Asymptotic form of spherical harmonics

The sperical harmonics by defination is given by.

$$Y_l^m(\theta,\varphi) = (-)^m \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta)\, e^{im\phi}.$$

The associated Legendre polynomials has an asymptotic form. (8.10.7 of Handbook of Mathematical functions with formulas.

$$P_l^m(\cos\theta) = \frac{\Gamma(l+m+1)}{\Gamma(l+\frac{3}{2})}\left(\frac{1}{2}\pi\sin\theta\right)^{-\frac{1}{2}}\cos\left[(l+\tfrac{1}{2})\theta-\tfrac{\pi}{4}+\tfrac{m}{2}\pi\right]$$

by using the relation of Gamma function.

$$\Gamma(n+1) = n! \qquad \text{eq. 6.1.6.}$$

The sperical harmonics beomes.

$$Y_l^m(\theta,\varphi) = (-)^m\sqrt{\frac{(2l+1)}{4\pi}\frac{\Gamma(l-m+1)}{\Gamma(l+m+1)}}\,\frac{\Gamma(l+m+1)}{\Gamma(l+\frac{3}{2})}\left(\frac{1}{2}\pi\sin\theta\right)^{-\frac{1}{2}}$$

$$\times\cos\left[(l+\tfrac{1}{2})\theta-\tfrac{\pi}{4}+\tfrac{m}{2}\pi\right]e^{im\phi} \qquad \frac{\sqrt{2}}{\sqrt{\pi\sin\theta}}.$$

$$= (-)^m\frac{1}{\pi}\sqrt{\frac{2l+1}{2}\frac{\Gamma(l-m+1)\Gamma(l+m+1)}{\Gamma(l+\frac{3}{2})^2}}\cos\left[(l+\tfrac{1}{2})\theta-\tfrac{\pi}{4}+\tfrac{m}{2}\pi\right]\frac{1}{\sqrt{\sin\theta}}e^{im\phi}$$

$$Y_l^m(\theta,\varphi) = \frac{B_{lm}\cos\left[(l+\frac{1}{2})\theta-\frac{\pi}{4}+\frac{m}{2}\pi\right]}{\sqrt{\sin\theta}}e^{im\phi}$$

$$B_{lm} = (-)^m \frac{1}{\pi} \sqrt{\frac{2l+1}{2} \frac{\Gamma(l-m+1)\Gamma(l+m+1)}{\Gamma(l+\frac{3}{2})^2}}$$

Now, we apply the asymptotic form of Gamma function. (eq. 6.1.41).

$$\ln\Gamma(z) \sim (z-\tfrac{1}{2})\ln z - z + \tfrac{1}{2}\ln(2\pi) + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} - \frac{1}{1680z^7} + \cdots$$

If we take $l \gg m$, make use of the above equation and ignore the small terms, one gets.

$$\ln\left[\frac{\pi}{(-)^m} \times B_{lm}\right] = \ln\left(\frac{(l+\frac{1}{2})\Gamma(l-m+1)\Gamma(l+m+1)}{\Gamma(l+\frac{3}{2})^2}\right)^{\frac{1}{2}}$$

$$= \frac{1}{2}\left[\ln(l+\tfrac{1}{2}) + \ln(\Gamma(l-m+1)) + \ln(\Gamma(l+m+1)) - 2\ln(\Gamma(l+\tfrac{3}{2}))\right]$$

$$= \frac{1}{2}\Big[\ln(l+\tfrac{1}{2}) + (l-m+\tfrac{1}{2})\ln(l-m+1) - (l-\cancel{m}+1) + \cancel{\tfrac{1}{2}\ln(2\pi)}$$

$$+ (l+m+\tfrac{1}{2})\ln(l+m+1) - (l+\cancel{m}+1) + \cancel{\tfrac{1}{2}\ln(2\pi)} - 2(l+1)\ln(l+\tfrac{3}{2})$$
$$+ (2l+\cancel{3}) - \cancel{\ln(2\pi)} \boxed{+1}\Big]$$

$$= \frac{1}{2}\Big[\ln(l+\tfrac{1}{2}) + (l-m+\tfrac{1}{2})\ln(l-m+1) + (l+m+\tfrac{1}{2})\ln(l+m+1)$$

$$- 2(l+1)\ln(l+\tfrac{3}{2}) + 1\Big]$$

$$= \frac{1}{2}\Big[\Big(\ln(l+\tfrac{1}{2}) - \ln(l+\tfrac{3}{2})\Big) + \Big((l+\tfrac{1}{2})\ln(l-m+1) - (l+\tfrac{1}{2})\ln(l+\tfrac{3}{2})\Big)$$

$$+\Big(m\ln(l+m+1) - m\ln(l-m+1)\Big) + \Big((l+\tfrac{1}{2})\ln(l+m+1) - (l+\tfrac{1}{2})\ln(l+\tfrac{3}{2})\Big)$$

$$+1\Big]$$

$$= \frac{1}{2}\Big[\ln\Big(1-\frac{1}{l+\frac{3}{2}}\Big) + (l+\tfrac{1}{2})\ln\Big(1-\frac{m+\frac{1}{2}}{l+\frac{3}{2}}\Big) + m\ln\Big(1+\frac{2m}{l-m+1}\Big).$$

$$+ (l+\tfrac{1}{2})\ln\Big(1+\frac{m-\frac{1}{2}}{l+\frac{3}{2}}\Big). + 1\Big]$$

We assumed l is large and $l \gg m$. then by using Series expansion of logarithm, $\ln(z) = \sum_{k=1}^{\infty} (-)^{k+1}\frac{(z-1)^k}{k}$.
and retain terms of the $\frac{1}{l}$ order, the above equation becomes.

$$\ln\Big(\frac{\pi}{(-)^m}B_{lm}\Big) = \frac{1}{2}\Big(-\frac{1}{l+\frac{3}{2}} - (l+\tfrac{1}{2})\Big(\frac{m+\frac{1}{2}}{l+\frac{3}{2}}\Big) + m\cdot\frac{2m}{l-m+1}$$

$$+ (l+\tfrac{1}{2})\Big(\frac{m-\frac{1}{2}}{l+\frac{3}{2}}\Big) + 1\Big).$$

$$= \frac{m^2}{l-m+1}$$

Then B_{lm} becomes

$$B_{lm} = \exp\Big(\frac{m^2}{l-m+1}\Big)\cdot\frac{(-)^m}{\pi}.$$

Finally the asymptotic form of spherical harmonics is given by

$$Y_l^m(\theta,\varphi) = \exp\left(\frac{m^2}{l-m+1}\right) \frac{(-)^m}{\pi} \frac{1}{\sqrt{\sin\theta}} \cos\left[\left(l+\tfrac{1}{2}\right)\theta - \frac{\pi}{4} + \frac{m}{2}\pi\right] e^{im\phi}$$

In the high energy limit we assume that the main contribution comes from the low m component. Then we drop the term $\exp\left(\frac{m^2}{l-m+1}\right)$ to keep high m component small.

This is the method widely used in semi-classical group, but the numerical results do not supported this formula. for example, with $l=100$ $m=1$ $\theta=30^\circ$ $\varphi=0^\circ$.

$$Y_l^m(\theta,\varphi)\Big|_{exact} = -0.45018217$$

$$Y_l^m(\theta,\varphi)\Big|_{asymptotic} = 0.450158158.$$

There is a sign difference in the numerical result. And $m>1$, the asymptotic form does not work fine even with $l>1000$, useful or useless ?????