Independent Set
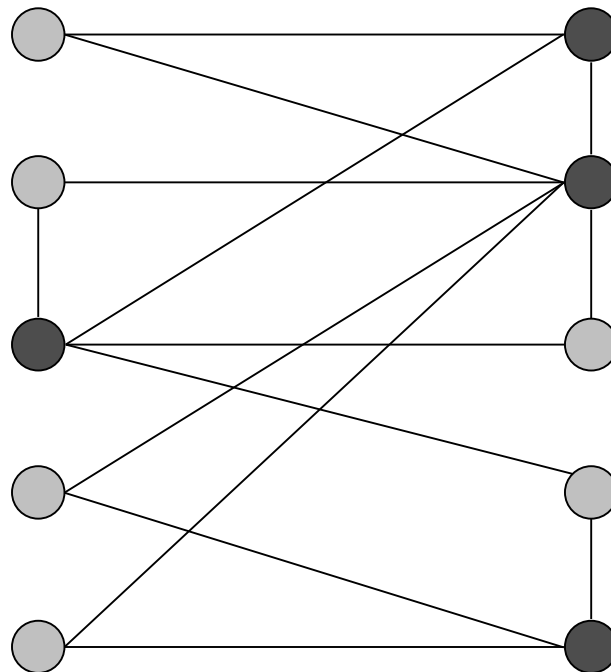
INDEPENDENT SET: Given a graph G = (V, E) and an integer k, is there a subset of vertices S ⊆ V such that |S| ≥ k, and for each edge at most one of its endpoints is in S?

Ex. Is there an independent set of size ≥ 6? Yes.
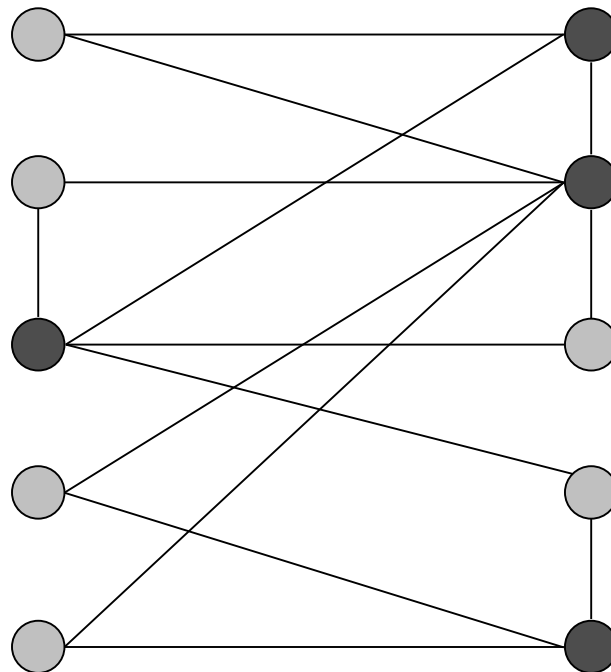Ex. Is there an independent set of size ≥ 7? No.

independent set

Vertex Cover

VERTEX COVER: Given a graph G = (V, E) and an integer k, is there a subset of vertices S ⊆ V such that |S| ≤ k, and for each edge, at least one of its endpoints is in S?

Ex. Is there a vertex cover of size ≤ 4? Yes.
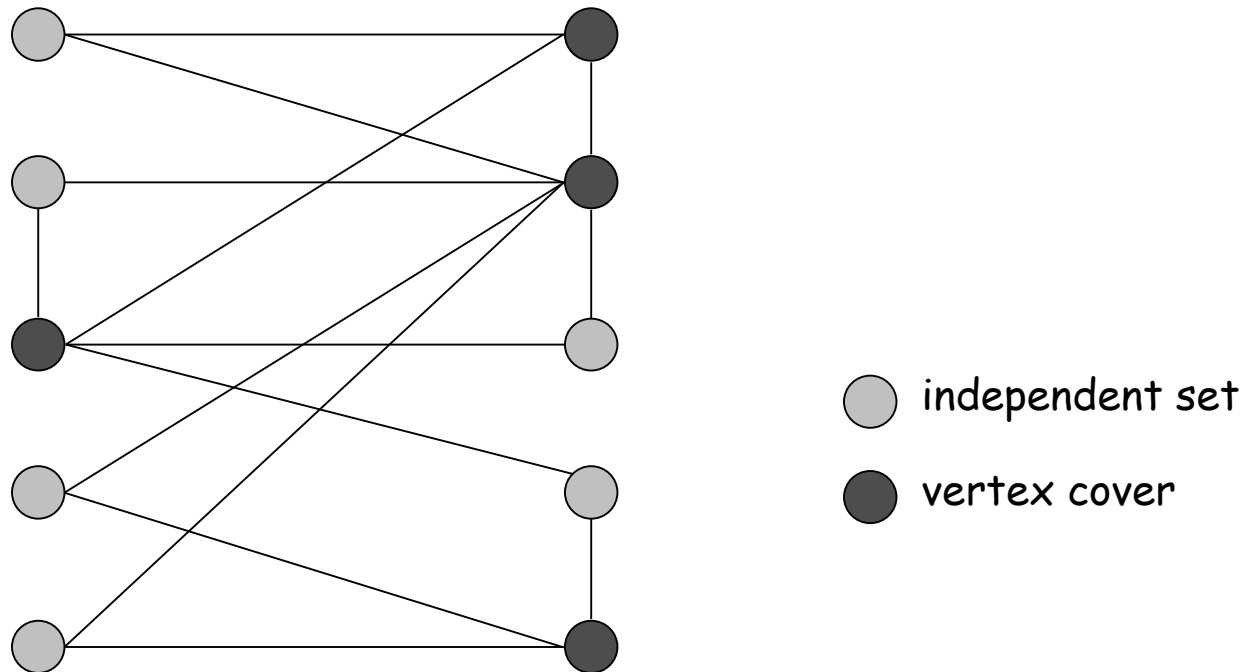Ex. Is there a vertex cover of size ≤ 3? No.

● vertex cover

Vertex Cover and Independent Set

Claim. VERTEX-COVER $\equiv_P$ INDEPENDENT-SET.
Pf. We show S is an independent set iff V − S is a vertex cover.

independent set

vertex cover

Vertex Cover and Independent Set

Claim. VERTEX-COVER $\equiv_P$ INDEPENDENT-SET.

Pf. We show S is an independent set iff V − S is a vertex cover.

$\Rightarrow$

- Let S be any independent set.
- Consider an arbitrary edge (u, v).
- S independent $\Rightarrow$ u $\notin$ S or v $\notin$ S $\Rightarrow$ u $\in$ V − S or v $\in$ V − S.
- Thus, V − S covers (u, v).

$\Leftarrow$

- Let V − S be any vertex cover.
- Consider two nodes u $\in$ S and v $\in$ S.
- Observe that (u, v) $\notin$ E since V − S is a vertex cover.
- Thus, no two nodes in S are joined by an edge $\Rightarrow$ S independent set. ▪

Set Cover

SET COVER: Given a set U of elements, a collection $S_1, S_2, \ldots, S_m$ of subsets of U, and an integer k, does there exist a collection of $\leq$ k of these sets whose union is equal to U?

Sample application.
- m available pieces of software.
- Set U of n capabilities that we would like our system to have.
- The ith piece of software provides the set $S_i \subseteq U$ of capabilities.
- Goal: achieve all n capabilities using fewest pieces of software.

Ex:

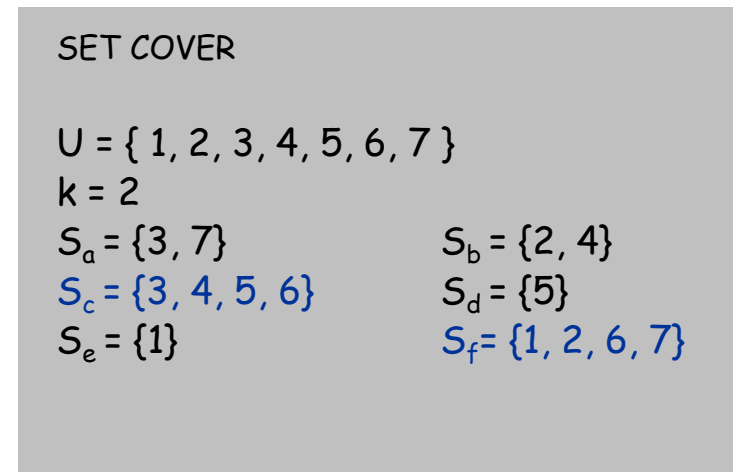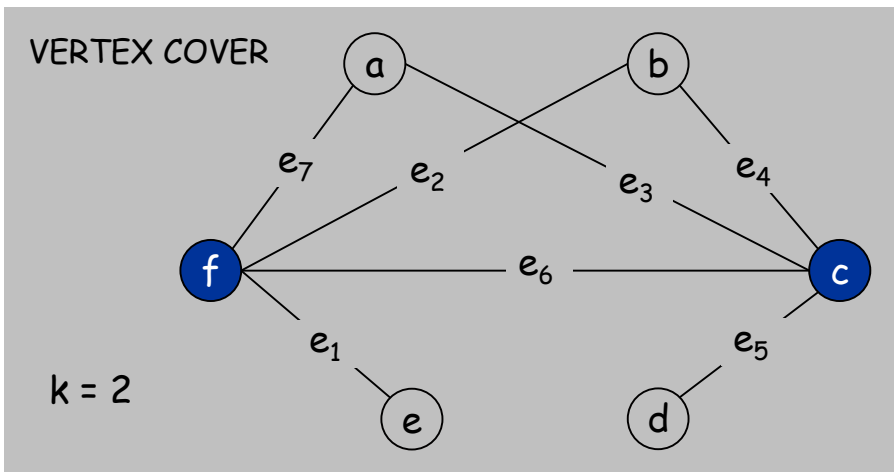|	
U = { 1, 2, 3, 4, 5, 6, 7 }	
k = 2	
S_1 = {3, 7}	S_4 = {2, 4}
S_2 = {3, 4, 5, 6}	S_5 = {5}
S_3 = {1}	S_6 = {1, 2, 6, 7}

Vertex Cover Reduces to Set Cover

Claim. VERTEX-COVER $\leq_P$ SET-COVER.

Pf. Given a VERTEX-COVER instance $G = (V, E)$, k, we construct a set cover instance whose size equals the size of the vertex cover instance.

Construction.

- Create SET-COVER instance:
  - $k = k$, $U = E$, $S_v = \{e \in E : e$ incident to $v\}$
- Set-cover of size $\leq k$ iff vertex cover of size $\leq k$. ▪

VERTEX COVER

$k = 2$

SET COVER

$U = \{1, 2, 3, 4, 5, 6, 7\}$
$k = 2$
$S_a = \{3, 7\}$ $S_b = \{2, 4\}$
$S_c = \{3, 4, 5, 6\}$ $S_d = \{5\}$
$S_e = \{1\}$ $S_f = \{1, 2, 6, 7\}$

Self-Reducibility

Decision problem. Does there $\textcolor{red}{\text{exist}}$ a vertex cover of size $\le$ k?
Search problem. $\textcolor{red}{\text{Find}}$ vertex cover of minimum cardinality.

Self-reducibility. Search problem $\le_P$ decision version.
- Applies to all (NP-complete) problems in this chapter.
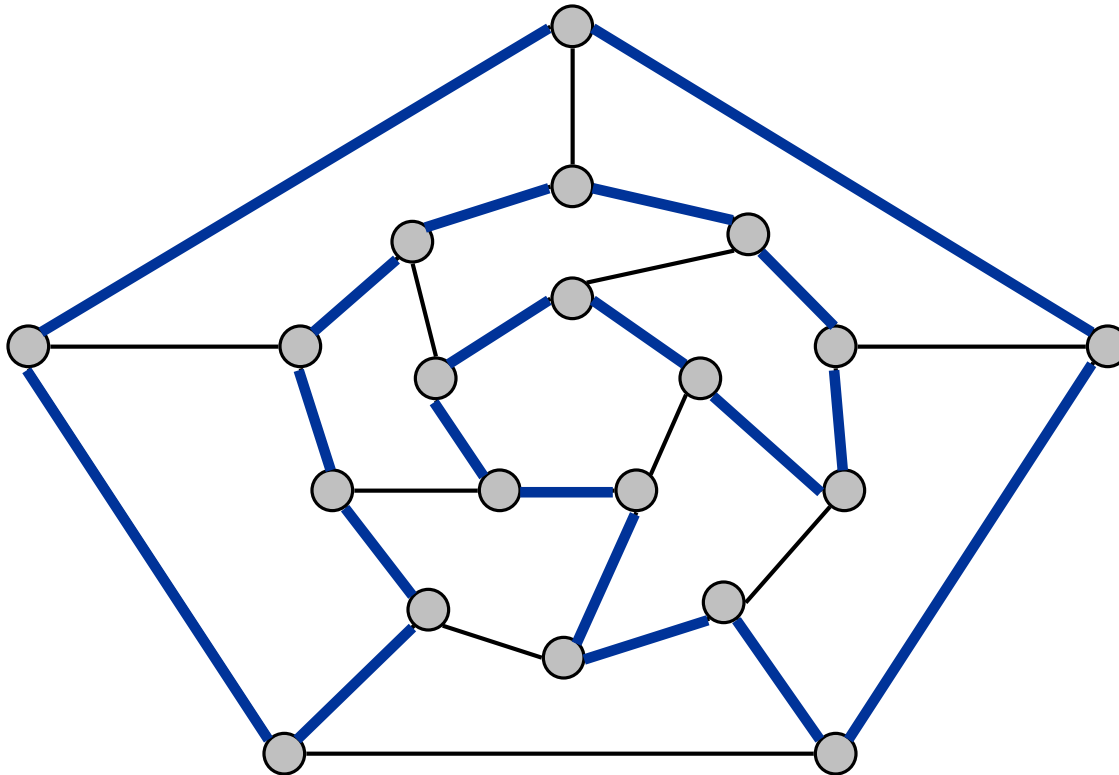- Justifies our focus on decision problems.

Ex: to find min cardinality vertex cover.
- (Binary) search for cardinality k* of min vertex cover.
- Find a vertex v such that $G - \{v\}$ has a vertex cover of size $\le$ k* - 1.
   - any vertex in any min vertex cover will have this property
- Include v in the vertex cover.
- Recursively find a min vertex cover in $G - \{v\}$.

↑

delete v and all incident edges

Hamiltonian Cycle

HAM-CYCLE: given an undirected graph G = (V, E), does there exist a simple cycle Γ that contains every node in V.

YES: vertices and faces of a dodecahedron.

Directed Hamiltonian Cycle

DIR-HAM-CYCLE: given a digraph $G = (V, E)$, does there exists a simple directed cycle Γ that contains every node in V?

Claim. DIR-HAM-CYCLE $\leq_P$ HAM-CYCLE.

Pf. Given a directed graph $G = (V, E)$, construct an undirected graph G' with 3n nodes.

G

G'
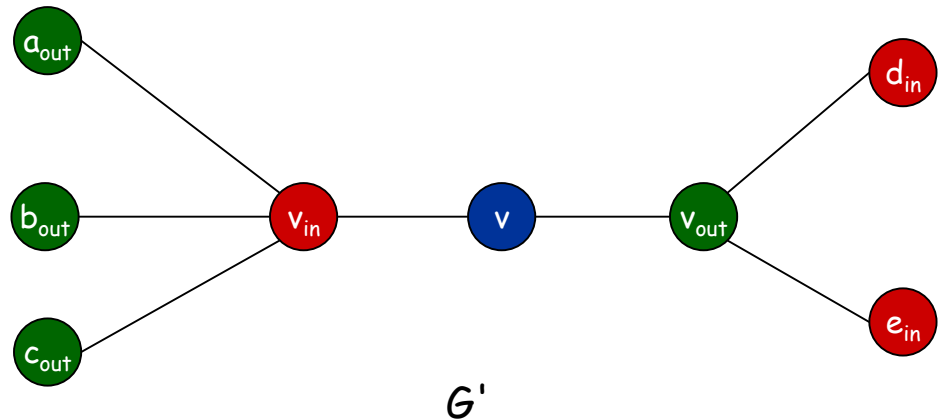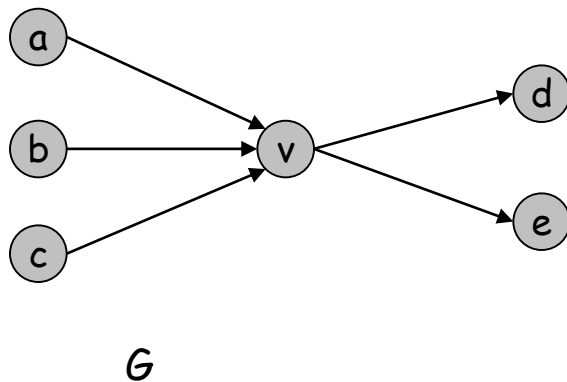
Directed Hamiltonian Cycle

Claim. G has a Hamiltonian cycle iff G' does.

Pf. $\Rightarrow$

- Suppose G has a directed Hamiltonian cycle Γ.
- Then G' has an undirected Hamiltonian cycle (same order).

Pf. $\Leftarrow$

- Suppose G' has an undirected Hamiltonian cycle Γ'.
- Γ' must visit nodes in G' using one of following two orders:

     …, B, G, R, B, G, R, B, G, R, B, …

     …, B, R, G, B, R, G, B, R, G, B, …

- Blue nodes in Γ' make up directed Hamiltonian cycle Γ in G, or reverse of one. ▪

Longest Path

SHORTEST-PATH. Given a digraph G = (V, E), does there exists a simple path of length at most k edges?

LONGEST-PATH. Given a digraph G = (V, E), does there exists a simple path of length at least k edges?

Claim. 3-SAT $\leq_P$ LONGEST-PATH.

Pf. Redo proof for DIR-HAM-CYCLE done in class, ignoring back-edge from t to s.

Traveling Salesperson Problem

TSP. Given a set of n cities and a pairwise distance function d(u, v), is there a tour of length $\leq$ D?

HAM-CYCLE: given a graph G = (V, E), does there exists a simple cycle that contains every node in V?

Claim. HAM-CYCLE $\leq_P$ TSP.
Pf.

- Given instance G = (V, E) of HAM-CYCLE, create n cities with distance function

$$d(u, v) = \begin{cases} 1 & \text{if } (u, v) \in E \\ 2 & \text{if } (u, v) \notin E \end{cases}$$

- TSP instance has tour of length $\leq$ n iff G is Hamiltonian. ▪

Remark. TSP instance in reduction satisfies $\triangle$-inequality.

Partition

SUBSET-SUM. Given natural numbers $w_1, ..., w_n$ and an integer W, is there a subset that adds up to exactly W?

PARTITION. Given natural numbers $v_1, ..., v_m$, can they be partitioned into two subsets that add up to the same value?

$$\frac{1}{2} \Sigma_i \, v_i$$

Claim. SUBSET-SUM $\leq_P$ PARTITION.

Pf. Let $W, w_1, ..., w_n$ be an instance of SUBSET-SUM.

- Create instance of PARTITION with $m = n+2$ elements.
    - $v_1 = w_1, v_2 = w_2, ..., v_n = w_n,$ $v_{n+1} = 2 \Sigma_i \, w_i - W,$ $v_{n+2} = \Sigma_i \, w_i + W$

- There exists a subset that sums to W iff there exists a partition since two new elements cannot be in the same partition. ▪

$v_{n+1} = 2 \Sigma_i \, w_i - W$	W	subset A
$v_{n+2} = \Sigma_i \, w_i + W$	$\Sigma_i \, w_i - W$	subset B