CS3230 Lecture 3

NUS National University of Singapore
School of Computing

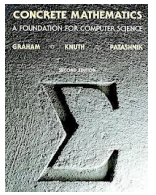
**"Asymptotics, Summation,
Divide & Conquer Algorithms,
Recurrences and Master Theorem"**

- **Lecture Topics and Readings**
  - **Asymptotic, Summation** [CLRS]-C3
  - **Divide and Conquer Algorithms** [CLRS]-C2
  - **Recurrences, Master Theorem** [CLRS]-C4

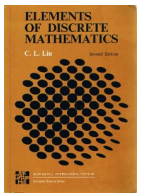
***Algorithms & Discrete Math
have very happy, stable marriage!***

Early pioneers of Math for CS

**Don Knuth
Stanford**



**C. L. Liu,
MIT, UIUC,
NTHU (tw)**



Elements of Discrete Math, 1977

"Elements of Discrete Mathematics," C. L. Liu, McGraw-Hill, (1977), has the most important topics in DM for CS.

Relations & Functions

Graphs & Planar Graphs

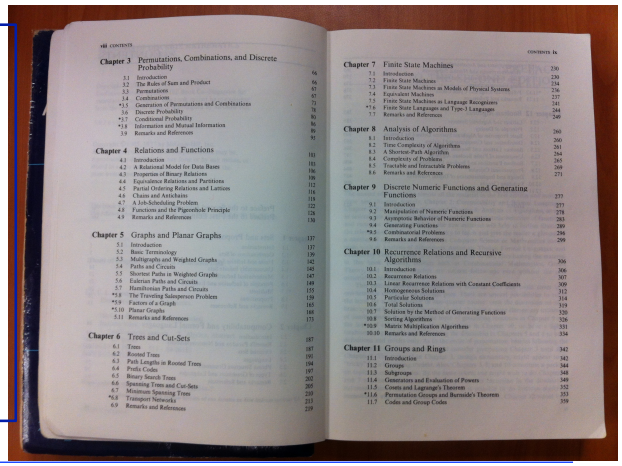
Trees and Cut-Sets

Finite State Machines

Analysis of Algorithms

Recurrence Relations

. . .



David Plaisted

**David Plaisted
UIUC 1978-84
UNC, 1985--**

Spring 1980, LeongHW took CS373 by Plaisted;

Summer 1984, Plaisted: UIUC ----> UNC

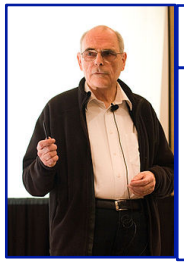
Spring 1985 @UIUC: CS373 by Plaisted

Dave Liu recommend PhD student LeongHW to the Dept Head for CS373

Herbrand Award, 2010
(for distinguished contribution to Automated Reasoning)

http://www.cs.miami.edu/~geoff/Conferences/CADE/HerbrandAward.html

Volker Strassen



Volker Strassen

Knuth Prize, 2009
"seminal and influential contributions
To the design and analysis
of efficient algorithms"

http://en.wikipedia.org/wiki/Volker_Strassen

Thank you.

Q & A

NUS National University of Singapore
School of Computing

CS3230 Lecture 3

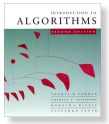
NUS National University of Singapore
School of Computing

**"Asymptotics, Summation,
Divide & Conquer Algorithms,
Recurrences and Master Theorem"**

- **Lecture Topics and Readings**
  - **Asymptotics, Summation** **[CLRS]-C3**
  - **Divide and Conquer Algorithms** **[CLRS]-C2**
  - **Recurrences, Master Theorem** **[CLRS]-C4**

***Algorithms & Discrete Math
have very happy, stable marriage!***

***Start by
revising Mergesort
(Focus on Analysis)***

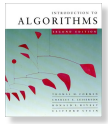


Merge sort

MERGE-SORT $A[1 \ldots n]$

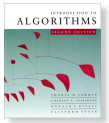
1. If $n = 1$, done.
2. Recursively sort $A[\,1 \ldots \lceil n/2 \rceil\,]$ and $A[\,\lceil n/2 \rceil + 1 \ldots n\,]$.
3. "***Merge***" the 2 sorted lists.

Key subroutine: **MERGE**



Merging two sorted arrays

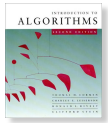
20 12
13 11
7 9
2 1



Merging two sorted arrays

20 12
13 11
7 9
2 1

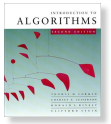
1



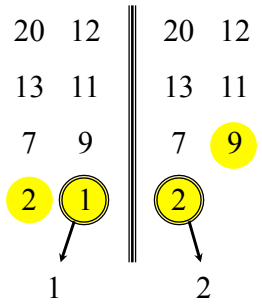
Merging two sorted arrays

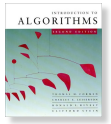
20 12 | 20 12
13 11 | 13 11
7 9 | 7 9
2 1 | 2

1

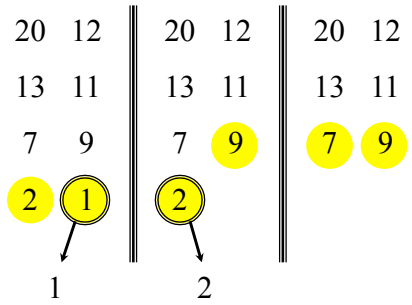


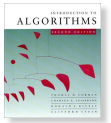
Merging two sorted arrays



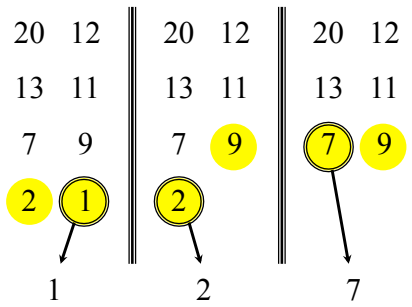


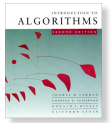
Merging two sorted arrays



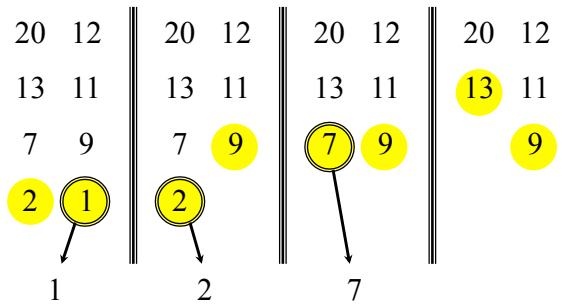


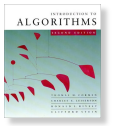
Merging two sorted arrays



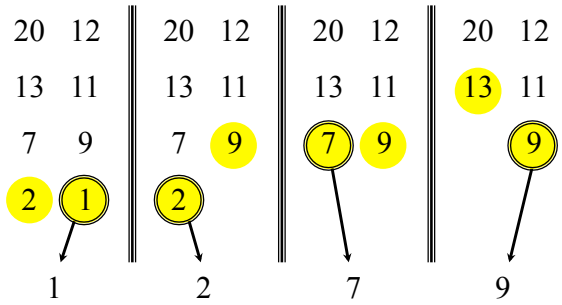


Merging two sorted arrays

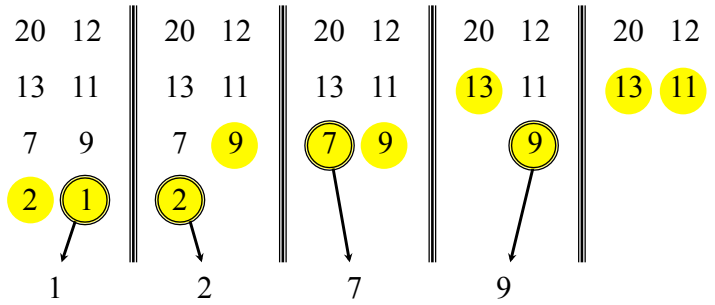




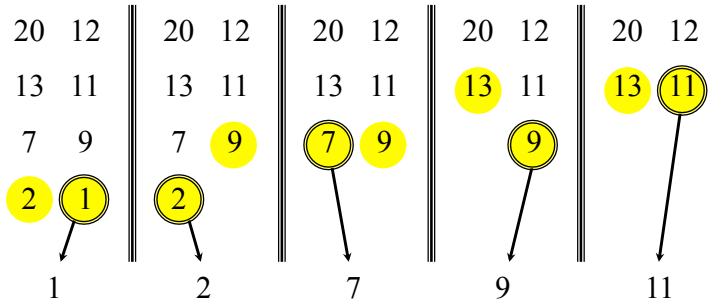
Merging two sorted arrays

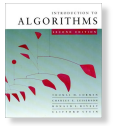


Merging two sorted arrays

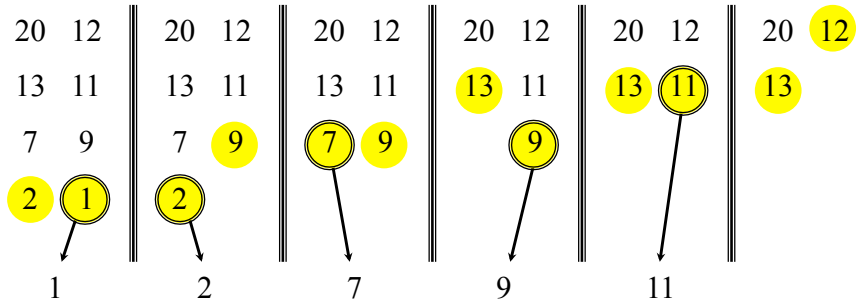


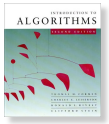
Merging two sorted arrays



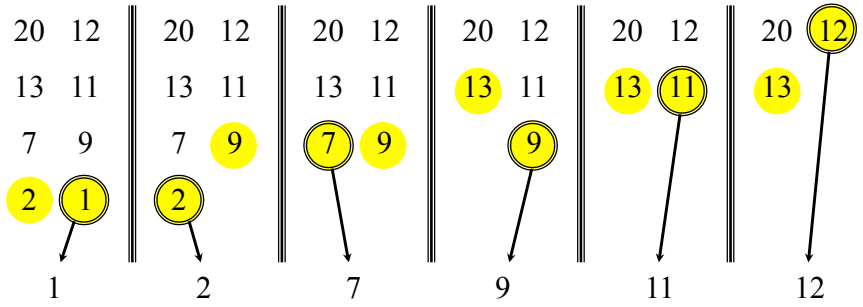


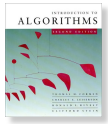
Merging two sorted arrays



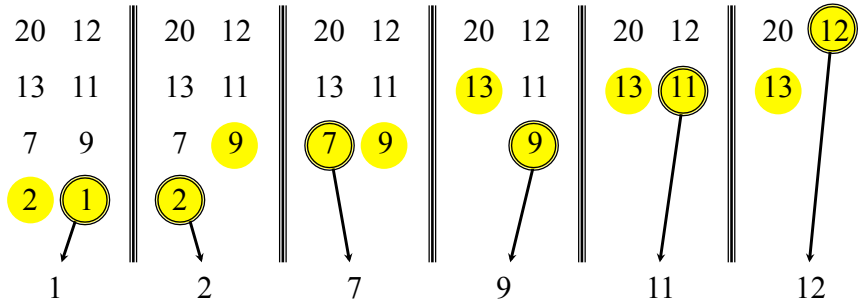


Merging two sorted arrays

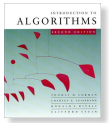




Merging two sorted arrays



Time = $\Theta(n)$ to merge a total of n elements (linear time).



Analyzing merge sort

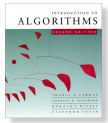
$T(n)$
$\Theta(1)$
$2T(n/2)$
$\Theta(n)$

Abuse

MERGE-SORT $A[1 \ldots n]$

1. If $n = 1$, done.
2. Recursively sort $A[\,1 \ldots \lceil n/2 \rceil\,]$ and $A[\,\lceil n/2 \rceil + 1 \ldots n\,]$.
3. ***"Merge"*** the 2 sorted lists

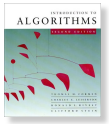
Sloppiness: Should be $T(\,\lceil n/2 \rceil\,) + T(\,\lfloor n/2 \rfloor\,)$, but it turns out not to matter asymptotically.



Recurrence for merge sort

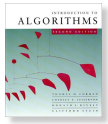
$$T(n) = \begin{cases} \Theta(1) \text{ if } n = 1; \\ 2T(n/2) + \Theta(n) \text{ if } n > 1. \end{cases}$$

- We shall usually omit stating the base case when $T(n) = \Theta(1)$ for sufficiently small n, but only when it has no effect on the asymptotic solution to the recurrence.



Solving recurrences

- The analysis of merge sort required us to solve a recurrence.
- Recurrences are like solving integrals, differential equations, etc.
  - Learn a few tricks.
- Applications of recurrences to divide-and-conquer algorithms.



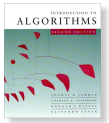
Substitution method

Recursion Tree method
basis of the Master Theorem

The most general method:

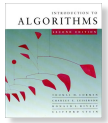
1. ***Guess*** the form of the solution.
2. ***Verify*** by induction.
3. ***Solve*** for constants.

Not covered in CS3230; those interested see example and details in CLRS



Recursion tree

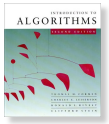
Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.



Recursion tree

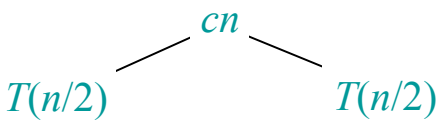
Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

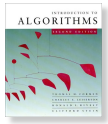
$T(n)$



Recursion tree

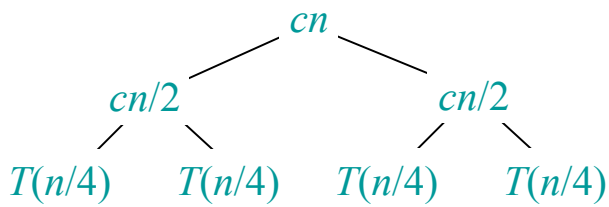
Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

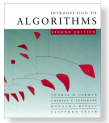




Recursion tree

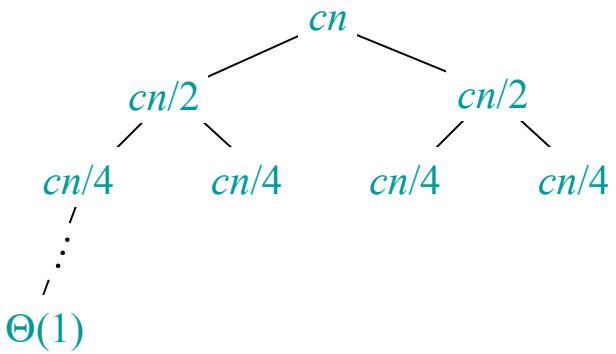
Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

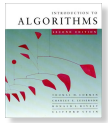




Recursion tree

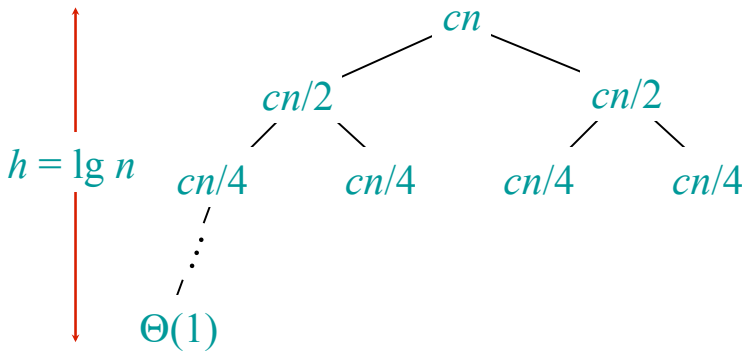
Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

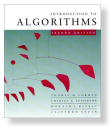




Recursion tree

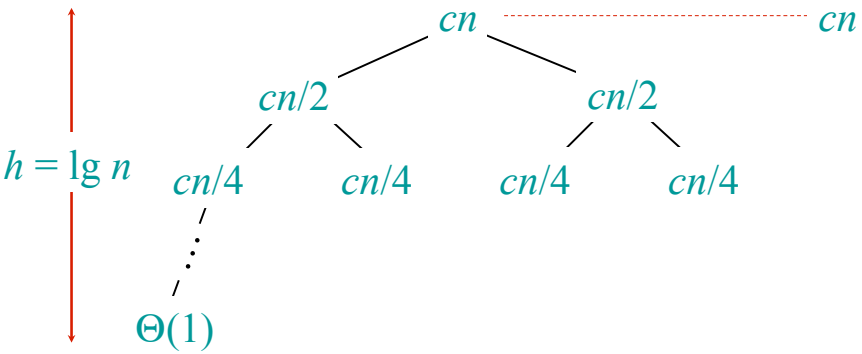
Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

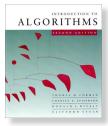




Recursion tree

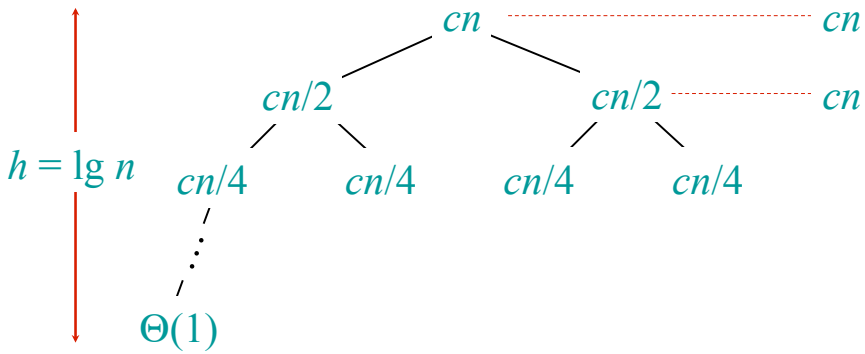
Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

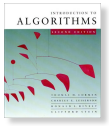




Recursion tree

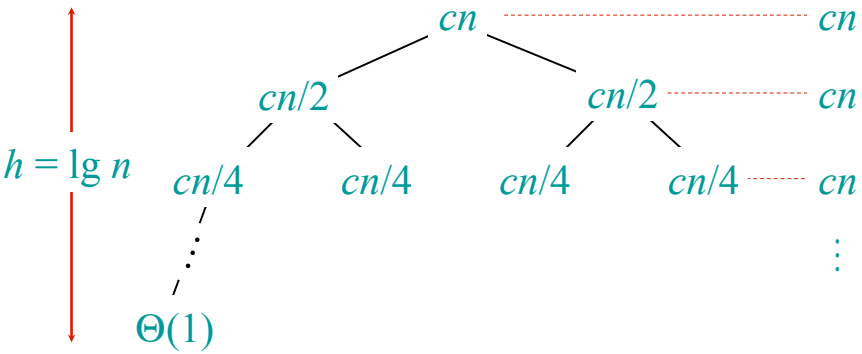
Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

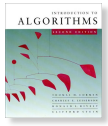




Recursion tree

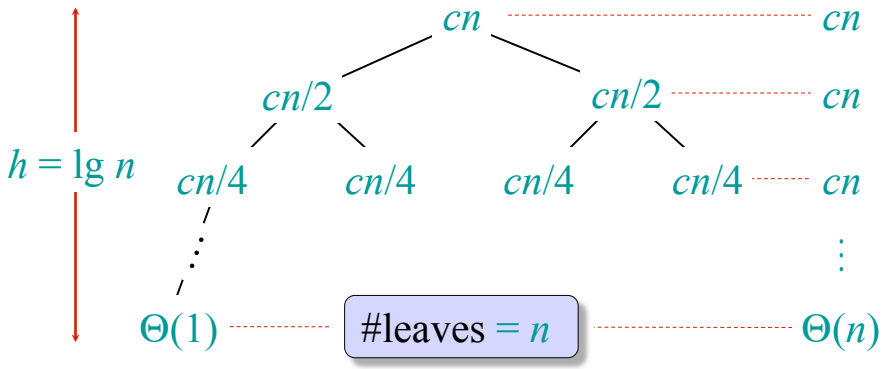
Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

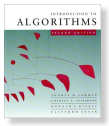




Recursion tree

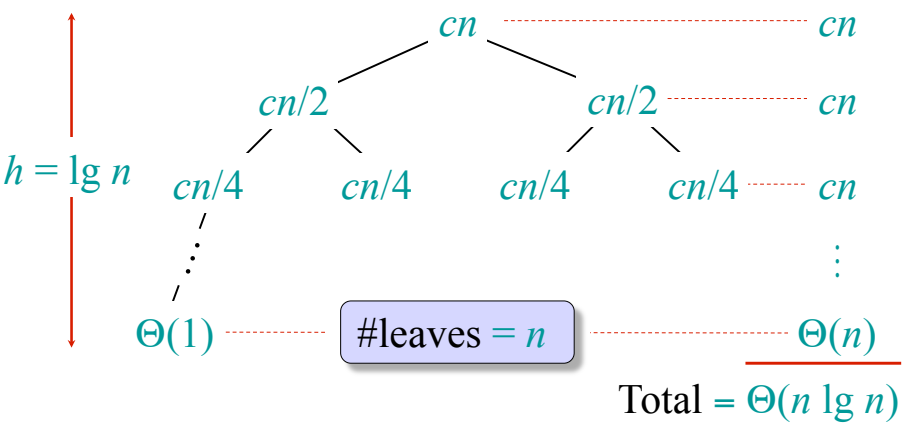
Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

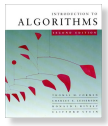




Recursion tree

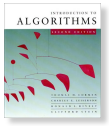
Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.





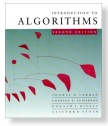
Recursion-tree method

- A recursion tree models the costs (time) of a recursive execution of an algorithm.
- The recursion-tree method can be unreliable, just like any method that uses ellipses (…).
- The recursion-tree method promotes intuition, however.
- The recursion tree method is good for generating guesses for the substitution method.



Example of recursion tree

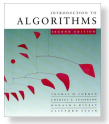
Solve $T(n) = T(n/4) + T(n/2) + n^2$:



Example of recursion tree

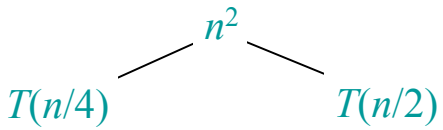
Solve $T(n) = T(n/4) + T(n/2) + n^2$:

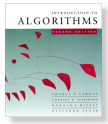
$T(n)$



Example of recursion tree

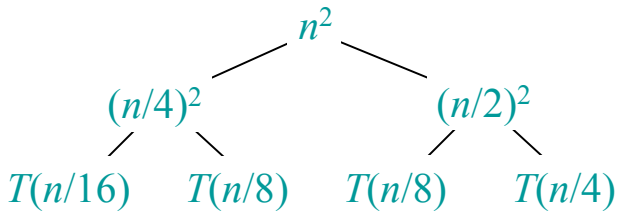
Solve $T(n) = T(n/4) + T(n/2) + n^2$:

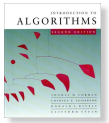




Example of recursion tree

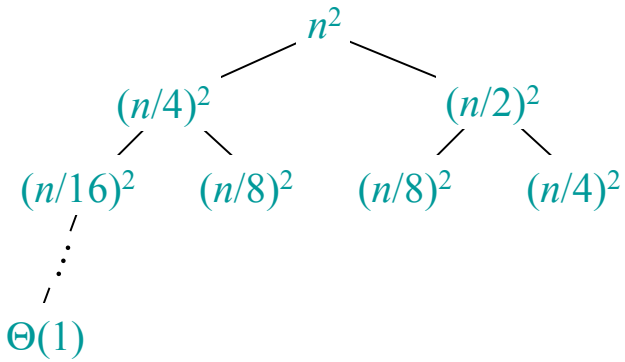
Solve $T(n) = T(n/4) + T(n/2) + n^2$:

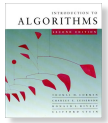




Example of recursion tree

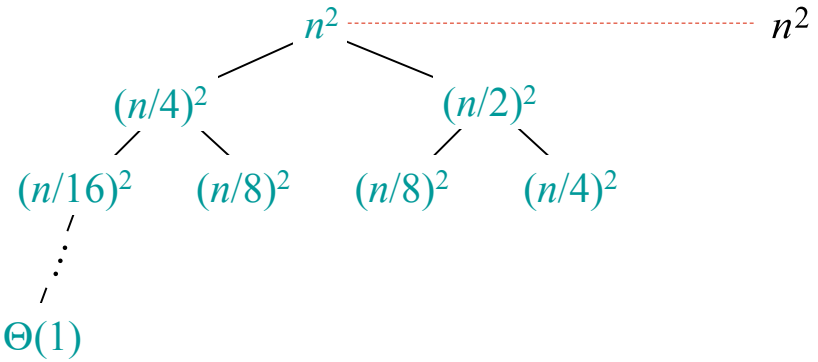
Solve $T(n) = T(n/4) + T(n/2) + n^2$:

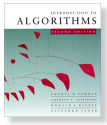




Example of recursion tree

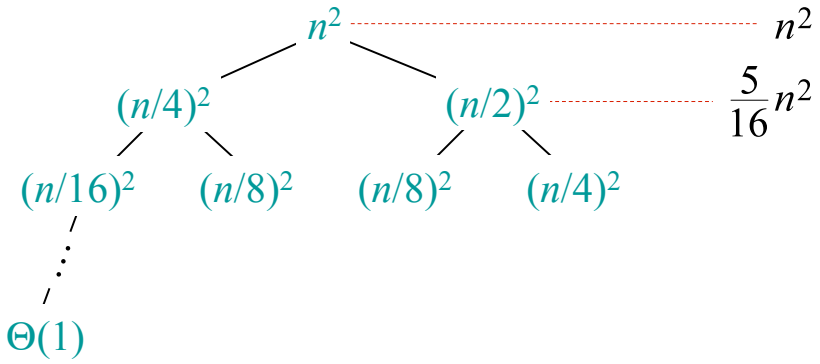
Solve $T(n) = T(n/4) + T(n/2) + n^2$:

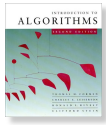




Example of recursion tree

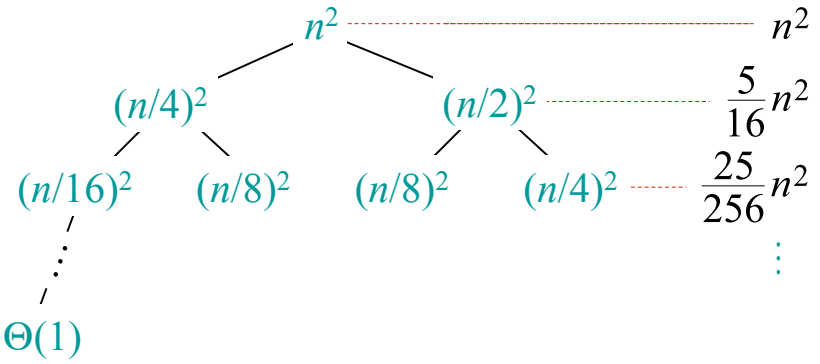
Solve $T(n) = T(n/4) + T(n/2) + n^2$:

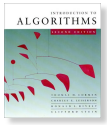




Example of recursion tree

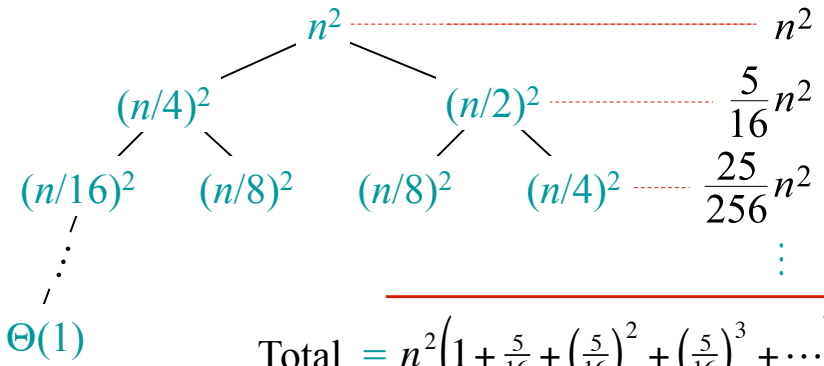
Solve $T(n) = T(n/4) + T(n/2) + n^2$:



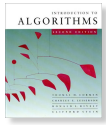


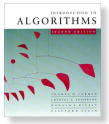
Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



$$\text{Total} = n^2\left(1 + \tfrac{5}{16} + \left(\tfrac{5}{16}\right)^2 + \left(\tfrac{5}{16}\right)^3 + \cdots\right)$$
$$= \Theta(n^2) \quad \textit{geometric series}$$



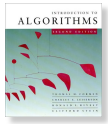


The master method

The master method applies to recurrences of the form

$$T(n) = a\,T(n/b) + f(n)\ ,$$

where $a \geq 1$, $b > 1$, and f is asymptotically positive.

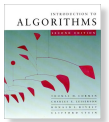


Three common cases

Compare $f(n)$ with $n^{\log_b a}$:

1. $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially slower than $n^{\log_b a}$ (by an n^{ε} factor).

   Solution: $T(n) = \Theta(n^{\log_b a})$.



Three common cases

Compare $f(n)$ with $n^{\log_b a}$:

1. $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially slower than $n^{\log_b a}$ (by an n^{ε} factor).

   Solution: $T(n) = \Theta(n^{\log_b a})$.

2. $f(n) = \Theta(n^{\log_b a} \lg^k n)$ for some constant $k \geq 0$.
   - $f(n)$ and $n^{\log_b a}$ grow at similar rates.

   Solution: $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$.

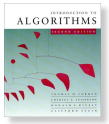
Three common cases (cont.)

Compare $f(n)$ with $n^{\log_b a}$:

3. $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially faster than $n^{\log_b a}$ (by an n^{ε} factor),

   and $f(n)$ satisfies the ***regularity condition*** that $a f(n/b) \leq c f(n)$ for some constant $c < 1$.

   Solution: $T(n) = \Theta(f(n))$.



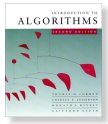
Examples

Ex. $T(n) = 4T(n/2) + n$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n.$

Case 1: $f(n) = O(n^{2-\varepsilon})$ for $\varepsilon = 1$.

$\therefore\ T(n) = \Theta(n^2).$



Examples

Ex. $T(n) = 4T(n/2) + n$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n.$

Case 1: $f(n) = O(n^{2-\varepsilon})$ for $\varepsilon = 1$.

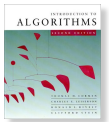
$\therefore\ T(n) = \Theta(n^2).$

Ex. $T(n) = 4T(n/2) + n^2$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n^2.$

Case 2: $f(n) = \Theta(n^2 \lg^0 n)$, that is, $k = 0$.

$\therefore\ T(n) = \Theta(n^2 \lg n).$



Examples

Ex. $T(n) = 4T(n/2) + n^3$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n^3.$

Case 3: $f(n) = \Omega(n^{2+\varepsilon})$ for $\varepsilon = 1$

and $4(n/2)^3 \le cn^3$ (reg. cond.) for $c = 1/2$.

$\therefore\ T(n) = \Theta(n^3).$

Examples

Ex. $T(n) = 4T(n/2) + n^3$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n^3.$

Case 3: $f(n) = \Omega(n^{2+\varepsilon})$ for $\varepsilon = 1$

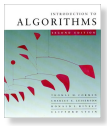
and $4(n/2)^3 \le cn^3$ (reg. cond.) for $c = 1/2$.

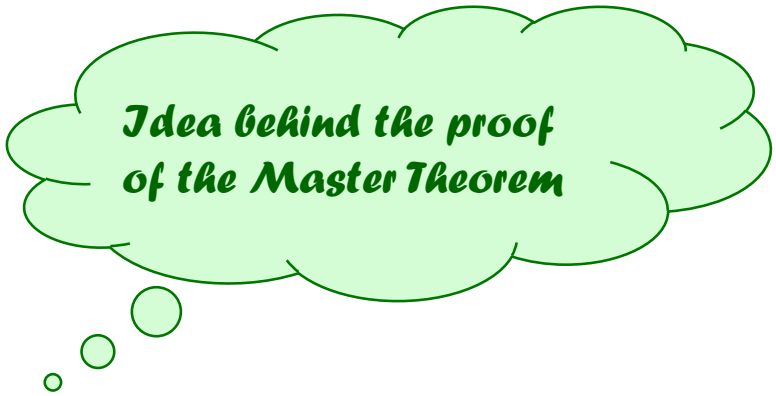
$\therefore\ T(n) = \Theta(n^3).$

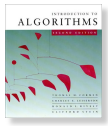
Ex. $T(n) = 4T(n/2) + n^2/\lg n$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n^2/\lg n.$

Master method does not apply. In particular, for every constant $\varepsilon > 0$, we have $n^\varepsilon = \omega(\lg n)$.

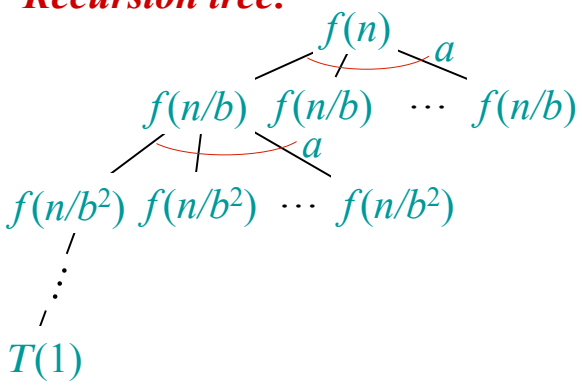


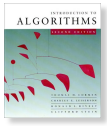




Idea of master theorem

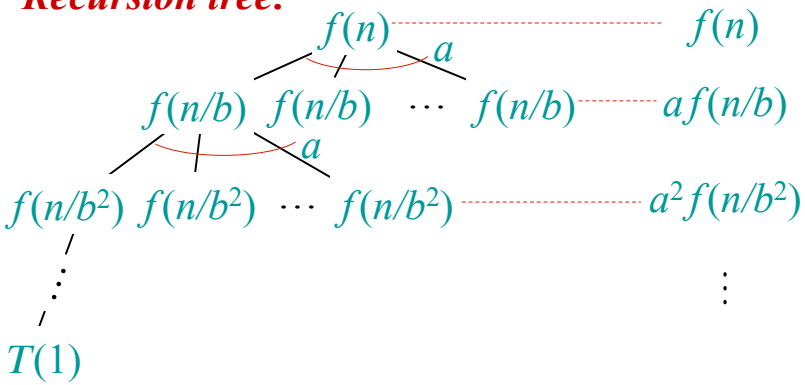
Recursion tree:

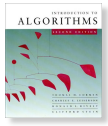




Idea of master theorem

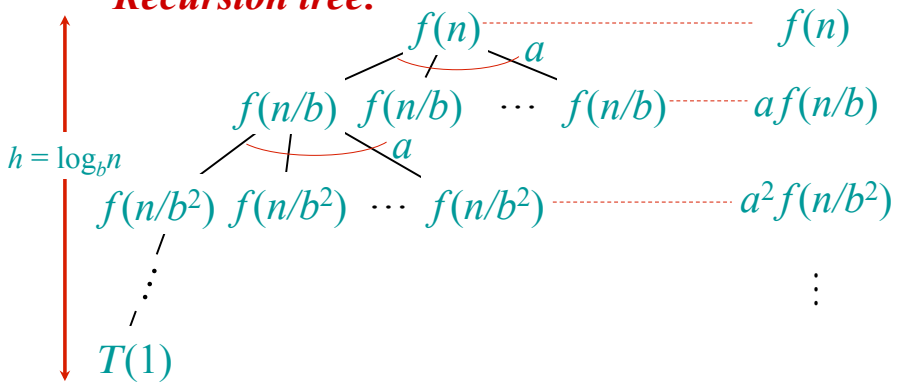
Recursion tree:

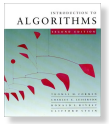




Idea of master theorem

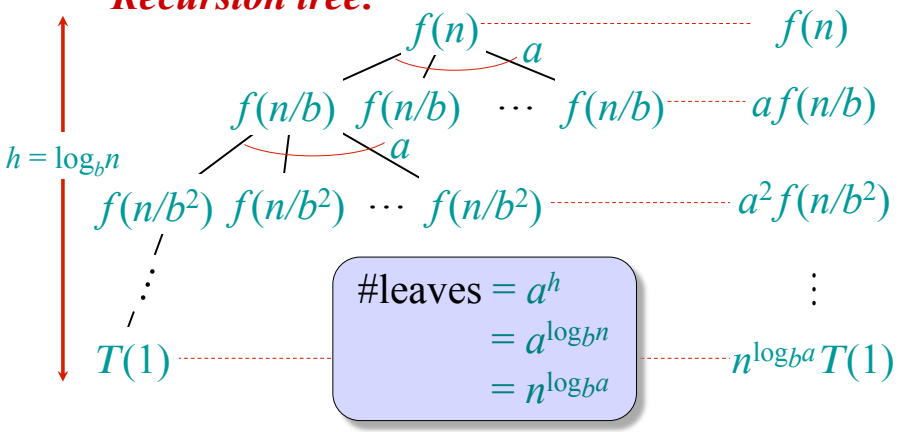
Recursion tree:

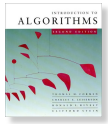




Idea of master theorem

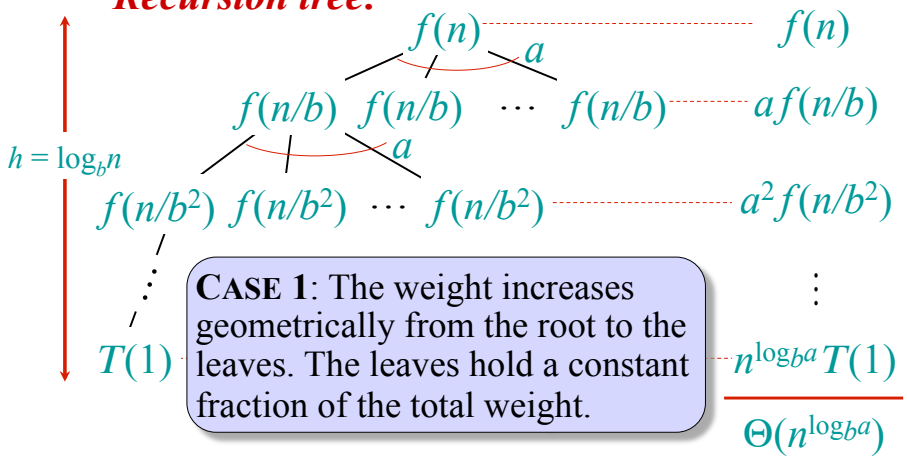
Recursion tree:

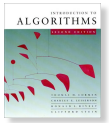




Idea of master theorem

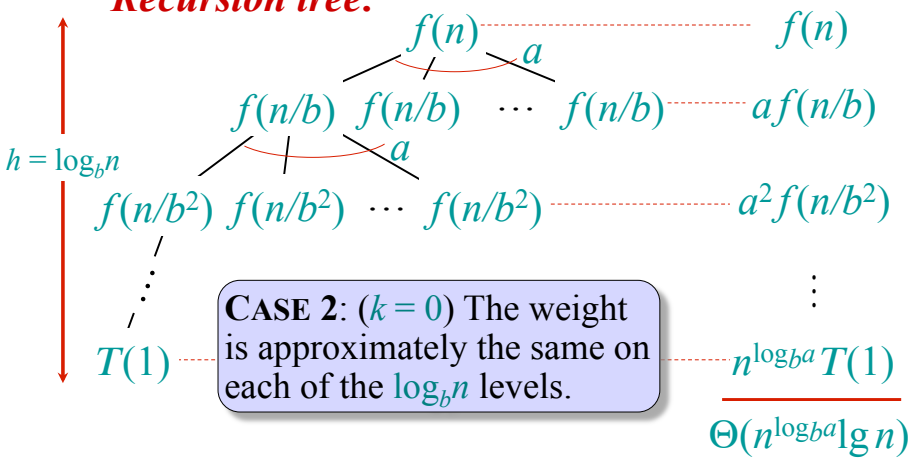
Recursion tree:

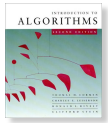




Idea of master theorem

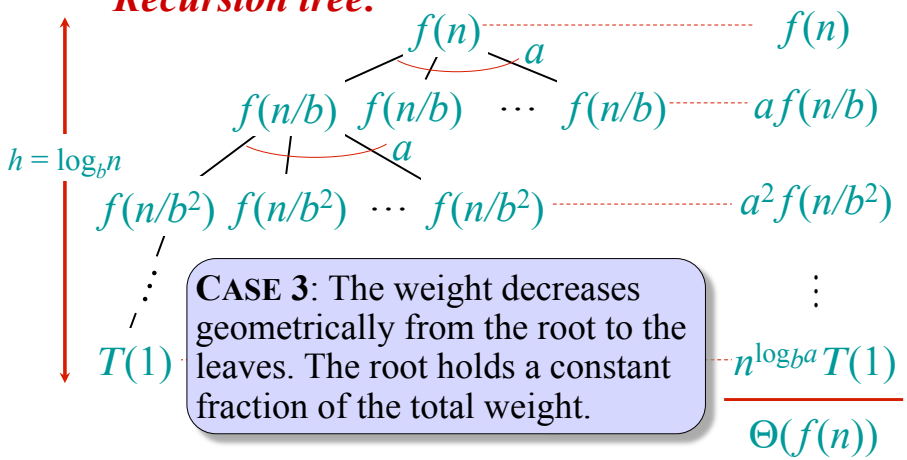
Recursion tree:

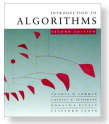


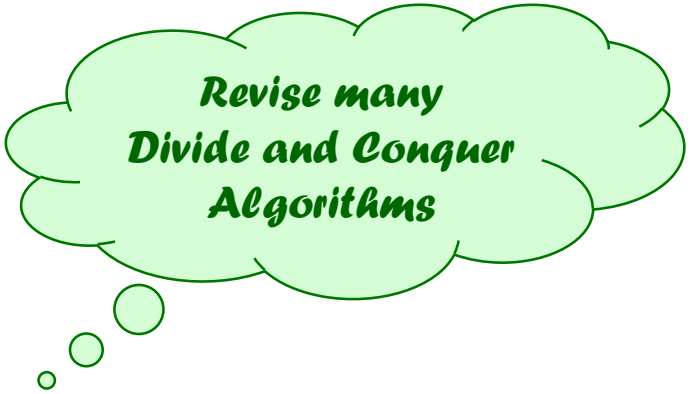


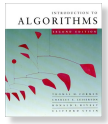
Idea of master theorem

Recursion tree:



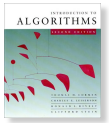






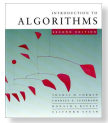
The divide-and-conquer design paradigm

1. ***Divide*** the problem (instance) into subproblems.
2. ***Conquer*** the subproblems by solving them recursively.
3. ***Combine*** subproblem solutions.



Merge sort

1. ***Divide:*** Trivial.
2. ***Conquer:*** Recursively sort 2 subarrays.
3. ***Combine:*** Linear-time merge.



Merge sort

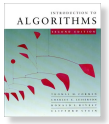
1. ***Divide:*** Trivial.
2. ***Conquer:*** Recursively sort 2 subarrays.
3. ***Combine:*** Linear-time merge.

$$T(n) = 2\,T(n/2) + \Theta(n)$$

 # subproblems

subproblem size

work dividing and combining



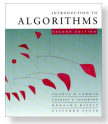
Master theorem (Mergesort)

$$T(n) = 2T(n/2) + \Theta(n)$$

- 2: # subproblems
- $T(n/2)$: subproblem size
- $\Theta(n)$: work dividing and combining

Merge sort: $a = 2$, $b = 2$ $\Rightarrow$ $n^{\log_b a} = n^{\log_2 2} = n$

$\Rightarrow$ CASE 2 ($k = 0$) $\Rightarrow$ $T(n) = \Theta(n \lg n)$.



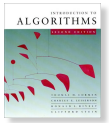
Binary search

Find an element in a sorted array:

1.Divide: Check middle element.

2.Conquer: Recursively search 1 subarray.

3.Combine: Trivial.



Binary search

Find an element in a sorted array:

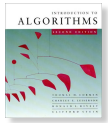
1.Divide: Check middle element.

2.Conquer: Recursively search 1 subarray.

3.Combine: Trivial.

Example: Find 9

3	5	7	8	9	12	15



Binary search

Find an element in a sorted array:

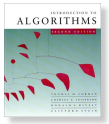
1.Divide: Check middle element.

2.Conquer: Recursively search 1 subarray.

3.Combine: Trivial.

Example: Find 9

3	5	7	8	9	12	15



Binary search

Find an element in a sorted array:

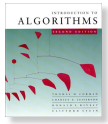
1.Divide: Check middle element.

2.Conquer: Recursively search 1 subarray.

3.Combine: Trivial.

Example: Find 9

3 5 7 8 9 12 15



Binary search

Find an element in a sorted array:

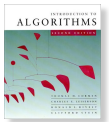
1.Divide: Check middle element.

2.Conquer: Recursively search 1 subarray.

3.Combine: Trivial.

Example: Find 9

3 5 7 8 9 12 15



Binary search

Find an element in a sorted array:

1.Divide: Check middle element.

2.Conquer: Recursively search 1 subarray.

3.Combine: Trivial.

Example: Find 9

3 5 7 8 9 12 15

Binary search

Find an element in a sorted array:

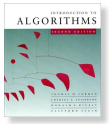
1.Divide: Check middle element.

2.Conquer: Recursively search 1 subarray.

3.Combine: Trivial.

Example: Find 9

3 5 7 8 9 12 15



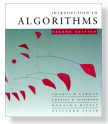
Recurrence for binary search

$$T(n) = 1\,T(n/2) + \Theta(1)$$

 # subproblems

subproblem size

work dividing and combining



Recurrence for binary search

$$T(n) = 1\,T(n/2) + \Theta(1)$$

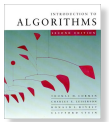
 # subproblems

subproblem size

work dividing and combining

$n^{\log_b a} = n^{\log_2 1} = n^0 = 1 \Rightarrow$ CASE 2 ($k = 0$)

$\Rightarrow T(n) = \Theta(\lg n)$.



Powering a number

Problem: Compute a^n, where $n \in \boldsymbol{N}$.

Naive algorithm: $\Theta(n)$.

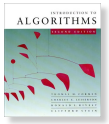
Powering a number

Problem: Compute a^n, where $n \in \boldsymbol{N}$.

Naive algorithm: $\Theta(n)$.

Divide-and-conquer algorithm:

$$a^n = \begin{cases} a^{n/2} \cdot a^{n/2} & \text{if } n \text{ is even;} \\ a^{(n-1)/2} \cdot a^{(n-1)/2} \cdot a & \text{if } n \text{ is odd.} \end{cases}$$



Powering a number

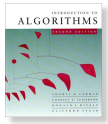
Problem: Compute a^n, where $n \in \mathbf{N}$.

Naive algorithm: $\Theta(n)$.

Divide-and-conquer algorithm:

$$a^n = \begin{cases} a^{n/2} \cdot a^{n/2} & \text{if } n \text{ is even;} \\ a^{(n-1)/2} \cdot a^{(n-1)/2} \cdot a & \text{if } n \text{ is odd.} \end{cases}$$

$$T(n) = T(n/2) + \Theta(1) \Rightarrow T(n) = \Theta(\lg n) .$$

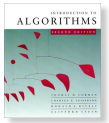


Matrix multiplication

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$c_{11} = (a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31} + \ldots + a_{1n} \cdot b_{n1})$$

$$c_{ij} = (a_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + a_{i3} \cdot b_{3j} + \ldots + a_{in} \cdot b_{nj})$$

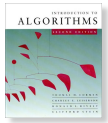


Matrix multiplication

Input: $A = [a_{ij}]$, $B = [b_{ij}]$. } $i, j = 1, 2, \ldots, n$.
Output: $C = [c_{ij}] = A \cdot B$. }

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

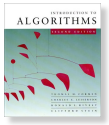
$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$



Standard algorithm

```
for i ← 1 to n
    do for j ← 1 to n
            do c_ij ← 0
               for k ← 1 to n
                   do c_ij ← c_ij + a_ik · b_kj
```

Running time = $\Theta(n^3)$



Divide-and-conquer algorithm

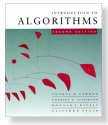
IDEA:

$n \times n$ matrix = 2×2 matrix of $(n/2)\times(n/2)$ submatrices:

$$\begin{bmatrix} r & s \\ t & u \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$C = A \cdot B$$

$$\left.\begin{aligned} r &= ae + bg \\ s &= af + bh \\ t &= ce + dg \\ u &= cf + dh \end{aligned}\right\} \begin{array}{l} 8 \text{ mults of } (n/2)\times(n/2) \text{ submatrices} \\ 4 \text{ adds of } (n/2)\times(n/2) \text{ submatrices} \end{array}$$



Divide-and-conquer algorithm

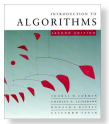
IDEA:

$n \times n$ matrix = 2×2 matrix of $(n/2)\times(n/2)$ submatrices:

$$\begin{bmatrix} r & s \\ t & u \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

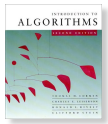
$$C = A \cdot B$$

$$\left.\begin{aligned} r &= ae + bg \\ s &= af + bh \\ t &= ce + dh \\ u &= cf + dg \end{aligned}\right\} \begin{array}{l} \text{recursive} \\ 8 \text{ mults of } (n/2)\times(n/2) \text{ submatrices} \\ 4 \text{ adds of } (n/2)\times(n/2) \text{ submatrices} \end{array}$$



Standard algorithm

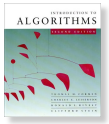
```
for i ← 1 to n
    do for j ← 1 to n
           do c_ij ← 0
              for k ← 1 to n
                  do c_ij ← c_ij + a_ik · b_kj
```

Analysis of D&C algorithm

$$T(n) = 8\,T(n/2) + \Theta(n^2)$$

- 8: # submatrices
- $n/2$: submatrix size
- $\Theta(n^2)$: work adding submatrices

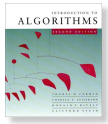


Analysis of D&C algorithm

$$T(n) = 8\,T(n/2) + \Theta(n^2)$$

 # submatrices — *submatrix size* — *work adding submatrices*

$$n^{\log_b a} = n^{\log_2 8} = n^3 \Rightarrow \text{CASE 1} \Rightarrow T(n) = \Theta(n^3).$$



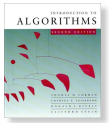
Analysis of D&C algorithm

$$T(n) = 8\,T(n/2) + \Theta(n^2)$$

 # submatrices — *submatrix size* — *work adding submatrices*

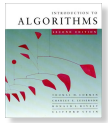
$$n^{\log_b a} = n^{\log_2 8} = n^3 \Rightarrow \text{CASE 1} \Rightarrow T(n) = \Theta(n^3).$$

No better than the ordinary algorithm.



Strassen's idea

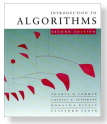
- Multiply 2×2 matrices with only 7 recursive mults.



Strassen's idea

- Multiply 2×2 matrices with only 7 recursive mults.

$$\begin{aligned}
P_1 &= a \cdot (f - h) \\
P_2 &= (a + b) \cdot h \\
P_3 &= (c + d) \cdot e \\
P_4 &= d \cdot (g - e) \\
P_5 &= (a + d) \cdot (e + h) \\
P_6 &= (b - d) \cdot (g + h) \\
P_7 &= (a - c) \cdot (e + f)
\end{aligned}$$

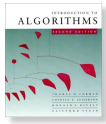


Strassen's idea

- Multiply 2×2 matrices with only 7 recursive mults.

$$P_1 = a \cdot (f - h)$$
$$P_2 = (a + b) \cdot h$$
$$P_3 = (c + d) \cdot e$$
$$P_4 = d \cdot (g - e)$$
$$P_5 = (a + d) \cdot (e + h)$$
$$P_6 = (b - d) \cdot (g + h)$$
$$P_7 = (a - c) \cdot (e + f)$$

$$r = P_5 + P_4 - P_2 + P_6$$
$$s = P_1 + P_2$$
$$t = P_3 + P_4$$
$$u = P_5 + P_1 - P_3 - P_7$$



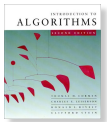
Strassen's idea

- Multiply 2×2 matrices with only 7 recursive mults.

$$P_1 = a \cdot (f - h)$$
$$P_2 = (a + b) \cdot h$$
$$P_3 = (c + d) \cdot e$$
$$P_4 = d \cdot (g - e)$$
$$P_5 = (a + d) \cdot (e + h)$$
$$P_6 = (b - d) \cdot (g + h)$$
$$P_7 = (a - c) \cdot (e + f)$$

$$r = P_5 + P_4 - P_2 + P_6$$
$$s = P_1 + P_2$$
$$t = P_3 + P_4$$
$$u = P_5 + P_1 - P_3 - P_7$$

7 mults, 18 adds/subs.
Note: No reliance on commutativity of mult!

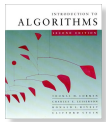


Strassen's idea

- Multiply 2×2 matrices with only 7 recursive mults.

$$P_1 = a \cdot (f - h)$$
$$P_2 = (a + b) \cdot h$$
$$P_3 = (c + d) \cdot e$$
$$P_4 = d \cdot (g - e)$$
$$P_5 = (a + d) \cdot (e + h)$$
$$P_6 = (b - d) \cdot (g + h)$$
$$P_7 = (a - c) \cdot (e + f)$$

$$\begin{aligned} r &= P_5 + P_4 - P_2 + P_6 \\ &= (a + d)(e + h) \\ &\quad + d(g - e) - (a + b)h \\ &\quad + (b - d)(g + h) \\ &= ae + ah + de + dh \\ &\quad + dg - de - ah - bh \\ &\quad + bg + bh - dg - dh \\ &= ae + bg \end{aligned}$$

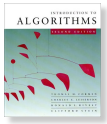


Strassen's algorithm

1.Divide: Partition A and B into $(n/2)\times(n/2)$ submatrices. Form terms to be multiplied using + and – .

2.Conquer: Perform 7 multiplications of $(n/2)\times(n/2)$ submatrices recursively.

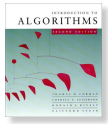
3.Combine: Form C using + and – on $(n/2)\times(n/2)$ submatrices.



Strassen's algorithm

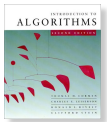
1. ***Divide:*** Partition A and B into $(n/2)\times(n/2)$ submatrices. Form terms to be multiplied using $+$ and $-$.
2. ***Conquer:*** Perform 7 multiplications of $(n/2)\times(n/2)$ submatrices recursively.
3. ***Combine:*** Form C using $+$ and $-$ on $(n/2)\times(n/2)$ submatrices.

$$T(n) = 7\,T(n/2) + \Theta(n^2)$$



Analysis of Strassen

$$T(n) = 7\,T(n/2) + \Theta(n^2)$$



Analysis of Strassen

$$T(n) = 7\,T(n/2) + \Theta(n^2)$$

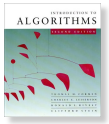
$n^{\log_b a} = n^{\log_2 7} \approx n^{2.81} \Rightarrow$ CASE 1 $\Rightarrow T(n) = \Theta(n^{\lg 7})$.

Analysis of Strassen

$$T(n) = 7\,T(n/2) + \Theta(n^2)$$

$n^{\log_b a} = n^{\log_2 7} \approx n^{2.81} \Rightarrow$ CASE 1 $\Rightarrow T(n) = \Theta(n^{\lg 7})$.

The number 2.81 may not seem much smaller than 3, but because the difference is in the exponent, the impact on running time is significant. In fact, Strassen's algorithm beats the ordinary algorithm on today's machines for $n \geq 32$ or so.



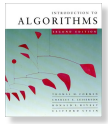
Analysis of Strassen

$$T(n) = 7\,T(n/2) + \Theta(n^2)$$

$n^{\log_b a} = n^{\log_2 7} \approx n^{2.81} \Rightarrow$ CASE 1 $\Rightarrow T(n) = \Theta(n^{\lg 7})$.

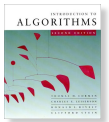
The number 2.81 may not seem much smaller than 3, but because the difference is in the exponent, the impact on running time is significant. In fact, Strassen's algorithm beats the ordinary algorithm on today's machines for $n \geq 32$ or so.

Best to date (of theoretical interest only): $\Theta(n^{2.376\cdots})$.



Conclusion

- Divide and conquer is just one of several powerful techniques for algorithm design.
- Divide-and-conquer algorithms can be analyzed using recurrences and the master method (so practice this math).
- The divide-and-conquer strategy often leads to efficient algorithms.



Why study algorithms and performance?

- Algorithms help us to understand ***scalability***.
- Performance often draws the line between what is feasible and what is impossible.
- Algorithmic mathematics provides a ***language*** for talking about program behavior.
- Performance is the ***currency*** of computing.
- The lessons of program performance generalize to other computing resources.
- Speed is fun!

Thank you.

Q & A

NUS National University of Singapore

School of Computing