Algorithmic Paradigms

Divide-and-conquer. Break up a problem into two sub-problems, solve each sub-problem independently, and combine solution to sub-problems to form solution to original problem.

Greed. Build up a solution incrementally, myopically optimizing some local criterion.

Dynamic programming. Break up a problem into a series of overlapping sub-problems, and build up solutions to larger and larger sub-problems.

Dynamic Programming History

Bellman. Pioneered the systematic study of dynamic programming in the 1950s.

Etymology.

- Dynamic programming = planning over time.
- Secretary of Defense was hostile to mathematical research.
- Bellman sought an impressive name to avoid confrontation.
  - "it's impossible to use dynamic in a pejorative sense"
  - "something not even a Congressman could object to"

Reference: Bellman, R. E. *Eye of the Hurricane, An Autobiography.*

Dynamic Programming Applications

Areas.

- Bioinformatics.
- Control theory.
- Information theory.
- Operations research.
- Computer science: theory, graphics, AI, systems, ….

Some famous dynamic programming algorithms.

- Viterbi for hidden Markov models.
- Unix diff for comparing two files.
- Smith-Waterman for sequence alignment.
- Bellman-Ford for shortest path routing in networks.
- Cocke-Kasami-Younger for parsing context free grammars.

Knapsack Problem

Knapsack Problem

Knapsack problem.
- Given n objects and a "knapsack."
- Item i weighs $w_i > 0$ kilograms (integer) and has value $v_i > 0$.
- Knapsack has capacity of W kilograms (integer).
- Goal: fill knapsack so as to maximize total value.

Ex: { 3, 4 } has value 40.

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

Greedy: repeatedly add item with maximum ratio v_i / w_i.
Ex: { 5, 2, 1 } achieves only value = 35 $\Rightarrow$ greedy not optimal.

Dynamic Programming: False Start

Def. OPT(i) = max profit subset of items 1, …, i.

- Case 1: OPT does not select item i.
  - OPT selects best of { 1, 2, …, i-1 }
- Case 2: OPT selects item i.
  - accepting item i does not immediately imply that we will have to reject other items
  - without knowing what other items were selected before i, we don't even know if we have enough room for i

Conclusion. Need more sub-problems!

Dynamic Programming: Adding a New Variable

Def. OPT(i, w) = max profit subset of items 1, …, i with weight limit w.

- Case 1: OPT does not select item i.
  - OPT selects best of { 1, 2, …, i-1 } using weight limit w
- Case 2: OPT selects item i.
  - new weight limit = $w - w_i$
  - OPT selects best of { 1, 2, …, i–1 } using this new weight limit

optimal substructure

$$OPT(i,w) = \begin{cases} 0 & \text{if } i = 0 \\ OPT(i-1,w) & \text{if } w_i > w \\ \max\{OPT(i-1,w),\ v_i + OPT(i-1,w-w_i)\} & \text{otherwise} \end{cases}$$

Knapsack Problem

Knapsack. Fill up an n-by-W array.

```
Input: n, w1,…,wN, v1,…,vN

for w = 0 to W
       M[0, w] = 0
       S[0, w] = empty set
\\ S[,] is a two dimensional array of sets

for i = 1 to n
   for w = 1 to W
       if (wi > w)
              M[i, w] = M[i-1, w]
       elseif (vi + M[i-1, w-wi] ≥  M[i-1, w])
              M[i, w] = vi + M[i-1, w-wi]
              S[i,w] = S[i-1,w-wi] U {i}
       else
              M[i, w] = M[i-1,w]
              S[i, w] = S[i-1,w]

return M[n, W] \\ optimal value of knapsack
return S[n, W]  \\ objects in an optimal knapsack
```

Knapsack Algorithm

W + 1 →

n + 1 ↓

	0	1	2	3	4	5	6	7	8	9	10	11
ϕ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7
{ 1, 2, 3 }	0	1	6	7	7	18	19	24	25	25	25	25
{ 1, 2, 3, 4 }	0	1	6	7	7	18	22	24	28	29	29	40
{ 1, 2, 3, 4, 5 }	0	1	6	7	7	18	22	28	29	34	34	40

OPT: { 4, 3 }
value = 22 + 18 = 40

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

9

Knapsack Problem: Running Time

Running time. $\Theta(n\ W)$.
- Not polynomial in input size!
- Decision version of Knapsack is NP-complete. [We have noted this already!]

Knapsack approximation algorithm. There exists a polynomial algorithm that produces a feasible solution that has value within 0.01% of optimum. [For your next course in algorithms !]

10

Dynamic Programming Summary

Recipe.
- Characterize structure of problem.
- Recursively define value of optimal solution.
- Compute value of optimal solution.
- Construct optimal solution from computed information.

Dynamic programming techniques.
- Adding a new variable: knapsack.
- Binary/multi-way choice: weighted interval scheduling.
- Dynamic programming over intervals: RNA secondary structure.

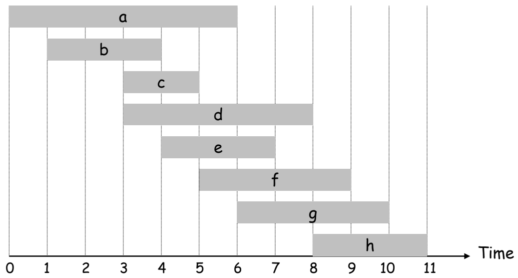
CKY parsing algorithm for context-free grammar has similar structure

11

Weighted Interval Scheduling

Weighted Interval Scheduling

Weighted interval scheduling problem.
- Job j starts at s_j, finishes at f_j, and has weight or value v_j .
- Two jobs compatible if they don't overlap.
- Goal: find maximum weight subset of mutually compatible jobs.

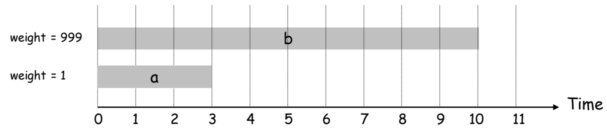


13

Unweighted Interval Scheduling Review

Recall. Greedy algorithm works if all weights are 1.
- Consider jobs in ascending order of finish time.
- Add job to subset if it is compatible with previously chosen jobs.

Observation. Greedy algorithm can fail spectacularly if arbitrary weights are allowed.

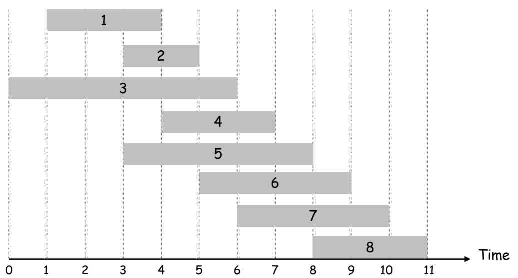


14

Weighted Interval Scheduling

Notation. Label jobs by finishing time: $f_1 \leq f_2 \leq \ldots \leq f_n$.
Def. p(j) = largest index i < j such that job i is compatible with j.

Ex: p(8) = 5, p(7) = 3, p(2) = 0.



15

Dynamic Programming: Binary Choice

Notation. OPT(j) = value of optimal solution to the problem consisting of job requests 1, 2, ..., j.

- Case 1: OPT selects job j.
  - can't use incompatible jobs { p(j) + 1, p(j) + 2, ..., j - 1 }
  - must include optimal solution to problem consisting of remaining compatible jobs 1, 2, ..., p(j)

optimal substructure

- Case 2: OPT does not select job j.
  - must include optimal solution to problem consisting of remaining compatible jobs 1, 2, ..., j-1

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max \{ v_j + OPT(p(j)),\ OPT(j-1) \} & \text{otherwise} \end{cases}$$

16

Weighted Interval Scheduling

```
Input: n, s1,…,sn , f1,…,fn , v1,…,vn

Sort jobs by finish times so that f1 ≤ f2 ≤ ... ≤ fn.

Compute p(1), p(2), …, p(n)

Compute-Opt {
   M[0] = 0
   for j = 1 to n
      M[j] = max(vj + M[p(j)], M[j-1])
}

Return M[n]
```

17

Summary

Greedy Algorithms:
- Huffman code [CLRS, Ch16.3]

Dynamic Programming:
- Knapsack
- Weighted interval scheduling

Next Lecture. Dynamic Programming:
- RNA secondary structure
- (Tutorial problem) Longest common subsequence [CLRS] Ch15.4

18