CS3230 : Design and Analysis of Algorithms (Spring 2015)

Tutorial Set #1

[SOLUTION SKETCHES]

[DO NOT give to future students: Let them have a chance to learn.]

Note (by LHW): Only for this first tutorial, I will be kind and explain in great details. For future tutorials, you will need to "grow up", as the explanation will be more like what we expect a CS3230 student to be able to understand (and fill in the gaps, if necessary).

Routine Practice Problems *-- do not turn these in -- but make sure you know how to do them.*

R1-(a) Show, by definition, that $f(n) = O(n)$, where $f(n) = 119n$.

ANSWER: We need to find c and n_0, and show that

$f(n) \le c\, n$ for all $n \ge n_0$. (**)

So, we first write down what we know

$$\begin{aligned} f(n) &= 119n \\ &\le 119n, \quad \text{for all } n \ge 1, \quad \text{(Here, we are lucky; we got it } \textit{for free}\text{)} \end{aligned}$$

We choose $c = 119$, $n_0 = 1$. Then (**) holds.

Hence, $f(n) = 119n = O(n)$.

R1-(a') Is $f(n) = O(n^2)$? **YES** (Proof by definition is similar)

$f(n) = 119n \le 119n^2$, for all $n \ge 1$, (Again, got it *almost free*)

Is $f(n) = O(n^{819})$? **YES**

$f(n) = 119n \le 119n^{819}$, for all $n \ge 1$, (Again, got it *almost free*)

R1-(c) Show, by definition, that $h(n) = O(n^2)$, where $h(n) = 26n^2 + 119n$.

ANSWER:

$$\begin{aligned} h(n) &= 26n^2 + 119n \\ &\le 26n^2 + 119n^2, \quad \text{for all } n \ge 1 \quad (119n \le 119n^2) \\ &= 145n^2, \quad \text{for all } n \ge 1, \end{aligned}$$

We choose $c = 145$, $n_0 = 1$, we have $h(n) \le 145n^2$, for all $n \ge 1$,

Hence, $h(n) = 26n^2 + 119n = O(n^2)$.

(**Note:** You do *not* have to find the smallest c and the smallest n_0.
Any c and n_0 that work will be equally good. I find an easy one that works.
If you want a smaller value for c, you will need to work hard and may also have to use a bigger n_0. Remember, the burden of proof is on YOU to prove your result.)

R1-(c') Is $h(n) = O(n^3)$? **YES** Is $h(n) = O(n^{145})$? **YES** **(DIY)**

R1(b), (b'): DIY (Do It Yourself) Similar to 1(a), (a').
R1(d), (d'): DIY. Similar to (c), (c').

R2. Will only show R1-(c). DIY for the rest.

R2-(c): Show, by definition, that $h(n) = \Theta(n^2)$, where $h(n) = 26n^2 + 119n$.

ANSWER:

Upper Bound:

For R1(c) above, we already know, $h(n) \leq 145\, n^2$, for all $n \geq n_0$=1,

Lower Bound:

Next, we need to do the "lower bound" proof. Which is also easy.

Since $h(n) = 26n^2 + 119n \geq 26n^2$, for all $n \geq 1$, (throw away $119n$)

Put these together, we have

$26n^2 \leq h(n) \leq 836n^2$, for all $n \geq 1$,

We choose $C_1 = 26$, $C_2 = 145$, $n_0 = 1$.

Hence, $h(n) = 26n^2 + 119n = \Theta(n^2)$.

R2-(c') Is $h(n) = \Theta(n)$? **NO** Is $h(n) = \Theta(n^3)$? **NO** Is $g(n) = \Theta(n^{2039})$? **NO**

R3. From R1 and R2, we see that O is only an upper bound and so it is less precise, but Θ is a more precise (lower and upper) bound on the time complexity.

D1: (No brainer analysis of algorithms) Bob designs an algorithm Bobal, and finds that it requires $(3n^2 + n)$ instructions to run. Use the definition to show $(3n^2 + n)$ is $O(n^2)$, $\Omega(n^2)$, and $\Theta(n^2)$. Hence, algorithm Bobal runs in time $\Theta(n^2)$.
[Hint: For all $n \geq 1$, we know that $n^2 \geq n$.]

Upper Bound:

$(3n^2 + n) \leq 3n^2 + n^2$, for all $n \geq 1$ $(n \leq n^2)$

$= 4n^2$, for all $n \geq 1$,

We choose $c = 4$, $n_0 = 1$, we have $(3n^2 + n) \leq 4n^2$, for all $n \geq 1$,

Hence, $(3n^2 + n) = O(n^2)$. [**]

Lower Bound:

$(3n^2 + n) \geq 3n^2$ for all $n \geq 1$ (Throw away n)

We choose $c = 3$, $n_0 = 1$, we have $(3n^2 + n) \geq 3n^2$, for all $n \geq 1$,

Hence, $(3n^2 + n) = \Omega(n^2)$. [***]

Combining [**] and [***], we have $(3n^2 + n) = \Theta(n^2)$
and algorithm Bobal runs in time $\Theta(n^2)$.

D2. [Simple Proving O, Ω, Θ by definition] Let $f(n) = 16n^3 - 6n + 121$

(a) Prove the following by using the definitions of O, Ω, Θ. Namely, find the respective constants c, c_1, c_2, and the positive integer n_0.

(i) $f(n) = O(n^3)$ (ii) $f(n) = \Omega(n^3)$ (iii) $f(n) = \Theta(n^3)$

[Answer Sketches]

(i) $f(n) = 16n^3 - 6n + 12 \leq 16n^3 + 12$ for all $n \geq 1$. (throw away $-6n$)
$\leq 16n^3 + 12n^3$ for all $n \geq 1$. ($12 \leq 12n^3$)
$\leq 28n^3$ for all $n \geq 1$.
Choose $c = 28$ and $n_0=1$. Therefore, by definition, $f(n) = O(n^3)$.

(ii) $f(n) = 16n^3 - 6n + 12 \geq 16n^3 - 6n$ for all $n \geq 1$. (throw away 12)
$\geq 15n^3 + (n^3 - 6n)$ for all $n \geq 1$. (algebra)
$\geq 15n^3$ for all $n \geq 3$. ($(n^3 - 6n) > 0$ when $n \geq 3$)
Choose $c = 15$ and $n_0=3$. Therefore, by definition, $f(n) = \Omega(n^3)$.

(iii) Note that (i) + (ii) gives (iii).

(b) **Sum Rule: (see T2 first, come back to this later)**
$f(n) = 16n^3 - 6n + 12$ cannot be directly applied here.

Polynomial Rule:
$f(n) = 16n^3 - 6n + 12$ is a polynomial of degree 3. Hence $f(n) = \Theta(n^3)$.
Note: $f(n) = \Theta(n^3)$ implies $f(n) = O(n^3)$ and $f(n) = \Omega(n^3)$

Limit Theorem and L' Hopital's rule:
lim (n→inf) f(n)/n3 = $(16n^3 - 6n + 12) / n^3$ = lim $16 - 6/n^2 + 12/n^3) = 16$
Thus, $f(n) = \Theta(n^3)$.

D3: (Two Important Processes in CS) [SS by HW]

(a),(b) **[DIY -- Do it with a Table (Excel table, maybe?)]**
IMPORTANT that you do it yourself. You will learn from doing it.

(c) Trivial, will NOT do it here;

(d) Most of the time $h(n) = d(n)$, except in some "boundary" cases, they differ by 1?
Q: Where are those boundary cases?

(e) $h(n) = \lceil \lg(n+1) \rceil$ and $d(n) = \lceil \lg n \rceil$. (They are equal MOST of the time.)
(Note: $\lceil \lg(n+1) \rceil = 1 + \lfloor \lg n \rfloor$)

D4. Solution sketch to HW1-S1. (Refer to earlier solution)

D5. Solution sketch to HW1-S3. (Refer to earlier solution)