# Probability Homework 5:
## PMFs, Expectations and Variances of Discrete Distributions

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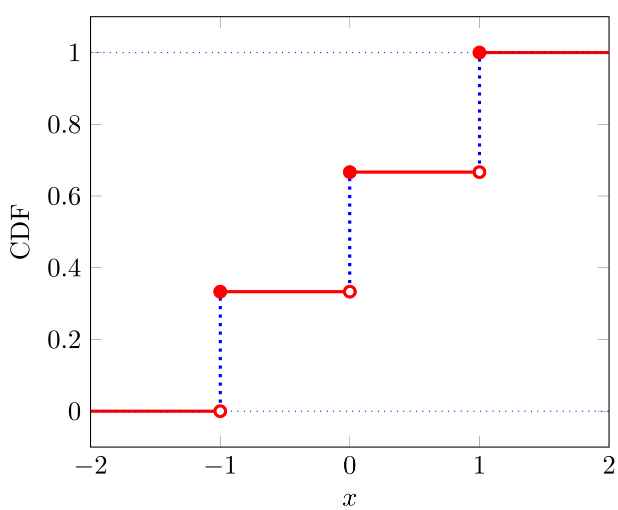
**Question 1**: Given $p_X(x)$ (the PMF), write out the formula for CDF $F(x)$ and sketch its graph:
$p_X(x) = 1/3,$ for $x = -1, 0, 1,$ zero elsewhere

**Solution**:

Let X be a discrete random variable with the following PMF:

$$p_X(x) = \begin{cases} 1/3 & \text{for } x = -1 \\ 1/3 & \text{for } x = 0 \\ 1/3 & \text{for } x = 1 \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

$$F_X(x) = \begin{cases} 0 & \text{if } x < -1 \\ 1/3 & \text{if } -1 \le x < 0 \\ 2/3 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x \ge 1 \end{cases} \tag{1.2}$$



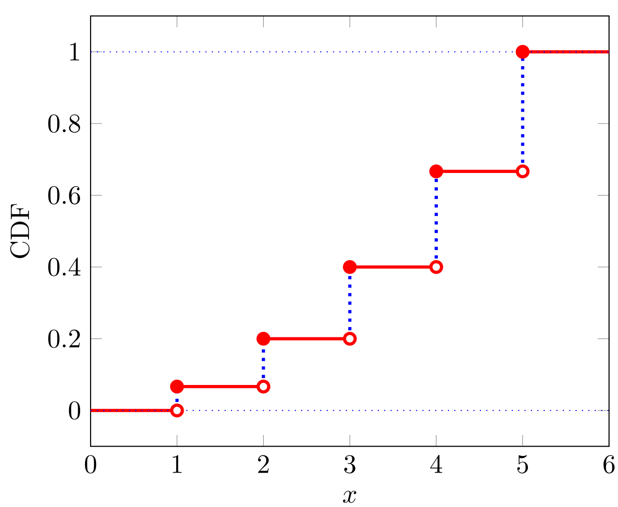
**Question 2**: Given $p_X(x)$ (the PMF), write out the formula for CDF $F(x)$ and sketch its graph: $p_X(x) = {}^{x}/_{15}$, for $x = 1, 2, 3, 4, 5$, zero elsewhere

**Solution**:

Let X be a discrete random variable with the following PMF:

$$p_X(x) = \begin{cases} 1/15 & \text{for } x = 1 \\ 2/15 & \text{for } x = 2 \\ 3/15 & \text{for } x = 3 \\ 4/15 & \text{for } x = 4 \\ 5/15 & \text{for } x = 5 \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

$$F_X(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1/15 & \text{if } 1 \le x < 2 \\ 3/15 & \text{if } 2 \le x < 3 \\ 6/15 & \text{if } 3 \le x < 4 \\ 10/15 & \text{if } 4 \le x < 5 \\ 1 & \text{if } x \ge 5 \end{cases} \tag{2.2}$$



**Question 3**: Let X be a discrete random variable with the following PMF:

$$P_X(k) = \begin{cases} 0.1 & \text{for } k = 0 \\ 0.4 & \text{for } k = 1 \\ 0.3 & \text{for } k = 2 \\ 0.2 & \text{for } k = 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $E(X)$.

**Solution**:

$$E(X) = \sum_x x \times Pr(X = x) = (0 \cdot 0.1) + (1 \cdot 0.4) + (2 \cdot 0.3) + (3 \cdot 0.2) = \mathbf{1.6} \tag{3.1}$$

(b) Find $Var(X)$.

**Solution**:

$$\begin{aligned} Var(X) = \sum_x [x - E(X)]^2 \times Pr(X = x) = E(X^2) - \mu^2 \\ = (0^2 \cdot 0.1) + (1^2 \cdot 0.4) + (2^2 \cdot 0.3) + (3^2 \cdot 0.2) - 1.6^2 \\ = 3.4 - 2.56 = \mathbf{0.84} \end{aligned} \tag{3.2}$$

(c) If $Y = (X - 2)^2$, find $E(Y)$.

**Solution**:

If $Y = (X - 2)^2$, then

$$\begin{aligned} E(Y) = E[(X - 2)^2] = E[(X^2 - 4X) + 4] \\ = E(X^2) - 4E(X) + 4 = 3.4 - 4 \cdot 1.6 + 4 = \mathbf{1} \end{aligned} \tag{3.3}$$

**Question 4**: Suppose that $p(x) = \frac{1}{5}$, for $x = 1, 2, 3, 4, 5$, zero elsewhere, is the PMF of the discrete-type random variable $X$. Compute $E(X)$ and $E(X^2)$. Use these two results to find $E[(X + 2)^2]$.

**Solution**:

First,the PMF of the variable X is:

$$P_X(x) = \begin{cases} 1/5 & \text{for } x = 1 \\ 1/5 & \text{for } x = 2 \\ 1/5 & \text{for } x = 3 \\ 1/5 & \text{for } x = 4 \\ 1/5 & \text{for } x = 5 \\ 0 & \text{otherwise} \end{cases} \tag{4.1}$$

Then, the $E(X)$ and $E(X^2)$ are:

$$E(X) = \sum_x x \times Pr(X = x) = (1 \cdot 1/5) + (2 \cdot 1/5) + (3 \cdot 1/5) + (4 \cdot 1/5) + (5 \cdot 1/5) = \mathbf{3} \tag{4.2}$$

$$E(X^2) = \sum_x x^2 \times Pr(X = x) = (1^2 \cdot 1/5) + (2^2 \cdot 1/5) + (3^2 \cdot 1/5) + (4^2 \cdot 1/5) + (5^2 \cdot 1/5) = \mathbf{11} \tag{4.3}$$

Thus, the $E[(X + 2)^2]$ is:

$$E[(X + 2)^2] = E(X^2 + 4X + 4) = E(X^2) + 4E(X) + 4 = 11 + 4 \cdot 3 + 4 = \mathbf{27} \tag{4.4}$$

**Question 5**: Find the expectation and variance of $p(x) = \frac{3!}{x!(3-x)!}\left(\frac{1}{2}\right)^3, x = 0, 1, 2, 3$, zero elsewhere.

**Solution**:

Let X be a discrete random variable with the following PMF:

$$p_X(x) = \begin{cases} 1/8 & \text{for } x = 0 \\ 3/8 & \text{for } x = 1 \\ 3/8 & \text{for } x = 2 \\ 1/8 & \text{for } x = 3 \\ 0 & \text{otherwise} \end{cases} \tag{5.1}$$

Then, the expectation of X is:

$$E(X) = \sum_x x \times Pr(X = x) = (0 \cdot {}^1\!/\!_8) + (1 \cdot {}^3\!/\!_8) + (2 \cdot {}^3\!/\!_8) + (3 \cdot {}^1\!/\!_8) = \mathbf{1.5} \tag{5.2}$$

The variance of X is:

$$\begin{aligned} Var(X) &= \sum_x [x - E(X)]^2 \times Pr(X = x) = E(X^2) - [E(X)]^2 \\ &= (0^2 \cdot {}^1\!/\!_8) + (1^2 \cdot {}^3\!/\!_8) + (2^2 \cdot {}^3\!/\!_8) + (3^2 \cdot {}^1\!/\!_8) - 1.5^2 \\ &= 3 - 2.25 = \mathbf{0.75} \end{aligned} \tag{5.3}$$

**Question 6**: Let $X$ be the damage incurred (in \$) in a certain type of accident during a given year. Possible $X$ values are 0, 1000, 5000, and 10000, with probabilities 0.8, 0.1, 0.08, and 0.02, respectively. A particular company offers a \$500 deductible policy. If the company wishes its expected profit per client to be \$100, what premium amount should it charge?

**Solution**:

First, for X=0, the premium charge the company should charge is:

$$\textbf{Premium}_0 = \$0 + \$100 = \mathbf{\$100} \tag{6.1}$$

With a deductible of \$500, for X among (1000, 5000, 10000), the actual claim amount the company should pay is:

$$\begin{aligned} \textbf{Claim}_{1,000} &= \$1,000 - \$500 = \$500 \\ \textbf{Claim}_{5,000} &= \$5,000 - \$500 = \$4,500 \\ \textbf{Claim}_{10,000} &= \$10,000 - \$500 = \$9,500 \end{aligned} \tag{6.2}$$

This is equal to Since the company wishes its expected profit per client to be \$100, there will be an extra \$100 on each claim amount:

$$\begin{aligned} \textbf{Total Claim}_{1,000} &= \$1000 - \$500 + \$100 = \mathbf{\$600} \\ \textbf{Total Claim}_{5,000} &= \$5,000 - \$500 + \$100 = \mathbf{\$4{,}600} \\ \textbf{Total Claim}_{10,000} &= \$10,000 - \$500 + \$100 = \mathbf{\$9{,}600} \end{aligned} \tag{6.3}$$

Then, the PMF of Y is:

$$P_Y(X) = \begin{cases} 0.8 & \text{for } X = \$100 \\ 0.1 & \text{for } X = \$600 \\ 0.08 & \text{for } X = \$4,600 \\ 0.02 & \text{for } X = \$9,600 \\ 0 & \text{otherwise} \end{cases} \tag{6.4}$$

The expected premium charge is:

$$E(Y) = 0.8 \cdot \$100 + 0.1 \cdot \$600 + 0.08 \cdot \$4,600 + 0.02 \cdot \$9,600 = \mathbf{\$700} \tag{6.5}$$

**Question 7**: Two fair six-sided dice are tossed independently. Let $M$ be the maximum of the two tosses. What is the PMF of $M$?

**Solution**:

The possible values of M are:
$(1, 2, 3, 4, 5, 6)$

Here is a list of all possible events under each value of M:

| M | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Event | (1,1) | (1,2) | (1,3) | (1,4) | (1,5) | (1,6) |
| | | (2,2) | (2,3) | (2,4) | (2,5) | (2,6) |
| | | (2,1) | (3,3) | (3,4) | (3,5) | (3,6) |
| | | | (3,2) | (4,4) | (4,5) | (4,6) |
| | | | (3,1) | (4,3) | (5,5) | (5,6) |
| | | | | (4,2) | (5,4) | (6,6) |
| | | | | (4,1) | (5,3) | (6,5) |
| | | | | | (5,2) | (6,4) |
| | | | | | (5,1) | (6,3) |
| | | | | | | (6,2) |
| | | | | | | (6,1) |

The PMF of M is:

$$P(M) = \begin{cases} 1/36 & \text{for } M = 1 \\ 3/36 & \text{for } M = 2 \\ 5/36 & \text{for } M = 3 \\ 7/36 & \text{for } M = 4 \\ 9/36 & \text{for } M = 5 \\ 11/36 & \text{for } M = 6 \end{cases} \tag{7.1}$$

**Question 8**: Airlines sometimes overbook flights. Suppose that for a plane with 50 seats, 55 passengers have tickets. Define the random variable $Y$ as the number of ticketed passengers who actually show up for the flight. The probability mass function of $Y$ appears in the following table:

| y | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 | 54 | 55 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| p(y) | 0.05 | 0.10 | 0.12 | 0.14 | 0.25 | 0.17 | 0.06 | 0.05 | 0.03 | 0.02 | 0.01 |

(a) What is the probability that the flight will accommodate all ticketed passengers who show up?

**Solution**:

$$\begin{aligned} P(Y \leq 50) &= P(Y = 45) + P(Y = 46) + P(Y = 47) \\ &\quad + P(Y = 48) + P(Y = 49) + P(Y = 50) \\ &= 0.05 + 0.10 + 0.12 + 0.14 + 0.25 + 0.17 \\ &= \mathbf{0.83} \end{aligned} \tag{8.1}$$

(b) What is the probability that not all ticketed passengers who show up can be accommodated?

**Solution**:

$$\begin{aligned} P(Y > 50) &= P(Y = 51) + P(Y = 52) + P(Y = 53) \\ &\quad + P(Y = 54) + P(Y = 55) \\ &= 0.06 + 0.05 + 0.03 + 0.02 + 0.01 \\ &= \mathbf{0.17} \end{aligned} \tag{8.2}$$

(c) If you are the first person on the standby list (which means you will be the first one to get on the plane if there are any seats available after all ticketed passengers have been accommodated), what is the probability that you will be able to take the flight? What is this probability if you are the third person on the standby list?