

Fig. 1.4 Velocities $+V$ and $-V$ give motion forward and back, ending at $f(6) = 0$.

The v -graph shows velocities $+V$ and $-V$. The distance starts up with slope $+V$ and reaches $f = 3V$. Then the car starts backward. The distance goes down with slope $-V$ and returns to $f = 0$ at $t = 6$.

Notice what that means. The total area “under” the v -graph is zero! A negative velocity makes the distance graph go *downward* (negative slope). The car is moving backward. *Area below the axis in the v -graph is counted as negative.*

FUNCTIONS

This forward-back example gives practice with a crucially important idea—the concept of a “**function**.” We seize this golden opportunity to explain functions:

The number $v(t)$ is the value of the function v at the time t .

The time t is the **input** to the function. The velocity $v(t)$ at that time is the **output**. Most people say “ v of t ” when they read $v(t)$. The number “ v of 2” is the velocity when $t = 2$. The forward-back example has $v(2) = +V$ and $v(4) = -V$. The function contains the whole history, like a memory bank that has a record of v at each t .

It is simple to convert forward-back motion into a formula. Here is $v(t)$:

$$v(t) = \begin{cases} +V & \text{if } 0 < t < 3 \\ ? & \text{if } t = 3 \\ -V & \text{if } 3 < t < 6 \end{cases}$$

The right side contains the instructions for finding $v(t)$. The input t is converted into the output $+V$ or $-V$. The velocity $v(t)$ depends on t . In this case the function is “discontinuous,” because the needle jumps at $t = 3$. *The velocity is not defined at that instant.* There is no $v(3)$. (You might argue that v is zero at the jump, but that leads to trouble.) The graph of f has a corner, and we can’t give its slope.

The problem also involves a second function, namely the distance. The principle behind $f(t)$ is the same: $f(t)$ is the **distance at time t** . It is the net distance forward, and again the instructions change at $t = 3$. In the forward motion, $f(t)$ equals Vt as before. In the backward half, a calculation is built into the formula for $f(t)$:

$$f(t) = \begin{cases} Vt & \text{if } 0 \leq t \leq 3 \\ V(6 - t) & \text{if } 3 \leq t \leq 6 \end{cases}$$

At the switching time the right side gives two instructions (one on each line). This would be bad except that they agree: $f(3) = 3V$.† The distance function is “con-

†A function is only allowed *one value* $f(t)$ or $v(t)$ at each time t .

tinuous.” There is no jump in f , even when there is a jump in v . After $t = 3$ the distance decreases because of $-Vt$. At $t = 6$ the second instruction correctly gives $f(6) = 0$.

Notice something more. The functions were given by graphs before they were given by formulas. The graphs tell you f and v at every time t —sometimes more clearly than the formulas. The values $f(t)$ and $v(t)$ can also be given by tables or equations or a set of instructions. (In some way all functions are instructions—the function tells how to find f at time t .) Part of knowing f is knowing all its inputs and outputs—its **domain** and **range**:

The domain of a function is the set of inputs. The range is the set of outputs.

The domain of f consists of all times $0 \leq t \leq 6$. The range consists of all distances $0 \leq f(t) \leq 3V$. (The range of v contains only the two velocities $+V$ and $-V$.) We mention now, and repeat later, that every “linear” function has a formula $f(t) = vt + C$. Its graph is a line and v is the slope. The constant C moves the line up and down. It adjusts the line to go through any desired starting point.

SUMMARY: MORE ABOUT FUNCTIONS

May I collect together the ideas brought out by this example? We had two functions v and f . One was *velocity*, the other was *distance*. Each function had a *domain*, and a *range*, and most important a *graph*. For the f -graph we studied the slope (which agreed with v). For the v -graph we studied the area (which agreed with f). Calculus produces functions in pairs, and the best thing a book can do early is to show you more of them.

$$\begin{array}{l} \text{in} \\ \text{the} \\ \text{domain} \end{array} \left\{ \begin{array}{l} \text{input } t \rightarrow \text{function } f \rightarrow \text{output } f(t) \\ \text{input } 2 \rightarrow \text{function } v \rightarrow \text{output } v(2) \\ \text{input } 7 \rightarrow f(t) = 2t + 6 \rightarrow f(7) = 20 \end{array} \right\} \begin{array}{l} \text{in} \\ \text{the} \\ \text{range} \end{array}$$

Note about the definition of a function. The idea behind the symbol $f(t)$ is absolutely crucial to mathematics. Words don’t do it justice! By definition, a function is a “rule” that assigns one member of the range to each member of the domain. Or, a function is a set of pairs $(t, f(t))$ with no t appearing twice. (These are “ordered pairs” because we write t before $f(t)$.) Both of those definitions are correct—but somehow they are too passive.

In practice what matters is the active part. The number $f(t)$ is produced from the number t . We read a graph, plug into a formula, solve an equation, run a computer program. The input t is “mapped” to the output $f(t)$, which changes as t changes. Calculus is about the *rate of change*. This rate is our other function v .

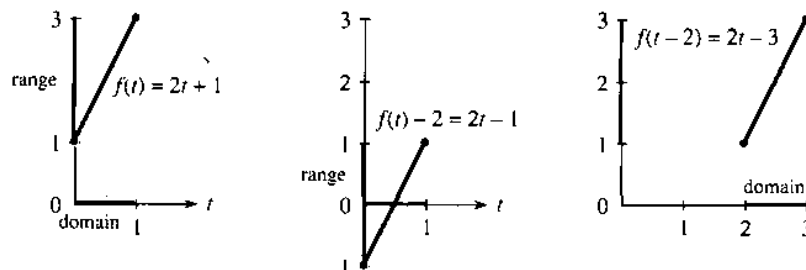


Fig. 1.5 Subtracting 2 from f affects the range. Subtracting 2 from t affects the domain.

It is quite hard at the beginning, and not automatic, to see the difference between $f(t) - 2$ and $f(t - 2)$. Those are both new functions, created out of the original $f(t)$. In $f(t) - 2$, we subtract 2 from all the distances. That moves the whole graph *down*. In $f(t - 2)$, we subtract 2 from the time. That moves the graph *over to the right*. Figure 1.5 shows both movements, starting from $f(t) = 2t + 1$. The formula to find $f(t - 2)$ is $2(t - 2) + 1$, which is $2t - 3$.

A graphing calculator also moves the graph, when you change the viewing window. You can pick any rectangle $A \leq t \leq B$, $C \leq f(t) \leq D$. The screen shows that part of the graph. But on the calculator, *the function $f(t)$ remains the same*. It is the axes that get renumbered. In our figures the axes stay the same and the function is changed.

There are two more basic ways to change a function. (We are always creating new functions—that is what mathematics is all about.) Instead of subtracting or adding, we can *multiply* the distance by 2. Figure 1.6 shows $2f(t)$. And instead of shifting the time, we can *speed it up*. The function becomes $f(2t)$. Everything happens twice as fast (and takes half as long). On the calculator those changes correspond to a “zoom”—on the f axis or the t axis. We soon come back to zooms.

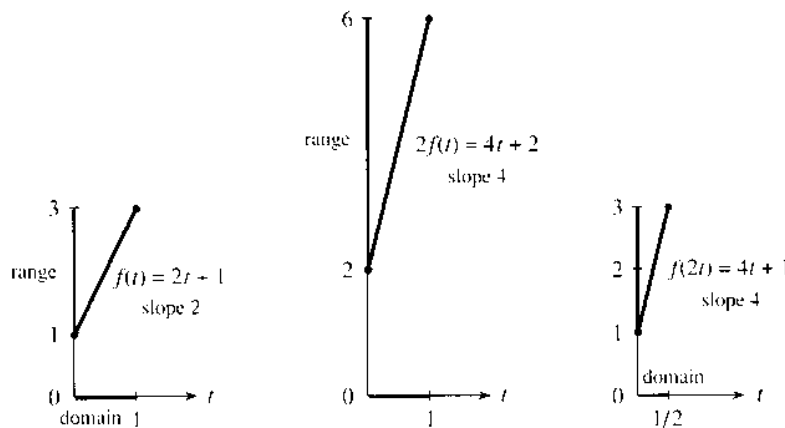


Fig. 1.6 Doubling the distance or speeding up the time doubles the slope.

1.1 EXERCISES

Each section of the book contains read-through questions. They allow you to outline the section yourself—more actively than reading a summary. This is probably the best way to remember the important ideas.

Starting from $f(0) = 0$ at constant velocity v , the distance function is $f(t) = \underline{a}$. When $f(t) = 55t$ the velocity is $v = \underline{b}$. When $f(t) = 55t + 1000$ the velocity is still \underline{c} and the starting value is $f(0) = \underline{d}$. In each case v is the \underline{e} of the graph of f . When \underline{f} is negative, the graph of \underline{g} goes downward. In that case area in the t -graph counts as \underline{h} .

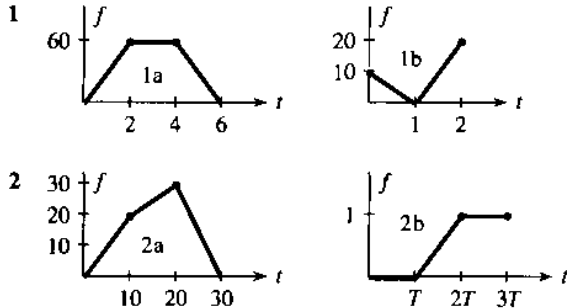
Forward motion from $f(0) = 0$ to $f(2) = 10$ has $v = \underline{i}$. Then backward motion to $f(4) = 0$ has $v = \underline{j}$. The distance function is $f(t) = 5t$ for $0 \leq t \leq 2$ and then $f(t) = \underline{k}$

(not $-5t$). The slopes are \underline{l} and \underline{m} . The distance $f(3) = \underline{n}$. The area under the v -graph up to time 1.5 is \underline{o} . The domain of f is the time interval \underline{p} , and the range is the distance interval \underline{q} . The range of $v(t)$ is only \underline{r} .

The value of $f(t) = 3t + 1$ at $t = 2$ is $f(2) = \underline{s}$. The value 19 equals $f(\underline{t})$. The difference $f(4) - f(1) = \underline{u}$. That is the change in distance, when $4 - 1$ is the change in \underline{v} . The ratio of those changes equals \underline{w} , which is the \underline{x} of the graph. The formula for $f(t) + 2$ is $3t + 3$ whereas $f(t + 2)$ equals \underline{y} . Those functions have the same \underline{z} as f : the graph of $f(t) + 2$ is shifted \underline{A} and $f(t + 2)$ is shifted \underline{B} . The formula for $f(5t)$ is \underline{C} . The formula for $5f(t)$ is \underline{D} . The slope has jumped from 3 to \underline{E} .

The set of inputs to a function is its F. The set of outputs is its G. The functions $f(t) = 7 + 3(t - 2)$ and $f(t) = vt + C$ are H. Their graphs are I with slopes equal to J and K. They are the same function, if $v = \underline{L}$ and $C = \underline{M}$.

Draw the velocity graph that goes with each distance graph.



3 Write down three-part formulas for the velocities $v(t)$ in Problem 2, starting from $v(t) = 2$ for $0 < t < 10$.

4 The distance in 1b starts with $f(t) = 10 - 10t$ for $0 \leq t \leq 1$. Give a formula for the second part.

5 In the middle of graph 2a find $f(15)$ and $f(12)$ and $f(t)$.

6 In graph 2b find $f(1.4T)$. If $T = 3$ what is $f(4)$?

7 Find the *average speed* between $t = 0$ and $t = 5$ in graph 1a. What is the speed at $t = 5$?

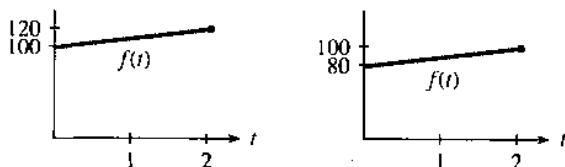
8 What is the average speed between $t = 0$ and $t = 2$ in graph 1b? The average speed is zero between $t = \frac{1}{2}$ and $t = \underline{\hspace{2cm}}$.

9 (recommended) A car goes at speed $v = 20$ into a brick wall at distance $f = 4$. Give two-part formulas for $v(t)$ and $f(t)$ (before and after), and draw the graphs.

10 Draw any reasonable graphs of $v(t)$ and $f(t)$ when

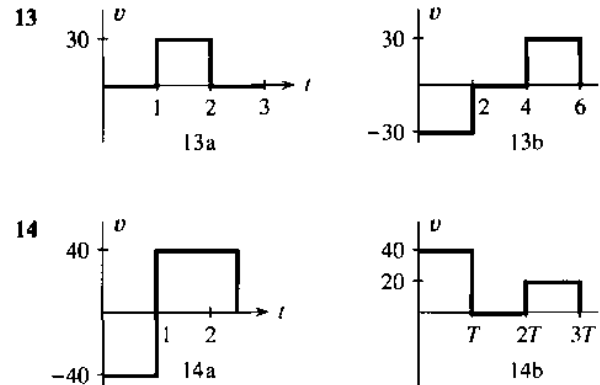
- the driver backs up, stops to shift gear, then goes fast;
- the driver slows to 55 for a police car;
- in a rough gear change, the car accelerates in jumps;
- the driver waits for a light that turns green.

11 Your bank account earns simple interest on the opening balance $f(0)$. What are the interest rates per year?



12 The earth's population is growing at $v = 100$ million a year, starting from $f = 5.2$ billion in 1990. Graph $f(t)$ and find $f(2000)$.

Draw the distance graph that goes with each velocity graph. Start from $f = 0$ at $t = 0$ and mark the distance.



15 Write down formulas for $v(t)$ in Problem 14, starting with $v = -40$ for $0 < t < 1$. Find the average velocities to $t = 2.5$ and $t = 3T$.

16 Give 3-part formulas for the areas $f(t)$ under $v(t)$ in 13.

17 The distance in 14a starts with $f(t) = -40t$ for $0 \leq t \leq 1$. Find $f(t)$ in the other part, which passes through $f = 0$ at $t = 2$.

18 Draw the velocity and distance graphs if $v(t) = 8$ for $0 < t < 2$, $f(t) = 20 + t$ for $2 \leq t \leq 3$.

19 Draw rough graphs of $y = \sqrt{x}$ and $y = \sqrt{x-4}$ and $y = \sqrt{x} - 4$. They are "half-parabolas" with infinite slope at the start.

20 What is the break-even point if x yearbooks cost \$1200 + 30x to produce and the income is 40x? The slope of the cost line is (cost per additional book). If it goes above you can't break even.

21 What are the domains and ranges of the distance functions in 14a and 14b—all values of t and $f(t)$ if $f(0) = 0$?

22 What is the range of $v(t)$ in 14b? Why is $t = 1$ not in the domain of $v(t)$ in 14a?

Problems 23–28 involve *linear functions* $f(t) = vt + C$. Find the constants v and C .

23 What linear function has $f(0) = 3$ and $f(2) = -11$?

24 Find *two* linear functions whose domain is $0 \leq t \leq 2$ and whose range is $1 \leq f(t) \leq 9$.

25 Find the linear function with $f(1) = 4$ and slope 6.

26 What functions have $f(t+1) = f(t) + 2$?

27 Find the linear function with $f(t+2) = f(t) + 6$ and $f(1) = 10$.

28 Find the only $f = vt$ that has $f(2t) = 4f(t)$. Show that every $f = \frac{1}{2}at^2$ has this property. To go times as far in twice the time, you must accelerate.

29 Sketch the graph of $f(t) = |5 - 2t|$ (absolute value) for $|t| \leq 2$ and find its slopes and range.

30 Sketch the graph of $f(t) = 4 - t - |4 - t|$ for $2 \leq t \leq 5$ and find its slope and range.

31 Suppose $v = 8$ up to time T , and after that $v = -2$. Starting from zero, when does f return to zero? Give formulas for $v(t)$ and $f(t)$.

32 Suppose $v = 3$ up to time $T = 4$. What new velocity will lead to $f(T) = 30$ if $f(0) = 0$? Give formulas for $v(t)$ and $f(t)$.

33 What function $F(C)$ converts Celsius temperature C to Fahrenheit temperature F ? The slope is _____, which is the number of Fahrenheit degrees equivalent to 1°C .

34 What function $C(F)$ converts Fahrenheit to Celsius (or Centigrade), and what is its slope?

35 What function converts the weight w in grams to the weight $f(w)$ in kilograms? Interpret the slope of $f(w)$.

36 (Newspaper of March 1989) Ten hours after the accident the alcohol reading was .061. Blood alcohol is eliminated at .015 per hour. What was the reading at the time of the accident? How much later would it drop to .04 (the maximum set by the Coast Guard)? The usual limit on drivers is .10 percent.

Which points between $t = 0$ and $t = 5$ can be in the domain of $f(t)$? With this domain find the range in 37–42.

37 $f(t) = \sqrt{t-1}$ 38 $f(t) = 1/\sqrt{t-1}$

39 $f(t) = |t-4|$ (absolute value) 40 $f(t) = 1/(t-4)^2$

41 $f(t) = 2^t$ 42 $f(t) = 2^{-t}$

43 (a) Draw the graph of $f(t) = \frac{1}{2}t + 3$ with domain $0 \leq t \leq 2$. Then give a formula and graph for

(b) $f(t) + 1$ (c) $f(t + 1)$
(d) $4f(t)$ (e) $f(4t)$.

44 (a) Draw the graph of $U(t) = \text{step function} = \{0 \text{ for } t < 0, 1 \text{ for } t \geq 0\}$. Then draw

(b) $U(t) + 2$ (c) $U(t + 2)$
(d) $3U(t)$ (e) $U(3t)$.

45 (a) Draw the graph of $f(t) = t + 1$ for $-1 \leq t \leq 1$. Find the domain, range, slope, and formula for

(b) $2f(t)$ (c) $f(t - 3)$ (d) $-f(t)$ (e) $f(-t)$.

46 If $f(t) = t - 1$ what are $2f(3t)$ and $f(1 - t)$ and $f(t - 1)$?

47 In the forward-back example find $f(\frac{1}{2}T)$ and $f(\frac{3}{2}T)$. Verify that those agree with the areas "under" the v -graph in Figure 1.4.

48 Find formulas for the outputs $f_1(t)$ and $f_2(t)$ which come from the input t :

(1) inside = input + 3 (2) inside \leftarrow input + 6
output = inside + 3 output \leftarrow inside + 3

Note BASIC and FORTRAN (and calculus itself) use = instead of \leftarrow . But the symbol \leftarrow or \Rightarrow is in some ways better. The instruction $t \leftarrow t + 6$ produces a new t equal to the old t plus six. The equation $t = t + 6$ is not intended.

49 Your computer can add and multiply. Starting with the number 1 and the input called t , give a list of instructions to lead to these outputs:

$$f_1(t) = t^2 + t \quad f_2(t) = f_1(f_1(t)) \quad f_3(t) = f_1(t + 1).$$

50 In fifty words or less explain what a *function* is.

The last questions are challenging but possible.

51 If $f(t) = 3t - 1$ for $0 \leq t \leq 2$ give formulas (with domain) and find the slopes of these six functions:

(a) $f(t + 2)$ (b) $f(t) + 2$ (c) $2f(t)$
(d) $f(2t)$ (e) $f(-t)$ (f) $f(f(t))$.

52 For $f(t) = vt + C$ find the formulas and slopes of

(a) $3f(t) + 1$ (b) $f(3t + 1)$ (c) $2f(4t)$
(d) $f(-t)$ (e) $f(t) - f(0)$ (f) $f(f(t))$.

53 (hardest) The forward-back function is $f(t) = 2t$ for $0 \leq t \leq 3$, $f(t) = 12 - 2t$ for $3 \leq t \leq 6$. Graph $f(f(t))$ and find its *four-part* formula. First try $t = 1.5$ and 3.

54 (a) Why is the letter **X** not the graph of a function?
(b) Which capital letters are the graphs of functions?
(c) Draw graphs of their slopes.

1.2 Calculus Without Limits

The next page is going to reveal one of the key ideas behind calculus. The discussion is just about numbers—functions and slopes can wait. The numbers are not even special, they can be any numbers. The crucial point is to look at their differences:

$$\begin{array}{ccccccc} \text{Suppose the numbers are } f = & 0 & 2 & 6 & 7 & 4 & 9 \\ \text{Their differences are } v = & & 2 & 4 & 1 & -3 & 5 \end{array}$$

The differences are printed in between, to show $2 - 0 = 2$ and $6 - 2 = 4$ and $7 - 6 = 1$.

Notice how $4 - 7$ gives a negative answer -3 . The numbers in f can go up or down, the differences in v can be positive or negative. The idea behind calculus comes when you **add up those differences**:

$$2 + 4 + 1 - 3 + 5 = 9$$

The sum of differences is 9. This is the last number on the top line (in f). Is this an accident, or is this always true? If we stop earlier, after $2 + 4 + 1$, we get the 7 in f . Test any prediction on a second example:

$$\begin{array}{rcccccc} \text{Suppose the numbers are } f = & 1 & 3 & 7 & 8 & 5 & 10 \\ \text{Their differences are } v = & & 2 & 4 & 1 & -3 & 5 \end{array}$$

The f 's are increased by 1. **The differences are exactly the same**—no change. The sum of differences is still 9. But the last f is now 10. That prediction is not right, we don't always get the last f .

The first f is now 1. The answer 9 (the sum of differences) is $10 - 1$, **the last f minus the first f** . What happens when we change the f 's in the middle?

$$\begin{array}{rcccccc} \text{Suppose the numbers are } f = & 1 & 5 & 12 & 7 & 10 \\ \text{Their differences are } v = & & 4 & 7 & -5 & 3 \end{array}$$

The differences add to $4 + 7 - 5 + 3 = 9$. This is still $10 - 1$. No matter what f 's we choose or how many, the sum of differences is controlled by the first f and last f . If this is always true, there must be a clear reason why **the middle f 's cancel out**.

$$\text{The sum of differences is } (5 - 1) + (12 - 5) + (7 - 12) + (10 - 7) = 10 - 1.$$

The 5's cancel, the 12's cancel, and the 7's cancel. It is only $10 - 1$ that doesn't cancel. This is the key to calculus!

1B The differences of the f 's add up to $(f_{\text{last}} - f_{\text{first}})$.

EXAMPLE 1 The numbers grow linearly: $f = 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7$
 Their differences are constant: $v = 1 \quad 1 \quad 1 \quad 1 \quad 1$

The sum of differences is certainly 5. This agrees with $7 - 2 = f_{\text{last}} - f_{\text{first}}$. The numbers in v remind us of constant velocity. The numbers in f remind us of a straight line $f = vt + C$. This example has $v = 1$ and the f 's start at 2. The straight line would come from $f = t + 2$.

EXAMPLE 2 The numbers are squares: $f = 0 \quad 1 \quad 4 \quad 9 \quad 16$
 Their differences grow linearly: $v = 1 \quad 3 \quad 5 \quad 7$

$1 + 3 + 5 + 7$ agrees with $4^2 = 16$. It is a beautiful fact that the first j odd numbers always add up to j^2 . The v 's are the odd numbers, the f 's are perfect squares.

Note The letter j is sometimes useful to tell which number in f we are looking at. For this example the zeroth number is $f_0 = 0$ and the j th number is $f_j = j^2$. This is a part of algebra, to give a formula for the f 's instead of a list of numbers. We can also use j to tell which difference we are looking at. The first v is the first odd number $v_1 = 1$. The j th difference is the j th odd number $v_j = 2j - 1$. (Thus v_4 is $8 - 1 = 7$.) It is better to start the differences with $j = 1$, since there is no zeroth odd number v_0 .

With this notation the j th difference is $v_j = f_j - f_{j-1}$. Sooner or later you will get comfortable with subscripts like j and $j - 1$, but it can be later. The important point is that the sum of the v 's equals $f_{\text{last}} - f_{\text{first}}$. We now connect the v 's to slopes and the f 's to areas.