Hidden Markov Models (HMM)

Tutorial

The evaluation, decoding and learning problems
(following Rabiner, 1989)

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Puppy Platone Example ☺

Platone is a 2 month old adorable Collie. Platone fills his days learning with his best friend and master Alex and from his simple yet interesting world.

Most of the day he is playing or exploring. This suggests that he is active and entertained or bored and looking for something new to do.

Platone also barks at times for attention from Alex when he is hungry or needs something. Then he gets tired and sleepy quite often and just lies down on the floor. Alex is learning to understand his puppy and being a computer science geek^^ he decides to model Platone using a HMM …

☺ Puppy Platone Example ☺

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What are the states?

What are the emissions?

What are the model parameters?

☺ Puppy Platone Example ☺ - BREAKING IT DOWN

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What are the states?

[Active, Bored, Hungry, Sleepy]

What are the emissions?

[Play, Explore, Bark, Lie Down]

What are the model parameters?

A = matrix 4states x 4states

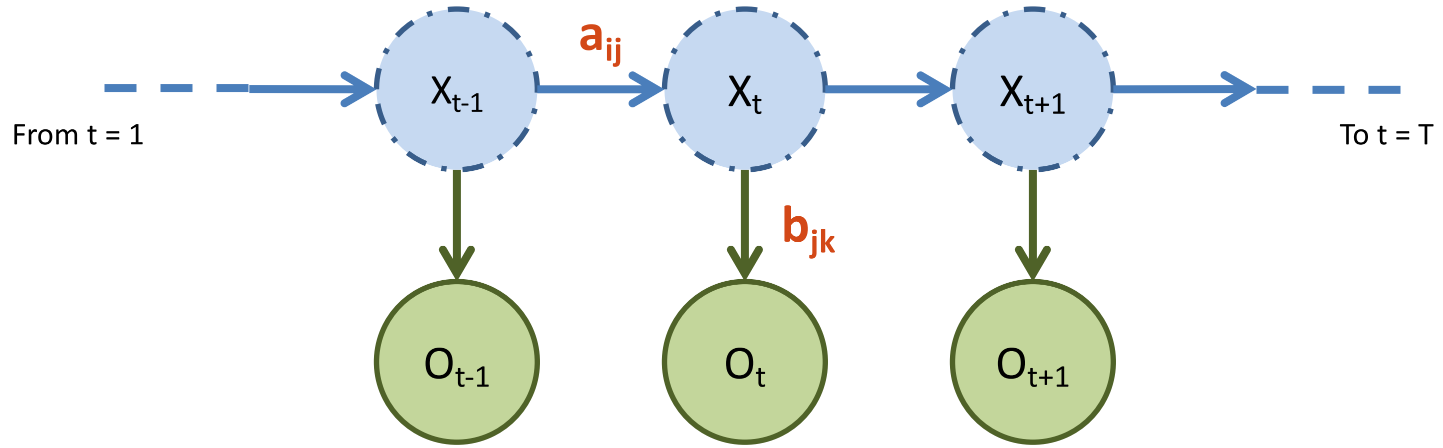
B = matrix 4states x 4emissions

q = matrix 4states x 1 initial probabilities

TERMINOLOGY:

HMM: $\lambda = (A, B, \pi)$

Time instants	t *in* {1,2 . . . T}
Hidden States / States / Emitters	X_t
Outputs / Emissions / Observations / Visible States	O_t
All possible states / states set	X_t *in* {1,2 . . . N}
All possible emissions / emissions set	O_t *in* {1,2 . . . K}
Initial state distribution / Initial state probabilities	q_i in q *or* π_i in π
Transition probabilities / State transition probabilities	a_{ij} *in* row-stochastic matrix A
Emission probabilities / Observation probabilities	b_{jk} *in* row-stochastic matrix B



Markov model

- The (first order) Markov assumption

–The distribution $p(X_t)$ depends only on the distribution $p(X_{t-1})$

–The present (current state) can be predicted using local knowledge of the past (state at the previous step)

HMM PROBLEMS:

HMM PROBLEMS:

0. Predicting most likely current emission

1. Evaluation Problem = ?

2. Decoding Problem = ?

3. Learning Problem =?

PROBLEM 0: Most likely current emission

Given:

- A, B, q
- Distribution of hidden states at time (t)

Unknown:

- The exact time instance
- Emission sequence until time (t)
- Hidden state sequence until time (t)

Find:

- Most likely next emission at time (t)

PROBLEM 0: Most likely current emission (Puppy Platone Example)

Given:

A =

$X_t \mid X_{t+1}$	**A**	**B**	**H**	**S**
A	0.6	0.1	0.1	0.2
B	0.0	0.3	0.2	0.5
H	0.8	0.1	0.0	0.1
S	0.2	0.0	0.1	0.7

B =

$X_t \mid O_t$	**p**	**e**	**b**	**l**
A	0.6	0.2	0.1	0.1
B	0.1	0.4	0.1	0.4
H	0.0	0.0	0.7	0.3
S	0.0	0.0	0.1	0.9

$P(X_t) =$

A	**B**	**H**	**S**
0.4	0.2	0.1	0.3

Find:

- $P(O_t \mid A, B, P(X_t))$

Solution:

- On the board …

PROBLEM 1: Evaluation

Given:

- A, B, q
- Emission sequence **O** = $\{O_1, O_2 \ldots O_T\}$

Unknown:

- Hidden state sequence **X** = $\{X_1, X_2 \ldots X_T\}$ that actually produced **O**.

To Find:

- Probability that the given sequence **O** occurred regardless of which **X** produced the sequence.

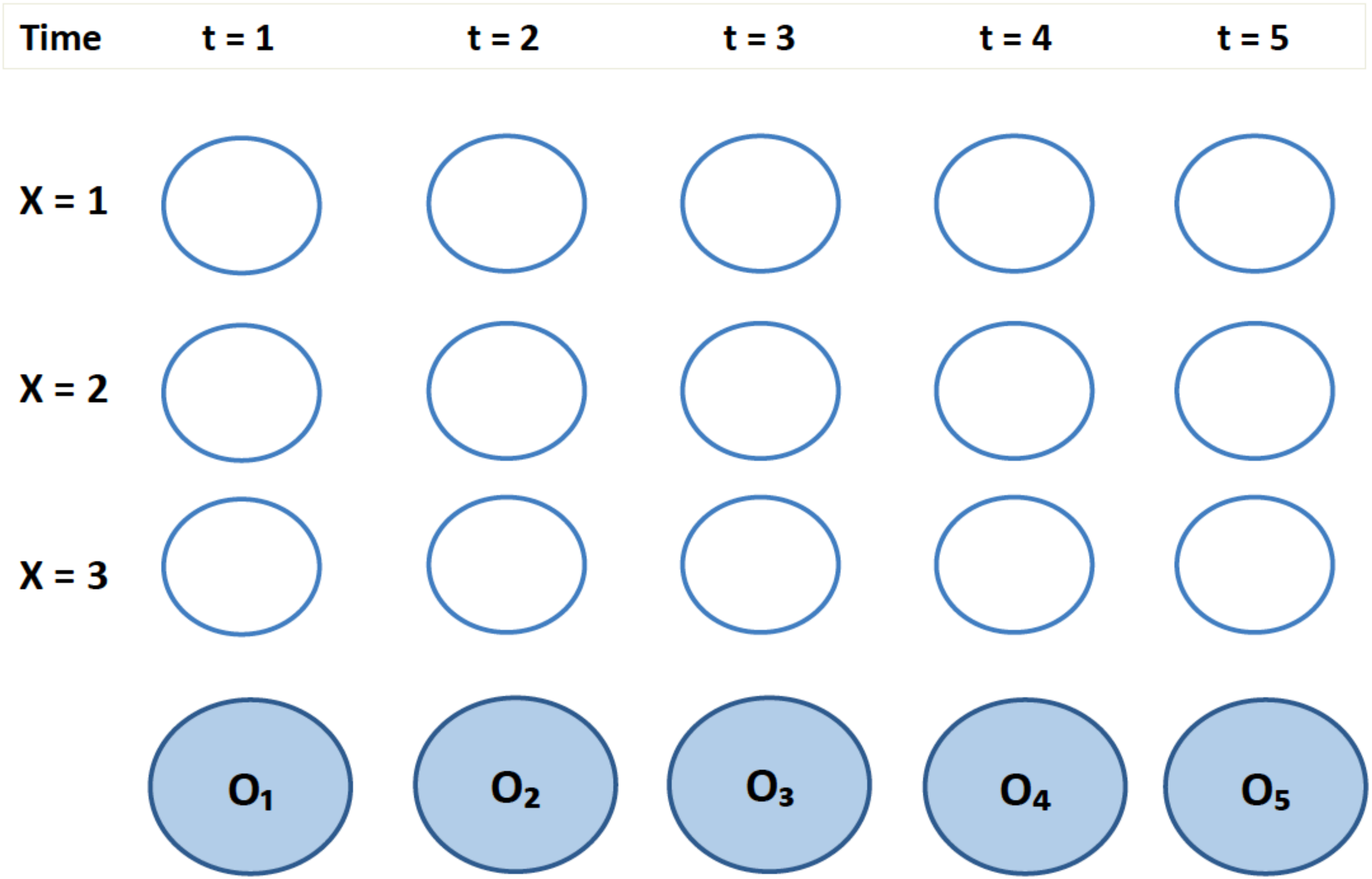
FORWARD ALGORITHM: (α - PASS)

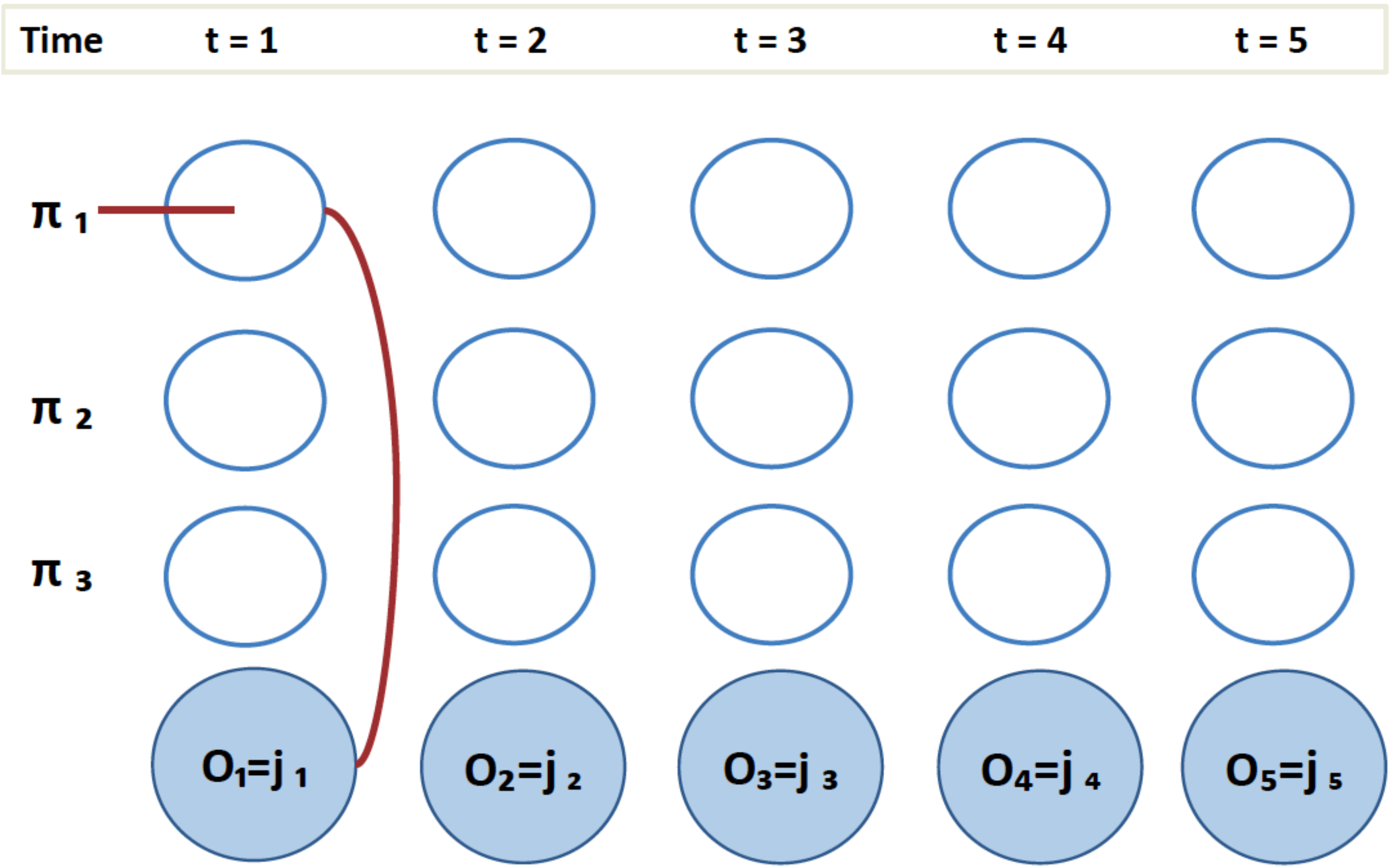
$\alpha_t(i)$ = Probability that the model is in the hidden state $X_t(i)$ (i *in* [1,2,…,N])

&&

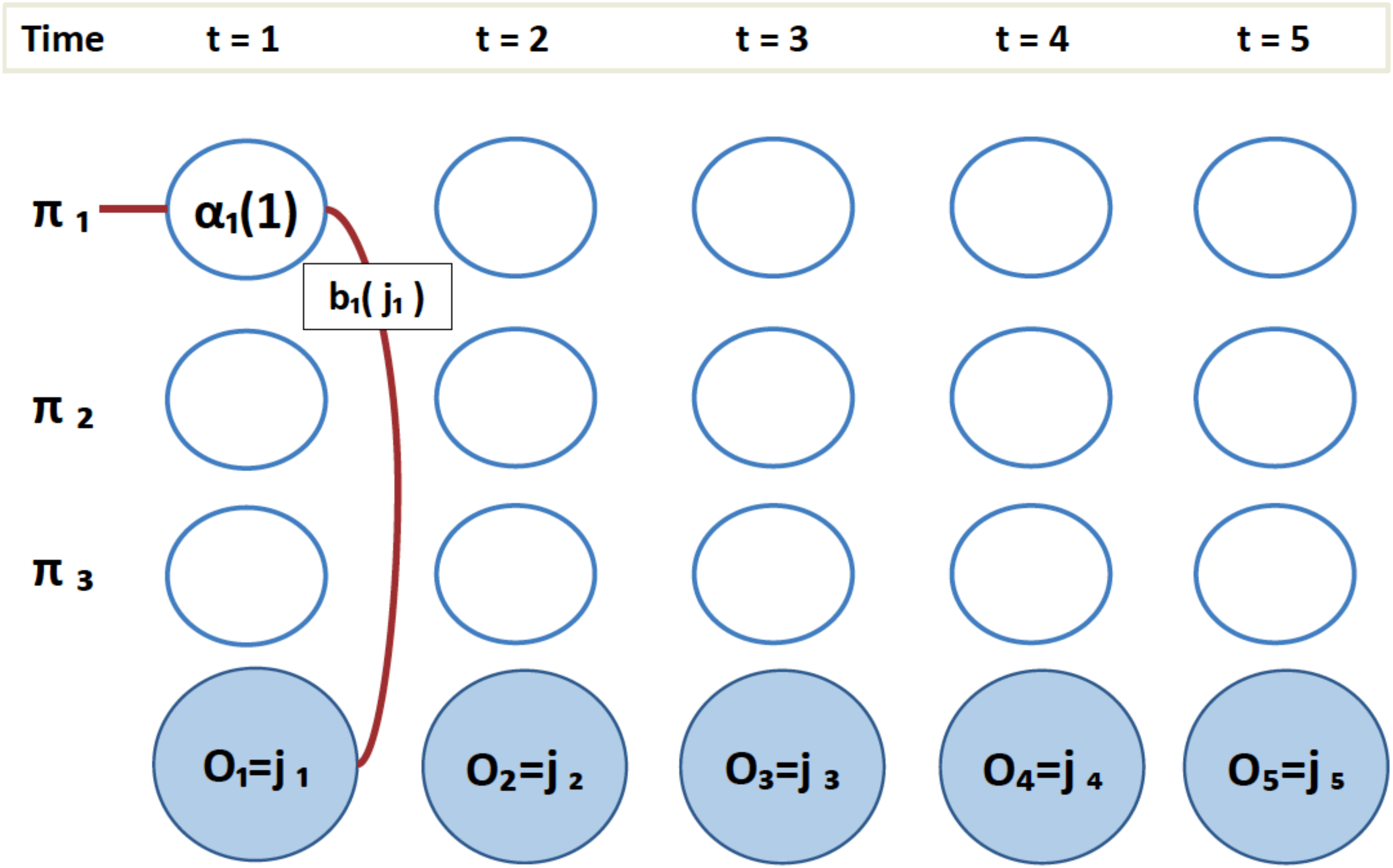
has generated the emission sequence up to O_t, where O_t has taken a value $O_t(k)$ (k *in* [1,2,…,K]) according to the emission sequence already observed.

- Introduce: $\alpha_t(i) = p(O_{1:t}, X_t = i|\lambda) \quad \forall \quad t = 1, \ldots, T$
- Initialize as: $\alpha_1(i) = \pi_i b_i(O_1)$
- For 2≤t≤T: $\alpha_t(i) = [\sum_{j=1}^{N} \alpha_{t-1}(j) a_{ji}] b_i(O_t)$
- Which gives us: $p(O_{1:T}|\lambda) = \sum_{i=1}^{N} p(O_{1:T}, X_T = i|\lambda) = \sum_{i=1}^{N} \alpha_T(i)$

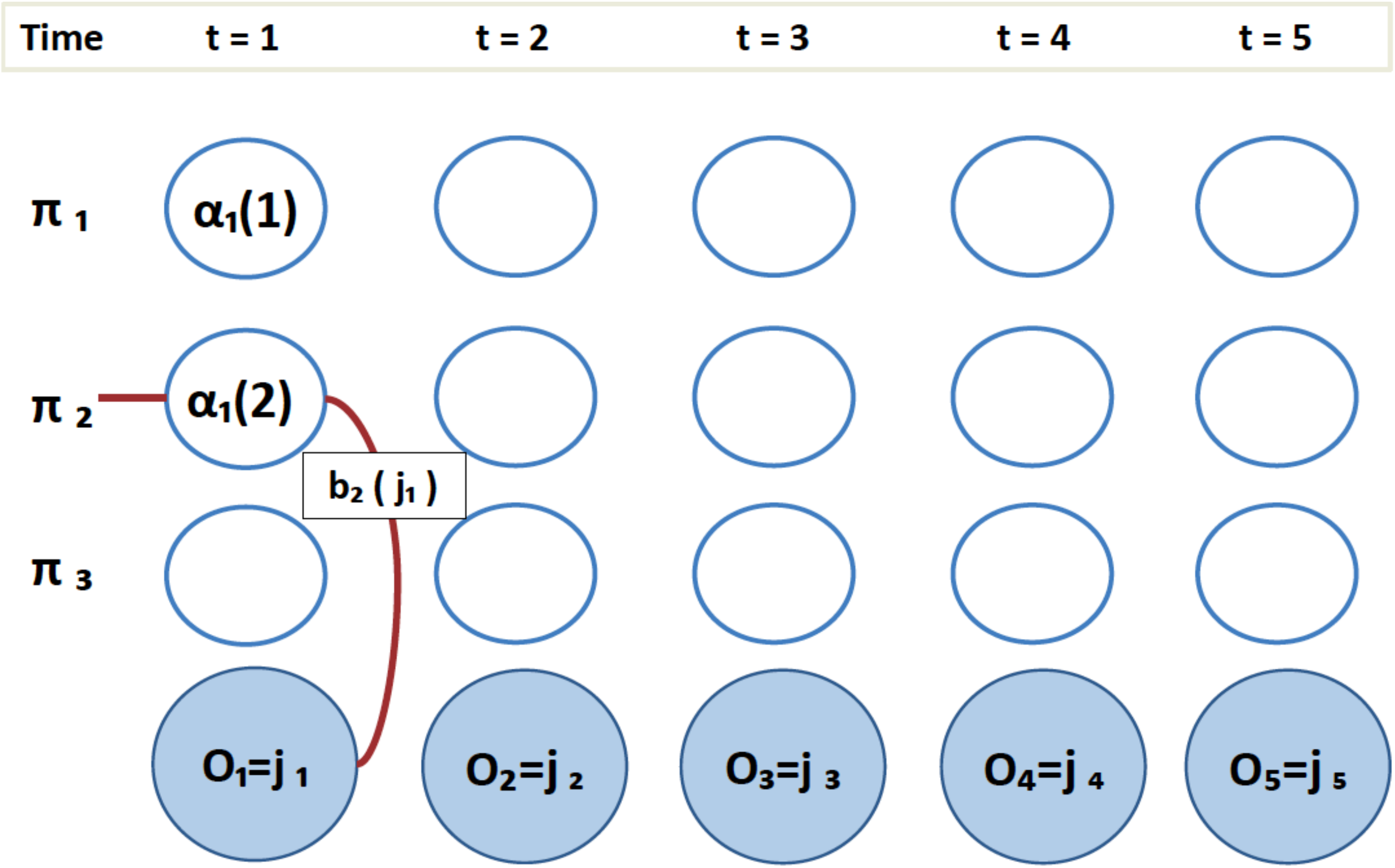

Time
t = 1
t = 2
t = 3
t = 4
t = 5
X = 1
X = 2
X = 3
O1
O2
O3
O4
O5



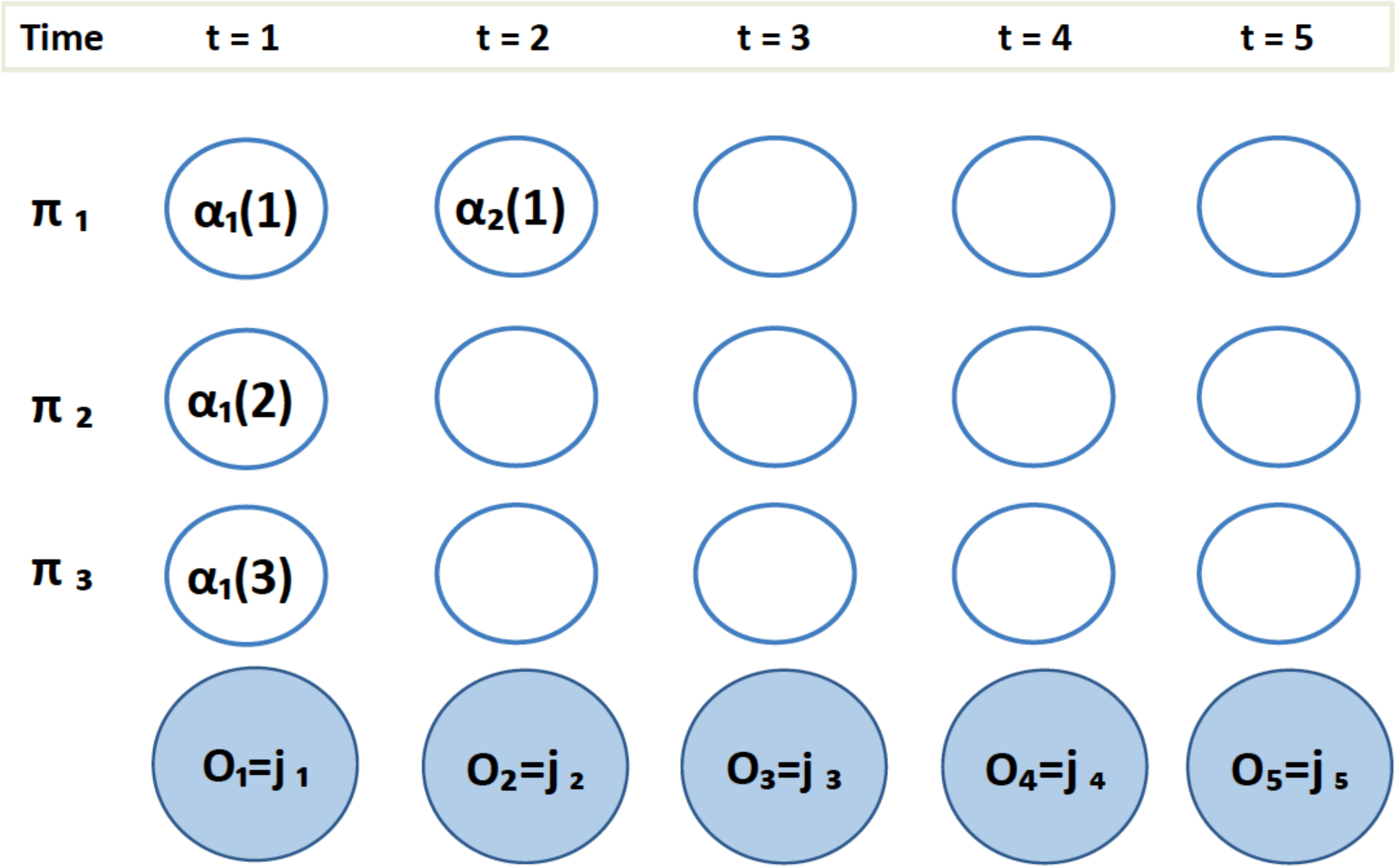
$$\alpha_1(1) = P(\, O_1 = j_1 \mid X_1 = 1)\; P(X_1 = 1)$$



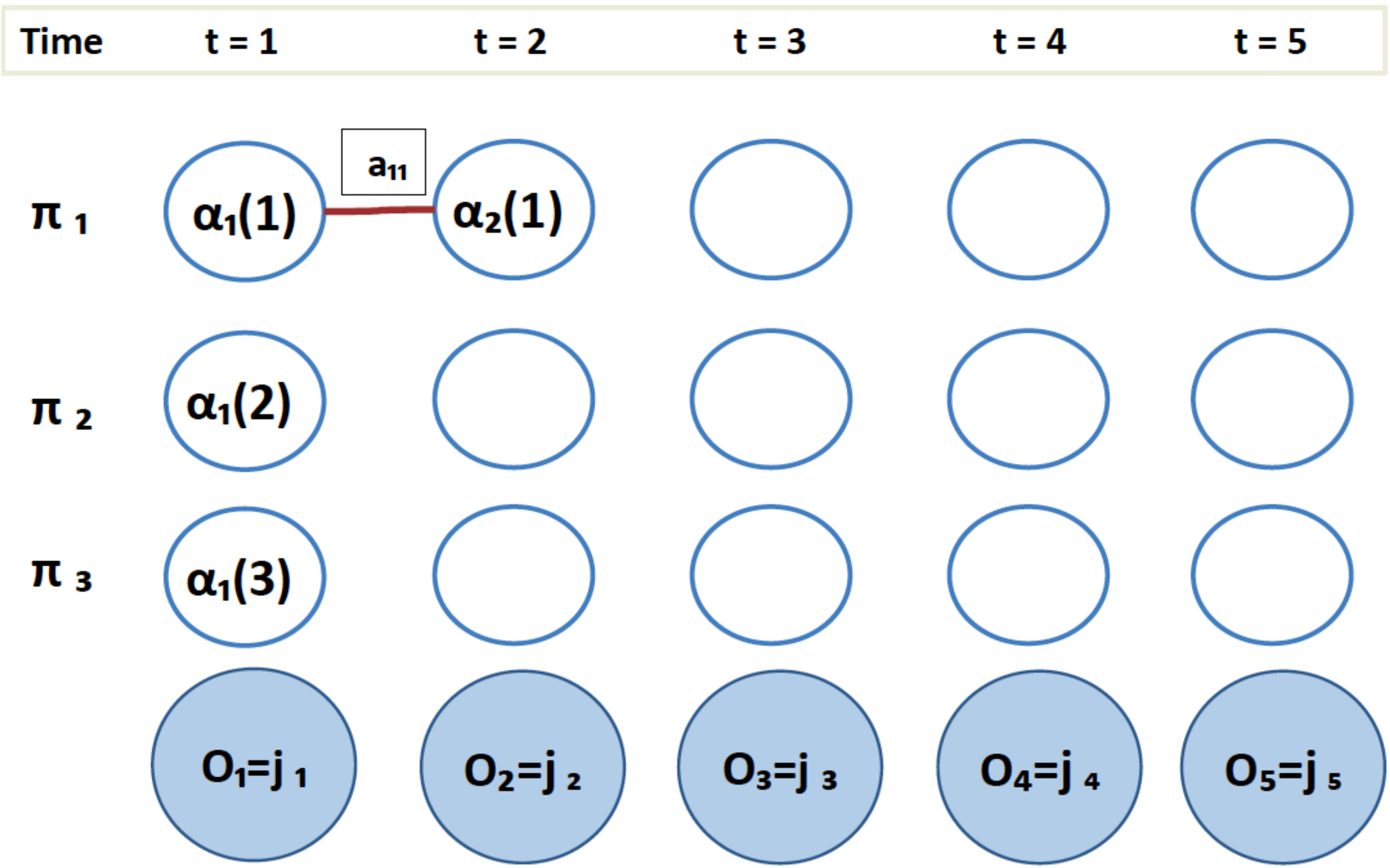
$$\alpha_1(1) = P(O_1 = j_1 \mid X_1 = 1)\, P(X_1 = 1) = b_1(j_1)\, \pi_1$$



$$\alpha_1(2) = P(O_1=j_1 \mid X_1 = 2)\, P(X_1 = 2) = b_2(j_1)\, \pi_2$$

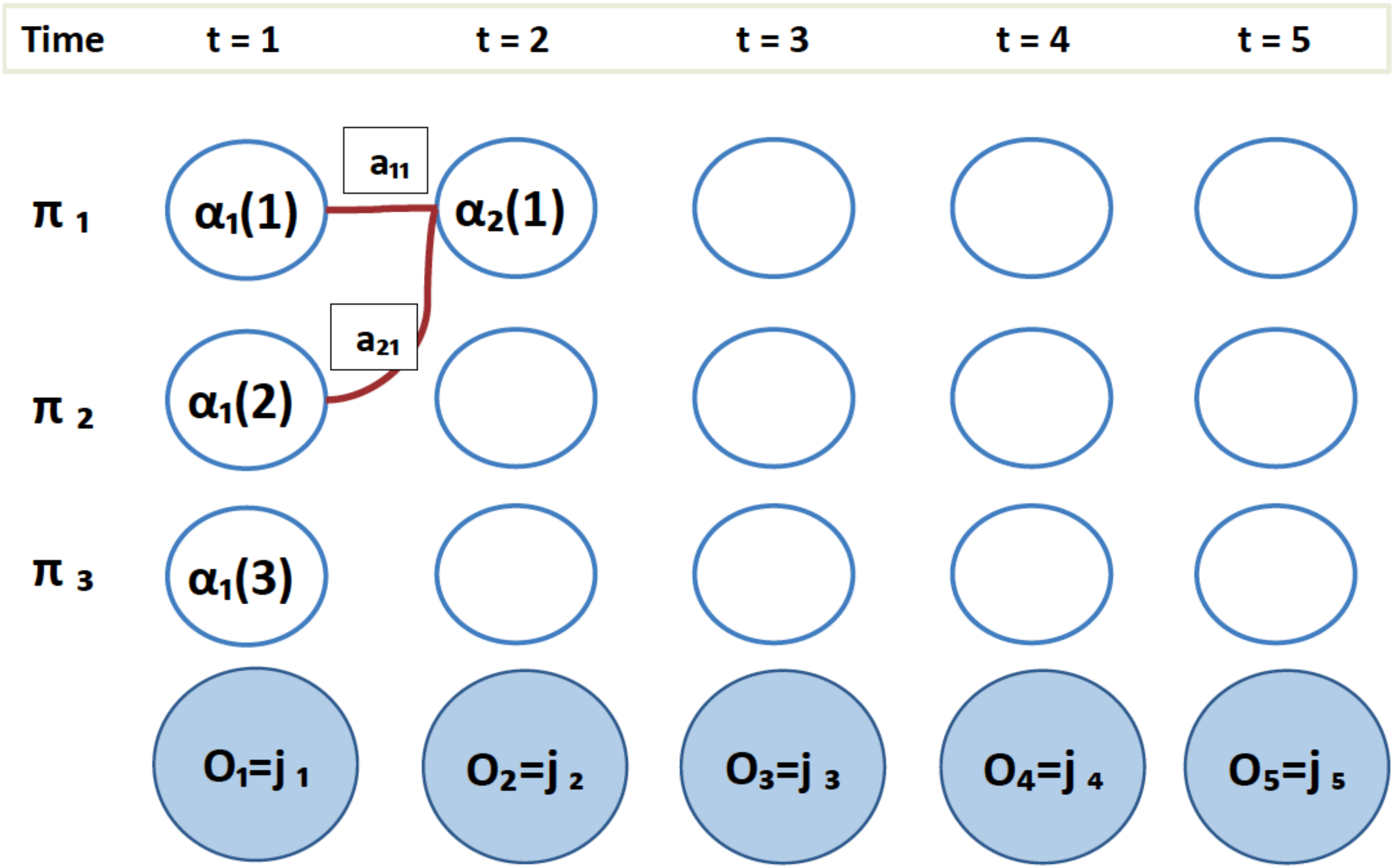


$$\alpha_2(1) = \left(\sum_i P(X_2 = 1 | X_1 = i)\, P(O_1=j_1, X_1 = i)\right) P(O_2=j_2 \mid X_2 = 1)$$



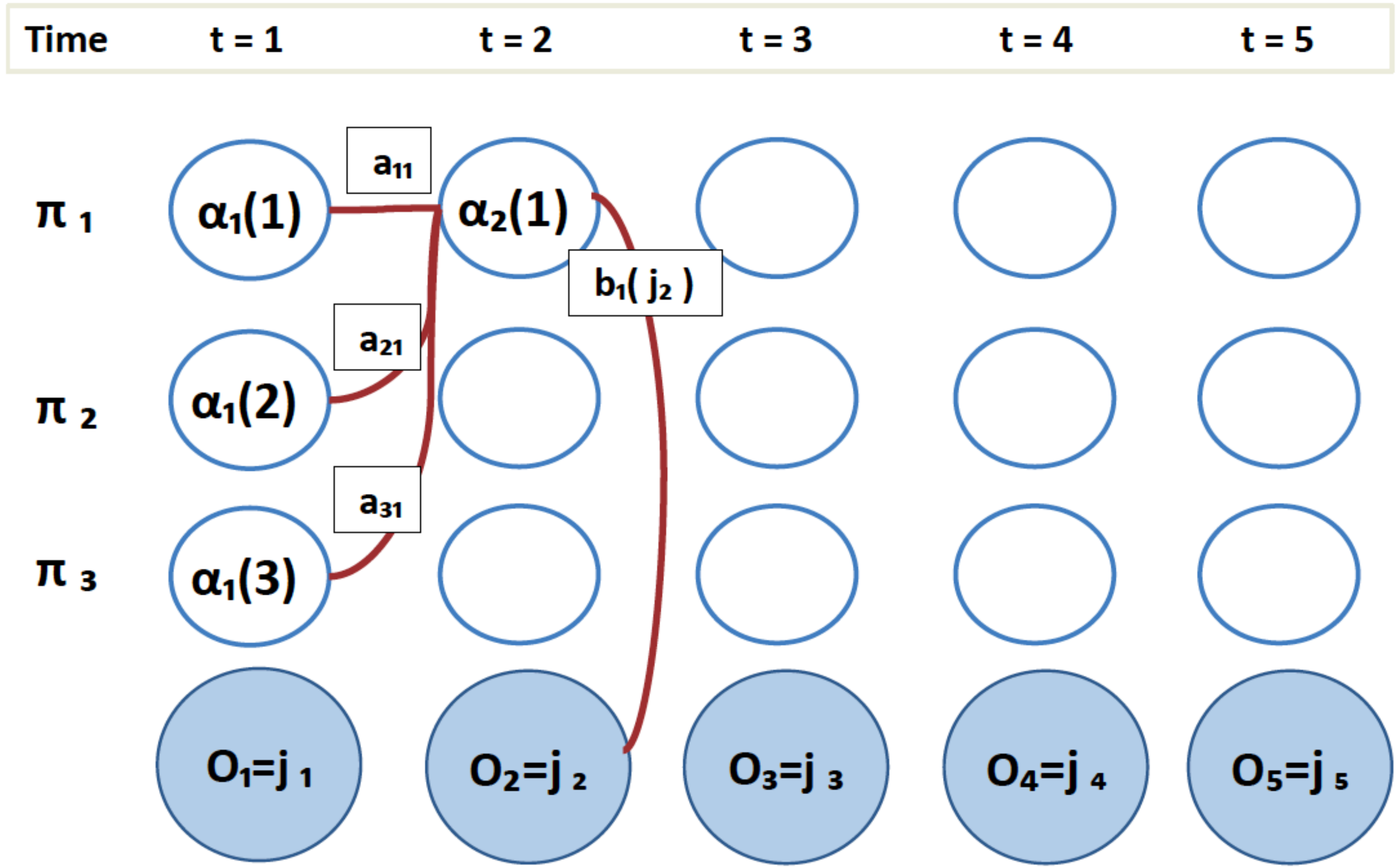
$$\alpha_2(1) = \left(\textstyle\sum_i P(X_2 = 1 \mid X_1 = i)\, P(O_1=j_1, X_1 = i)\right) P(O_2=j_2 \mid X_2 = 1)$$

$$= \left(\textstyle\sum_i a_{i1}\, \alpha_1(i)\right) b_1(j_2)$$



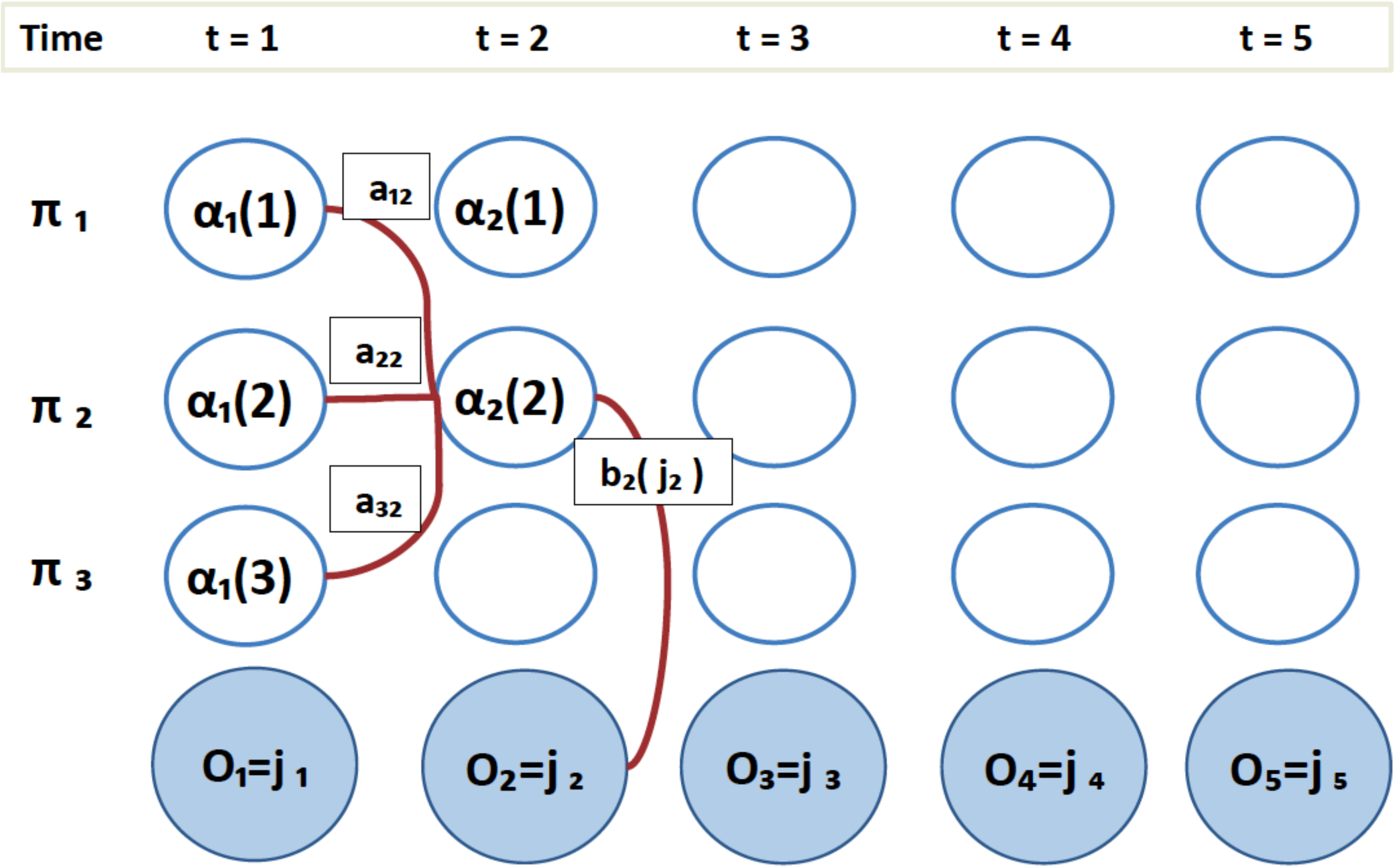
$$\alpha_2(1) = \left(\textstyle\sum_i P(X_2 = 1 \mid X_1 = i)\, P(O_1 = j_1, X_1 = i)\right) P(O_2 = j_2 \mid X_2 = 1)$$

$$= \left(\textstyle\sum_i a_{i1}\, \alpha_1(i)\right) b_1(j_2)$$



$$\alpha_2\,(1) \;=\; \left(\textstyle\sum_i P(\,X_2 = 1 \,|\, X_1 = i)\; P(O_1{=}j_1\,,\, X_1 = i)\,\right) P(O_2{=}j_2 \mid X_2 = 1)$$

$$= \left(\textstyle\sum_i a_{i1}\, \alpha_1(i)\,\right)\; b_1(\,j_2\,)$$



$$\alpha_2(2) = \left(\sum_i P(X_2 = 2 \mid X_1 = i)\, P(O_1=j_1, X_1 = i)\right) P(O_2=j_2 \mid X_2 = 2)$$

$$= \left(\sum_i a_{i2}\, \alpha_1(i)\right) b_2(j_2)$$

Time	t = 1	t = 2	t = 3	t = 4	t = 5
π_1	$\alpha_1(1)$	$\alpha_2(1)$	$\alpha_3(1)$	. . .	$\alpha_5(1)$
π_2	$\alpha_1(2)$	$\alpha_2(2)$	. . .		$\alpha_5(2)$
π_3	$\alpha_1(3)$	$\alpha_2(3)$			$\alpha_5(3)$
	$O_1=j_1$	$O_2=j_2$	$O_3=j_3$	$O_4=j_4$	$O_5=j_5$

$$P(O_{1..5}) \;=\; \sum_i \alpha_5(i)$$

FORWARD ALGORITHM: (α - PASS)

$\alpha_t(i)$ = Probability that the model is in the hidden state $X_t(i)$ (i *in* [1,2,…,N])

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has generated the emission sequence up to O_t, where O_t has taken a value $O_t(k)$ (k *in* [1,2,…,K]) according to the emission sequence already observed.

- Introduce: $\alpha_t(i) = p(O_{1:t}, X_t = i|\lambda) \quad \forall \quad t = 1, \ldots, T$
- Initialize as: $\alpha_1(i) = \pi_i b_i(O_1)$
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- Which gives us: $p(O_{1:T}|\lambda) = \sum_{i=1}^{N} p(O_{1:T}, X_T = i|\lambda) = \sum_{i=1}^{N} \alpha_T(i)$

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PROBLEM 1: Evaluation (Puppy Platone Example)

Given:

A =

X_t | X_{t+1}	**A**	**B**	**H**	**S**
A	0.6	0.1	0.1	0.2
B	0.0	0.3	0.2	0.5
H	0.8	0.1	0.0	0.1
S	0.2	0.0	0.1	0.7

B =

X_t | O_t	**p**	**e**	**b**	**l**
A	0.6	0.2	0.1	0.1
B	0.1	0.4	0.1	0.4
H	0.0	0.0	0.7	0.3
S	0.0	0.0	0.1	0.9

q =

A	**B**	**H**	**S**
0.5	0.0	0.0	0.5

O = { **l, p, p, b** }

Find:

- P(**O** | A, B, q)

Solution:

- Pen and paper exercise…

PROBLEM 2: Decoding

Given:

- Emission sequence **O** = $\{O_1, O_2 \dots O_T\}$
- A, B, q

To Find:

- Hidden state sequence **X*** = $\{X_1, X_2 \dots X_T\}$ that most likely produced **O**.
- Probability of occurrence of **X***

VITERBI ALGORITHM: (δ – Pass)

$\delta_t(i)$ = Probability of having been through state sequence **X*** = $\{X_1, X_2 \ldots X_t\}$,
as ***back-tracked*** by Viterbi algorithm
&&
having generated the observation sequence up to O_t.

VITERBI ALGORITHM: (δ – Pass)

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having generated the observation sequence up to O_t.

- Solved using dynamic programming (DP) with an algorithm called Viterbi.
- Initialize: $\delta_1(i) = \pi_i b_i(O_1), \;\; i = 1, \dots, N$
- For each t>1: $\delta_t(i) = \max_{j \in \{1,\dots,N\}} [\delta_{t-1}(j) a_{ji} b_i(O_t)]$
- Probability of best path: $\max_{j \in \{1,\dots,N\}} [\delta_T(j)]$
- Find path by keeping book of preceding states and trace back from highest-scoring final state.

PROBLEM 1: Evaluation (Puppy Platone Example)

Given:

A =

X_t | X_{t+1}	**A**	**B**	**H**	**S**
A	0.6	0.1	0.1	0.2
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X_t | O_t	**p**	**e**	**b**	**l**
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S	0.0	0.0	0.1	0.9

q =

A	**B**	**H**	**S**
0.5	0.0	0.0	0.5

O = { **b, p, l, e** }

Find:

- P(**X*** | A, B, q)

Solution:

- Pen and paper exercise…

VITERBI ALGORITHM: (δ – Pass)

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- Solved using dynamic programming (DP) with an algorithm called Viterbi.
- Initialize: $\delta_1(i) = \pi_i b_i(O_1), \;\; i = 1, \ldots, N$
- For each t>1: $\delta_t(i) = \max_{j \in \{1,\ldots,N\}} [\delta_{t-1}(j) a_{ji} b_i(O_t)]$
- Probability of best path: $\max_{j \in \{1,\ldots,N\}} [\delta_T(j)]$
- Find path by keeping book of preceding states and trace back from highest-scoring final state.

PROBLEM 3: Learning

Given:

- Emission sequence **O** = $\{O_1, O_2 \dots O_T\}$ (T is a few orders larger than usually seen in text book problems)
- Initial guesses of A, B (maybe)

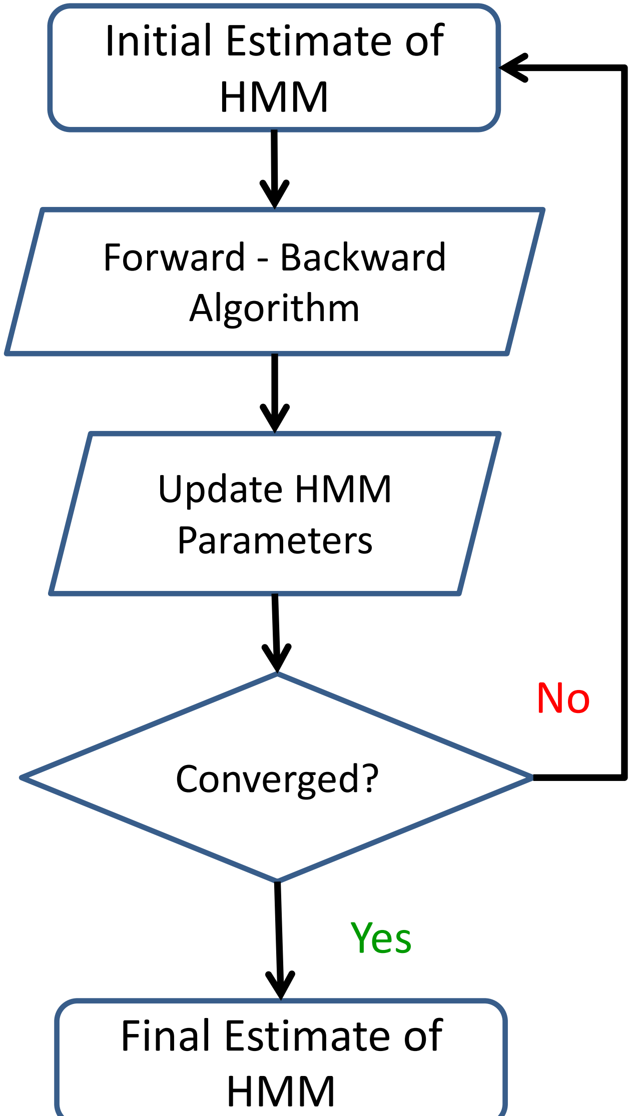
Unknown:

- Hidden state sequence **X** = $\{X_1, X_2 \dots X_T\}$ that actually produced **O**.

To Find:

- A, B

BAUM-WELCH ALGORITHM:



- Given an observation sequence $O_{1:T}$, the number of states, N, and the number of observation outcomes, M.

1. Initialize $\lambda=(A,B,\pi)$
2. Compute $\alpha_t(i)$, $\beta_t(k)$, $\gamma_t(i,j)$ and $\gamma_t(i)$
3. Re-estimate the model $\lambda=(A,B,\pi)$
4. Repeat from 2 until $p(O|\lambda)$ levels out

Will be introduced in the following slides

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BACKWARD ALGORITHM: (β - PASS)

$\beta_t(i)$ = Probability that the model is in the hidden state $X_t(i)$ (i *in* [1,2,…,N])

&&

will generate the remainder of the emission sequence, from O_{t+1} to O_T, as specified by the emission sequence **O**.

- Assume measurement sequence $O_{1:T}$
- Introduce: $\beta_t(i) = p(O_{t+1:T} | X_t = i, \lambda)$
- Initialize: $\beta_T(i) = 1, \forall i = 1, \ldots, N$
- For t<T: $\beta_t(i) = \sum_{j=1}^{N} a_{ij} b_j(O_{t+1}) \beta_{t+1}(j)$

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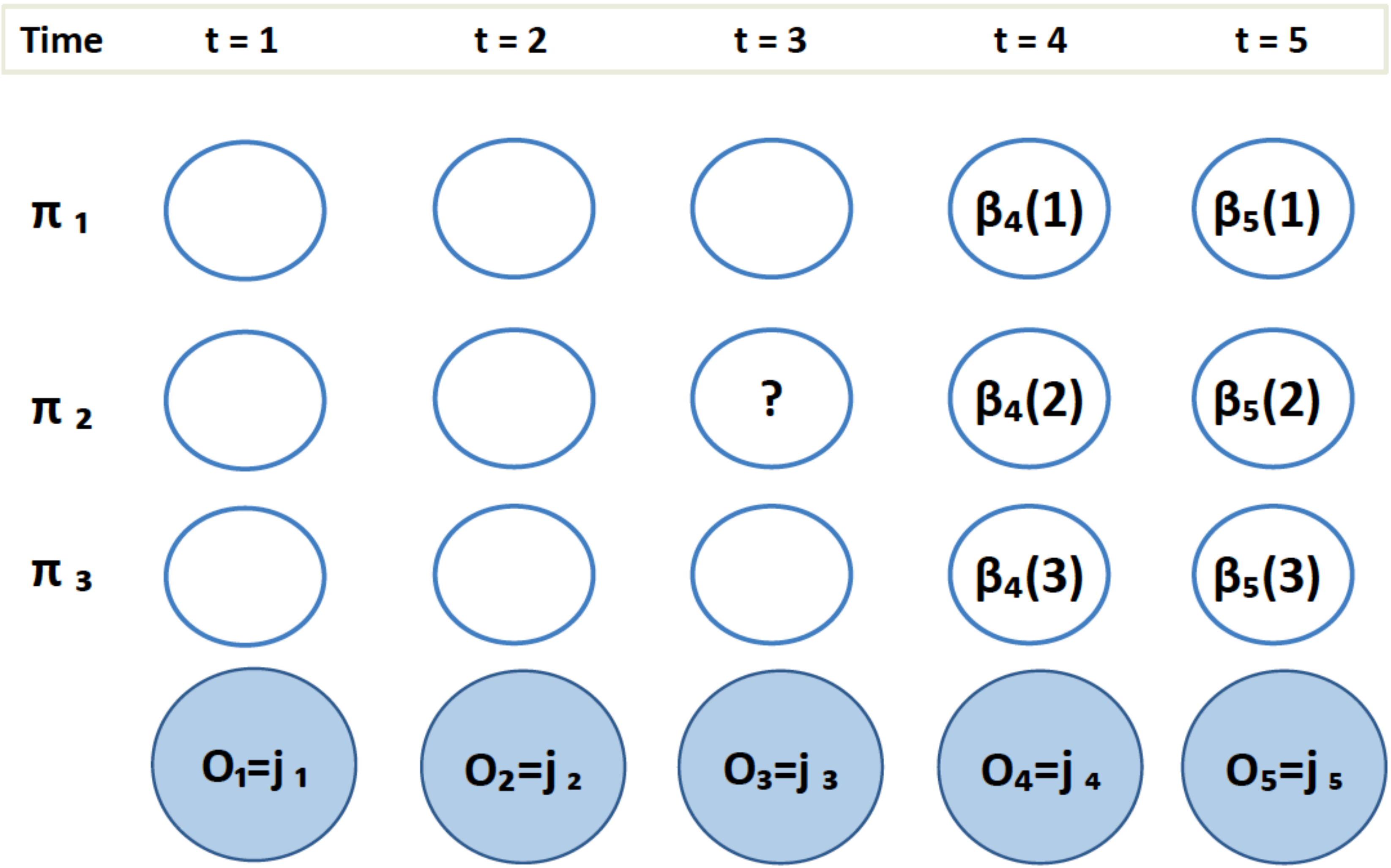
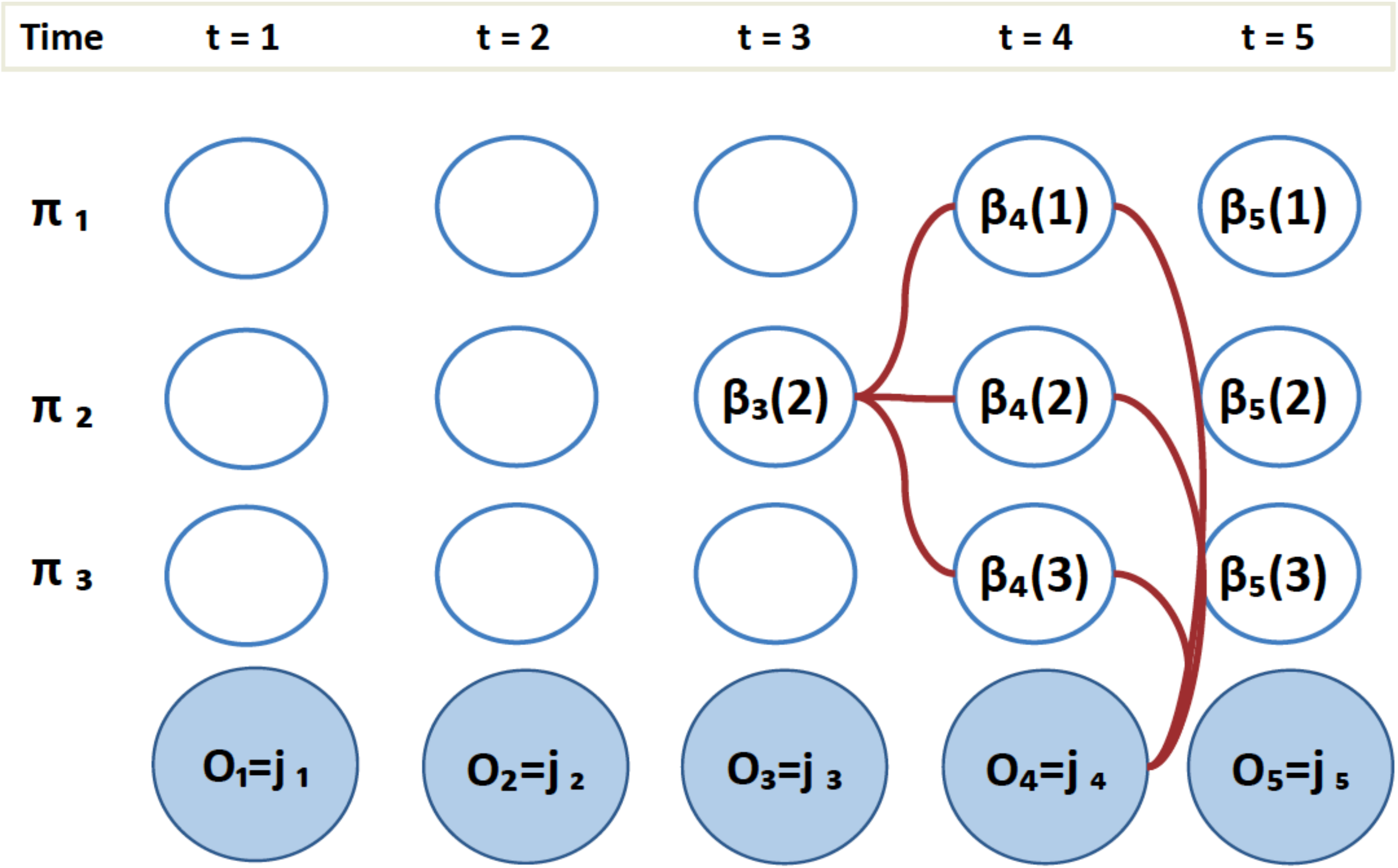
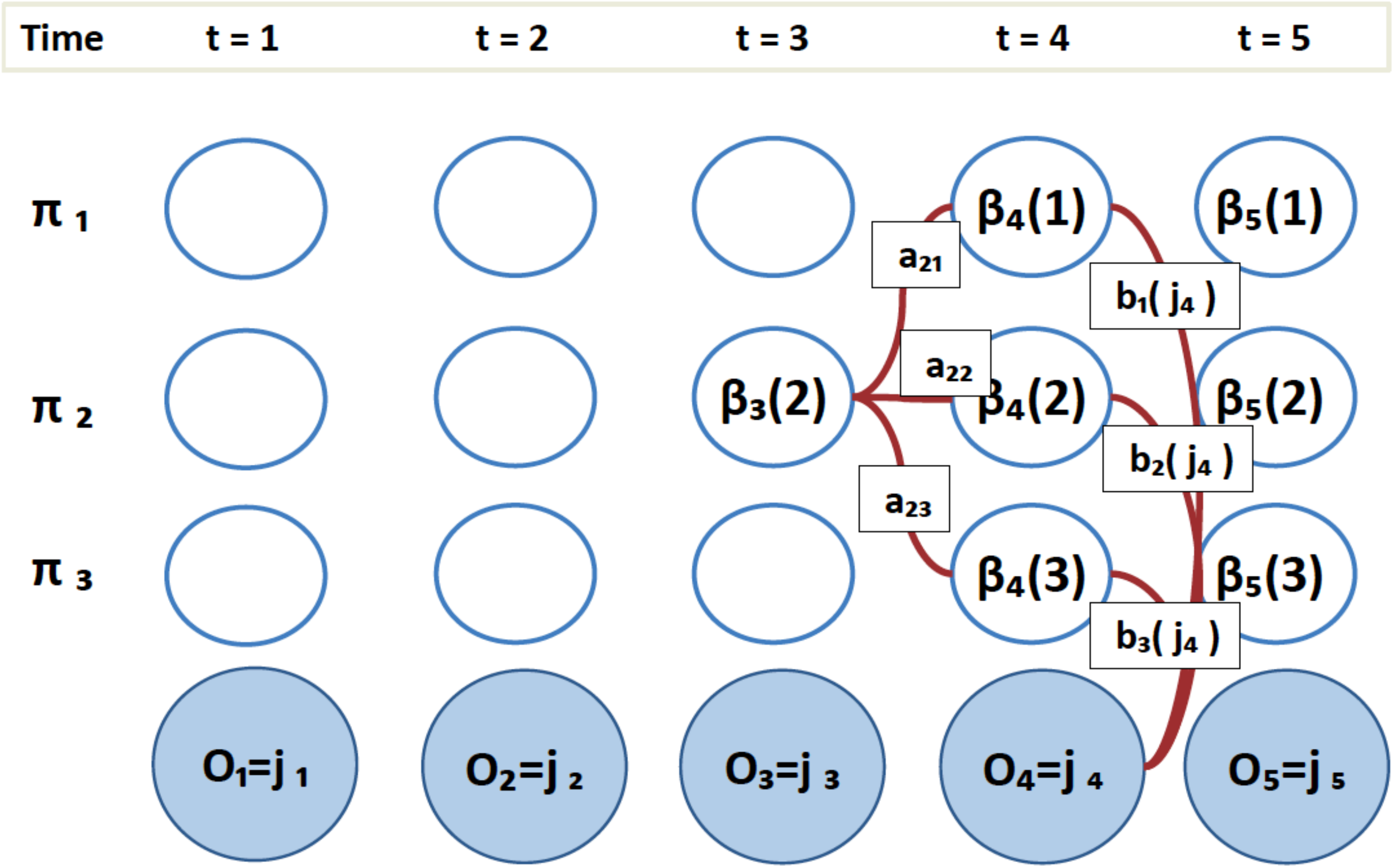


$$\beta_3(2) = P(O_{4..5} \mid X_3 = 2)$$



$$\beta_3(2) = \sum_i P(X_4 = i \mid X_3 = 2)\, P(O_4=j_4 \mid X_4 = i)\, P(O_5 \mid X_4 = i)$$



$$\beta_3(2) = \sum_i a_{2i}\, b_i(j_4)\, \beta_4(i)$$

BACKWARD ALGORITHM: (β - PASS)

$\beta_t(i)$ = Probability that the model is in the hidden state $X_t(i)$ (i *in* [1,2,…,N])

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will generate the remainder of the emission sequence, from O_{t+1} to $O_{T,}$ as specified by the emission sequence **O**.

- Assume measurement sequence $O_{1:T}$
- Introduce: $\beta_t(i) = p(O_{t+1:T} | X_t = i, \lambda)$
- Initialize: $\beta_T(i) = 1, \forall i = 1, \ldots, N$
- For t<T: $\beta_t(i) = \sum_{j=1}^{N} a_{ij} b_j(O_{t+1}) \beta_{t+1}(j)$

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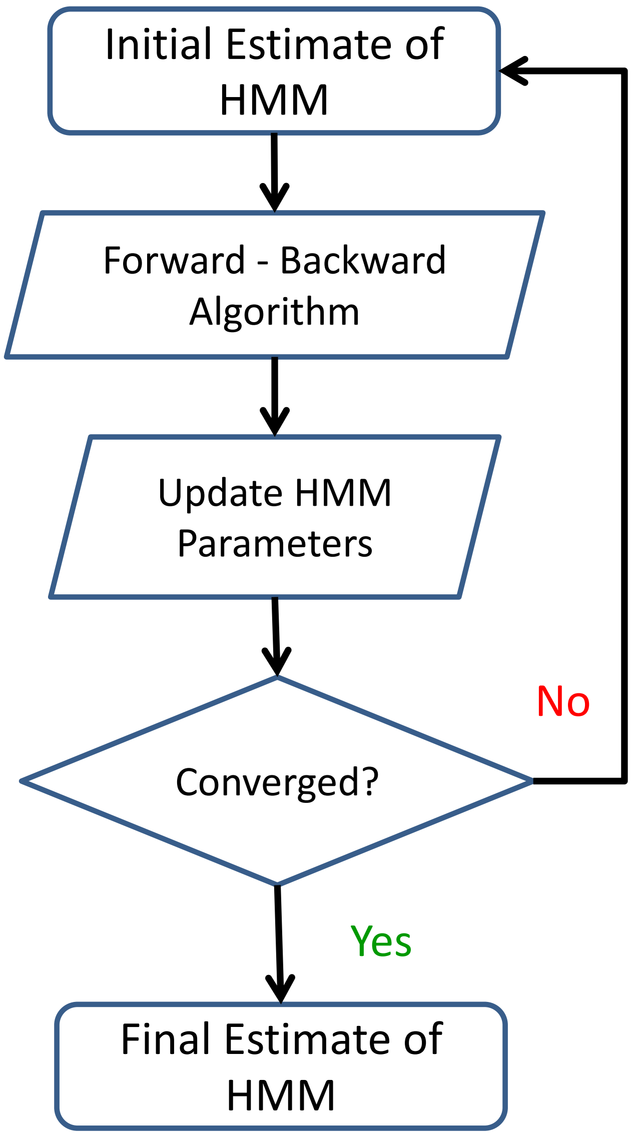
BAUM-WELCH ALGORITHM:



- Given an observation sequence $O_{1:T}$, the number of states, N, and the number of observation outcomes, M.

1. Initialize $\lambda=(A,B,\pi)$
2. Compute $\alpha_t(i)$, $\beta_t(k)$, $\gamma_t(i,j)$ and $\gamma_t(i)$
3. Re-estimate the model $\lambda=(A,B,\pi)$
4. Repeat from 2 until $p(O|\lambda)$ levels out

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GAMMA CALCULATIONS: (1)

1) **Di – Gamma Function**

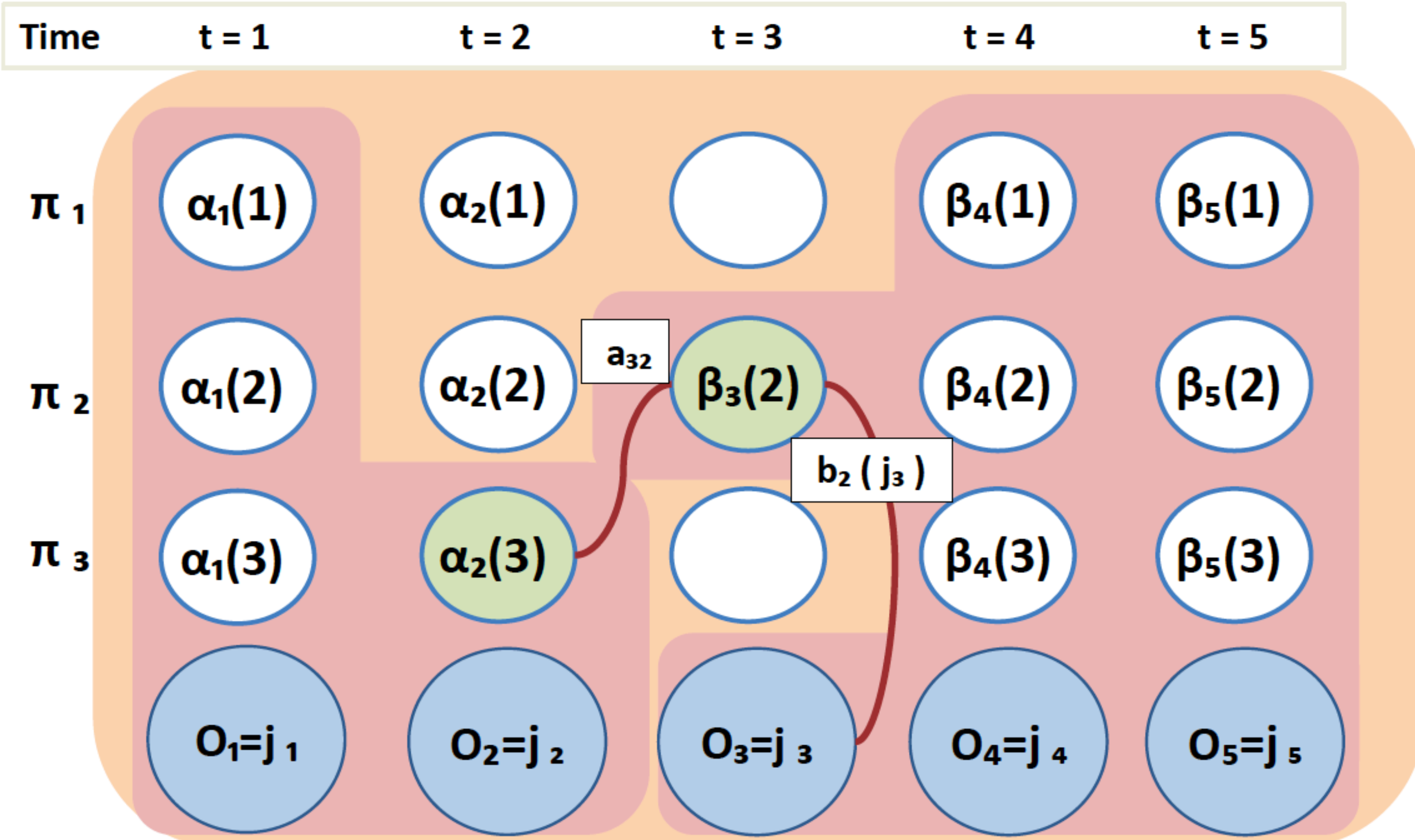
$$\gamma_t(i,j) = \frac{\alpha_t(i)\, a_{ij}\, b_j(O_{t+1})\, \beta_{t+1}(j)}{\sum_{i=1}^{N} \alpha_T(i)} \quad = p(X_t = i, X_{t+1} = j \mid O_{1:T}, \lambda)$$

Interpretation: Given the entire observation sequence and current estimate of the HMM, what is the probability that at time (t) the hidden state is (X_t=i) && at time (t+1) the hidden state is (X_{t+1}=j)?

2) **Gamma Function** (Marginalizing out X_{t+1})

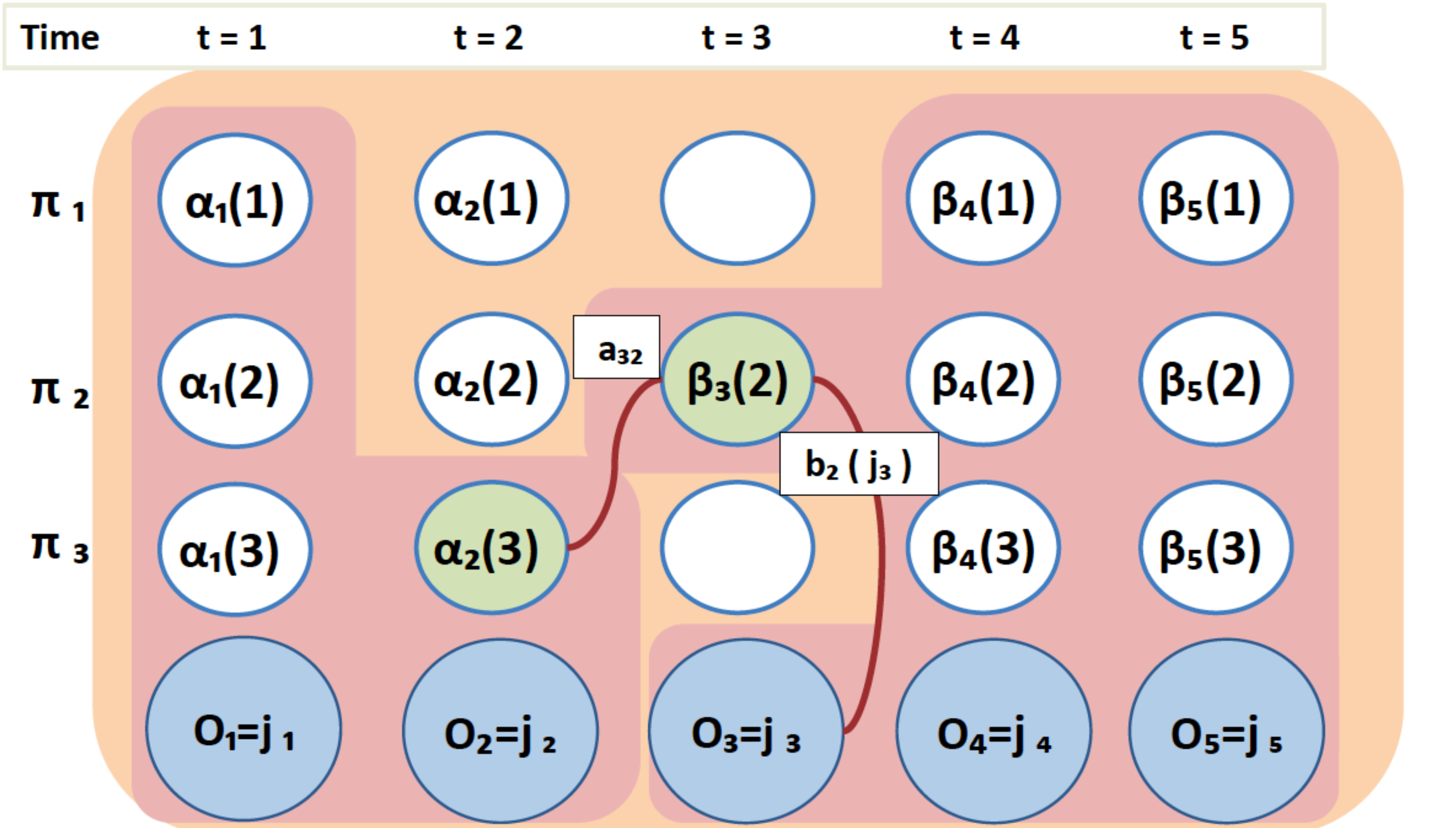
$$\gamma_t(i) = \sum_{j=1}^{N} \gamma_t(i,j) \quad = p(X_t = i \mid O_{1:T}, \lambda)$$

Interpretation: Given the entire observation sequence and current estimate of the HMM, what is the probability that at time (t) the hidden state is (X_t=i)?



$$P(X_2 = 3 , X_3 = 2 \mid O_{1..5}) = \frac{\alpha_2(3)\, a_{32}\, b_2 (j_3)\, \beta_3(2)}{\sum_i \alpha_5 (i)}$$

Normalization



$$\Upsilon_2(3,2) = \frac{\alpha_2(3)\, a_{32}\, b_2(j_3)\, \beta_3(2)}{\sum_i \alpha_5(i)}$$

GAMMA CALCULATIONS: (1)

1) **Di – Gamma Function**

$$\gamma_t(i,j) = \frac{\alpha_t(i)\, a_{ij}\, b_j(O_{t+1})\, \beta_{t+1}(j)}{\sum_{i=1}^{N} \alpha_T(i)} \quad = p(X_t = i, X_{t+1} = j | O_{1:T}, \lambda)$$

Interpretation: Given the entire observation sequence and current estimate of the HMM, what is the probability that at time (t) the hidden state is (X_t=i) && at time (t+1) the hidden state is (X_{t+1}=j)?

2) **Gamma Function** (Marginalizing out X_{t+1})

$$\gamma_t(i) = \sum_{j=1}^{N} \gamma_t(i,j) \quad = p(X_t = i | O_{1:T}, \lambda)$$

Interpretation: Given the entire observation sequence and current estimate of the HMM, what is the probability that at time (t) the hidden state is (X_t=i)?

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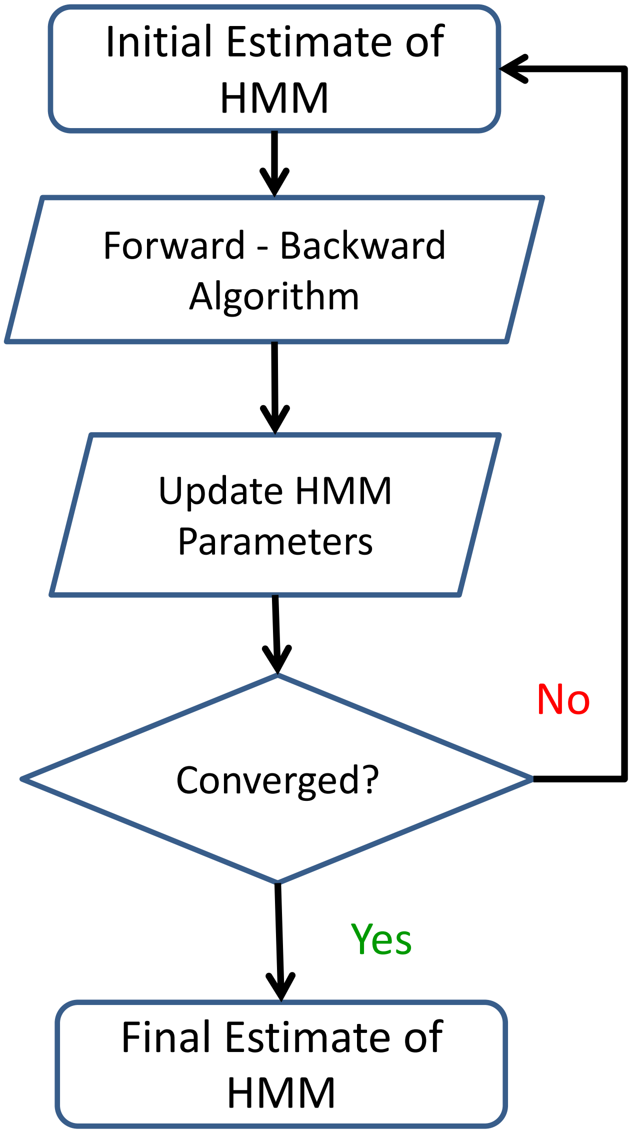
BAUM-WELCH ALGORITHM:



- Given an observation sequence $O_{1:T}$, the number of states, N, and the number of observation outcomes, M.

1. Initialize $\lambda=(A,B,\pi)$
2. Compute $\alpha_t(i)$ ✓, $\beta_t(k)$ ✓, $\gamma_t(i,j)$ ✓ and $\gamma_t(i)$ ✓
3. Re-estimate the model $\lambda=(A,B,\pi)$
4. Repeat from 2 until $p(O|\lambda)$ levels out

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GAMMA CALCULATIONS: (2)

A) Transition estimates

$$a_{ij} = \frac{\sum_{t=1}^{T-1} \gamma_t(i,j)}{\sum_{t=1}^{T-1} \gamma_t(i)} \quad \forall \ i,j=1,\ldots,N \quad = \frac{\text{E \{\# of transitions from state (i) to state (j)\}}}{\text{E \{\# of transitions from state (i) to state (don't care)\}}}$$

B) Emission estimates

$$b_j(k) = \frac{\sum_{\substack{t=1,2,\ldots,T-1 \\ O_t=k}} \gamma_t(i)(j)}{\sum_{t=1}^{T-1} \gamma_t(i)(j)} \quad \forall \quad j=1,\ldots,N, \quad k=1,\ldots,K$$

$$= \frac{\text{E \{\# of emission (k) from state (i)\}}}{\text{E \{\# of transitions from state (i) to state (don't care)\}}}$$

q) Initial state probabilities

$$\pi_i = \gamma_1(i) \quad \forall \quad i=1,\ldots,N$$

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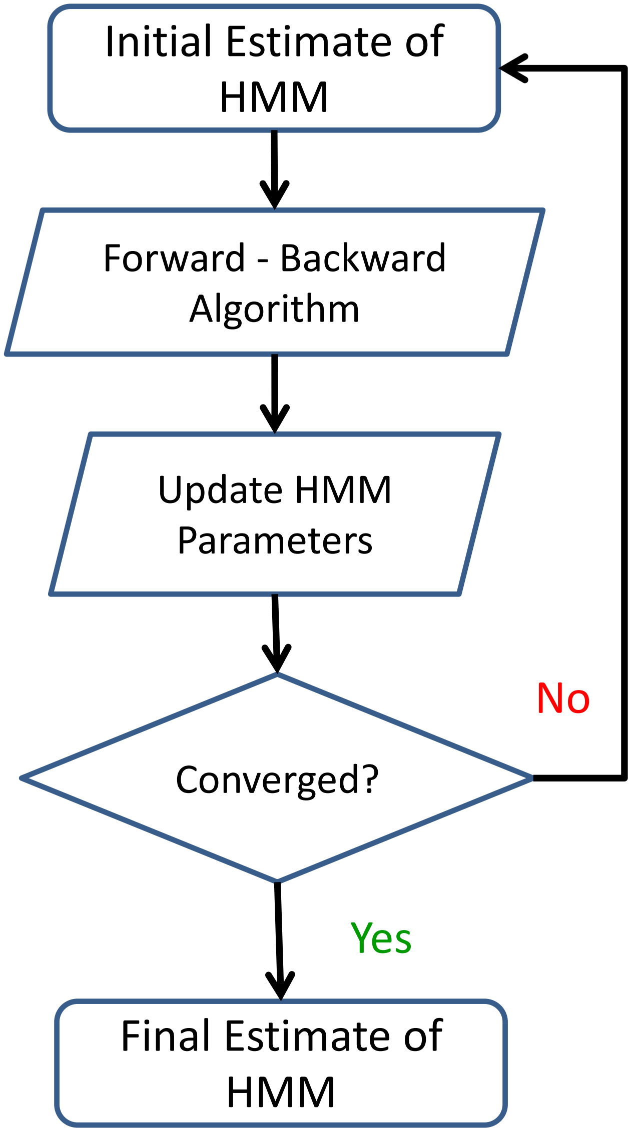
BAUM-WELCH ALGORITHM:



- Given an observation sequence $O_{1:T}$, the number of states, N, and the number of observation outcomes, M.

1. Initialize $\lambda=(A,B,\pi)$
2. Compute $\alpha_t(i)$, $\beta_t(k)$, $\gamma_t(i,j)$ and $\gamma_t(i)$
3. Re-estimate the model $\lambda=(A,B,\pi)$
4. Repeat from 2 until $p(O|\lambda)$ levels out

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GENERAL TIPS AND POINTERS:

1) Using natural logarithms and summations instead of direct products. (**Stamp-implementation tutorial)**

2) Forward algorithm: The $\alpha_t(i)$ s can be normalized across hidden states at every time instant for numerical stability. Algorithm is modified slightly!

3) Type of matrix initializations are important for convergence using Baum-Welch Algorithm. Types = Flat / Random Row-Stochastic / Suggestive

4) ASSIGNMENT THINK → (Optional – NOT FOR PASSING GRADE!)

   - Why do you need to compare matrices in A1?
   - Why is this not a trivial problem?
   - How will you solve it?

QUESTIONS?