Last lecture

- Definition of P, NP, EXP.
- Circuit Sat: First NP complete problem.
- Polynomial time reductions: CIRCUIT-SAT $\leq_P$ 3-SAT.

Next

- Examples of many NP complete problems via different types of polynomial time reductions.
- Definition of Co-NP and relationship between P, NP, Co-NP.

NP and Computational Intractability

Basic genres.

- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3D-MATCHING, 3-COLOR.
- Numerical problems: SUBSET-SUM, KNAPSACK.

Reduction By Simple Equivalence

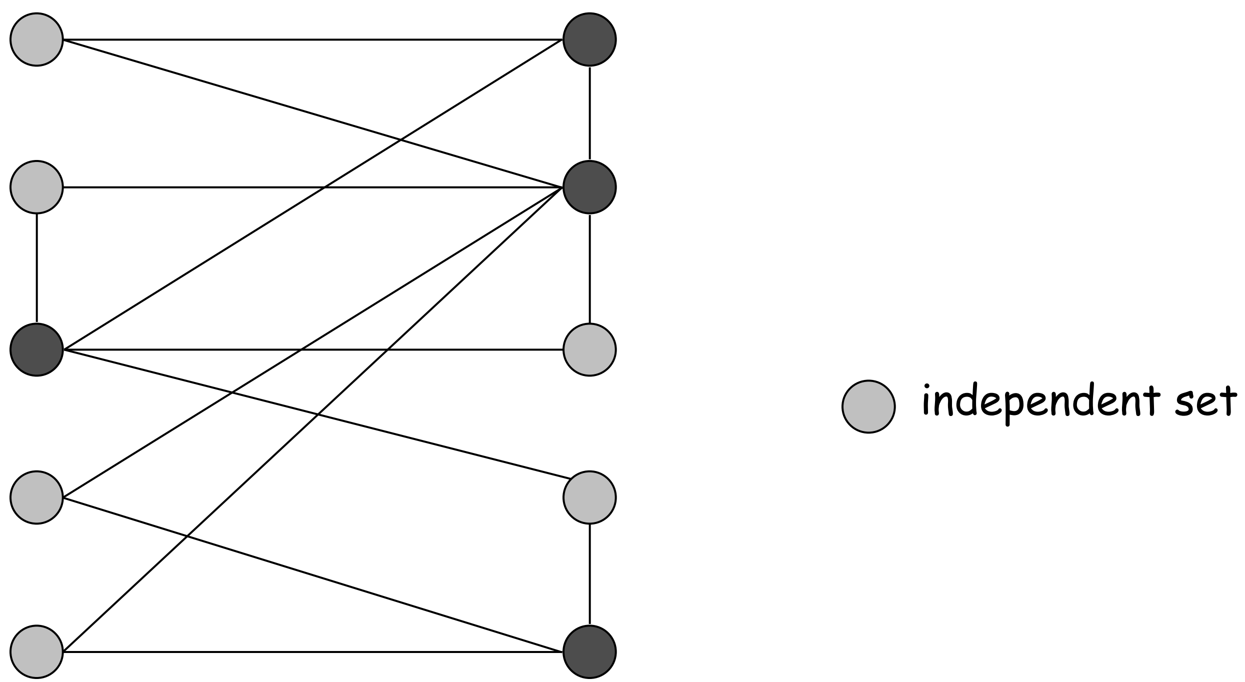
Basic reduction strategies.

- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction by encoding with gadgets.

Independent Set

INDEPENDENT SET [CLRS Chapter 34.5]: Given a graph $G = (V, E)$ and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \geq k$, and for each edge at most one of its endpoints is in S?

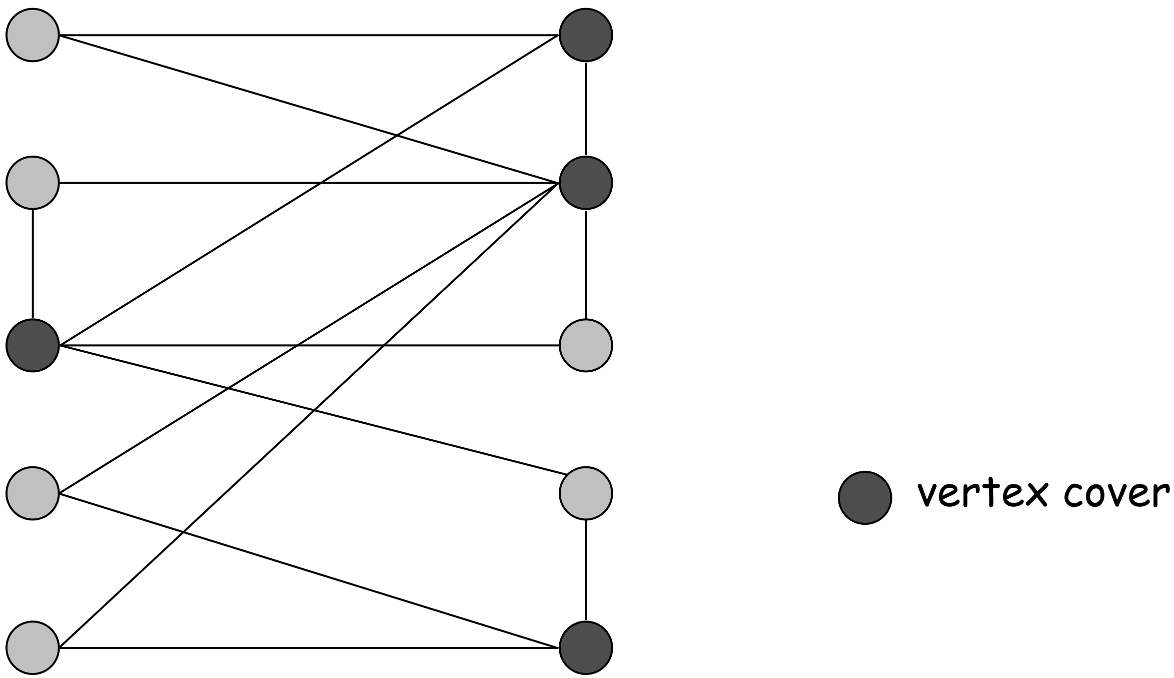
Ex. Is there an independent set of size ≥ 6? Yes.
Ex. Is there an independent set of size ≥ 7? No.



Vertex Cover

VERTEX COVER [CLRS Chapter 34.5]: Given a graph $G = (V, E)$ and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \leq k$, and for each edge, at least one of its endpoints is in S?

Ex. Is there a vertex cover of size ≤ 4? Yes.
Ex. Is there a vertex cover of size ≤ 3? No.



Vertex Cover and Independent Set

Claim. VERTEX-COVER $\equiv_P$ INDEPENDENT-SET.

Pf. Tutorial question.

Reduction from Special Case to General Case

Basic reduction strategies.

- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction by encoding with gadgets.

Set Cover

SET COVER: Given a set U of elements, a collection $S_1, S_2, \ldots, S_m$ of subsets of U, and an integer k, does there exist a collection of $\leq k$ of these sets whose union is equal to U?

Sample application.

- m available pieces of software.
- Set U of n capabilities that we would like our system to have.
- The ith piece of software provides the set $S_i \subseteq U$ of capabilities.
- Goal: achieve all n capabilities using fewest pieces of software.

Ex:

$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$

$k = 2$

$S_1 = \{3, 7\}$	$S_4 = \{2, 4\}$
$S_2 = \{3, 4, 5, 6\}$	$S_5 = \{5\}$
$S_3 = \{1\}$	$S_6 = \{1, 2, 6, 7\}$

Vertex Cover Reduces to Set Cover

Claim. VERTEX-COVER $\le_P$ SET-COVER.
Pf. Tutorial question.

Polynomial-Time Reduction

Basic strategies.

- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction by encoding with gadgets.

8.2 Reductions via "Gadgets"

Basic reduction strategies.

- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction via "gadgets."

Satisfiability

Literal: A Boolean variable or its negation. x_i or $\overline{x_i}$

Clause: A disjunction of literals. $C_j = x_1 \vee \overline{x_2} \vee x_3$

Conjunctive normal form: A propositional formula Φ that is the conjunction of clauses. $\Phi = C_1 \wedge C_2 \wedge C_3 \wedge C_4$

SAT: Given CNF formula Φ, does it have a satisfying truth assignment?

3-SAT: SAT where each clause contains exactly 3 literals.

each corresponds to a different variable

Ex: $\left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(x_2 \vee x_3\right) \wedge \left(\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}\right)$

Yes: x_1 = true, x_2 = true x_3 = false.

3 Satisfiability Reduces to Independent Set

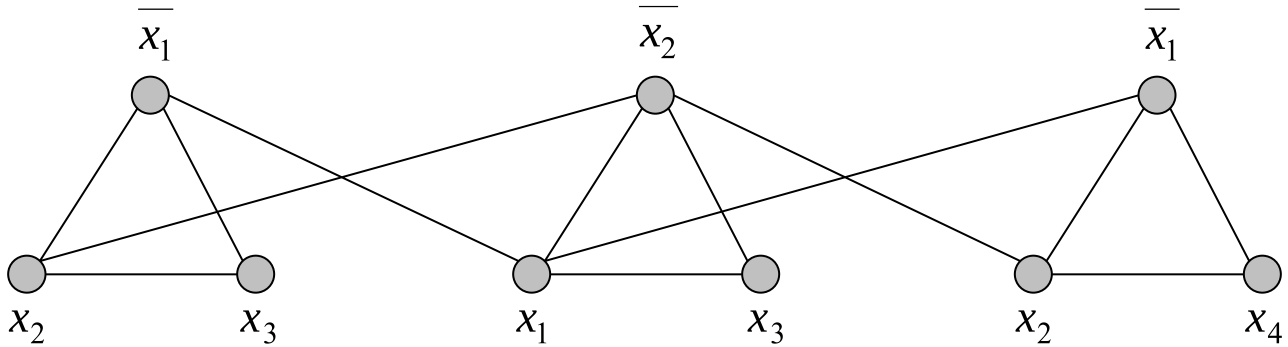
Claim. 3-SAT $\leq_P$ INDEPENDENT-SET.

Pf. Given an instance Φ of 3-SAT, we construct an instance (G, k) of INDEPENDENT-SET that has an independent set of size k iff Φ is satisfiable.

Construction.

- G contains 3 vertices for each clause, one for each literal.
- Connect 3 literals in a clause in a triangle.
- Connect literal to each of its negations.

G



k = 3

$$\Phi = \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_4\right)$$

3 Satisfiability Reduces to Independent Set

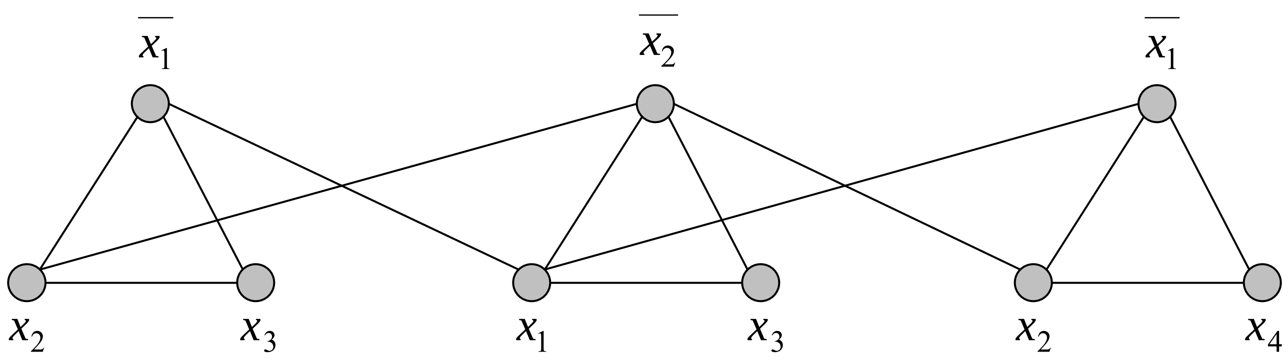
Claim. G contains independent set of size k = $|\Phi|$ iff Φ is satisfiable.

Pf. $\Rightarrow$ Let S be independent set of size k.

- S must contain exactly one vertex in each triangle.
- Set these literals to true. $\leftarrow$ and any other variables in a consistent way
- Truth assignment is consistent and all clauses are satisfied.

Pf $\Leftarrow$ Given satisfying assignment, select one true literal from each triangle. This is an independent set of size k. ▪

G



k = 3

$$\Phi = \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_4\right)$$

Composing reductions

Transitivity. If $X \leq_P Y$ and $Y \leq_P Z$, then $X \leq_P Z$.
Pf. Tutorial question.

Ex: 3-SAT $\leq_P$ INDEPENDENT-SET $\leq_P$ VERTEX-COVER $\leq_P$ SET-COVER.

Self-Reducibility

Decision problem. Does there exist a vertex cover of size $\leq$ k?
Search problem. Find vertex cover of minimum cardinality.

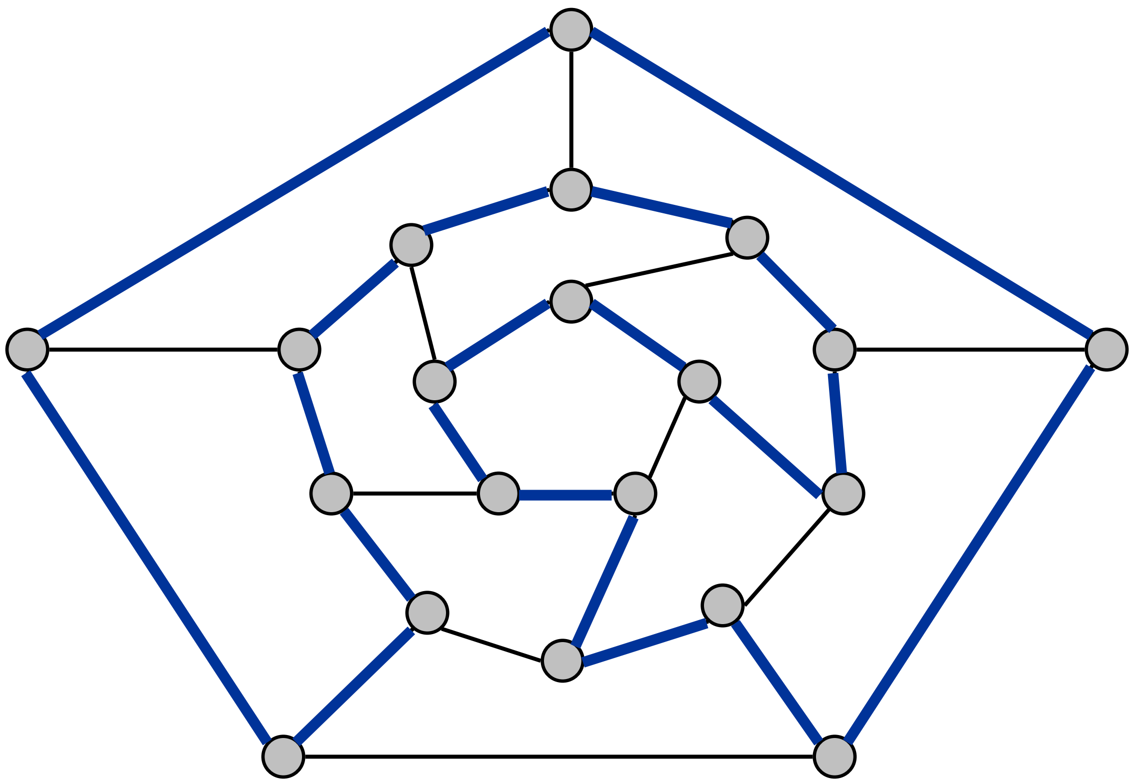
Self-reducibility. Search problem $\leq_P$ decision version.

- Applies to all (NP-complete) problems that we will see.
- Justifies our focus on decision problems.

Tutorial question: Show it for vertex cover.

Hamiltonian Cycle

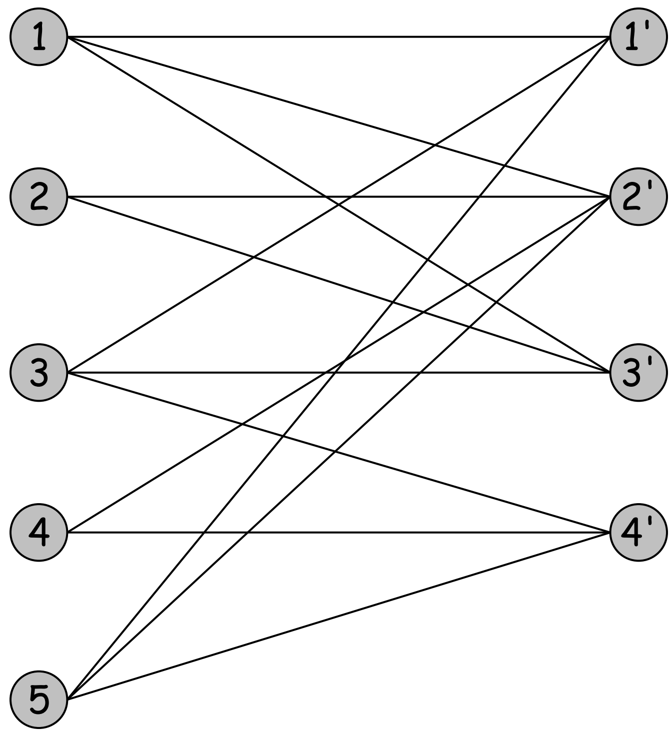
HAM-CYCLE [CLRS Chapter 34.5.3]: given an undirected graph $G = (V, E)$, does there exist a simple cycle Γ that contains every node in V.



YES: vertices and faces of a dodecahedron.

Hamiltonian Cycle

HAM-CYCLE: given an undirected graph $G = (V, E)$, does there exist a simple cycle Γ that contains every node in V.



NO: bipartite graph with odd number of nodes.

Directed Hamiltonian Cycle

DIR-HAM-CYCLE: given a digraph G = (V, E), does there exists a simple directed cycle Γ that contains every node in V?

Claim. DIR-HAM-CYCLE $\leq_P$ HAM-CYCLE.

Pf. Tutorial problem.

3-SAT Reduces to Directed Hamiltonian Cycle

Claim. 3-SAT $\leq_P$ DIR-HAM-CYCLE.

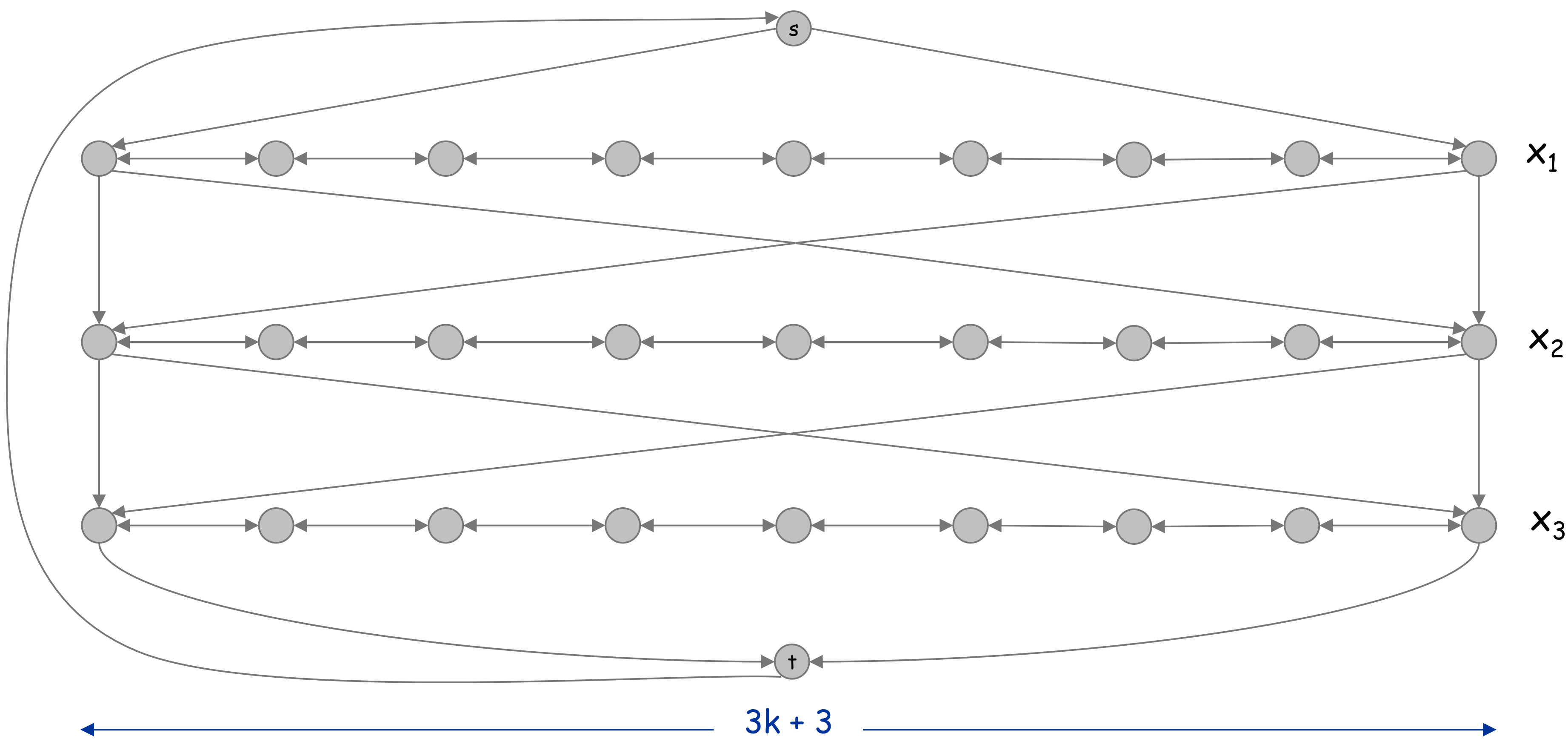
Pf. Given an instance Φ of 3-SAT, we construct an instance of DIR-HAM-CYCLE that has a Hamiltonian cycle iff Φ is satisfiable.

Construction. First, create graph that has 2^n Hamiltonian cycles which correspond in a natural way to 2^n possible truth assignments.

3-SAT Reduces to Directed Hamiltonian Cycle

Construction. Given 3-SAT instance Φ with n variables x_i and k clauses.

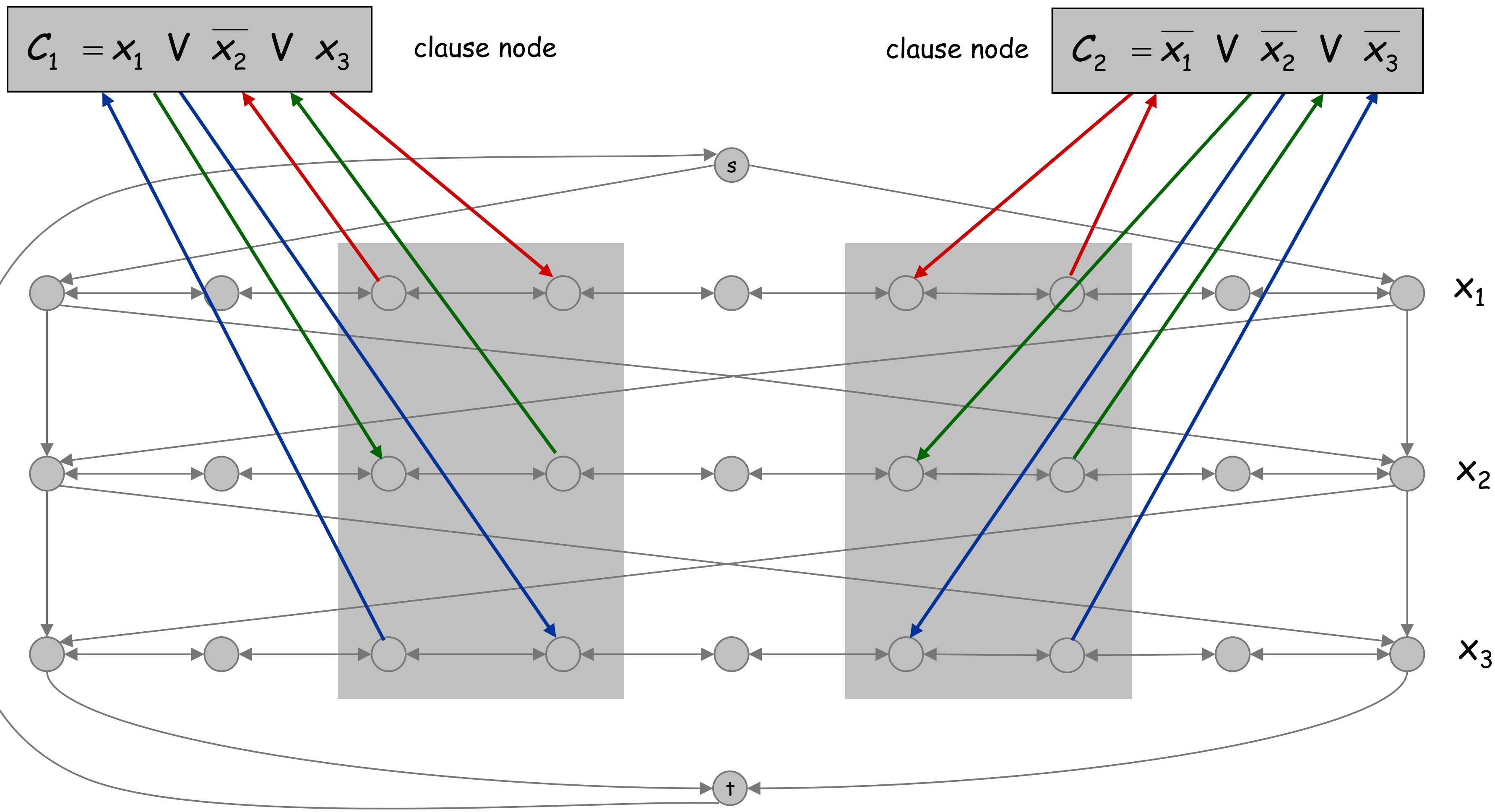
- Construct G to have 2^n Hamiltonian cycles.
- Intuition: traverse path i from left to right $\Leftrightarrow$ set variable $x_i = 1$.



3-SAT Reduces to Directed Hamiltonian Cycle

Construction. Given 3-SAT instance Φ with n variables x_i and k clauses.

- For each clause: add a node and 6 edges.



3-SAT Reduces to Directed Hamiltonian Cycle

Claim. Φ is satisfiable iff G has a Hamiltonian cycle.

Pf. $\Rightarrow$

- Suppose 3-SAT instance has satisfying assignment x^*.
- Then, define Hamiltonian cycle in G as follows:
  - if $x^*_i = 1$, traverse row i from left to right
  - if $x^*_i = 0$, traverse row i from right to left
  - for each clause C_j , there will be at least one row i in which we are going in "correct" direction to splice node C_j into tour

3-SAT Reduces to Directed Hamiltonian Cycle

Claim. Φ is satisfiable iff G has a Hamiltonian cycle.

Pf. $\Leftarrow$

- Suppose G has a Hamiltonian cycle Γ.
- If Γ enters clause node C_j, it must depart on mate edge.
  - thus, nodes immediately before and after C_j are connected by an edge e in G
  - removing C_j from cycle, and replacing it with edge e yields Hamiltonian cycle on $G - \{ C_j \}$
- Continuing in this way, we are left with Hamiltonian cycle Γ' in $G - \{ C_1, C_2, \ldots, C_k \}$.
- Set $x^*_i = 1$ iff Γ' traverses row i left to right.
- Since Γ visits each clause node C_j, at least one of the paths is traversed in "correct" direction, and each clause is satisfied. ▪

Longest Path

SHORTEST-PATH. Given a digraph $G = (V, E)$, does there exists a simple path of length at most k edges?

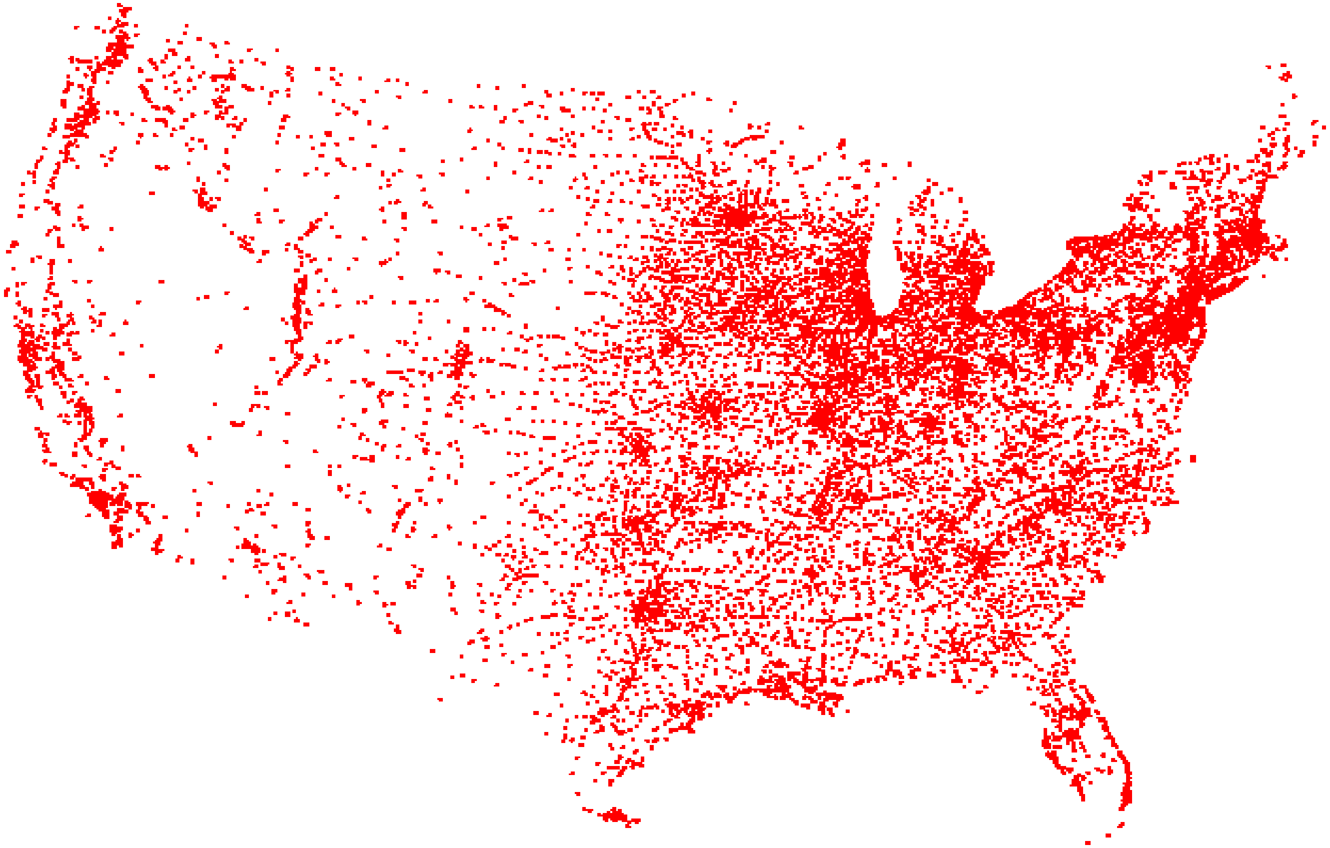
LONGEST-PATH. Given a digraph $G = (V, E)$, does there exists a simple path of length at least k edges?

Claim. 3-SAT $\leq_P$ LONGEST-PATH.

Pf. Tutorial problem .

Traveling Salesperson Problem

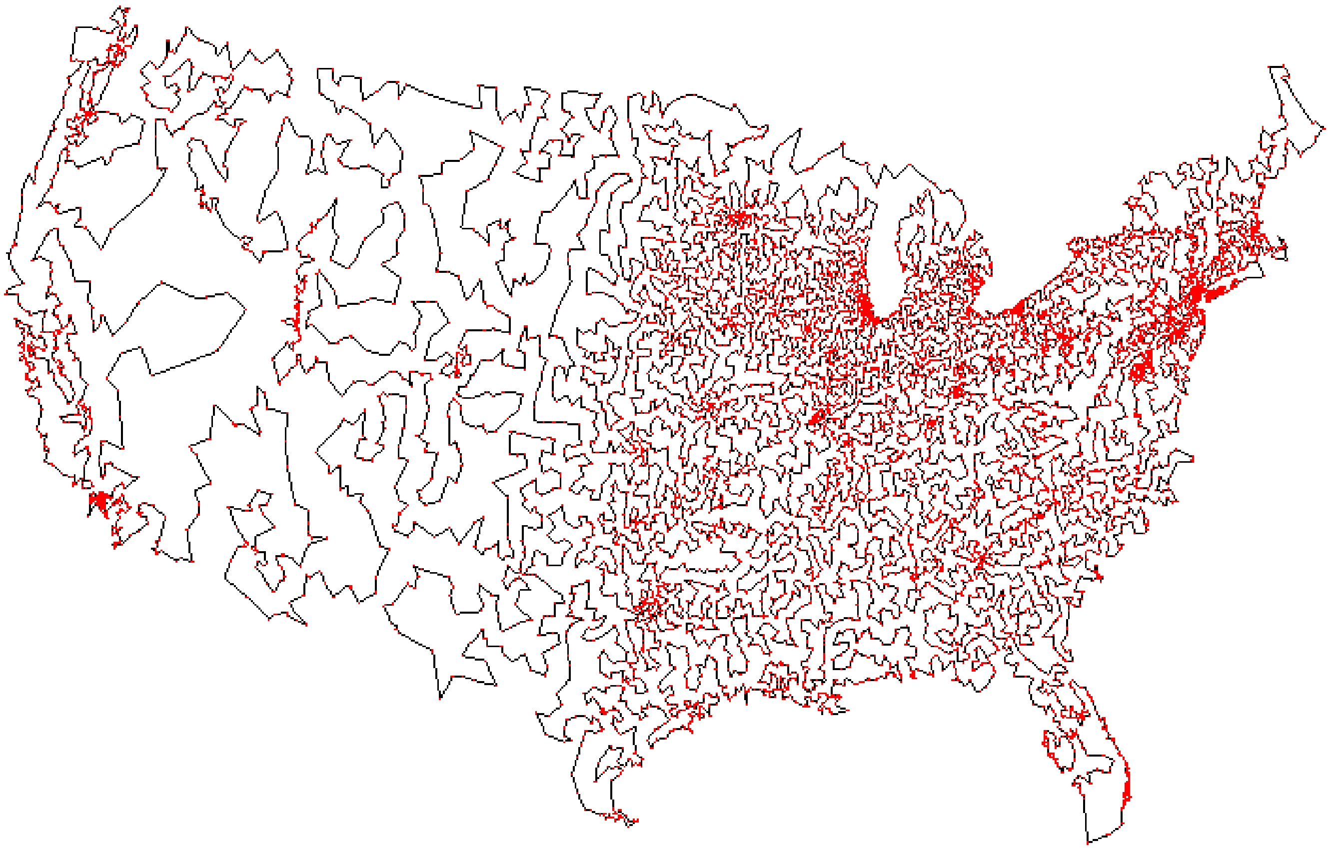
TSP [CLRS Chapter 34.5.4]: Given a set of n cities and a pairwise distance function d(u, v), is there a tour of length $\leq$ D?



All 13,509 cities in US with a population of at least 500
Reference: http://www.tsp.gatech.edu

Traveling Salesperson Problem

TSP. Given a set of n cities and a pairwise distance function d(u, v), is there a tour of length $\leq$ D?



Optimal TSP tour
Reference: http://www.tsp.gatech.edu

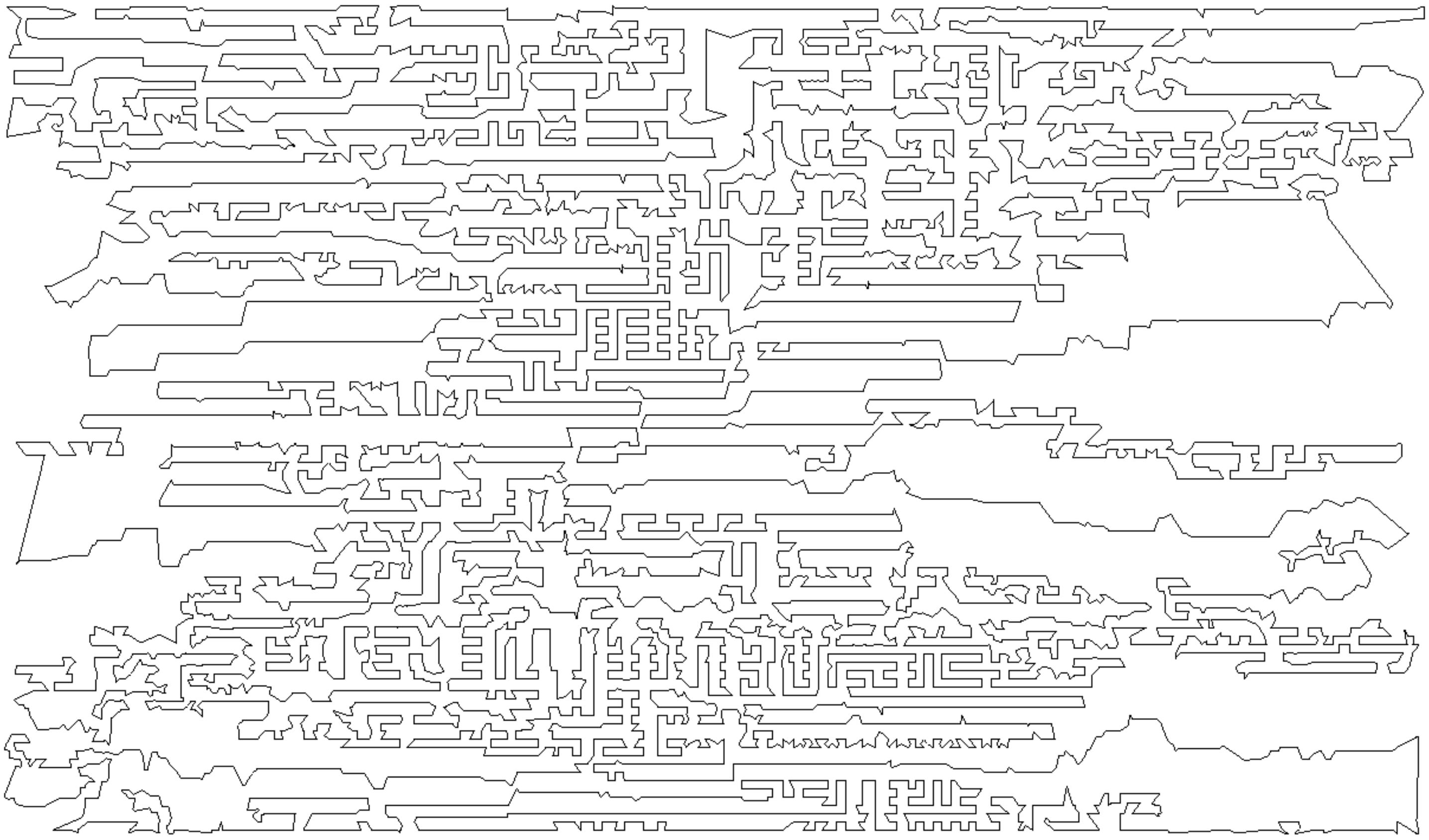
Traveling Salesperson Problem

TSP. Given a set of n cities and a pairwise distance function d(u, v), is there a tour of length $\leq$ D?

11,849 holes to drill in a programmed logic array
Reference: http://www.tsp.gatech.edu

Traveling Salesperson Problem

TSP. Given a set of n cities and a pairwise distance function d(u, v), is there a tour of length ≤ D?



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Traveling Salesperson Problem

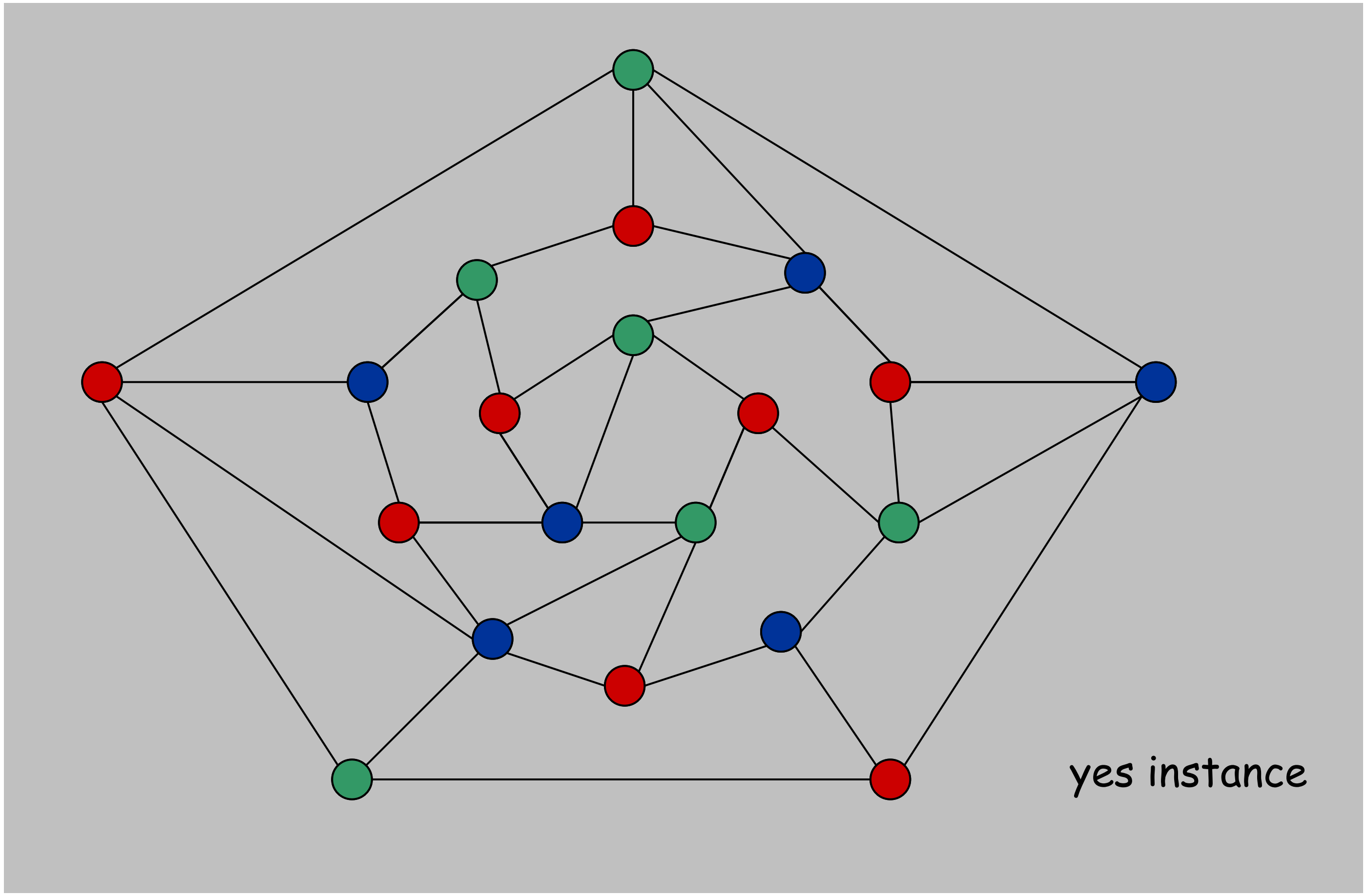
TSP. Given a set of n cities and a pairwise distance function d(u, v), is there a tour of length $\leq$ D?

HAM-CYCLE: given a graph G = (V, E), does there exists a simple cycle that contains every node in V?

Claim. HAM-CYCLE $\leq_P$ TSP.
Pf. Tutorial problem.

3-Colorability

3-COLOR: Given an undirected graph G does there exists a way to color the nodes red, green, and blue so that no adjacent nodes have the same color?



Register Allocation

Register allocation. Assign program variables to machine register so that no more than k registers are used and no two program variables that are needed at the same time are assigned to the same register.

Interference graph. Nodes are program variables names, edge between u and v if there exists an operation where both u and v are "live" at the same time.

Observation. [Chaitin 1982] Can solve register allocation problem iff interference graph is k-colorable.

Fact. 3-COLOR $\leq_P$ k-REGISTER-ALLOCATION for any constant $k \geq 3$.

3-Colorability

Claim. 3-SAT $\le_P$ 3-COLOR.

Pf. Given 3-SAT instance Φ, we construct an instance of 3-COLOR that is 3-colorable iff Φ is satisfiable.

Construction.

i. For each literal, create a node.
ii. Create 3 new nodes T, F, B; connect them in a triangle, and connect each literal to B.
iii. Connect each literal to its negation.
iv. For each clause, add gadget of 6 nodes and 13 edges.

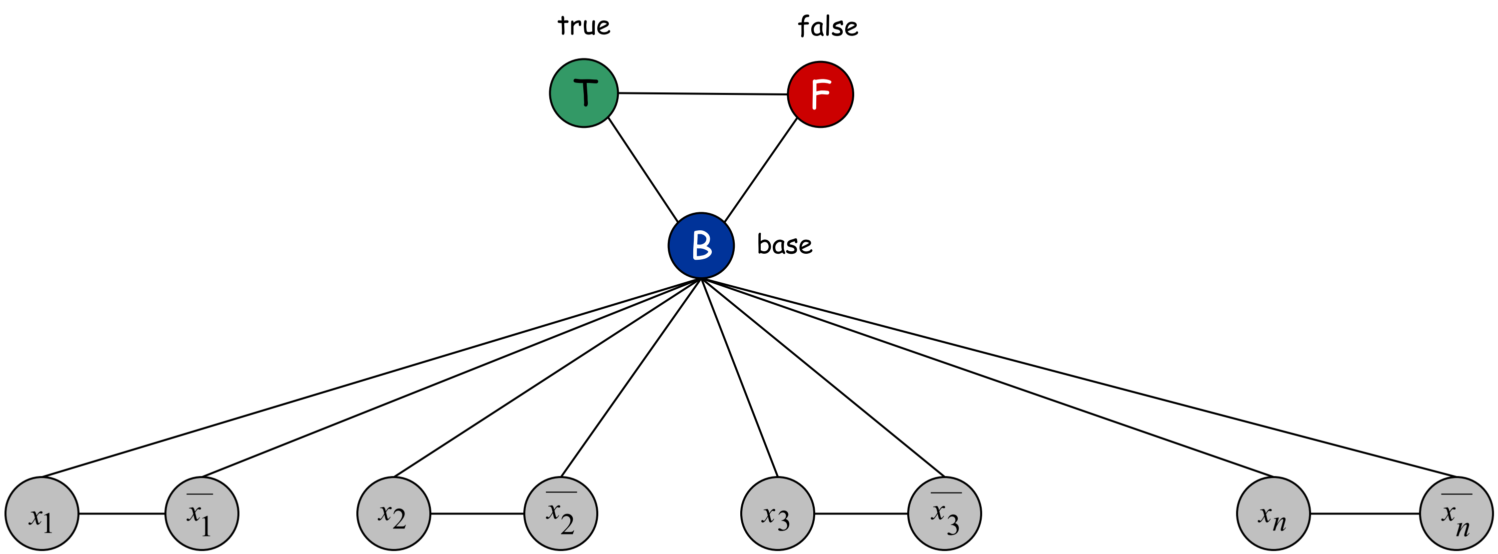
to be described next

3-Colorability

Claim. Graph is 3-colorable iff Φ is satisfiable.

Pf. $\Rightarrow$ Suppose graph is 3-colorable.

- Consider assignment that sets all T literals to true.
- (ii) ensures each literal is T or F.
- (iii) ensures a literal and its negation are opposites.



3-Colorability

Claim. Graph is 3-colorable iff Φ is satisfiable.

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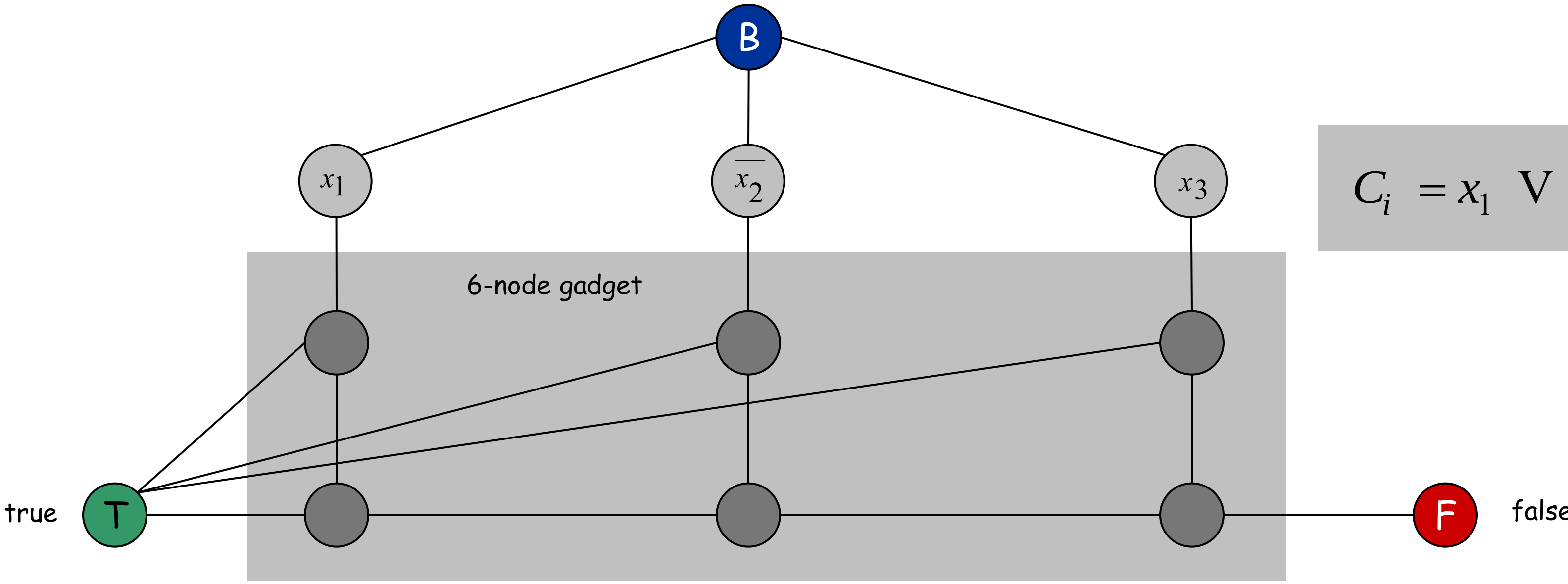
- Consider assignment that sets all T literals to true.
- (ii) ensures each literal is T or F.
- (iii) ensures a literal and its negation are opposites.
- (iv) ensures at least one literal in each clause is T.



$$C_i = x_1 \vee \overline{x_2} \vee x_3$$

3-Colorability

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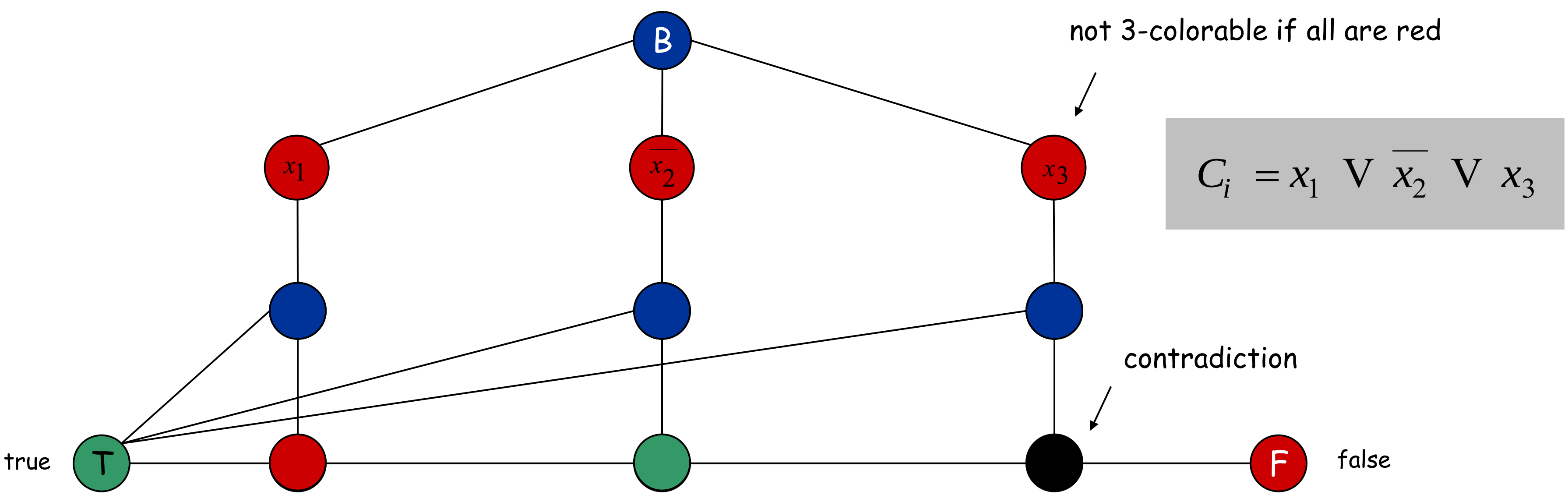
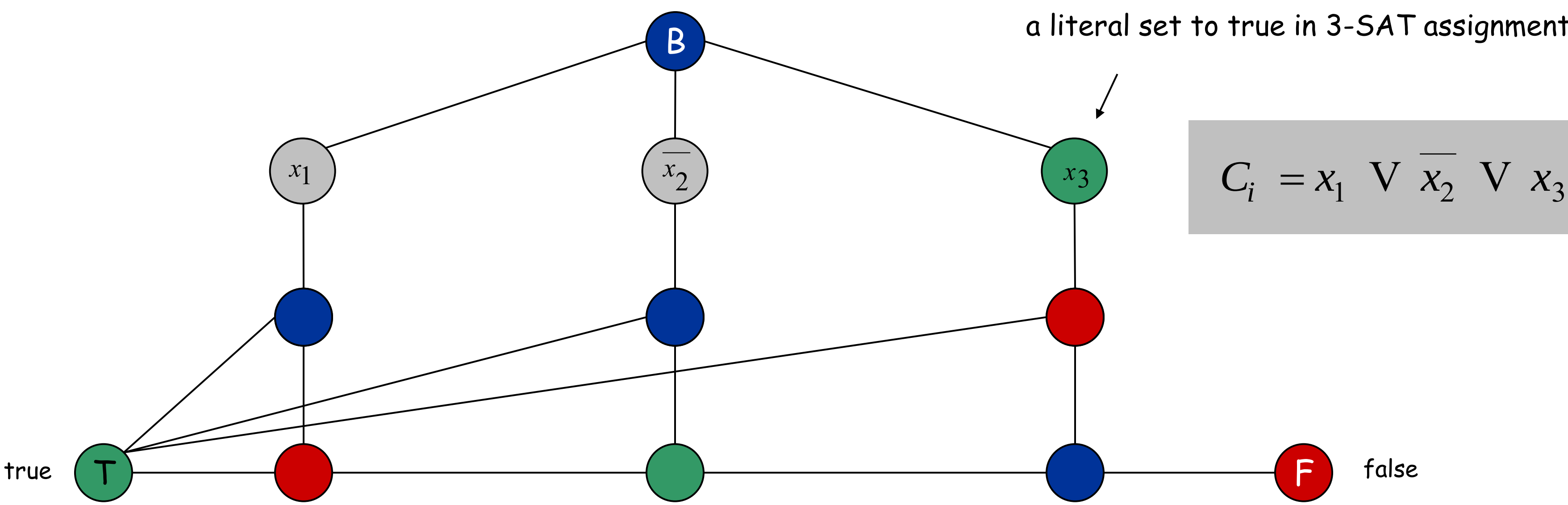


3-Colorability

Claim. Graph is 3-colorable iff Φ is satisfiable.

Pf. $\Leftarrow$ Suppose 3-SAT formula Φ is satisfiable.

- Color all true literals T.
- Color node below green node F, and node below that B.
- Color remaining middle row nodes B.
- Color remaining bottom nodes T or F as forced. ▪



$$C_i = x_1 \vee \overline{x_2} \vee x_3$$

Subset Sum

SUBSET-SUM. [CLRS Chapter 34.5.5]: Given natural numbers $w_1, \ldots, w_n$ and an integer W, is there a subset that adds up to exactly W?

Ex: { 1, 4, 16, 64, 256, 1040, 1041, 1093, 1284, 1344 }, W = 3754.
Yes. 1 + 16 + 64 + 256 + 1040 + 1093 + 1284 = 3754.

Remark. With arithmetic problems, input integers are encoded in binary. Polynomial reduction must be polynomial in binary encoding.

Claim. 3-SAT $\leq_P$ SUBSET-SUM.
Pf. Given an instance Φ of 3-SAT, we construct an instance of SUBSET-SUM that has solution iff Φ is satisfiable.

Subset Sum

Construction. Given 3-SAT instance Φ with n variables and k clauses, form 2n + 2k decimal integers, each of n+k digits, as illustrated below.

Claim. Φ is satisfiable iff there exists a subset that sums to W.

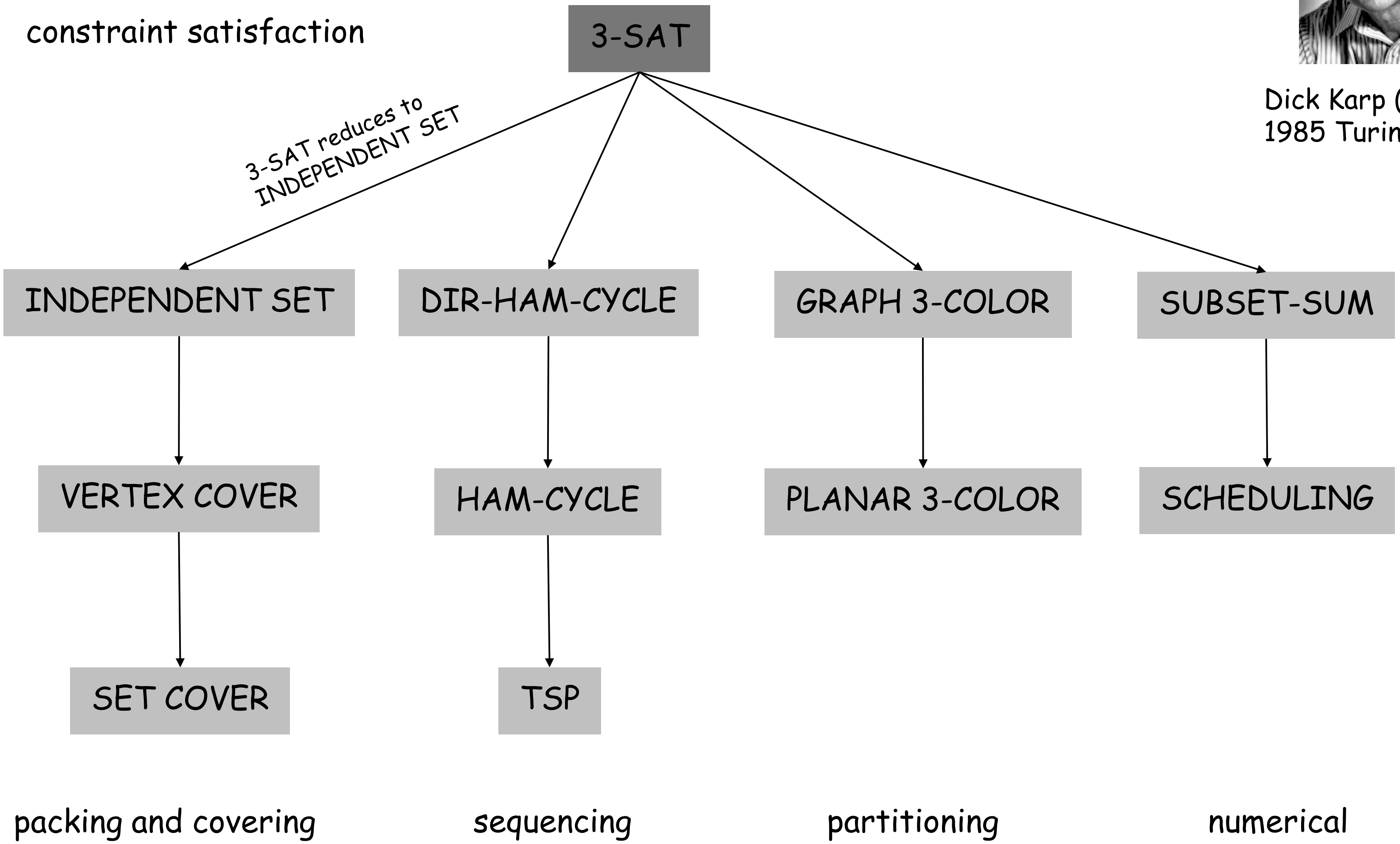
Pf. No carries possible.

$$C_1 = \bar{x} \vee y \vee z$$
$$C_2 = x \vee \bar{y} \vee z$$
$$C_3 = \bar{x} \vee \bar{y} \vee \bar{z}$$

	x	y	z	C_1	C_2	C_3	
x	1	0	0	0	1	0	100,010
¬ x	1	0	0	1	0	1	100,101
y	0	1	0	1	0	0	10,100
¬ y	0	1	0	0	1	1	10,011
z	0	0	1	1	1	0	1,110
¬ z	0	0	1	0	0	1	1,001
dummies to get clause columns to sum to 4	0	0	0	1	0	0	100
	0	0	0	2	0	0	200
	0	0	0	0	1	0	10
	0	0	0	0	2	0	20
	0	0	0	0	0	1	1
	0	0	0	0	0	2	2
W	1	1	1	4	4	4	111,444

8.10 A Partial Taxonomy of Hard Problems

Polynomial-Time Reductions



Dick Karp (1972)
1985 Turing Award

Partition

SUBSET-SUM. Given natural numbers $w_1, \ldots, w_n$ and an integer W, is there a subset that adds up to exactly W?

PARTITION. Given natural numbers $v_1, \ldots, v_m$, can they be partitioned into two subsets that add up to the same value?

$\frac{1}{2} \Sigma_i v_i$

Claim. SUBSET-SUM $\leq_P$ PARTITION.

Pf. Tutorial question.

Summary

Basic reduction strategies.

- Simple equivalence: INDEPENDENT-SET $\equiv_P$ VERTEX-COVER.
- Special case to general case: VERTEX-COVER $\leq_P$ SET-COVER.
- Encoding with gadgets:
  - 3-SAT $\leq_P$ INDEPENDENT-SET.
  - 3-SAT $\leq_P$ DIR-HAM-CYCLE.
  - 3-SAT $\leq_P$ 3-COLOR.
  - 3-SAT $\leq_P$ SUBSET-SUM.