CS3230 Lecture 2

NUS National University of Singapore
School of Computing

"Asymptotics, Summation in Analysis of Algorithms"

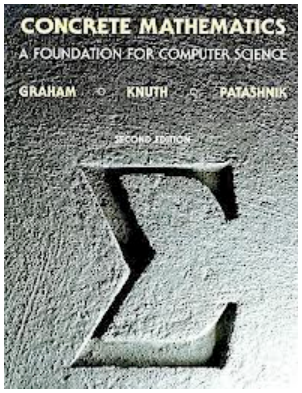
- **Lecture Topics and Readings**
  - **Simple Analysis of Algorithms (AA-P)** **[CLRS]-C2**
  - **Asymptotic Notations O, Θ, Ω, o, ω,** **[CLRS]-C3.1**
  - **Summation and Bounding** **[CLRS]-App.A**
- **Assignment:** T1 to start next week.

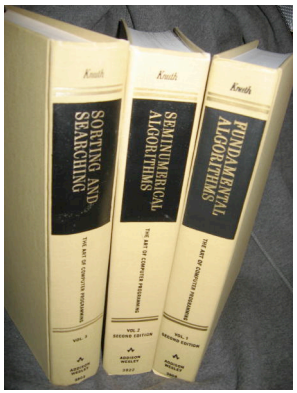
Algorithms & Discrete Math
have very happy, stable marriage!

Hon Wai Leong, NUS

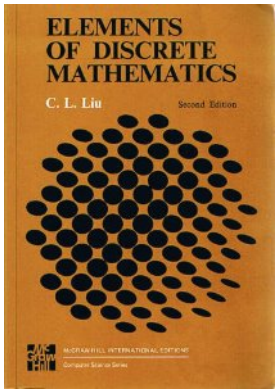
Early pioneers of Math for CS

Don Knuth
Stanford





C. L. Liu,
MIT, UIUC,
NTHU (tw)



Visiting Dave in Taipei (Feb 2006)

Thank you.

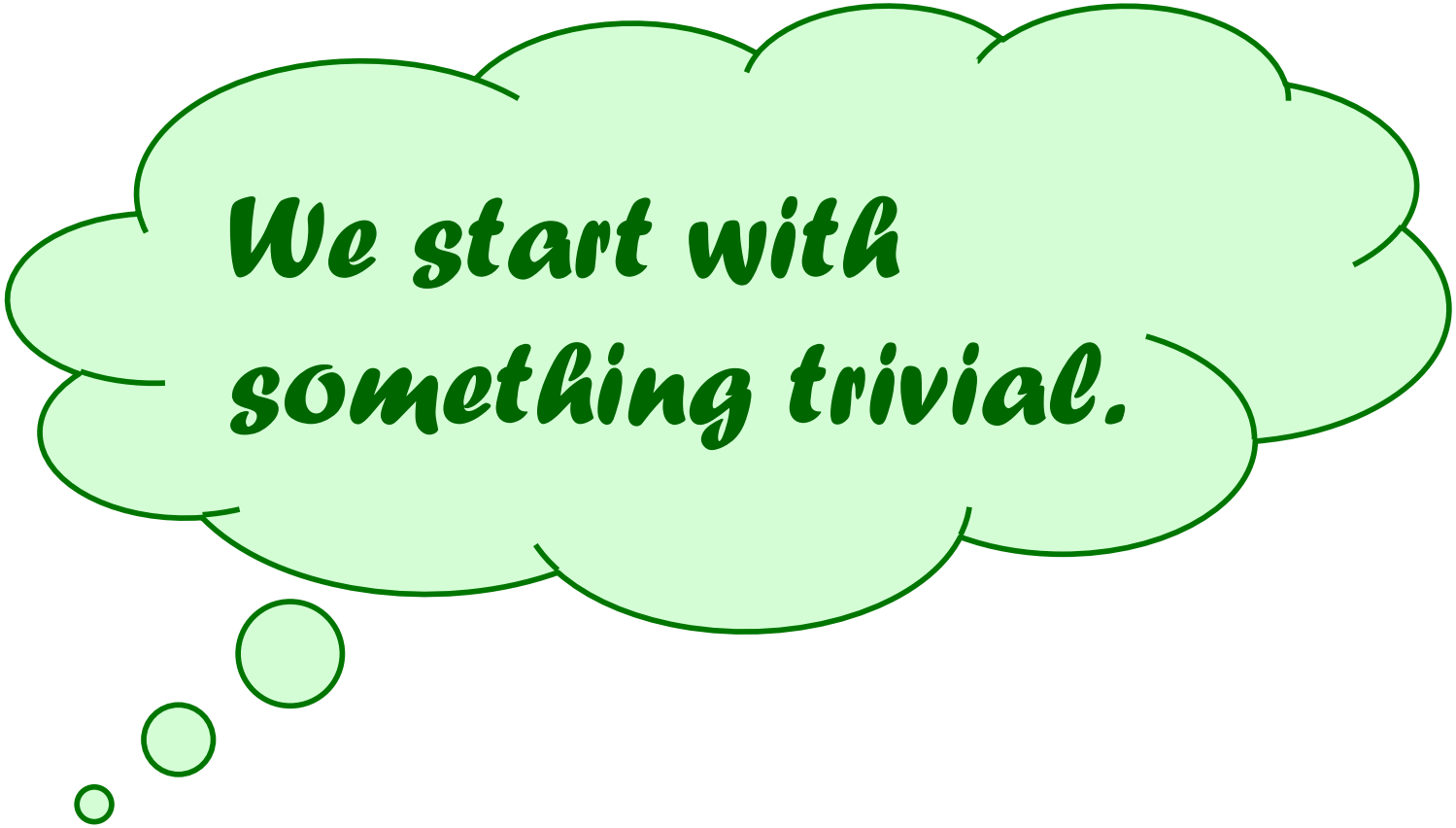
Q & A

CS3230 Algorithm Analysis

"Math for AA"

- Lecture Topics and Readings
  - Simple Algorithm Analysis — CS1020/2010
  - Algorithm Analysis Pattern — [LHW] A&E
  - Asymptotic Notations — [CLRS]-C3
  - Functions & Summations (Review) — [CLRS]-C3, App

Quickly get up to speed.



Trivial Algorithm Analysis

```
PASS-FAIL (Mark)   ▷
    Print Mark
    if Mark < 40
        then print "Student Fails"
        else print "Student Passes"
```

"pseudocode"

AA. Simple. Constant time.
(*always* take the same amount of time)
Does not vary with n.

Trivial Algorithm Analysis

```
MPG ()
    Read StartMiles, EndMiles, GasUsed
    Distance ← (EndMiles – StartMiles)
    Average ← Distance / GasUsed)
    Print  "Average mileage is ", Average
    if Average >= 25
        then Print " Mileage is good"
        then Print " Mileage is not good"
    Print "Bye Bye"
```

"pseudo-code"

AA. Simple. Constant time.
(Does not vary with n.)

Trivial Algorithm Analysis

```
TEST (A, n)      ▷ A[1 . . n]
    Print A[1]
    key ← A[2] * 2
    if A[1] > 0  then Print "first number is positive"
    Print A[n]    (* print last element of array *)
```

"pseudo-code"

AA: Simple. Constant time.
Does not vary with n.

$$T(n) \geq 1, \quad T(n) \leq 1000, \quad T(n) = \Theta(1)$$

Lower bound

Upper bound

Guaranteed running time

Trivial Algorithm Analysis

```
ARRAY-SUM (A, N)        ▷ A[1 . . N]
    Sum-sf ← 0
    for j ← 1 to N
        do  Sum-sf ← Sum-sf + A[j]
    Sum ← Sum-sf
```

"pseudo-code"

AA: Simple. Linear time.
Directly proportional to n.

$$T(n) \geq n, \quad T(n) \leq 10{,}000n, \quad T(n) = \Theta(n)$$

Lower bound

Upper bound

Guaranteed running time

Trivial Algorithm Analysis

```
ALL-PRODUCT (A, n, M, n)    ▷ A[1..n], M[1..n; 1..n]
    (* compute all products *)
    for k ← 1 to n
        for j ← 1 to n
            do M[k, j] ← A[k] * A[j]
```

"pseudo-code"

AA: Simple. Quadratic time.
Proportional to n^2.

$$T(n) \geq n^2, \quad T(n) \leq 10{,}000n^2, \quad T(n) = \Theta(n^2)$$

Lower bound

Upper bound

Guaranteed running time

Testing * operation in a CPU

Q: How to test that the "*" operation of your CPU is correct?

A: Check exhaustively. for all a, b
Check $a * b = c$

Q: How long will it take?

A: Any guesses?

Testing * operation in a CPU

Q: How long will it take?

Assume we use a 100G-Flop CPU
can take 100B operations per sec.

a is a 32-bit number (2^{32} cases)

b is a 32-bit number (2^{32} cases)

So, (a * b) there are (2^{64} cases)

Time taken = (2^{64} / 100×10^9) sec

≈ 6 years!

Testing * operation in a CPU

Q: What if we have (n lg n) algorithm?

Assume we use a 100G-Flop CPU
can take 100B operations per sec.

a is a 32-bit number (2^{32} cases)
b is a 32-bit number (2^{32} cases)

When $n = 2^{32}$, with (n lg n) algorithm

Time taken = (2^{32} * 32 / 100x10^9) sec

< 2 sec!

Moral of the story

Analysis of algorithms
help us make *predictions*.

Analysis of algorithms
help us *prepare for the worst case*.

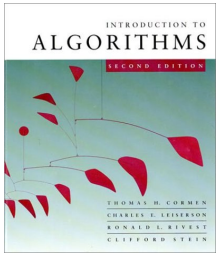
Story of Algorithms in Action

Credit card processing centre in SG (Sci Park):

- monitor showing servers load for diff. countries,
- blue, green, yellow, orange (send SMS alert),
- RED (URGENT Alert! ➔ deploy more servers!)

Note: ***Picture is NOT the real thing.***
But it gives the rough idea
and "demos" my point.

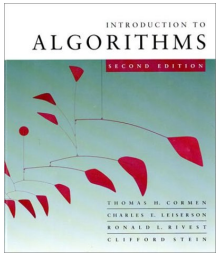
Picture is NOT very good.
Will find a better one.



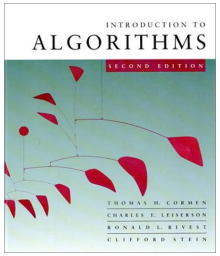
Why study algorithms and performance?

- Algorithms help us to understand ***scalability***.
- Performance often draws the line between what is feasible and what is impossible.
- Algorithmic mathematics provides a ***language*** for talking about program behavior.
- Performance is the ***currency*** of computing.
- The lessons of program performance generalize to other computing resources.
- Speed is fun!

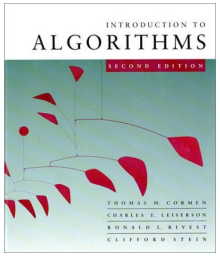
Application in Web-Service



Introduction to some fun AA patterns



Now analyze some algorithms and fit into AA patterns



The problem of sorting

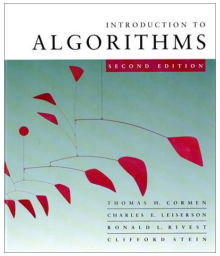
Input: sequence $\langle a_1, a_2, \ldots, a_n \rangle$ of numbers.

Output: permutation $\langle a'_1, a'_2, \ldots, a'_n \rangle$ such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$.

Example:

Input: 8 2 4 9 3 6

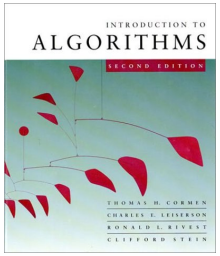
Output: 2 3 4 6 8 9



Insertion sort

"pseudocode"

```
INSERTION-SORT (A, n)      ▷ A[1 . . n]
    for j ← 2 to n
        do key ← A[ j]
           i ← j – 1
           while i > 0 and A[i] > key
               do A[i+1] ← A[i]
                  i ← i – 1
           A[i+1] = key
```



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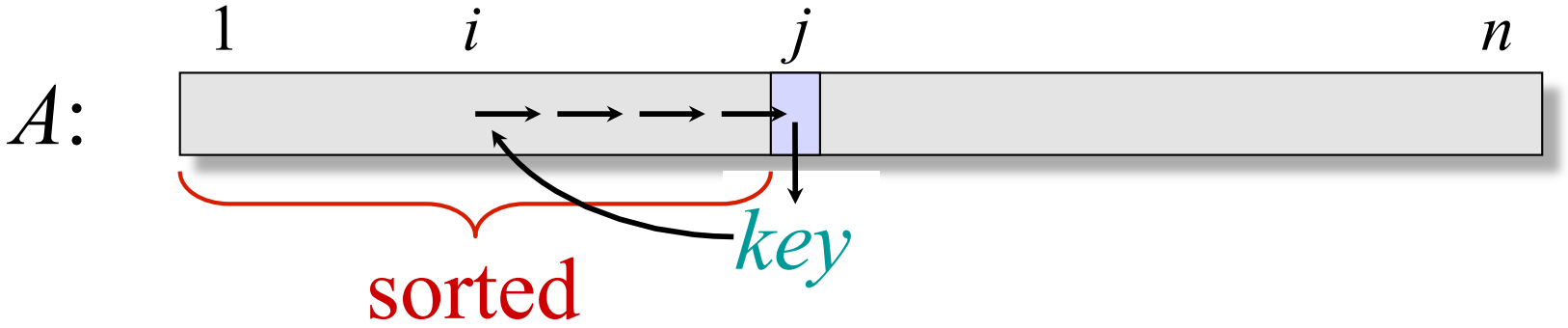
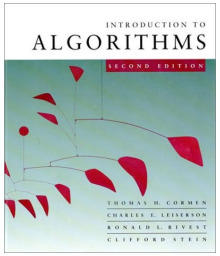
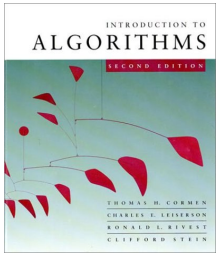




Example of insertion sort

8 2 4 9 3 6



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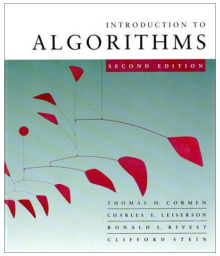

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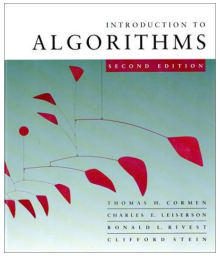

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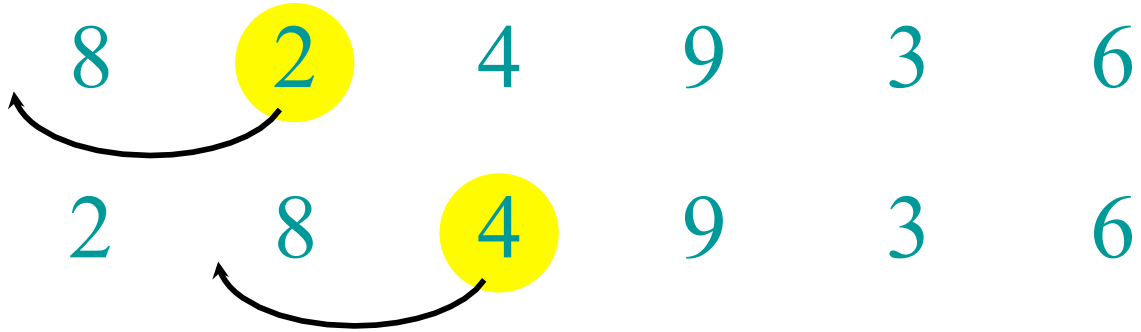

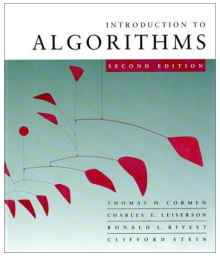

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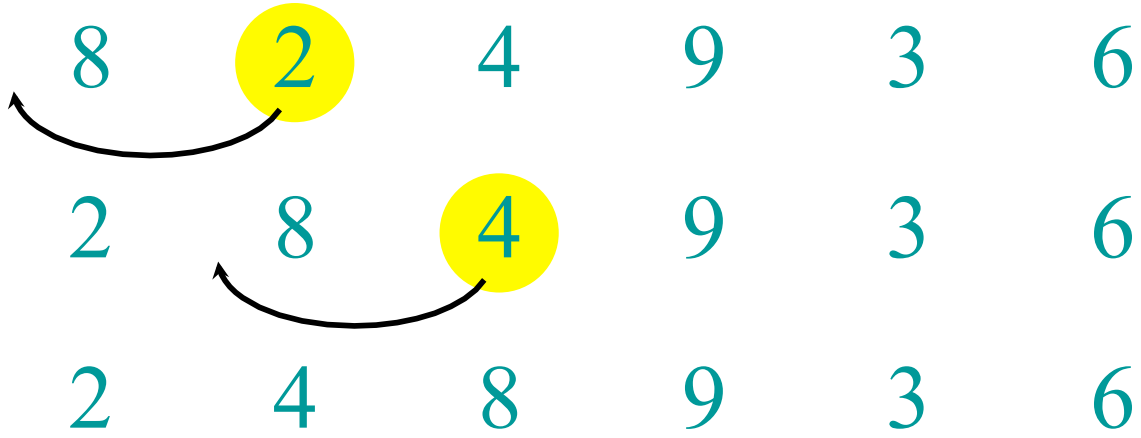

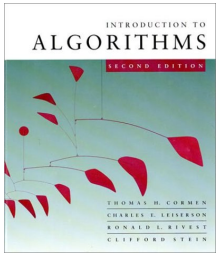

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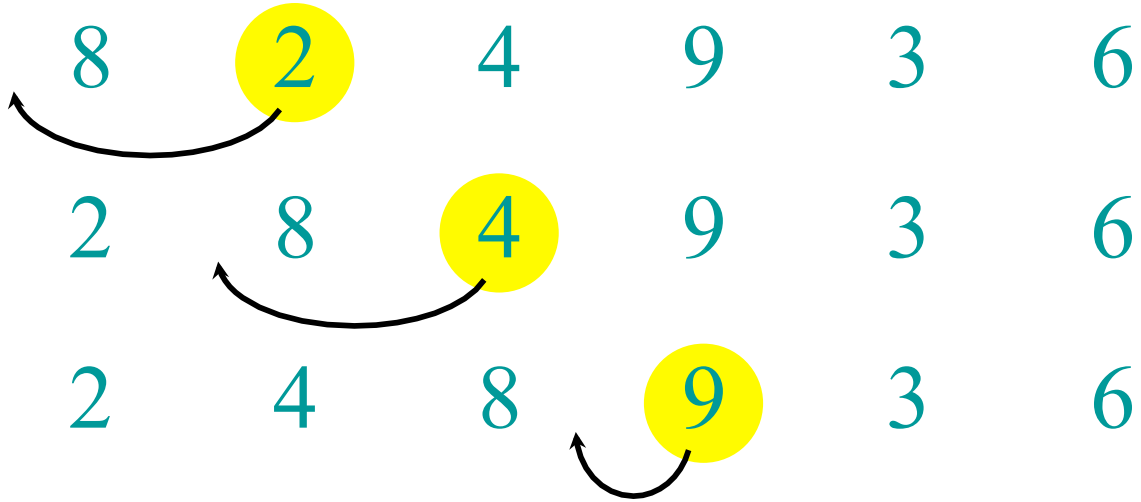

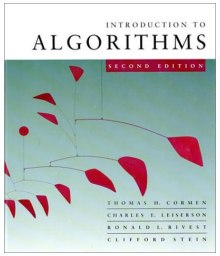

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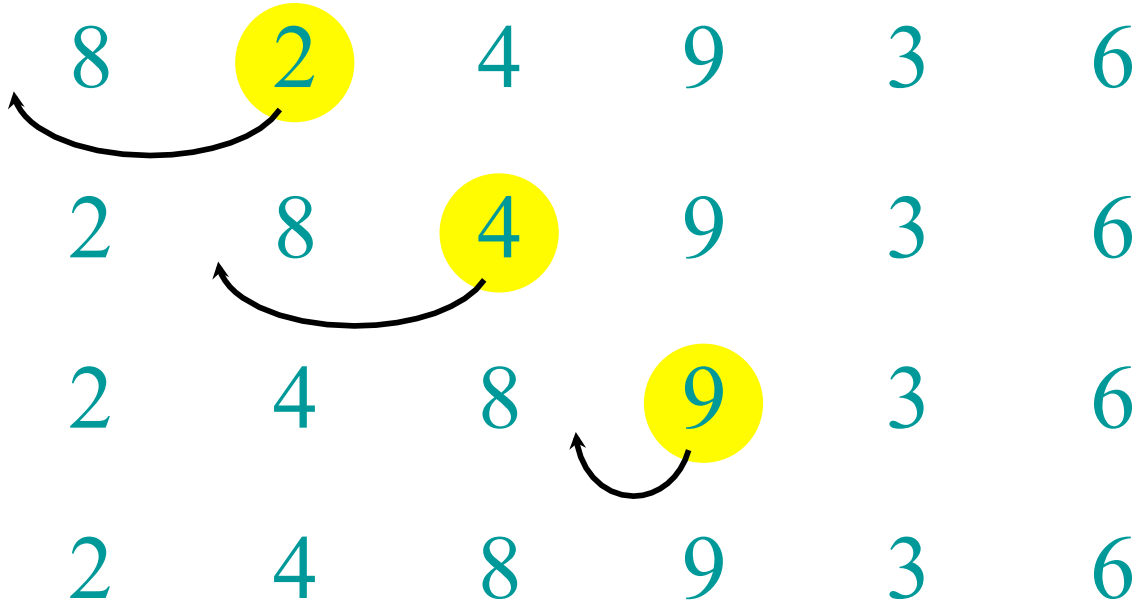

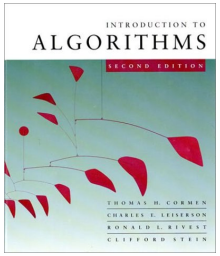

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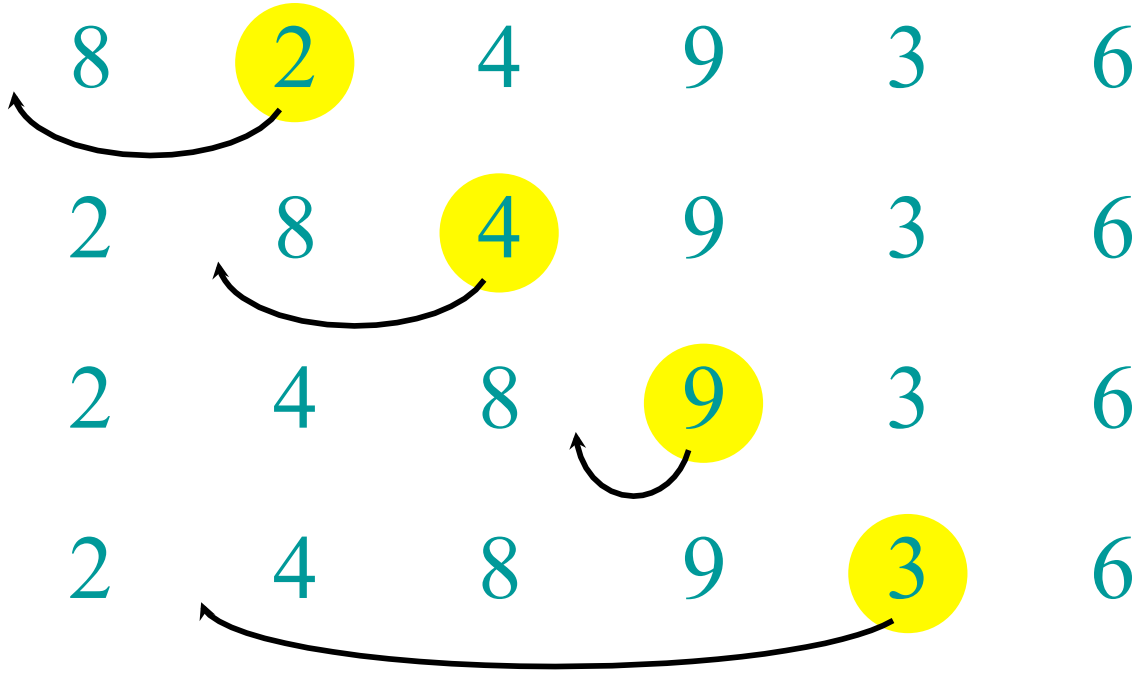

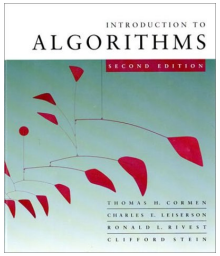

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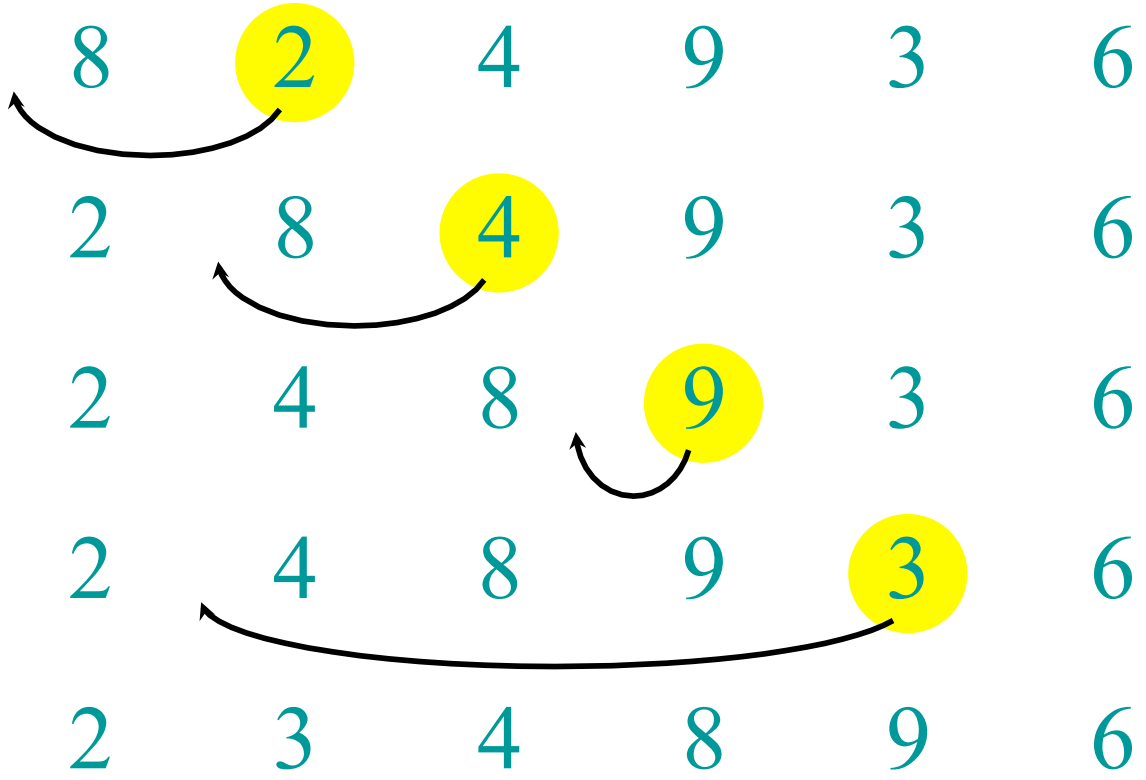

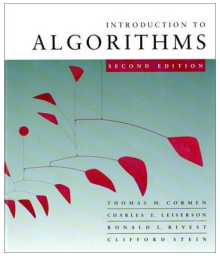

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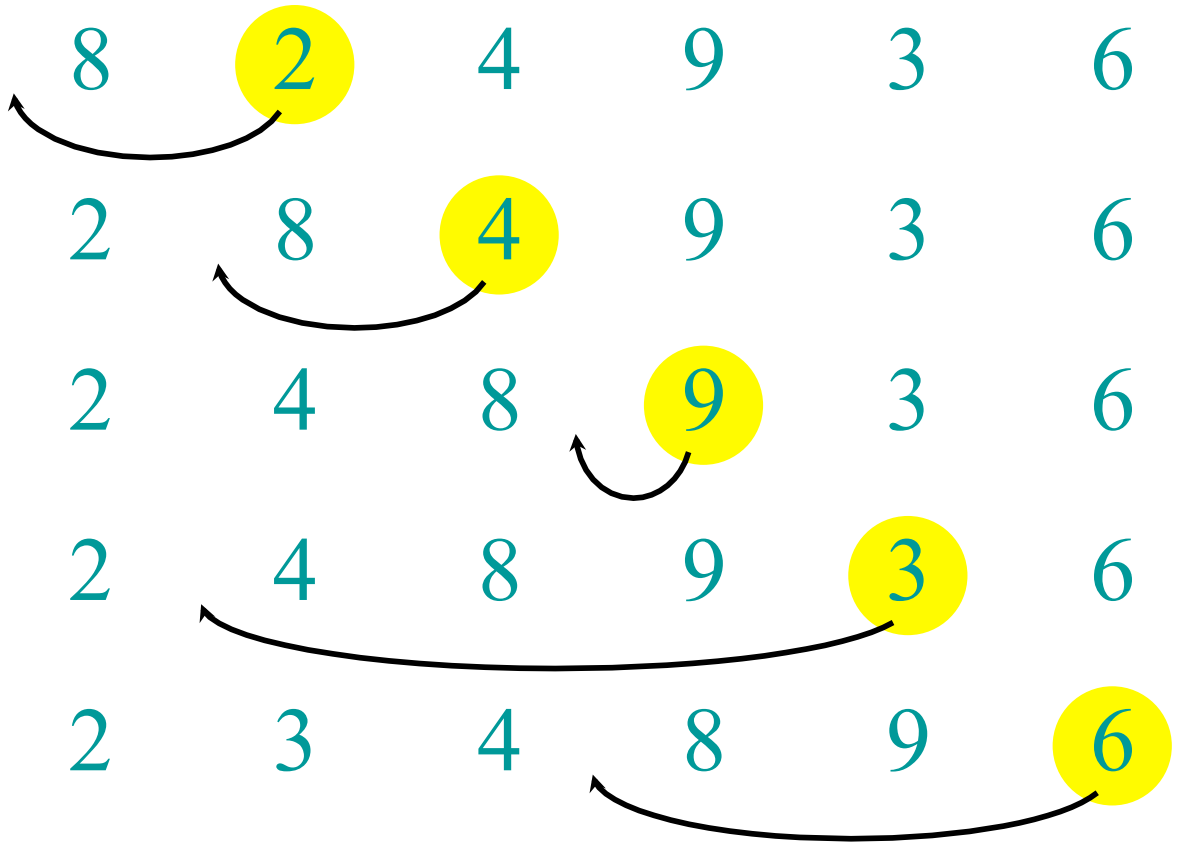

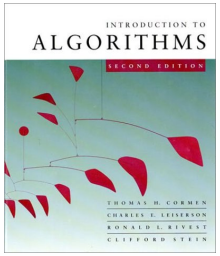

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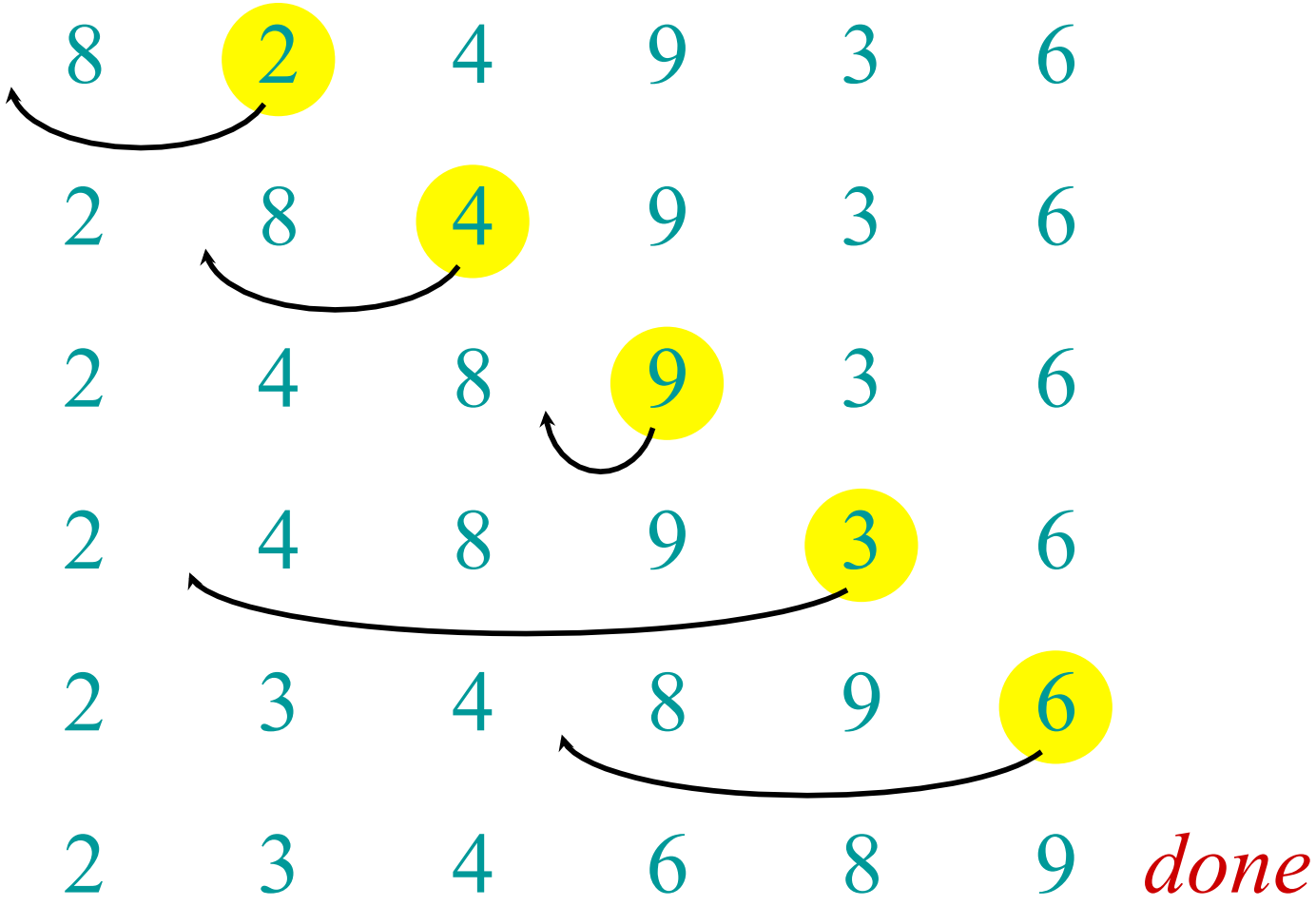
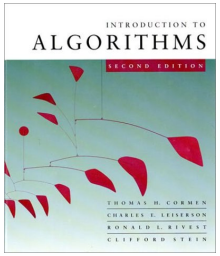
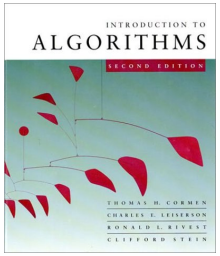




Running time

- The running time depends on the input: an already sorted sequence is easier to sort.
- Parameterize the running time by the size of the input, since short sequences are easier to sort than long ones.
- Generally, we seek upper bounds on the running time, because everybody likes a guarantee.



Kinds of analyses

Worst-case: (usually)

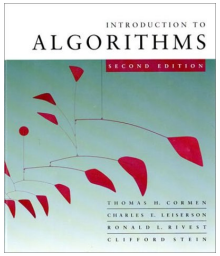
- $T(n)$ = maximum time of algorithm on any input of size n.

Average-case: (sometimes)

- $T(n)$ = expected time of algorithm over all inputs of size n.
- Need assumption of statistical distribution of inputs.

Best-case: (bogus)

- Cheat with a slow algorithm that works fast on *some* input.



Machine-independent time

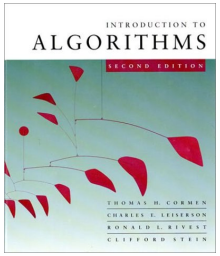
What is insertion sort's worst-case time?

- It depends on the speed of our computer:
  - relative speed (on the same machine),
  - absolute speed (on different machines).

BIG IDEA:

- Ignore machine-dependent constants.
- Look at ***growth*** of $T(n)$ as $n \to \infty$.

"Asymptotic Analysis"



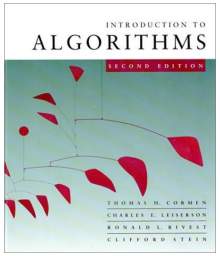
Θ-notation

Math:

$\Theta(g(n)) = \{ f(n) :$ there exist positive constants c_1, c_2, and n_0 such that $0 \le c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_0 \}$

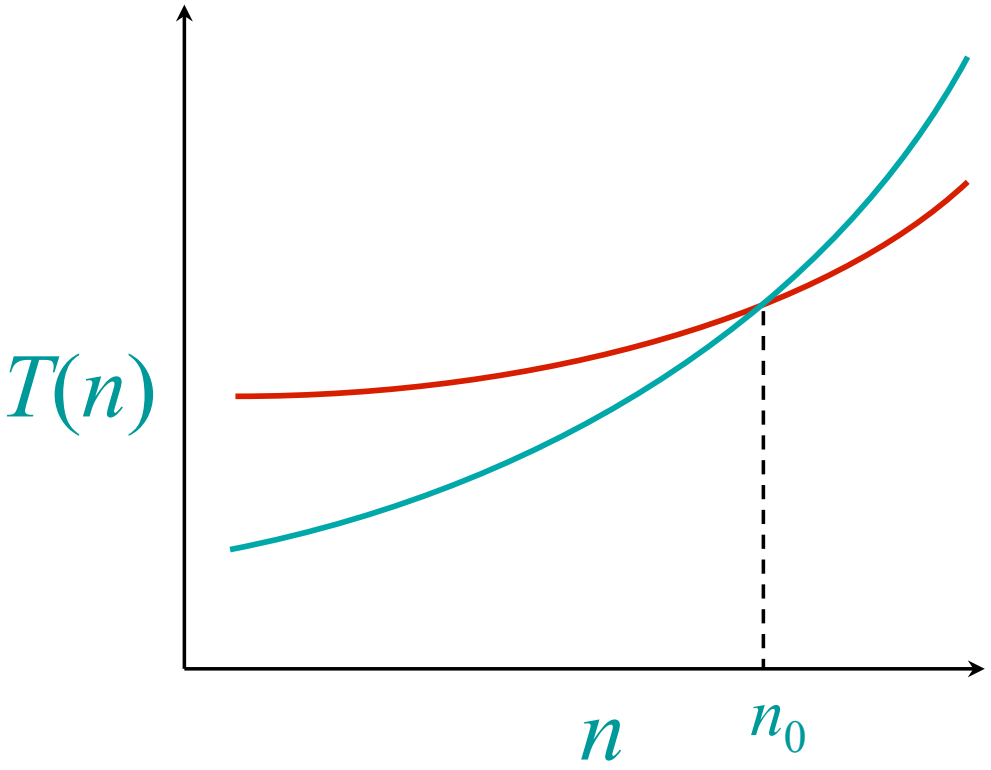
Engineering:

- Drop low-order terms; ignore leading constants.
- Example: $3n^3 + 90n^2 - 5n + 6046 = \Theta(n^3)$

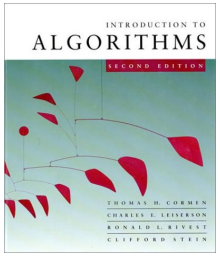


Asymptotic performance

When n gets large enough, a $\Theta(n^2)$ algorithm *always* beats a $\Theta(n^3)$ algorithm.



- We shouldn't ignore asymptotically slower algorithms, however.
- Real-world design situations often call for a careful balancing of engineering objectives.
- Asymptotic analysis is a useful tool to help to structure our thinking.

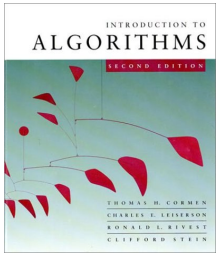


Comparing $100n^2$ vs $0.01n^3$

n	$100n^2$	$0.01n^3$	Diff	Ratio
1	100	0.01	-99.99	10000
10	10000	10	-9990	1000
100	1000000	10000	-990000	100
1000	100000000	10000000	-90000000	10
10000	10000000000	10000000000	0	1
10001	10002000100	10003000300	1000200.01	0.9999
50000	2.5E+11	1.25E+12	1E+12	0.2000

Higher order terms dominates
(ignore lower order terms)

Can ignore
constant factors



Insertion sort analysis

Worst case: Input reverse sorted.

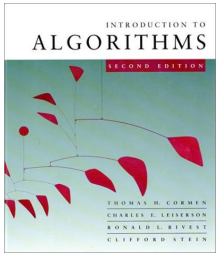
$$T(n) = \sum_{j=2}^{n} \Theta(j) = \Theta(n^2) \qquad \text{[arithmetic series]}$$

Average case: All permutations equally likely.

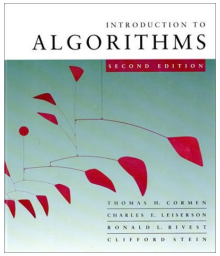
$$T(n) = \sum_{j=2}^{n} \Theta(j/2) = \Theta(n^2)$$

Is insertion sort a fast sorting algorithm?

- Moderately so, for small n.
- Not at all, for large n.



What's the AA pattern for Insertion Sort?



More sorting algorithm: Selection Sort

Sorting: Problem and Algorithms

- **Problem: Sorting**
  - Take a list of n numbers and rearrange them in increasing order
- **Algorithms:**
  - Selection Sort $\Theta(n^2)$
  - Insertion Sort $\Theta(n^2)$
  - Bubble Sort $\Theta(n^2)$
  - Merge Sort $\Theta(n \lg n)$
  - Quicksort $\Theta(n \lg n)$**

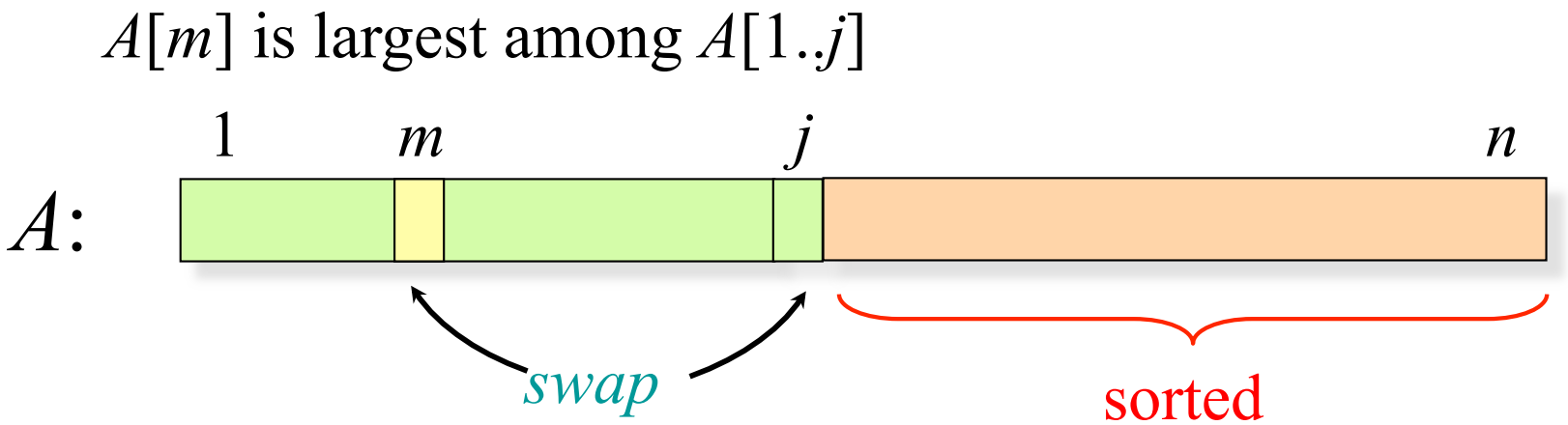
Some slides from UIT2201

Not covered in the course UIT2201

** *average case*

Selection sort

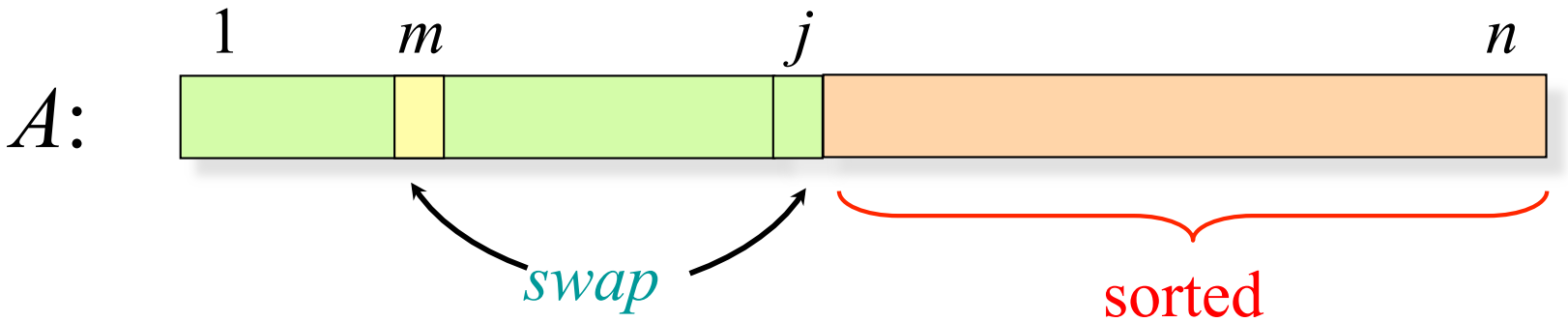
- **Idea behind the algorithm…**
  - **Repeatedly**
    - *find the largest number in unsorted section*
    - *Swap it to the end (the sorted section)*
  - **Re-uses the Find-Max algorithm**



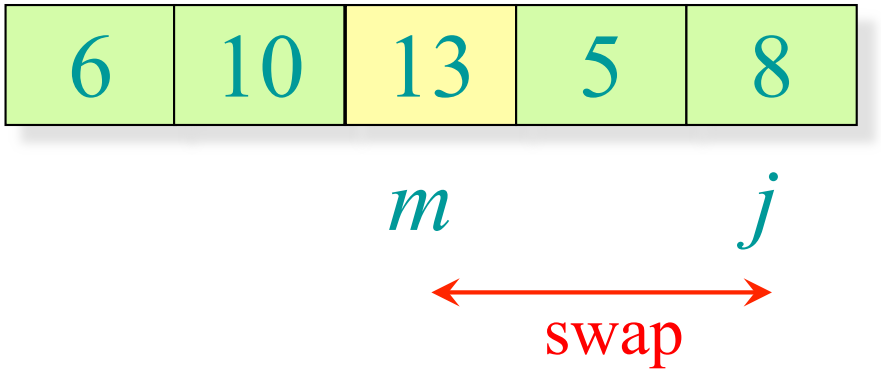
Selection Sort Algorithm (pseudo-code)

```
Selection-Sort(A, n);
begin
  j ← n;
  while (j > 1) do
    m ← Find-Max(A,j);
    swap(A[m],A[j]);
    j ← j - 1;
  endwhile
end;
```

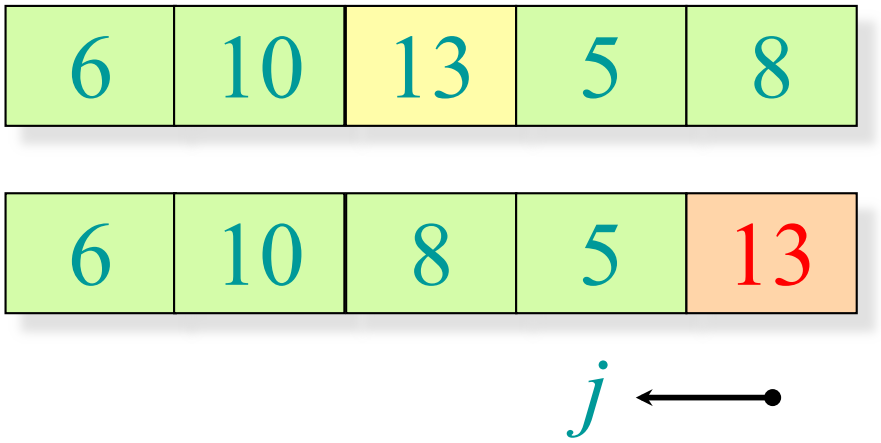
$A[m]$ is largest among $A[1..j]$



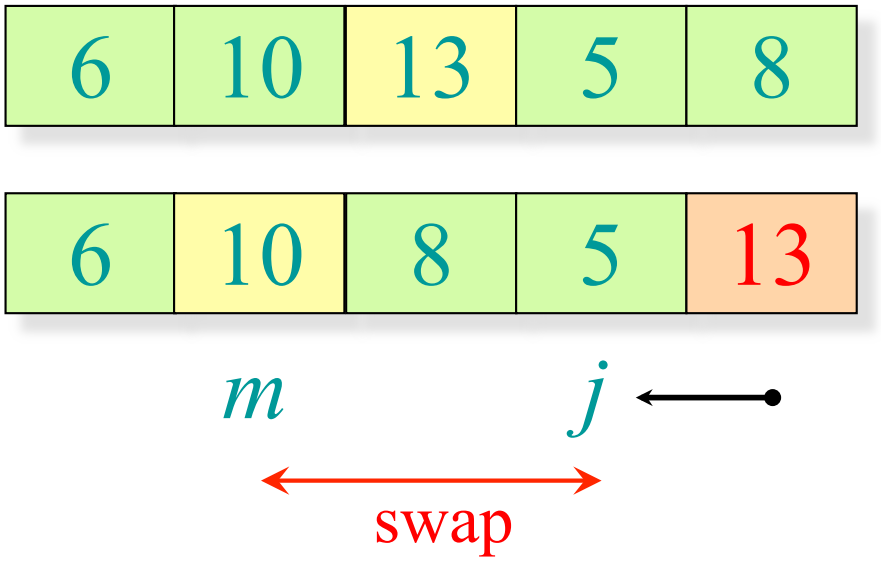
Example of selection sort



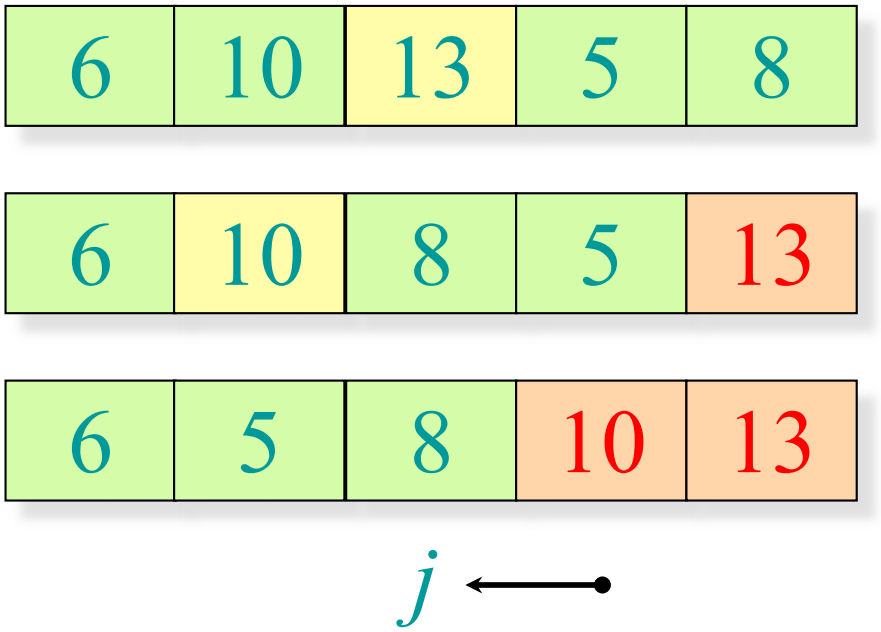
Example of selection sort



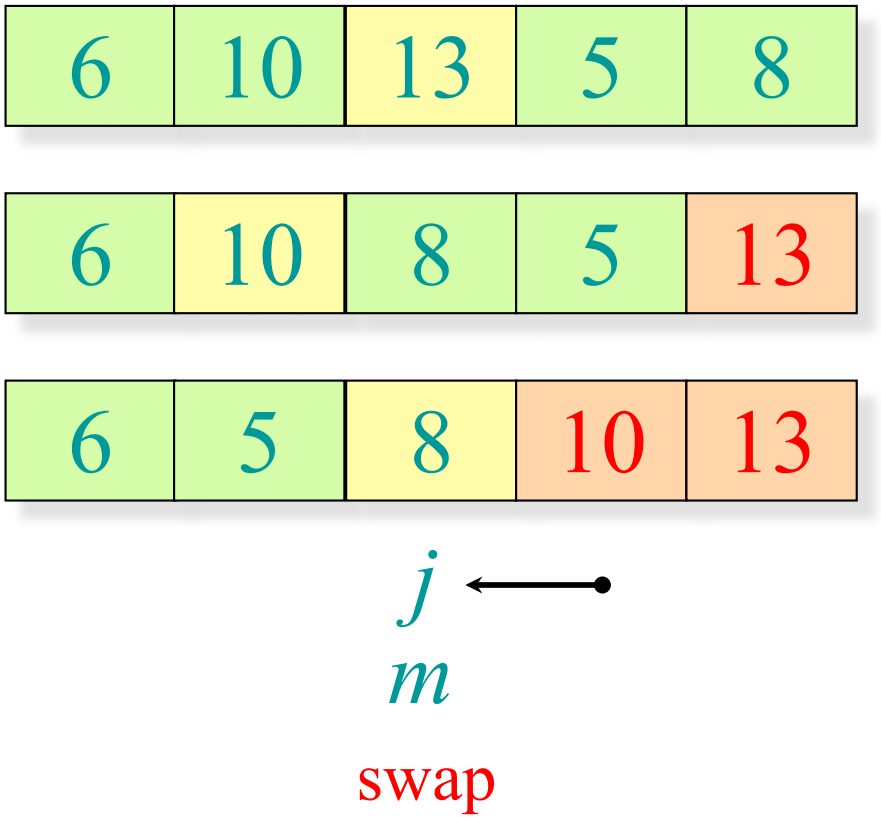
Example of selection sort



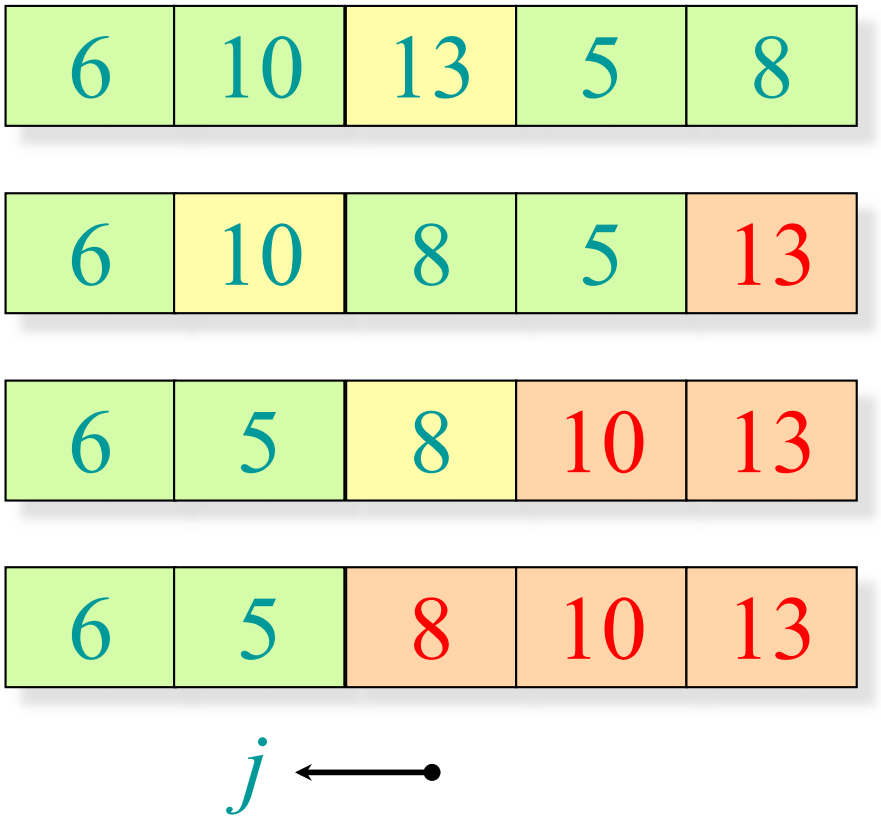
Example of selection sort



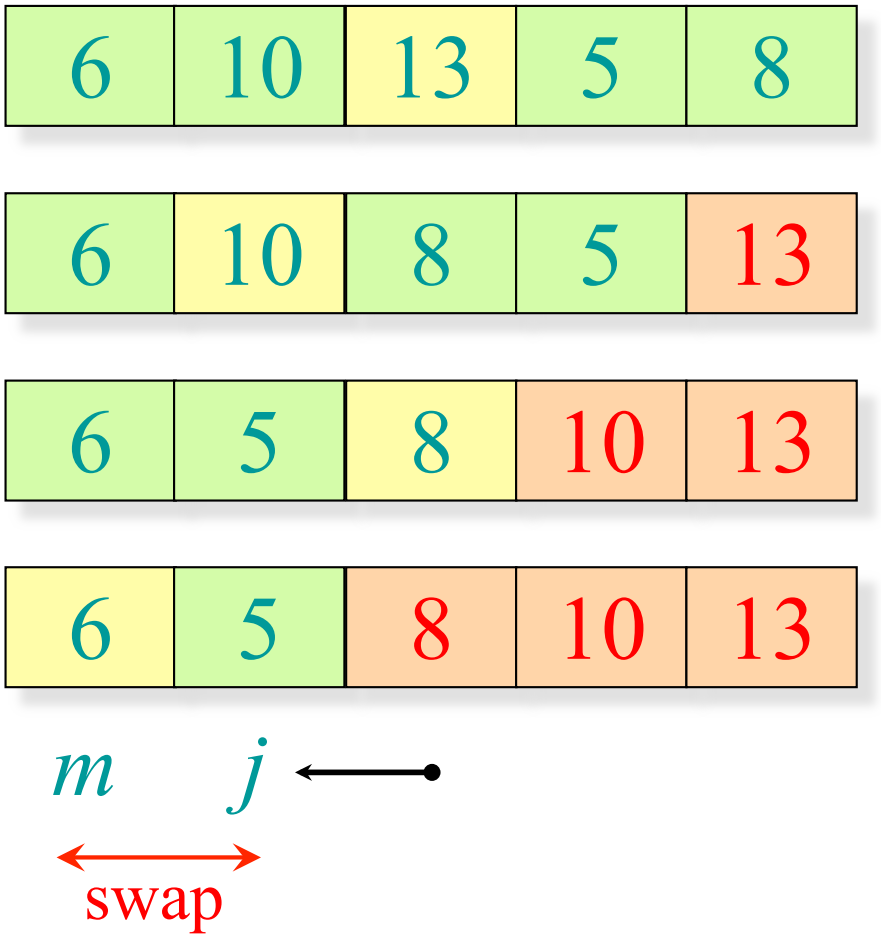
Example of selection sort



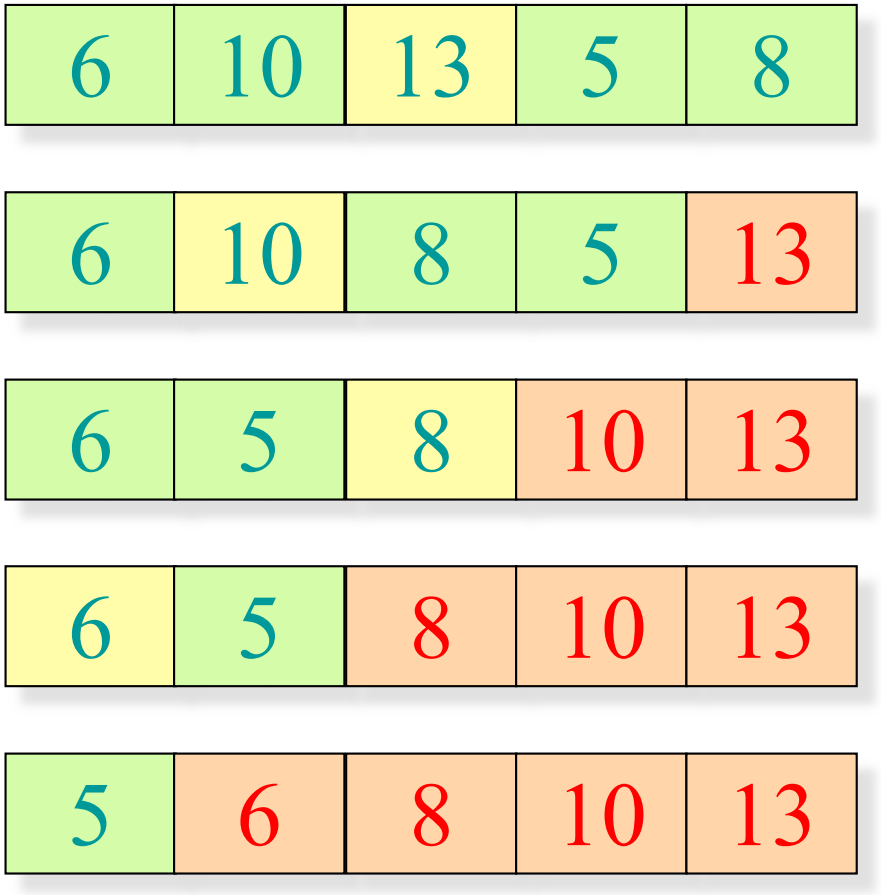
Example of selection sort



Example of selection sort



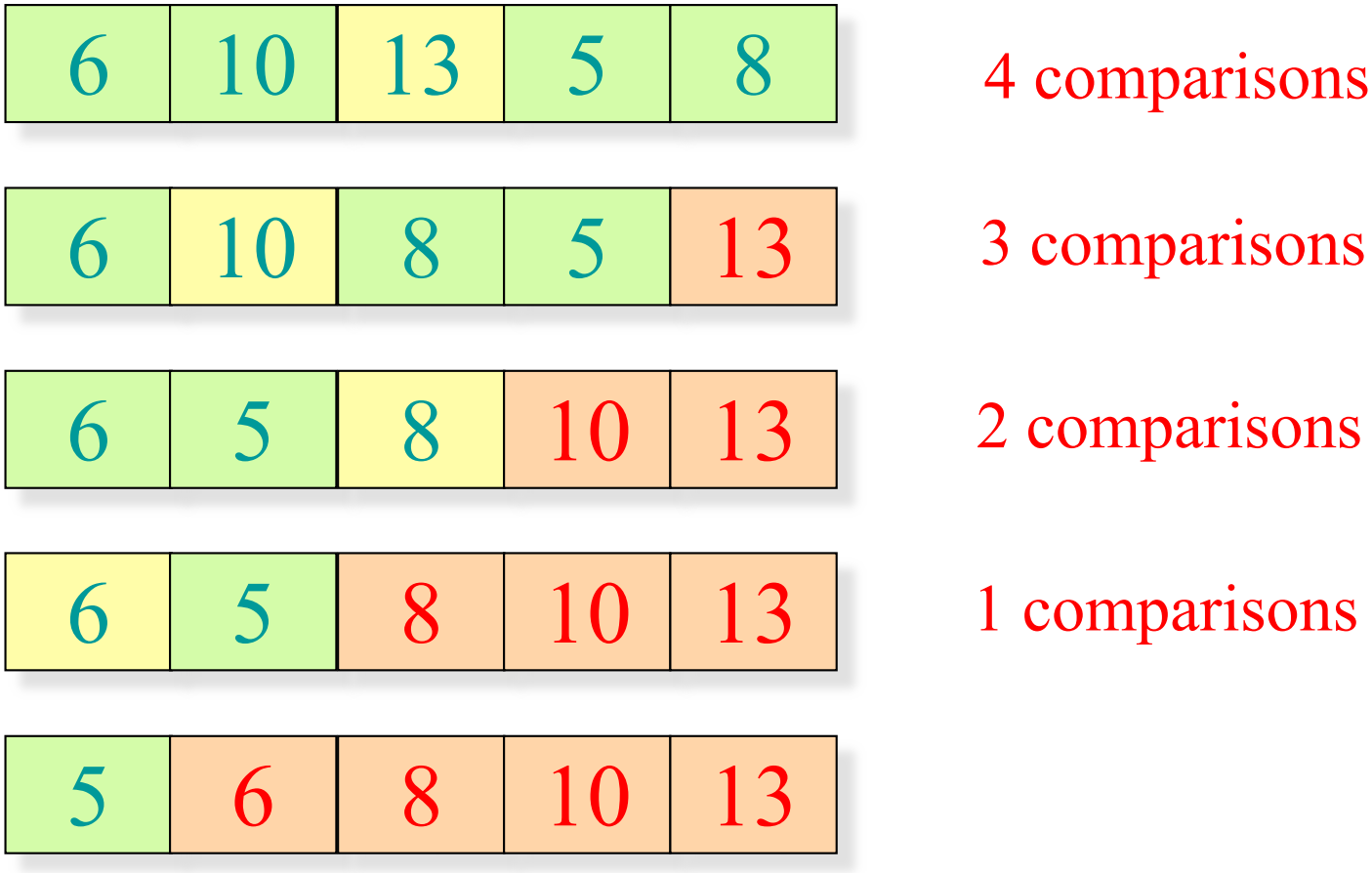
Example of selection sort



Done. j ⟵

What about the time complexity?

- **Dominant operation: *comparisons***



Analysis of Selection Sort

Find-Max for *j* numbers takes (*j*-1) comparisons

- **When sorting *n* numbers,**
  - (*n*–1) comparisons in iteration 1 (when $j = n$)
  - (*n*–2) comparisons in iteration 2 (when $j = n–1$)
  - (*n*–3) comparisons in iteration 3 (when $j = n–2$)
  - . . . .
  - 2 comparisons in iteration (*n*-2) (when $j = 3$)
  - 1 comparisons in iteration (*n*-1) (when $j = 2$)

- **Total number of comparisons:**
  - Cost = $(n–1) + (n–2) + \ldots + 2 + 1 = n(n–1)/2$
    $= \Theta(n^2)$

Analysis of Selection Sort: Summary

- **Time complexity: $\Theta(n^2)$**
  - **Comparisons:** $n(n\text{-}1)/2$
  - **Exchanges:** n (swapping largest into place)
  - **Overall time complexity:** $\Theta(n^2)$
- **Space complexity: $\Theta(n)$**
  - **$\Theta(n)$ – space for input sequence, plus a few variables.**

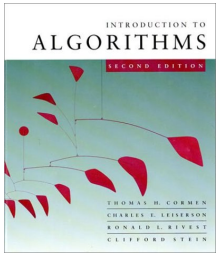
Selection Sort:

Time complexity: $T(n) = \Theta(n^2)$

Space complexity: $S(n) = \Theta(n)$

AA Pattern

What is the AA Pattern for Selection Sort?



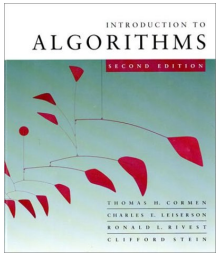
Merge sort

MERGE-SORT $A[1 \ldots n]$

1. If $n = 1$, done.
2. Recursively sort $A[\,1 \ldots \lceil n/2 \rceil\,]$ and $A[\,\lceil n/2 \rceil + 1 \ldots n\,]$.
3. "***Merge***" the 2 sorted lists.

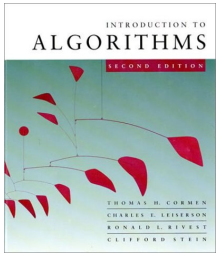
Key subroutine: **MERGE**

Mergesort is $\Theta(n \lg n)$
(shown later, also covered in CS2010/CS2020)

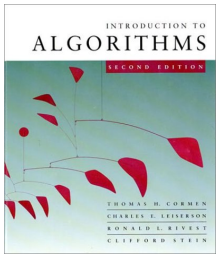


Observations about Sorting

- $\Theta(n \lg n)$ grows more slowly than $\Theta(n^2)$.
- Therefore, merge sort asymptotically beats insertion sort in the worst case.
- In practice, merge sort beats insertion sort for $n > 30$ or so.
- Go test it out for yourself!



Graph Pattern?



Unweighted graphs

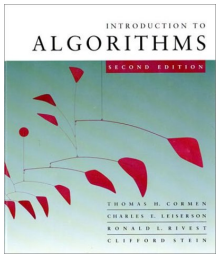
Suppose that $w(u, v) = 1$ for all $(u, v) \in E$.
Can use BFS to find shortest paths.

- Use a simple FIFO queue.

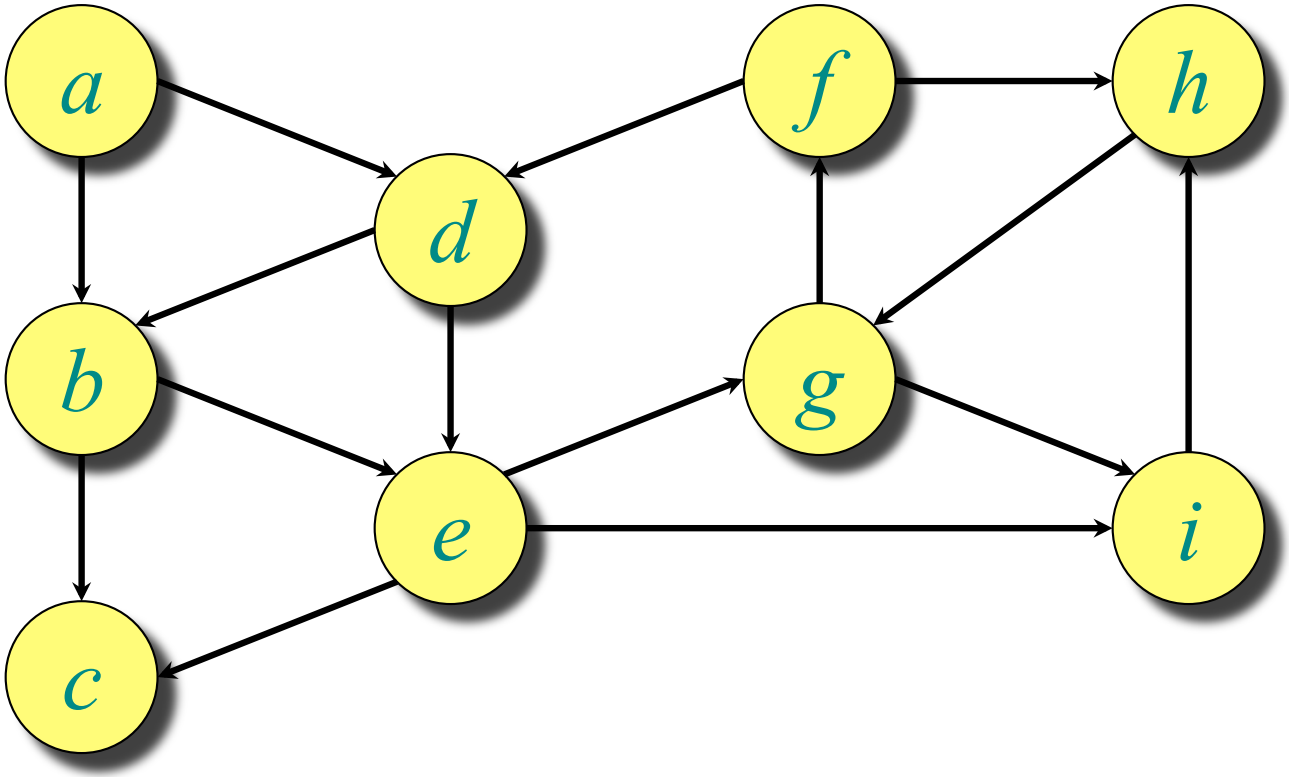
Breadth-first search

while $Q \neq \varnothing$
  do $u \leftarrow$ DEQUEUE(Q)
    for each $v \in Adj[u]$
      do if $d[v] = \infty$
        then $d[v] \leftarrow d[u] + 1$
          ENQUEUE(Q, v)

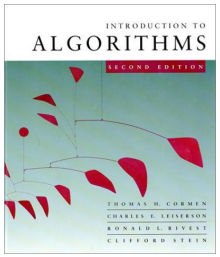
Analysis: Time $= O(V + E)$.



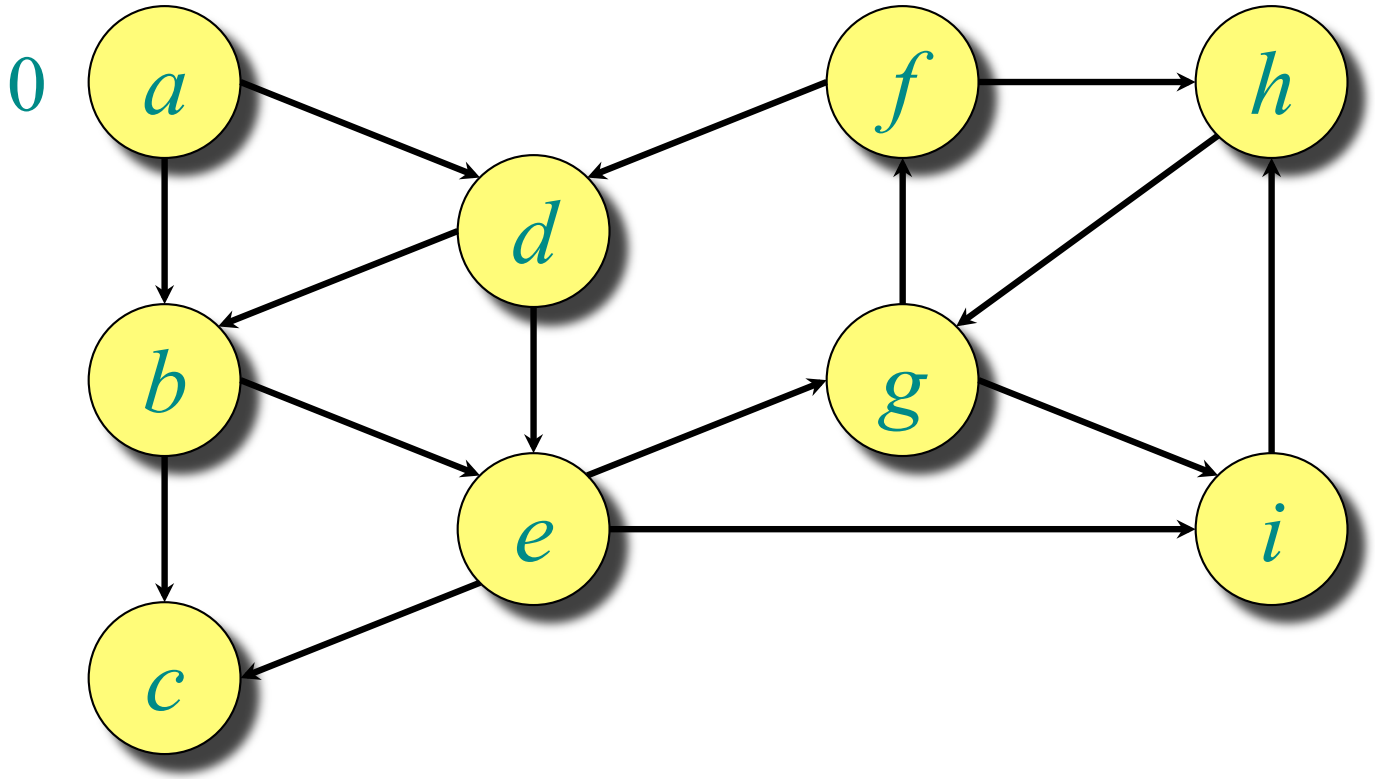
Example of breadth-first search



Q:

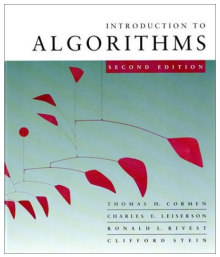


Example of breadth-first search

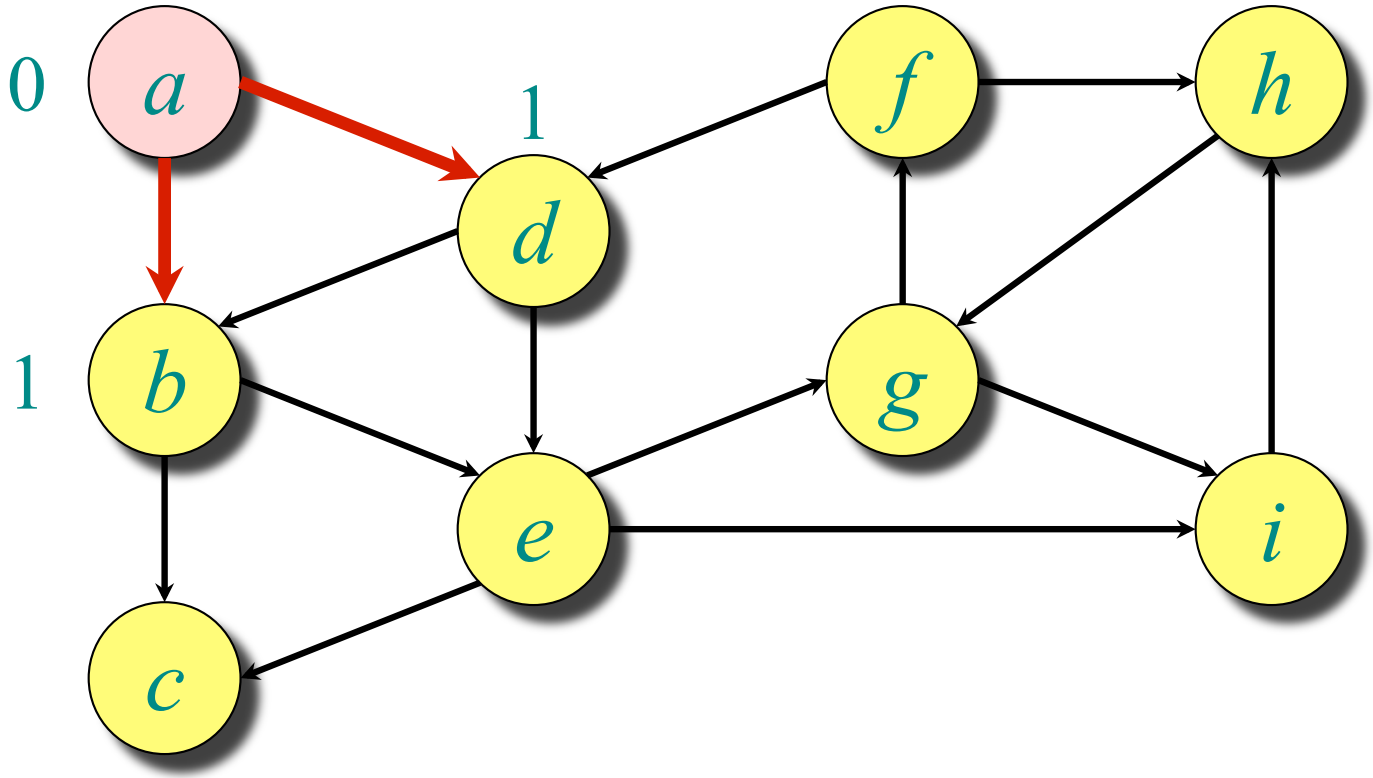


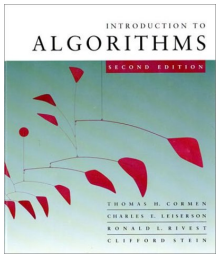
0

Q: a



Example of breadth-first search





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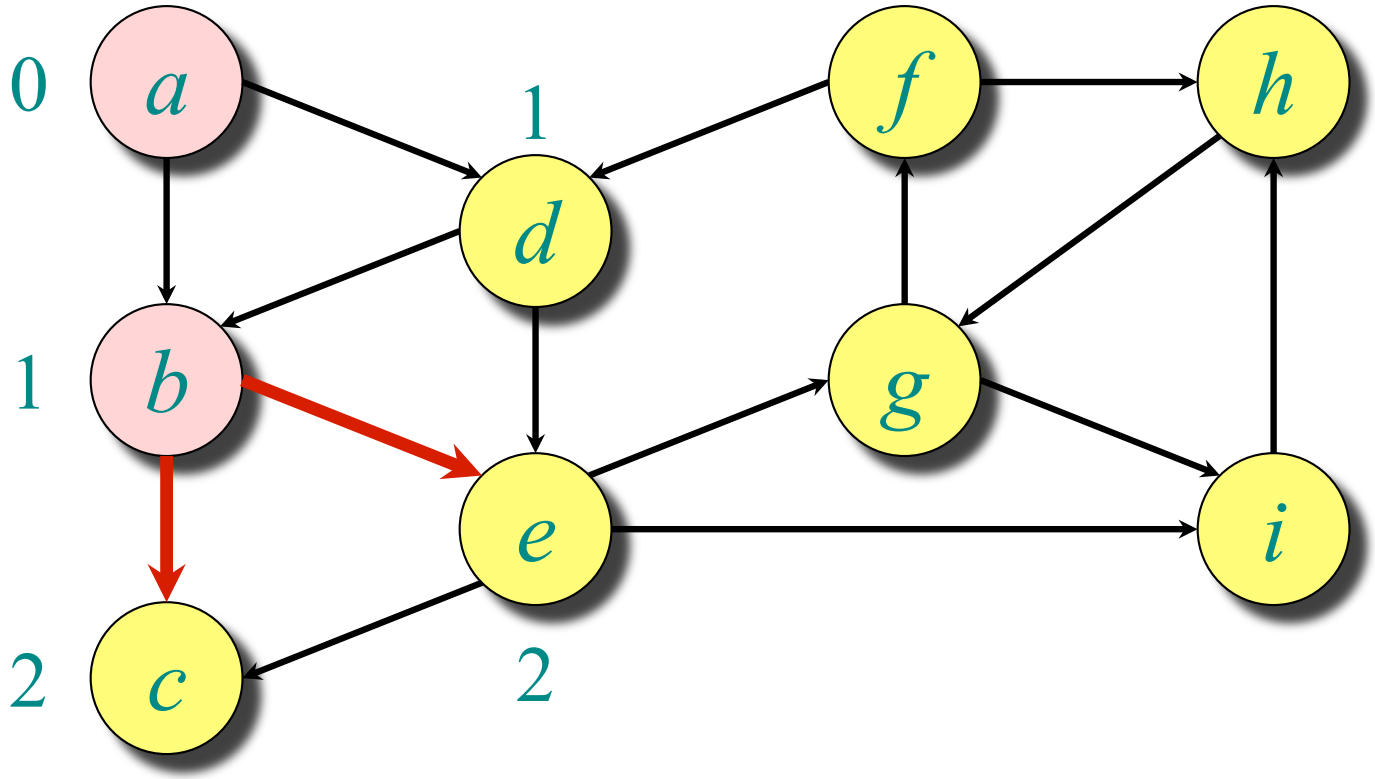

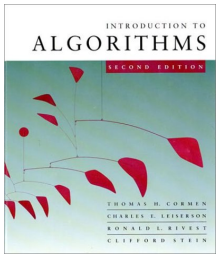

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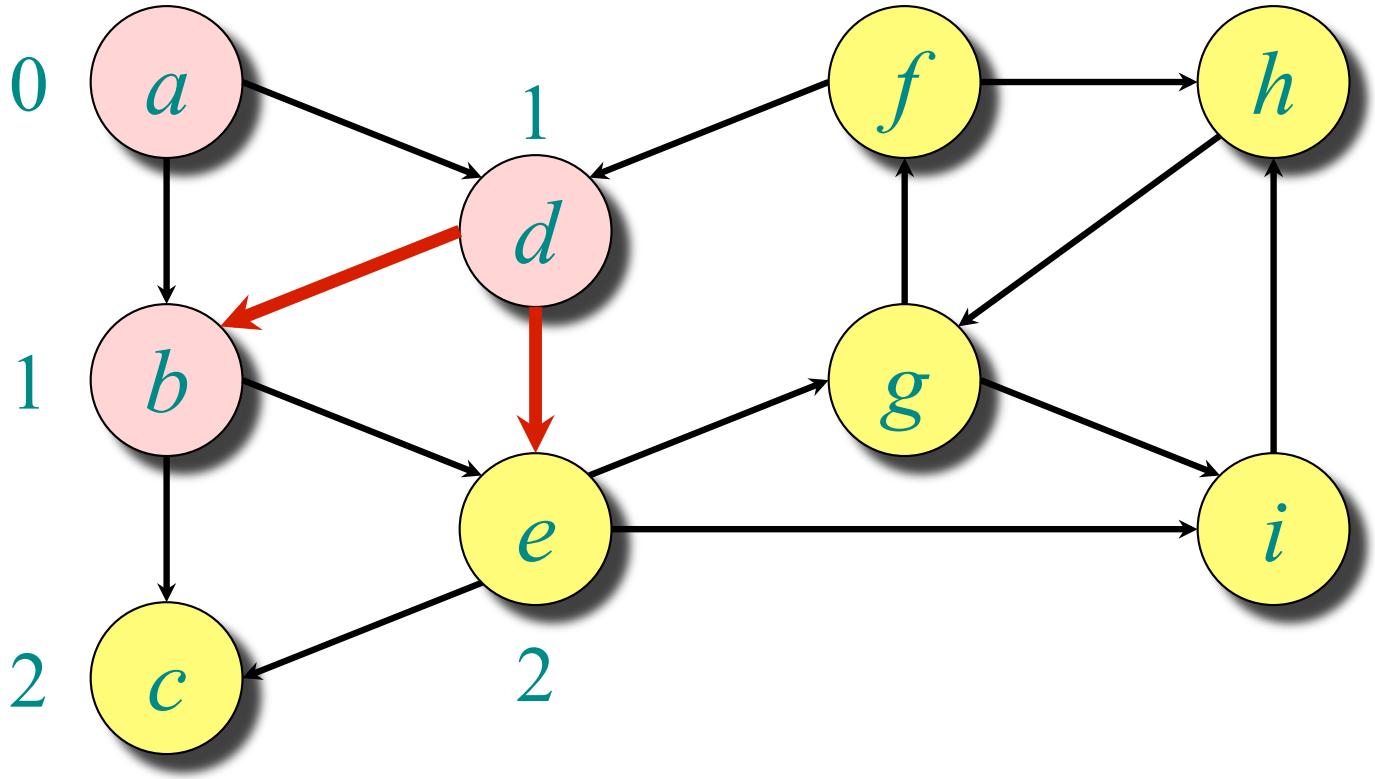
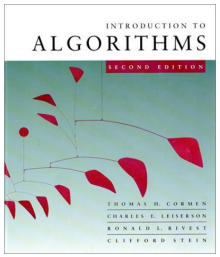
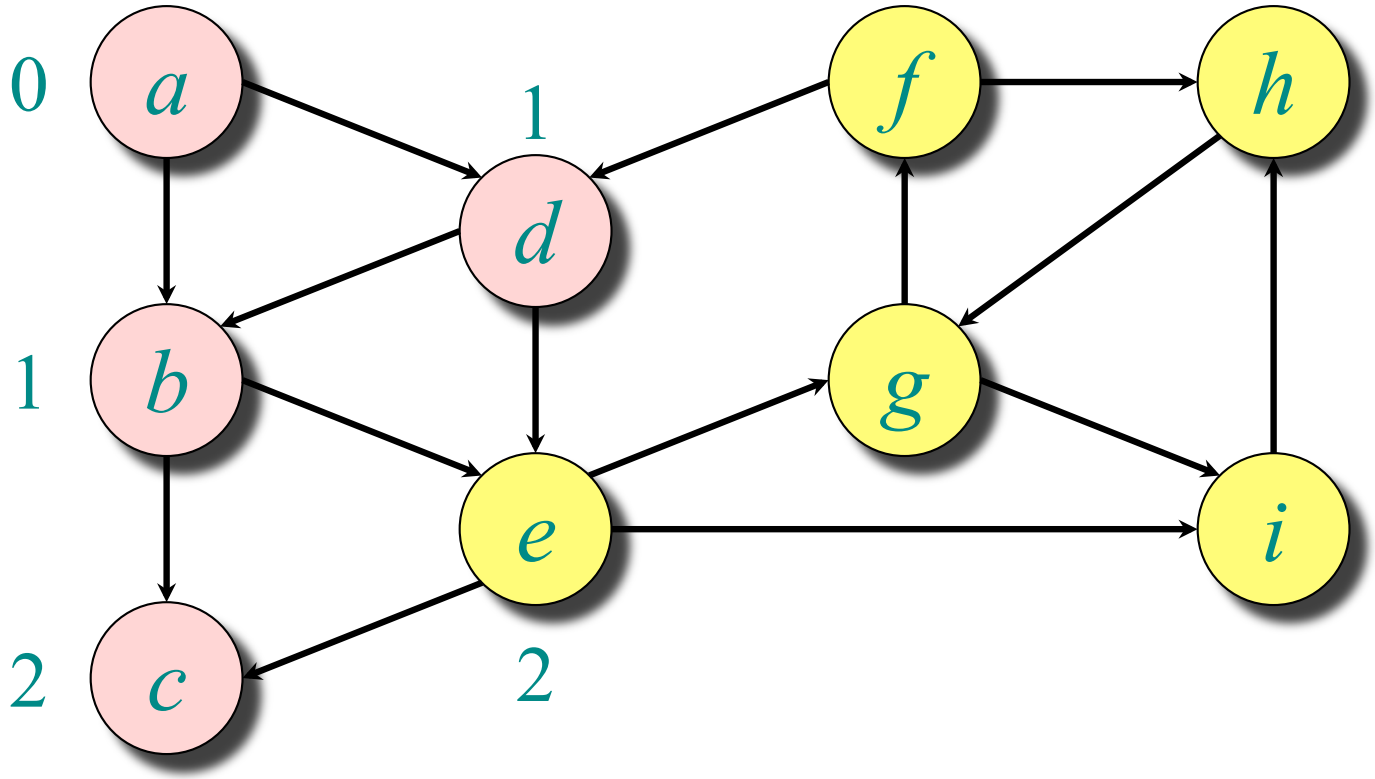


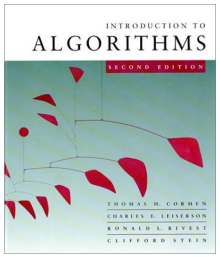
2 2

Q: a b d c e



Example of breadth-first search





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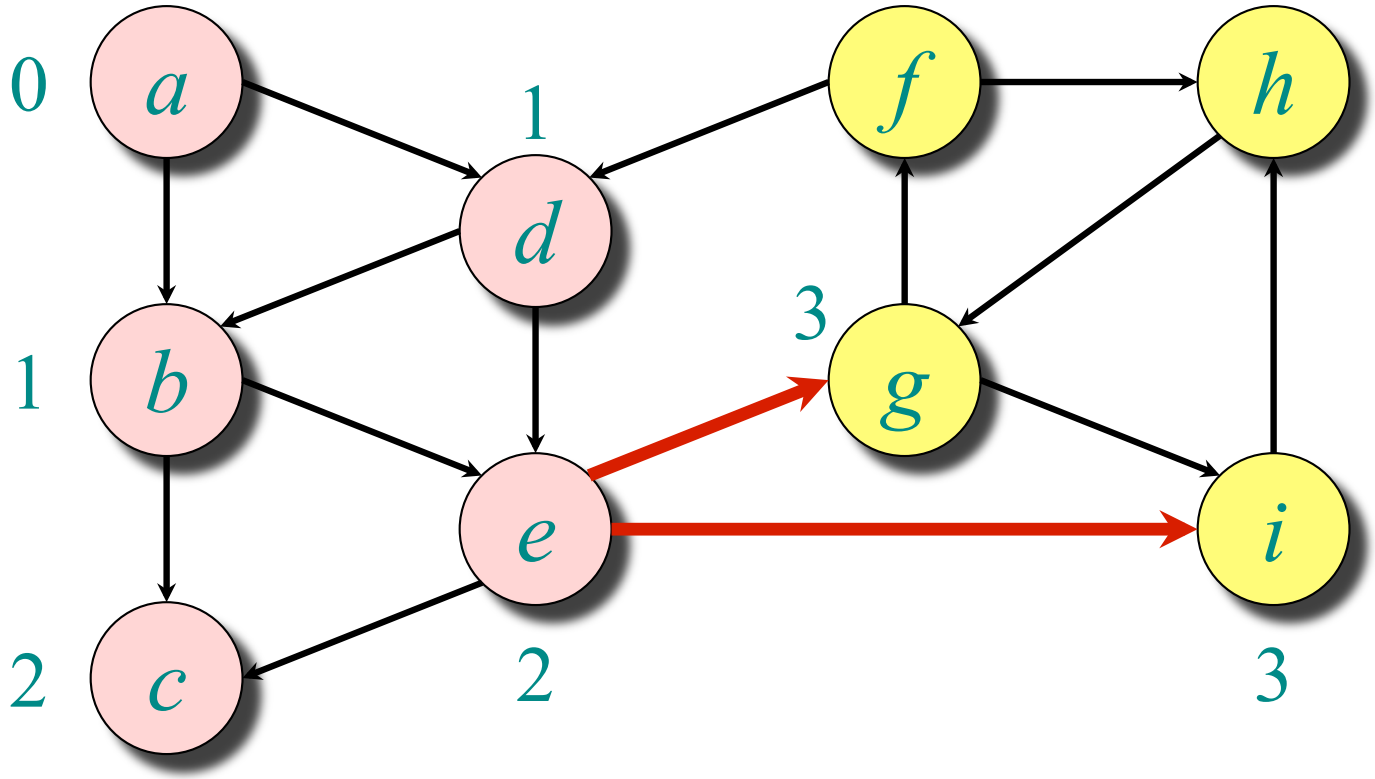
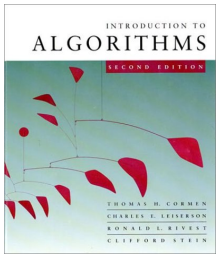
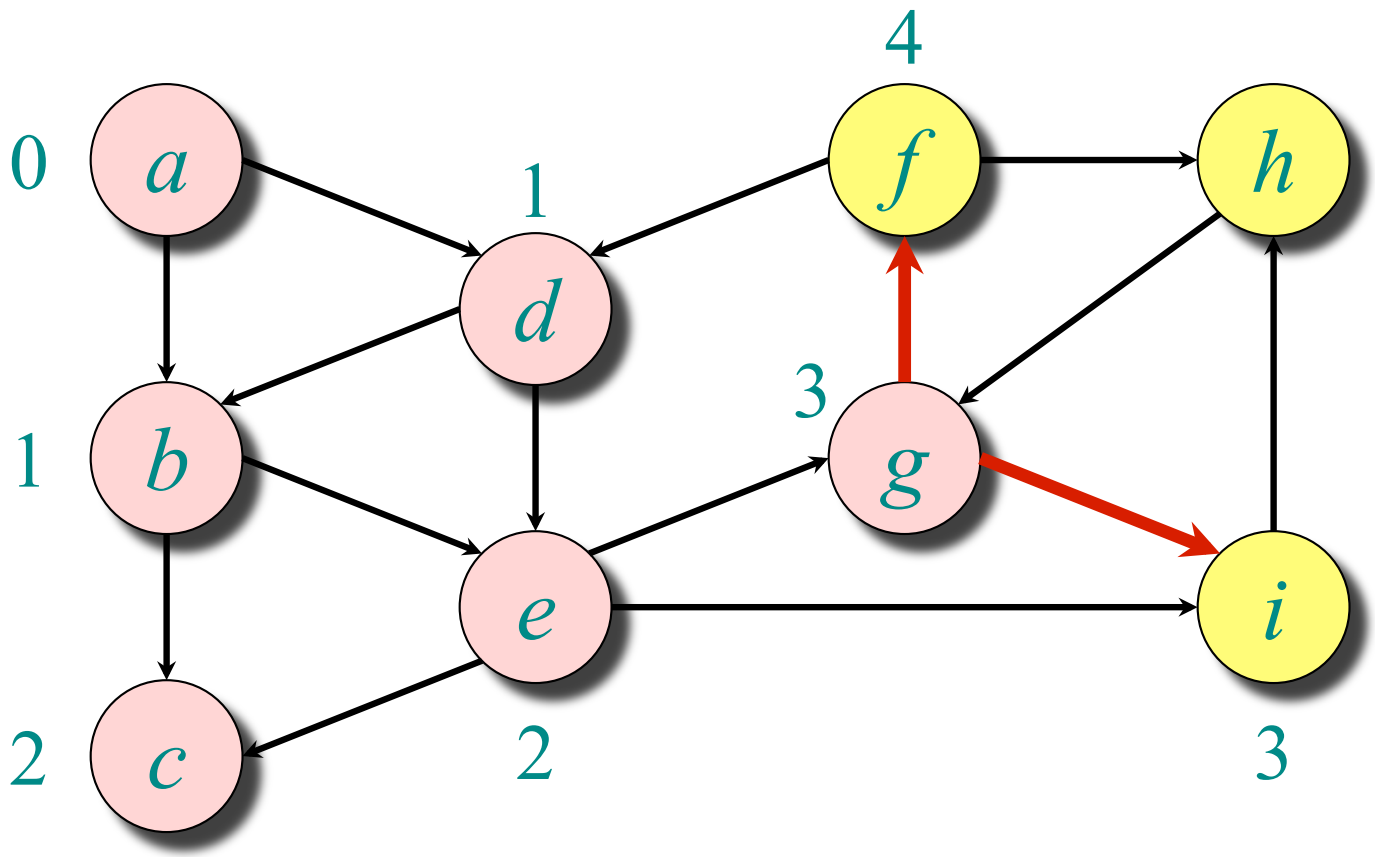


3 3

Q: a b d c e g i

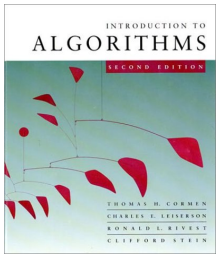


Example of breadth-first search



3 4

Q: *a b d c e g i f*



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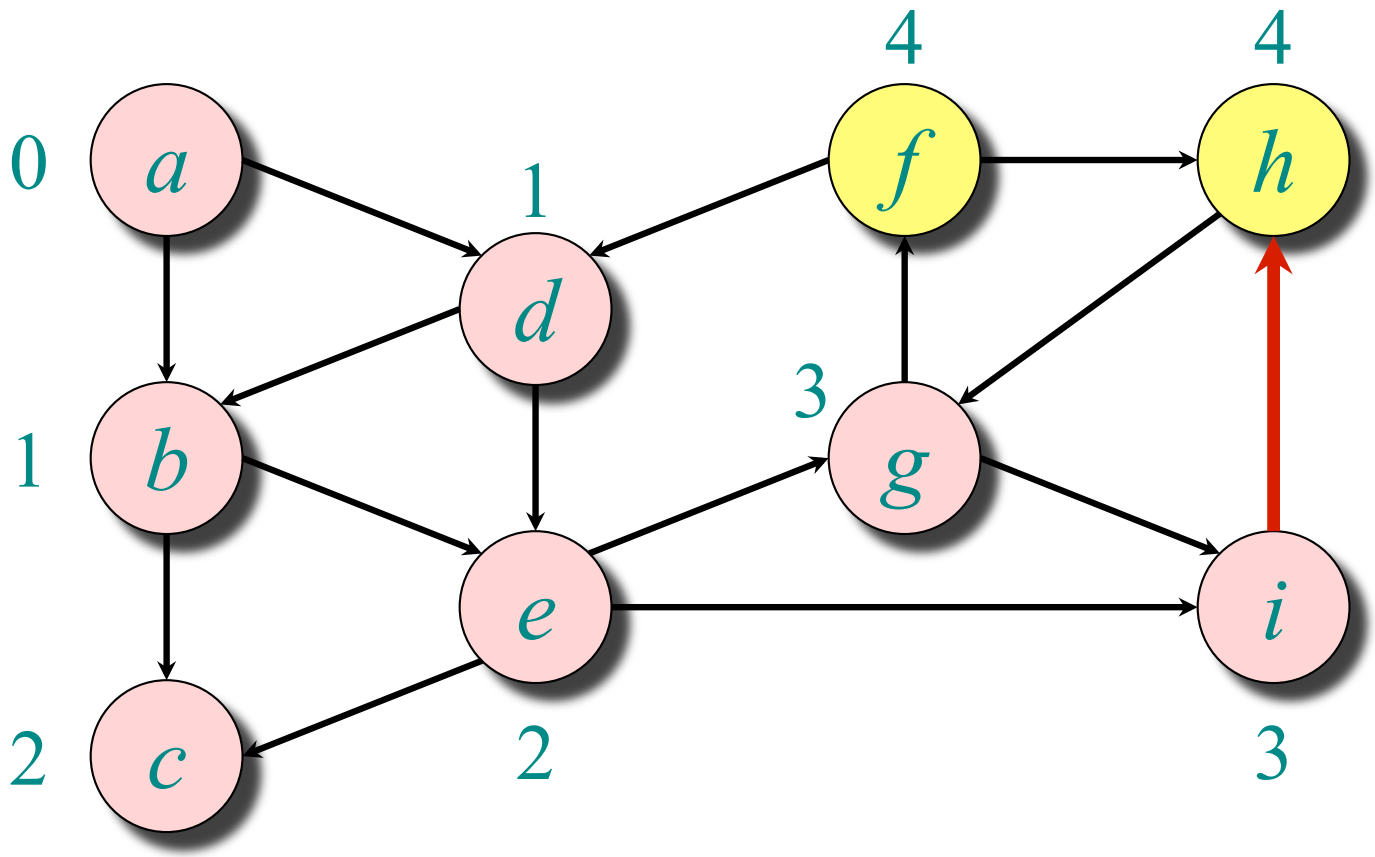
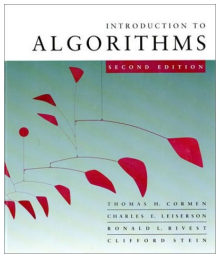
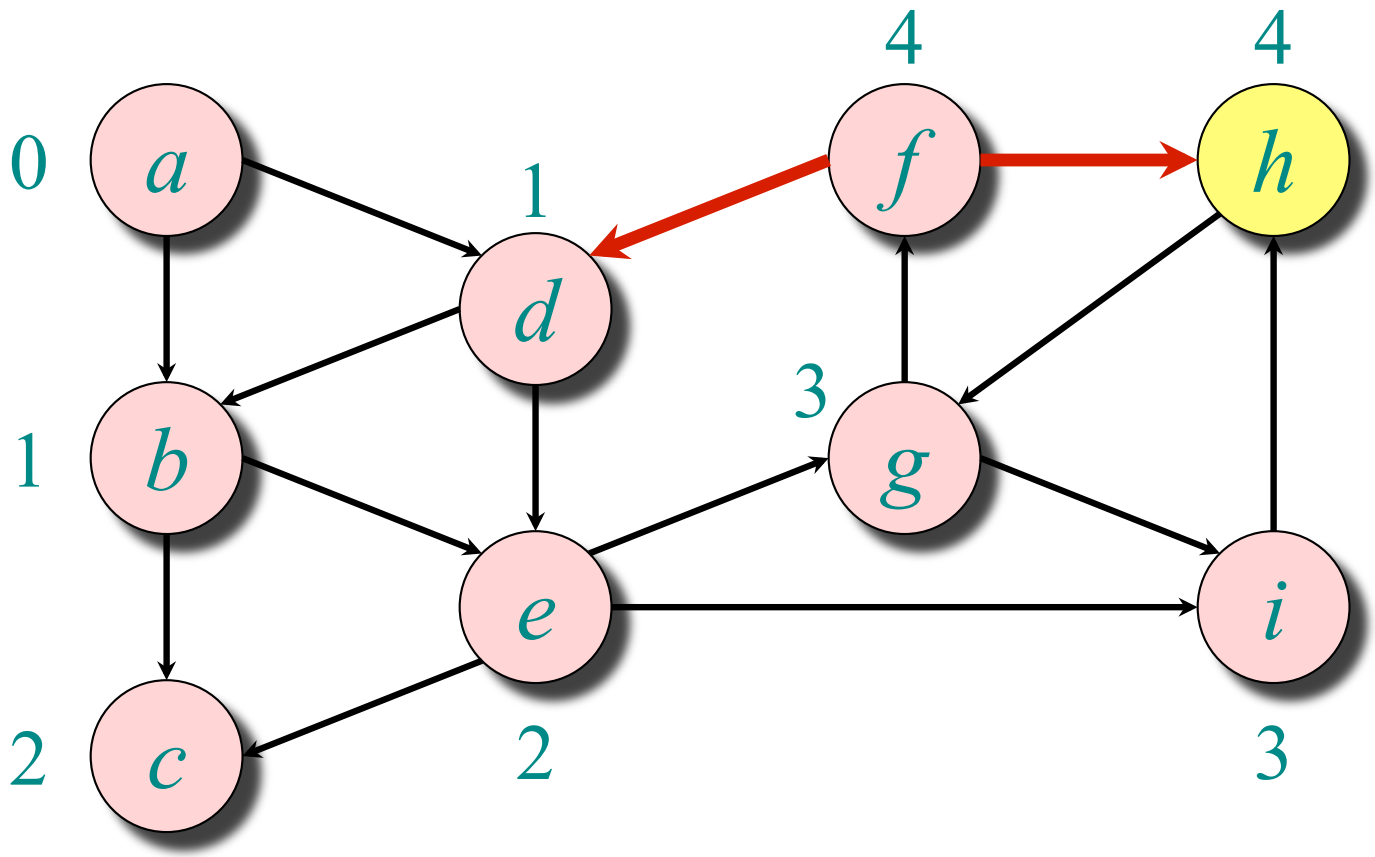


4 4

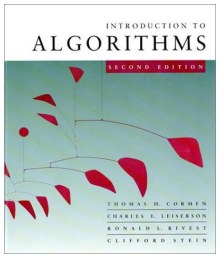
Q: a b d c e g i f h



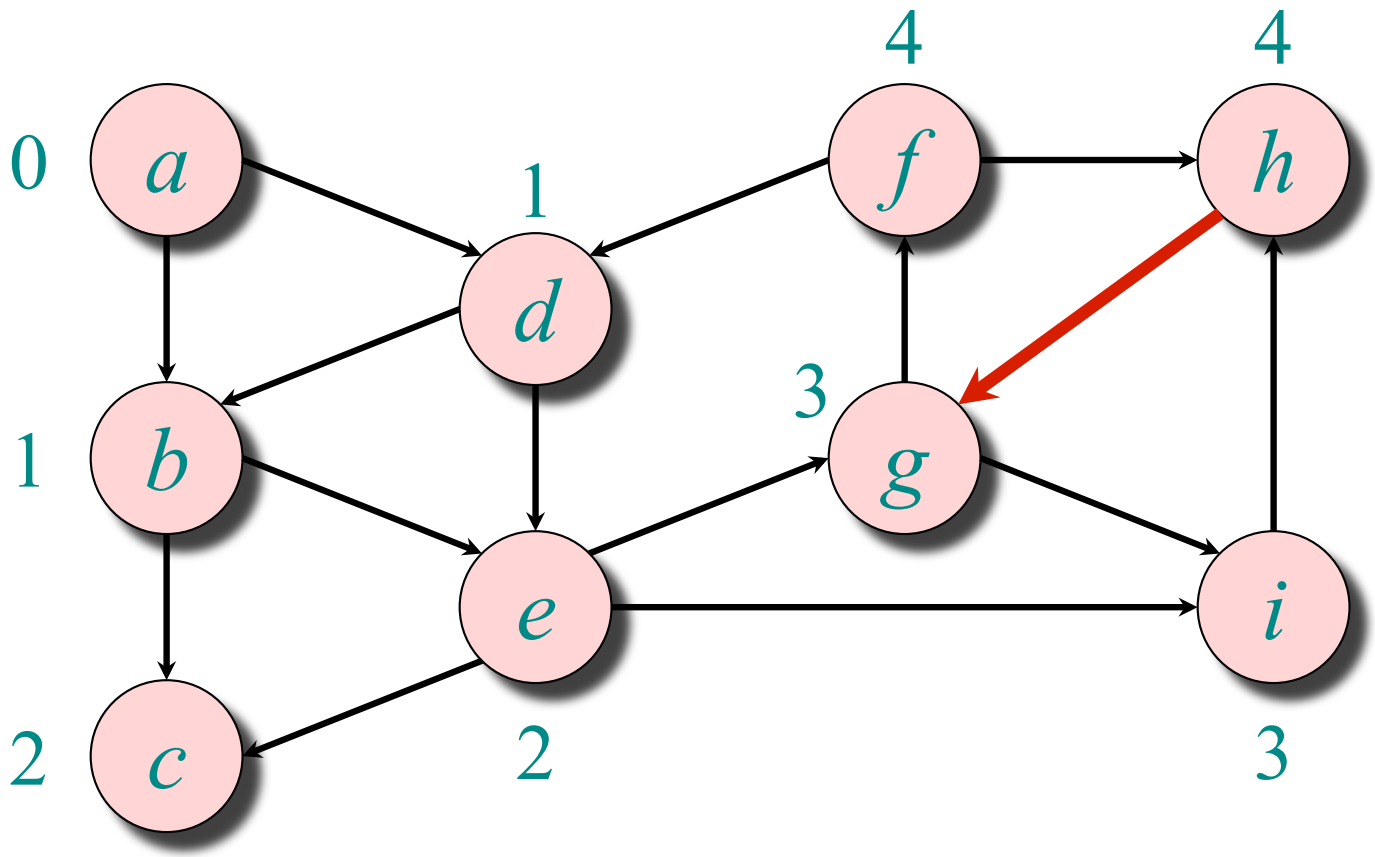
Example of breadth-first search



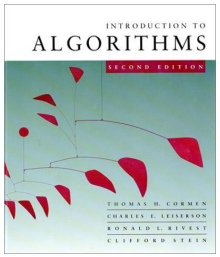
Q: a b d c e g i f h



Example of breadth-first search



Q: a b d c e g i f h



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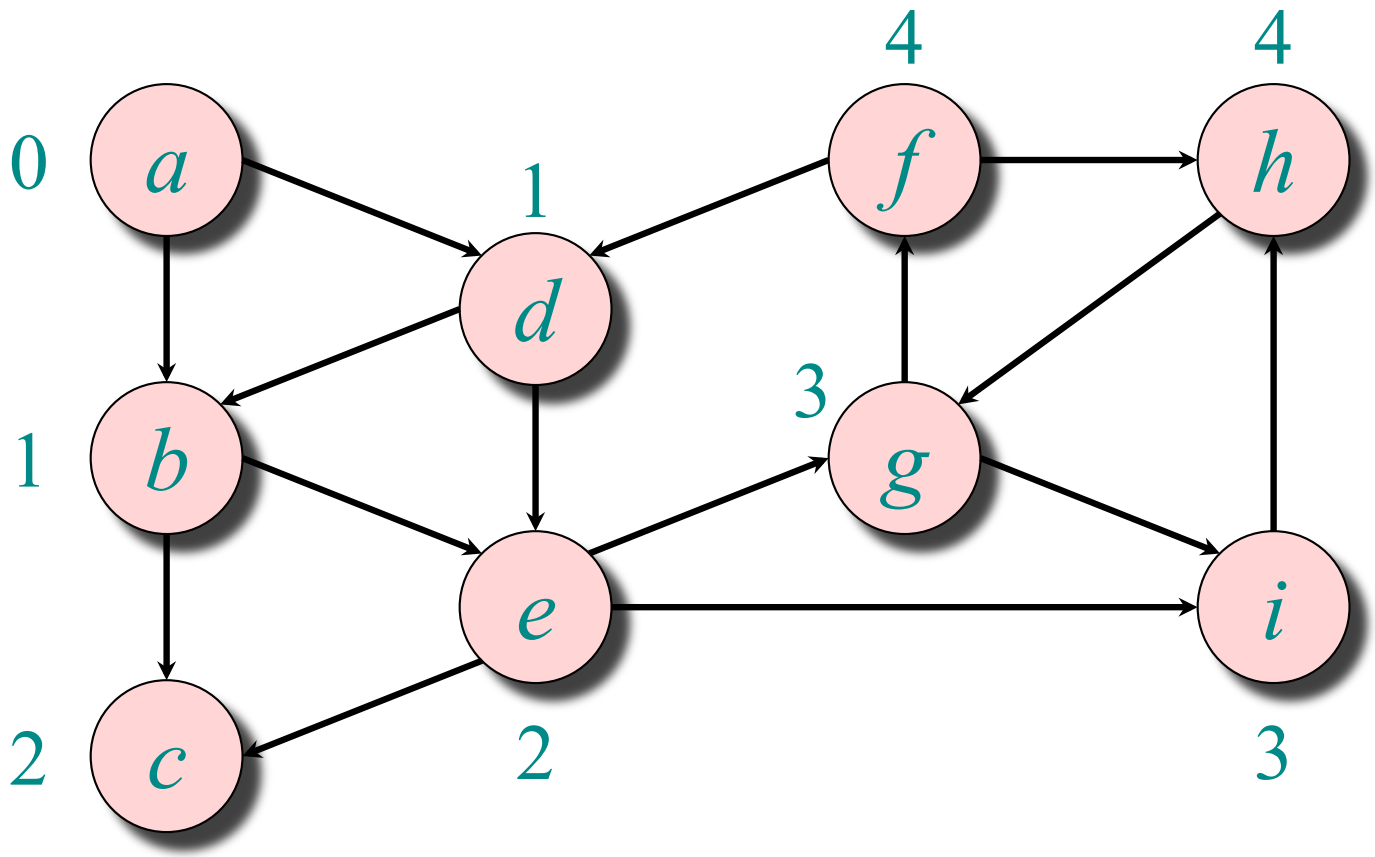


Q: *a b d c e g i f h*

Thank you.

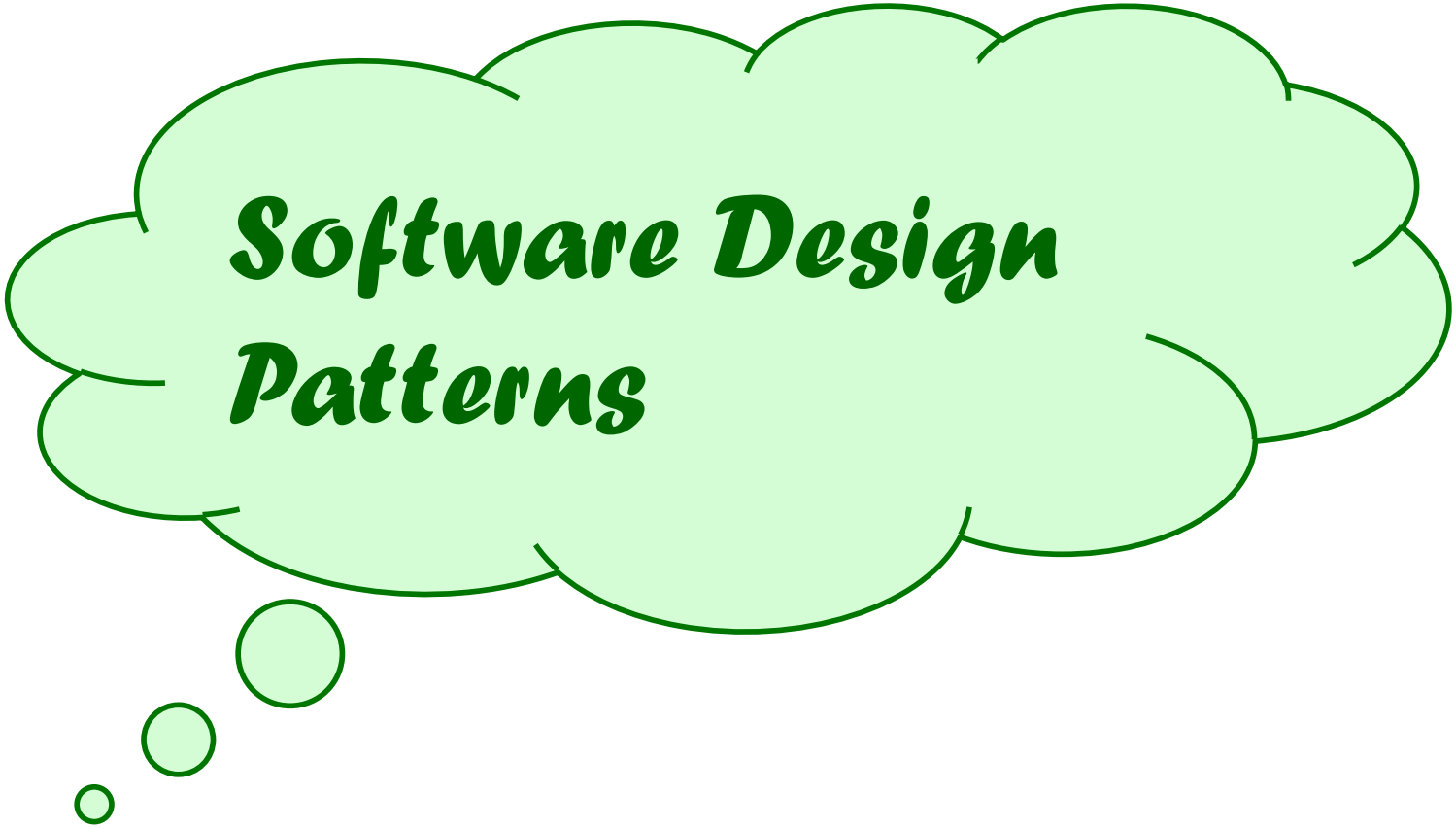
Q & A

CS3230 Algorithm Analysis

"Math for AA"

- **Lecture Topics and Readings**
  - Simple Algorithm Analysis — CS1020/2010
  - Algorithm Analysis Pattern — [LHW] A&E
  - Asymptotic Notations — [CLRS]-C3
  - Functions & Summations (Review) — [CLRS]-C3, App

Quickly get up to speed.

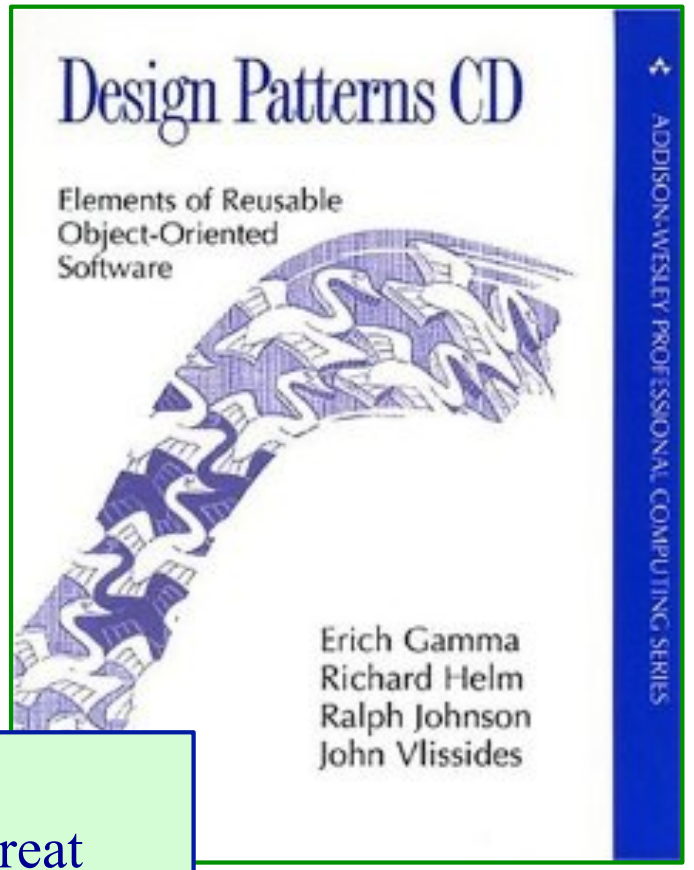


Software Design Patterns

- **Started with GoF book, 1995**
  - **Gave list of 23 patterns**

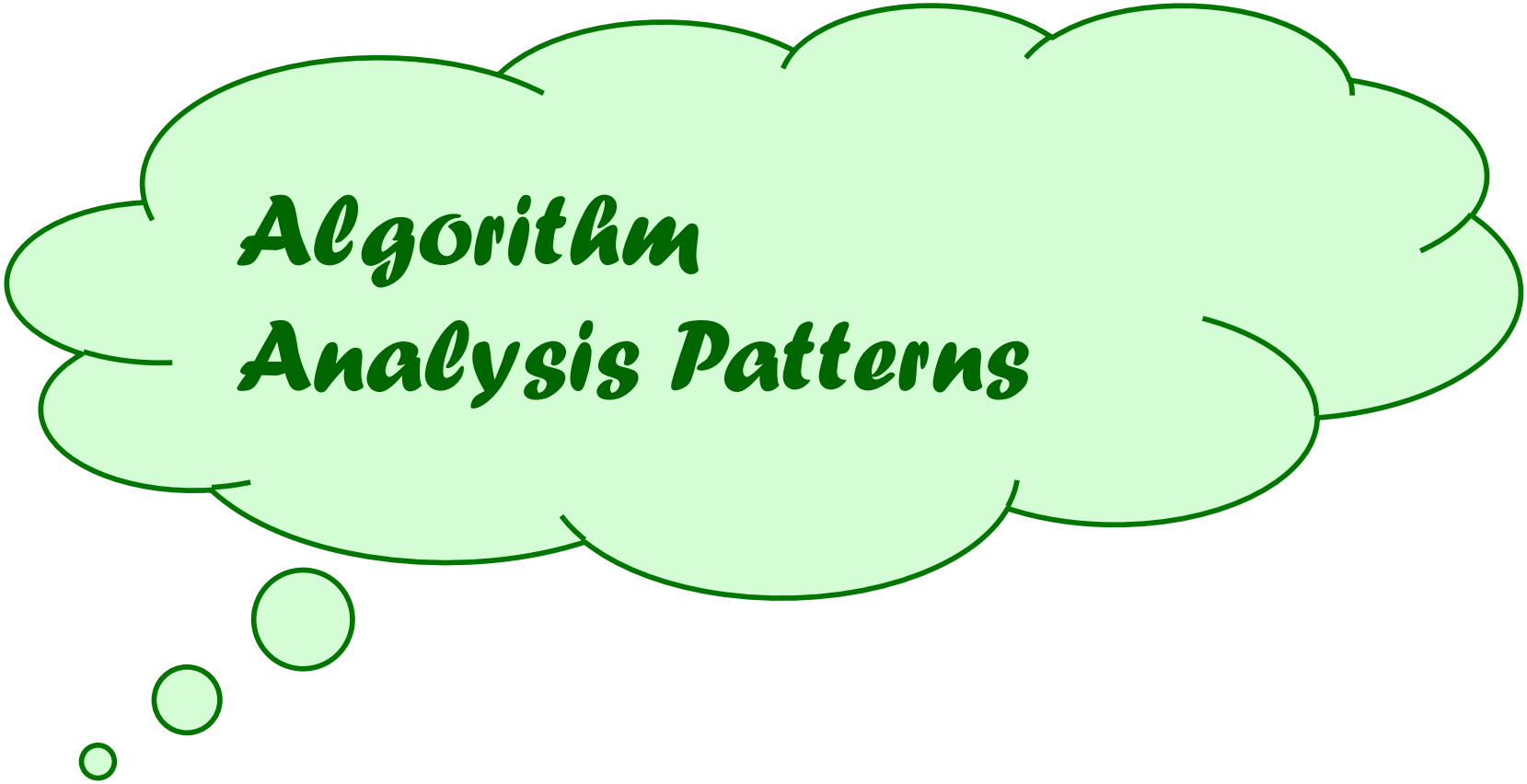
Design Patterns CD: Elements of Reusable Object-Oriented Software (Professional Computing) [CD-ROM]

Erich Gamma (Author), Richard Helm (Author), Ralph Johnson (Author), John Vlissides (Author)



"Published in 1995, Design Patterns: Elements of Reusable Object-Oriented Software has elicited a great deal of praise from the press and readers. The 23 patterns contained in the book have become an essential resource for anyone developing reusable software designs."

How about…

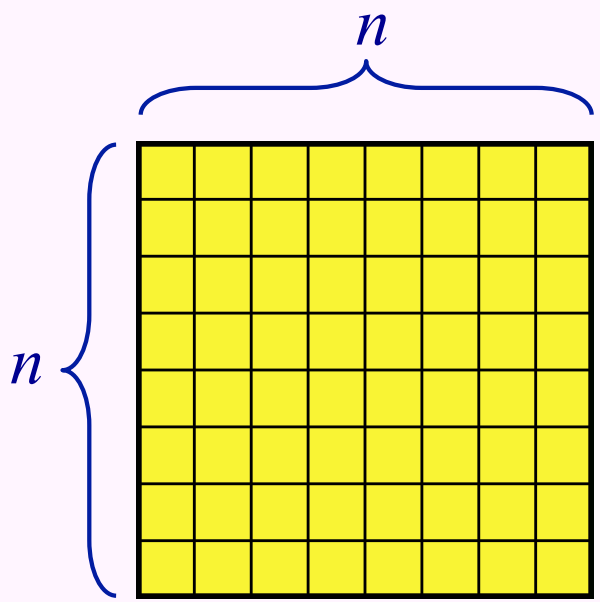


Algorithm-Analysis Pattern?

Algorithm Analysis Pattern

AA-Pattern: 2D-Box

$$T(n) = \sum_{k=1}^{n} cn = \sum_{k=1}^{n} \left(\sum_{j=1}^{n} c \right) = \Theta(n^2)$$



$T(n) = n^2$

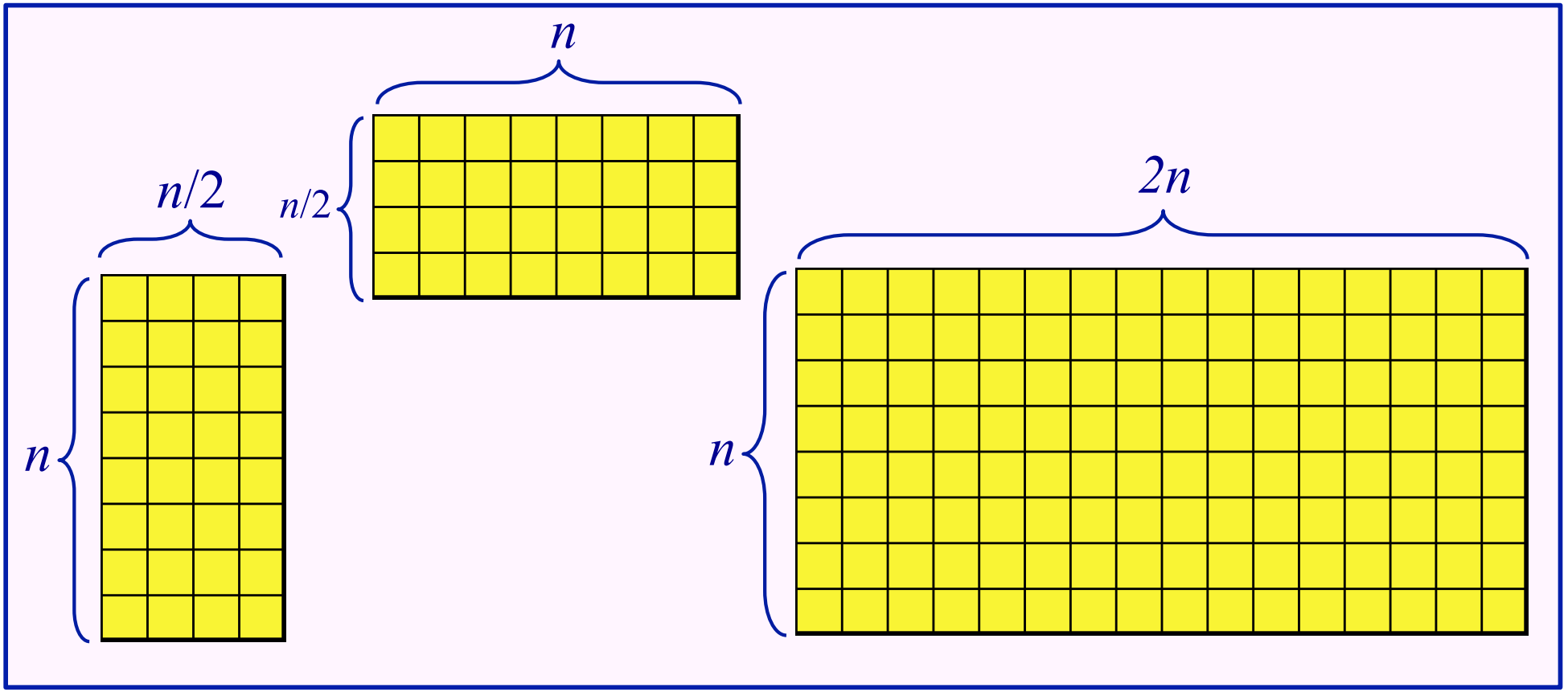
Code Pattern

```
MATRIX-SUM (M, n, n) ▷ M[1..n; 1..n]
    Sumsf ← 0
    for k ← 1 to n
        for j ← 1 to n
            do Sumsf ← Sumsf + M[k, j]
    Sum ← Sumsf
```

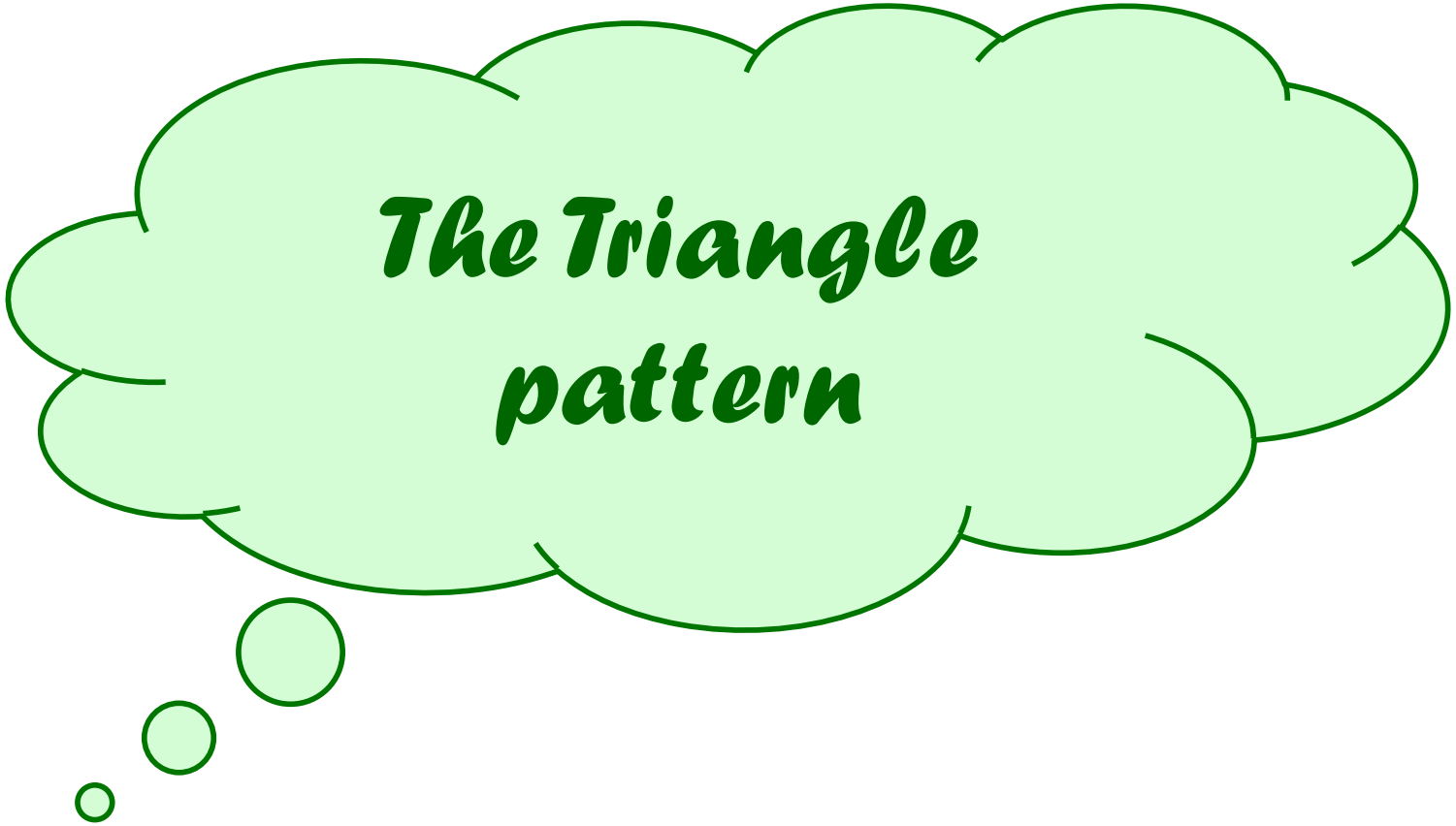
2D-Box Pattern (variants)

AA-Pattern: 2D-Box

$$T(n) = \sum_{k=1}^{dn} cn = \sum_{k=1}^{dn} \left(\sum_{j=1}^{n} c \right) = \Theta(n^2)$$



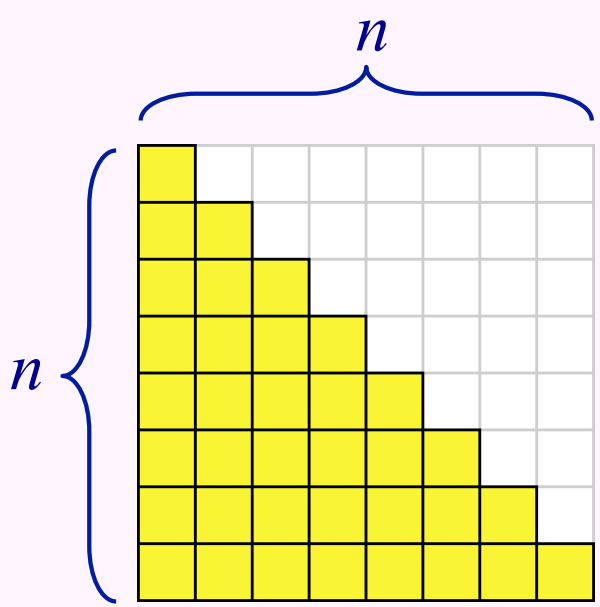
Algorithm-Analysis Pattern?



Algorithm Analysis Pattern

AA-Pattern: Triangle

$$T(n) = \sum_{k=1}^{n} ck = \sum_{k=1}^{n} \left(\sum_{j=1}^{k} c \right) = \Theta(n^2)$$



$T(n) = n(n+1)/2$

Code Pattern

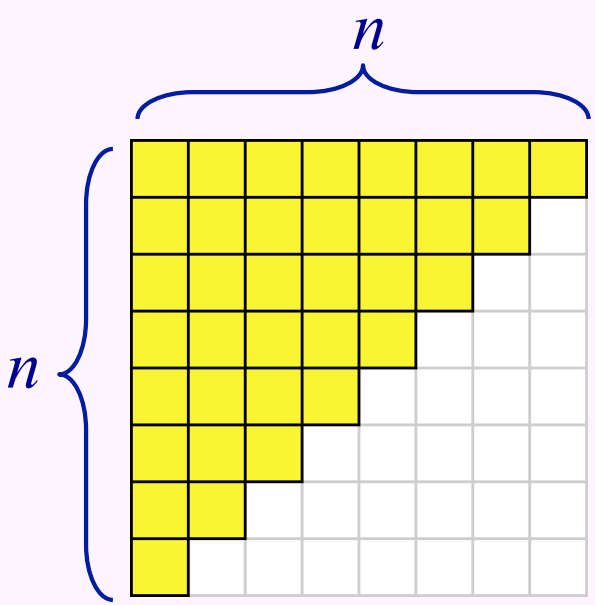
```
TRIANGLE-SUM (M, n, n)  ▷ M[1..n; 1..n]
    Sumsf ← 0
    for k ← 1 to n
        for j ← 1 to k
            do Sumsf ← Sumsf + M[k, j]
    Sum ← Sumsf
```


Hon Wai Leong, NUS

Triangle Pattern (variants)

AA-Pattern: Triangle

$$T(n) = \sum_{k=1}^{n} ck = \sum_{k=1}^{n} \left(\sum_{j=1}^{k} c \right) = \Theta(n^2)$$



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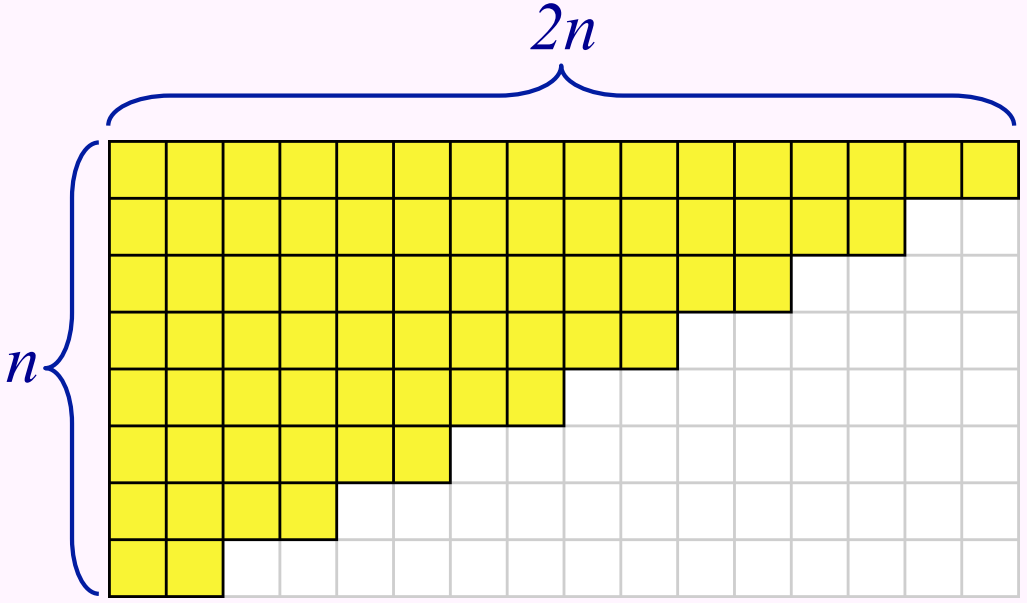

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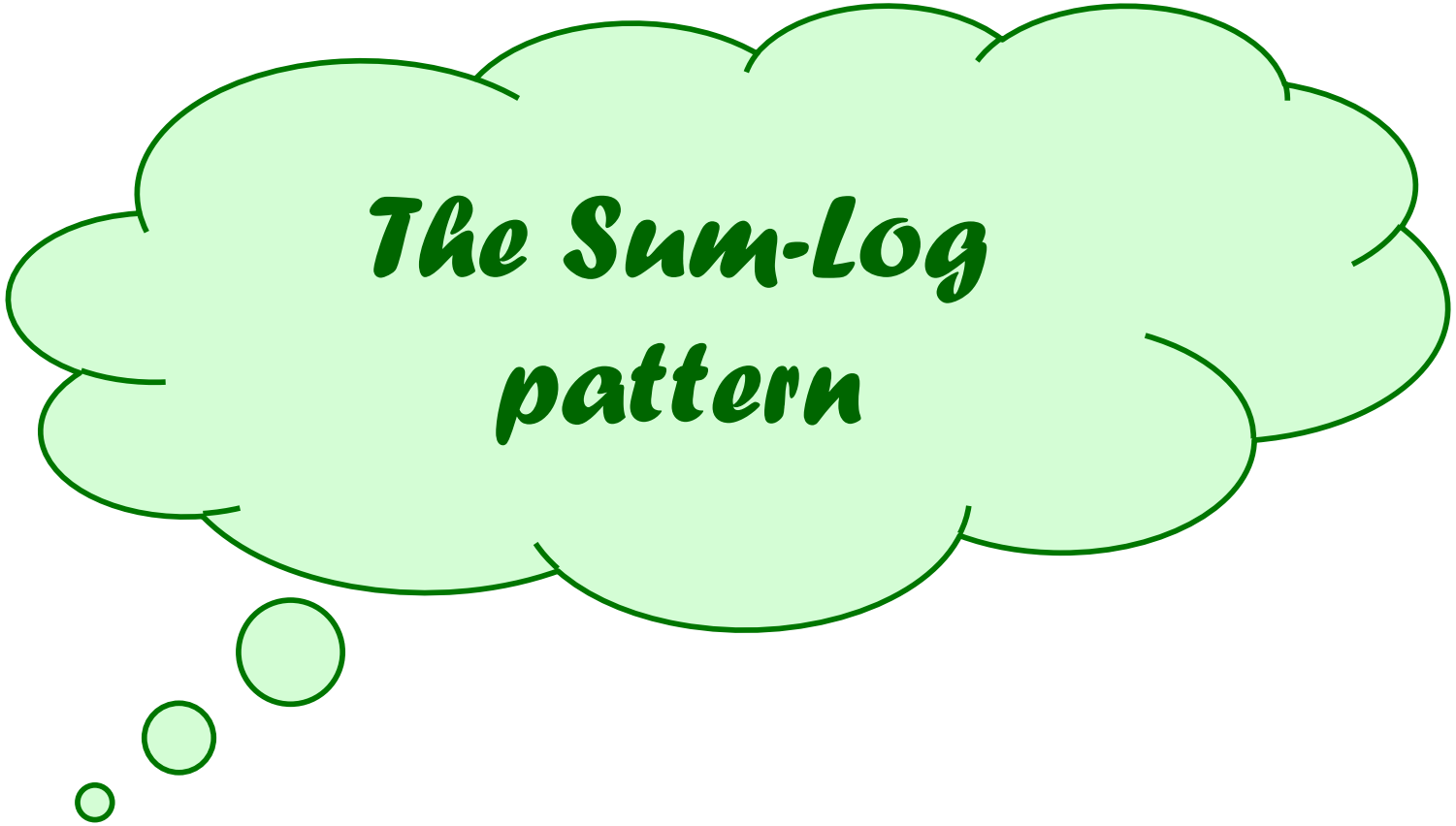
Algorithm-Analysis Pattern?

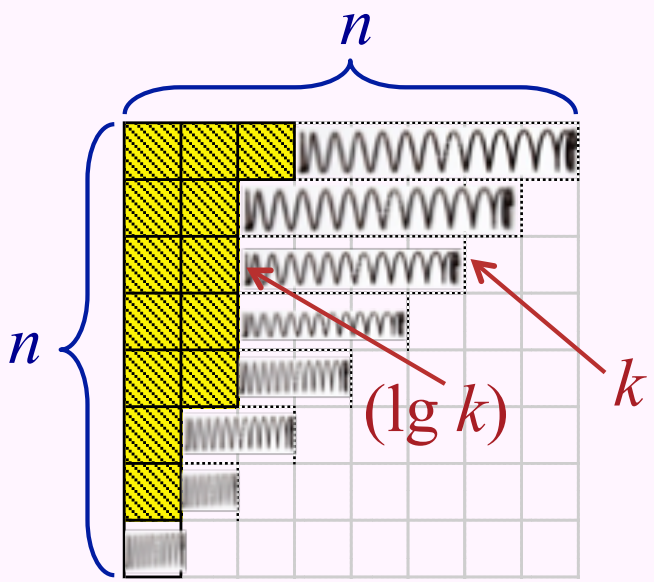


Sum-Log Pattern

AA-Pattern: Sum-Log

$$T(n) = \sum_{k=1}^{n} (\lg k) = \Theta(n \lg n)$$

$$= (\lg 1) + (\lg 2) + \ldots + (\lg n)$$



$T(n) = \Theta(n \lg n)$

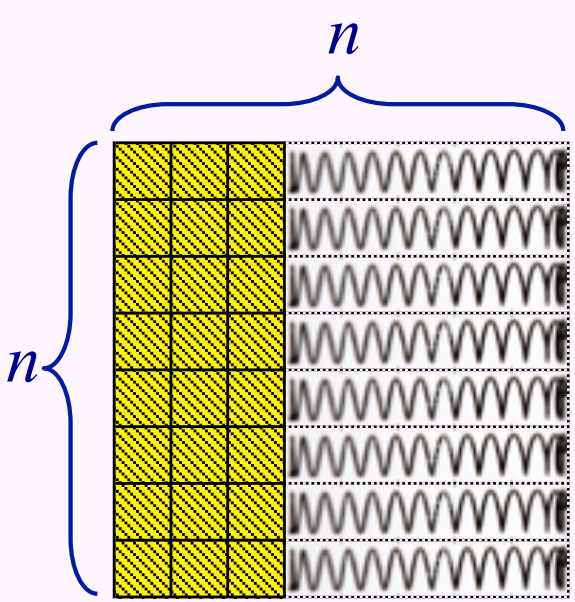
Code Pattern

```
SUM-LOG (A, n, X)      ▷ A[1..n]
    (* bin-search A[1..k], k=1,2,…,n *)
    for k ← n downto 1
        Binary-Search(A, 1, k, X)
```

Sum-Log Pattern (variants)

AA-Pattern: Sum-Log

$$T(n) = \sum_{k=1}^{n} (\lg k) = \Theta(n \lg n)$$
$$= (\lg 1) + (\lg 2) + \ldots + (\lg n)$$



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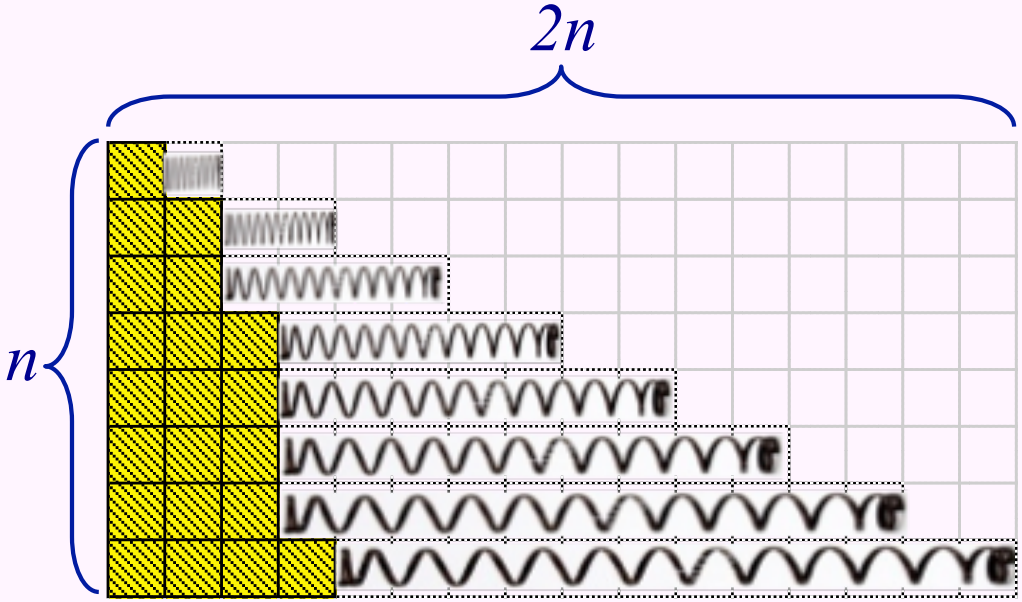

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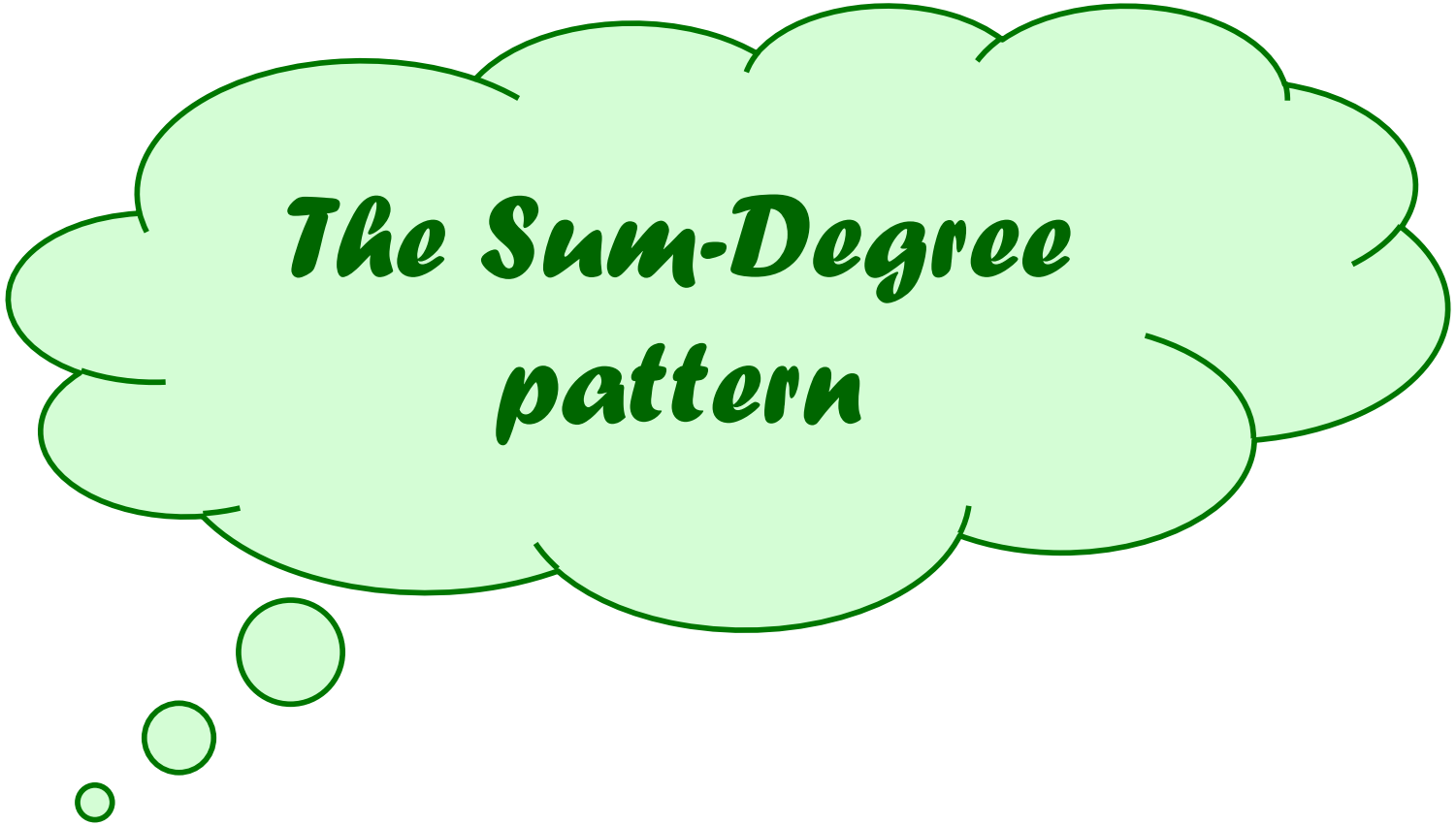
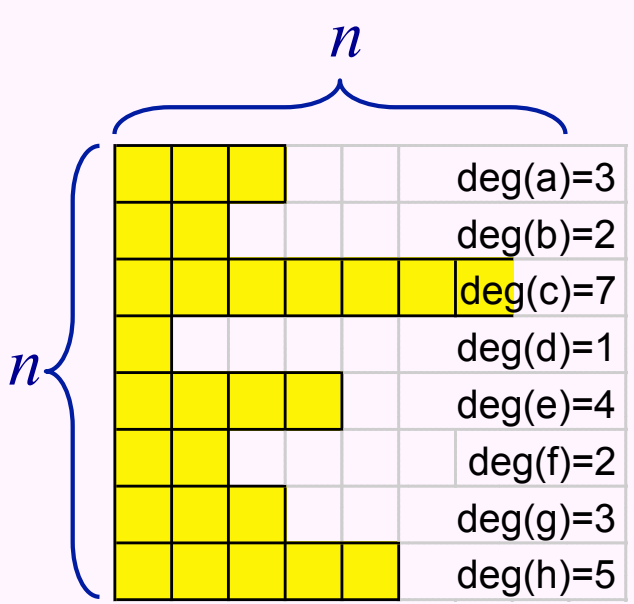
Algorithm-Analysis Pattern?



Sum-Degree Pattern

AA-Pattern:
Sum-Degree

$$T(n,m) = \sum_{k=1}^{n} \deg(v_k) = 2m = \Theta(n+m)$$
$$= \deg(v_1) + \deg(v_2) + \ldots + \deg(v_n)$$



$T(n,m) = \Theta(n+m)$

Code Pattern

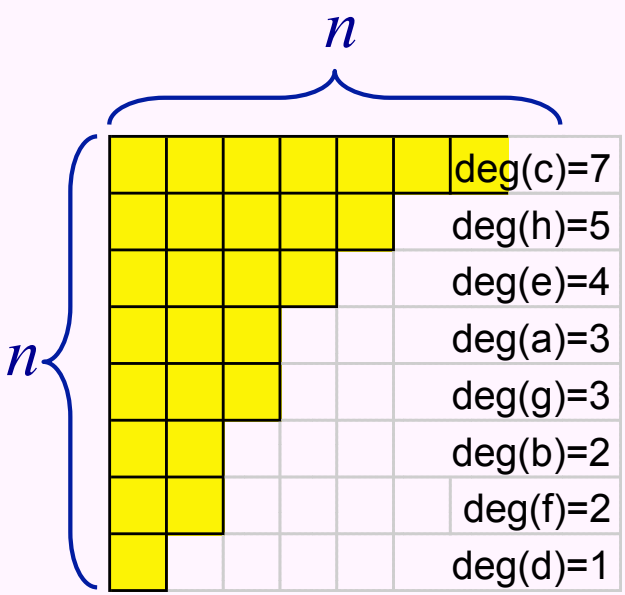
```
SUM-DEGREE (G, n, m)    ▷ G=(V,E)
    (* V has n nodes, E has m edges *)
    for each v in G do
        for each w in Adj(v)
            Print details of (v, w)
```

Sum-Degree Pattern (variants)

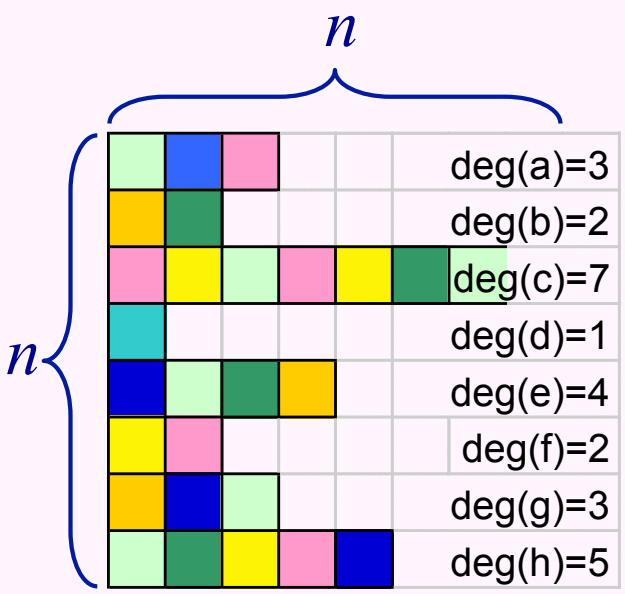
AA-Pattern: Sum-Degree

$$T(n,m) = \sum_{k=1}^{n} \deg(v_k) = 2m = \Theta(m)$$

$$= \deg(v_1) + \deg(v_2) + \ldots + \deg(v_n)$$



$T(n) = \Theta(n+m)$



$T(n) = \Theta(n+m)$

HW for myself and students…

This is for CS3230 staff and students:

Anyone interested in volunteering to help in this, please email me and say you like to help.

Thanks in advance.

- For each AA-pattern, list the different algorithms that has that AA-pattern.
- Do it for all the algorithms covered in CS1020, CS2010, CS2020.
- Discover new AA-patterns

HW for myself and students…

- **IDEA by HW:**
  - Create flash-card of each AA-pattern.
  - From page has definition of pattern, back page has variant pattern and corresponding algorithms;

Looking for people who are good with artistic design to prototype some
nice, cool, AA-pattern flash-cards

Thank you.

Q & A

Asymptotic Notation, Review of Functions & Summations

(Notes from Prof. David Plaisted,
UNC, Chapel Hill)

READ: [CLRS]-Ch.2-3, App-A

Asymptotic Notation, Review of Functions & Summations

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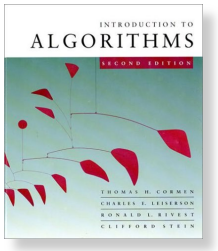
Asymptotic Complexity

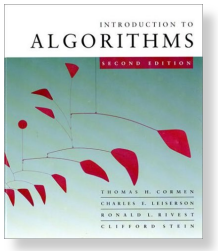
- Running time of an algorithm as a function of input size n **for large n**.
- Expressed using only the **highest-order term** in the expression for the exact running time.
  - Instead of exact running time, say $\Theta(n^2)$.
- Describes behavior of function in the limit.
- Written using ***Asymptotic Notation***.



Asymptotic notation

O-notation (upper bounds):

We write $f(n) = O(g(n))$ if there exist constants $c > 0$, $n_0 > 0$ such that $0 \le f(n) \le cg(n)$ for all $n \ge n_0$.



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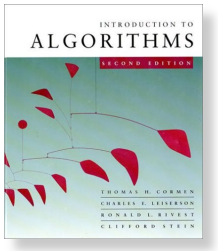
EXAMPLE: $2n^2 = O(n^3)$ $\quad (c = 1, n_0 = 2)$



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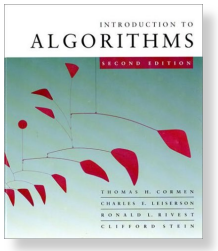
functions, not values



Asymptotic notation

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EXAMPLE: $2n^2 = O(n^3)$ $(c = 1, n_0 = 2)$

functions, not values

funny, "one-way" equality

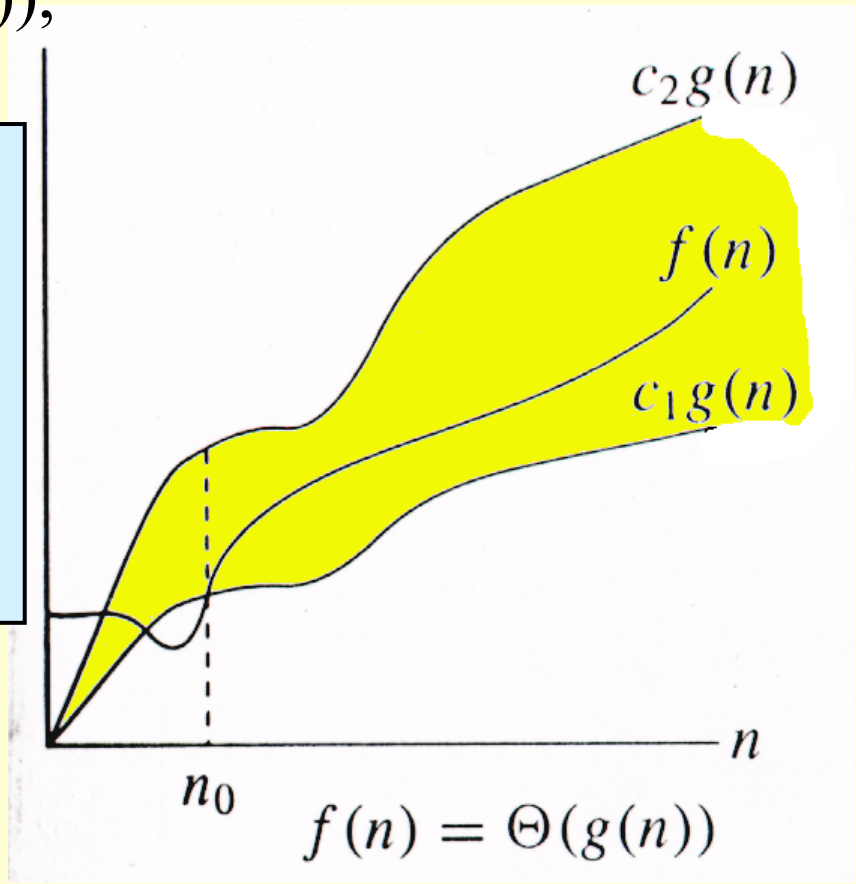
Asymptotic Notation

- Θ, O, Ω, o, ω
- Defined for functions over the natural numbers.
  - **Ex:** $f(n) = \Theta(n^2)$.
  - Describes how $f(n)$ grows in comparison to n^2.
- Define a ***set*** of functions; in practice used to compare two function sizes.
- The notations describe different rate-of-growth relations between the defining function and the defined set of functions.

Θ-notation

For function $g(n)$, we define $\Theta(g(n))$, big-Theta of $g(n)$, as the set:

$\Theta(g(n)) = \{f(n)$:
∃ positive constants c_1, c_2, and n_0,
such that $\forall n \geq n_0$,
we have $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$
$\}$



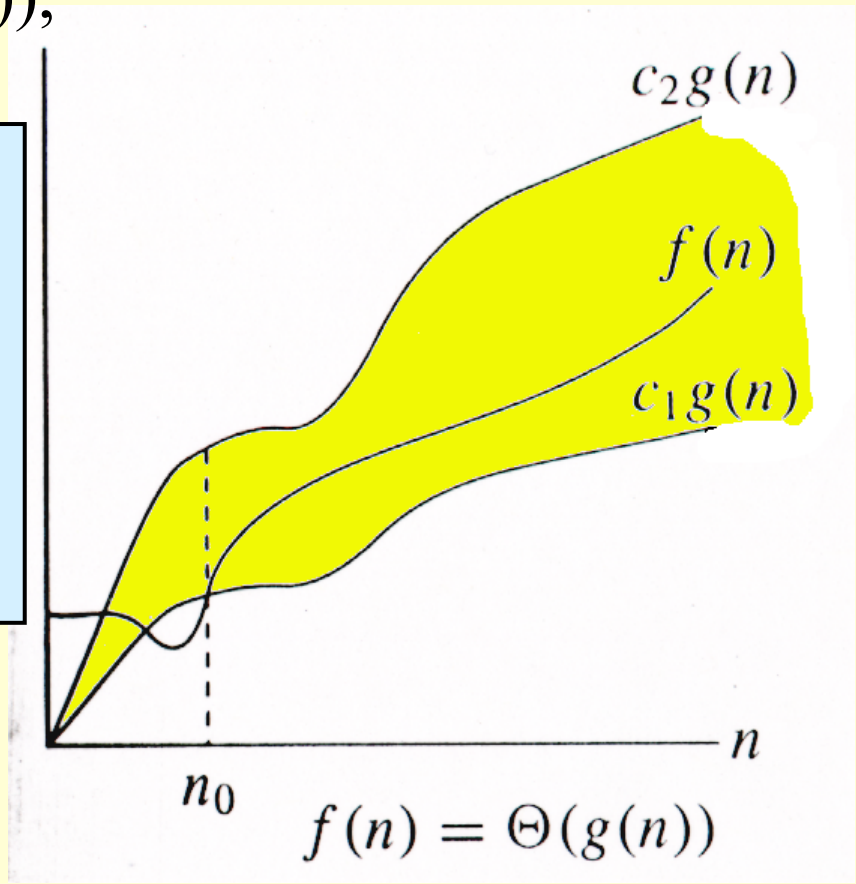
Intuitively: Set of all functions that have the same *rate of growth* as $g(n)$.

$g(n)$ is an *asymptotically tight bound* for $f(n)$.

Θ-notation

For function $g(n)$, we define $\Theta(g(n))$, big-Theta of n, as the set:

$\Theta(g(n)) = \{f(n)$:
∃ positive constants c_1, c_2, and n_0,
such that $\forall n \geq n_0$,
we have $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$
$\}$



Technically, $f(n) \in \Theta(g(n))$.
Older usage, $f(n) = \Theta(g(n))$.
I'll accept either…

***f*(*n*) and *g*(*n*) are nonnegative, for large *n*.**

Example

$\Theta(g(n)) = \{f(n) :$ ∃ positive constants c_1, c_2, and n_0, such that $\forall n \geq n_0$, $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)\}$

- $10n^2 - 3n = \Theta(n^2)$
- What constants for n_0, c_1, and c_2 will work?
- Make c_1 a little smaller than the leading coefficient, and c_2 a little bigger.
- ***To compare orders of growth, look at the leading term.***
- Exercise: Prove that $n^2/2 - 3n = \Theta(n^2)$

Example

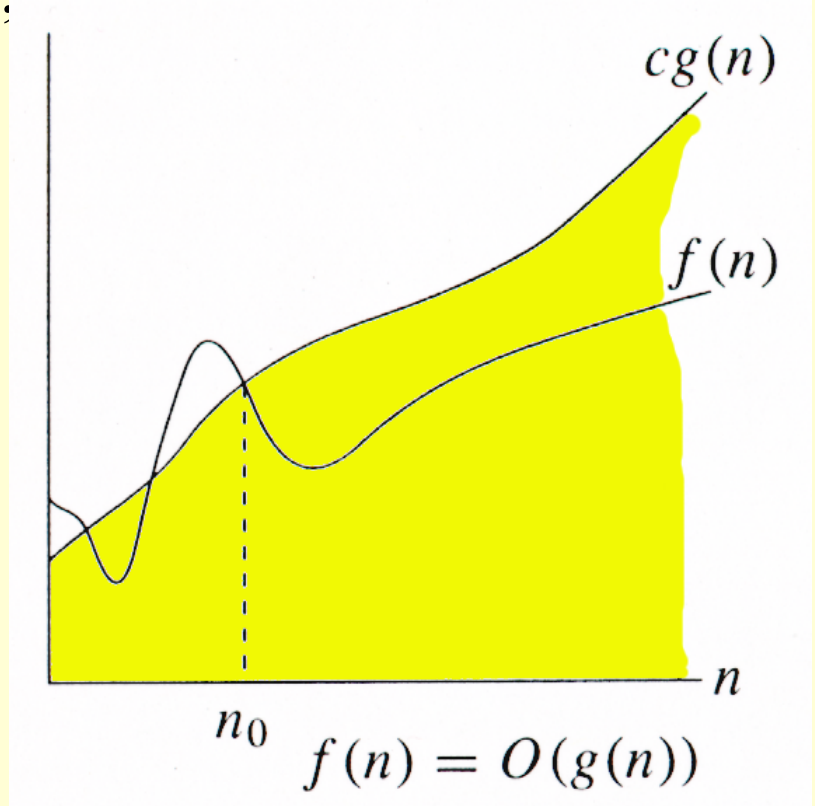
$\Theta(g(n)) = \{f(n) : \exists$ positive constants c_1, c_2, and n_0, such that $\forall n \geq n_0$, $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)\}$

- Is $3n^3 \in \Theta(n^4)$??
- How about $2^{2n} \in \Theta(2^n)$??

O-notation

For function $g(n)$, we define $O(g(n))$, big-O of n, as the set:

$O(g(n)) = \{f(n)$:
∃ positive constants c and n_0,
such that $\forall n \geq n_0$,
we have $0 \leq f(n) \leq cg(n)\}$



Intuitively: Set of all functions whose *rate of growth* is the same as or lower than that of $g(n)$.

$g(n)$ is an *asymptotic upper bound* for $f(n)$.

$f(n) = \Theta(g(n)) \Rightarrow f(n) = O(g(n))$.
$\Theta(g(n)) \subset O(g(n))$.

Examples

$O(g(n)) = \{f(n)$: ∃ positive constants c and n_0, such that $\forall n \geq n_0$, we have $0 \leq f(n) \leq cg(n)\}$

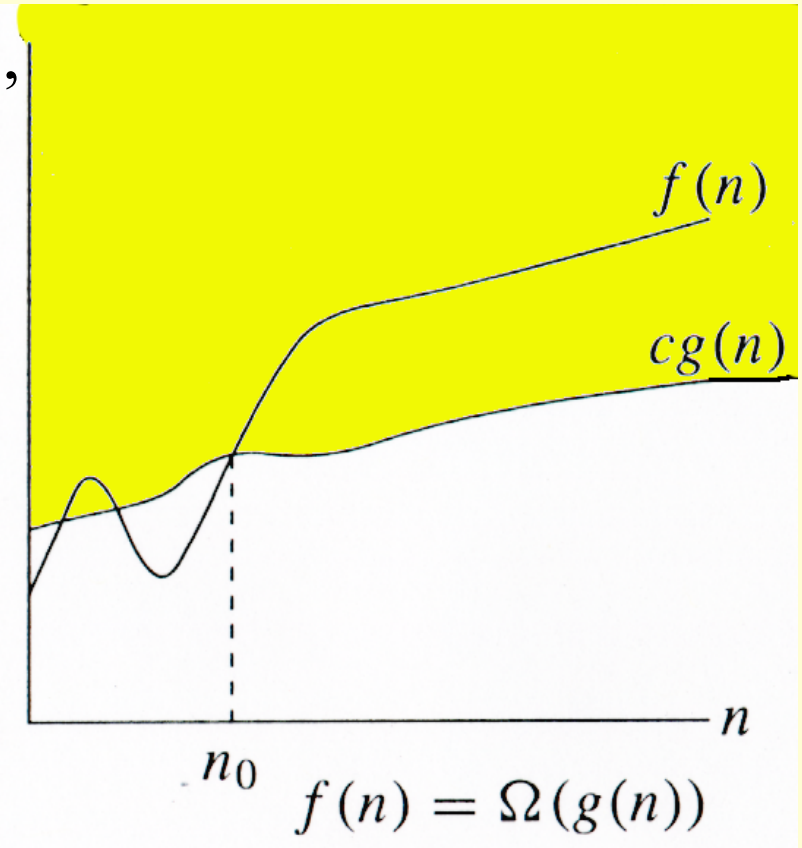
- Any linear *function* $an + b$ is in $O(n^2)$. **How?**
- Show that $3n^3 = O(n^4)$ for appropriate c and n_0.

Ω -notation

For function $g(n)$, we define $\Omega(g(n))$, big-Omega of n, as the set:

$\Omega(g(n)) = \{f(n)$:
∃ positive constants c and n_0,
such that $\forall n \geq n_0$,
we have $0 \leq cg(n) \leq f(n)\}$

Intuitively: Set of all functions whose *rate of growth* is the same as or higher than that of $g(n)$.



$g(n)$ is an *asymptotic lower bound* for $f(n)$.

$f(n) = \Theta(g(n)) \Rightarrow f(n) = \Omega(g(n))$.
$\Theta(g(n)) \subset \Omega(g(n))$.

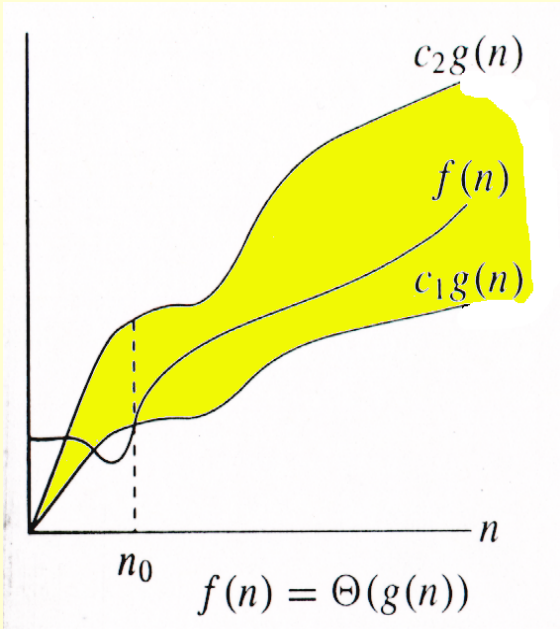
Example

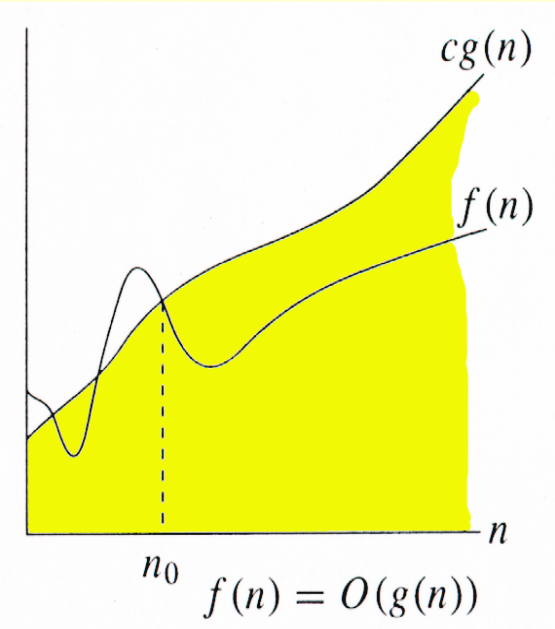
$\Omega(g(n)) = \{f(n) : \exists$ positive constants c and n_0, such that $\forall n \geq n_0$, we have $0 \leq cg(n) \leq f(n)\}$

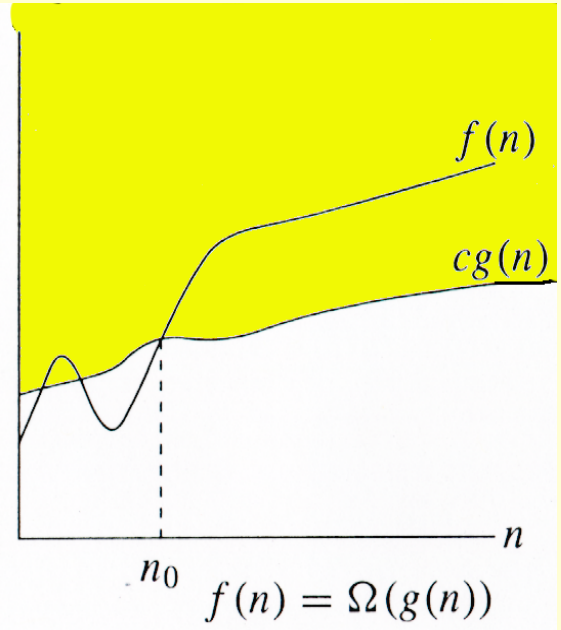
- Eg: $\sqrt{n} = \Omega(\lg n)$

  Q: How to choose c and n_0. (Your HW)

Relations Between Θ, *O*, Ω







Relations Between Θ, Ω, O

Theorem : For any two functions $g(n)$ and $f(n)$,

$f(n) = \Theta(g(n))$ iff

$f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

- I.e., $\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$
- In practice, asymptotically tight bounds are obtained from asymptotic upper and lower bounds.

Running Times

- "Running time is $O(f(n))$" $\Rightarrow$ Worst case is $O(f(n))$
- $O(f(n))$ bound on the worst-case running time $\Rightarrow$ $O(f(n))$ bound on the running time of every input.
- $\Theta(f(n))$ bound on the worst-case running time $\not\Rightarrow$ $\Theta(f(n))$ bound on the running time of every input.
- "Running time is $\Omega(f(n))$" $\Rightarrow$ Best case is $\Omega(f(n))$
- Can still say "Worst-case running time is $\Omega(f(n))$"
  - Means worst-case running time is given by some unspecified function $g(n) \in \Omega(f(n))$.

Example

- ***Insertion sort*** takes $\Theta(n^2)$ in the worst case, so sorting (as a *problem*) is $O(n^2)$. **Why?**

- Any sort algorithm must look at each item, so sorting is $\Omega(n)$.

- In fact, using (e.g.) merge sort, sorting is $\Theta(n \lg n)$ in the worst case.

  - Later, we will prove that we cannot hope that any comparison sort to do better in the worst case.

Asymptotic Notation in Equations

- Can use asymptotic notation in equations to replace expressions containing lower-order terms.
- For example,

  $4n^3 + 3n^2 + 2n + 1 = 4n^3 + 3n^2 + \Theta(n)$

  $= 4n^3 + \Theta(n^2) = \Theta(n^3)$. **How to interpret?**
- In equations, $\Theta(f(n))$ always stands for an ***anonymous function*** $g(n) \in \Theta(f(n))$
  - In the example above, $\Theta(n^2)$ stands for $3n^2 + 2n + 1$.

o-notation (little-*o*)

For a given function $g(n)$, the set little-*o*:

$$o(g(n)) = \{f(n): \forall\, c > 0, \exists\, n_0 > 0 \text{ such that } \forall\, n \geq n_0, \text{ we have } 0 \leq f(n) < cg(n)\}.$$

$f(n)$ becomes insignificant relative to $g(n)$ as n approaches infinity:

$$\lim_{n \to \infty} [f(n) / g(n)] = 0$$

$g(n)$ is an ***upper bound*** for $f(n)$ that is not asymptotically tight.

Observe the difference in this definition from previous ones. **Why?**

ω -notation

For a given function $g(n)$, the set little-omega:

$$\omega(g(n)) = \{f(n): \forall\, c > 0, \exists\, n_0 > 0 \text{ such that } \forall\, n \geq n_0, \text{ we have } 0 \leq cg(n) < f(n)\}.$$

$f(n)$ becomes arbitrarily large relative to $g(n)$ as n approaches infinity:

$$\lim_{n \to \infty} [f(n) / g(n)] = \infty.$$

$g(n)$ is a ***lower bound*** for $f(n)$ that is not asymptotically tight.

Comparison of Functions

$$f \leftrightarrow g \approx a \leftrightarrow b$$

$$f(n) = O(g(n)) \approx a \leq b$$
$$f(n) = \Omega(g(n)) \approx a \geq b$$
$$f(n) = \Theta(g(n)) \approx a = b$$
$$f(n) = o(g(n)) \approx a < b$$
$$f(n) = \omega(g(n)) \approx a > b$$

Limits

- $\lim_{n \to \infty} [f(n) / g(n)] = 0 \quad \Rightarrow f(n) \in o(g(n))$
- $\lim_{n \to \infty} [f(n) / g(n)] < \infty \quad \Rightarrow f(n) \in O(g(n))$
- $0 < \lim_{n \to \infty} [f(n) / g(n)] < \infty \quad \Rightarrow f(n) \in \Theta(g(n))$
- $0 < \lim_{n \to \infty} [f(n) / g(n)] \quad \Rightarrow f(n) \in \Omega(g(n))$
- $\lim_{n \to \infty} [f(n) / g(n)] = \infty \quad \Rightarrow f(n) \in \omega(g(n))$
- $\lim_{n \to \infty} [f(n) / g(n)]$ undefined $\Rightarrow$ can't say

Properties

- **Transitivity**

$$f(n) = \Theta(g(n)) \;\&\; g(n) = \Theta(h(n)) \Rightarrow f(n) = \Theta(h(n))$$
$$f(n) = O(g(n)) \;\&\; g(n) = O(h(n)) \Rightarrow f(n) = O(h(n))$$
$$f(n) = \Omega(g(n)) \;\&\; g(n) = \Omega(h(n)) \Rightarrow f(n) = \Omega(h(n))$$
$$f(n) = o\,(g(n)) \;\&\; g(n) = o\,(h(n)) \Rightarrow f(n) = o\,(h(n))$$
$$f(n) = \omega(g(n)) \;\&\; g(n) = \omega(h(n)) \Rightarrow f(n) = \omega(h(n))$$

- **Reflexivity**

$$f(n) = \Theta(f(n))$$
$$f(n) = O(f(n))$$
$$f(n) = \Omega(f(n))$$

Properties

- **Symmetry**

$$f(n) = \Theta(g(n)) \quad iff \quad g(n) = \Theta(f(n))$$

- **Complementarity**

$$f(n) = O(g(n)) \quad iff \quad g(n) = \Omega(f(n))$$

$$f(n) = o(g(n)) \quad iff \quad g(n) = \omega((f(n))$$

Common Functions

Monotonicity

- $f(n)$ is
  - **monotonically increasing** if $m \le n \Rightarrow f(m) \le f(n)$.
  - **monotonically decreasing** if $m \ge n \Rightarrow f(m) \ge f(n)$.
  - **strictly increasing** if $m < n \Rightarrow f(m) < f(n)$.
  - **strictly decreasing** if $m > n \Rightarrow f(m) > f(n)$.

Exponentials

- **Useful Identities:**

$$a^{-1} = \frac{1}{a}$$

$$(a^m)^n = a^{mn}$$

$$a^m a^n = a^{m+n}$$

- **Exponentials and polynomials**

$$\lim_{n\to\infty} \frac{n^b}{a^n} = 0$$

$$\Rightarrow n^b = o(a^n)$$

Logarithms

$x = \log_b a$ is the exponent for $a = b^x$.

Natural log: $\ln a = \log_e a$

Binary log: $\lg a = \log_2 a$

$\lg^2 a = (\lg a)^2$

$\lg \lg a = \lg (\lg a)$

$$a = b^{\log_b a}$$

$$\log_c (ab) = \log_c a + \log_c b$$

$$\log_b a^n = n \log_b a$$

$$\log_b a = \frac{\log_c a}{\log_c b}$$

$$\log_b (1/a) = -\log_b a$$

$$\log_b a = \frac{1}{\log_a b}$$

$$a^{\log_b c} = c^{\log_b a}$$

Logarithms and exponentials – Bases

- If the base of a logarithm is changed from one constant to another, the value is altered by a constant factor.
  - **Ex:** $\log_{10} n * \mathbf{\log_2 10} = \log_2 n$.
  - Base of logarithm is not an issue in asymptotic notation.
- Exponentials with different bases differ by a exponential factor (not a constant factor).
  - **Ex:** $2^n = \mathbf{(2/3)^n} * 3^n$.

Polylogarithms

- **For $a \geq 0$, $b > 0$,** $\lim_{n \to \infty} (\lg^a n / n^b) = 0$,
  so $\lg^a n = o(n^b)$, and $n^b = \omega(\lg^a n)$
  - Prove using L'Hopital's rule repeatedly

- $\lg(n!) = \Theta(n \lg n)$
  - Prove using Stirling's approximation (in the text) for $\lg(n!)$.

Exercise

Express functions in A in asymptotic notation using functions in B.

A	B	
$\mathbf{5n^2 + 100n}$	$\mathbf{3n^2 + 2}$	$\mathbf{A \in \Theta(B)}$

$A \in \Theta(n^2)$, $n^2 \in \Theta(B) \Rightarrow A \in \Theta(B)$

$\mathbf{\log_3(n^2)}$	$\mathbf{\log_2(n^3)}$	$\mathbf{A \in \Theta(B)}$

$\log_b a = \log_c a / \log_c b$; $A = 2\lg n / \lg 3$, $B = 3\lg n$, $A/B = 2/(3\lg 3)$

$\mathbf{n^{\lg 4}}$	$\mathbf{3^{\lg n}}$	$\mathbf{A \in \omega(B)}$

$a^{\log b} = b^{\log a}$; $B = 3^{\lg n} = n^{\lg 3}$; $A/B = n^{\lg(4/3)} \rightarrow \infty$ as $n \rightarrow \infty$

$\mathbf{\lg^2 n}$	$\mathbf{n^{1/2}}$	$\mathbf{A \in o(B)}$

$\lim_{n\rightarrow\infty} (\lg^a n / n^b) = 0$ (here $a = 2$ and $b = 1/2$) $\Rightarrow A \in o(B)$

Summations – Review

Review on Summations

- Why do we need summation formulas?

  For computing the running times of iterative constructs (loops). (CLRS – Appendix A)

  Example: Maximum Subvector

  Given an array $A[1 \ldots n]$ of numeric values (can be positive, zero, and negative) determine the subvector $A[i \ldots j]$ ($1 \leq i \leq j \leq n$) whose sum of elements is maximum over all subvectors.

1	-2	2	2

Review on Summations

```
MaxSubvector(A, n)
        maxsum ← 0;
        for i ← 1 to n
            do for j = i to n
                        sum ← 0
                        for k ← i to j
                            do sum += A[k]
                        maxsum ← max(sum, maxsum)
        return maxsum
```

- $T(n) = \sum_{i=1}^{n} \sum_{j=i}^{n} \sum_{k=i}^{j} 1$
- NOTE: This is not a simplified solution. What *is* the final answer?

Review on Summations

- **Constant Series:** For integers a and b, $a \le b$,

$$\sum_{i=a}^{b} 1 = b - a + 1$$

- **Linear Series (Arithmetic Series):** For $n \ge 0$,

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

- **Quadratic Series:** For $n \ge 0$,

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Review on Summations

- **Cubic Series:** For $n \geq 0$,

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

- **General Case:** For $n \geq 0$,

$$\sum_{k=1}^{n} k^s = \frac{1}{(s+1)} n^{s+1} + \Theta(n^s) = \Theta(n^{s+1})$$

Review on Summations

- **Geometric Series:** For real $x \neq 1$,

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n = \frac{(x^{n+1} - 1)}{(x - 1)}$$

For $|x| < 1$, $\sum_{k=0}^{\infty} x^k = \frac{1}{(1 - x)}$

Review on Summations

- **Linear-Geometric Series:** For $n \geq 0$, real $c \neq 1$,

$$\sum_{k=1}^{n} kc^k = c + 2c^2 + \cdots + nc^n = \frac{nc^{n+2} - (n+1)c^{n+1} + c}{(c-1)^2}$$

- **Harmonic Series:** n^{th} harmonic number, $n \in \mathrm{I}^+$,

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

$$= \sum_{k=1}^{n} \frac{1}{k} = \ln(n) + O(1)$$

Review on Summations

- **Telescoping Series:**

$$\sum_{k=1}^{n} a_k - a_{k-1} = a_n - a_0$$

- **Differentiating Series:** For $|x| < 1$,

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}$$

Review on Summations

- **Approximation by integrals:**
  - For monotonically increasing $f(n)$

$$\int_{m-1}^{n} f(x)dx \le \sum_{k=m}^{n} f(k) \le \int_{m}^{n+1} f(x)dx$$

  - For monotonically decreasing $f(n)$

$$\int_{m}^{n+1} f(x)dx \le \sum_{k=m}^{n} f(k) \le \int_{m-1}^{n} f(x)dx$$

- **How?**

For Monotone Increasing Function

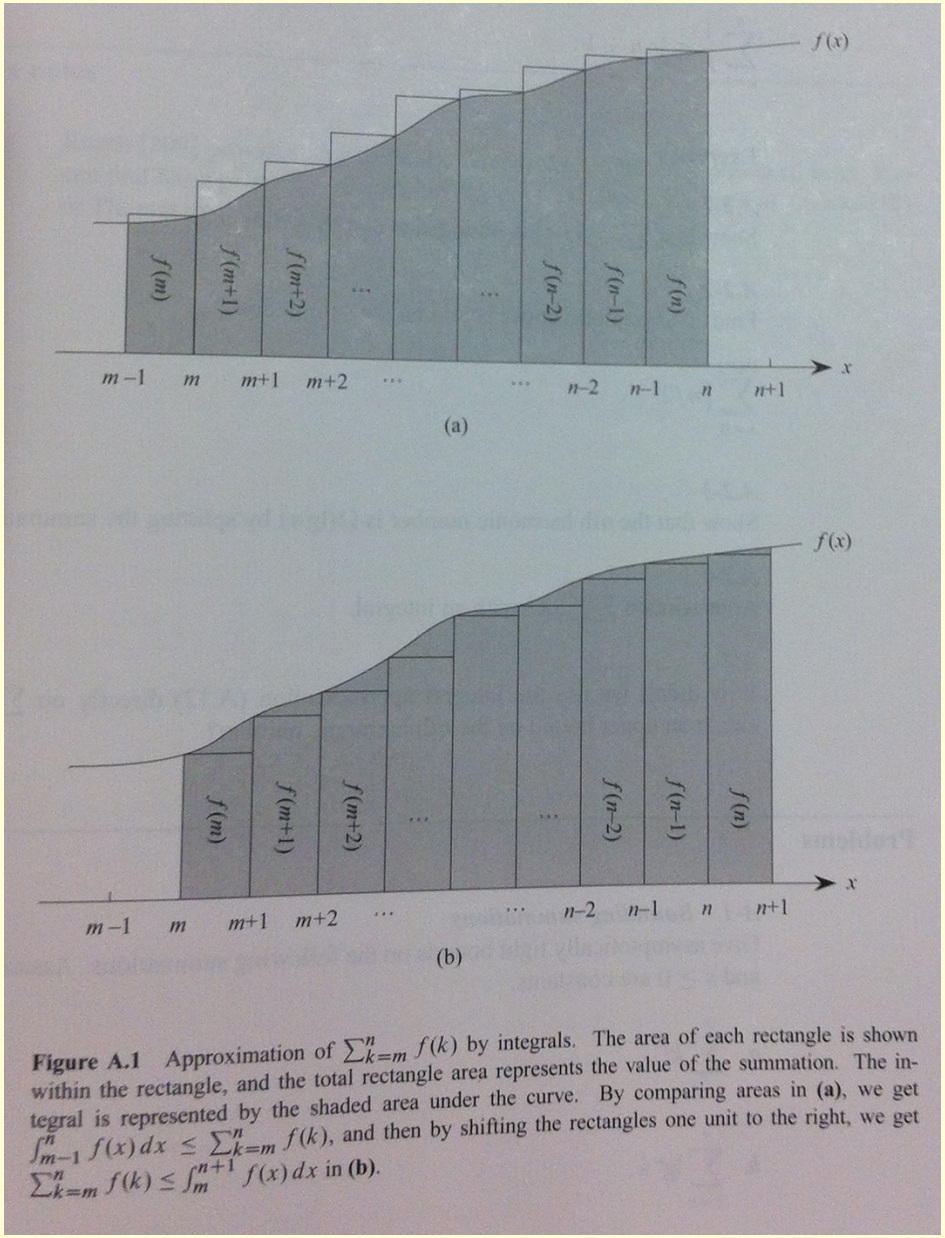


Figure A.1 Approximation of $\sum_{k=m}^{n} f(k)$ by integrals. The area of each rectangle is shown within the rectangle, and the total rectangle area represents the value of the summation. The integral is represented by the shaded area under the curve. By comparing areas in **(a)**, we get $\int_{m-1}^{n} f(x)\,dx \le \sum_{k=m}^{n} f(k)$, and then by shifting the rectangles one unit to the right, we get $\sum_{k=m}^{n} f(k) \le \int_{m}^{n+1} f(x)\,dx$ in **(b)**.

$$\int_{m-1}^{n} f(x)dx \le \sum_{k=m}^{n} f(k)$$

$$\sum_{k=m}^{n} f(k) \le \int_{m}^{n+1} f(x)dx$$

Notes from D. Plaisted: Reused for CS3230

Review on Summations

- ***n*th harmonic number**

$$\sum_{k=1}^{n} \frac{1}{k} \geq \int_{1}^{n+1} \frac{1}{x} dx = [\ln x]_1^{n+1} = \ln(n+1)$$

$$\sum_{k=2}^{n} \frac{1}{k} \leq \int_{1}^{n} \frac{1}{x} dx = [\ln x]_1^{n} = \ln n$$

$$\Rightarrow \sum_{k=1}^{n} \frac{1}{k} \leq (\ln n) + 1$$

Reading Assignment

- Chapter 2-3, App-A of [CLRS].