quad f(2) = 1 > 0$

$\Rightarrow [1.5,2]$ has root in it

(1) $x = \sqrt{x+1} = g_1(x)$ as $1.5 < \sqrt{1.5+1} \le g_1(x) \le \sqrt{2+1} < 2$

$$\left|g_1{}'(x)\right| = \frac{1}{2\sqrt{x+1}} \le \frac{1}{2\sqrt{1.5+1}} = \frac{1}{2\sqrt{2.5}} \approx \frac{1}{3.162}$$

(2) $x = 1 + \frac{1}{x} = g_2(x)$ as $1.5 = 1 + \frac{1}{2} \le g_2(x) \le 1 + \frac{1}{1.5} < 2$

$$\left|g_2{}'(x)\right| = -\frac{1}{x^2} \le \frac{1}{1.5^2} = \frac{1}{2.25}$$

Thus, any x_0 in [1.5, 2] will converge to the fixed point

- §4.3 Newton’s Method

§ 4.3 Newton's method

- Contents
  - 1. Iteration formula
  - 2. Geometrical meaning of Newton's method
  - 3. convergence of Newton's method

Solving $f(x)=0$ by Newton's method

§ 4.3 Newton’s method

■ Newton’s method

One special case of the iteration method， which owns its intrinsic form

Question： solve f(x)=0

Formula(algorithm):

$$\begin{cases} 1、x = g(x) \\ 2、x_{k+1} = g(x_k) \end{cases}$$

§ 4.3 Newton's method

■ Idea of the iteration

(1) Replace the original equation by an approximate equation (find g(x))

(2) Linearize the nonlinear equation

idea 1 : $f(x) = 0,$

$$x = x \pm f(x) = g(x),$$

$$g'(x) = 1 \pm f'(x).$$

§ 4.3 Newton's method

idea 2:

$$-\frac{f(x)}{f'(x)}=0, \Rightarrow x=x-\frac{f(x)}{f'(x)}=g(x),$$

$$g'(x)=1-\frac{f'(x)^2-f(x)f''(x)}{f'(x)^2}=\frac{f(x)f''(x)}{f'(x)^2}$$

$$\approx g'(x^*)=\frac{f(x^*)f''(x^*)}{f'(x^*)^2}=0$$

§ 4.3 Newton's method

Formula of Newton's method

$$x = x - \frac{f(x)}{f'(x)} = g(x),$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

§ 4.3 Newton's method

- formula

——Taylor expansion（important）

Suppose x_k is an approximate root of f(x)=0, and calculate the Taylor expansion of f(x) at x_k :

$$f(x)=f(x_k)+f'(x_k)(x-x_k)+\frac{f''(x_k)}{2!}(x-x_k)^2+\cdots$$

$$f(x)\approx f(x_k)+f'(x_k)(x-x_k)$$

$f(x)=0$; nearly $f(x_k)+f'(x_k)(x-x_k)=0$

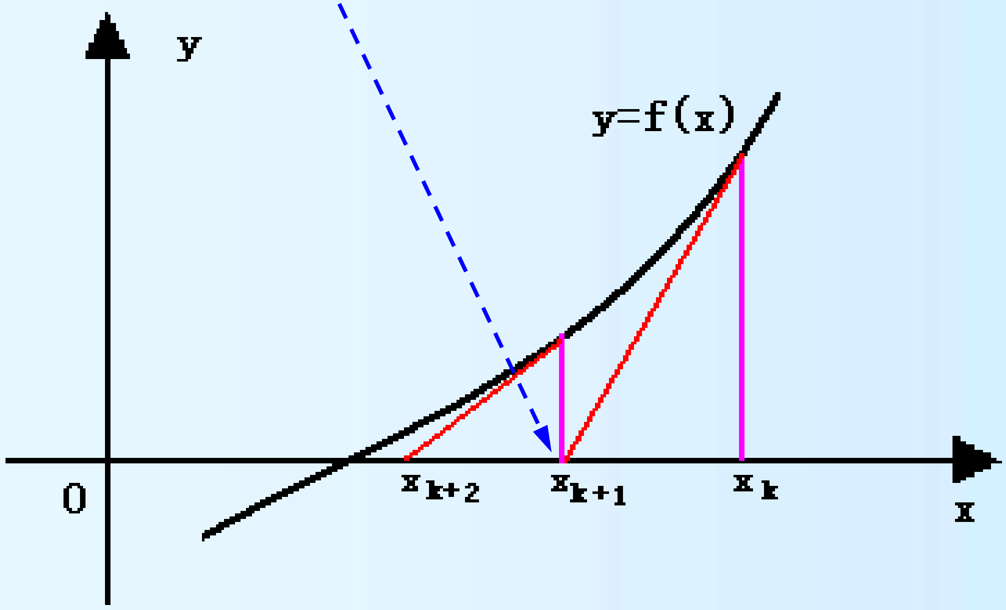
set $f'(x)\neq 0$, then find x noted as x_{k+1}, thus

$$x_{k+1}=x_k-\frac{f(x_k)}{f'(x_k)} \qquad (k=0,1,2,\cdots)$$

■2. Geometrical meaning of Newton method

Calculate the intersection point of the tangent line $y=f(x_k)+f'(x_k)(x-x_k)$ and $y=0$, finding $x=x_{k+1}$, thus

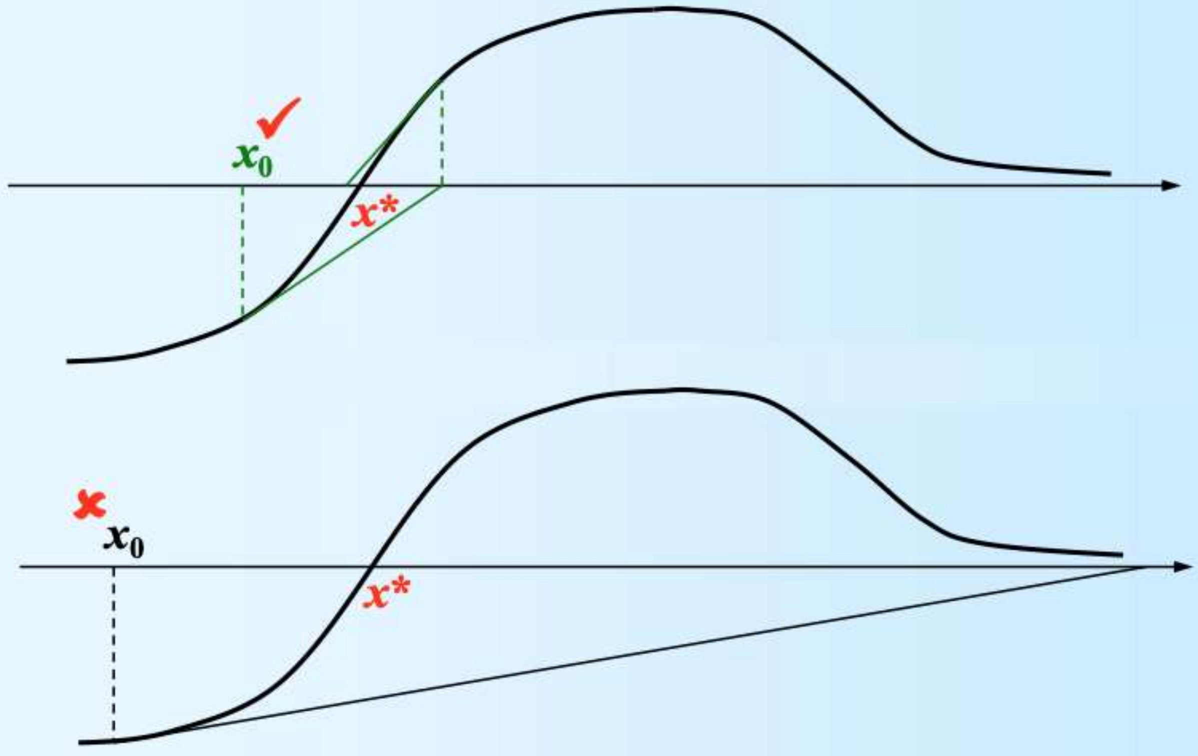
$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$



In fact, Newton's method is called tangent method as the tangent line through point x_k intersects with $y=0$ at the point x_{k+1}

§ 4.3 Newton's method

Initial point x_0 is very important for Newton's method



§ 4.2 fixed-point iteration

- Global convergence of Newton's method：

suppose $f(x) \in C^2[a,b]$, if

(1) $f(a)f(b)<0$; (root existence)

(2) f'' keeps the sign in [a,b] and $f'(x) \neq 0$; (uniqueness)

(3) choose $x_0 \in [a,b]$ that $f(x_0)f''(x_0)>0$;

thus the sequence $\{x_k\}$ generated by the Newton's method will converge to the only root of $f(x)$ in [a,b].

■**Prove:**

suppose $f'(x)>0$, $f''(x)>0$ and $f(x_0)>0$ (similar for other case)

taking the Taylor expansion of $f(\alpha)$ at x_k

$$f(\alpha)=f(x_k)+f'(x_k)(\alpha-x_k)+\frac{f''(\xi_k)}{2!}(\alpha-x_k)^2=0$$

$$\therefore \alpha = x_k-\frac{f(x_k)}{f'(x_k)}-\frac{f''(\xi_k)}{2f'(x_k)}(\alpha-x_k)^2$$

$$=x_{k+1}-\frac{f''(\xi_k)}{2f'(x_k)}(\alpha-x_k)^2\le x_{k+1}$$

the sequence $\{x_{k+1}\}$ has the lower bound α

and $x_{k+1}=x_k-\dfrac{f(x_k)}{f'(x_k)}<x_k$, $\{x_{k+1}\}$ monotonic decrease

$\therefore \{x_{k+1}\}$ converge $\lim\limits_{n\to\infty} x_{k+1}=\bar{x}$,

so we have $\bar{x}=\bar{x}-\dfrac{f(\bar{x})}{f'(\bar{x})}$, $f(\bar{x})=0$, $\bar{x}=\alpha$

§ 4.3 Newton's method

- **Local convergence**

suppose $f(x) \in C^2[a,b]$, if x^* is root of $f(x)$ in [a,b] and $f(x^*) \neq 0$, thus there exist the neighborhood $S_\delta(x^*)$ of x^* that when $x_0 \in S_\delta(x^*)$ the sequence of Newton's method $\{x_k\}$ converges to x^*, and

$$\lim_{k\to\infty} \frac{x^* - x_{k+1}}{(x^* - x_k)^2} = -\frac{f''(x^*)}{2f'(x^*)}$$

§ 4.3 Newton's method

■ **Prove :**

In fact, Newton's method is a particular fixed-point iteration where

$$g(x) = x - \frac{f(x)}{f'(x)}, |g'(x)| = \left| \frac{f''(x) f(x)}{f'(x)^2} \right|$$

then

$$|g'(x^*)| = \left| \frac{f''(x^*) f(x^*)}{f'^2(x^*)} \right| = 0 < 1 \quad \text{convergence}$$

§ 4.3 Newton's method

- **Prove :**

take the Taylor expansion:

$$0 = f(x^*) = f(x_k) + f'(x_k)(x^* - x_k) + \frac{f''(\xi_k)}{2!}(x^* - x_k)^2$$

$$\Rightarrow \quad x^* = \underbrace{x_k - \frac{f(x_k)}{f'(x_k)}}_{x_{k+1}} - \frac{f''(\xi_k)}{2!\,f'(x_k)}(x^* - x_k)^2$$

/*simple root */
converges fast

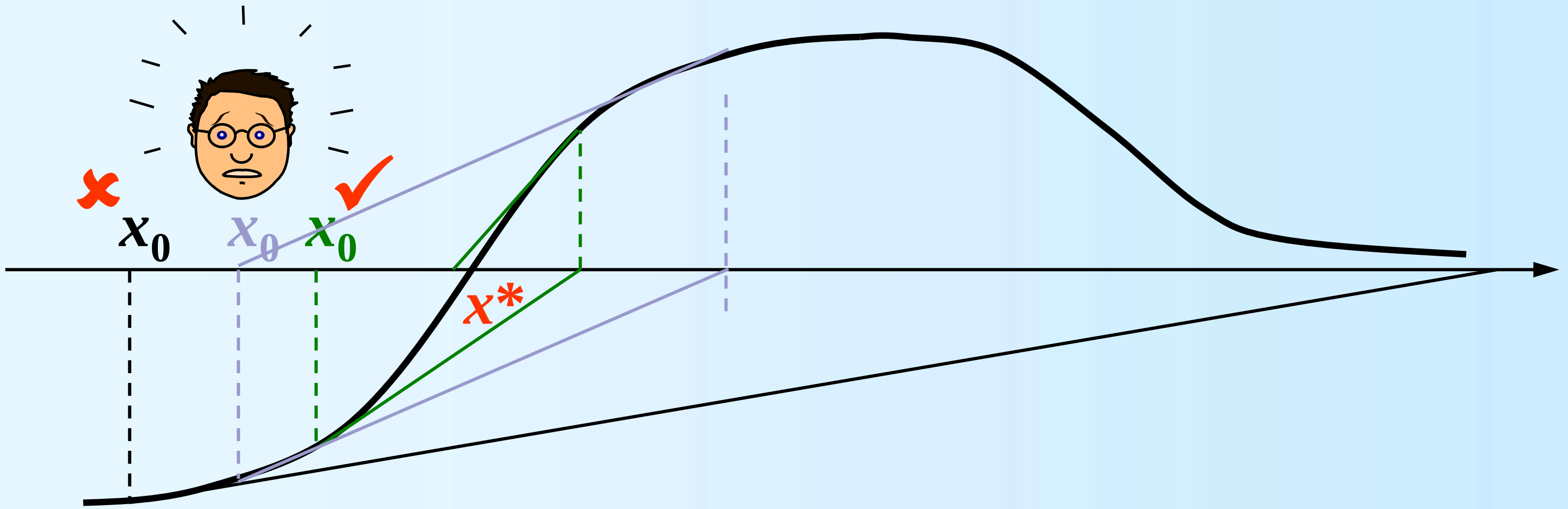
$$\Rightarrow \quad \frac{x^* - x_{k+1}}{(x^* - x_k)^2} = -\frac{f''(\xi_k)}{2f'(x_k)}$$ as long as $f'(x^*) \neq 0$, $k \to \infty$

x_0 affects the convergence of Newton's method:

1. Take x_0 satisfying $f(x_0)f''(x_0)>0$, which can increase the convergence speed

2. Usually use bisection method to get x_0.

3. If $f'(x_0)\approx 0$ or iteration times reach the maximum, then stop the iteration.



§ 4.3 Newton's method

- Example：

  find the root x^* of $f(x)=e^{-x/4}(2-x)-1=0$.

§ 4.3 Newton's method

- Example：

  find the root x^* of $f(x)=e^{-x/4}(2-x)-1=0$.

Solution：

Obviously, $f(0)*f(2)<0$, there exists root in [0,2]

taking the derivative $f'(x)= e^{-x/4}(x-6)/4$, and the Newton's formula:

$$x_{k+1} = x_k - \frac{e^{-x_k/4}(2-x_k)-1}{e^{-x_k/4}(x_k-6)/4} \quad (k=0,1,2,\cdots)$$

§ 4.3 Newton's method

Solution：

Obviously, $f(0)*f(2)<0$, there exists root in [0,2]

taking the derivative $f'(x)= e^{-x/4}(x-6)/4$, and the Newton's formula:

$$x_{k+1} = x_k - \frac{e^{-x_k/4}(2-x_k)-1}{e^{-x_k/4}(x_k-6)/4} \quad (k=0,1,2,\cdots)$$

take $x_0=1.0$ and $x_0=8.0$, then see the result in the following

§ 4.3 Newton's method

k	x_k
0	1.0
1	−1.155999
2	0.189438
3	0.714043
4	0.782542
5	0.783595
6	0.783596

k	x_k
0	8.0
1	34.778107
2	869.1519
	divergence

finding that x^*=0.783596, and $f(x_6)$=-3.8*10^{-8}

Newton's method depends on the initial point !

§ 4.3 Newton's method

- Example：

Calculate $x = \sqrt{c}$ (c>0) by Newton's method

$$\Leftrightarrow$$

find the positive root of equation $x^2 = c$

§ 4.3 Newton's method

- Example：

  Calculate $x = \sqrt{c}$ (c>0) by Newton's method

  $\Leftrightarrow$ find the positive root of equation $x^2 = c$

Solution:

take $f(x) = x^2 - c$, and the Newton's formula gives:

$$x_{k+1} = x_k - \frac{x_k^2 - c}{2x_k} = \frac{1}{2}\left(x_k + \frac{c}{x_k}\right)$$

§ 4.3 Newton's method

- Example：

Find the root x^* of x-sin$x$=0.5, where the numerical precision is 0.0001

Example : find the root x^* of x-sin$x$=0.5, where the numerical precision is 0.0001

Solution : $f(x)=x-\sin x-0.5$

$\because f(1)=-0.34<0$, $f(2)=0.591>0$

$\therefore$ there exist a root in [1,2].

$f'(x)=1-\cos x$, $f''(x)=\sin x$

$\therefore x_{k+1}=x_k-f(x_k)/f'(x_k)=x_k-(x_k-\sin x_k-0.5)/(1-\cos x_k)$

$f'(2)>0$, $f''(2)=\sin 2>0$

i.e. $f(2)*f''(2)>0$, thus chose $x_0=2$

$x_1=x_0-(x_0-\sin x_0-5)/(1-\cos x_0)=1.5829$

$x_2=1.5009$, $x_3=1.4973$, $x_4=1.4973$

$|x_4-x_3|=0<0.0001$ then take $x=x_4$ as the approximation

§ 4.3 Newton's method

The shortage of Newton's method

(1) Result depends on the initial value x_0, bad x_0 will not lead to convergence. (multiple root)

(2) Derivative is not easy to calculate. (high dimension)

§ 4.3 Newton's method

Improvement and extension

1. Expand the convergence range

2. Approximately calculate the derivative

§ 4.3 Newton's method

Improvement： simplified Newton method

Original Newton's method need to calculate $f'(x_k)$, if use a constant c to replace $f'(x)$, thus we have

$$\boldsymbol{x_{k+1}=x_k-f(x_k)/c}$$

for the convergence purpose, it needs to satisfies:

(1) $g(x) = x-f(x)/c$; (2) $g'(x) = 1-f'(x)/c$;

$$(3)\ |g'(x)| = |1-f'(x)/c| < 1$$

i.e., $0<f'(x)/c<2$, take $c*f'(x)>0$， and $f'(x)/c<2$, which is the fixed gradient of the tangent equation.

§ 4.3 Simplified Newton's method

Example： find the root of x-sin$x$=0.5, x^* is close to1.57.

$f'(1.5)=1-\cos 1.5\approx 0.9$

take c=0.9

$f'(1.5)/c<2$

$x_{k+1}=x_k-(x_k-\sin x_k-0.5)/0.9$

take $x_0=2$

through $x_1 \ldots\ldots x_5=1.497209, x_6=1.49730$, 6 times iteration

it converges slowly but amount of computation is reduced

§ 4.3 Newton's method (Improvement)

■ Newton decent method

Usually the convergence of Newton's method depends on the x_0. If we constrain the Newton iteration process with the monotonic condition, that is

$$\left|f(x_{k+1})\right| < \left|f(x_k)\right|$$

This method is called Newton descent method

§ 4.3 Newton descent method

The Newton’s method combines with the descent method, to guarantees the drop of function value and to accelerate the convergence speed in each iteration. This is called Newton descent method:

$$x_{k+1} = x_k - \lambda \frac{f(x_k)}{f'(x_k)}$$

where λ ($0<\lambda<1$) is the descent parameter.

§ 4.3 Newton descent method

The selection of descent parameter λ is a step-by-step exploration process. It starts from λ=1 then repeatedly halve it for trail calculation, that is

$$\lambda = 1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots$$

Until some suitable λ is found that satisfies the monotonic condition:

$$|f(x_{k+1})| < |f(x_k)|$$

Otherwise, the descent fails, choose new x_0

§ 4.3 Newton descent method

- Example：

Find the root x^* of $f(x) \equiv x^3 - x - 1 = 0$ near the point x=1.5

§ 4.3 Newton descent method

- Example: Find the root x^* of $f(x) \equiv x^3 - x - 1 = 0$ near the point x=1.5

Newton's method:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^3 - x_k - 1}{3x_k^2 - 1},$$

$$x_0 = 0.6, x_1 = 0.6 - \frac{0.6^3 - 0.6 - 1}{3 \times 0.6^2 - 1} \approx 17.9,$$

§ 4.3 Newton descent method

- Example: Find the root x^* of $f(x) \equiv x^3 - x - 1 = 0$ near the point x=1.5

Newton descent method:

$$x_{k+1} = x_k - \lambda \frac{f(x_k)}{f'(x_k)} = x_k - \lambda \frac{x_k^3 - x_k - 1}{3x_k^2 - 1},$$

take $\lambda_5^1 = \frac{1}{2^5}$, $x_1 = x_0 - \frac{1}{2^5} \times \frac{f(x_0)}{f'(x_0)}$

$$= 0.6 - \frac{1}{32} \times \frac{f(0.6)}{f'(0.6)} = 1.140625$$

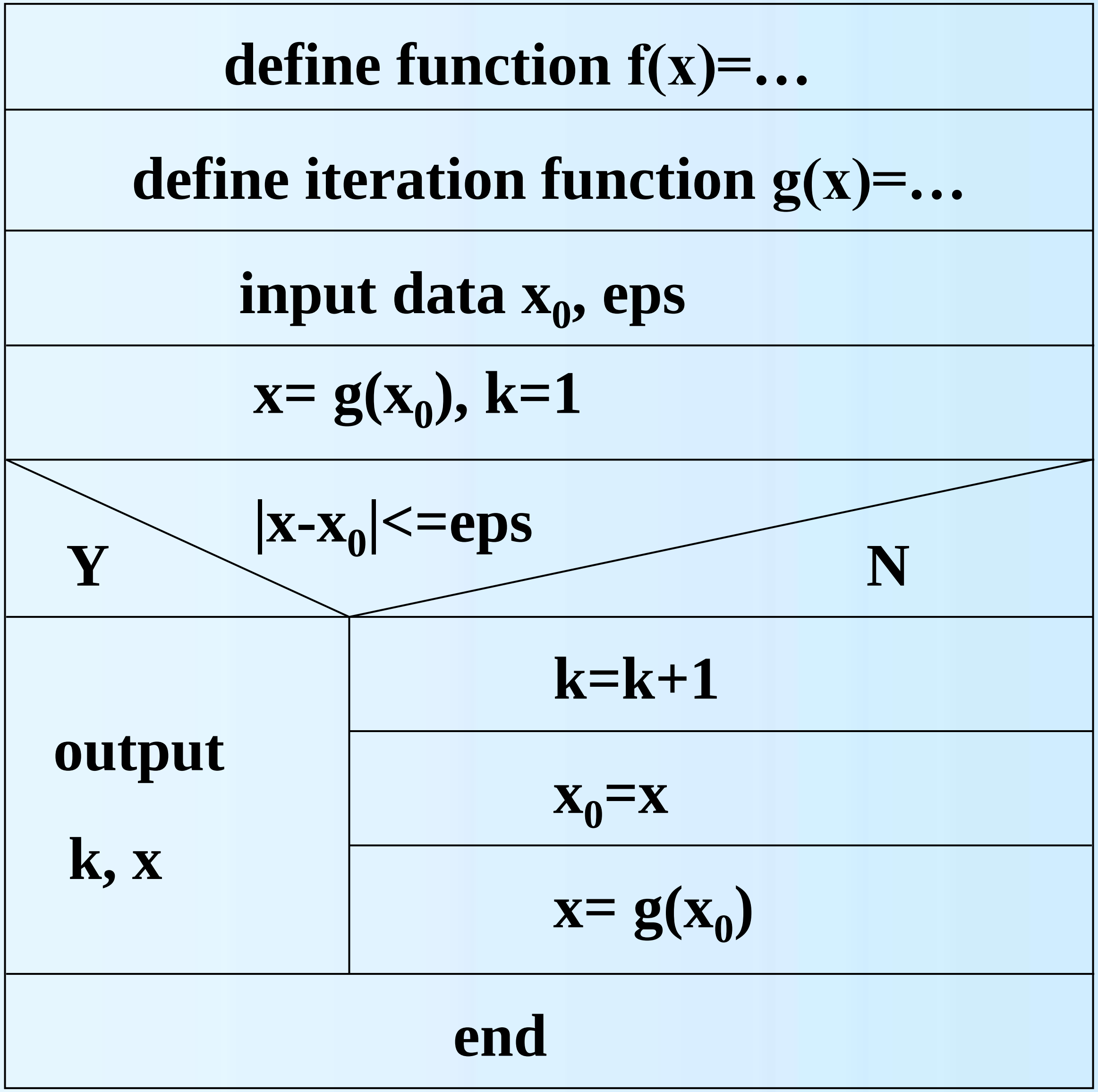
$f(x)$=-0.656643, and $|f(x_1)|<|f(x_0)|$ convergence

§ 4.3 Newton descent method

k	λ	x_k	$f(x_k)$
0		0.6	-1.384
1	$1/2^5$	1.140625	-0.6566
2	1	1.36681	0.1866
3	1	1.326280	0.00667
4	1	1.324720	8.711×10^{-6}

As shown in the table, the Newton descent method falls into the local convergence range for the first time, and the accuracy is quite high by the fourth time.

Process diagram of Newton's method



§ 4.3 Newton's method

Improvement and extension

1. Expand the convergence range

2. Approximately calculate the derivative

- §4.4 Secant Method

§ 4.4 Secant method

- Contents
  - 1. significance of secant method
  - 2. basic idea of secant method
  - 3. geometric significance of secant method
  - 4. algorithm of secant method

§ 4.4 Secant method

- 1. significance of secant method

(1) Although Newton's method converges fast, the derivative $f'(x_k)$ must be calculated every iteration.

(2)When $f(x)$ is complex, it is inconvenient to calculate the derivative $f'(x_k)$.

(3) Meanwhile the convergence of the numerical iteration is often linear without the calculation of derivative.

§ 4.4 Secant method

- 1. significance of secant method

- The secant method is a root finding method without derivative calculation.
- In the iteration process, the secant section method uses the function value at x_{k-1} in the previous step and the function value at x_k to construct the iteration function, which can improve the convergence speed.

§ 4.4 Secant method

- Basic idea

To avoid the calculation of derivative $f'(x_k)$, use finite difference

$$\frac{f(x_k)-f(x_{k-1})}{(x_k-x_{k-1})}$$

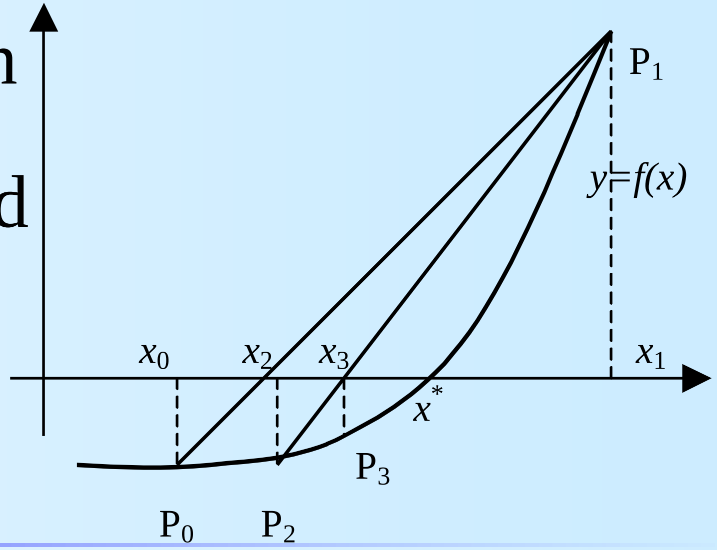
to approximate derivative $f'(x_k)$ in Newton's method, so that

$$x_{k+1}=x_k-\frac{f(x_k)}{f(x_k)-f(x_{k-1})}(x_k-x_{k-1}) \qquad (k=1,2,\cdots)$$

is called the secant iteration formula. This is secant method

§ 4.4 Secant method (geometric significance)

The geometric meaning of secant method is to use the secant line of two points $P_0(x_0,f(x_0))$ and $P_1(x_1,f(x_1))$ on the curve to replace the curve. Using the intersection x_2 of secant line and X-axis as the approximation of the root. Then, construct the secant line of point P_1 and $P_2(x_2,f(x_2))$ to find the intersection x_3 , following the same step to find the point x_4, then x_5, and so on… until the iteration stops



§ 4.4 Secant method

- Example :

use the secant method to calculate the root of equation x^3-3x+1=0

- Solution:

suppose $f(x)= x^3$-3x+1, then by the secant method

$$x_{k+1} = x_k - \frac{f(x_k)}{f(x_k) - f(x_{k-1})}(x_k - x_{k-1})$$

§ 4.4 Secant method

- Example :

- Solution: suppose $f(x)= x^3-3x+1$, then by the secant method

$$x_{k+1} = x_k - \frac{f(x_k)}{f(x_k) - f(x_{k-1})}(x_k - x_{k-1})$$

take the initial point $x_0=0.5$ and $x_1=0.4$, then we can get

x0=0.5; x1=0.4;	x4 = 0.3472965093
x2 = 0.3430962343	x5 = 0.3472963553
x3 = 0.3473897274	x6 = 0.3472963553

5 times iteration

precision=10^{-8}

§ 4.4 Secant method

- Example :

use the secant method to find the root of equation $x=e^{-x}$ near the initial point x_0

- Solution: take $x_0 = 0.5$ and $x_1 = 0.6$

use the iteration formula

$$x_{k+1} = x_k - \frac{(x_k - e^{-x_k})}{(x_k - x_{k-1}) - (e^{-x_k} - e^{-x_{k-1}})}(x_k - x_{k-1})$$

find the approximation of root $x_4 \approx 0.56714$

§ Convergence order of iteration method

- Convergence rate
  - An iteration method with practical value, should not only converge, but also converges fast. The convergence rate of iteration process refers to the decline rate of the iterative error when it approaches convergence. Specifically, if the error $e_k = x - x_k$ satisfies

$$\frac{e_{k+1}}{e_k{}^p} \to C \quad (C \neq 0, \text{ constant}) \qquad k \to \infty$$

Thus, the iterative process is said to be pth-order convergent

§ Convergence order of iterative method

- In particular,
  - $p = 1$ ($0 < C < 1$) is called Linear convergence
  - $p > 1$ is called superlinear convergence ,
  - when $p = 2$ is called quadratic convergence

    (or square convergence) .
- The lager p, the faster$\{ x_k \}$ converges to x^* .

  The value of p is one of the symbol to measure the quality of an iteration process.

§ Convergence order of iterative method

- Convergence order of common-used iteration
  - general iteration method： $p = 1$, linear convergence
  - Newton method： $p = 2$， square convergence
    （ linear convergence when x^* is a double root）
  - secant method： p=1.618, superlinear convergence