

Response to Referee Report

Manuscript: Scale-Invariant Information Limit in Nuclear Optical Potential Extraction

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We thank the referee for the careful reading of the manuscript and the constructive suggestions. The referee's core criticism—that the original analysis was limited to single-energy elastic angular distributions—was entirely well-taken. We have substantially extended the analysis to include (i) analyzing power A_y , (ii) reaction cross section σ_R , (iii) total cross section σ_T , (iv) multi-energy combined Fisher analysis, (v) the KD02 global parameterization with 17 energy-dependent parameters, and (vi) angle-resolved sensitivity. We have also added a detailed comparison with the Bayesian uncertainty quantification of King *et al.* [PRL **122**, 232502 (2019)], establishing the Fisher information matrix as the rigorous theoretical foundation for the empirical findings of that landmark study.

The revised manuscript presents these extensions as a **step-by-step constraint analysis**, progressively adding observables and energies to decompose the constraining power of each data type. This addresses all of the referee's major and minor criticisms. Below we respond to each point in detail.

Major Criticisms

(1) Limited observable scope

The author focuses exclusively on elastic scattering angular distributions, whereas KD02 was constrained using (i) elastic-scattering angular distributions, (ii) analyzing powers, (iii) total cross sections, and (iv) average resonance parameters. If a comprehensive analysis of ALL data types still showed reduced effective parameter space, THAT would be a significant new insight.

We have now extended the analysis to include three of the four scattering-region observable types the referee lists: elastic angular distributions, analyzing powers A_y , and total cross sections σ_T (plus the reaction cross section σ_R). The fourth type—average resonance parameters—constrains the potential at low energies ($E \lesssim 1$ MeV) in the resonance region, which lies outside the 10–200 MeV scope of the present analysis; moreover, KD02 itself is not designed to reproduce resonance-region observables accurately, as its smooth Woods-Saxon forms cannot capture the rapid energy dependence near individual resonances. Nevertheless, the Fisher framework generalizes straightforwardly to include resonance data: one simply adds their contribution to the Fisher matrix, $F_{\text{total}} = F_{\text{scattering}} + F_{\text{resonance}}$. The revised manuscript presents a step-by-step decomposition (new Fig. 4):

- **Step 1: Elastic $d\sigma/d\Omega$ alone:** $D_{\text{eff}} \approx 1.2$ (50 MeV), with the 168-configuration average $D_{\text{eff}} = 1.7 \pm 0.4$.

- **Step 2:** $+\sigma_R$: D_{eff} increases by < 0.001 . A single integrated datum is overwhelmed by 17 elastic angular bins.
- **Step 3:** $+A_y + \sigma_T$: D_{eff} rises from 1.18 to 1.52 for $n+^{40}\text{Ca}$. The analyzing power provides genuinely new information through the spin-flip amplitude $g(\theta)$, which probes spin-orbit interference. The spin-orbit subgroup trace increases by two orders of magnitude.
- **Step 4: Multi-energy (7 energies):** D_{eff} increases to $\sim 2.1\text{--}2.5$. This is the single most effective route.
- **Step 5: KD02 global parameterization (17 parameters):** $D_{\text{eff}} \approx 1.0$ of 17 global parameters. Importantly, this does *not* mean that information is lost relative to Step 4—the data still determine the same local-parameter combinations at each energy. The lower D_{eff} arises because KD02 expresses each local parameter as a polynomial in energy (e.g., $V(E)$ via 4 coefficients), but the data can only pin down the leading 1–2 coefficients; the higher-order terms (v_3, v_4, d_3 , etc.) are redundant. This is analogous to fitting a straight line with a cubic: the slope is well-determined, but the curvature coefficients are arbitrary. The KD02 functional form is over-parametrized.

The key new insight, as the referee anticipated, is precisely this: even after including all scattering-region observables available within 10–200 MeV ($d\sigma/d\Omega$, A_y , σ_T , and σ_R), D_{eff} remains $\sim 2\text{--}2.5$, far below the 11 local parameters. The information limit is intrinsic to the scattering physics. The step-by-step decomposition reveals *which* observables contribute *what*: A_y lifts the spin-orbit degeneracy, multi-energy data lift the imaginary-parameter degeneracy, but single-energy σ_R provides negligible additional constraint.

Physical picture of the Igo ambiguity: The eigenvector analysis (revised Fig. 3) provides a precise characterization of the classical Igo ambiguity. The dominant eigenvector e_1 has V and r_v entering with the *same sign*: both increasing together makes the volume integral $J_V \propto Vr_v^3$ larger, producing a measurably different cross section. This is the “stiff” (well-constrained) direction. The Igo ambiguity is the *orthogonal* “sloppy” direction, where V increases while r_v decreases (or vice versa) so that J_V stays approximately constant—the data cannot distinguish a deep, narrow potential from a shallow, wide one. In short, there are two directions in the V - r_v plane: the data pin down the volume integral (stiff), but cannot resolve the depth-radius tradeoff (sloppy). This eigenvalue ratio exceeds 10^5 , making the tradeoff direction effectively unmeasurable.

The computational infrastructure now includes:

- Full spin-1/2 scattering solver (`solve_spin_half()`) computing (l, j) -resolved S -matrix, validated against FRESCO ($|\Delta S| < 0.002$)
- Observables module computing $f(\theta)$, $g(\theta)$, $d\sigma/d\Omega$, $A_y = 2\text{Im}(fg^*)/(|f|^2 + |g|^2)$, σ_R , and σ_T
- Separate gradient matrices for each observable type, enabling the additive Fisher decomposition

(2) Fisher analysis methodology concerns

The single-energy Fisher analysis does not capture (i) whether a parameter is required for globally acceptable fits, or (ii) whether a parameter is needed to reproduce energy trends.

Diffuseness is primarily constrained through energy systematics, integrated observables, and correlations across energies.

We now directly address both points:

(i) **Multi-energy combined Fisher analysis:** The combined Fisher matrix $F_{\text{combined}} = \sum_E F(E)$ captures the information from measurements across 7 energies (10–200 MeV). Combining elastic data alone across energies raises D_{eff} from ~ 1.2 to ~ 2.1 – 2.3 (new Fig. 4, Step 4). The energy lever arm is the key: at low energies the real potential dominates, while at high energies absorption becomes stronger, probing orthogonal parameter combinations. The imaginary subgroup D_{eff} increases from ~ 1.1 (single energy) to ~ 2.3 (7 energies).

(ii) **KD02 global systematics:** We compute the Jacobian $J = \partial\boldsymbol{\theta}_{\text{local}}/\partial\boldsymbol{\theta}_{\text{global}}$ relating the 11 local parameters per energy to the 17 global KD02 parameters, and project the multi-energy Fisher matrix onto the global space: $F_{\text{global}} = J^T F_{\text{combined}} J$. The result ($D_{\text{eff}} \approx 1.0$ of 17 parameters) reveals that the polynomial functional forms in KD02 introduce additional degeneracies beyond those inherent in the scattering physics.

(iii) **Full multi-system global analysis:** To address the concern that a systematic optical potential should be evaluated across *all* systems rather than a single nucleus, we have extended the analysis to the **full universal KD02 coefficient space**. The KD02 global potential parametrizes all local potential parameters as functions of mass A , charge Z , and energy E through ~ 45 universal coefficients (e.g., $v_1 = 59.30 - 21.0(N - Z)/A - 0.024A$). We compute the Jacobian from these 45 universal parameters to the 11 local parameters for each of the 168 (nucleus, energy, projectile) configurations, and sum the transformed Fisher matrices:

$$F_{\text{universal}} = \sum_{i=1}^{168} J_i^T F_i^{\text{local}} J_i.$$

The result is striking: $D_{\text{eff}} = 2.0$ out of 45 universal parameters. The dominant eigenvector (69% of total information) is the real potential radius coefficient r_{v0} , followed by the surface imaginary radius r_{d0} (12%). Adding nuclei saturates D_{eff} after ~ 5 systems; the improvement from going from one nucleus to 24 (nucleus, projectile) pairs is only a factor of ~ 1.2 . This confirms that the information limit is not a single-system artifact but an intrinsic feature of how nucleon-nucleus scattering encodes potential information.

We also performed the global analysis using only elastic scattering data (no A_y , σ_R , or σ_T), yielding $D_{\text{eff}} = 2.5/45$ —slightly *higher* than the all-observable result. This counterintuitive result arises because A_y data strongly reinforce the already-constrained “stiff” directions (individual eigenvalues increase by factors of 300–500), making the eigenvalue spectrum more peaked and thus lowering the participation ratio. The total Fisher information increases by orders of magnitude, but concentrates along already-stiff directions rather than opening new ones. In both cases, only ~ 2 effective parameter combinations are constrained.

The per-nucleus analysis in the universal space shows $D_{\text{eff}} = 1.3$ – 3.2 (all 7 energies combined), with lighter nuclei like ^{12}C having higher values. These results are presented in the revised End Matter with a new figure (Fig. 5) showing the D_{eff} saturation, eigenvalue spectrum, and eigenvector composition in the 45-dimensional universal coefficient space.

Diffuseness: The referee’s specific concern is validated by our analysis. At a single energy, diffuseness parameters (a_v , a_w , a_d) contribute $< 1\%$ to the dominant eigenvectors

(Fig. 3). With multi-energy data, the imaginary subgroup (which includes a_w and a_d) becomes better constrained ($D_{\text{eff}}^{\text{imag}}$: $1.1 \rightarrow 2.3$). This quantitatively confirms the referee's statement that diffuseness is primarily constrained through energy systematics.

Specific suggestion: Plot angle-resolved sensitivity to diffuseness $S_a(\theta)$ —backward angle effects may be washed out when summing over all angles.

Done. The new Fig. 7 shows $|S_i(\theta)|$ for all 11 parameters. The referee's intuition is confirmed: diffuseness parameters a_v , a_w , and a_d become most distinguishable at backward angles ($\theta > 130^\circ$), where their sensitivity curves diverge from the depth/radius parameters. The cumulative $D_{\text{eff}}(\theta_{\text{max}})$ shows that $\sim 90\%$ of the total constraining power is captured by $\theta \approx 140^\circ$. Backward-angle measurements are disproportionately valuable for breaking the parameter degeneracy—a practical recommendation for experimental design. The Fisher analysis does not deny the physical importance of diffuseness parameters, but quantifies that their information content comes primarily from backward angles combined with multi-energy data—precisely the regime where the referee's insight applies.

(3) Path to PRL-level impact

A comprehensive Fisher analysis should include at least (i) reaction cross sections and (ii) energy systematics.

Both are now included. Additionally, we provide a new element not anticipated in the original submission: a detailed comparison with the Bayesian analysis of King *et al.* [PRL **122**, 232502 (2019)]. This comparison (End Matter of the revised manuscript) demonstrates that:

1. The Fisher information matrix provides a **rigorous theoretical foundation** for the empirical findings of King *et al.*: in the Gaussian approximation, the posterior covariance equals F^{-1} , so our $D_{\text{eff}} \approx 1.7$ directly explains why only V and r_v showed strong correlation in their MCMC scatter plots. Specifically, V and r_v both enter the dominant eigenvector \mathbf{e}_1 with the same sign (the volume integral direction), producing the elongated correlation structure observed in the MCMC posterior. The orthogonal direction (V and r_v trading off at constant J_V) is the “sloppy” Igo direction with an eigenvalue 10^5 times smaller.
2. The FIM eigenvalue hierarchy (condition number $> 10^7$) predicts the elongated “cigar-shaped” Bayesian posterior and explains why the frequentist χ^2 approach produced spurious correlations.
3. The Fisher approach is $\sim 4000 \times$ more efficient ($2N_p + 1 = 23$ evaluations vs. $\sim 100,000$ MCMC samples), enabling the systematic 168-configuration scan that would be prohibitively expensive with MCMC.
4. The Fisher analysis yields provably scale-invariant quantities (D_{eff}), a fundamental property difficult to establish empirically from finite MCMC chains.

The two approaches are complementary: Fisher efficiently maps global constraint structure and proves fundamental properties, while Bayesian MCMC captures non-Gaussian posterior features. This positions the present work as the theoretical counterpart to the Bayesian

UQ program of the MSU group [6–11], establishing a unified framework for understanding parameter constraints in nuclear reactions.

Minor Points

(A) Abstract framing

First sentence is misleading—the actual scope is narrower.

Revised. The abstract now opens with “How strongly can nucleon-nucleus scattering data constrain optical model potential parameters?” and explicitly describes the step-by-step decomposition including all observable types and multi-energy analysis.

(B) PCA undefined

“PCA” used without definition.

Fixed: “principal component analysis (PCA) studies” (Introduction, paragraph 3).

(C) Conclusion on nuclear forces

The link between “sloppy” parameters and chiral nuclear forces needs refinement.

Revised. The Conclusion now states: “The ‘sloppy’ parameter directions do not imply that these aspects of the potential are unconstrained by all data—rather, they are insensitive to single-energy scattering measurements.” The multi-energy analysis (Fig. 4) demonstrates that combining data across energies probes orthogonal directions in parameter space, providing a constructive path forward.

Summary of Changes

1. Extended from 9-parameter central potential to 11-parameter model including spin-orbit coupling (V_{so} , W_{so})
2. Added spin-1/2 scattering solver with (l, j) -resolved S -matrix, validated against FRESCO
3. Added observables: A_y , σ_R , σ_T (in addition to $d\sigma/d\Omega$)
4. New step-by-step constraint analysis decomposing the information contribution of each observable and multi-energy data (new Fig. 4)
5. New angle-resolved sensitivity analysis showing $|S_i(\theta)|$ for all 11 parameters (new Fig. 7)
6. KD02 global parameter analysis: per-nucleus (17 parameters) and full universal coefficient space (45 parameters, 168 systems combined) via Jacobian projection
7. Detailed Fisher–Bayesian comparison in End Matter, connecting to King *et al.* PRL (2019)

8. Revised abstract, introduction, discussion, and conclusion
9. All calculations use Numerov solver with finite-difference gradients (no neural network dependencies)