AP Calculus BC

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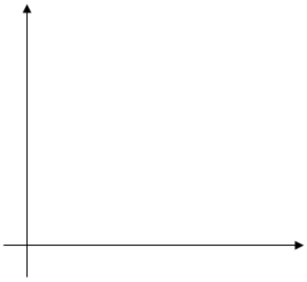
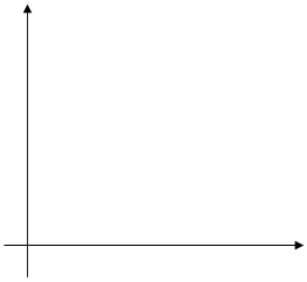
Name :

Table of Contents

I. Limit and Continuity

1. Limits of Function Values

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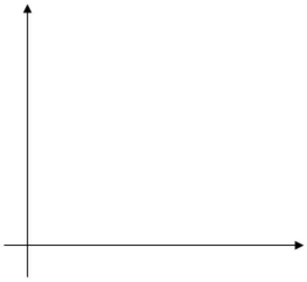
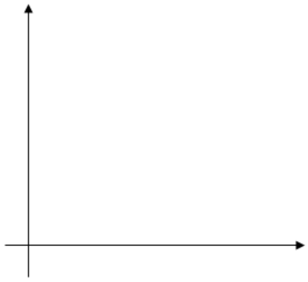

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⇨ x ______________ to a (   )

⇨ How to find limits? ① ______________________

② ____

Exceptions]

Ex) Find the limits of following functions as $x \to 1$

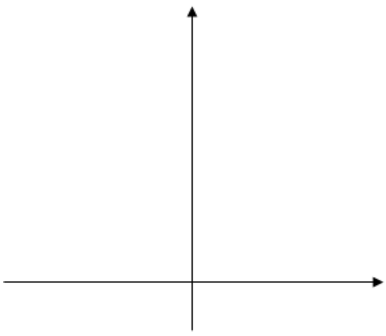
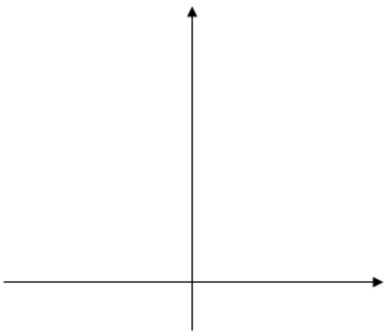
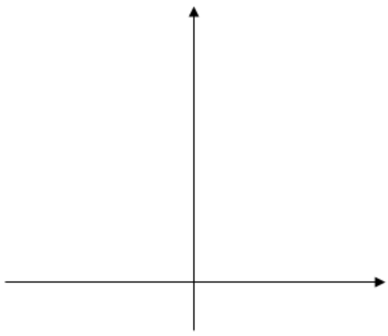
(a) $f(x) = x + 1$

(b) $g(x) = \dfrac{x^2 - 1}{x - 1}$

(c) $h(x) = \begin{cases} \dfrac{x^2-1}{x-1} & x \neq 1 \\ 1 & x = 1 \end{cases}$

M1>

M2>

$\lim_{x \to 1} f(x) =$

$\lim_{x \to 1} g(x) =$

$\lim_{x \to 1} h(x) =$

Ex) Find the limits of following functions as $x \to \infty$ and $x \to -\infty$.

(a) $f(x) = 2$

(b) $g(x) = \dfrac{1}{x}$

- **Limit does not exist when**

① ____________________ $\neq$ ____________________

② y **value diverges to** ________

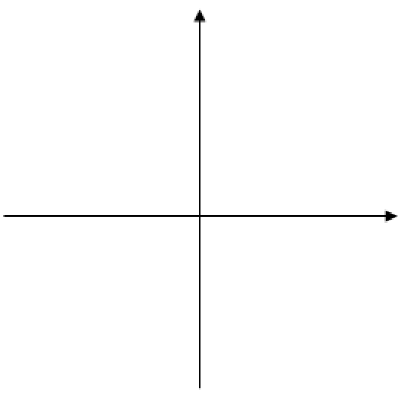
③ ______________________

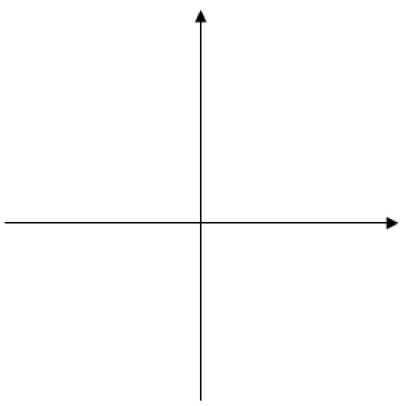
Ex) Find the limits of the following functions as $x \to 0$.

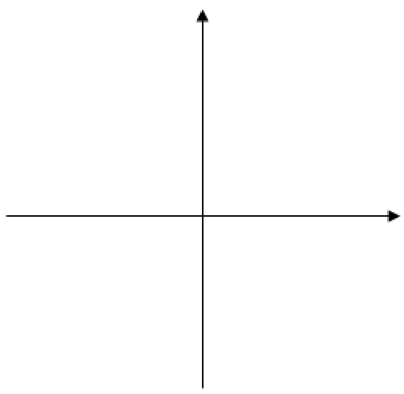
(a) $g(x) = \begin{cases} 3x+1 & x < 0 \\ -x+2 & x \geq 0 \end{cases}$

(b) $h(x) = \begin{cases} \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

(c) $h(x) = \begin{cases} -\dfrac{1}{x} & x < 0 \\ \dfrac{1}{x} & x > 0 \end{cases}$

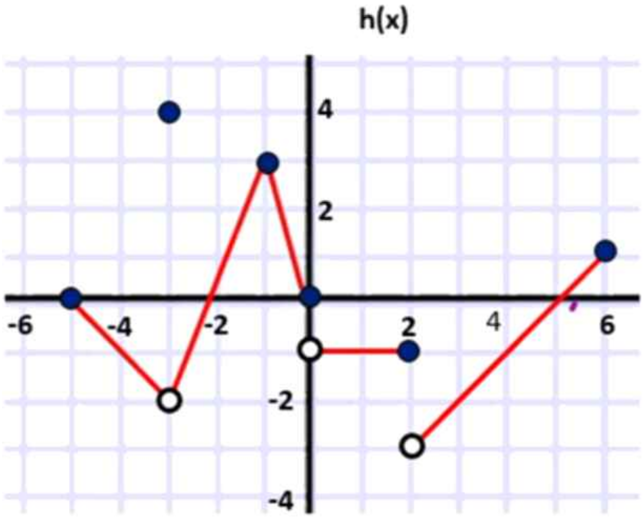






cf> $\lim_{x \to \infty} \sin x =$

Past Test Question 1) Using the graph of h(x), answer problems 1-6 (1 pt each)



1. $\lim_{x \to -3} h(x) =$

2. $h(-3) =$

3. $\lim_{x \to 2} h(x) =$

4. $h(2) =$

5. $\lim_{x \to 4} h(x) =$

6. $h(4) =$

Past Test Question 2)

7. $\lim_{x \to 3} g(x) =$

x	2.9	2.99	2.999	2.9999	3	3.0001	3.001	3.01	3.1
$g(x)$	1.8	1.1	1.01	1.001	Undef	-1.001	-1.01	-1.1	-1.8

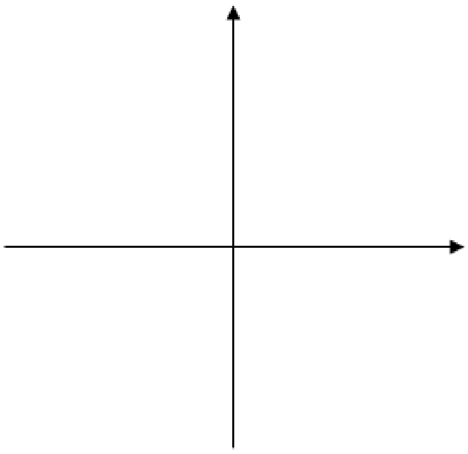
8. $\lim_{x \to 1} f(x) =$

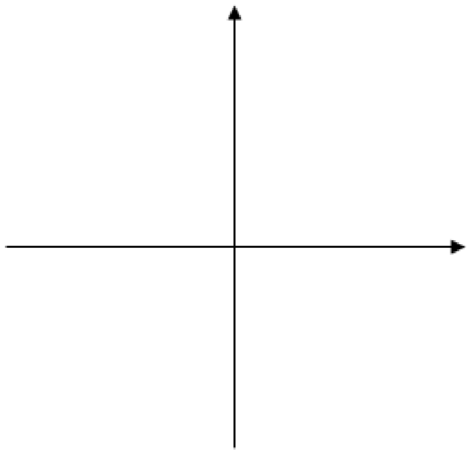
x	0.5	0.8	0.9	0.99	1	1.01	1.1	1.2	1.5
$f(x)$	4.5	4.08	4.02	4.002	3	4.002	4.02	4.08	4.5

2. Asymptotes

- If $\lim_{x \to \infty} f(x) = b$ or $\lim_{x \to -\infty} f(x) = b$, then ________ is a __________ asymptote for $y = f(x)$
- If $\lim_{x \to a-} f(x) = +\infty$ or $\lim_{x \to a+} f(x) = -\infty$ then ________ is a _________ asymptote for $y = f(x)$

  $\lim_{x \to a+} f(x) = +\infty$ or $\lim_{x \to a-} f(x) = -\infty$





Ex) Find the horizontal and vertical asymptotes of the curve $f(x) = \frac{x+3}{x+2}$.

cf> $\lim_{x \to \infty} f(x) =$

$\lim_{x \to a+} f(x) = +\infty$ ⇨ $a =$

Ex) Find the horizontal and vertical asymptotes of the curve $y=\dfrac{-2x+1}{x+3}$.

Ex) Find the horizontal and vertical asymptotes of the curve $y=\dfrac{3x-3}{2x+3}$ and sketch the graph.

3. Theorems on Limits

• For real numbers L, M, c, k, if $\lim_{x\to c} f(x)=L$ & $\lim_{x\to c} g(x)=M$,

(1) Sum and Difference Rule :

(2) Product Rule :

(3) Constant Multiple Rule :

(4) Quotient Rule :

(5) Power Rule :

(6) Sandwich Theorem / Rule (Squeeze Rule)

If $f(x) \le g(x) \le h(x)$ **for all real numbers** x **&** ______________________________ **,**

then ______________

Ex) Find the following limits.

(a) $\lim_{x \to 2} (x^2 + 4x - 3)$

(b) $\lim_{x \to 1} \frac{x^4 + x^2 - 1}{x^2 + 5}$

(c) $\lim_{x \to -2} \sqrt{4x^2 - 3}$

Ex) Use the Squeeze theorem, where appropriate, to evaluate the given limit.

① $\lim_{x \to 1} f(x)$, **where** $3x - 2 \le f(x) \le x^3$

② $\lim_{x \to \infty} f(x)$, **where** $\frac{1}{x} < f(x) < \frac{2}{x}$

③ $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) =$

④ $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) =$

4. $\frac{0}{0}$ Limits

① Factorization

② Rationalization

③ If $\lim_{x \to a} \frac{f(x)}{g(x)} = L$ & $\lim_{x \to a} g(x) = 0$, then __________________

④ If $\lim_{x \to a} \frac{f(x)}{g(x)} = L \ (L \neq 0)$ & $\lim_{x \to a} f(x) = 0$, then __________________

Ex) Find the following limits.

(a) $\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}$

(b) $\lim_{x \to 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}$

(c) $\lim_{x \to 2} \frac{x - 2}{x^2 + ax + b} = 1 \Rightarrow a, b$?

(d) $\lim_{x \to 4} \frac{4\sqrt{9 + x^2} - 5x}{ax + b} = -2 \Rightarrow a, b$?

5. $\frac{\infty}{\infty}$ Limits

(1) Limit of a Quotient of Polynomials

If $P(x), Q(x)$ are a ________ with respect to x, $\lim_{x \to \infty} \frac{P(x)}{Q(x)}$ can be found by comparing the ____________ and their ________ .

① (numerator's degree) (denomiator's degree) ⇨ Limit : ____

② (numerator's degree) (denomiator's degree) ⇨ Limit : ____

③ (numerator's degree) (denomiator's degree) ⇨ Limit : ____________

※ If $x \to -\infty$, then substitute ______ , ______

Ex) Find the following limits.

(a) $\lim_{x \to \infty} \frac{3-x}{4+x+x^2}$

(b) $\lim_{x \to \infty} \frac{4x^4+5x+1}{37x^3-9}$

(c) $\lim_{x \to \infty} \frac{x^3-4x^2+7}{3-6x-2x^3}$

(d) $\lim_{x \to -\infty} \frac{4+x^2-3x^3}{x+7x^3}$

※ If $P(x)$ or $Q(x)$ is not a polynomial and $\lim_{x \to \infty} \frac{P(x)}{Q(x)} = \frac{\infty}{\infty}$, investigate which one approaches to ∞ faster.

Tip] High-Base Exponential function Low-Base Exponential function

High degree Polynomial function Low degree Polynomial function

Ex) $\lim_{x \to \infty} \frac{2^x}{1000x^{999}} =$

$\lim_{x \to \infty} \frac{5^x+2^{x+1}}{-2 \cdot 5^x - 3^x} =$

⇨ Divide the numerator and denominator with the ________ increasing function

6 $\infty - \infty$ Limits : Factorization, Rationalization, Substitution (when $x \to -\infty$)

Ex) $\lim\limits_{x \to -\infty} (\sqrt{ax^2 - bx} + 2x) = 1 \Rightarrow a,\ b$ **?**

7. Existence of Limit

- **If the limit of** $f(x) = \begin{cases} 2x^2 + a & (x \leq 1) \\ 3x - 2 & (x > 1) \end{cases}$ **at** $x = 1$ **exist, what is** a**?**

8. Other Basic Limits

① $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$ = ______ ⇨ cf> $\lim_{\theta \to 0} \frac{\theta}{\sin\theta}$ = , $\lim_{\triangle \to 0} \frac{\sin\triangle}{\triangle} =$

$\therefore \lim_{\theta \to 0} \frac{\sin a\theta}{b\theta} =$

② $\lim_{\theta \to 0} \frac{\tan\theta}{\theta}$ = ______ ⇨ cf> $\lim_{\theta \to 0} \frac{\theta}{\tan\theta}$ = , $\lim_{\triangle \to 0} \frac{\tan\triangle}{\triangle} =$

$\therefore \lim_{\theta \to 0} \frac{\tan a\theta}{b\theta} =$

※ $\lim_{\theta \to 0} \frac{\cos\theta}{\theta}$:

③ $\lim_{x \to \infty} (1+\frac{1}{x})^x = \lim_{x \to 0} (1+x)^{\frac{1}{x}} =$ ※ e :

⇨ $\lim_{\triangle \to \infty} (1+\frac{1}{\triangle})^{\triangle} = \lim_{\triangle \to 0} (1+\triangle)^{\frac{1}{\triangle}} =$

④ $\lim_{x \to 0} \frac{\ln(1+x)}{x} =$ ⇨ cf> $\lim_{x \to 0} \frac{\ln(1+nx)}{mx} =$

proof>

⑤ $\lim_{x \to 0} \frac{\log_a(1+x)}{x} =$ ⇨ cf> $\lim_{x \to 0} \frac{\log_a(1+nx)}{mx} =$

proof>

⑥ $\lim_{x \to 0} \frac{e^x-1}{x}$ = ______ ⇨ cf> $\lim_{\triangle \to 0} \frac{e^{\triangle}-1}{\triangle} =$ $\therefore \lim_{x \to 0} \frac{e^{nx}-1}{mx} =$

proof > $e^x - 1 = t$ ⇨

⑦ $\lim_{x \to 0} \frac{a^x - 1}{x} =$ ______ ⇨ **cf>** $\lim_{\triangle \to 0} \frac{a^{\triangle} - 1}{\triangle} =$ $\therefore \lim_{x \to 0} \frac{a^{nx} - 1}{mx} =$

proof> $a^x - 1 = t$ ⇨

Ex) Find the limits.

(a) $\lim_{x \to 0} \frac{\sin 3x}{2x}$ **(b)** $\lim_{x \to 0} \frac{\tan \frac{1}{2} x}{-3x}$

(c) $\lim_{x \to 0} \frac{e^{3x} - 1}{2x}$ **(d)** $\lim_{x \to 0} \frac{\ln(1-x)}{-x}$

9. Continuity

- **Definition**

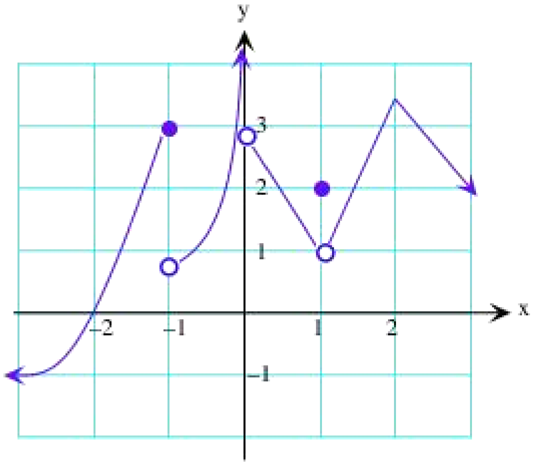
⇨ $f(x)$ **is continuous at** $x = a$ **if** ①

②

③

⇨ **If not,** $f(x)$ **is discontinuous as** $x = a$

Ex) Whether the function graphed is continuous on $[-3, 3]$. If not, where does it fail to be continuous and why?



10. Properties of Continuous Functions

⇨ If $f(x), g(x)$ is continuous at $x=a$, the following functions are also continuous at $x=a$

(1) Sums & Differences :

(2) Product :

(3) Constant multiples :

(4) Quotients : ( )

※ If $f(x)$ is continuous at $x=a$, and $g(x)$ is continuous at $f(a)$, then $g(f(x))$ is continuous at $x=a$.

※ Normal polynomial functions, exponential functions, trigonometric functions are continuous at $(-\infty, \infty)$.

Ex) If $f(x)=\begin{cases} -\dfrac{1}{2}x^2+a & (x \le 1) \\ b & (x=1) \\ 3x-2 & (x>1) \end{cases}$ is continuous at all real numbers, what is a, b?

11. Interval

① **Open Interval :** ________ = ______________

② **Closed Interval :** _______ = _______________

Ex) $(-\infty,\ a]$ $(-a,\ \infty)$ $(-\infty,\ \infty)$

$(a,\ b)$ $(-2,\ 5]$

12. Intermediate Value Theorem (IVT)

____

____

____

____

- **Application of Intermediate Value Theorem**

Ex) For $f(x)=x^2-2x-6$, show that there is at least one solution in $(1,5)$.

sol>

13. Extreme Value Theorem

I. Limit and Continuity

- **Find the limit.**

1. $\lim_{x \to -1} (x+3) =$

2. $\lim_{x \to 2} (x^2 - 1) =$

3. $\lim_{x \to 1} \frac{1}{x} =$

4. $\lim_{x \to 3} \sqrt{4x-3} =$

5. $\lim_{x \to 0} (-\frac{1}{x^2}) =$

6. $\lim_{x \to 1} \frac{1}{|x-1|} =$

7. $\lim_{x \to \infty} (-x+5) =$

8. $\lim_{x \to -\infty} x^2 =$

9. $\lim_{x \to \infty} \frac{1}{x+1} =$

10. $\lim_{x \to -\infty} (3 - \frac{1}{x}) =$

[11-14] $f(x) = \begin{cases} x^2 - 4x + 3 & (x \neq 2) \\ 0 & (x = 2) \end{cases}$

11. $\lim_{x \to 0} f(x) =$

12. $\lim_{x \to 1} f(x) =$

13. $\lim_{x \to 2} f(x) =$

14. $\lim_{x \to -\infty} f(x) =$

15. $\lim_{x \to 0+} \frac{1}{x} =$

16. $\lim_{x \to 0-} \frac{1}{x} =$

17. $\lim_{x \to 1} (2x+5) =$

18. $\lim_{x \to 3} (x-1)(2x^2 - 3x - 1) =$

19. $\lim_{x \to 2} \frac{2x-1}{x+1} =$

20. $\lim_{x \to 1} \frac{\sqrt{1+x} - \sqrt{1-x}}{x} =$

21. $\lim_{x \to 0} \frac{x(x^2+2)}{x} =$

22. $\lim_{x \to -1} \frac{2x^2 - x - 3}{x+1} =$

23. $\lim_{x \to 1} \frac{(x-1)(x^2+x+2)}{(x^2-1)} =$

24. $\lim_{x \to 2} \frac{x^3 - 8}{x-2} =$

25. $\lim_{x \to 9} \frac{\sqrt{x}-3}{x-9} =$

26. $\lim_{x \to 3} \frac{2x-6}{\sqrt{x+1}-2} =$

27. $\lim_{x \to \infty} \frac{x^2-1}{3x^3+x-1} =$

28. $\lim_{x \to -\infty} \frac{4x-1}{3x+2} =$

29. $\lim_{x \to \infty} \frac{2x^2+3}{4x+3}$

30. $\lim_{x \to \infty} \frac{(x+1)(2x-1)}{3x^2+x-1}$

31. $\lim_{x \to \infty} \frac{2x}{\sqrt{x^2+3}+4}$

32. $\lim_{x \to -\infty} (x^3+3x^2+2x-1)$

33. $\lim_{x \to \infty} (\sqrt{x^2+1}-x)$

34. $\lim_{x \to \infty} (\sqrt{x^2-3x}-\sqrt{x^2+3x})$

35. $\lim_{x \to 0} \frac{1}{x}\left(\frac{1}{x+\sqrt{2}}-\frac{1}{\sqrt{2}}\right)$

36. $\lim_{x \to 2} \frac{1}{x-2}\left(2x-\frac{5x+2}{x+1}\right)$

- **Determine the value of a, b.**

37. $\lim_{x \to -1} \frac{x^2+ax+b}{x+1} = 3$

38. $\lim_{x \to 2} \frac{a\sqrt{x-1}+b}{x-2} = 1$

39. $\lim_{x \to 2} \frac{1}{x-2}\left(\frac{1}{x+a}-\frac{1}{b}\right) = -\frac{1}{9}$

- **Write the domain where the function is continuous.**

1. $f(x)=x^2$

2. $f(x)=|x-1|$

3. $f(x)=\dfrac{1}{|x-3|}$

4. $f(x)=\begin{cases} \dfrac{x^2-1}{x-1} & (x\neq 1) \\ 2 & (x=1) \end{cases}$ **cf>** $f(1)=2$

5. $y=\sqrt{5-x}$

6. $y=3$

7. $y=\dfrac{x-1}{2x+1}$

II. Differentiation

1. Average Rate of Change

• The average rate of change of $f(x)$ in $[a, b]$ is

• $=$

$=$

$=$

Ex) Find the Average Rate of Change of $y=2x^2+1$ when x changes from -1 to 2.

Ex) Find the Average Rate of Change of $y=-e^x+2$ when x changes from 1 to 3.

2. Derivative

$=$ ______________________________________ $=$ ______________________________

$=$ ______________________________________

$f'(a)=$ $\qquad$ $=$

$=$ $\qquad$ $=$ $\qquad$ $\Leftrightarrow$

- **General Derivative**

Ex) $\lim_{h \to 0} \frac{f(1+h)-f(1)}{h} =$

$\lim_{h \to 0} \frac{f(-2-h)-f(-2)}{-h} =$

$\lim_{h \to 0} \frac{f(3+h)-f(3-h)}{h} =$

$\lim_{n \to \infty} n\left\{f(-1+\frac{1}{n})-f(-1-\frac{1}{n})\right\}=$

Ex) Find the derivative of $f(x)=x^2$**.**

M1 >

M2 >

3. Formulas (1)

① $y=c$ ⇨

② $y=x^n$ ⇨ **cf>** $y=\frac{1}{x}$

③ $y=ax$ ⇨ $y=\sqrt{x}$

④ $y=f(x)g(x)$ ⇨ **(Product Rule)**

⑤ $y=\frac{f(x)}{g(x)}$ ⇨ **(Quotient Rule)**

Ex) Find the derivatives.

① $y=x^2$

② $y=3$

③ $f(x)=3x^3+1$

④ $y=(x+1)(x+2)$

⑤ $f(x)=(x+1)(2x+1)(3x-3)$

⑥ $y=\frac{1}{x+1}$

⑦ $y=\frac{x-1}{2x+1}$

⑧ $y=-\frac{3+x}{2x-1}$

Ex) Find the values or simplify the expressions using $f(x)=x^2$ for the following questions.

① $\lim_{h \to 0} \frac{f(1-2h)-f(1+3h)}{h} =$

② $\lim_{h \to 0} \frac{f(a+5h)-f(a)}{4h} =$

③ $\lim_{h \to 0} \frac{f(2+mh)-f(2-nh)}{h} = 8 \ \rightarrow m+n\,?$

④ $\lim_{h \to 0} \frac{1}{h}\left\{\frac{1}{f(a+h)} - \frac{1}{f(a)}\right\}$

⑤ $\lim_{h \to 0} \frac{f(a+4h)-f(a+h^2)}{h}$

⑥ $\lim_{x \to a} \frac{f(x^2)-f(a^2)}{x-a}$

⑦ $\lim_{x \to 1} \frac{\sqrt{f(x)}-\sqrt{f(1)}}{\sqrt{x}-1}$

3. Formulas (2)

① $y = \sin x$ ⇨

② $y = \cos x$ ⇨

③ $y = \tan x$ ⇨

④ $y = \ln x$ ⇨

⑤ $y = \log_a x$ ⇨

⑥ $y = e^x$ ⇨

⑦ $y = a^x$ ⇨

⑧ $y = \csc x$ ⇨

⑨ $y = \sec x$ ⇨

⑩ $y = \cot x$ ⇨

(See textbook for extra formulas)

Ex) Find the derivatives for the following functions using Quotient Rule.

① $y = \sec x$

② $y = \csc x$

③ $y = \cot x$

3. Formulas (3)

① $y=\sin^{-1}x$ ⇨ cf> $y=\sin^{-1}ax$

② $y=\cos^{-1}x$ ⇨ $y=\cos^{-1}ax$

③ $y=\tan^{-1}x$ ⇨ $y=\tan^{-1}ax$

4. The Chain Rule

- If $f(x)$ and $g(x)$ is both differentiable and $F(x)=f(g(x))$, then ____________________
- If $y=f(u)$, $u=g(x)$, then
- $y=\{f(x)\}^n$ $\rightarrow$
- $y=\ln f(x)$ $\rightarrow$
- $y=\dfrac{1}{f(x)}$ $\rightarrow$

Ex) Find the derivatives for the following functions.

① $y=(5x^3-x^4)^7$ ② $y=\sin(x^2+x)$ ③ $y=\sqrt{x^2+1}$

④ $y=\tan(5-\sin 2x)$ ⑤ $y=\ln 2x$ ⑥ $y=\dfrac{1}{2x^2+1}$

Ex) If $y = \tan u$, $u = v - \frac{1}{v}$, $v = \ln x$, what is the value of $\frac{dy}{dx}$ at $x = e$?

5. Differentiability and Continuity

$f(x)$ **is continuous at** $x = a$ $f(x)$ **is differentiable at** $x = a$

- $f(x)$ **is** <u>**not**</u> **differentiable when**

① ____________________ ② ____________________

③ ____________________ ④ ____________________

- $f(x)$ **is differentiable at** $x = a$ $\Leftrightarrow$ ①

② = =

Ex) For $f(x)\begin{cases} g(x) & (x \le a) \\ h(x) & (x > a) \end{cases}$, what is the condition for being $f(x)$ differentiable at all real numbers?

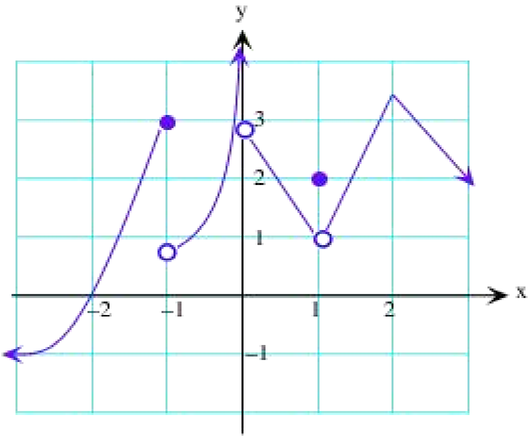
Ex) If $f(x) = \begin{cases} -x^2 + ax & (x \ge 1) \\ \dfrac{1}{2}x^2 + x + b & (x < 1) \end{cases}$ is differentiable on all real numbers x, find a, b.

- **Normal polynomial functions, trigonometric functions, exponential functions are differentiable at all points. (interval에 따라 나뉘어진 function 제외)**

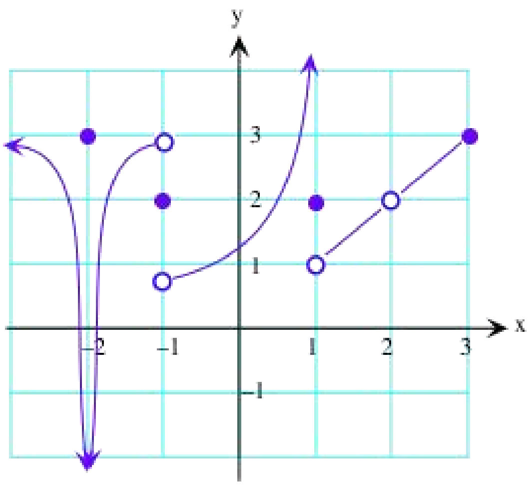
Ex) Investigate the continuity and differentiability of $f(x) = |x|$ at $x = 0$.

Ex) Find the points where $f(x)$ is ⓐ discontinuous, ⓑ not differentiable.

①



②



6. Estimating a Derivative

① Forward Difference Quotient $f'(a) \approx$

② Backward Difference Quotient $f'(a) \approx$

③ Symmetric Difference Quotient $f'(a) \approx$

Ex) For the function $f(x) = x^4$, approximate $f'(1)$ using the symmetric difference quotient with $h = 0.01$.

Ex) For the function $f(x) = |x|$, approximate $f'(0)$ using the symmetric difference quotient with $h = 0.01$.

※ Note : Although the approximation of $f'(0)$ for $f(x) = |x|$ above is 0, $f'(0)$ does not actually exist for $f(x) = |x|$. Therefore, differentiability should be checked before the evaluation.

7. Derivatives of Parametrically Defined Functions

- If $x=f(t),\ y=g(t)$ are differentiable with respect to t

$$\frac{dy}{dx}= \qquad\qquad \text{and} \quad \frac{d^2y}{dx^2}=$$

Ex) If $x=2t+3,\ y=t^2-1$, find the value of $\frac{dy}{dx}$ at $t=6$, and $\frac{d^2y}{dx^2}$ at $t=2$.

Ex) If $y=x^2+2$ and $u=2x-1$, then $\frac{dy}{du}=$?

Ex) If $x=t^2-1$ and $y=2e^t$, find the value of $\frac{dy}{dx}$ at $t=1$ and find the value of $\frac{d^2y}{dx^2}$.

8. Implicit Differentiation

⇨ For Implicit function $f(x, y) = 0$, if y is a differentiable with respect to x, the derivative can be found by the following procedure.

① Differentiate $f(x, y) = 0$ with respect to x

② Assume y as a function defined by using x, and use the chain rule.

③ Solve the equation with respect to $\frac{dy}{dx}$ (y').

Ex) If $x^2 + xy + y^3 = 0$, then in terms of x and y, $\frac{dy}{dx}$ = ?

Ex) If $xy^2 + 2xy = 8$, then, at the point $(1, 2)$, y' is ?

Ex) Find $\frac{dy}{dx}$ using implicit differentiation on the equation $\frac{y}{x+y} = x^3 - 1$. Then find the value of $\frac{dy}{dx}$ when $x = 1$.

9. Derivative of the Inverse of a Function

- If $y=f(x)$ is differentiable and it's inverse function is $x=g(y)$,

______________________________ ______________________ (   )

Proof)

Ex) Let $y=f(x)=\sqrt{x^3+2x+1}$, and let $g(x)$ be the inverse function. Evaluate $g'(2)$.

10. L'HOSPITAL's Rule

- For open interval I ($a \in I$), if $f,\ g$ is differentiable and $g'(x) \neq 0$,

  $\lim_{x \to a} \frac{f(x)}{g(x)}$ is a indeterminate form if $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ or

  $\lim_{x \to a} f(x) = \pm\infty,\ \lim_{x \to a} g(x) = \pm\infty$.

  Then, if $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists, then ______________________________

Ex) Find $\lim_{x \to 0} \frac{e^x - 1}{x}$.

Ex) Find $\lim_{x \to 3} \frac{x^2 - 2x - 3}{x - 3}$.

Ex) Find $\lim_{h \to 0} \frac{f(2+h) - f(2-h)}{h}$.

Ex) Find $\lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$

Ex) Find $\lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$.

Ex) Find $\lim_{x \to 0} \frac{x^2 + 2x + 1}{2x + 1}$.

11. The Mean Value Theorem

12. Rolle's Theorem

Ex) For $f(x) = x^3 - 1$, find all values of c that satisfies the MVT in [-1, 1].

II. Differentiation

- **Find the derivatives for the following functions.**

1. $y = -x^8$

2. $y = 6$

3. $y = \frac{3}{2x^2}$

4. $y = 2\sqrt[3]{x}$

5. $y = \frac{1}{\sqrt[5]{x^3}}$

6. $y = 3x^6$

7. $y = -5x + 2$

8. $y = -x^2 + 2x + 4$

9. $y = \frac{1}{2}x^2 - x + 4$

10. $y = (x-3)(2x-1)$

11. $y = -5x(x^2-1)$

12. $y = (x^2 - 4x + 5)(3x + 7)$

13. $y = x(x+1)(x+2)$

14. $y = (x-5)(2x+1)(-x+7)$

15. $y = -3e^x$

16. $y = e^{x+1}$

17. $y = (x+1)e^x$

18. $y = x^3e^x$

19. $y=3 \cdot 2^{x}$

20. $y=5^{2x+1}$

21. $y=4x \cdot 3^{x}$

22. $y=2^{x}(x-1)$

23. $y=\ln 2x$

24. $y=x\ln 3x$

25. $y=\ln x^{5}$

26. $y=e^{x}\ln x$

27. $y=x^{2}\ln x+2x$

28. $y=2\log x-x$

29. $y=\log_{2}5x$

30. $y=x\log_{5}2x$

31. $y=(\log_{3}x)^{2}$

32. $y=x+2\cos x$

33. $y=\sin x-\ln x$

34. $y=3\sin x-\cos x$

35. $y=e^{x}\cos x$

36. $y=x^{2}\sin x$

37. $y=\sin x\cos x$

38. $y=x\cos x$

39. $y=\dfrac{1}{x+1}$

40. $y=\dfrac{2x-3}{3x+2}$

41. $y=\dfrac{x^2+3}{x-1}$

42. $y=\dfrac{x+1}{e^x}$

43. $y=\dfrac{\ln x}{x}$

44. $y=\dfrac{\cos x}{1+\sin x}$

45. $y=-\dfrac{1}{x^3}$

46. $y=\dfrac{x^2-6}{x^4}$

- **Find the derivatives for the following functions.**

1. $y=(3x-2)^4$

2. $y=(2x^2-x+5)^2$

3. $y=(x+2)^2(3x^2-1)$

4. $y=(x-5)^2(x^2+1)^3$

5. $y=\sin^2 x$

6. $y=\sin 2x$

7. $y=x\tan 2x$

8. $y=2\sin^2 x-8\sin^2\frac{x}{2}$

9. $y=\frac{1}{(3-x)^4}$

10. $y=\frac{(2x-5)^2}{x+1}$

11. $y=e^{x^2+3x}$

12. $y=\sin^4 x$

13. $y=\cos(\sin x)$

14. $y=x\ln|x|$

15. $y=\log_3(4x+3)$

16. $y=\ln(e^x-1)$

17. $y=\ln\cos x$

- **Find the derivatives for the following functions.**

1. $y = \frac{1}{\sqrt[3]{x^2}}$

2. $y = \sqrt{x}(1+x^2)$

3. $y = x^{\sqrt{2}}$

4. $y = x^{-e}$

5. $y = \sqrt{3x^2-1}$

6. $y = (x^2-1)\sqrt{x-2}$

7. $y = \frac{5x^2}{\sqrt{2x+1}}$

III. Application of Derivative

1. Tangents and Normals

- The equation of the tangent to the curve $y=f(x)$ at point $P(x_1, y_1)$ is

  ____

- The line through P that is perpendicular to the tangent, called the __________ to the curve at P, has slope . Its equation is

  ____

cf > perpendicular : (slope 1) $\times$ (slope 2) =

Ex) Find the equation of the tangent and the normal to the curve of $y=2x^2+2x+1$ at the point where $x=1$.

Ex) Find the equation of the tangent and the normal to the curve of $x^2y-x=y^3-8$ at the point where $x=0$.

Ex) Find the equation of the tangent to $F(t)=(\cos t,\ 2\sin^2 t)$ at the point where $t=\frac{\pi}{3}$.

Ex) [When the slope of a tangent is given] ⇨ Set ____________________________

Find the equation of the tangent to $y=x^3+1$ and parallel to $y=3x+5$.

Ex) [When Exterior point is given] ⇨ Set the ______________________________

Find the equation of the tangent to $y = x^2 + 2x - 1$ **drawn from** $(-1,\ -3)$

2. Increasing and Decreasing Function

- Assume $f(x)$ is defined in the interval I, and x_1, x_2 are random points from I

(1) If $x_1 < x_2$ & $f(x_1) < f(x_2)$, then $f(x)$ is ______________________ in I

(2) If $x_1 < x_2$ & $f(x_1) > f(x_2)$, then $f(x)$ is ______________________ in I

- If $f(x)$ is continuous at $[a, b]$ and differentiable at (a, b)

(1) If __________ for all $x \in (a,b)$, then $f(x)$ is ______________________ in $[a, b]$

(2) If __________ for all $x \in (a,b)$, then $f(x)$ is ______________________ in $[a, b]$

(3) If $f(x)$ is increasing in $[a, b]$, then ______________ for (a, b)

(4) If $f(x)$ is decreasing in $[a, b]$, then _______________ for (a, b)

Ex) Which of the following statements are true?

I. If $f'(x) \geq 0$ for for all $x \in (a,b)$, then $f(x)$ is increasing in $[a, b]$

II. If $f(x)$ is decreasing in $[a, b]$, then $f'(x) < 0$ for all $x \in (a, b)$

III. $f(x) = x^2$ is increasing in $[0, 1]$

Ex) Identify the interval on which $f(x) = x^3 - 12x - 5$ is increasing and decreasing.

Ex) Identify the interval on which $f(x) = x^2 e^x$ is increasing and decreasing.

3. Global (Absolute) Maximum and Minimum

- For $f(x)$ (D: domain)

(1) If $f(x) \le f(c)$ $(c \in D)$ for all $x \in D$, then $f(x)$ has a ____________________ at ______

(2) If $f(c) \le f(x)$ $(c \in D)$ for all $x \in D$, then $f(x)$ has a ____________________ at ______

* ________________ : Absolute Maximum, Absolute Minimum

________________ ________________ ________________ ________________

⇨ Investigate interior points (local extrema), endpoints to find absolute extrema

4. Local Maximum and Local Minimum

(1) If $f(x) \le f(c)$ for all x in a certain interval including c, then $f(x)$ has a

________________ at $x = c$

If $f'(c) = 0$ and the sign of $f'(x)$ changes from $+$ to $-$, then $f(x)$ has a

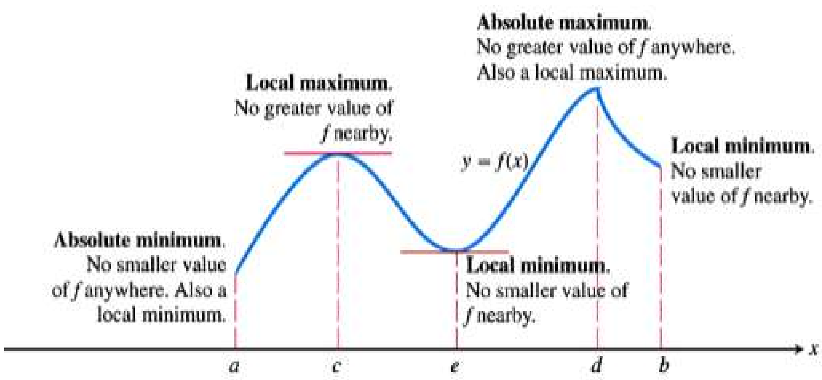
________________ at $x = c$

(2) If $f(x) \ge f(c)$ for all x in a certain interval including c, then $f(x)$ has a

________________ at $x = c$

If $f'(c) = 0$ and the sign of $f'(x)$ changes from $-$ to $+$, then $f(x)$ has a

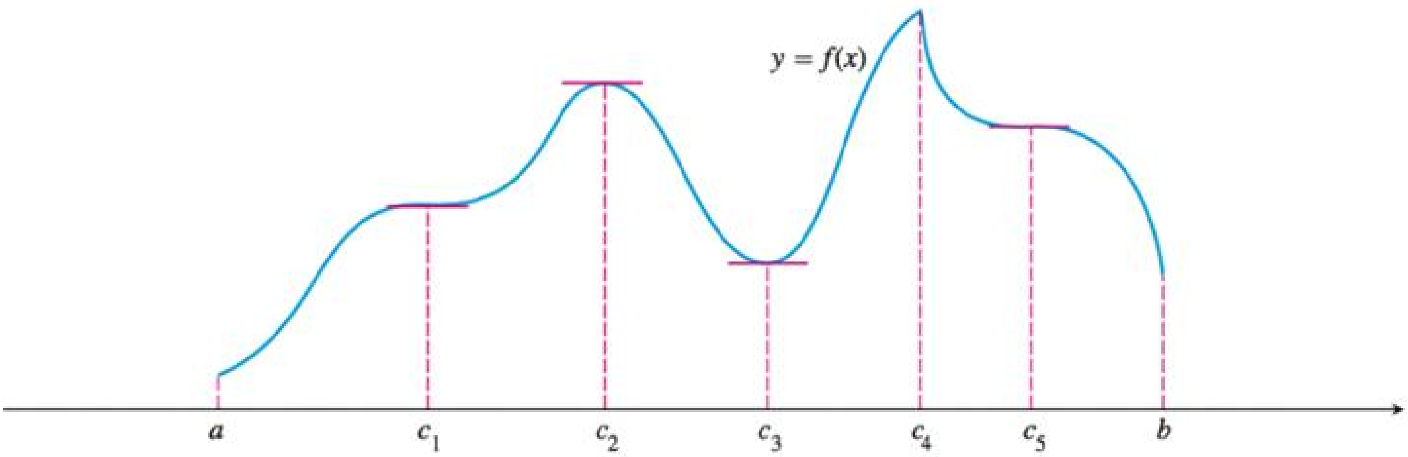
________________ at $x = c$



* ________________ / ________________ : Local Maximum & Local Minimum

(3) If $f(x)$ has a local max/mim at $x = c$ and $f(x)$ is differentiable at $x = c$, then ________

(4) If ____________ and the ________ of $f'(x)$ changes at $x = c$ then $f(x)$ has a local max or min at $x = c$



Ex) Answer the following questions.

(a) Mark the intervals that has $f'(x) < 0$

(b) Mark the intervals that has $f'(x) > 0$

(c) Mark all points that satisfies $f'(x) = 0$

(d) Mark all local extremas on the graph.

(e) Mark absolute extremas on the graph.

(f) Mark the critical points.

Ex) Which of the following statements are true?

I. If $f'(c) = 0$, then $f(x)$ has a local maximum or local minimum at $x = c$

II. A corner or cusp cannot be a local maximum or local minimum.

III. At corner or cusp, $f'(x)$ cannot be defined.

⇨ Generally, if $f'(c) = 0$, then $f(x)$ has a local max/min at $x = c$ unless it is not a squared form

5. How to sketch polynomial functions

① Find the derivative

② Find the points where $f'(x)=0$

③ Find local max and local min

④ Connect local extrema smoothly

(⑤ Find $x, y-$intercept if needed)

• Sketch the following graph.

① $y=x^3$, $y=-x^3$

② $y=ax^3+bx^2+cx+d$

③ $y=ax^4+bx^3+cx^2+dx+e$

④ $y=x^3+6x^2-15x$

⑤ $y=-x^3+3x^2+9x+1$

⑥ $y = -x^4 + 2x^2 + 12$

⑦ $y = 3x^4 - 20x^3 + 42x^2 - 36x + 3$

- **Sketch the following graph and find the absolute extrema in the given interval.**

① $f(x) = \frac{1}{3}(x-1)^2(x+1) \qquad [0,\ 5]$

② $f(x) = x^2 - 2x + 3 \qquad [0,\ 3]$

③ $f(x) = -x^3 + 3x^2 \qquad [-1,\ 1]$

⑤ $f(x) = -3x^4 + 8x^3 - 6x^2 - 1 \qquad [0,\ 2]$

④ $f(x) = x^4 - 4x^3 - 2x^2 + 12x - 3 \qquad [-2,\ 4]$

⑥ $f(x) = \frac{1}{4}x^4 + \frac{2}{3}x^3 - \frac{1}{2}x^2 - 2x \qquad [-2,\ 0]$

6. Concavity & Inflection Point

(1) $f(x)$ **is concave up** $\Leftrightarrow$ ______________ $\Leftrightarrow$ $f'(x)$ **is** _______________

(2) $f(x)$ **is concave down** $\Leftrightarrow$ _____________ $\Leftrightarrow$ $f'(x)$ **is** ______________

Ex) $y = x^2$

Ex) $y = x^3$

(3) Inflection point

- Inflection point is a point where __________ and the __________________ changes

  (Generally inflection point is where $f''(x) = 0$, unless it is not a squared form)

  (Also, inflection point can be a point where $f''(x)$ does not exist)

Ex 1) $f(x) = 2x^3 - 9x^2 + 12x - 6$

Ex 2) $y = \dfrac{1}{12}x^4 - \dfrac{1}{3}x^3 - \dfrac{3}{2}x^2 + 3$

Ex 3) $y = x^4$

Ex 4) $y = x^{\frac{1}{3}}$

Ex) Which of the following statements are true?

I. If $f''(c)=0$, then $f(x)$ has an inflection point at $x=c$

II. If $f(x)$ has an inflection point at $x=c$, then $f''(c)=0$

III. If $f''(c)=0$ and the sign of $f''(x)$ changes from $+$ to $-$, then $f(x)$ has an inflection point at $x=c$.

7. How to Sketch a Graph $y=f(x)$

(exponential, trigonometric, fraction, etc.)

① Investigate the function whether it is odd/even function

② Find the asymptotes using L'Hospital Rule

③ Find $x, y-$intercepts if possible / needed

④ Find the points where $f'(x)=0$ (Local Extremas)

⑤ Find the points where $f''(x)=0$ (Concavity / Inflection points)

⑥ Connect the identified points smoothly considering the concavity

Ex) $f(x)=\dfrac{x}{e^x}$

Ex) $f(x)=x^2e^{-x}$

Ex) $y = \dfrac{x}{x^2+1}$

Ex) $y = \dfrac{\ln x^2}{x}$

Ex) $y = x\ln x$

8. Local Linear Approximations

⇨ **The Linearization of $f(x)$ at $x=a$ is**

______________________________ ⇔ ______________________________

Ex) Find the linearization of $f(x)=\sqrt{x}$ at $x=4$. Then, approximate $\sqrt{4.01}$.

Ex) Approximate $\sin 46^\circ$ using local linear approximation.

9. Differentials

- $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ ⇨

$\therefore$

• How to solve Differentials Problems

① Set $f(x)$ ⇨ ② Set x and $\triangle x$ ⇨ ③ Use the differentials formula

Ex) Use differentials to approximate $\sqrt{9.01}$. cf> Local Linearization?

Ex) Use differentials to approximate $(3.98)^4$.

Ex) Use differentials to approximate $\sin 46^\circ$.

Ex) The radius of a circle is increased from 3 to 3.04. Estimate the change in area.

Ex) The radius of a sphere is measured to be 4cm with an error of $\pm 0.01cm$. Use differentials to approximate the error in the surface area.

10. Velocity and Acceleration Vectors

(1) Motion along a Number Line (Straight line)

If P moves along a number line and if the position x of P at time t is $x=f(t)$, the velocity and acceleration of x at time t is,

Position Velocity Acceleration

① Velocity : $v=$

$v>0$: $\Leftrightarrow$ $\Leftrightarrow$ Moving to a __________ direction

$v<0$: $\Leftrightarrow$ $\Leftrightarrow$ Moving to a __________ direction

$v=0$: _______________________ or ________

cf> $|v|=$

=

② Acceleration : $a=$

$a>0$: $\Leftrightarrow$ $\Leftrightarrow$

$a<0$: $\Leftrightarrow$ $\Leftrightarrow$

$a=0$: $\Leftrightarrow$

Ex) A particle moves according to the equation $x=2t^3-3t^2+6$ (x : position, t : time)

(a) Find the velocity and acceleration at time $t=2$.

(b) At what time t does the particle change its' direction?

(2) Motion along a Plane Curve

If P moves along a curve that is defined as a parametric equation $P(t)=(x(t),\, y(t))$,

• Position Vector : A Vector that has a starting point at origin and endpoint at P

⇨ Horizontal Component (x) + Vertical Component (y)

⇨ $\overrightarrow{R}=(x,\, y)$ or $R = xi + yj$

Position Vector　　　　Velocity Vector　　　　Acceleration Vector

cf > Magnitude of Velocity = ___________ :

Magnitude of Acceleration :

cf > At Rest $\Leftrightarrow$, Change Direction $\Leftrightarrow$

$\Leftrightarrow$ $\Leftrightarrow$

Ex) A particle moves according to the equations $x=3\cos t,\ y=2\sin t$

(a) Find the velocity and acceleration vectors at any time t.

(b) When is the speed a maximum? A minimum? The Values?

Ex) A point moves on the $x-$axis in such a way that its velocity at time t ($t>0$) is given by $v=\frac{\ln t}{t}$. At what value of t does v attain its maximum?

Ex) The position of a particle moving in the $xy-$plane is given by the parametric equations $x=t^3-3t^2$ and $y=2t^3-3t^2-12t$. For what values of t is the particle at rest?

(A) -1 only

(B) 0 only

(C) 2 only

(D) -1 and 2 only

(E) $-1,\ 0,$ and 2

11. Rate of Change of the Length / Volume / Area by time

Suppose the length, area, volume of an object at time t, as l (length), S (Area), V (Volume). If the change of length, area, volume during the time $\triangle t$ is $\triangle l$, $\triangle S$, $\triangle V$, the rate of change of each quantity at time t is

① Rate of Change (Length) :

② Rate of Change (Area) :

③ Rate of Change (Volume) :

[STEP 1] Define $l(t)$, $S(t)$, $V(t)$ according to the condition.

[STEP 2] Find the derivative. (with respect to t)

[STEP 3] Put in t value.

Ex) If the length and height of a rectangle at time t is $t+1$ and $2t+1$, what is the rate of change of the area S, of the rectangle at $t=1$.

Ex) The rate of change of the volume, V, of water in a tank with respect to time, t, is directly proportional to the square root of the volume. Which of the following is a differential equation that describes this relationship?

(A) $V(t)=k\sqrt{t}$

(B) $V(t)=k\sqrt{V}$

(C) $\frac{dV}{dt}=k\sqrt{t}$

(D) $\frac{dV}{dt}=\frac{k}{\sqrt{V}}$

(E) $\frac{dV}{dt}=k\sqrt{V}$

IV. Polar Curves

1. Polar Coordinates

r : ______________________________

θ : ______________________________

2. Cartesian Coordinate & Polar Coordinate

$x =$

$y =$

$\therefore \ x^2 + y^2 =$

$\tan\theta =$

Ex) $(1,\ \frac{5}{4}\pi)$ $(2,\ -\frac{2}{3}\pi)$ $(-3,\ \frac{3}{4}\pi)$

Ex) Convert polar coordinates to Cartesian coordinates or vice versa.

① $(2,\ \frac{\pi}{3})$

② $(1,\ -1)$ $(r > 0,\ 0 < \theta < 2\pi)$

③ $(\sqrt{2},\ \frac{7}{4}\pi)$

④ $(-1,\ \sqrt{3})\quad (r>0,\ 0<\theta<2\pi)$

3. Polar Curve

Ex) Sketch the following polar curve.

① $r=2$

② $r=-1$

③ $\theta=1$

④ $r=2\cos\theta$

M1>

M2> Chart

θ	r

Ex) Sketch the following polar curves.

① $r = \sin\theta$

② $r = 1 + \sin\theta$

③ $r = \sin 2\theta$

④ $r = \cos 2\theta$

⑤ $r = \sin\dfrac{\theta}{2}$

⑥ $r = 1 + 2\sin\theta$

4. Slope of a Polar Curve

- The slope of a polar curve $r=f(\theta)$ can be defined as below

⇨ $x=r\cos\theta=f(\theta)\cos\theta \quad \rightarrow \quad \frac{dx}{d\theta}=$

$y=r\sin\theta=f(\theta)\sin\theta \quad \rightarrow \quad \frac{dy}{d\theta}=$

$\therefore \ \frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=$

Ex) Find the slope of the cardioid $r=2(1+\cos\theta)$ at $\theta=\frac{\pi}{6}$.

Ex) Find the equation of a tangent line to the curve $r=1+\sin\theta$ at $\theta=\frac{\pi}{3}$.

Ex) When is the tangent line to the curve $r = 1 + \sin\theta$ horizontal or vertical? ($0 \le \theta \le 2\pi$)

Note* Area and Length in polar coordinates ⇨ Integration

V. Indefinite Integral

1. Indefinite Integral (Anti-derivative)

- For $f(x)$ that satisfies $F'(x)=f(x)$,

$$F(x) \qquad\qquad\qquad f(x)$$

$$\int f(x)dx = F(x) + C$$

Ex) $\int 1\,dx =$

$\int 2x dx =$

2. Indefinite Integral and Differentiation

① $\frac{d}{dx}\int f(x)dx =$

② $\int \left\{\frac{d}{dx}f(x)\right\}dx =$

※ $\frac{d}{dx}\int f(x)dx \qquad \int \left\{\frac{d}{dx}f(x)\right\}dx$

Ex) Find $f(x)$.

① $\int f(x)dx = 3x + C$

② $\int xf(x)dx = \frac{1}{2}x^2 - 3x + C$

Ex) Evaluate the expression.

① $\frac{d}{dx}\int (x^2+x)dx =$

② $\int \left\{\frac{d}{dx}(x^2+x)\right\}dx =$

3. General Rules of Integration

① $\int x^n dx =$ (n , $C:$)

② $\int a dx =$, $\int y\,dx =$

③ $\int \frac{1}{x} dx =$ → $\int \frac{b}{x+a} dx =$ $\int \frac{f'(x)}{f(x)} dx =$

④ $\int e^x dx =$

⑤ $\int a^x dx =$ (,)

⑥ $\int \sin x\,dx =$

⑦ $\int \cos x dx =$

⑧ $\int \tan x\,dx =$

⑨ $\int \sec^2 x\,dx =$

⑩ $\int \csc^2 x\,dx =$

⑪ $\int \sec x \tan x dx =$

⑫ $\int \csc x \cot x\,dx =$

⑬ $\int \frac{1}{x^2+a^2} dx =$

⑭ $\int \frac{1}{\sqrt{a^2-x^2}} dx =$

(See textbook for extra formulas)

• Additional Formulas

① $\int \sin^2 x\, dx \ =$

② $\int \cos^2 x\, dx \ =$

③ $\int \tan^2 x\, dx \ =$

④ $\int \cot^2 x\, dx \ =$

⑤ $\int \ln x\, dx \ =$

⑥ $\int \frac{f'(x)}{f(x)} dx =$

⑦ $\int f(ax)dx = \frac{1}{a}\int f(u)du$

Ex) $\int \sin 2x\, dx =$

4. General Properties of Indefinite Integration

① $\int kf(x)dx =$

② $\int \{f(x)+g(x)\}dx =$

③ $\int \{f(x)-g(x)\}dx =$

(Integral constant: _____)

Ex) $\int (x+\cos x+\sin x+e^x)\, dx$

5. Integration by Substitution

- If the range of $u=g(x)$ is I and $f(x)$ is continuous in I, then

$$\int f(g(x))g'(x)dx=\int f(u)du$$

Ex) (a) $\int \sin 2x\,dx$

(b) $\int \cos(7\theta+5)d\theta$

(c) $\int x^2\sin(x^3)dx$

(d) $\int \frac{\ln x}{x}dx$

(e) $\int \cos^2 x\,dx$

(f) $\int \frac{1}{\cos^2 2x}dx$

Ex) Prove that $\int \tan x\,dx=\ln|\sec x|+C$.

6. Integration by Parts

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

$$\uparrow \quad \uparrow$$

Proof> $\{f(x)g(x)\}' =$

Tip] Order of easy-integration $(g'(x))$ **function**

Exponential > Trigonometric > Polynomial > Logarithmic function

Ex) $\int x\cos x\,dx$

$\int xe^x\,dx$

$\int \ln x\,dx$

$\int (2x+1)(x-1)^4 dx$

$\int x^2 e^x dx$

$\int \ln(x+1)\,dx$

7. Integration by Partial Fraction

- For integrals such as $\int \frac{g(x)}{f(x)}dx$ can be evaluated by

(1) If $f'(x) = g(x)$, then $\int \frac{f'(x)}{f(x)}dx =$

(2) If $f'(x) \neq g(x)$

(a) If (numerator's degree) $\geq$ (denominator's degree), then

① Try Factorization → Cancel

② If can't, divide the fraction into quotient and remainder

(b) If (numerator's degree) $\leq$ (denominator's degree)

① Divide the fraction into partial fractions

- Partial Fraction Formulas

1. $\frac{1}{AB} =$

2. $\frac{(1st\ \text{degree or } below)}{(x+a)(x+b)}$

3. $\frac{(2nd\,\text{degree or } below)}{(x+a)(x+b)(x+c)} =$

4. $\frac{(2nd\,\text{degree or } below)}{(x+a)(x^2+bx+c)} =$

5. $\frac{(2nd\,\text{degree or } below)}{(x+a)(x+b)^2} =$

6. $\frac{(3rd\,\text{degree or } below)}{(x+a)(x+b)^3} =$

7. $\frac{(3rd\,\text{degree or } below)}{(x+a)^2(x+b)^2} =$

Ex) $\int \frac{x^3+1}{x+1}dx$

$$\int \frac{x^3+2x+3}{x}dx$$

$$\int \frac{x^2+1}{x-1}dx$$

$$\int \frac{x+1}{x^2-3x+2}dx$$

V. Indefinite Integral

1. Find $f(x)$.

(1) $\int f(x)dx = 2x^2 + 5x + C$

(2) $\int f(x)dx = -e^{-x} + 2x + C$

(3) $\int f(x)dx = x^3 + 2x^2 - 4x + C$

(4) $\int f(x)dx = 2\tan 2x - 3^x + C$

(5) $\int xf(x)dx = x^3 + x^2 + C$

(6) $\int (x-1)f(x)dx = \frac{1}{3}x^3 - x + C$

2. Solve the integral.

① $\frac{d}{dx}\int x^3 dx$ =

② $\int (\frac{d}{dx}x^3)dx$ =

③ $\int 3dx$

④ $\int x^5 dx$

⑤ $\int x^{10} dx$

⑥ $\int (3x+2)dx$

⑦ $\int (2x^3 - x + 1)dx$

⑧ $\int (x+1)^2 dx$

⑨ $\int (x+1)(x^2 - x + 1)dx$

⑩ $\int (2x+1)^2 dx - \int (2x-1)^2 dx$

⑪ $\int (x+1)^3 dx + \int (x-1)^3 dx$

⑫ $\int \frac{x^2}{x+1}dx - \int \frac{1}{x+1}dx$

3. Solve the integral.

① $\int \frac{3}{x}dx$

② $\int \frac{1}{x^2}dx$

③ $\int \sqrt[3]{x^2}\,dx$

④ $\int x\sqrt{x}\,dx$

⑤ $\int (x^2\sqrt{x}-\frac{2}{x^3})dx$

⑥ $\int (x-3+\frac{4}{x^5})dx$

⑦ $\int \frac{x^3-x^2-2x}{x}dx$

⑧ $\int \frac{\sqrt{x}+1}{x}dx$

⑨ $\int (\sqrt{x}-\frac{1}{\sqrt{x}})^2 dx$

⑩ $\int (\sqrt[4]{x}-1)(\sqrt[4]{x}+1)dx$

⑪ $\int (3e^x+2^x)dx$

⑫ $\int (e^{x+2}-5^{x+1})dx$

⑬ $\int (3^x+1)^2 dx$

⑭ $\int (2\sin x+4\cos x)dx$

⑮ $\int (1-\cot x)\sin x\,dx$

⑯ $\int (\cos x+\tan x)\sec x\,dx$

⑰ $\int \frac{2-\cos^2 x}{\cos^2 x}dx$

⑱ $\int \frac{\sin^2 x + 3}{\sin^2 x} dx$

⑲ $\int \tan^2 x\, dx$

⑳ $\int (\cot^2 x - 1) dx$

4. Solve the integral.

① $\int (4x-1)^3 dx$

② $\int \frac{1}{(5x+1)^2} dx$

③ $\int \sqrt{3-x}\, dx$

④ $\int e^{-2x+3} dx$

⑤ $\int \cos(2x-1) dx$

⑥ $\int x(x^2+1)^2 dx$

⑦ $\int \frac{3x^2}{\sqrt{x^3-1}} dx$

⑧ $\int 2x e^{x^2} dx$

⑨ $\int \frac{\ln x}{x} dx$

⑩ $\int \sin^2 x \cos x\, dx$

⑪ $\int \frac{x}{x^2+3}dx$

⑫ $\int \frac{2x-1}{x^2-x+2}dx$

⑬ $\int \frac{e^x}{e^x-1}dx$

⑭ $\int \frac{\csc^2 x}{\cot x}dx$

⑮ $\int \tan x dx$

⑯ $\int (x+1)\sqrt{x^2+2x+2}\,dx$

⑰ $\int \frac{e^x}{\sqrt{e^x+3}}dx$

⑱ $\int \frac{\sin^3 x}{1+\cos x}dx$

5. Solve the integral.

① $\int \frac{x^2+4}{x-1}dx$

② $\int \frac{1}{x^2+2x}dx$

③ $\int \frac{x-3}{x^2-1}dx$

6. Solve the integral.

① $\int xe^{-x}dx$

② $\int \ln x dx$

③ $\int x\cos x dx$

④ $\int x\cos 2x dx$

⑤ $\int (x-1)e^{x}dx$

⑥ $\int x^{2}\sin x\, dx$

⑦ $\int (\ln x)^{2}dx$

VI. Definite Integral

1. Definition of Definite Integrals

• If $f(x)$ is continuous on $[a, b]$,

(1) ____ ($\triangle x =$, $x_k =$)

(a : ______________ , b : ________________)

• $\int_a^a f(x)dx =$

• Evaluating Definite Integral Using the Definition of Definite Integral

① Divide the area into ___ pieces

② Find the sum of the area S_n

③ Find the limit of S_n as $n \to \infty$

Ex) $\int_0^1 x^2\,dx =$

Ex) $\int_1^2 (1+x)^2\,dx =$

2. Relationship between Derivative and Integral (The First Fundamental Theorem of Calculus)

- If $f(x)$ is continuous on $[a, b]$, then the function defined by $S(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b), and ____________________
- i.e. ______________________________________

Ex) $\dfrac{d}{dx}\int_a^x t^3\,dt =$

$\dfrac{d}{dx}\int_0^{x^2} (t-2)dt =$

$\dfrac{d}{dx}\int_{x-1}^{x+1} (t-2)dt =$

$\dfrac{d}{dx}\int_{2x}^{x^2} \sin x\,dt =$

Ex) What is the derivative of $h(x) = \int_2^{x^2} \dfrac{1}{1+t^2}\,dt$?

Ex) Find the derivative of $\int_1^{x^2} \cos t\,dt$.

3. (The Second) Fundamental Theorem of Calculus (FTC)

- If $f(x)$ is a continuous function on $[a, b]$, then

______________________________ where F is the anti-derivative of f, i.e. $F' = f$

Proof > If $S(x) = \int_a^x f(t)dt$, then

Ex) $\int_1^3 x^2 dx =$

$\int_2^4 (e^x + 1)dx =$

4. Properties of Definite Integrals

(1) $\frac{d}{dx}\int_a^x f(x) =$

(2) Constant Multiple : $\int_a^b kf(x)\,dx =$ (k is a constant)

(3) Sum and Difference : $\int_a^b f(x) \pm g(x)\,dx =$

(4) Zero Width Interval : $\int_a^a f(x)dx =$

(5) Order of Integration : $\int_a^b f(x)dx =$

(6) Additivity : $\int_a^c f(x)dx + \int_c^b f(x)dx =$

Ex) Suppose that $\int_{-1}^{1} f(x)dx = 5,\ \int_{1}^{4} f(x)dx = -2,\ \int_{-1}^{1} h(x)dx = 7$

(a) $\int_{4}^{1} f(x)dx$ **(b)** $\int_{-1}^{1} 2f(x)+3h(x)\,dx$ **(c)** $\int_{-1}^{4} f(x)dx$

(7) For even function $f(x)$, i.e. $f(x)=f(-x)$, $\int_{-a}^{a} f(x)dx =$

(8) For odd function $f(x)$, i.e. $f(-x)=-f(x)$, $\int_{-a}^{a} f(x)dx =$

Ex) $\int_{-a}^{a} (x^2+\cos x)dx =$ $\int_{-a}^{a} (x^3+x-\sin x)dx =$

(9) For periodic function $f(x)$ (period : p) , $\int_{a}^{b} f(x)dx =$,

$\int_{a}^{a+p} f(x)dx =$

Ex) For $f(x+2)=f(x)$, if $\int_{0}^{2} f(x)dx = 2$, $\int_{0}^{8} f(x)dx =$?

5. Functions defined by Definite Integral

① $\frac{d}{dx}\int_a^x f(t)dt =$

② $\frac{d}{dx}\int_a^{x+a} f(t)dt =$ $\frac{d}{dx}\int_x^{x+a} f(t)dt =$

③ $\lim_{x \to 0} \frac{1}{x}\int_a^{x+a} f(t)dt =$

④ $\lim_{x \to a} \frac{1}{x-a}\int_a^x f(t)dt =$

Ex) Find the limit.

① $\lim_{x \to 0} \frac{1}{x}\int_0^x (t+\cos t)dt$

② $\lim_{x \to 1} \frac{1}{x-1}\int_1^x (x+2)(x+3)dx$

③ $\lim_{x \to 1} \frac{1}{x-1}\int_1^x (e^t+2)dt$

Ex) Evaluate.

① $\frac{d}{dx}\int_x^{x+2} (t^2+t)dt =$

② $\frac{d}{dx}\int_{x-1}^{x^2} (t^2+t)dt =$

6. Approximations of the Definite Integral

- $\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{k=0}^{n-1} f(x_k) \triangle x = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k) \triangle x$

- Approximations of the Definite Integral $\int_a^b f(x)dx$

① **Riemann Sum :** $\sum_{k=1}^{n} f(c_k) \triangle x$

($a = x_0 < x_1 < \cdots \ \cdots < x_{n-1} < x_n = b$, $\triangle x = \frac{b-a}{n}$,

$x_k = a + k\triangle x$, $x_{k-1} \le c_k \le x_k$)

② **Right Sum :** $\sum_{k=1}^{n} f(x_k) \triangle x =$

(Right Riemann Sum)

③ **Left Sum :** $\sum_{k=0}^{n-1} f(x_k) \triangle x =$

(Left Riemann Sum)

④ **Midpoint Sum :** $\sum_{k=0}^{n-1} f(\frac{x_k + x_{k+1}}{2}) \triangle x$

Ex) Approximate $\int_0^2 x^3\,dx$ by using four subintervals and calculating (a) the left sum, (b) the right sum, (c) the midpoint sum. (d) Evaluate the integral exactly.

Ex) Approximate $\int_0^1 x^2\,dx$ by using (a) the left sum, (b) the right sum, (c) the midpoint sum. ($\triangle x = \frac{1}{4}$)

- **Trapezoidal Rule**

⇨ **If** $\triangle x = \dfrac{b-a}{n}$, $x_k = a + k\triangle x$**,**

$$\int_a^b f(x)dx \approx$$

$$=$$

Ex) Approximate $\int_0^2 x^3\,dx$ by using four subintervals and Trapezoidal Rule.

- Comparing Approximating Sums

※ Concave Up

※ Concave Down

① Increasing, Concave Up

② Decreasing, Concave Up

③ Increasing, Concave Down

④ Decreasing, Concave Down

6. Average Value

- **If $y = f(x)$ is continuous on $[a, b]$, then the average value M for $f(x)$ on $[a, b]$ is**

- **The Mean Value Theorem for Integrals**

⇨

- **Max - Min Inequality**

⇨

Ex) Find the average value of $f(x) = \sqrt{4 - x^2}$ on the interval $[-2, 2]$.

7. Integration by Substitution (Definite Integral)

- **If the range of $u=g(x)$ is I and $f(x)$ is continuous in I, then**

$$\int_a^b f(g(x))g'(x)dx = \int \qquad f(u)du$$

Ex) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin 2x\,dx \;=$

$\int_{\sqrt{e}}^{e^3} \frac{\ln x}{x} dx$

① $\sqrt{a^2-x^2}$ **Integral ⇨ Substitute** $x=$

Ex) $\int_{-2}^{2} \sqrt{4-x^2}\,dx =$

Ex) $\int_{-3}^{3} \frac{1}{\sqrt{9-x^2}}\,dx =$

② $\dfrac{1}{a^2+x^2}$ Integral ⇨ Substitute $x=$

Ex) $\displaystyle\int_{-2}^{2}\frac{1}{4+x^2}\,dx=$

Ex) $\displaystyle\int_{-3}^{3}\frac{1}{x^2+9}\,dx=$

8. Integration by Parts (Definite Integral)

$$\int_a^b f(x)g'(x)dx=[f(x)g(x)]\;-\int\;f'(x)g(x)dx$$

Ex) $\displaystyle\int_0^{\frac{\pi}{2}} x\cos x\,dx$

Ex) $\displaystyle\int_0^2 x^2e^x\,dx=$

9. Definite Integral and Series

- $\int_a^b f(x)dx=$ ($\triangle x=$, $x_k=$)

(1) $\lim_{n\to\infty}\sum_{k=1}^{n} f(\frac{k}{n})\frac{1}{n}=$

(2) $\lim_{n\to\infty}\sum_{k=1}^{n} f(\frac{p}{n}k)\frac{p}{n}=$

(3) $\lim_{n\to\infty}\sum_{k=1}^{n} f(a+\frac{b-a}{n}k)\frac{b-a}{n}=$

(4) $\lim_{n\to\infty}\sum_{k=1}^{n} f(a+\frac{p}{n}k)\frac{p}{n}=$

[Tips] When changing Series → Definite Integral

① Substitute the entire part inside $f(\quad)$ as x

② For $\sum_{k=1}^{n}$, put in $k=1$ and determine the value of $f(\quad)$ when $n\to\infty$, ... ,

put in $k=n$ and determine the limit of $f(\quad)$ when $n\to\infty$

③ If the increment in ② per 1 unit increase of k = the exterior part of $f(\quad)$, then it is = ____________________

(If not, a constant is multiplied to ____________________)

Ex) $\lim_{n\to\infty}\sum_{k=1}^{n} f(1+\frac{k}{n})\frac{1}{n}=$

$\lim_{n\to\infty}\sum_{k=1}^{n} f(2+\frac{k}{n})\frac{1}{n}=$

Ex) $\lim_{n \to \infty} \sum_{k=1}^{n} \left(\frac{3k}{n}\right)^2 \frac{1}{n} =$

[Tips 2] Different expressions of Definite Integral

$$\lim_{n \to \infty} \sum_{k=1}^{n} f\left(a + \frac{p}{n}k\right) \frac{p}{n} =$$

① **When substituting** $x = a + \frac{p}{n}k$ **from** $f\left(a + \frac{p}{n}k\right)$

② **When substituting** $x = \frac{p}{n}k$ **from** $f\left(a + \frac{p}{n}k\right)$

③ **When substituting** $x = \frac{k}{n}$ **from** $f\left(a + \frac{p}{n}k\right)$

Ex) Change the given series into definite integral.

$$\lim_{n\to\infty}\sum_{k=1}^{n} f\left(3+\frac{2k}{n}\right)\frac{2}{n}=$$

$$\lim_{n\to\infty}\sum_{k=1}^{n} f\left(2+\frac{k}{n}\right)\frac{3}{n}=$$

$$\lim_{n\to\infty}\sum_{k=1}^{n} \left(1+\frac{3k}{n}\right)^2\frac{2}{n}=$$

VI. Definite Integral

1. Evaluate $\int_0^1 x^3\,dx$ using the definition of Definite Integral.

sol> Let $f(x)=x^3$. Since $f(x)$ is continuous at $[0,1]$, then $\triangle x=$, $x_k=k\triangle x=$

$\therefore \int_0^1 x^3dx =$

2. [The 1st Fundamental Theorem of Cal.]

① $\frac{d}{dx}\int_1^x (t^2-2t)dt \ =$

② $\frac{d}{dx}\int_2^x (-t^3+4t+2)dt \ =$

③ $\frac{d}{dx}\int_{2x^2}^{x+2} (e^t-t^2+1)dt =$

④ $\lim_{x\to 0}\frac{1}{x}\int_a^{x+a} f(t)dt =$

⑤ $\lim_{x\to 0}\frac{1}{x-a}\int_a^{x} f(t)dt =$

3. Evaluate the integral.

① $\int_1^4 \sqrt{x}\,dx$

② $\int_1^{e^2}\frac{1}{x}dx$

③ $\int_1^2 (2x^2+3x-1)dx$

④ $\int_0^{\ln 2} e^{3x}dx$

⑤ $\int_{\frac{\pi}{2}}^{\pi}\sin^2\frac{x}{2}dx$

⑥ $\int_0^1 (x-1)(x^2+x+1)dx$

⑦ $\int_{-1}^3 (3x^2+x-2)dx-\int_{-1}^3 (x+3)dx$

⑧ $\int_0^1 (x+1)^2dx+\int_1^0 (x-1)^2\,dx$

⑨ $\int_{-1}^{0}(3\sqrt{x}+2)dx+\int_{0}^{2}(3\sqrt{x}+2)dx$

⑩ $\int_{-2}^{-1}(x^2-4x+5)dx+\int_{-1}^{1}(y^2-4y+5)dy$

⑪ $\int_{0}^{\pi}(\cos x+1)dx+\int_{\pi}^{2\pi}(\cos y+1)dy$

⑫ $\int_{-1}^{2}(5x^4-6x-1)dx-\int_{3}^{2}(5x^4-6x-1)dx$

⑫ $\int_{0}^{\pi}(\sin x+1)^2dx+\int_{\pi}^{0}(\sin x-1)^2dx$

⑬ $\int_{-1}^{3}(e^x+1)dx+\int_{-1}^{3}(e^x-1)dx$

⑭ $\int_{-3}^{3}(-5x^3+3x^2+4x-2)dx$

⑮ $\int_{-1}^{1}(x^2+\sin 2x)dx$

⑯ $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos x+\sin x)dx$

⑰ $\int_{-2}^{2}(e^x+e^{-x})dx$

⑱ For periodic function $f(x)$ that satisfies $f(x+3)=f(3)$, what is $\int_1^{10} f(x)dx$ if $\int_1^4 f(x)dx=2$?

4. Evaluate the integral.

① $\int_0^2 \sqrt{x+2}\,dx$

② $\int_0^1 (2x+3)^3dx$

③ $\int_1^2 x(x^2-1)^2dx$

④ $\int_1^2 \frac{x}{2x^2+1}dx$

⑤ $\int_0^2 xe^{x^2}dx$

⑥ $\int_0^1 \frac{\ln x}{x}dx$

⑦ $\int_0^1 \cos^3 x \sin x\,dx$

⑧ $\int_0^{\frac{\pi}{2}} \frac{\cos x}{2+\sin x}\,dx$

⑨ $\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx$

⑩ $\int_0^{\sqrt{3}} \frac{1}{1+x^2}\,dx$

⑪ $\int_1^e \ln x\,dx$

⑫ $\int_0^2 xe^x\,dx$

5. Find $f(x)$ if the following equation satisfies with all real number x.

① $\int_{-\pi}^{x} f(t)dt = \cos x + 1$

② $\int_0^x f(t)dt = e^{2x} - x - 1$

③ $\int_1^x f(t)dt = \ln x + 2x - 2 \quad (x > 0)$

VII. Application of Integration

1. Area

(1) Area between the curve and $x-$axis

- If $f(x)$ is continuous on $[a, b]$, then the area enclosed by $y=f(x)$ and $x-$axis and $x=a$, $x=b$ is

$S =$ ________________

⇨ If $f(x) \geq 0$: $S=$

If $f(x) \leq 0$: $S=$

Tip] ∴ Find the ______________ and divide the interval!

Ex) What is the area of the region R that lies above the $x-$axis, below the graph of $y=1-x^2$, and between the vertical lines $x=0$ and $x=1$?

Ex) What is the area of the region that is enclosed by the curve $y=f(x)$ and $x-$axis?

(a) $f(x)=-(x+1)(x-2)$

(b) $f(x)=x^2-x$

Ex) What is the area of the region that is enclosed by the curve $y = x^3 - 3x^2 - x + 3$ and $x-$axis?

(2) Area between curves

- If $f(x)$, $g(x)$ is continuous on $[a, b]$, then the area between $y = f(x)$ and $y = g(x)$ and $x = a$, $x = b$ is

$S =$ ____________________

⇨ If $f(x) \geq g(x)$: $S =$

If $f(x) \leq g(x)$: $S =$

Tip] ∴ Find the ________________ and divide the interval!

Ex) Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Ex) Find the area of the region enclosed by $y=x^2$ and $y=-x^2+2$.

Ex) Find the area of the region enclosed by $y=x^3-x$ and $y=x$.

(3) Area between curve and $y-$axis

- **If $g(y)$ is continuous on $[c,d]$, then the area enclosed by $x=g(y)$ and $y-$axis and $y=c$, $y=d$ is**

$S=$ ____________________

⇨ **If $g(y)=x\geq 0$: $S=$**

If $g(y)=x\leq 0$: $S=$

Tip] ∴ Find the ____________________ and divide the interval!

Ex) Find the area of the region enclosed by the curve $x=3y-y^2$ and $y-$axis.

Ex) Find the area of the region enclosed by the curve $y=\frac{1}{2}x^2\ (x\geq 0)$ and $y-$axis, $y=2$.

M1>

M2>

Ex) Find the area of the region enclosed by the curve $y=\sqrt{x}$ and $y=1$, $y=2$.

M1>

M2>

(4) Area between curves (with respect to y)

- If $f(y)$, $g(y)$ is continuous on $[c, d]$, then the area between $x = f(y)$ and $x = g(y)$ and $y = c$, $y = d$ is

$S =$ ____________________

⇨ If $f(y) \geq g(y)$: $S =$

If $f(y) \leq g(y)$: $S =$

Tip] ∴ Find the ________________ and divide the interval!

Ex) Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the $x-$axis and the line $y = x - 2$.

M1>

M2>

Ex) Find the area that is enclosed by $y = \ln x$ and $x-$axis, with $y = x$ and $y = 1$.

Ex) Find the area that is enclosed by $y = e^{x}$, $y = e^{-x}$ and $y = 2$.

(5) Region Bounded by a Polar Curve

Region A expressed by a polar curve $r = f(\theta)$, $\alpha \le \theta \le \beta$ is

____

____

cf> Region A between $r = f(\theta)$, $r = g(\theta)$ $\alpha \le \theta \le \beta$ is ____________________________

Ex) Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.

2. Arc Length

(1) Length of Parametrically Defined Curve

- If point $P(x, y)$ moves on a plane with the position defined by $x = f(t),\ y = g(t)$ at time t, then the length L that point P have traveled from time $t = a$ to $t = b$ is

cf> For $r = f(\theta)$ ($a < \theta < b$), ________________________________

Ex) Find the length of the curve defined by $x = \cos^3 t,\ y = \sin^3 t,\ 0 \le t \le \frac{\pi}{2}$.

Ex) Find the length of the polar curve $r = \cos\theta \quad (0 \le \theta \le 2\pi)$.

Ex) Find the length of the curve defined by $x = \sin t + 3$, $y = 1 - \cos t \quad (0 \le t \le \pi)$.

3. Length of a Curve $y=f(x)$

- If $f(x)$ is continuous and differentiable on $[a,\ b]$ then the length of a curve $y=f(x)$ at $[a,\ b]$ is

Ex) Find the length of the curve $y=\dfrac{e^{x}+e^{-x}}{2}$ $(-1 \le x \le 1)$.

Ex) Find the length of the curve $y=\dfrac{4\sqrt{2}}{3}x^{\frac{3}{2}}-1,\quad 0 \le x \le 1$.

Ex) Find the length of the curve $y=\dfrac{1}{6}x^{3}+\dfrac{1}{2x}$ $(1 \le x \le 3)$.

VII. Application of Integration

1. Find the area enclosed by the following curve and $x-$axis.

① $y=-(x+1)(x+2)$

② $y=x^2-x$

④ $y=2e^x,\ x-$axis, $x=-1$, $x=2$

⑤ $y=-\ln x,\ x-$axis, $x=e$

⑥ $y=e^x$, $y-$axis, $y=e$

2. Find the area enclosed by the following curve and vertical / horizontal lines.

① $y=\sqrt{x}$, $x-$axis, $x=4$

② $y=\dfrac{1}{x}$, $x-$axis, $x=1,\ x=3$

③ $y=\dfrac{1}{x-1}$, $y-$axis, $y=1$, $y=3$

⑦ $y=\sin x\ (0\le x\le \pi)$, $x-$axis

⑧ $y=\dfrac{1}{2}x^2\ (x\le 0)$, $y-$axis, $y=2$

⑨ $y=\cos x$, $x=\frac{\pi}{4}$, $x=\pi$

⑩ $y=\ln(x-1)$, $y-$axis, $y=-1$, $y=1$

4. Find the area between the following curves.

① $y=-2x^2$, $y=-x-1$

② $y=x^2$, $y=\sqrt{x}$

③ $y=x^3-x$, $y=x$

④ For $0\le x\le\frac{\pi}{4}$, $y=\sin x$, $y=\cos x$, $y-$axis

⑤ $y=e^x$, $y=e^{-x}$, $y=2$

⑥ $y=x^2-2,\ y=3x+2$

⑦ $y=x^2,\ y=-x^2+2$

⑧ $y=x^2-5x+6,\ y=-x^2+3x$

VII. Application of Integration - 3. Volume

(1) Volumes by Slicing About an $x-$axis

- If the area of a cross section of a solid, sliced perpendicular to the $x-$axis at point x is defined by $A(x)$, then the volume of a solid V is

Ex) A curved wedge is cut from a cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first planed at a 45° angle at the center of the cylinder. Find the volume of the wedge.

Ex) A solid has a base that is enclosed by the curve $|y| = \sin x$. If the cross section of the solid that is perpendicular to the $x-$axis is always a square, what is the volume of the solid? $(0 \le x \le \pi)$

(2) Volumes by Slicing About an $y-$axis

- If the area of a cross section of a solid, sliced perpendicular to the $x-$axis at point x is defined by $A(x)$, then the volume of a solid V is

(3) Volumes by Rotation about $x-$axis (Disk Method)

- The Volume of a solid V resulting by rotating the region enclosed by $y=f(x)$ and $x-$axis and $x=a$, $x=b$ ($a<b$) about $x-$axis is

Ex) The region between the curve $y=\sqrt{x}$, $0\le x\le 4$, and $x-$axis is revolved about the $x-$axis to generate a solid. Find its volume.

Ex) Find the volume of a solid that results when the region between the curve $y=\sqrt{x}$, $1\le x\le 4$ and $y=1$, is revolved around $y=1$.

Ex) Write the expression of the volume of a solid that results when the region between the curve $y=x(x-1)(x+1)$, $0\le x\le 2$ and $x-$axis, is revolved around the $x-$axis.

(4) Volumes by Rotation about $y-$axis (Disk Method)

• The Volume of a solid V resulting by rotating the region enclosed by $x=g(y)$ and $y-$axis and $y=a$, $y=b$ $(a<b)$ about $y-$axis is

Ex) Find the volume of the solid generated by revolving the region between the parabola $x=y^2+1$ $(0 \le y \le \sqrt{2})$ and

(1) the $y-$axis, $x-$axis and $y=\sqrt{2}$ about $y-$axis

(2) the line $x=3$ and $x-$axis about $x=3$

(5) Method of Washers

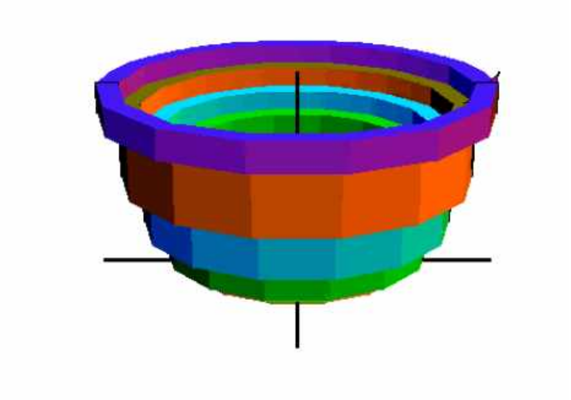
- The Volume of a solid V resulting by rotating the region enclosed by $y=f(x)$, $y=g(x)$ ($f(x)\geq g(x)\geq 0$) and $x=a$, $x=b$ ($a<b$) about $x-$axis is

- The Volume of a solid V resulting by rotating the region enclosed by $x=f(y)$, $x=g(y)$ ($f(y)\geq g(y)\geq 0$) and $y=a$, $y=b$ ($a<b$) about $y-$axis is

Ex) The region bounded by the curve $y=x^2+1$ and the line $y=-x+3$ is revolved about the $x-$axis to generate a solid. Find the volume of the solid.

Ex) The region bounded by the parabola $y=x^2$ and the line $y=2x$ in the first quadrant is revolved about the $y-$axis to generate a solid. Find the volume of the solid.

(6) Shell Method (when rotated about $y-$axis)



Ex) Use the shell method to find the volume generated by the region bounded by $y=\frac{1}{x}$, $y=0$, $x=1$ and $x=4$ about the $y-$axis.

Ex) Use the shell method to find the volume generated by the region bounded by $y=x$ and $y=x^2$ about the $y-$axis.

(7) Shell method (when rotated about $x-$axis)

Ex) Determine the volume of the solid obtained when the region bounded by $x = y^2 - 4y + 4$, $x = 0$, $x = 1$ is rotated about the $x-$axis.

4. Rate of change

5. Parametric Equation

Ex) A particle $P(x, y)$ **moves along a curve so that** $\frac{dx}{dt} = 2\sqrt{x}$**,** $\frac{dy}{dt} = \frac{1}{x}$ $(t \geq 0)$**. At** $t = 0$**,** $x = 1,\ y = 0$**.**

(a) Find the parametric equations of motion.

6. Motion along a straight line

- Let a particle P has a velocity $v(t)$ at time t and position x_0 at time $t=a$.

(1) Position of particle P at time t :

(2) Displacement of particle P from $t=a$ to $t=b$:

(3) Travel Distance of particle P from $t=a$ to $t=b$:

Position Velocity Acceleration

Ex) A particle moves along the $x-$axis so that its velocity at time t is given by $v(t)=4t-2t^2$. The initial position of the particle is $x=3$.

(a) Find the position of the particle when $t=6$.

(b) Find the displacement of the particle from $t=0$ and $t=3$.

(c) Find the total distance covered between $t=0$ and $t=3$.

7. Motion along a plane curve

- Let the position of a particle P is defined by a parametric equation $x = f(t),\ y = g(t)$.

(1) Position Vector :

(2) Velocity Vector :

(3) Speed (Magnitude) :

(4) Acceleration Vector :

(5) Distance (Length) traveled from $t = t_1$ to $t = t_2$:

⇔ Length of a parametric curve from $t = t_1$ to $t = t_2$

Ex) A particle $P(x,\ y)$ moves along a curve according to a parametric equation $x = (t+1)^2$, $y = \dfrac{t}{t+1}$ at time t. Find its position, velocity, speed and acceleration at $t = 1$.

Ex) Find the distance that a particle traveled according to $x = \cos^3 t,\ y = \sin^3 t$ from $t = 0$ to $t = \dfrac{\pi}{2}$.

8. Improper Integrals

(1) Type I Improper Integrals (when intervals include infinity)

① If $f(x)$ is continuous at $[a, \infty)$, then $\int_a^{\infty} f(x)dx =$

② If $f(x)$ is continuous at $(-\infty, b]$, then $\int_{-\infty}^{b} f(x)dx =$

③ If $f(x)$ is continuous at $(-\infty, \infty)$, then $\int_{-\infty}^{\infty} f(x)dx =$

$=$

⇨ $\int_{-\infty}^{\infty} f(x)dx =$ converges when and both converges.

Ex) Is the area under the curve $y = \frac{\ln x}{x^2}$ from $x = 1$ to $x = \infty$ finite? If so, what is it?

Ex) Evaluate $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

(2) Type II Improper Integrals (Integration of discontinuous functions)

① If $f(x)$ is continuous on $(a, b]$ and discontinuous at $x=a$,

then $\int_a^b f(x)dx =$

② If $f(x)$ is continuous on $[a, b)$ and discontinuous at $x=b$,

then $\int_a^b f(x)dx =$

③ If $f(x)$ is discontinuous at $x=c$ and continuous at $[a, c),\ (c, b]$,

then $\int_a^b f(x)dx =$

$=$

⇨ $\int_a^b f(x)dx$ converges when $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ ____________________.

Ex) Investigate the convergence of $\int_0^1 \frac{1}{1-x}dx$.

Ex) Investigate the convergence of $\int_{-1}^2 \frac{1}{x}dx$

Ex) Evaluate $\displaystyle\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx.$

Ex) Evaluate $\displaystyle\int_2^\infty \frac{x+3}{(x-1)(x^2+1)}\,dx.$

(3) The Comparison Test

- **For $f(x),\ g(x)$ that is continuous at $[a,\ \infty)$ and $0 \le f(x) \le g(x)$,**

① If $\int_a^{\infty} g(x)dx$ converges, then $\int_a^{\infty} f(x)dx$ ________________

② If $\int_a^{\infty} f(x)dx$ diverges, then $\int_a^{\infty} g(x)dx$ ________________

cf> If $f(x) \ge g(x)$ on $[a,\ b]$ ⇨

If $f(x) \ge 0$ on $[a,\ b]$ ⇨

Ex) Show that $\int_1^{\infty} \frac{\sin^2 x}{x^2}$ converges.

Ex) Show that $\int_1^{\infty} \frac{1}{\sqrt{x^2 - 0.1}} dx$ diverges.

VIII. Differential Equation

1. Differential Equation

⇨ Includes Derivative

Ex) $v(t) = \frac{dx}{dt} = 3t^2 - 2t$ …… (1)

$x(t) =$ …… (2) : ____________________

If the initial position $x(0) = 1$, then

$x(t) =$ …… (3) : ____________________

2. Separable Differential Equation

① Assemble x terms with x, and y terms with y

② Solve the Integral on both sides

③ Organize it with respect to y $(y = f(x))$

④ (if needed) Find the particular solution (integral constant) by using initial value

Ex) Solve the differential equation $\frac{dy}{dx} = \frac{4x}{y}$, $y(0) = 2$.

Ex) Solve the differential equation $\frac{dy}{dx} = (1 + y^2)e^x$.

Ex) Solve the differential equation $\frac{dy}{dx} = \frac{y^2}{x}$ and $y(1) = 3$.

3. Application of Separable Differential Equation

(1) Exponential Growth and Exponential Decay

⇨ If the rate of change (of y) is proportionate to the ________________ y as time (x) goes, it is called ________________________ or ______________________

__________________________ ⇨ $k > 0$:

$k < 0$:

⇨ By solving separable differential equation,

Ex) A city had a population of 10,000 in 1980 and 13,000 in 1990. Assuming an exponential growth rate, estimate the city's population in 2000.

Ex) Bacteria in a certain culture increase at a rate proportional to the number present. If the number of bacteria doubles in three hours, in how many hours will the number of bacteria triple?

(a) $\ln 2$ (b) $\ln 3$ (c) $\frac{\ln 3}{\ln 2}$ (d) $\frac{2\ln 3}{\ln 2}$ (e) $\frac{3\ln 3}{\ln 2}$

Ex) A puppy weighs 2.0 pounds at birth and 3.5 pounds two months later. If the weight of the puppy during its first 6 months is increasing at a rate proportional to its weight, then how much will the puppy weigh when it is 3 months old?

(2) Restricted Growth

⇨ If the rate of change (of y) is proportionate to the difference between the ____________________(y) and a ______________

____

($k, A > 0$)

$f(t)$ is increasing	$f(t)$ is decreasing
$f'(t) =$ ______________	$f'(t) =$ ______________
$f(t) =$ ______________	$f(t) =$ ______________
A is an __________ on the size of f	A is an __________ on the size of f

Ex) According to Newton's law of cooling, a hot object cools at a rate proportional to the difference between its own temperature and that of its environment. Fresh from the pot, a cup of tea is initially 212 degrees. After six minutes of sitting in a 68 degree room, its temperature has dropped to 190 degrees. How many minutes will it take for the tea to be drinkable, which is when its temperature has reached 150 degrees?

(3) Logistic Growth

⇨ If the rate of change (of y) is proportionate to the ____________________ and also to the difference between the ____________________(y) and a ____________________

____

⇨ By solving separable differential equation,

⇨ A is a ____________________ on the size of f

Ex) Population Growth considering the land & food

Ex) Because a flu-like virus is spreading through a population of 50,000 at a rate proportional both to the number of people already infected and to the number still uninfected. If 100 people were infected yesterday and 130 are infected today

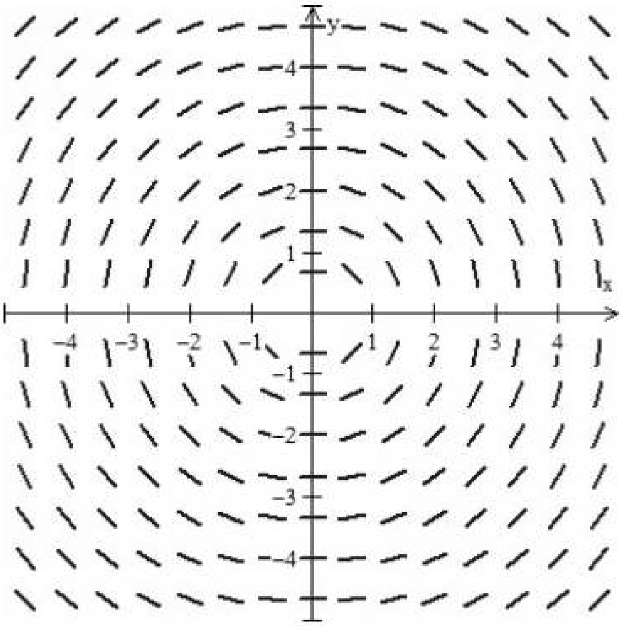
(a) Write an expression for the number of people $N(t)$ infected after t days

(b) Determine how many will be infected a week from today

Ex) The population $P(t)$ of a species satisfies the logistic differential equation $\frac{dP}{dt} = P(2 - \frac{P}{5000})$, where the initial population $P(0) = 3000$ and t is time in years. What is $\lim_{t \to \infty} P(t)$?

4. Slope Field (Vector Field / Direction Field)

⇨ Displays the general solution of a differential equation in a graph



- $y' = -\dfrac{x}{y}$

⇨ At $(1,1)$ → $y' =$

(a, b) → $y' =$

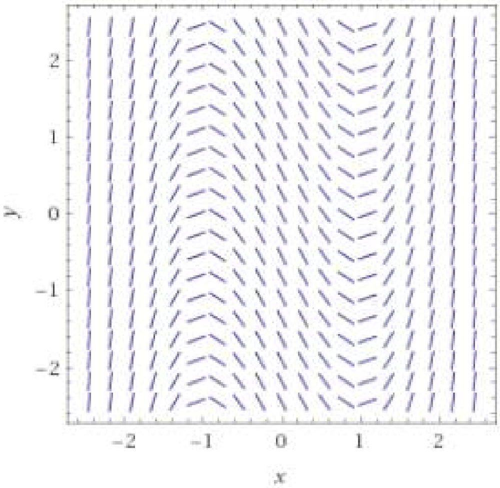
⇨ General Solution :

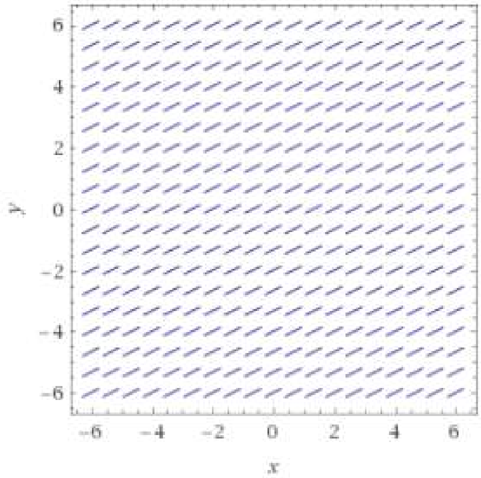
⇨ Solution Curve : passes particular point

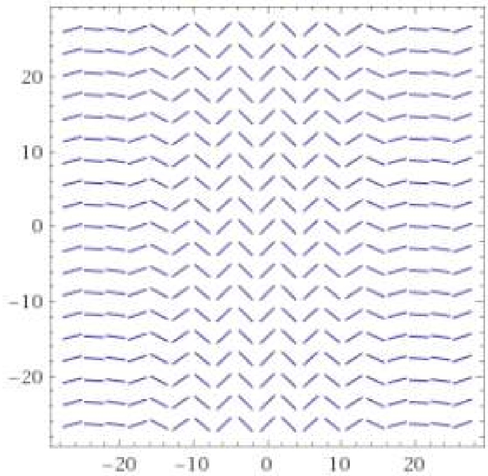
Ex) $(4, -3)$ passing solution curve?

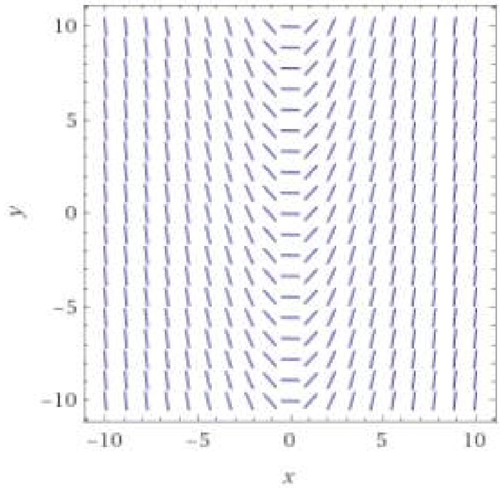
Ex) **Choose the corresponding slope fields for each differential equation. The graph indicates a particular solution that passes** $(0, 0)$.

(1) $y' = 2x^2 - 2$ **(2)** $y' = \frac{1}{2}$ **(3)** $y' = \cos x$ **(4)** $y' = x$









5. Euler's Method

⇨ Numerical Calculation Method

⇨ Starting from initial value, continue to find for new points

Ex) Find the first three approximations y_1, y_2, y_3 using Euler's method for the initial value problem $y' = 1 + y$. $y(0) = 1$, starting at $x_0 = 0$ with $dx = 0.1$.

IX. Sequences and Series

1. Infinite Sequence

$\lim_{n \to \infty} a_n$ ______________ ⇨ a_n ______________

a_n

$\lim_{n \to \infty} a_n$ ______________ ⇨ a_n ______________

Ex) Which sequence diverges?

(a) $a_n = \sqrt{n}$ (b) $a_n = \frac{1}{n}$ © $a_n = (-1)^{n+1} \times \frac{1}{n}$

2. Infinite Series

- $\sum_{k=1}^{n} a_k =$
- $\sum_{k=1}^{\infty} a_k =$ $\qquad\qquad =$

⇨ If $\sum_{k=1}^{\infty} a_k = \lim_{n \to \infty} \sum_{k=1}^{n} a_k =$, then the inifinite series ______________ .

(If not, the series ______________)

3. Theorems

(1) If $\sum_{k=1}^{\infty} a_k$ converges, then

※ If $\lim_{n \to \infty} a_n = 0$, then $\sum_{k=1}^{\infty} a_k$ converges ( ) Ex) $a_n =$

If $\lim_{n \to \infty} a_n \neq 0$, then $\sum_{k=1}^{\infty} a_k$ diverges ( )

(2) If $\sum_{k=1}^{\infty} a_k$, $\sum_{k=1}^{\infty} b_k$ converges, then also converges.

(3) If $\sum_{k=1}^{\infty} a_k$ converges, then also converges. ($c \neq 0$)

(4) The convergence/divergence does not change even finite terms are omitted from the original series. $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=m}^{\infty} a_k$ both converges or diverges.

(5) The subset (consisting of the terms of a) of a converging series also converges.

4. Convergence Test for Infinite Series

(1) The nth term Test

: If $\lim_{n \to \infty} a_n$ does not converges, or does not converges to 0, then $\sum_{k=1}^{\infty} a_k$ __________ .

Ex) Investigate the convergence of the following series.

(a) $\sum_{n=1}^{\infty} n^2$ (b) $\sum_{n=1}^{\infty} \frac{n+1}{n}$ (c) $\sum_{n=1}^{\infty} (-1)^{n+1}$

(2) The Geometric Series Test

- Geometric Series

$=$

• $\sum_{k=1}^{\infty} a_k =$ $=$

$=$

Ex) Show that geometric series $1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots\frac{1}{3^{n-1}}+\cdots$ **converges.**

cf> $1+2+2^2+\cdots+2^{n-1}+\cdots$

(3) The Integral Test

If ___________ for all real numbers x, and $f(x)$ is a _______________ function, then $\sum_{n=1}^{\infty}a_n$ ($a_n=f(n)$) converges only when ______________ converges

Ex) $\sum_{n=1}^{\infty}\frac{1}{n^2}$ ⇨ **Converge?**

(4) $p-$series Test

$=$ $(p>1)$

- $\sum_{k=1}^{\infty} a_k =$ $=$

$=$ $(p \le 1)$

cf> $p=1$ **: Harmonic Series**

Ex) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\cdots$

Ex) Show that the $p-$**series** $\sum_{k=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots+\frac{1}{n^p}+\cdots$ **(**p **is a real constant) converges if** $p>1$**, and diverges if** $p \le 1$**.**

⇨ ____________________ **로 Prove**

(a) $p>1$

(b) $p<1$

(c) $p=1$:

(5) The Comparison Test

For $\sum_{n=1}^{\infty} a_n$ where $a_n \geq 0$ for all natural numbers,

(1) If ______________ and ________________________ , then $\sum_{n=1}^{\infty} a_n$ _____________

(2) If ______________ and ________________________ , then $\sum_{n=1}^{\infty} a_n$ _____________

Ex) Investigate the convergence.

(a) $\sum_{n=1}^{\infty} \frac{5}{5n-1}$

(b) $\sum_{n=0}^{\infty} \frac{1}{n!}$

(c) $1+\frac{1}{2+\sqrt{1}}+\frac{1}{4+\sqrt{2}}+\frac{1}{8+\sqrt{3}}+\cdots+\frac{1}{2^n+\sqrt{n}}+\cdots$

(6) The Limit Comparison Test

For $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ where $a_n > 0$, $b_n > 0$ for all natural numbers,

① If and converges, then also converges.

② If and diverges, then also diverges.

③ If , then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converges or diverges.

Ex) Investigate then convergence of the following series.

(a) $\sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$

(b) $\sum_{n=2}^{\infty} \frac{1+n\ln n}{n^2+5}$

(c) $\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$

(7) Ratio Test

For $\sum_{n=1}^{\infty} a_n$ where $a_n > 0$ for all natural numbers, suppose $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$.

① If ________ , then $\sum_{n=1}^{\infty} a_n$ ______________

② If _________ or __________ , then $\sum_{n=1}^{\infty} a_n$ ______________

③ If __________ , then the convergence cannot be investigated.

Ex) Investigate the convergence.

(a) $\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}$

(b) $\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$

(c) $\sum_{n=1}^{\infty} \frac{4^n n! n!}{(2n)!}$

(8) The Root Test

For $\sum_{n=1}^{\infty} a_n$ where $a_n \geq 0$ for all natural numbers, suppose $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$.

① If ________ , then $\sum_{n=1}^{\infty} a_n$ ______________

② If _________ or __________ , then $\sum_{n=1}^{\infty} a_n$ ______________

③ If __________ , then the convergence cannot be investigated.

Ex) Investigate the convergence.

(a) $\sum_{n=0}^{\infty} \frac{n^2}{2^n}$ (b) $\sum_{n=0}^{\infty} \frac{2^n}{n^2}$ © $\sum_{n=0}^{\infty} (\frac{1}{n+1})^n$

(9) Alternating Series Test

- **Alternating Series** : Alternating Signs

  Ex) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$

- **Alternating Series Test**

  For $\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots$, the series converges when all 3 conditions below are satisfied.

  ① For all natural numbers n, ____________

  ② For all natural numbers n, ____________ (   )

  ③ ________________

  Ex) Show that the alternating harmonic series converges.

- **Alternating Series Estimating Theorem**

For converging alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$, the subset of the infinite series

$S_n = u_1 - u_2 + \cdots + (-1)^{n+1}u_n$ approximates the value L ($\sum_{n=1}^{\infty}(-1)^{n+1}u_n = L$).

⇨ $|L - S_n| =$

Ex) How many terms must be summed to approximate to three decimal places the value of $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2^n}$?

(10) Absolute Convergent Test

① If the series $\sum_{n=1}^{\infty}|a_n|$ converges, then also converges.

② $\sum_{n=1}^{\infty}a_n$ is ______________________ if converges.

③ $\sum_{n=1}^{\infty}a_n$ is ______________________ if converges but diverges.

Ex) Determine whether the series converges. If it converges, determine whether it is absolutely convergent or conditionally convergent.

(a) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$ (b) $\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$ (c) $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$

5. Power Series

(1) Definition

- $\sum_{n=0}^{\infty} a_n(x-c)^n =$

  (c : center of the series)

- $\sum_{n=0}^{\infty} a_n x^n =$

  (   )

(2) The Radius of Convergence of a Power Series

If the power series   converges at $|x-c|<r$ / diverges at $|x-c|>r$, then $r=$________________________.

- Interval of Convergence : Set of converging x values

⇨ How to find the Interval of Convergence

① Conuct ____________________ for the absolute value of the series

② Find the Radius

③ Check the convergence at the endpoints

Ex) Find all x for which $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}$ converges.

(3) Functions defined by Power Series

$$f(x)=\sum_{k=0}^{\infty}a_k(x-c)^k=$$

① $f(x)$ is continuous at ______________________________

② The new series attained by differentiating each terms of the power series converges to ________ . (within the radius of convergence)

$$f'(x)=$$

③ The new series attained by integrating each terms of the power series converges to ________ . (within the radius of convergence)

$$\int_a^x f(t)dt=$$

- How to Express the sum of Polynomial functions in another function

Ex) $f(x)=x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\cdots, \ -1\le x\le 1$

Ex) $\dfrac{1}{1+t}=1-t+t^2-t^3+\cdots \quad (-1<t<1)$

(4) Taylor Series and Maclaurin Series

: How to express a function into a sum of polynomial functions

$$f(x)=\sum_{k=0}^{\infty}a_k(x-a)^k=a_0+a_1(x-a)+a_2(x-2)^2+\cdots+a_n(x-a)^n+\cdots$$

$f'(x)=$

$f''(x)=$

$f'''(x)=$

$\vdots$

⇨ **Put in** $x=a$

$f'(a)=$, $f''(a)=$, $f'''(a)=$, $f^{(n)}(a)=$ $\cdots$

- **Taylor Series at** $x=a$

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k=$$

- **Maclaurin Series** : Taylor Series at ________

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k=$$

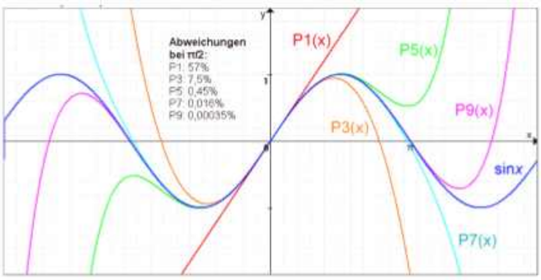
Ex) Find the Taylor Series generated by $f(x)=\frac{1}{x}$ at $a=2$. Where, if anywhere, does the series converge to $\frac{1}{x}$?

(5) Taylor and Maclaurin Polynomials

: At $x=a$, $f(x)$ can be approximated as a Taylor Polynomial(P_n)

$$f(x) \approx P_n(x) =$$

Ex) Find the Taylor series and Taylor polynomials generated by $f(x)=\cos x$ at $x=0$.



- **Formulas (Optional)**

$\cos x =$

$\sin x =$

$e^x =$

(6) Taylor's Formula with Remainder; Lagrange Error Bound

- If a Taylor Series is not given as an alternating series ⇨ **Lagrange Method**

If $f(x)$ and it's derivative until $n+1$ th term is continuous on $|x-a|<r$, then there exist $R_n(x)$ such that

$f(x) =$

$R_n(x) \ =$

$(a < c < x)$

⇨ Error Bound

$\therefore \ R_n(x)$ <

Ex) Find the Maclaurin Series for $\ln(1+x)$ and the associated Lagrange Error Bound.

Ex) Calculate e with an error of less than 10^{-6}.