Proof of the convexity adjustment formula

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1 Problem

This is a conclusion in pg 740, John Hull's book (10th edition)

Let y_f be the forward bond yield observed today for a forward contract with maturity T, y_T be the bond yield at time T, B_T be the price of the bond at time T and let σ_y be the volatility of the forward bond yield.

Suppose that $B_T = g(y_T)$ then expanding using a taylor series yields,

$$B_T = G(y_f) + (y_T - y_f)G'(y_f) + 0.5 * G''(y_f)(y_T - y_f)^2$$

and then taking expectations we get

$$E_T(B_T) = G(y_f) + E_T(y_T - y_f)G'(y0) + 0.5G''(y_f)E_T(y_T - y_f)^2$$

as we are working in the risk neutral world, $E_T(B_T) = G(y_f)$,

and so

$$E_T(y_T - y_f)G'(y_f) + 0.5G''(y_f)E_T(y_T - y_f)^2$$

Now apparently $E_T[(y_T - y_f)^2]$ is approximately equal to $\sigma_y^2 y_f^2 T$, but cannot see why this approximation is true.

2 Solution

Let's assume the bond yield y follows the geometric brownian motion, i.e.,

$$dy = aydt + \sigma_y ydW.$$

The variable t starts with y_f and end with y_T, therefore, y_T can be approximated by the below formula.

$$y_T - y_f \approx ay_f T + \sigma_y y_f \Delta W_T + \textbf{Higher Order Terms}$$

Furthermore, by dropping off the higher order terms in the square difference, we have

$$E_T(y_T - y_f)^2 \approx \sigma_y^2 y_f^2 T$$