A brief Introduction to Particle Filters

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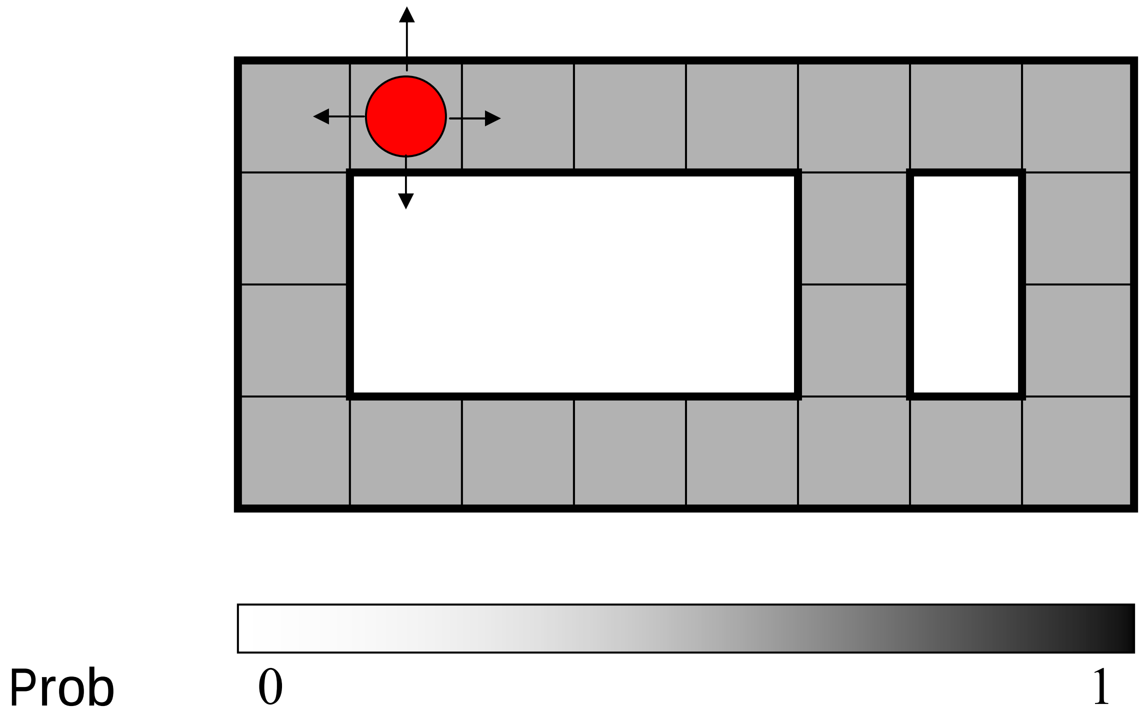
Agenda

- Problem Statement
- Classical Approaches
- Particle Filters
  - Theory
  - Algorithms
- Applications

Problem Statement

- Tracking the state of a system as it evolves over time
- We have: Sequentially arriving (noisy or ambiguous) observations
- We want to know: Best possible estimate of the hidden variables

Illustrative Example: Robot Localization

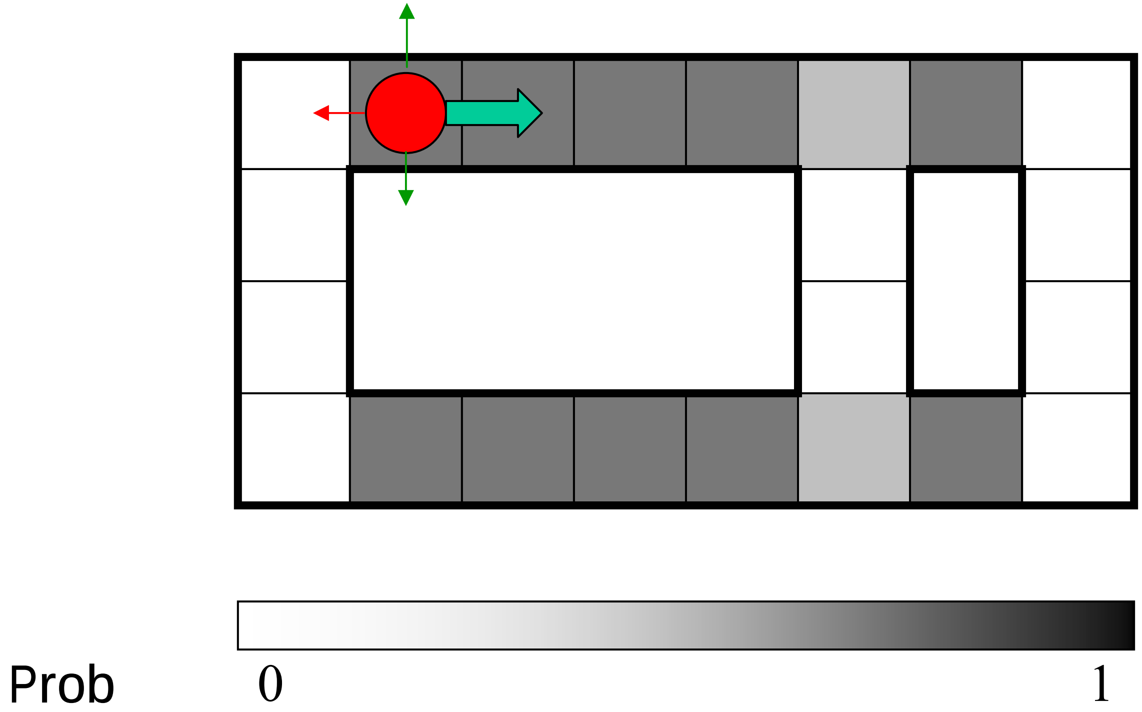


t=0

Sensory model: never more than 1 mistake

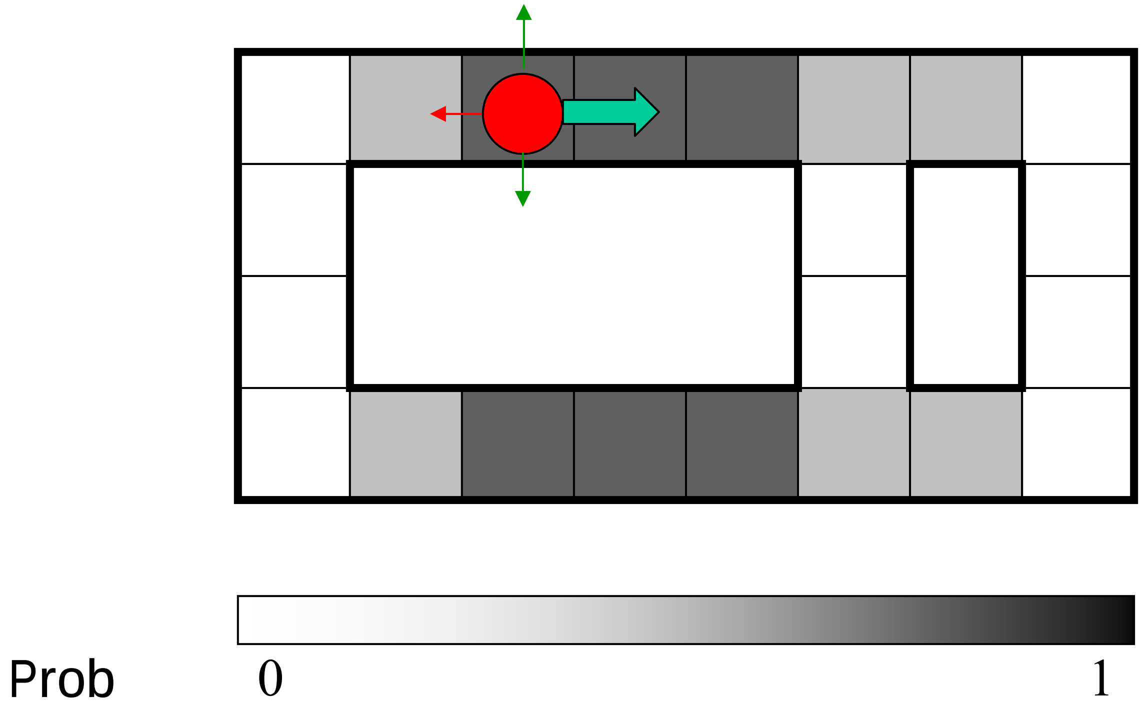
Motion model: may not execute action with small prob.

Illustrative Example: Robot Localization



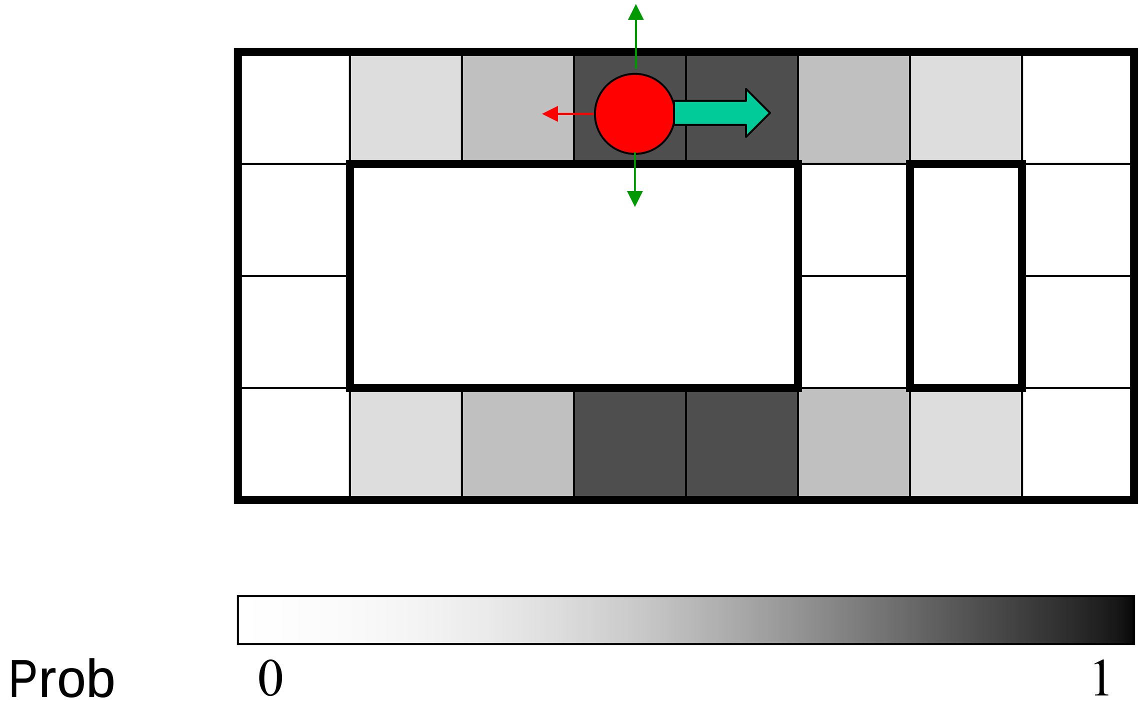
t=1

Illustrative Example: Robot Localization



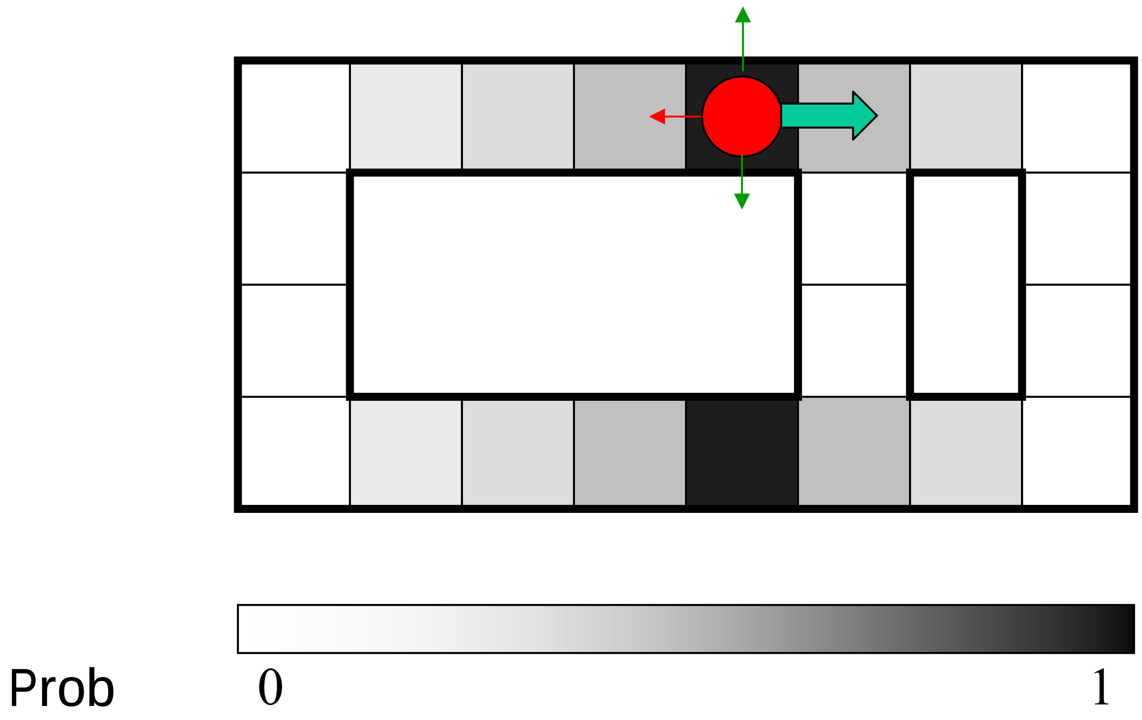
t=2

Illustrative Example: Robot Localization



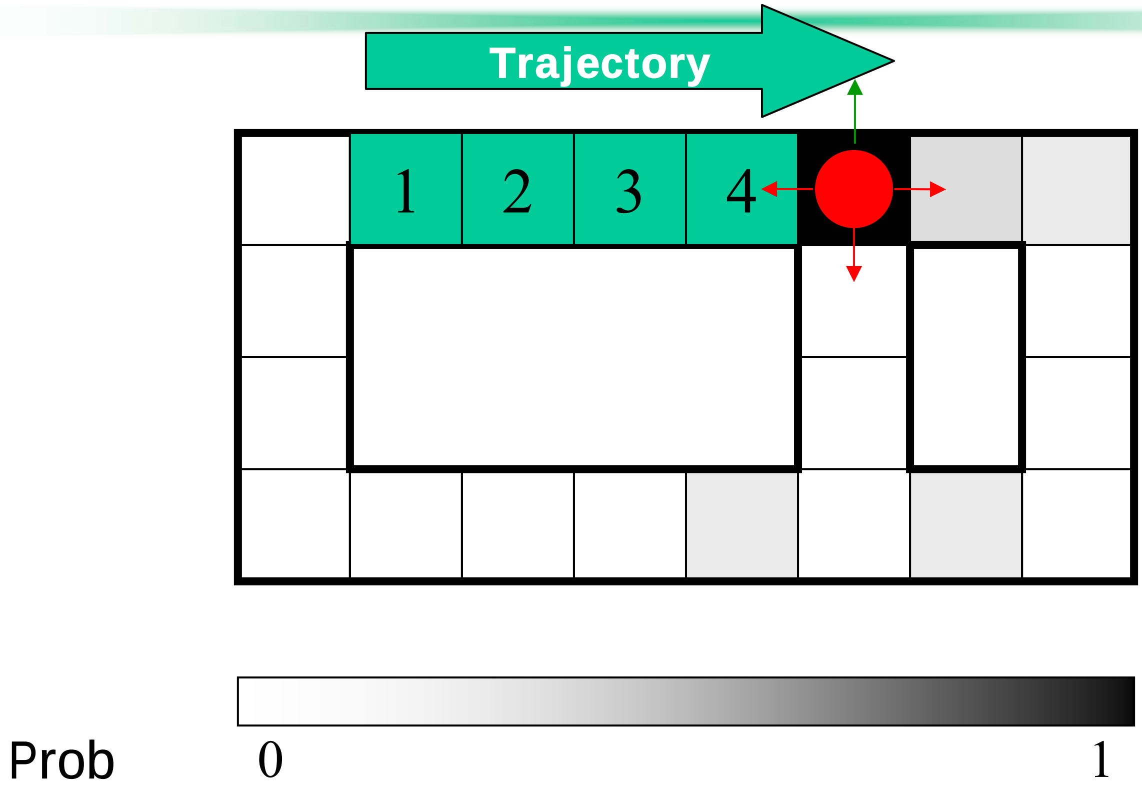
t=3

Illustrative Example: Robot Localization



t=4

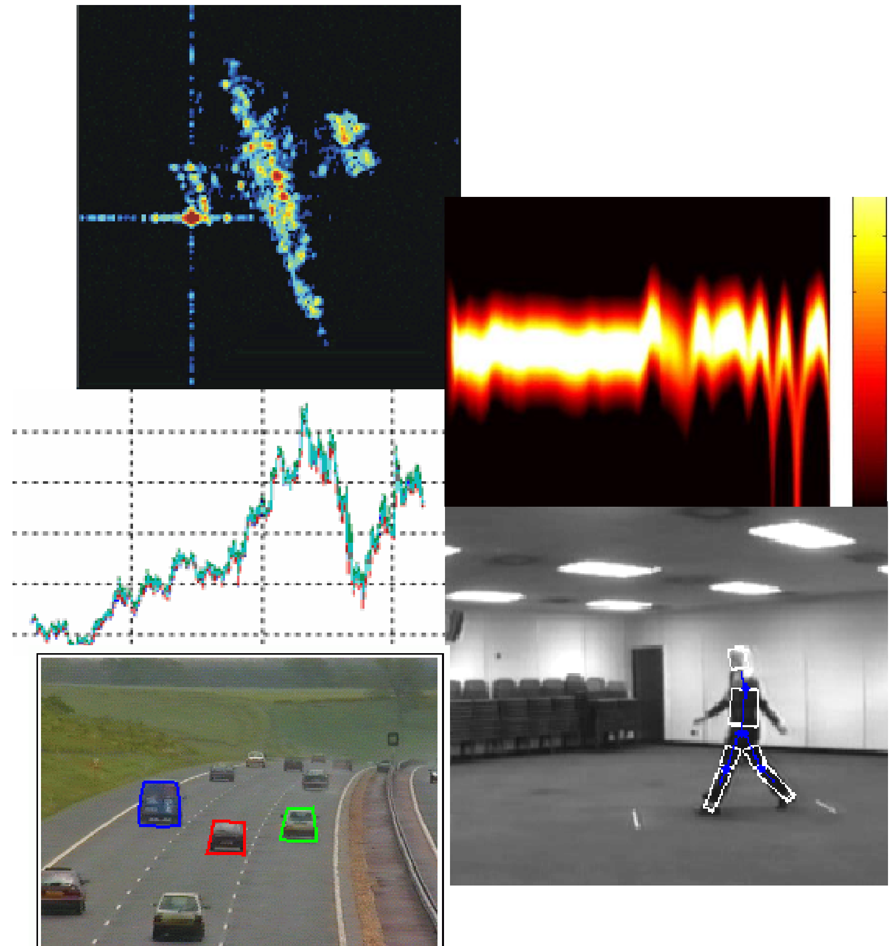
Illustrative Example: Robot Localization



t=5

Applications

- Tracking of aircraft positions from radar
- Estimating communications signals from noisy measurements
- Predicting economical data
- Tracking of people or cars in surveillance videos



Bayesian Filtering / Tracking Problem

- Unknown State Vector $x_{0:t} = (x_0, \ldots, x_t)$
- Observation Vector $z_{1:t}$
- Find PDF $p(x_{0:t} \mid z_{1:t})$ ... *posterior distribution*
- or $p(x_t \mid z_{1:t})$ ... *filtering distribution*

- Prior Information given:
  - $p(x_0)$ ... prior on state distribution
  - $p(z_t \mid x_t)$ ... sensor model
  - $p(z_t \mid x_{t-1})$ ... Markovian state-space model

Sequential Update

- Storing all incoming measurements is inconvenient
- Recursive filtering:
  - Predict next state pdf from current estimate
  - Update the prediction using sequentially arriving new measurements

- Optimal Bayesian solution: recursively calculating exact posterior density

Bayesian Update and Prediction

- Prediction

$$p(x_t \mid z_{1:t-1}) = \int p(x_t \mid x_{t-1})\, p(x_{t-1} \mid z_{1:t-1})\, dx_{t-1}$$

- Update

$$p(x_t \mid z_{1:t}) = \frac{p(z_t \mid x_t)\, p(x_t \mid z_{1:t-1})}{p(z_t \mid z_{1:t-1})}$$

$$p(z_t \mid z_{1:t-1}) = \int p(z_t \mid x_t)\, p(x_t \mid z_{1:t-1})\, dx_t$$

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Kalman Filter

- Optimal solution for linear-Gaussian case
- Assumptions:
  - State model is known linear function of last state and Gaussian noise signal
  - Sensory model is known linear function of state and Gaussian noise signal
  - Posterior density is Gaussian

Kalman Filter: Update Equations

$$x_t = F_t x_{t-1} + v_{t-1} \quad v_{t-1} \sim N(0, Q_{t-1})$$

$$z_t = H_t x_t + n_t \qquad n_t \sim N(0, R_t)$$

$$F_t, H_t : \text{known matrices}$$

$$p(x_{t-1} \mid z_{1:t-1}) = N(x_{t-1} \mid m_{t-1|t-1}, P_{t-1|t-1})$$

$$p(x_t \mid z_{1:t-1}) = N(x_t \mid m_{t|t-1}, P_{t|t-1})$$

$$p(x_t \mid z_{1:t}) = N(x_t \mid m_{t|t}, P_{t|t})$$

$$m_{t|t-1} = F_t \, m_{t-1|t-1}$$

$$P_{t|t-1} = Q_{t-1} + F_t P_{t-1|t-1} F_t^T$$

$$m_{t|t} = m_{t|t-1} + K_t (z_t - H_t m_{t|t-1})$$

$$P_{t|t} = P_{t|t-1} - K_t H_t P_{t|t-1}$$

$$S_t = H_t P_{t|t-1} H_t^T + R_t$$

$$K_t = P_{t|t-1} H_t^T S_t^{-1}$$

Limitations of Kalman Filtering

- Assumptions are too strong. We often find:
  - Non-linear Models
  - Non-Gaussian Noise or Posterior
  - Multi-modal Distributions
  - Skewed distributions
- Extended Kalman Filter:
  - local linearization of non-linear models
  - still limited to Gaussian posterior

Grid-based Methods

- Optimal for discrete and finite state space
- Keep and update an estimate of posterior pdf for every single state
- No constraints on posterior (discrete) density

Limitations of Grid-based Methods

- Computationally expensive
- Only for finite state sets
- Approximate Grid-based Filter
  - divide continuous state space into finite number of cells
  - Hidden Markov Model Filter
  - Dimensionality increases computational costs dramatically

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Many different names...

Particle Filters

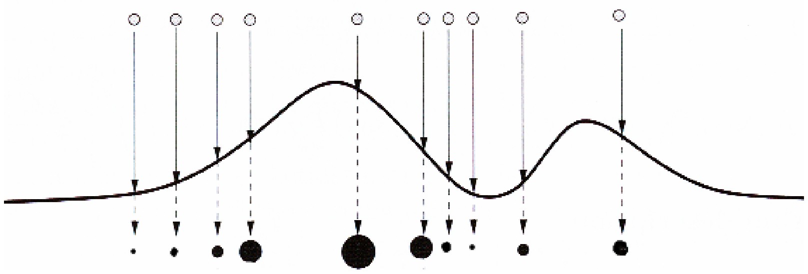
- (Sequential) Monte Carlo filters
- Bootstrap filters
- Condensation
- Interacting Particle Approximations
- Survival of the fittest
- ...

Sample-based PDF Representation

- Monte Carlo characterization of pdf:
  - Represent posterior density by a set of random i.i.d. samples (**particles**) from the pdf $p(x_{0:t}|z_{1:t})$
  - For larger number N of particles equivalent to functional description of pdf
  - For $N \rightarrow \infty$ approaches optimal Bayesian estimate

Sample-based PDF Representation

- Regions of high density
  - Many particles
  - Large weight of particles
- Uneven partitioning



- Discrete approximation for continuous pdf

$$P_N(x_{0:t} \mid z_{1:t}) = \sum_{i=1}^{N} w_t^i \, \delta(x_{0:t} - x_{0:t}^i)$$

Importance Sampling

- Draw N samples $x_{0:t}^{(i)}$ from Importance sampling distribution $\pi(x_{0:t}|z_{1:t})$
- Importance weight $$w(x_{0:t}) = \frac{p(x_{0:t} \mid z_{1:t})}{\pi(x_{0:t} \mid z_{1:t})}$$
- Estimation of arbitrary functions f_t:

$$\hat{I}_N(f_t) = \sum_{i=1}^{N} f_t(x_{0:t}^{(i)})\, \widetilde{w}_t^{(i)}, \quad \widetilde{w}_t^{(i)} = \frac{w(x_{0:t}^{(i)})}{\sum_{j=1}^{N} w(_{0:t}^{(j)})}$$

$$\hat{I}_N(f_t) \underset{N\to\infty}{\overset{a.s.}{\to}} I(f_t) = \int f_t(x_{0:t})\, p(x_{0:t} \mid y_{1:t})\, dx_{0:t}$$

Sequential Importance Sampling (SIS)

- Augmenting the samples

$$\pi(x_{0:t} \mid z_{1:t}) \quad = \pi(x_{0:t-1} \mid z_{1:t-1})\,\pi(x_t \mid x_{0:t-1}, z_{1:t}) =$$
$$= \pi(x_{0:t-1} \mid z_{1:t-1})\,\pi(x_t \mid x_{t-1}, z_t)$$

$$x_t^{(i)} \sim \pi(x_t \mid x_{t-1}^{(i)}, z_t)$$

- Weight update

$$w_t^{(i)} \propto w_{t-1}^{(i)} \frac{p(z_t \mid x_t^{(i)})\, p(x_t^{(i)} \mid x_{t-1}^{(i)})}{\pi(x_t^{(i)} \mid x_{t-1}^{(i)}, z_t)}$$

Degeneracy Problem

- After a few iterations, all but one particle will have negligible weight
- Measure for degeneracy: *Effective sample size*

$$N_{eff} = \frac{N}{1 + Var(w_t^{*i})}$$

w_t^* ... true weights at time t

- Small N_{eff} indicates severe degeneracy
- Brute force solution: Use very large N

Choosing Importance Density

- Choose π to minimize variance of weights
- Optimal solution: $\pi_{opt}(x_t \mid x_{t-1}^{(i)}, z_t) = p(x_t \mid x_{t-1}^{(i)}, z_t)$

$$\Rightarrow w_t^{(i)} \propto w_{t-1}^{(i)}\, p(z_t \mid x_{t-1}^{(i)})$$

- Practical solution $\pi(x_t \mid x_{t-1}^{(i)}, z_t) = p(x_t \mid x_{t-1}^{(i)})$

$$\Rightarrow w_t^{(i)} \propto w_{t-1}^{(i)}\, p(z_t \mid x_t^{(i)})$$

  - importance density = prior

Resampling

- Eliminate particles with small importance weights
- Concentrate on particles with large weights

- Sample N times *with replacement* from the set of particles $x_t^{(i)}$ according to importance weights $w_t^{(i)}$
- „*Survival of the fittest*“

Sampling Importance Resample Filter: Basic Algorithm

- 1. INIT, $t=0$
  - for $i=1,\ldots, N$: sample $x_0^{(i)} \sim p(x_0)$; $t:=1$;
- 2. IMPORTANCE SAMPLING
  - for $i=1,\ldots, N$: sample $x_t^{(i)} \sim p(x_t|x_{t-1}^{(i)})$
    - $x_{0:t}^{(i)} := (x_{0:t-1}^{(i)}, x_t^{(i)})$
  - for $i=1,\ldots, N$: evaluate importance weights $w_t^{(i)}=p(z_t|x_t^{(i)})$
  - Normalize the importance weights
- 3. SELECTION / RESAMPLING
  - resample with replacement N particles $x_{0:t}^{(i)}$ according to the importance weights
  - Set $t:=t+1$ and go to step 2

Variations

- Auxiliary Particle Filter:
  - resample at time t-1 with one-step lookahead (re-evaluate with new sensory information)
- Regularisation:
  - resample from continuous approximation of posterior $p(x_t|z_{1:t})$

Visualization of Particle Filter

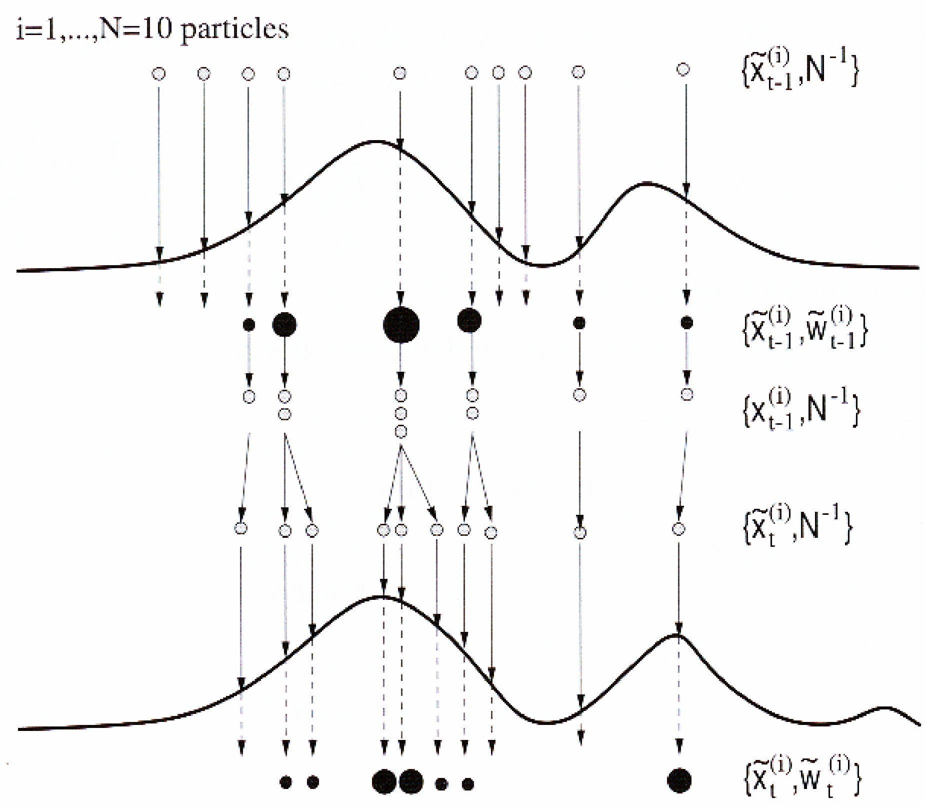
unweighted measure

compute importance weights $\Rightarrow p(x_{t-1}|z_{1:t-1})$

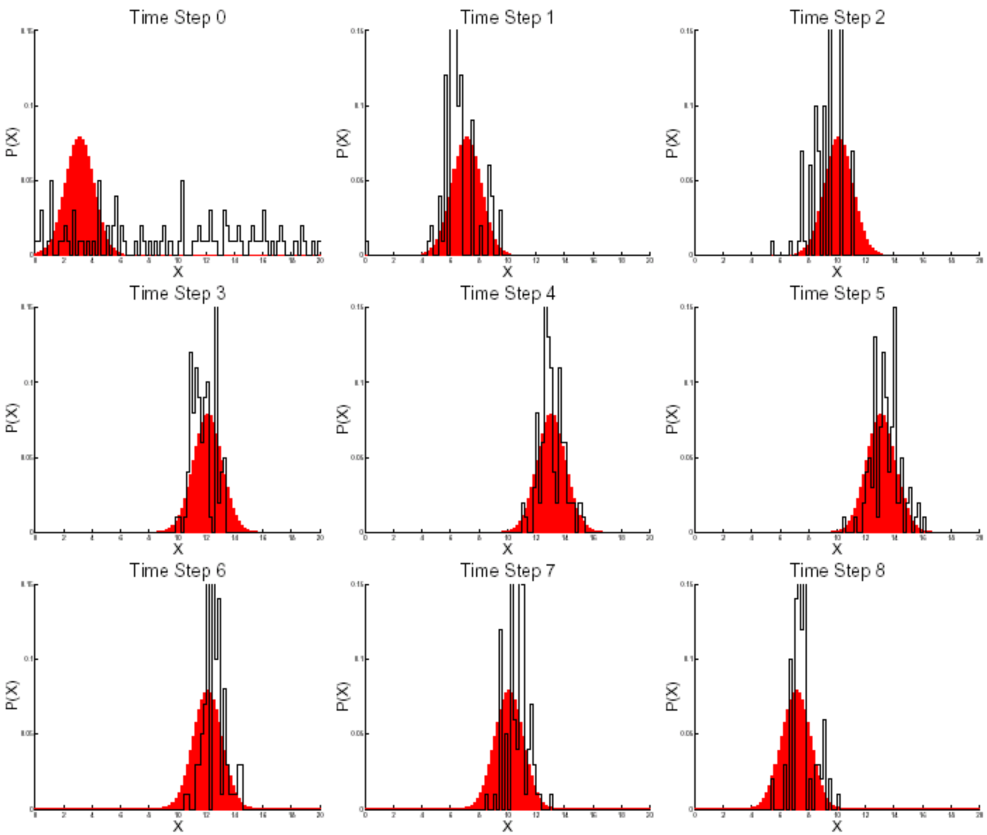
resampling

move particles

predict $p(x_t|z_{1:t-1})$

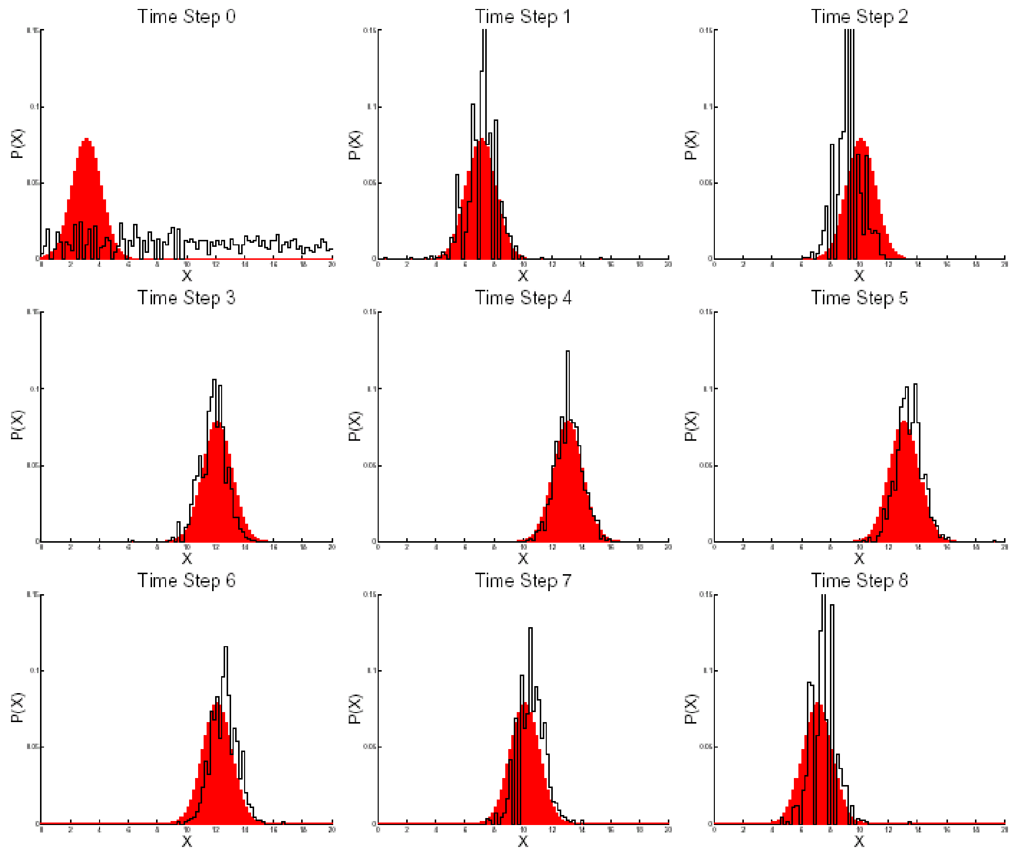


Particle Filter Demo 1



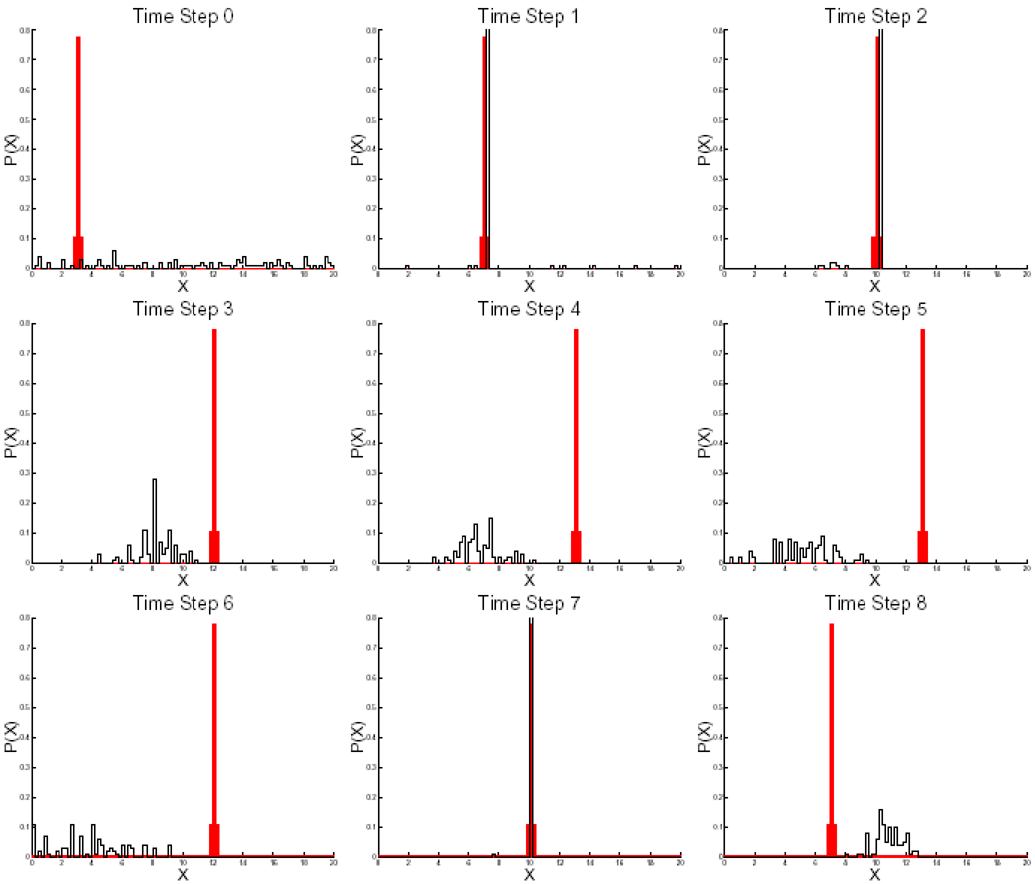
moving Gaussian + uniform, N=100 particles

Particle Filter Demo 2



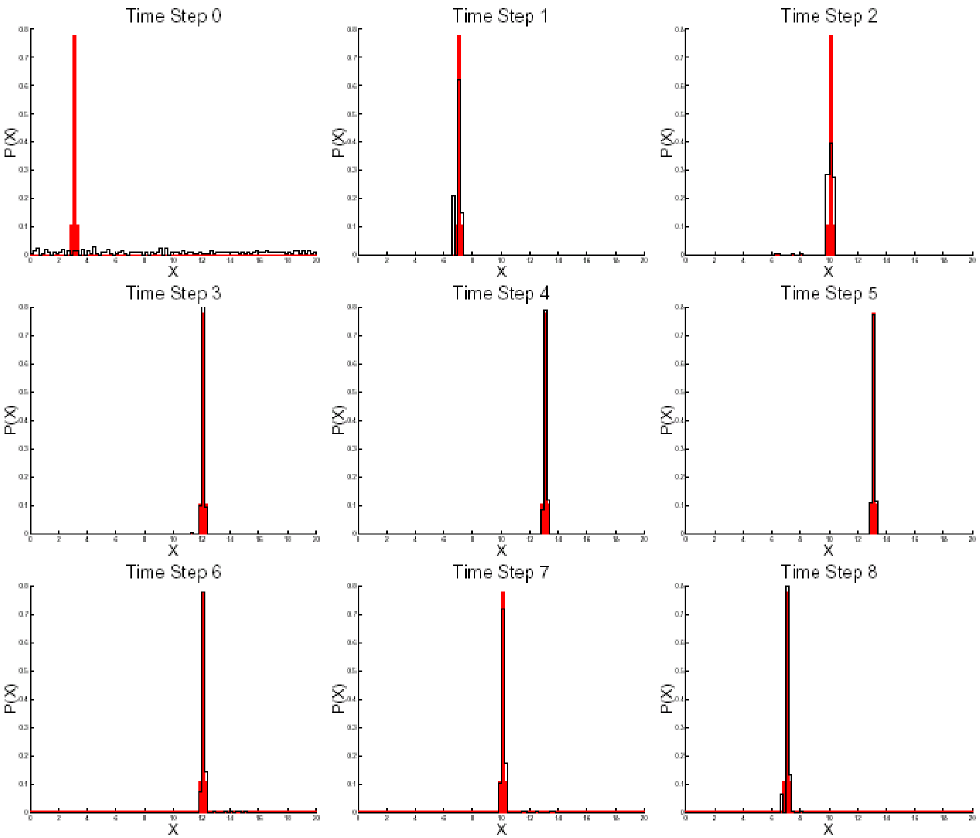
moving Gaussian + uniform, N=1000 particles

Particle Filter Demo 3



moving (sharp) Gaussian + uniform, N=100 particles

Particle Filter Demo 4

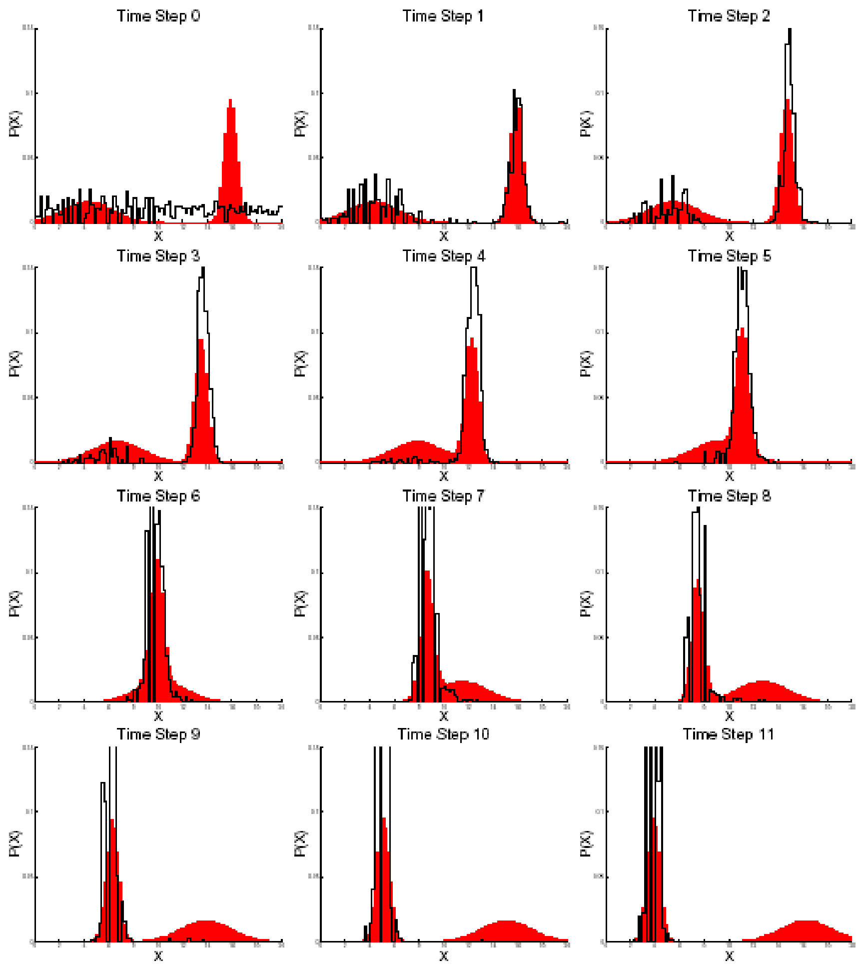


moving (sharp) Gaussian + uniform, N=1000 particles

Particle Filter Demo 5

mixture of two Gaussians,

filter loses track of smaller and less pronounced peaks



Obtaining state estimates from particles

- Any estimate of a function $f(x_t)$ can be calculated by discrete PDF-approximation

$$E\left[f(x_t)\right] = \frac{1}{N}\sum_{j=1}^{N} w_t^{(j)} f(x_t^{(j)})$$

- Mean: $$E\left[x_t\right] = \frac{1}{N}\sum_{j=1}^{N} w_t^{(j)} x_t^{(j)}$$
- MAP-estimate: particle with largest weight
- Robust mean: mean within window around MAP-estimate

Pros and Cons of Particle Filters

+ Estimation of full PDFs
+ Non-Gaussian distributions
  + e.g. multi-modal
+ Non-linear state and observation model
+ Parallelizable

- Degeneracy problem
- High number of particles needed
- Computationally expensive
- Linear-Gaussian assumption is often sufficient

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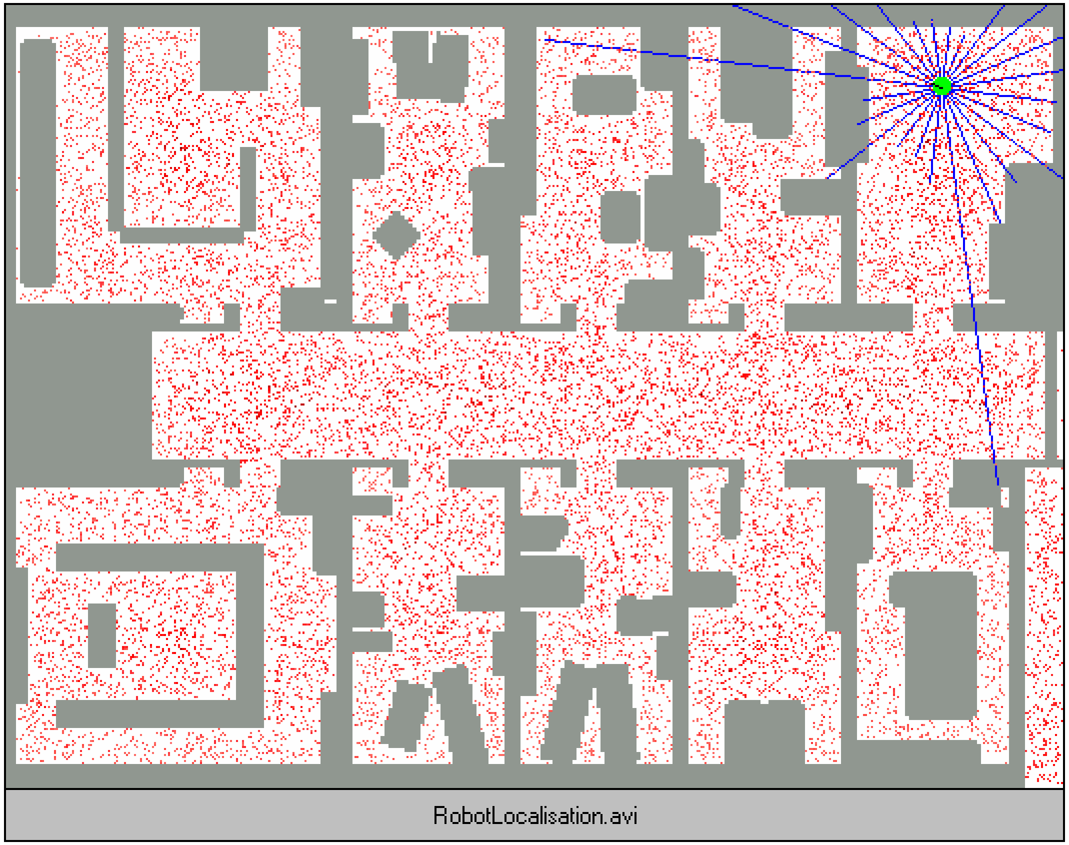
Mobile Robot Localization

- Animation by Sebastian Thrun, Stanford
- http://robots.stanford.edu



Positioning Systems[1]

- Track car position in given road map
- Track car position from radio frequency measurements
- Track aircraft position from estimated terrain elevation
- Collision Avoidance (Prediction)
- Replacement for GPS

1: Gustafsson, et.al.: Particle Filters for Positioning, Navigation and Tracking. IEEE Transactions on Signal Processing Vol. 50, 2002

Model Estimation

- Tracking with multiple motion-models
  - Discrete hidden variable indicates active model (manoever)
- Recovery of signal from noisy measurements
  - even if signal may be absent (e.g. synaptic currents)
  - mixture model of several hypotheses
- Neural Network model selection [de Freitas][1]
  - estimate parameters and architecture of RBF network from input-output pairs
  - on-line classification (time-varying classes)

1: de Freitas, et.al.: Sequential Monte Carlo Methods for Neural Networks. in: Doucet, et.al.: Sequential Monte Carlo Methods in Practice, Springer Verlag, 2001

Other Applications

- Visual Tracking
  - e.g. human motion (body parts)
- Prediction of (financial) time series
  - e.g. mapping gold price → stock price
- Quality control in semiconductor industry
- Military applications
  - Target recognition from single or multiple images
  - Guidance of missiles

Possible Uses for our Group

- Reinforcement Learning
  - POMDPs
  - Estimating Opponent States
- RoboCup: Multi-robot localization and tracking
- Representation of PDFs
- Prediction Tasks
- Preprocessing of Visual Input
- Identifying Neural Network Layouts or other Hidden System Parameters
- Applications in Computational Neuroscience (?)

- Other suggestions?

Sources

- Doucet, de Freitas, Gordon: *Sequential Monte Carlo Methods in Practice*, Springer Verlag, 2001
- Arulampalam, Maskell, Gordon, Clapp: *A Tutorial on Particle Filters for on-line Non-linear / Non-Gaussian Bayesian Tracking*, IEEE Transactions on Signal Processing, Vol. 50, 2002

Thank you!