CS3230
Tutorial 11

1. Show that the question of determining whether a graph $G = (V, E)$ has a simple cycle of size at least k is NP-complete.

   Ans: (a) Certificate would be a cycle of size at least k.

   Verifier just checks whether the cycle is indeed simple (that is, it does not repeat a vertex), and all the edges in the cycle are indeed in the graph. This clearly can be done in polynomial time.

   (b) Hamiltonian circuit problem can be reduced to the above problem by using the same graph and taking $k = n$ (the number of nodes in the graph).

2. Consider the following problem called vertex cover.

   Input: An undirected graph $G = (V, E)$, and a number k.

   Question: Does there exists a vertex cover of size k? That is, does there exist $V' \subseteq V$, $|V'| \leq k$ such that, for each edge $(u, v) \in E$, at least one of u, v is in V'.

   Show that the above problem is NP complete.

   Ans:

   (a) To show that the problem is in NP, the certificates would be of the form V' of cardinality at most k which form a cover.

   Verification would be to check that indeed $V' \subseteq V$, $|V'| \leq k$, and for every edge $(u, v) \in E$, at least one of u, v is in V'.

   (b) Two methods.

   First method: Note that for a given graph $G = (V, E)$, V' is a vertex cover iff $V - V'$ is an independent set. Thus, there exists a vertex cover of size $\leq k$ iff there exists an independent set of size $\geq |V| - k$. As independent set problem is NP-hard, we immediately get that vertex cover problem is NP-hard.

   Second method:

   Direct reduction from 3-SAT:

   Suppose (U, C) is a 3-SAT problem.

   Consider the vertex cover problem constructed as follows:

   Suppose $U = \{x_1, x_2, \ldots, x_n\}$, and $C = \{c_1, c_2, \ldots, c_m\}$, where $c_i = \ell_i^1 \vee \ell_i^2 \vee \ell_i^3$.

   Then, let $V = \{u_i, w_i : 1 \leq i \leq n\} \cup \{a_i^1, a_i^2, a_i^3 : 1 \leq i \leq m\}$.

Below, let $b_i^r = u_j$, if $\ell_i^r = x_j$; $b_i^r = w_j$, if $\ell_i^r = \neg x_j$.

$E = E1 \cup E2 \cup E3$, where

$E1 = \{(u_i, w_i) : 1 \leq i \leq n\}$,

$E2 = \{(a_i^1, a_i^2), (a_i^1, a_i^3), (a_i^2, a_i^3) : 1 \leq i \leq m\}$

and

$E3 = \{(a_i^r, b_i^r) : 1 \leq i \leq m, 1 \leq r \leq 3\}$.

$k = 2m + n$.

Now, if the 3-SAT problem (U, C) is satisfiable, then fix one satisfying assignment $Q(\cdot)$. Let $R(i) \in \{1, 2, 3\}$ be such that $\ell_i^{R(i)}$ is true (in case of several possible $R(i)$, choose arbitrarily).

Let $V' = \{u_i : Q(x_i) \text{ is true }\} \cup \{w_i : Q(x_i) \text{ is false }\} \cup \{a_i^r : r \neq R(i)\}$.

Then clearly $|V'| = n + 2m$, and each edge in $E1, E2$ is covered. Furthermore, the edges in $E3$ are also covered as $b_i^{R(i)}$ is chosen in V' since $\ell_i^{R(i)}$ is true.

On the other hand, if vertex cover problem has solution V', then clearly, V' must have exactly two vertices from each of $\{a_i^1, a_i^2, a_i^3\}$ and one vertex from each of $\{u_i, w_i\}$. Then, let x_i be true iff $u_i \in V'$.

Now, each clause c_i has a true literal (if a_i^r is not chosen in V', then b_i^r must be chosen, and thus the literal ℓ_i^r is true).

3. 3-Colorability

   Input: An undirected graph $G = (V, E)$.

   Question: Is there a mapping $color : V \to \{1, 2, 3\}$ such that for all $(v, w) \in E$, $color(v) \neq color(w)$?

   Show that the above problem is NP-complete.

   Ans: Easy to show that graph 3-colorability problem is in NP.

   To show that it is NP-hard, consider reduction from 3-SAT.

   Let (U, C) be a 3-SAT problem.

   Suppose $U = \{x_1, x_2, \ldots, x_n\}$ and $C = \{c_1, c_2, \ldots, c_m\}$, where $c_i = (\ell_i^1 \vee \ell_i^2 \vee \ell_i^3)$.

   Then, construct the following 3-color problem:

   $V = \{u_i, w_i : 1 \leq i \leq n\} \cup \{C0, C2\} \cup \{a_i^1, a_i^2, a_i^3, a_i^4, a_i^5, a_i^6 : 1 \leq i \leq m\}$.

   Let $b_i^r = u_j$, if $\ell_i^r = x_j$, and $b_i^r = w_j$, if $\ell_i^r = \neg x_j$.

   $E = \{(C0, C2)\} \cup E1 \cup E2 \cup E3 \cup E4$, where

   $E1 = \{(u_i, w_i), (u_i, C2), (w_i, C2) : 1 \leq i \leq n\}$

   $E2 = \{(a_i^6, C0), (a_i^6, C2)\}$

   $E3 = \{(a_i^1, a_i^2), (a_i^1, a_i^4), (a_i^2, a_i^4), (a_i^4, a_i^5), (a_i^3, a_i^5), (a_i^3, a_i^6), (a_i^5, a_i^6) : 1 \leq i \leq m\}$

   $E4 = \{(b_i^r, a_i^r) : 1 \leq i \leq m, 1 \leq r \leq 3\}$

Now, if the 3-SAT problem (U, C) is satisfiable, then fix one such satisfying assignment. Color the vertices in V as follows.

u_i is colored 1, and w_i is colored 0 iff x_i is true.

u_i is colored 0, and w_i is colored 1 iff x_i is false.

$C0$ is colored 0.

$C2$ is colored 2.

a_i^4 is colored as ($color(b_i^1)$ OR $color(b_i^2)$) and a_i^6 is colored as 1 (that is ($color(a_i^4)$ OR $color(b_i^3)$).

One of a_i^1, a_i^2 is colored 2 and the other is colored $1 - color(a_i^4)$ (note that this can be done as at least one of b_i^1, b_i^2 is the same color as a_i^4). Similarly, one of a_i^3, a_i^5 is colored 2 and the other is colored $1 - color(a_i^6)$ (note that this can be done as at least one of a_i^4, b_i^3 is the same color as a_i^6).

On the other hand, if the coloring is possible, then without loss of generality assume that the color of $C2$ is 2, and color of $C0$ is 0. This implies that color of one of u_i, w_i is 0 and the other 1. Consider the truth assignment $Q(x_i)$ =true iff u_i is colored 1.

Now, if b_i^r, $1 \leq r \leq 3$ are all colored 0, then a_i^4 and a_i^6 must also be colored 0. But this will cause a conflict with the edge $(a_i^6, C0)$. Thus, at least one of b_i^r, $1 \leq r \leq 3$ is colored 1, and thus at least one literal in c_i is true.

4. Not-All-Equal SAT (NAESAT).

   Input: A set of variables V, and a set C of clauses (you may assume each clause has exactly three literals).

   Question: Is there a truth assignment to the variables so that each clause has at least one true literal and at least one false literal?

   Show that the above problem is NP complete.

   Ans: It is easy to see the NAESAT is in NP: Certificates would be truth assignment which make each clause have at least one true and at least one false literal. Verifier checks if the truth assignment indeed makes at least one literal true and one literal false in each clause.

   To show that NAESAT is NP-hard, we reduce 3-SAT to NAESAT.

   Suppose (U, C) is an instance of 3SAT. Suppose the clauses in C are $c_1, \ldots, c_m$.

   Let $U' = U \cup \{w_i, x_i : 1 \leq i \leq m\} \cup \{T\}$, where x_i, w_i and T are new variables.

   Suppose the i-th clause c_i is $(l_1^i \vee l_2^i \vee l_3^i)$. Then, let $C' = \{(l_1^i \vee l_2^i \vee w_i), (l_1^i \vee l_3^i \vee x_i), (x_i \vee w_i \vee T) : 1 \leq i \leq m\}$.

   We claim that (U, C) is satisfiable iff (U', C') is in NAESAT.

   Suppose (U, C) is satisfiable: Fix a truth assignment $A(.)$ to variables in U which satisfies all clauses in C.

   Consider the following truth assignment to variables in U'. $A'(v) = A(v)$, for $v \in U$. $A'(T)$ =True. For, $1 \leq i \leq m$, let w_i be false iff at least one of l_1^i or l_2^i is true. For, $1 \leq i \leq m$, let x_i be false iff at least one of l_1^i or l_3^i is true. Note that at least one of x_i

and w_i is false. It is now easy to verify that the truth assignment A' witnesses that each clause in C' has at least one true literal and at least one false literal.

Now suppose that there exists a truth assignment A' to variables in U' such that each clause in C' has at least one true literal and at least one false literal. Without loss of generality assume that $A'(T)$ is true (otherwise just flip the truth assignment of each variable). We claim that A' restricted to variables in U is a satisfying truth assignment for C. To see this note that both x_i and w_i cannot be true (otherwise all three literals in (x_i, w_i, T) are true). If w_i is false, then at least one of l_1^i, l_2^i is true. If x_i is false, then at least one of l_1^i, l_3^i is true. Thus C is satisfiable.