Probability Homework 2:
Counting

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September 20, 2019

Question 1: *The Organizational Development Journal (Summer 2006) reported on the results of a survey of human resource officers (HROs) at major firms. The focus of the study was employee behaviour, namely absenteeism, promptness to work, and turnover. The study found that 55% of the HROs had problems with absenteeism. Also, 41% of the HROs had problems with turnover. Suppose that 22% of the HROs had problems with both absenteeism and turnover.*

(a) Find the probability that a HRO selected from the group surveyed had problems with employee absenteeism or employee turnover.

Solution:

First, assume that the probability of HROs having problem with absenteeism is A, and the probability of HROs having problem with turnover is B. The question can be interpreted as:

$$P(A) = 0.55$$
$$P(B) = 0.41$$
$$P(A \cap B) = 0.22$$

Then, according to the union and intersection rule:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Therefore, the probability, in this case, is:

$$P(A \cup B) = 0.55 + 0.41 - 0.22 = \mathbf{0.74} \tag{1.1}$$

(b) Find the probability that a HRO selected from the group surveyed did not have problems with employee absenteeism.

Solution:

The probability, in this case, is:

$$P(\overline{A}) = 1 - P(A) = \mathbf{0.45} \tag{1.2}$$

(c) Find the probability that a HRO selected from the group surveyed did not have problems with employee absenteeism nor with employee turnover.

Solution:

The probability, in this case, is:

$$P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B) = \mathbf{0.26} \tag{1.3}$$

Question 2: *The Journal of Accounting Research (March 2008) ... For each of the following, describe each of the events in terms of unions, intersections and complements of events A, B and C.*

(a) The analyst makes an early forecast and is only concerned with accuracy.

Solution:

$$A \cap B$$

(b) The analyst is not only concerned with accuracy.

Solution:

$$\overline{A}$$

(c) The analyst is from a small brokerage firm or makes an early forecast.

Solution:

$$C \cup B$$

(d) The analyst makes a late forecast and is not only concerned with accuracy.

Solution:

$$\overline{B} \cap \overline{A}$$

Question 3: *Consider an experiment that consists of flipping 3 fair coins. In this case the sample space can be written as: ...*

(a) Write out the possible outcomes for each of these 3 events by listing the sample points that comprise each event.

Solution:

$$A = \{(\mathbf{T}, \mathbf{T}, H), (\mathbf{T}, H, \mathbf{T}), (H, \mathbf{T}, \mathbf{T})\}$$
$$B = \{(\mathbf{T}, \mathbf{T}, H), (\mathbf{T}, \mathbf{T}, \mathbf{T})\}$$
$$C = \{(\mathbf{T}, \mathbf{T}, \mathbf{T})\}$$

(b) What sample points define $A \cap B$, $A \cup C$, and $(A \cap B) \cup C$?

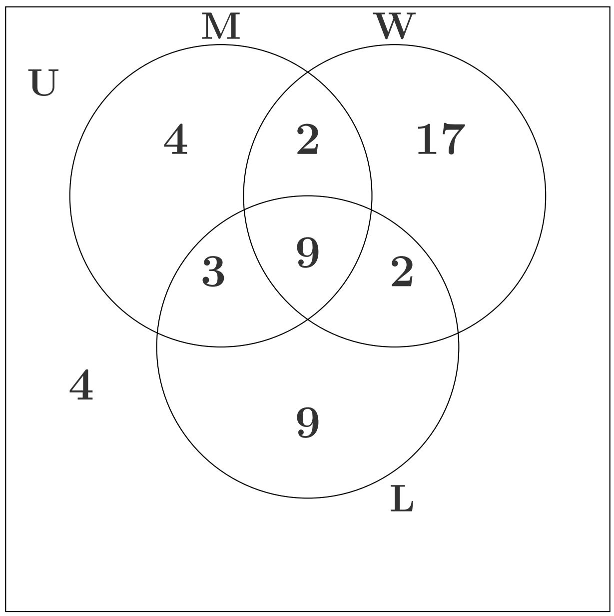
Solution:

$$A \cap B = \{(\mathbf{T}, \mathbf{T}, H)\}$$
$$A \cup C = \{(\mathbf{T}, \mathbf{T}, H), (\mathbf{T}, H, \mathbf{T}), (H, \mathbf{T}, \mathbf{T}), (\mathbf{T}, \mathbf{T}, \mathbf{T})\}$$
$$(A \cap B) \cup C = \{(\mathbf{T}, \mathbf{T}, H), (\mathbf{T}, \mathbf{T}, \mathbf{T})\}$$

Question 4: *Three of the major commercial computer operating systems are Windows, Mac OS, and Red Hat Linux Enterprise. A Computer Science professor selects 50 of her students and asks which of these three operating systems they use. The results for the 50 students are summarized below ...*

(a) Let W denotes Windows, M denotes Mac OS, L denotes Red Hat Linux Enterprise, and U denotes the sample space. Use the above information to complete a three-way Venn diagram.

Solution:



(b) Calculate the following: (i) The number of people in $(\overline{W} \cap \overline{M})$

Solution:

$$(\overline{W} \cap \overline{M}) = 4 + 9 = \mathbf{13}$$

(ii) $Pr(\overline{W} \cup \overline{M})$

Solution:

$$Pr(\overline{W} \cup \overline{M}) = 1 - Pr(W \cap M) = 1 - \frac{11}{50} = \mathbf{0.78}$$

(iii) The number of people in $(\overline{W} \cup \overline{M} \cup \overline{L})$

Solution:

$$(\overline{W} \cup \overline{M} \cup \overline{L}) = \overline{W \cap M \cap L} = 50 - 9 = \mathbf{41}$$

Question 5: *The 2000 census allowed each person to choose the most appropriate option from a long list of races and ethnicity. ...*

(a) Verify that the table gives a legitimate assignment of probabilities by showing that Kolmogorov's first two axioms hold.

Solution:
For the Kolmogorov's First Axiom, let E be an event in the sample space, Ω:

$$P(E) \geq 0, \forall E \in \Omega$$

In this case, the table shows probabilities that are all greater or equal to zero. The Kolmogorov's First Axiom holds. For the Kolmogorov's Second Axiom:

$$P(\Omega) = 1$$

In this case, the sum of probabilities in each cell equals to 1:

$$0.000 + 0.003 + 0.060 + 0.062 + 0.036 + 0.121 + 0.691 + 0.027 = \mathbf{1}$$

The Kolmogorov's Second Axiom holds.

Thus, the table gives a legitimate assignment of probabilities.

(b) Let A be the event that a randomly chosen American is Hispanic. Find $P(A)$.

Solution:

$$P(A) = 0.000 + 0.003 + 0.060 + 0.062 = \mathbf{0.125}$$

(c) Let B be the event that the person chosen is white. Describe $\overline{B}$ in words and find $P(\overline{B})$.

Solution:

$\overline{B}$ is the event that the person chosen is not white.
$$P(\overline{B}) = 1 - (0.060 + 0.691) = \mathbf{0.249}$$

(d) Express the event "a person is a non-Hispanic white" in terms of events A and B. What is the probability of this event?

Solution:
"A person is a non-Hispanic white" means:

$$\overline{A} \cap B$$
$$P(\overline{A} \cap B) = \mathbf{0.691}$$

Question 6: *A slot machine at a casino consists of three reels which spin when the button is pushed, so that one image on each reel is shown at the end. If each of the three reels has five equally likely images on it, and each reel functions independently of the other two, find the following probabilities.*

(a) How many sample points are there? What is the probability of each sample point?

Solution:
If we sample by outcomes, the total number of sample points is:

$$5 \cdot 5 \cdot 5 = 125 \tag{6.1}$$

Since three reels function independently and images have equal chances to be shown, the probability of each sample point is:

$$\frac{1}{125} = \mathbf{0.008} \tag{6.2}$$

(b) Find the probability that the same image appears on each of the three reels.

Solution:
First, the total number of sample points is:

$$5 \cdot 5 \cdot 5 = 125 \tag{6.3}$$

Second, there are 5 scenarios when same image appears on each of the three reels:

Table 1: Scenarios of Three-in-a-row

Reel	1	2	3
Case 1	a	a	a
Case 2	b	b	b
Case 3	c	c	c
Case 4	d	d	d
Case 5	e	e	e

Therefore, the possibiliy is:

$$\frac{5}{125} = \mathbf{0.04} \tag{6.4}$$

(c) Find the probability that at least one of three images is different.

Solution:
Since:

$$P(\text{at least 1 of 3 is diff.}) = 1 - P(\text{all 3 are the same})$$

The probability, in this case, is:

$$1 - 0.04 = \mathbf{0.96} \tag{6.5}$$

Question 7: *There are six roads from A to B and three roads from B to C. In how many ways can one go from A to C via B?*

Solution:
First, there are **six** ways from A to B.
Second, there are **three** ways from B to C.
According to the Multiplication Rule, the total number of ways one can go from A to C via B is:

$$6 \cdot 3 = \mathbf{18} \tag{7.1}$$

Question 8: *There are several fair dice. What is the probability that every possible number appears?*

(a) In the case that six dice are tossed.

Solution:
First, assume that the fair dice has **six** faces.

Table 2: Values of a Fair Dice

Face	1	2	3	4	5	6
Value	1	2	3	4	5	6

In this case, the probability that every possible number appears is:

$$\frac{6!}{6^6} = \frac{720}{46,656} \approx \mathbf{0.0154} \tag{8.1}$$

(b) In the case that seven dice are tossed.

Solution:
In this case, the probability that every possible number appears is:

$$\frac{\left(\frac{7\cdot 6}{2\cdot 1}\right) \cdot 6!}{6^7} = \frac{15,120}{279,936} \approx \mathbf{0.054} \tag{8.2}$$

Question 9: *Count the number of license plates in the following situations.*

(a) How many license plates with 3 letters followed by 3 digits exist?

Solution:

Table 3: Letters and Numbers for License Plates

Letters	ABCDEFGHIJKLMNOPQRSTUVWXYZ
Numbers	1 2 3 4 5 6 7 8 9 0

First, there are 26 letters and 10 numbers available for license plates:
Second, the number of ways for arranging the first three slots (letters only) is:

$$26 \cdot 26 \cdot 26 = 17,576 \tag{9.1}$$

Third, the number of ways for arranging the last three slots (numbers only) is:

$$10 \cdot 10 \cdot 10 = 1,000 \tag{9.2}$$

Finally, combine two arrangements:

The number of license plates with 3 letters followed by 3 digits is:

$$17,576 \cdot 1,000 = \mathbf{17{,}576{,}000} \tag{9.3}$$

b) How many license plates with 3 letters followed by 3 digits exist if exactly one of the digits is the number 1?

Solution:
First, the number of ways for arranging the first three slots (letters only) is:

$$26 \cdot 26 \cdot 26 = 17,576 \tag{9.4}$$

Second, for the last three slots, if we put 1 to each slot, the total number of arrangements is:

$$9 \cdot 9 \cdot 3 = 243 \tag{9.5}$$

Finally, combine all steps:

The number of license plates, in this case, is:

$$17,576 \cdot 243 = \mathbf{4{,}270{,}968} \tag{9.6}$$

c) How many license plates are there that start with three letters followed by 4 digits (no repetitions)?

Solution:
First, the number of ways for arranging the first three slots (letters only) is:

$$26 \cdot 25 \cdot 24 = 15,600 \tag{9.7}$$

Second, for the last four slots, if there is no repetitions, the total number of arrangements is:

$$10 \cdot 9 \cdot 8 \cdot 7 = 5,040 \tag{9.8}$$

Finally, combine all steps:

The number of license plates, in this case, is:

$$15,600 \cdot 5,040 = \mathbf{78{,}624{,}000} \tag{9.9}$$

Question 10: *Suppose we have the fictional word 'DALDERFARG',*

a) How many ways are there to **arrange** all of the letters?

Solution:
First, count the letters in the word **'DALDERFARG'**:

Table 4: Letter Count of the word 'DALDERFARG'

Letter	D	A	L	E	R	F	G
Count	2	2	1	1	2	1	1

Second, calculate the number of ways of arranging 10 unique letters:

$$10!$$

Finally, solve the over-counting of letter D, A and R.
The number of ways to arrange all of the letters is:

$$\frac{10!}{2!\,2!\,2!} = \mathbf{453{,}600} \tag{10.1}$$

b) What is the probability that the 1$^{\text{st}}$ letter is the same as the 2$^{\text{nd}}$ letter?

Solution:
First, according to **Table 1**, for the first two letters, there are only three combinations:

$$\begin{gathered} D\,D\ ... \\ A\,A\ ... \\ R\,R\ ... \end{gathered}$$

Second, the number of ways of the arrangement of the remaining eight letters, if all unique, is:

$$8!$$

Third, solve the over-counting:

$$\frac{8!}{2!\,2!} \tag{10.2}$$

Finally, combine all steps:

$$\frac{3\,\cdot\,8!}{2!\,2!} = 30,240 \tag{10.3}$$

The probability, in this case, is:

$$\frac{30,240}{453,600} \approx \mathbf{0.0667} \tag{10.4}$$